Theorem [Equivalence Preservation]

Contextual equivalence is preserved via translation.

If $\Gamma \vdash e \cong^{ctx} e' : \tau$ and $\Gamma \vdash e : \tau \rightsquigarrow e^+$

$\Gamma \vdash e' : \tau \rightsquigarrow e'^+$

where $\Gamma^+ \vdash e^+ : E^0 \tau^+$

$\Gamma^+ \vdash e'^+ : E^0 \tau^+$,

then $\cdot ; \Gamma^+ \vdash e^+ \cong^{ctx} e'^+ : E^0 \tau^+$.

proof. By $[\cong^{ctx} \Leftrightarrow \approx_{\beta\eta}]$, $\Gamma \vdash e \cong^{ctx} e' : \tau$ iff $\Gamma \vdash e \approx_{\beta\eta} e' : \tau$.

By "logical relation soundness / completeness" w.r.t contextual equivalence"

$\Delta ; \Gamma \vdash e \simeq^{lr} e' : \sigma$ iff $\Delta ; \Gamma \vdash e \cong^{ctx} e' : \sigma$

So if we show $\Gamma \vdash e \cong^{axm} e' : \tau$, by transitivity

$$\begin{array}{ccccc} \Gamma \vdash e & \cong^{axm} & e' : \tau & & \\ \downarrow^{+} & & \downarrow^{+} & \downarrow^{+} & \downarrow^{\div} \\ \cdot ; \Gamma^+ \vdash e^+ & \simeq^{lr} & e'^+ : E^0 \tau^+ & & \end{array}$$

we've shown

$$\begin{array}{ccccc} \Gamma \vdash e & \cong^{ctx} & e' : \tau & & \\ \downarrow^{+} & & \downarrow^{+} & \downarrow^{+} & \downarrow^{\div} \\ \cdot ; \Gamma^+ \vdash e^+ & \cong^{ctx} & e'^+ : E^0 \tau^+ & & \end{array}$$

By [$\approx_{\beta\eta}$ implies $\simeq^{lr}$] lemma below,

we showed $\Gamma \vdash e \approx_{\beta\eta} e' : \tau \Longrightarrow \cdot ; \Gamma^+ \vdash e^+ \simeq^{lr} e'^+ : E^0 \tau^+$.

$\square$

Axiomatic equivalence implies logical relation.

$\Gamma \vdash e \simeq^{axm} e' : \tau \quad \Rightarrow \quad \cdot ; \Gamma^+ \vdash e^+ \simeq^{lr} e'^+ : E^0 \tau^+$

proof.

Case $\dfrac{\Gamma, x:\tau \vdash e_1 \simeq^{axm} e_2 : \tau'}{\Gamma \vdash \lambda x:\tau.e_1 \simeq^{axm} \lambda x:\tau.e_2 : \tau \to \tau'}$:

We need to show

$\cdot ; \Gamma^+ \vdash \lambda x:\tau^+.e_1^+ \simeq^{lr} \lambda x:\tau^+.e_2^+ : E^0(\tau^+ \to E^0 \tau'^+)$

1. $\cdot ; \Gamma^+ \vdash \lambda x:\tau^+.e_1^+ \leq \lambda x:\tau^+.e_2^+ : E^0(\tau^+ \to E^0 \tau'^+)$
2. $\cdot ; \Gamma^+ \vdash \lambda x:\tau^+.e_2^+ \leq \lambda x:\tau^+.e_1^+ : E^0(\tau^+ \to E^0 \tau'^+)$

prove 1.

Consider arbitrary $(\gamma, \gamma') \in \mathcal{G}_k[\![\Gamma^+]\!]_\phi$ for $k \geq 0$.

We need to show $(\gamma(\lambda x:\tau^+.e_1^+), \gamma'(\lambda x:\tau^+.e_2)) \in \mathcal{E}_k[\![E^0(\tau^+ \to E^0 \tau'^+)]\!]_\phi$.

