Theorem [Equivalence Reflection]

$$\cdot ; \Gamma^+ \vdash e_1^+ \cong^{ctx} e_2^+ : \tau^{\div} \implies \Gamma \vdash e_1 \cong^{ctx} e_2 : \tau$$

Indirect proof. Suppose $\Gamma \vdash e_1 \not\cong^{ctx} e_2 : \tau$, which means there exists some C such that

- $\cdot \vdash C[e_1] : bool$ and $\cdot \vdash C[e_2] : bool$
- $C[e_1] \mapsto^* true$ and $C[e_2] \mapsto^* false$

By [Context translation] lemma there exist a C such that

- $\cdot \vdash C[e_1] : bool \rightsquigarrow C[e_1^+]$
- $\cdot \vdash C[e_2] : bool \rightsquigarrow C[e_2^+]$.

By [Semantics preservation -bool] lemma, it must be that

$C[e_1^+] \mapsto^* true$ and $C[e_2^+] \mapsto^* false$

But e_1^+ and e_2^+ are contextually equivalent from our assumption.

The fact that there exists a target context C that distinguishes e_1^+, e_2^+ is a contradiction.

$\square$

Lemma [Semantics Preservation - bool]

Let $\cdot \vdash e : \text{bool} \rightsquigarrow e^+$

If $e \xmapsto{*} \text{true}$ then $e^+ \xmapsto{*} \text{true}$ and

if $e \xmapsto{*} \text{false}$ then $e^+ \xmapsto{*} \text{false}$.

proof. By [Semantics Preservation] lemma, we know

$$\cdot \vdash e \;{}_S\!\leq_T\; e^+ : \text{bool}.$$

This means $(e, e^+) \in \mathcal{E}_{ST}[\![\text{bool}]\!]$.

This means there exist v, v' such that

- $e \xrightarrow{*} v$
- $e^+ \xrightarrow{*} v'$
- $(v, v') \in \mathcal{V}_{ST}[\![\text{bool}]\!]$

So it must be that

$$v = \text{true} \quad \text{and} \quad v' = \text{true}.$$

or

$$v = \text{false} \quad \text{and} \quad v' = \text{false}.$$

$\square$

Crosslanguage Logical Relation ${}_S\tilde{\approx}_T$

Closed logical relation

$$\mathcal{V}_{ST}[\![\text{unit}]\!] = \{((),())\}$$

$$\mathcal{V}_{ST}[\![\text{bool}]\!] = \{(\text{true},\text{true}),\ (\text{false},\text{false})\}$$

$$\mathcal{V}_{ST}[\![\tau \to \tau']\!] = \{(\lambda x{:}\tau.e,\ \lambda x{:}\tau^+.e) \mid \cdot \vdash \lambda x{:}\tau^+.e : (\tau \to \tau')^{\div} \wedge \forall (v,v) \in \mathcal{V}_{ST}[\![\tau]\!].\ (e[v/x],\ e[v/x]) \in \mathcal{E}_{ST}[\![\tau']\!]\}$$

$$\mathcal{E}_{ST}[\![\tau]\!] = \{(e,e) \mid \exists v.\ e \mapsto^* v \wedge \exists v.\ e \mapsto^* v \wedge (v,v) \in \mathcal{V}_{ST}[\![\tau]\!]\}$$

$$\mathcal{G}_{ST}[\![\Gamma, x{:}\tau]\!] = \{(\gamma[x \mapsto v],\ \gamma[x \mapsto v]) \mid (\gamma,\gamma) \in \mathcal{G}_{ST}[\![\Gamma]\!] \wedge (v,v) \in \mathcal{V}_{ST}[\![\tau]\!]\}$$

