Theorem [$\lambda^{1,b,\to,\mu,0,\bullet}$ LR : Soundness w.r.t Contextual Equivalence]

We show $\leq \subseteq \lesssim^{ctx}$.

Lemma [$\lambda^{1,b,\to,\mu,0,\bullet}$ Context Compatibility : Empty]

$$\frac{\Gamma \leq \Gamma_0}{\Gamma_0 \vdash [\cdot] \leq [\cdot] : (\Gamma \triangleright E^{\delta}\tau) \rightsquigarrow E^{\delta}\tau}$$

Proof.

Consider arbitrary e and e' such that

$$\Gamma \vdash e \leq e' : E^{\delta}\tau$$

We are required to show $\Gamma_0 \vdash [e] \leq [e'] : E^{\delta}\tau \equiv \Gamma_0 \vdash e \leq e' : E^{\delta}\tau$

Consider arbitrary k, γ_0, and γ_0' such that

- $k \geq 0$
- $(\gamma_0, \gamma_0') \in \mathcal{G}_k[\![\Gamma_0]\!]$

We are required to show $(\gamma_0(e), \gamma_0'(e')) \in \mathcal{E}_k[\![E^{\delta}\tau]\!]$.

There are two cases to prove.

<u>Pure case</u> : 1. $\delta = 0$

<u>Impure case</u> : 2. $\delta = \bullet$

We prove 1 first and apply the [Pure implies Impure-closed] lemma to the pure case to prove 2.

case $\delta = 0$

Let $\gamma = \gamma_0|_{dom(\Gamma)}$ and $\gamma' = \gamma_0'|_{dom(\Gamma)}$.

Note that

- $(\gamma, \gamma') \in \mathcal{G}_k[\![\Gamma]\!]$, which follows from
  - $(\gamma_0, \gamma_0') \in \mathcal{G}_k[\![\Gamma]\!]$ and $\Gamma \subseteq \Gamma_0$.
- $(\gamma_0(e), \gamma_0'(e')) \in \mathcal{E}_k[\![E^0\tau]\!] \equiv (\gamma(e), \gamma'(e)) \in \mathcal{E}_k[\![E^0\tau]\!]$, which follows from

- $FV(e) \subseteq dom(\Gamma)$ and $FV(e') \subseteq dom(\Gamma)$

Hence, it suffices to show that $(\gamma(e), \gamma'(e)) \in \mathcal{E}_k[\![E^{\circ}\tau]\!]$.

Instantiate $\Gamma \vdash e \leq e' : E^{\circ}\tau$ with k, γ, γ'.

Note that

- $k \geq 0$
- $(\gamma, \gamma') \in \mathcal{G}_k[\![\Gamma]\!]$

Thus, $(\gamma(e), \gamma'(e')) \in \mathcal{E}_k[\![E^{\circ}\tau]\!]$.

Case $\delta = \bullet$

By the [pure implies impure-closed] applied to $(\gamma(e), \gamma'(e')) \in \mathcal{E}_k[\![E^{\circ}\tau]\!]$, we have $(\gamma(e), \gamma'(e')) \in \mathcal{E}_k[\![E^{\bullet}\tau]\!]$.

Lemma [$\lambda^{1,b_1\to,\mu,\circ,\bullet}$ Context Compatibility : If Cee]

$$\frac{\Gamma_0 \vdash C \leq C' : (\Gamma \triangleright E^{\delta}\tau) \rightsquigarrow E^{\delta_0}\mathrm{bool} \qquad \Gamma_0 \vdash e_1 \leq e_1' : E^{\delta_1}\tau' \qquad \Gamma_0 \vdash e_2 \leq e_2' : E^{\delta_2}\tau'}{\Gamma_0 \vdash \mathrm{if}\ C\ e_1\ e_2 \leq \mathrm{if}\ C'\ e_1'\ e_2' : (\Gamma \triangleright E^{\delta}\tau) \rightsquigarrow E^{\delta_0 \wedge \delta_1 \wedge \delta_2}\tau'}$$

Proof.

