International Journal of Control
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tcon20

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Published online: 08 Nov 2010.

To cite this article: J. F. Whidborne & R. S. H. Istepanian (2000) Finite word length stability issues in an l_1 framework, International Journal of Control, 73:2, 166-176, DOI: 10.1080/002071700219876

To link to this article: http://dx.doi.org/10.1080/002071700219876

INT. J. CONTROL, 2000, VOL. 73, NO. 2, 166–176

Finite word length stability issues in an ℓ_1 framework

J. F. WHIDBORNE†¶, J. WU‡ and R. S. H. ISTEPANIAN§

The paper addresses the digital controller structure problem for the closed loop stability of a feedback digital control system subject to finite word length (FWL). A new method of maximizing the stability subject to perturbations in the digital controller implementation is proposed. The approach is based on structured perturbation theory in an ℓ_1 framework, and unlike some previous approaches, can be simply extended to consider closed loop nominal performance and closed loop robust performance and stability. The method is demonstrated with application examples.

1. Introduction

For the production of mass volume products, such as automobiles and electronic consumer goods, it is imperative to keep component costs low. For this reason, and for advantages of speed, memory requirements, physical space and simplicity; fixed point processors with as small a word length as possible are preferred for the implementation of digital control in such products. This results in the problem of determining the controller realization which can use least possible word length, whilst maintaining closed loop stability and performance. Over the years, many results have been reported in the literature dealing with FWL controller implementation and their relevant parameterization issues (e.g. Morony *et al.* 1980, Morony 1983, Middleton and Goodwin 1990, Williamson 1991, Gevers and Li 1993, Fialho and Georgiou 1994, Madievski *et al.* 1995, Skelton *et al.* 1998). Solutions based on closed loop eigenvalue sensitivity measures have been proposed recently by Li and Gevers (1996), Istepanian *et al.* (1998 a, b) and Li (1998). The main limitation of these approaches is that only closed loop stability with optimal finite precision controller implementation can be considered, the closed loop performance cannot be analysed without augmenting the stability measures. In addition, the method does not provide an exact stability margin, but is only accurate if the perturbations in the controller parameters are small.

In this paper, a new approach using a measure based on ℓ_1 small gain stability theory (Dahleh and Khammash 1993, Khammash and Pearson 1993) is proposed. The use of an ℓ_1 framework approach has been suggested previously by Dahleh and Diaz-Bobillo (1995, p. 57). The advantage of the proposed approach is that, unlike the eigenvalue sensitivity-based approaches, it can be simply extended to consider both closed loop nominal performance and closed loop robust performance and stability (Khammash and Pearson 1993). The approach is illustrated with two application examples; a study of the implementation of a PID controller for a steel rolling mill problem (Hori 1996), and an H_∞-optimal controller for a fluid power system (Njabeleke *et al.* 1997). The results for problems are compared with results using measures based on the eigenvalue sensitivities proposed by Istepanian *et al.* (1998) and Li (1998).

The next section of the paper introduces the finite word length stability problem. Section 3 presents some results from ℓ_1 small gain stability theory, for further details, see Khammash and Pearson (1993) and Dahleh and Khammash (1993). In § 4, the results are extended to the problem of finite word length controller implementation. The problem of determining the controller realization which minimizes the required word length is addressed in § 5. In § 6, the theory is applied to the steel rolling mill and fluid power system controller structures, and comparisons made with results using measures based on the eigenvalue sensitivities. In the final section, some conclusions are drawn. The eigenvalue sensitivity measures are presented in the appendix.

2. The finite word length stability problem

2.1. *Standard discrete time feedback control system*

Consider the standard discrete time control system shown in figure 1. Let the discrete-time linear state space plant $G(z)$ be

$$G(z) = C_g(zI - A_g)^{-1}B_g + D_g \tag{1}$$

where $A_g \in \mathbb{R}^{n_g \times n_g}$, $B_g \in \mathbb{R}^{n_g \times n_u}$, $C_k \in \mathbb{R}^{n_y \times n_g}$ and $D_k \in \mathbb{R}^{n_y \times n_u}$, and the discrete-time linear state space controller be

First received 25 September 1998. Revised 23 May 1999.

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International Journal of Control ISSN 0020–7179 print/ISSN 1366–5820 online © 2000 Taylor & Francis Ltd
http://www.tandf.co.uk/journals/tf/00207179.html

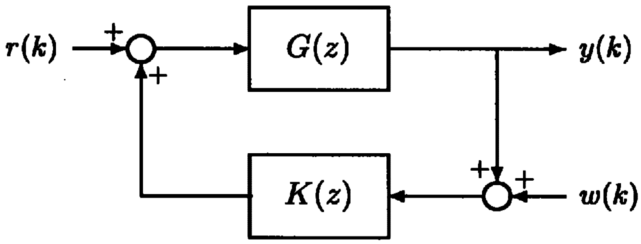

Figure 1. Discrete time control system.