$\equiv (\lambda x:\tau^+.\gamma(e_1^+), \lambda x:\tau^+.\gamma(e_1^+)) \in \mathcal{E}_k[\![E^0(\tau^+ \to E^0 \tau'^+)]\!]_\phi$

need to show $\exists$ e_f, e_f' such that

$\lambda x:\tau^+.\gamma(e_1^+) \longmapsto^* e_f$

$\lambda x:\tau^+.\gamma'(e_2^+) \longmapsto^* e_f'$

$(e_f, e_f') \in \mathcal{V}_k[\![\tau^+ \to E^0 \tau'^+]\!]_\phi$

Since $\lambda x:\tau^+.\gamma(e_1^+)$ and $\lambda x:\tau^+.\gamma'(e_2^+)$ are already values

let $e_f = \lambda x:\tau^+.\gamma(e_1^+)$ and $e_{f'} = \lambda x:\tau^+.\gamma'(e_2^+)$.

need to show $(\lambda x:\tau^+.\gamma(e_1^+), \lambda x:\tau^+.\gamma'(e_2^+)) \in \mathcal{V}_k[\![\tau^+ \to E^0\tau'^+]\!]\phi$

Pick arbitrary $j \le k$, $(v_1, v_2) \in \mathcal{V}_j[\![\tau^+]\!]\phi$.

need to show $(\gamma(e_1^+)[v_1/x], \gamma'(e_2^+)[v_2/x]) \in \mathcal{E}_j[\![E^0\tau'^+]\!]\phi$

The I.H. says

$$\Gamma, x:\tau \vdash e_1 \simeq^{axm} e_2 : \tau' \Rightarrow \Gamma^+, x:\tau^+ \vdash e_1^+ \simeq^{lr} e_2^+ : E^0\tau'^+$$

Hence, $\Gamma^+, x:\tau^+ \vdash e_1^+ \le e_2^+ : E^0\tau'^+$.

Instantiate this with $j, \gamma[x \mapsto v_1], \gamma'[x \mapsto v_2]$.

Note that $j \ge 0$ and $(\gamma[x \mapsto v_1], \gamma'[x \mapsto v_2]) \in \mathcal{G}_j[\![\Gamma^+, x:\tau^+]\!]\phi$,

which follows from

$(\gamma, \gamma') \in \mathcal{G}_j[\![\Gamma^+]\!]\phi$ and $(v_1, v_2) \in \mathcal{V}_j[\![\tau^+]\!]\phi$.

which follows from applying [Downward-closed] lemma to $(\gamma, \gamma') \in \mathcal{G}_k[\![\Gamma^+]\!]\phi$

Hence, $(\gamma[x \mapsto v_1](e_1^+), \gamma'[x \mapsto v_2](e_2^+)) \in \mathcal{E}_j[\![E^0\tau'^+]\!]\phi$.

$\equiv (\gamma(e_1^+)[v_1/x], \gamma'(e_2^+)[v_2/x]) \in \mathcal{E}_j[\![E^0\tau'^+]\!]\phi$.

Proof of 2 goes similar with

$$\Gamma \vdash \lambda x:\tau.e_1 \approx_{\beta\eta} \lambda x:\tau.e_2 : \tau \to \tau' \Rightarrow$$

$$\frac{\Gamma, x:\tau \vdash e_2 \approx_{\beta\eta} e_1 : \tau'}{\Gamma \vdash \lambda x:\tau.e_2 \approx_{\beta\eta} \lambda x:\tau.e_1 : \tau \to \tau'}$$

Case $\dfrac{\Gamma \vdash e_1 \approx_{\beta\eta} e_1' : \tau_1 \to \tau_2 \qquad \Gamma \vdash e_2 \approx_{\beta\eta} e_2' : \tau_1}{\Gamma \vdash e_1 e_2 \approx_{\beta\eta} e_1' e_2' : \tau_2}$:

proof. We need to show $\Gamma^+ \vdash e_1^+ e_2^+ \simeq^{er} e_1'^+ e_2'^+ : E^0 \tau_2^+$.

1. $\because \Gamma^+ \vdash e_1^+ e_2^+ \leq e_1'^+ e_2'^+ : E^0 \tau_2^+$
2. $\because \Gamma^+ \vdash e_1'^+ e_2'^+ \leq e_1^+ e_2^+ : E^0 \tau_2^+$

prove 1.