Consider arbitrary e and e' such that

$$\Gamma \vdash e \leq_e e' : E^{\delta}\tau$$

We are required to show $\Gamma_0 \vdash \mathrm{if}\ C[e]\ e_1\ e_2 \leq \mathrm{if}\ C'[e']\ e_1'\ e_2' : E^{\delta_0 \wedge \delta_1 \wedge \delta_2}\tau'$.

Instantiate $\Gamma_0 \vdash C \leq C' : (\Gamma \triangleright E^{\delta}\tau) \rightsquigarrow E^{\delta_0}\mathrm{bool}$ with e, e' noting that

$$\Gamma \vdash e \leq_e e' : E^{\delta}\tau.$$

We have $\Gamma_0 \vdash C[e] \leq C'[e'] : E^{\delta_0}\mathrm{bool}$

Applying the [Compatibility-If] lemma to

- $\Gamma_0 \vdash C[e] \leq_e C'[e'] : E^{\delta_0}\mathrm{bool}$

- $\Gamma_0 \vdash e_1 \leq_e e_1' : E^{\delta_1}\tau'$
- $\Gamma_0 \vdash e_2 \leq_e e_2' : E^{\delta_2}\tau'$

We conclude that $\Gamma_0 \vdash \text{if } C[e]\ e_1\ e_2 \leq \text{if } C'[e']\ e_1'\ e_2' : E^{\delta_0 \wedge \delta_1 \wedge \delta_2}\tau'$.

Lemma [$\lambda^{1,b_1 \to, \mu, \circ, \bullet}$ Context Compatibility : If eCe]

$$\frac{\Gamma_0 \vdash e_0 \leq e_0' : E^{\delta_0}\text{bool} \quad \Gamma_0 \vdash C \leq C' : (\Gamma \triangleright E^{\delta}\tau) \rightsquigarrow E^{\delta_1}\tau' \quad \Gamma_0 \vdash e_2 \leq e_2' : E^{\delta_2}\tau'}{\Gamma_0 \vdash \text{if } e_0\ C\ e_2 \leq \text{if } e_0'\ C'\ e_2' : (\Gamma \triangleright E^{\delta}\tau) \rightsquigarrow E^{\delta_0 \wedge \delta_1 \wedge \delta_2}\tau'}$$

Proof.

Consider arbitrary e, e' such that

- $\Gamma \vdash e \leq_e e' : E^{\delta}\tau$

We are required to show

$\Gamma_0 \vdash \text{if } e_0\ C[e]\ e_2 \leq \text{if } e_0'\ C'[e']\ e_2' : E^{\delta_0 \wedge \delta_1 \wedge \delta_2}\tau'$.

Instantiate $\Gamma_0 \vdash C \leq C' : (\Gamma \triangleright E^{\delta}\tau) \rightsquigarrow E^{\delta_1}\tau'$ with e and e' noting

$\Gamma \vdash e \leq_e e' : E^{\delta}\tau$.

Hence, $\Gamma_0 \vdash C[e] \leq C'[e'] : E^{\delta_1}\tau'$.

Applying the [Compatibility - If] lemma to

- $\Gamma_0 \vdash e_0 \leq e_0' : E^{\delta_0}\text{bool}$
- $\Gamma_0 \vdash C[e] \leq C'[e'] : E^{\delta_1}\tau'$
- $\Gamma_0 \vdash e_2 \leq_e e_2' : E^{\delta_2}\tau'$

We conclude that $\Gamma_0 \vdash \text{if } e_0\ C[e]\ e_2 \leq \text{if } e_0'\ C'[e']\ e_2' : E^{\delta_0 \wedge \delta_1 \wedge \delta_2}\tau'$.

Lemma [$\lambda^{!,b,\to,\mu,\circ,\bullet}$ Context Compatibility : If $e \in C$]

$$\frac{\Gamma_0 \vdash e_0 \leq e_0' : E^{\delta_0} bool \qquad \Gamma_0 \vdash e_1 \leq e_1' : E^{\delta_1}\tau' \qquad \Gamma_0 \vdash C \leq C' : (\Gamma \triangleright E^{\delta}\tau) \rightsquigarrow E^{\delta_2}\tau'}{\Gamma_0 \vdash \text{if } e_0\, e_1\, C \leq \text{if } e_0\, e_1\, C : (\Gamma \triangleright E^{\delta}\tau) \rightsquigarrow E^{\delta_0 \wedge \delta_1 \wedge \delta_2}\tau'}$$

Proof.