$$K(z) = C_k(zI - A_k)^{-1}B_k + D_k \tag{2}$$

where $A_k \in \mathbb{R}^{n_k \times n_k}$, $B_k \in \mathbb{R}^{n_k \times n_y}$, $C_k \in \mathbb{R}^{n_u \times n_k}$ and $D_k \in \mathbb{R}^{n_u \times n_y}$. The closed loop system, $S(G, K)$, shown in figure 1 is described by

$$\begin{bmatrix} zx_k \\ zx_g \end{bmatrix} = \bar{A}\begin{bmatrix} x_k \\ x_g \end{bmatrix} + \bar{B}\begin{bmatrix} w \\ r \end{bmatrix} \tag{3}$$

$$y = \bar{C}\begin{bmatrix} x_k \\ x_g \end{bmatrix} + \bar{D}\begin{bmatrix} w \\ r \end{bmatrix} \tag{4}$$

where

$$\bar{A} = \begin{bmatrix} A_k + B_k E^{-1} D_g C_k & B_k E^{-1} C_g \\ B_g \tilde{E}^{-1} C_k & A_g + B_g \tilde{E}^{-1} D_k C_g \end{bmatrix} \tag{5}$$

$$\bar{B} = \begin{bmatrix} B_k E^{-1} & B_k D_g \tilde{E}^{-1} \\ B_g D_k E^{-1} & B_g \tilde{E}^{-1} \end{bmatrix} \tag{6}$$

$$\bar{C} = [E^{-1} D_g C_k \quad E^{-1} C_g] \tag{7}$$

$$\bar{D} = [E^{-1} D_g D_k \quad E^{-1} D_g] \tag{8}$$

and where $E = I - D_g D_k$, $\tilde{E} = I - D_k D_g$.

2.2. *Finite word length effect*

Let (A_k, B_k, C_k, D_k) denote a canonical realization of a discrete time controller $K(z)$. The controller has been designed *a priori* with the assumption that it can be implemented accurately. However, with a FWL implementation, the parameters of the controller will have a finite precision due to the well-known FWL effects (Gevers and Li 1993).

Let $K(z)$ be parameterized as

$$K \stackrel{s}{=} \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right] \tag{9}$$

$$= \begin{bmatrix} p_{1,1} & p_{2,1} & \cdots & p_{n_k+n_y,1} \\ p_{1,2} & p_{2,2} & \cdots & p_{n_k+n_y,2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1,n_k+n_u} & p_{2,n_k+n_u} & \cdots & p_{n_k+n_y,n_k+n_u} \end{bmatrix} \tag{10}$$

where $K \in \mathbb{R}^{(n_k+n_y)\times(n_k+n_u)}$. Using an FWL implementation, each element, $p_{i,j}$ of K can be implemented to a finite accuracy. Thus, in implementation, each element of K may be perturbed by $\pm\delta$, where, for example (Gevers and Li 1993, p. 33), using a word length of B_s bits

$$|\delta| = 2^{-(B_s+1)} \tag{11}$$

In another words, the quantization step when the parameter is discretized is 2δ. Thus, the implemented controller K_Δ, is perturbed from the original controller, K, so that $K_\Delta = K + \Delta$ where

$$\Delta = \begin{bmatrix} \Delta_{1,1} & \Delta_{2,1} & \cdots & \Delta_{n_k+n_y,1} \\ \Delta_{1,2} & \Delta_{2,2} & \cdots & \Delta_{n_k+n_y,2} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{1,n_k+n_u} & \Delta_{2,n_k+n_u} & \cdots & \Delta_{n_k+n_y,n_k+n_u} \end{bmatrix} \tag{12}$$

and $\Delta \in \mathbb{R}^{(n_k+n_y)\times(n_k+n_u)}$. Each element $\Delta_{i,j}$ is bounded in magnitude by δ, that is

$$|\Delta_{i,j}| \leq \delta \; \forall i,j \tag{13}$$

The problem is to minimize the fractional word length, B_s, whilst ensuring that the closed loop system remains stable. In order to do this, it is required to know the minimum perturbation, $\Delta_{i,j}$, on each element of K which will cause the closed loop system, $S(G, K_\Delta)$ to be not stable.

2.3. *A FWL stability measure*

In this section, a finite precision stability measure is defined. Firstly, the maximum perturbation, $\Delta_{i,j}$, on the elements of K is defined as the ∞-norm

$$\mu(\Delta) := \max_{i,j} |\Delta_{i,j}| \leq \delta \tag{14}$$

A measure of the stability of the particular digital controller parameterization, K, in the face of perturbations on the controller parameters is defined as (Fialho and Georgiou 1994)

$$\mu_0 := \inf\{\mu(\Delta) : S(G, K_\Delta) \text{ is not stable}\} \tag{15}$$

Thus μ_0 is the maximal upper bound on the FWL perturbation for which stability is guaranteed.