Consider arbitrary $(\gamma, \gamma') \in \mathcal{G}_k[\![\Gamma^+]\!]_\phi$ for $k \geq 0$.

We need to show $(\gamma(e_1^+ e_2^+), \gamma'(e_1'^+ e_2'^+)) \in \mathcal{E}_k[\![E^0 \tau_2^+]\!]_\phi$.

$\equiv (\gamma(e_1^+)\gamma(e_2^+), \gamma'(e_1'^+)\gamma'(e_2'^+)) \in \mathcal{E}_k[\![E^0 \tau_2^+]\!]_\phi$

need to show $\exists$ e_f, e_f' such that

- $\gamma(e_1^+)\gamma(e_2^+) \xmapsto{*} e_f$
- $\gamma'(e_1'^+)\gamma'(e_2'^+) \xmapsto{*} e_f'$
- $(e_f, e_f') \in \mathcal{V}_k[\![\tau_2^+]\!]_\phi$

The I.H. (1) $\Gamma \vdash e_1 \approx_{\beta\eta} e_1' : \tau_1 \to \tau_2 \Rightarrow \because \Gamma^+ \vdash e_1 \simeq_{er} e_1'^+ : E^0(\tau_1^+ \to E^0 \tau_2^+)$

The I.H. (2) $\Gamma \vdash e_2 \approx_{\beta\eta} e_2' : \tau_2 \Rightarrow \because \Gamma^+ \vdash e_2 \simeq_{er} e_2'^+ : E^0 \tau^+$

Instantiate $\because \Gamma^+ \vdash e_1^+ \leq e_1'^+ : E^0(\tau_1^+ \to E^0 \tau_2^+)$ with k, γ, γ'

noting $k \geq 0$ and $(\gamma, \gamma') \in \mathcal{G}_k[\![\Gamma^+]\!]_\phi$.

Then $(\gamma(e_1^+), \gamma'(e_1'^+)) \in \mathcal{E}_k[\![E^0(\tau_1^+ \to E^0 \tau_2^+)]\!]_\phi$

Hence, there exist e_{f_1}, e_{f_1}' s.t.

- $\gamma(e_1^+) \xmapsto{*} e_{f_1}$
- $\gamma'(e_1'^+) \xmapsto{*} e_{f_1}'$
- $(e_{f_1}, e_{f_1}') \in \mathcal{V}_k[\![\tau_1^+ \to E^o \tau_2^+]\!]\phi$

Let $e_{f_1} = \lambda x : \tau_1^+ . e_1$ and $e_{f_1}' = \lambda x : \tau_1^+ . e_2$.

Similarly, instantiate $\cdot ; \Gamma^+ \vdash e_2^+ \leq e_2'^+ : E^o \tau_1^+$ with k, γ, γ'.

Then $(\gamma(e_2^+), \gamma'(e_2'^+)) \in \mathcal{E}_k[\![E^o \tau_1^+]\!]\phi$

Hence, there exist e_{f_2}, e_{f_2}' s.t.

- $\gamma(e_2^+) \xmapsto{*} e_{f_2}$
- $\gamma'(e_2'^+) \xmapsto{*} e_{f_2}'$
- $(e_{f_2}, e_{f_2}') \in \mathcal{V}_k[\![\tau_1^+]\!]\phi$

Let $e_{f_2} = v_2$ and $e_{f_2}' = v_2'$.

Thus so far we have

- $$\begin{aligned} \gamma(e_1^+)\gamma(e_2^+) &\xmapsto{*} e_{f_1}\, \gamma(e_2^+) \\ &\equiv (\lambda x : \tau_1^+ . e_1)\, \gamma(e_2^+) \\ &\xmapsto{*} (\lambda x : \tau_1^+ . e_1)\, e_{f_2} \\ &\equiv (\lambda x : \tau_1^+ . e_1)\, v_2 \\ &\mapsto e_1[v_2/x] \end{aligned}$$

- $$\begin{aligned} \gamma'(e_1'^+)\gamma'(e_2'^+) &\xmapsto{*} e_{f_1}'\, \gamma'(e_2^+) \\ &\equiv (\lambda x : \tau_1^+ . e_2)\, \gamma'(e_2^+) \\ &\xmapsto{*} (\lambda x : \tau_1^+ . e_2)\, e_{f_2}' \\ &\equiv (\lambda x : \tau_1^+ . e_2)\, v_2' \\ &\mapsto e_2[v_2/x] \end{aligned}$$