Considerate arbitrary e, e' such that

- $\Gamma \vdash e \leq_e e' : E^{\delta}\tau$

We are required to show

$$\Gamma_0 \vdash \text{if } e_0\, e_1\, C[e] \leq \text{if } e_0'\, e_1'\, C'[e'] : E^{\delta_0 \wedge \delta_1 \wedge \delta_2}\tau'.$$

Instantiate $\Gamma_0 \vdash C \leq C' : (\Gamma \triangleright E^{\delta}\tau) \rightsquigarrow E^{\delta_2}\tau'$ with e and e' noting

$$\Gamma \vdash e \leq_e e' : E^{\delta}\tau.$$

Hence, $\Gamma_0 \vdash C[e] \leq C'[e'] : E^{\delta_2}\tau'$.

Applying the [Compatibility - If] lemma to

- $\Gamma_0 \vdash e_0 \leq e_0' : E^{\delta_0} bool$
- $\Gamma_0 \vdash e_1 \leq e_1' : E^{\delta_1}\tau'$
- $\Gamma_0 \vdash C[e] \leq C'[e'] : E^{\delta_2}\tau'$

we conclude that $\Gamma_0 \vdash \text{if } e_0\, e_1\, C[e] \leq \text{if } e_0'\, e_1'\, C'[e'] : E^{\delta_0 \wedge \delta_1 \wedge \delta_2}\tau'$.

Lemma [$\lambda^{!,b,\to,\mu,\circ,\bullet}$ Context Compatibility : Function]

$$\frac{\Gamma_0, x:\tau_0 \vdash C \leq C' : (\Gamma, x:\tau_0 \triangleright E^{\delta}\tau) \rightsquigarrow E^{\delta_1}\tau_1}{\Gamma_0 \vdash \lambda x:\tau_0.\, C \leq \lambda x:\tau_0.\, C' : (\Gamma \triangleright E^{\delta}\tau) \rightsquigarrow (\tau_0 \to E^{\delta_1}\tau_1)}$$

Proof.

Considerate arbitrary e, e' such that

- $\Gamma \vdash e \leq_e e' : E^{\delta}\tau$

We are required to show

$$\Gamma_0 \vdash \lambda x{:}\tau_0.\, C[e] \quad \leq \quad \lambda x{:}\tau_0.\, C'[e'] : \tau_0 \to E^{\delta_1}\tau_1.$$

Instantiate $\Gamma_0, x{:}\tau_0 \vdash C \leq C' : (\Gamma, x{:}\tau_0 \triangleright E^{\delta}\tau) \rightsquigarrow E^{\delta_1}\tau_1$ with e, e'

noting that $\Gamma \vdash e \leq_e e' : E^{\delta}\tau$.

Hence, $\Gamma_0, x{:}\tau_0 \vdash C[e] \leq C'[e'] : E^{\delta_1}\tau_1$

Applying the [Compatibility-Function] lemma to

- $\Gamma_0, x{:}\tau_0 \vdash C[e] \leq C'[e'] : E^{\delta_1}\tau_1$

We conclude that $\Gamma_0 \vdash \lambda x{:}\tau_0.\, C[e] \leq \lambda x{:}\tau_0.\, C'[e'] : \tau_0 \to E^{\delta_1}\tau_1.$

Lemma [$\lambda^{1,b_1\to,\mu,\circ,\bullet}$ Context Compatibility : AppCe]

$$\frac{\Gamma_0 \vdash C \leq C' : (\Gamma \triangleright E^{\delta}\tau) \rightsquigarrow E^{\delta_0}(\tau_1 \to E^{\delta_2}\tau_2) \qquad \Gamma_0 \vdash e_2 \leq e_2' : E^{\delta_1}\tau_1}{\Gamma_0 \vdash C\, e_2 \quad \leq \quad C' e_2' \quad : \quad (\Gamma \triangleright E^{\delta}\tau) \rightsquigarrow E^{\delta_0 \wedge \delta_1 \wedge \delta_2}\tau_2}$$

Proof.