Open logical relation

$$\Gamma \vdash e\ {}_S\!\leq_T\ e^+ : \tau \quad \stackrel{\text{def}}{=} \quad \forall (\gamma,\gamma) \in \mathcal{G}_{ST}[\![\Gamma]\!].\ (\gamma(e),\ \gamma(e^+)) \in \mathcal{E}_{ST}[\![\tau]\!]$$

$$\Gamma \vdash e\ {}_S\tilde{\approx}_T\ e^+ : \tau \quad \stackrel{\text{def}}{=} \quad \Gamma \vdash e\ {}_S\!\leq_T\ e^+ : \tau \ \wedge\ \Gamma \vdash e^+\ {}_S\!\leq_T\ e : \tau$$

Lemma [Semantics Preservation]

If $\Gamma \vdash e : \tau \rightsquigarrow e^{+}$, then $\Gamma \vdash e \leq_{\tau} e^{+} : \tau$.

proof. Induction on the structure of e. Let $(\gamma, \gamhat) \in \mathcal{G}_{ST}[\![\Gamma]\!]$.

Case $\dfrac{}{\Gamma \vdash () : unit}$: We need to show

$$(\gamma(()), \gamhat(()^{+})) \in \mathcal{E}_{ST}[\![unit]\!]$$

$$\equiv ((), ()) \in \mathcal{E}_{ST}[\![unit]\!]$$

$$\downarrow^{*} \quad \downarrow^{*}$$

$$((), ()) \in \mathcal{V}_{ST}[\![unit]\!]$$

Case $\dfrac{}{\Gamma \vdash true : bool}$: We need to show

$$(\gamma(true), \gamhat(true^{+})) \in \mathcal{E}_{ST}[\![bool]\!]$$

$$\equiv (true, true) \in \mathcal{E}_{ST}[\![bool]\!],$$

$$\downarrow^{*} \quad \downarrow^{*}$$

$$(true, true) \in \mathcal{V}_{ST}[\![bool]\!]$$

Case $\dfrac{}{\Gamma \vdash false : bool}$: We need to show

$$(\gamma(false), \gamhat(false^{+})) \in \mathcal{E}_{ST}[\![bool]\!]$$

$$\equiv (false, false) \in \mathcal{E}_{ST}[\![bool]\!],$$

$$\downarrow^{*} \quad \downarrow^{*}$$

$$(false, false) \in \mathcal{V}_{ST}[\![bool]\!]$$

Case $\dfrac{\Gamma, x:\tau \vdash e : \tau'}{\Gamma \vdash \lambda x:\tau.\, e : \tau \to \tau'}$: We need to show

$$(\gamma(\lambda x:\tau.e),\ \gamma((\lambda x:\tau.e)^+)) \in \mathcal{E}_{ST}[\![\tau \to \tau']\!].$$

$$\equiv \quad (\lambda x:\tau.\ \gamma(e),\ \lambda x:\tau^+.\ \gamma(e^+)) \in \mathcal{E}_{ST}[\![\tau \to \tau']\!].$$

Since $\lambda x:\tau.\ \gamma(e) \mapsto^* \lambda x:\tau.\ \gamma(e)$ and

$\lambda x:\tau^+.\ \gamma(e^+) \mapsto^* \lambda x:\tau^+.\ \gamma(e^+)$,

We need to show $(\lambda x:\tau.\ \gamma(e),\ \lambda x:\tau^+.\ \gamma(e^+)) \in \mathcal{V}_{ST}[\![\tau \to \tau']\!]$.

Consider arbitrary v and v such that $(v, v) \in \mathcal{V}_{ST}[\![\tau]\!]$.

We need to show $(\gamma(e)[v/x],\ \gamma(e^+)[v/x]) \in \mathcal{E}_{ST}[\![\tau']\!]$.

The induction hypothesis says $\Gamma \vdash e \leq_{S<:T} e^+ : \tau'$.

Consider γ' and γ' such that

$$\gamma' = \gamma[x \mapsto v] \text{ and } \gamma' = \gamma[x \mapsto v], \text{ where}$$

$$(\gamma, \gamma) \in \mathcal{G}_{ST}[\![\Gamma]\!].$$

Then $(\gamma'(e),\ \gamma'(e^+)) \in \mathcal{E}_{ST}[\![\tau']\!]$.