Instantiate $(e_{f_1}, e_{f_1}') \in \mathcal{V}_k[\![\tau_1^+ \to E^0\tau_2^+]\!]\phi$

$$\begin{array}{ccc} \| & & \| \\ (\lambda x:\tau_1^+.\, e_1, & & \lambda x:\tau_1^+.\, e_2) \end{array} \in \mathcal{V}_k[\![\tau_1^+ \to E^0\tau_2^+]\!]\phi \text{ with } k, v_2, v_2'$$

Note that

- $k \leq k$
- $(v_2, v_2') \in \mathcal{V}_k[\![\tau_1^+]\!]\phi$

Hence, $(e_1[v_2/x], e_2[v_2'/x]) \in \mathcal{E}_k[\![E^0\tau_2^+]\!]\phi$

It follows that there exist e_{f_3}, e_{f_3}' such that

$$e_1[v_2/x] \longmapsto^* e_{f_3}$$

$$e_2[v_2'/x] \longmapsto^* e_{f_3}'$$

$$(e_{f_3}, e_{f_3}') \in \mathcal{V}_k[\![\tau_2^+]\!]\phi.$$

Let $e_f = e_{f_3}$ and $e_f = e_{f_3}'$, then we are done.

Case $\dfrac{}{\Gamma \vdash true \approx_{pq} true : bool}$:

proof. We need to show $\cdot ; \Gamma^+ \vdash true^+ \simeq^{lr} true^+ : E^0 bool^+$.

So it suffices to show

$$\cdot ; \Gamma^+ \vdash true^+ \leq true^+ : E^0 bool^+$$

$$\equiv \cdot ; \Gamma^+ \vdash \underset{\shortparallel}{true} \leq \underset{\shortparallel}{true} : \underset{\shortparallel}{E^0 bool}$$

Consider arbitrary $(\gamma, \gamma') \in \mathcal{G}_k[\![\Gamma^+]\!]_\phi$ for $k \geq 0$.

We need to show $(\gamma(true), \gamma'(true)) \in \mathcal{E}_k[\![E^0 bool]\!]_\phi$.

$$\equiv (true, true) \in \mathcal{E}_k[\![E^0 bool]\!]_\phi$$

Need to show $\exists$ $e_f, e_{f'}$ such that

- $true \xrightarrow{*} e_f$
- $true \xrightarrow{*} e_{f'}$
- $(e_f, e_{f'}) \in \mathcal{V}_k[\![bool]\!]_\phi$

Let $e_f = true$ and $e_{f'} = true$, then we are done.

Case $\dfrac{}{\Gamma \vdash false \approx_{pq} false : bool}$: Similar to true case.

Case _____ :

$$\Gamma, x:\tau \vdash x \approx_{\beta\eta} x : \tau$$

proof. We need to show $\Gamma^+, x:\tau^+ \vdash x^+ \simeq^{er} x^+ : E^\circ\tau^+$.

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \qquad\quad \|$$

It suffices to show $\Gamma^+, x:\tau^+ \vdash x \leq x : E^\circ\tau^+$.

Consider arbitrary $(\gamma[x\mapsto v_1], \gamma'[x\mapsto v_2]) \in \mathcal{G}_k[\![\Gamma^+, x:\tau^+]\!]\phi$ for $k \geq 0$.

Note that this mean $(\gamma, \gamma') \in \mathcal{G}_k[\![\Gamma^+]\!]\phi$ and $(v_1, v_2) \in \mathcal{V}_k[\![\tau^+]\!]\phi$

We need to show $(\gamma[x\mapsto v_1](x), \gamma'[x\mapsto v_2](x)) \in \mathcal{E}_k[\![E^\circ\tau^+]\!]\phi$.

$$\equiv (v_1, v_2) \in \mathcal{E}_k[\![E^\circ\tau^+]\!]\phi$$

need to show $\exists$ $e_f, e_{f'}$ such that

- $v_1 \xmapsto{*} e_f$
- $v_2 \xmapsto{*} e_{f'}$
- $(e_f, e_{f'}) \in \mathcal{V}_k[\![\tau^+]\!]\phi$

Let $e_f = v_1$ and $e_{f'} = v_2$.