Considerate arbitrary e, e' such that

- $\Gamma \vdash e \leq_e e' : E^{\delta}\tau$

We are required to show

$$\Gamma_0 \vdash C[e]\, e_2 \quad \leq \quad C'[e']\, e_2' \quad : \quad E^{\delta_0 \wedge \delta_1 \wedge \delta_2}\tau_2$$

Instantiate $\Gamma_0 \vdash C \leq C' : (\Gamma \triangleright E^{\delta}\tau) \rightsquigarrow E^{\delta_0}(\tau_1 \to E^{\delta_2}\tau_2)$, with e, e'

noting that $\Gamma \vdash e \leq_e e' : E^{\delta}\tau$.

Hence, $\Gamma_0 \vdash C[e] \leq C'[e'] : E^{\delta_0}(\tau_1 \to E^{\delta_2}\tau_2)$.

Applying the [Compatibility. App] lemma to

- $\Gamma_0 \vdash C[e] \leq C'[e'] : E^{\delta_0}(\tau_1 \to E^{\delta_2}\tau_2)$
- $\Gamma_0 \vdash e_2 \leq e_2' : E^{\delta_1}\tau_1$

We conclude that $\Gamma_0 \vdash C[e]\, e_2 \leq C'[e']\, e_2' : E^{\delta_0 \wedge \delta_1 \wedge \delta_2}\tau_2$.

Lemma [$\lambda^{1,b_1 \to, \mu, \circ, \bullet}$ Context Compatibility : AppeC]

$$\frac{\Gamma_0 \vdash e_1 \leq e_1' : E^{\delta_0}(\tau_1 \to E^{\delta_2}\tau_2) \qquad \Gamma_0 \vdash C \leq C' : (\Gamma \triangleright E^{\delta}\tau) \rightsquigarrow E^{\delta_1}\tau_1}{\Gamma_0 \vdash e_1 C \leq e_1' C' : (\Gamma \triangleright E^{\delta}\tau) \rightsquigarrow E^{\delta_0 \wedge \delta_1 \wedge \delta_2}\tau_2}$$

Proof.

Considerate arbitrary e, e' such that

- $\Gamma \vdash e \leq_e e' : E^{\delta}\tau$

We are required to show

$$\Gamma_0 \vdash e_1 C[e] \leq e_1' C'[e'] : E^{\delta_0 \wedge \delta_1 \wedge \delta_2}\tau_2$$

Instantiate $\Gamma_0 \vdash C \leq C' : (\Gamma \triangleright E^{\delta}\tau) \rightsquigarrow E^{\delta_1}\tau_1$ with e, e'

noting that $\Gamma \vdash e \leq_e e' : E^{\delta}\tau$.

Hence, $\Gamma_0 \vdash C[e] \leq C'[e'] : E^{\delta_1}\tau_1$

Applying the [Compatibility. App] lemma to

- $\Gamma_0 \vdash e_1 \leq e_1' : E^{\delta_0}(\tau_1 \to E^{\delta_2}\tau_2)$
- $\Gamma_0 \vdash C[e] \leq C'[e'] : E^{\delta_1}\tau_1$

We conclude that $\Gamma_0 \vdash e_1 C[e] \leq e_1 C'[e'] : E^{\delta_0 \wedge \delta_1 \wedge \delta_2}\tau_2$.

Lemma [$\lambda^{1,b_1\to,\mu,\circ,\bullet}$ Context Compatibility : Fold C]

$$\frac{\Gamma_0 \vdash C \leq C' : (\Gamma \triangleright E^{\delta} \tau) \rightsquigarrow E^{\delta_0} \tau'[\mu\alpha.\tau'/\alpha]}{\Gamma_0 \vdash \text{fold } C \leq \text{fold } C' : (\Gamma \triangleright E^{\delta} \tau) \rightsquigarrow E^{\delta_0} \mu\alpha.\tau'}$$

Proof.

