Definition: Contextual Approximation $\lesssim^{ctx}$ and equivalence $\simeq^{ctx}$

Let e and e' be expressions s.t. $\Gamma \vdash e : \sigma$

$\Gamma \vdash e' : \sigma$

$$\Gamma \vdash v \lesssim^{ctx} v' : \tau \overset{def}{=} \forall C, \sigma' \text{ s.t. } \cdot \vdash C : (\Gamma \triangleright \tau) \rightsquigarrow \sigma' \wedge \Gamma \vdash C[v] : \sigma' \Rightarrow (C[v] \Downarrow \Rightarrow C[v'] \Downarrow)$$

$$\Gamma \vdash v \simeq v' : \tau \overset{def}{=} \Gamma \vdash v \lesssim^{ctx} v' : \tau \wedge \Gamma \vdash v' \lesssim^{ctx} v : \tau$$

$$\Gamma \vdash e \lesssim^{ctx} e' : \sigma \overset{def}{=} \forall C, \sigma' \text{ s.t. } \cdot \vdash C : (\Gamma \triangleright \sigma) \rightsquigarrow \sigma' \wedge \Gamma \vdash C[e] : \sigma' \Rightarrow (C[e] \Downarrow \Rightarrow C[e'] \Downarrow)$$

$$\Gamma \vdash e \simeq^{ctx} e' : \sigma \overset{def}{=} \Gamma \vdash e \leq^{ctx} e' : \sigma \wedge \Gamma \vdash e' \leq^{ctx} e : \sigma$$

Definition: Logical relation for Contexts

$$\Gamma_1 \vdash C^v \leq C^{v'} : (\Gamma \triangleright \rho) \rightsquigarrow \tau_1 \overset{def}{=} \forall e, e' \text{ s.t. } \Gamma \vdash e \leq e' : \rho \wedge \Gamma_1 \vdash C^v[e] : \tau_1 \wedge \Gamma_1 \vdash C^v[e] : \tau_1 \Rightarrow \Gamma_1 \vdash C^v[v] \leq_v C^v[v'] : \tau_1$$

$$\Gamma_1 \vdash C^v \simeq^{lr} C^{v'} : (\Gamma \triangleright \rho) \rightsquigarrow \tau_1 \overset{def}{=} \Gamma_1 \vdash C^v \leq C^{v'} : (\Gamma \triangleright \tau) \rightsquigarrow \tau_1 \wedge \Gamma_1 \vdash C^{v'} \leq C^v : (\Gamma \triangleright \tau) \rightsquigarrow \tau_1.$$

$$\Gamma_1 \vdash C \leq C' : (\Gamma \triangleright \rho) \rightsquigarrow \sigma_1 \overset{def}{=} \forall e, e' \text{ s.t. } \Gamma \vdash e \leq_e e' : \rho. \wedge \Gamma_1 \vdash C[e] : \sigma_1 \wedge \Gamma_1 \vdash C'[e'] : \sigma_1 \Rightarrow \Gamma_1 \vdash C[e] \leq_e C[e'] : \sigma_1$$

$$\Gamma_1 \vdash C \simeq^{lr} C' : (\Gamma \triangleright \rho) \rightsquigarrow \sigma_1 \overset{def}{=} \Gamma_1 \vdash C \leq C' : (\Gamma \triangleright \sigma) \rightsquigarrow \sigma_1 \wedge \Gamma_1 \vdash C' \leq C : (\Gamma \triangleright \sigma) \rightsquigarrow \sigma_1$$