Since $(v_1, v_2) \in \mathcal{V}_k[\![\tau^+]\!]\phi$, we are done.

case $\dfrac{\Gamma \vdash e_0 \approx_{\beta\eta} e_0' : \mathrm{bool} \quad \Gamma \vdash e_1 \quad e_1' : \tau \quad \Gamma \vdash e_2 \approx_{\beta\eta} e_2' : \tau}{\Gamma \vdash \mathrm{if}\ e_0\ e_1\ e_2 \approx_{\beta\eta} \mathrm{if}\ e_0'\ e_1'\ e_2' : \tau}$:

proof. We need to show $\cdot ; \Gamma^+ \vdash \mathrm{if}\ e_0^+\ e_1^+\ e_2^+ \simeq^{lr} \mathrm{if}\ e_0'^+\ e_1'^+\ e_2'^+ : E^\circ \tau^+$.

This requires us to show

1. $\cdot ; \Gamma^+ \vdash \mathrm{if}\ e_0^+\ e_1^+\ e_2^+ \leq \mathrm{if}\ e_0'^+\ e_1'^+\ e_2'^+ : E^\circ \tau^+$

2. $\cdot ; \Gamma^+ \vdash \mathrm{if}\ e_0'^+\ e_1'^+\ e_2'^+ \leq \mathrm{if}\ e_0^+\ e_1^+\ e_2^+ : E^\circ \tau^+$

Will prove 1. first.

Pick arbitrary $k \geq 0$, $(\gamma, \gamma') \in \mathcal{G}_k[\![\Gamma^+]\!]\phi$.

Need to show $(\gamma(\mathrm{if}\ e_0^+\ e_1^+\ e_2^+), \gamma'(\mathrm{if}\ e_0'^+\ e_1'^+\ e_2'^+)) \in \mathcal{E}_k[\![E^\circ \tau^+]\!]\phi$.

$\equiv (\mathrm{if}\ \gamma(e_0^+)\ \gamma(e_1^+)\ \gamma(e_2^+), \mathrm{if}\ \gamma'(e_0'^+)\ \gamma'(e_1'^+)\ \gamma'(e_2'^+)) \in \mathcal{E}_k[\![E^\circ \tau^+]\!]\phi$.

Need to show $\exists$ e_f, e_f' such that

- $\mathrm{if}\ \gamma(e_0^+)\ \gamma(e_1^+)\ \gamma(e_2^+) \longmapsto^* e_f$
- $\mathrm{if}\ \gamma'(e_0'^+)\ \gamma'(e_1'^+)\ \gamma'(e_2'^+) \longmapsto^* e_f'$
- $(e_f, e_f') \in \mathcal{V}_k[\![\tau^+]\!]\phi$

The I.H. says

(1) $\Gamma \vdash e_0 \approx_{\beta\eta} e_0' : \mathrm{bool} \implies \cdot ; \Gamma^+ \vdash e_0^+ \simeq^{lr} e_0'^+ : E^\circ \mathrm{bool}^+$

(2) $\Gamma \vdash e_1 \approx_{\beta\eta} e_1' : \tau \implies \cdot ; \Gamma^+ \vdash e_1^+ \simeq^{lr} e_1'^+ : E^\circ \tau^+$

(3) $\Gamma \vdash e_2 \approx_{\beta\eta} e_2' : \tau \implies \cdot ; \Gamma^+ \vdash e_2^+ \simeq^{lr} e_2'^+ : E^\circ \tau^+$

From (1), $\cdot ; \Gamma^+ \vdash e_0^+ \leq e_0'^+ : E^0 bool^+$.

Instantiate this with $k, (\gamma, \gamma')$ noting $k \geq 0$, $(\gamma, \gamma') \in \mathcal{G}_k[\![\Gamma^+]\!]\phi$.

So $(\gamma(e_0^+), \gamma'(e_0'^+)) \in \mathcal{E}_k[\![E^0 bool^+]\!]\phi$.