Considerate arbitrary e, e' such that

- $\Gamma \vdash e \leq_e e' : E^{\delta} \tau$

We are required to show

$$\Gamma_0 \vdash \text{fold } C[e] \leq \text{fold } C'[e'] : E^{\delta_0} \mu\alpha.\tau'$$

Instantiate $\Gamma_0 \vdash C \leq C' : (\Gamma \triangleright E^{\delta} \tau) \rightsquigarrow E^{\delta_0} \tau'[\mu\alpha.\tau'/\alpha]$ with e, e' noting that $\Gamma \vdash e \leq_e e' : E^{\delta} \tau$.

Hence, $\Gamma_0 \vdash C[e] \leq C'[e'] : E^{\delta_0} \tau'[\mu\alpha.\tau'/\alpha]$.

Applying the [Compatibility-Fold] lemma to

- $\Gamma_0 \vdash C[e] \leq C'[e'] : E^{\delta_0} \tau'[\mu\alpha.\tau/\alpha]$

We conclude that $\Gamma_0 \vdash \text{fold } C[e] \leq \text{fold } C'[e'] : E^{\delta_0} \mu\alpha.\tau'$.

Lemma [$\lambda^{1,b_1\to,\mu,\circ,\bullet}$ Context Compatibility : Unfold C]

$$\frac{\Gamma_0 \vdash C \leq C' : (\Gamma \triangleright E^{\delta} \tau) \rightsquigarrow E^{\delta_0} \mu\alpha.\tau'}{\Gamma_0 \vdash \text{unfold } C \leq \text{unfold } C' : (\Gamma \triangleright E^{\delta} \tau) \rightsquigarrow E^{\bullet} \tau'[\mu\alpha.\tau'/\alpha]}$$

Proof.

Considerate arbitrary e, e' such that

- $\Gamma \vdash e \leq_e e' : E^{\delta}\tau$

We are required to show

$$\Gamma_0 \vdash \text{unfold } C[e] \leq \text{unfold } C'[e'] : E^{\bullet}\tau'[\mu\alpha.\tau'/\alpha]$$

Instantiate $\Gamma_0 \vdash C \leq C' : (\Gamma \triangleright E^{\delta}\tau) \rightsquigarrow E^{\delta_0}\mu\alpha.\tau'$ with e, e'

noting that $\Gamma \vdash e \leq_e e' : E^{\delta}\tau$.

Hence, $\Gamma_0 \vdash C[e] \leq C'[e'] : E^{\delta_0}\mu\alpha.\tau'$

Applying the [Compatibility-unfold] lemma to

- $\Gamma_0 \vdash C[e] \leq C'[e'] : E^{\delta_0}\mu\alpha.\tau'$

We conclude that $\Gamma_0 \vdash \text{unfold } C[e] \leq \text{unfold } C'[e'] : E^{\bullet}\tau'[\mu\alpha.\tau'/\alpha]$

Lemma [$\lambda^{!,b,\to,\mu,\otimes,\bullet}$ Context Compatibility : Empty$^{\vee}$]

$$\frac{}{\Gamma_0 \vdash [\cdot]^{\vee} \leq [\cdot]^{\vee} : (\Gamma \triangleright \tau) \rightsquigarrow \tau}$$

Proof.

Considerate arbitrary v, v' such that

- $\Gamma \vdash v \leq v' : \tau$

We are required to show

$$\Gamma_0 \vdash [v]^{\vee} \leq [v']^{\vee} : \tau$$

$$\equiv \Gamma_0 \vdash v \leq v' : \tau$$

Lemma [$\lambda^{1,b,\to,\mu,\oplus,\bullet}$ Context Compatibility : Fold C^v]

$$\frac{\Gamma_0 \vdash C^v \leq C^{v'} : (\Gamma \triangleright \tau) \rightsquigarrow \tau'[\mu\alpha.\tau'/\alpha]}{\Gamma_0 \vdash \text{fold } C^v \leq \text{fold } C^{v'} : (\Gamma \triangleright \tau) \rightsquigarrow \mu\alpha.\tau'}$$

Proof.

Consider arbitrary e, e' such that

- $\Gamma \vdash v \leq_v v' : \tau$

We are required to show

$$\Gamma_0 \vdash \text{fold } C^v[e]^v \leq \text{fold } C^{v'}[e']^v : \mu\alpha.\tau'$$

Instantiate $\Gamma_0 \vdash C^v \leq C^{v'} : (\Gamma \triangleright \tau) \rightsquigarrow \tau'[\mu\alpha.\tau'/\alpha]$ with v, v'

noting that $\Gamma \vdash v \leq_v v' : \tau$.