Notice $\equiv (\gamma(e)[v/x],\ \gamma(e^+)[v/x]) \in \mathcal{E}_{ST}[\![\tau']\!]$,

so we are done.

Case $\dfrac{\Gamma \vdash e_1 : \tau \to \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1\, e_2 : \tau'}$:

We need to show $(\gamma(e_1\, e_2), \gamma(e_1^+\, e_2^+)) \in \mathcal{E}_{ST}[\![\tau']\!]$.

$$\equiv (\gamma(e_1)\,\gamma(e_2), \gamma(e_1^+)\,\gamma(e_2^+)) \in \mathcal{E}_{ST}[\![\tau']\!].$$

We need to show $\exists\, v, v$ such that

- $\gamma(e_1)\,\gamma(e_2) \longmapsto^* v$
- $\gamma(e_1^+)\,\gamma(e_2^+) \longmapsto^* v$
- $(v, v) \in \mathcal{V}_{ST}[\![\tau']\!]$

The I.H. on $\Gamma \vdash e_1 : \tau \to \tau'$ says $\Gamma \vdash e_1 \lesssim_{ST} e_1^+ : \tau \to \tau'$.

So we know $(\gamma(e_1), \gamma(e_1^+)) \in \mathcal{E}_{ST}[\![\tau \to \tau']\!]$.

This means $\exists\, v_1, v_1$ such that

- $\gamma(e_1) \longmapsto^* v_1$
- $\gamma(e_1^+) \longmapsto^* v_1$
- $(v_1, v_1) \in \mathcal{V}_{ST}[\![\tau \to \tau']\!]$

Let $v_1 = \lambda x{:}\tau.\, e$ and $v_2 = \lambda x{:}\tau^+.\, e^+$ for some e.

The I.H. on $\Gamma \vdash e_2 : \tau$ says $\Gamma \vdash e_2 \lesssim_{ST} e_2^+ : \tau$.

So we know $(\gamma(e_2), \gamma(e_2^+)) \in \mathcal{E}_{ST}[\![\tau]\!]$.

This means $\exists$ v_2, v_2 such that

- $\gamma(e_2) \xmapsto{*} v_2$
- $\gamma(e_2^+) \xmapsto{*} v_2$
- $(v_2, v_2) \in \mathcal{V}_{ST}[\![\tau]\!]$

So by the operational semantics of the languages,

$$\begin{aligned}
\gamma(e_1)\,\gamma(e_2) &\xmapsto{*} v_1\,\gamma(e_2) \\
&\equiv (\lambda x{:}\tau.e)\,\gamma(e_2) \\
&\xmapsto{*} (\lambda x{:}\tau.e)\,v_2 \\
&\mapsto e[v_2/x]
\end{aligned}
\qquad
\begin{aligned}
\gamma(e_1^+)\,\gamma(e_2^+) &\xmapsto{*} v_1\,\gamma(e_2^+) \\
&\equiv (\lambda x{:}\tau^+.e^+)\,\gamma(e_2^+) \\
&\xmapsto{*} (\lambda x{:}\tau^+.e^+)\,v_2 \\
&\mapsto e^+[v_2/x]
\end{aligned}$$

We now need to show there exist v_f, v_f such that

(1) $e[v_2/x] \xrightarrow{*} v_f$

(2) $e^+[v_2/x] \xrightarrow{*} v_f$

(3) $(v_f, v_f) \in \mathcal{V}_{ST}[\![\tau']\!]$

Remember $(\lambda x{:}\tau.e, \lambda x{:}\tau^+.e^+) \in \mathcal{V}_{ST}[\![\tau \to \tau']\!]$.

Instantiate this with $(v_2, v_2) \in \mathcal{V}_{ST}[\![\tau]\!]$.

We have $(e[v_2/x], e^+[v_2/x]) \in \mathcal{E}_{ST}[\![\tau']\!]$, which implies (1), (2), and (3).

$\square$