So $\exists$ e_{f_0}, e_{f_0}' s.t.

- $\gamma(e_0^+) \longmapsto^* e_{f_0}$
- $\gamma'(e_0'^+) \longmapsto^* e_{f_0}'$
- $(e_{f_0}, e_{f_0}') \in \mathcal{V}_k[\![\overset{bool^+}{\overset{\shortparallel}{bool}}]\!]\phi$

So $e_{f_0} = e_{f_0}' = true$ or $e_{f_0} = e_{f_0}' = false$.

<u>Case</u> $e_{f_0} = e_{f_0}' = true$:

Note that $\text{if } \gamma(e_0^+)\, \gamma(e_1^+)\, \gamma(e_2^+) \longmapsto^* \text{if true } \gamma(e_1^+)\, \gamma(e_2^+)$

$\longmapsto \gamma(e_1^+)$

$\text{if } \gamma'(e_0'^+)\, \gamma'(e_1'^+)\, \gamma'(e_2'^+) \longmapsto^* \text{if true } \gamma(e_1'^+)\, \gamma(e_2'^+)$

$\longmapsto \gamma(e_1'^+)$

We need to show

- $\gamma(e_1^+) \longmapsto^* e_f$
- $\gamma(e_1'^+) \longmapsto^* e_{f'}$
- $(e_f, e_{f'}) \in \mathcal{V}_k[\![\tau^+]\!]\phi$

From I.H (2), $\cdot ; \Gamma^+ \vdash e_1^+ \leq e_1'^+ : E^0 \tau^+$.

Instantiate this with k, γ, γ' noting

- $k \geq 0$
- $(\gamma, \gamma') \in \mathcal{G}_k[\![\Gamma^+]\!]\phi$

So $(\gamma(e_1^+), \gamma'(e_1'^+)) \in \mathcal{E}_k[\![E^0\tau^+]\!]\phi$

So $\exists$ e_{f_1}, e_{f_1}' such that

- $\gamma(e_1^+) \xmapsto{*} e_{f_1}$
- $\gamma'(e_1'^+) \xmapsto{*} e_{f_1}'$
- $(e_{f_1}, e_{f_1}') \in \mathcal{V}_k[\![\tau^+]\!]\phi$

Let $e_f = e_{f_1}$ and $e_f' = e_{f_1}'$, then we are done.

<u>Case</u> $e_{f_0} = e_{f_0}' = \text{false}$:

Note that $\text{if } \gamma(e_0^+)\, \gamma(e_1^+)\, \gamma(e_2^+) \xmapsto{*} \text{if false } \gamma(e_1^+)\, \gamma(e_2^+)$

$\mapsto \gamma(e_2^+)$

$\text{if } \gamma'(e_0'^+)\, \gamma'(e_1'^+)\, \gamma'(e_2'^+) \xmapsto{*} \text{if false } \gamma(e_1'^+)\, \gamma(e_2'^+)$

$\mapsto \gamma(e_2'^+)$

We need to show

- $\gamma(e_2^+) \mapsto^* e_f$
- $\gamma(e_2'^+) \mapsto^* e_f'$
- $(e_f, e_f') \in \mathcal{V}_k[\![\tau^+]\!]\phi$

From I.H (3), $\cdot ; \Gamma^+ \vdash e_2^+ \leq e_2'^+ : E^0\tau^+$.

Instantiate this with k, γ, γ' noting

- $k \geq 0$
- $(\gamma, \gamma') \in \mathcal{G}_k[\![\tau^+]\!]_\phi$

So $(\gamma(e_2^+), \gamma'(e_2'^+)) \in \mathcal{E}_k[\![E^0\tau^+]\!]_\phi$

So $\exists$ e_{f_1}, e_{f_1}' such that

- $\gamma(e_2^+) \overset{*}{\longleftrightarrow} e_{f_2}$
- $\gamma'(e_2'^+) \overset{*}{\longmapsto} e_{f_2}'$
- $(e_{f_2}, e_{f_2}') \in \mathcal{V}_k[\![\tau^+]\!]_\phi$

Let $e_f = e_{f_2}$ and $e_{f'} = e_{f_2}'$, then we are done.