Hence, $\Gamma_0 \vdash C^v[v]^v \leq C^{v'}[v']^v : \tau'[\mu\alpha.\tau'/\alpha]$.

Applying the [Compatibility. Fold] lemma to

- $\Gamma_0 \vdash C^v[e]^v \leq C^{v'}[e']^v : \tau'[\mu\alpha.\tau/\alpha]$

We conclude that $\Gamma_0 \vdash \text{fold } C^v[v]^v \leq \text{fold } C^{v'}[v']^v : \mu\alpha.\tau'$.

Lemma [$\lambda^{1,b,\to,\mu,\oplus,\bullet}$ Context Compatibility : Whole]

$$\frac{\Gamma_0 \vdash C \leq C' : (\Gamma_1 \triangleright E^{\delta_0}\tau_1) \rightsquigarrow E^{\delta_1}\tau' \qquad \Gamma_1 \vdash C_1 \leq C_1' : (\Gamma \triangleright E^{\delta}\tau) \rightsquigarrow E^{\delta_0}\tau_1}{\Gamma_0 \vdash C[C_1[\cdot]] \leq C'[C_1'[\cdot]] : (\Gamma \triangleright E^{\delta}\tau) \rightsquigarrow E^{\delta_1}\tau'}$$

Proof.

Consider arbitrary e, e' such that

- $\Gamma \vdash e \leq_e e' : E^{\delta}\tau$

We are required to show $\Gamma_0 \vdash C[C_1[e]] \leq C'[C_1'[e']] : E^{\delta_1}\tau'$.

Instantiate $\Gamma_1 \vdash C_1 \leq C_1' : (\Gamma \triangleright E^{\delta} \tau) \rightsquigarrow E^{\delta_0} \tau$ with e, e' noting that

$\Gamma \vdash e \leq_e e' : E^{\delta} \tau$.

Hence, $\Gamma_1 \vdash C_1[e] \leq C_1'[e'] : E^{\delta_0} \tau$.

Instantiate $\Gamma_0 \vdash C \leq C' : (\Gamma_1 \triangleright E^{\delta_0} \tau) \rightsquigarrow E^{\delta_1} \tau'$ with $C_1[e], C_1'[e']$

noting that $\Gamma_1 \vdash C_1[e] \leq C_1'[e'] : E^{\delta_0} \tau$.

Hence, $\Gamma_0 \vdash C[C_1[e]] \leq C'[C_1'[e']] : E^{\delta_1} \tau'$.

Lemma [$\lambda^{1,b,\to,\mu,0,\bullet}$ Context Compatibility : whole-V]

$$\frac{\Gamma_0 \vdash C^v \leq C^{v\prime} : (\Gamma_1 \triangleright \tau_1) \rightsquigarrow \tau' \qquad \Gamma_1 \vdash C_1^v \leq C_1^{v\prime} : (\Gamma \triangleright \tau) \rightsquigarrow \tau_1}{\Gamma_0 \vdash C^v[C_1^v[\cdot]] \leq C^{v\prime}[C_1^{v\prime}[\cdot]] : (\Gamma \triangleright \tau) \rightsquigarrow \tau'}$$

Proof.

Consider arbitrary v, v' such that

- $\Gamma \vdash v \leq v' : \tau$

We are required to show $\Gamma_0 \vdash C^v[C_1^v[v]] \leq C^{v\prime}[C_1^{v\prime}[v']] : \tau'$.

Instantiate $\Gamma_1 \vdash C_1^v \leq C_1^{v\prime} : (\Gamma \triangleright \tau) \rightsquigarrow \tau_1$ with v, v' noting that

$\Gamma \vdash v \leq v' : \tau$.

Hence, $\Gamma_1 \vdash C_1^v[v]^v \leq C_1^{v\prime}[v']^v : \tau_1$.

