Definition : Ciu Equivalence

Let $\Gamma \vdash_e e : \sigma$ and $\Gamma \vdash_e e' : \sigma$.

$$\Gamma \vdash e \lesssim^{ciu} e' : \sigma \overset{def}{=} \forall \gamma, E, \sigma' \text{ such that}$$

$\vdash \gamma : \Gamma$

$\cdot \vdash E : (\cdot \triangleright \sigma) \rightsquigarrow \sigma'$.

$(E[\gamma(e)] \Downarrow \implies E[\gamma(e')] \Downarrow)$

Theorem. [Completeness w.r.t contextual equivalence]

We show $\lesssim^{ctx} \implies \lesssim^{ciu} \implies \lesssim^{lr}$

Lemma. [$\lesssim^{ctx}$ congruence]

If $\Gamma \vdash_e e \lesssim^{ctx} e' : \sigma$ and $\Gamma_1 \vdash C_1 : (\Gamma \triangleright \sigma) \rightsquigarrow \sigma_1$

then $\Gamma_1 \vdash C_1[e] \lesssim^{ctx} C_1[e'] : \sigma_1$.

proof. Consider arbitrary C and σ' such that

- $\cdot \vdash C : (\Gamma_1 \triangleright \sigma_1) \rightsquigarrow \sigma'$
- $C[C_1[e]] \Downarrow$

We need to show $C[C_1[e']] \Downarrow$

Instantiate $\Gamma \vdash_e e \lesssim^{ctx} e' : \sigma$ with $C[C_1[\cdot]]$ and σ', noting that

- $\cdot \vdash C[C_1[\cdot]] : (\Gamma \triangleright \sigma) \rightsquigarrow \sigma'$, which follows from

$$\frac{\cdot \vdash C : (\Gamma_1 \triangleright \sigma_1) \rightsquigarrow \sigma' \qquad \Gamma_1 \vdash C_1 : (\Gamma \triangleright \sigma) \rightsquigarrow \sigma_1}{\cdot \vdash C[C_1[\cdot]] : (\Gamma \triangleright \sigma) \rightsquigarrow \sigma'}$$

- $C[C_1[e]] \Downarrow$

Hence, $C[C_1[e']] \Downarrow$.

$\square$

Lemma. $[\lesssim^{ctx} \implies \lesssim^{ciu}]$

If $\Gamma \vdash e \lesssim^{ctx} e' : \sigma$, then $\Gamma \vdash e \lesssim^{ciu} e' : \sigma$.

Proof. Consider arbitrary γ, E, and σ' such that

- $\cdot \vdash \gamma : \Gamma$
- $\cdot \vdash E : (\cdot \triangleright \sigma) \rightsquigarrow \sigma'$
- $E[\gamma(e)] \Downarrow$

Hence $dom(\gamma) = dom(\Gamma)$.

Let $\Gamma = x_1 : \tau_1, x_2 : \tau_2, \dots, x_n : \tau_n$.

If $\gamma = \{x_1 \to v_1, x_2 \to v_2, \dots, x_n \to v_n\}$, then let

$$C_\gamma = (\lambda x_1 : \tau_1.\ \lambda x_2 : \tau_2.\ \dots\ \lambda x_n : \tau_n.\ [\cdot])\ v_1\ v_2 \cdots v_n.$$

Note that

$\cdot \vdash C_\gamma : (\Gamma \triangleright \sigma) \rightsquigarrow \sigma$, which follows from

$$\frac{\dfrac{x_1 : \tau_1, x_2 : \tau_2, \dots, x_n : \tau_n \vdash [\cdot] : (\Gamma \triangleright \sigma) \rightsquigarrow \sigma}{\cdot \vdash \lambda x_1 : \tau_1\ \lambda x_2 : \tau_2\ ..\ \lambda x_n : \tau_n.\ [\cdot] : (\tau_1 \to \dots \to \tau_n) \to \sigma} \qquad \cdot \vdash v_1, v_2, .. v_n : \tau_1, \tau_2, \dots, \tau_n}{\cdot \vdash (\lambda x_1 : \tau_1\ \lambda x_2 : \tau_2\ ..\ \lambda x_n : \tau_n.\ [\cdot])\ v_1\ v_2 \cdots v_n : (\Gamma \triangleright \sigma) \to \sigma}$$

Note

- $\cdot \vdash C_\gamma[e] \precsim^{ctx} C_\gamma[e'] : \sigma$, which follows from applying $\precsim^{ctx}$ congruence lemma to
  - $\Gamma \vdash e \precsim^{ctx} e' : \sigma$
  - $\cdot \vdash C_\gamma : (\Gamma \triangleright \sigma) \rightsquigarrow \sigma$

Instantiate this with E and σ' noting that

- $\cdot \vdash E : (\cdot \triangleright \sigma) \rightsquigarrow \sigma'$
- $E[C_\gamma[e]] \Downarrow$, which follows from
  - $E[C_\gamma[e]] \longmapsto^* E[\gamma(e)]$, which follows from the operational semantics and examination of C_γ
  - $E[\gamma(e)] \Downarrow$

Hence, $E[C_\gamma[e']] \Downarrow$

Since $E[C_\gamma[e']] \longmapsto^* E[\gamma(e')]$, it must be that $E[\gamma(e')] \Downarrow$.