Instantiate $\Gamma_0 \vdash C^v \leq C^{v\prime} : (\Gamma_1 \triangleright \tau_1) \rightsquigarrow \tau'$ with $C_1^v[v]^v, C_1^{v\prime}[v']^v$

noting that $\Gamma_1 \vdash C_1^v[v]^v \leq C_1^{v\prime}[v']^v : \tau$.

Hence, $\Gamma_0 \vdash C^v[C_1^v[v]^v] \leq C^{v\prime}[C_1^{v\prime}[v']^v] : \tau'$.

Lemma $[\lambda^{1,b,\to,\mu,\circ,\bullet}$ Empty 3$]$

$$\frac{\Gamma \subseteq \Gamma_0}{\Gamma_0 \vdash [\cdot]^{\vee} : (\Gamma \triangleright E^{\circ}\tau) \rightsquigarrow E^{\circ}\tau}$$

Proof.

Consider arbitrary e and e' such that

$$\Gamma \vdash e \leq e' : E^{\circ}\tau$$

We are required to show $\Gamma_0 \vdash [e] \leq [e'] : E^{\circ}\tau \equiv \Gamma_0 \vdash e \leq e' : E^{\circ}\tau$

Consider arbitrary k, γ_0, and γ_0' such that

- $k \geq 0$
- $(\gamma_0, \gamma_0') \in \mathcal{G}_k[\![\Gamma_0]\!]$

We are required to show $(\gamma_0(e), \gamma_0'(e')) \in \mathcal{E}_k[\![E^{\circ}\tau]\!]$.

Let $\gamma = \gamma_0|_{dom(\Gamma)}$ and $\gamma' = \gamma_0'|_{dom(\Gamma)}$.

Note that

- $(\gamma, \gamma') \in \mathcal{G}_k[\![\Gamma]\!]$, which follows from
  - $(\gamma_0, \gamma_0') \in \mathcal{G}_k[\![\Gamma]\!]$ and $\Gamma \subseteq \Gamma_0$.
- $(\gamma_0(e), \gamma_0'(e')) \in \mathcal{E}_k[\![E^{\circ}\tau]\!] \equiv (\gamma(e), \gamma'(e)) \in \mathcal{E}_k[\![E^{\circ}\tau]\!]$, which follows from
  - $FV(e) \subseteq dom(\Gamma)$ and $FV(e') \subseteq dom(\Gamma)$ & $\gamma = \gamma_0|_{dom(\Gamma)}$, $\gamma' = \gamma_0'|_{dom(\Gamma)}$

Hence, it suffices to show that $(\gamma(e), \gamma'(e)) \in \mathcal{E}_k[\![E^{\circ}\tau]\!]$.

Instantiate $\Gamma \vdash e \leq e' : E^{\circ}\tau$ with k, γ, γ'.

Note that

- $k \geq 0$
- $(\gamma, \gamma') \in \mathcal{G}_k[\![\Gamma]\!]$

Thus, $(\gamma(e), \gamma'(e')) \in \mathcal{E}_k[\![E^{\circ}\tau]\!]$.

Lemma [$\lambda^{\{b, \to, \mu, 0, \bullet\}}$ Context Reflexivity]

1. $$\frac{\Gamma' \vdash C^{v} : (\Gamma \triangleright \rho) \rightsquigarrow \tau'}{\Gamma' \vdash C^{v} \leq C^{v} : (\Gamma \triangleright \rho) \rightsquigarrow \tau'}$$

2. $$\frac{\Gamma' \vdash C : (\Gamma \triangleright \rho) \rightsquigarrow \sigma'}{\Gamma' \vdash C \leq C : (\Gamma \triangleright \rho) \rightsquigarrow \sigma'}$$

Proof.

By induction on $\Gamma' \vdash C^{v} : (\Gamma \triangleright \rho) \rightsquigarrow \tau'$ and $\Gamma' \vdash C : (\Gamma \triangleright \rho) \rightsquigarrow \sigma'$.

Each case follow from the corresponding compatibility lemma.

Theorem [Logical relation sound w.r.t Contextual Equivalence]

1. $\Gamma \vdash v \leq v' : \tau \implies \Gamma \vdash v \lesssim^{ctx} v' : \tau$
2. $\Gamma \vdash e \simeq e' : \sigma \implies \Gamma \vdash e \simeq^{ctx} e' : \sigma$

<u>Proof of 1.</u>

Consider arbitrary C and σ such that

- $\vdash C : (\Gamma \triangleright \tau) \rightsquigarrow \sigma$
- $C[v] \Downarrow$ which implies $\exists$ some value v_f and k such that

$$C[v] \xmapsto{k} v_f$$

we are required to show $C[v'] \Downarrow$.

Note that $\cdot \vdash C \leq C : (\Gamma \triangleright \tau) \rightsquigarrow \sigma$ by the [Context reflexivity] lemma applied to $\cdot \vdash C : (\Gamma \triangleright \tau) \rightsquigarrow \sigma$.

Instantiate $\cdot \vdash C \leq C : (\Gamma \triangleright \tau) \rightsquigarrow \sigma$ with v and v', noting that $\Gamma \vdash v \leq v' : \tau$.

Hence, $\cdot \vdash C[v] \leq C[v'] : \sigma$.

Instantiate this with $k+1$, ϕ, ϕ.

Note that

- $k+1 \geq 0$
- $(\phi, \phi) \in \mathcal{G}_{k+1}[\![\cdot]\!]$

Hence, $(C[v], C[v']) \in \mathcal{E}_{k+1}[\![\sigma]\!]$

Instantiate this with k and v_f. Note that

- $k < k+1$
- $C[v] \overset{k}{\longmapsto} v_f$

Therefore, $\exists\ v_f'$ such that $C[v] \overset{*}{\longmapsto} v_f$.

Proof of 2.

Consider arbitrary C and σ' such that

- $\vdash C : (\Gamma \triangleright \sigma) \rightsquigarrow \sigma'$
- $C[e] \Downarrow$ which implies $\exists$ some value v_f and k such that

$$C[e] \overset{k}{\longmapsto} v_f$$

We are required to show $C[e'] \Downarrow$.

Note that $\bullet \vdash C \leq C : (\Gamma \triangleright \sigma) \rightsquigarrow \sigma'$ by the [Context reflexivity] lemma applied to $\bullet \vdash C : (\Gamma \triangleright \sigma) \rightsquigarrow \sigma'$.

Instantiate $\bullet \vdash C \leq C : (\Gamma \triangleright \sigma) \rightsquigarrow \sigma'$ with e and e', noting that $\Gamma \vdash e \leq e' : \sigma$.

Hence, $\bullet \vdash C[e] \leq C[e'] : \sigma'$.

Instantiate this with $k+1$, ϕ, ϕ.

Note that

- $k+1 \geq 0$
- $(\phi, \phi) \in \mathcal{G}_{k+1}[\![\bullet]\!]$

Hence, $(C[e], C[e']) \in \mathcal{E}_{k+1}[\![\sigma']\!]$.

$\mathcal{E}_{k+1}[\![E^0 \tau']\!]$ for some τ'

$\mathcal{E}_{k+1}[\![E^\bullet \tau']\!]$ for some τ'

These are two cases to consider

✚ case $\delta = 0$:

We have $(C[e], C[e']) \in \mathcal{E}_{k+1}[\![E^0 \tau']\!]$ for some τ'.

So it follows that $\exists$ i, e_0, e_1' such that

So, it follows that $\exists\, j, e_f, e_f'$

- $j \leq k+1$
- $C[e] \xmapsto{j} e_f$
- $C[e'] \xmapsto{*} e_f'$
- $(e_f, e_f') \in \mathcal{V}_{k+1-j}[\![\tau']\!]$

Hence, $j = k$, $e_f = v_f$, and $C[e'] \Downarrow e_f'$.

✚ Case $\delta = \bullet$:

We have $(C[e], C[e']) \in \mathcal{E}_{k+1}[\![E^{\bullet}\tau']\!]$ for some τ'.

Instantiate this with k and v_f.

Note that

- $k < k+1$
- $C[e] \xmapsto{k} v_f$
- $\mathrm{irred}(v_f)$

Hence, $\exists\, e_f'$ such that $C[e'] \xmapsto{*} e_f' \wedge (v_f, e_f') \in \mathcal{V}_{k+1-k}[\![\tau']\!]$

So $C[e'] \Downarrow e_f'$.