2.4. *Optimal realization problem*

It is well known that a particular realization of $K(z)$ is not unique. If $K_0 = (A_{k_0}, B_{k_0}, C_{k_0}, D_{k_0})$ is a particular canonical realization of $K(z)$, then $K(T) = (T^{-1}A_{k_0}T, T^{-1}B_{k_0}, C_{k_0}T, D_{k_0})$ is an equivalent realization of $K_0(z)$ and the set, $\mathcal{K}$, of equivalent realizations of $K(z)$ is defined as $\mathcal{K} = \{K(T) : T \in \mathbb{R}^{n_k \times n_k}, \det(T) \neq 0\}$. In order to minimize the FWL effect and minimize the required word length, the stability measure, μ_0, should be maximized for all $K(T) \in \mathcal{K}$. However, the calculation of μ_0 is not tractable (Fialho and Georgiou 1994, Li

1998). Instead, if some lower bound on μ_0 is defined as μ_{lb} where

$$\mu_{lb}(K) \le \mu_0(K)\ \forall K(T) \in \mathcal{K} \tag{16}$$

then a subsidiary problem of

$$\max_{K(T)\in\mathcal{K}} \mu_{lb}(K) \tag{17}$$

can be used.

Two computable lower bounds on the stability measure, μ_0, have recently been proposed (Li 1998, Istepanian *et al.* 1998 a), these are given in the appendix as μ_1 and μ_2. Note that these two lower bounds are based on the sensitivities of the closed loop eigenvalues to perturbations in the controller parameters, and as such are only valid if the perturbations are small, which is usually the case for FWL implementation. Another lower bound, μ_{ℓ}, based on ℓ_∞ small gain stability is derived in this paper. This bound is valid for all constant perturbations of the controller parameters.

Based on $\mu_{lb}(K)$, from (11) and (14), one can compute integer B_{lb} as an upper bound on $B_s^{\min}$, the minimal word-length that can guarantee the closed loop system stability when subject to FWL perturbations, that is $B_{lb}(K) = \lceil -(1+\log_2 \mu_{lb}(K))\rceil$, where $\lceil\cdot\rceil$ is the ceiling operator.

3. ℓ_1/ℓ_∞ theory

3.1. *Preliminaries*

Let ℓ_∞ be the space of all bounded sequences of real numbers $x = \{x(k), k = 0, 1, \ldots\}$ such that

$$\|x\|_\infty = \sup_k |x(k)| < \infty \tag{18}$$

The impulse response of a stable linear time invariant (LTI) system belongs to the space ℓ_1 if and only if $\|x\|_1 = \sum_{k=0}^{\infty}|x(k)| < \infty$. The $\mathcal{A}$-norm of a z-transform of an ℓ_1 sequence is the ℓ_1-norm of that sequence. So for an LTI system, $G(z)$ with an impulse response $g(k)$

$$\|G\|_{\mathcal{A}} = \|g\|_1 = \sum_{k=0}^{\infty}|g(k)| < \infty \tag{19}$$

It is well known that this is a measure of the maximum amplitude gain of the systems, i.e.

$$\|G\|_{\mathcal{A}} = \sup_{x\in\ell_\infty} \frac{\|Gx\|_\infty}{\|x\|_\infty} \tag{20}$$

In this paper, we will be dealing with multi-input multi-output (MIMO) LTI systems with vectors of signals. Thus, given the the vector of bounded real sequences $x = \{x_0, x_1, x_2 \ldots, x_n\}, x_i \in \ell_\infty$, let ℓ_∞^n be the set of all such sequences and let

$$\|x\|_\infty = \max_i \|x_i\|_\infty \tag{21}$$

For a MIMO LTI system

$$G = \begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1n} \\ G_{21} & G_{22} & \cdots & G_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ G_{m1} & G_{m2} & \cdots & G_{mn} \end{bmatrix} \tag{22}$$

then

$$\|G\|_{\mathcal{A}} = \max_i \sum_j \|g_{ij}\|_1 = \sup_{x\in\ell_\infty^n} \frac{\|Gx\|_\infty}{\|x\|_\infty} \tag{23}$$

The spectral radius of an $n\times n$ matrix X is defined by $\rho(X) = \max_i|\lambda_i|$, where $\{\lambda_i\} \in \lambda(X)$, the set of eigenvalues of X. Note that the spectral radius is not a matrix norm, however, it does satisfy the homogeneous property (Horn and Johnson 1985, p. 313], i.e. $\rho(cX) = |c|\rho(X)$ for all scalar $c \in \mathbb{C}$.

The unstructured set of admissible perturbations $\Delta^{p\times q}$ is the set of all operators which map l_∞^q to l_∞^p with an induced ℓ_∞ norm less than or equal to unity. That is

$$\Delta^{p\times q} := \left\{\Delta : \Delta \text{ is causal and } \sup_{x\neq 0}\frac{\|\Delta x\|_\infty}{\|x\|_\infty} \le 1\right\} \tag{24}$$

The structured set of admissible perturbations $\mathcal{D}([(p_1, q_1), \ldots, (p_n, q_n)])$ is the set of all diagonal operators of the form $\Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_n)$ where $\Delta_i \in \Delta^{p_i\times q_i}$. Note that the perturbations are permitted to be time varying or non-linear.

3.2. ℓ_∞ *stability with structured uncertainty*

Consider the system of figure 2. Let $\Delta \in \mathcal{D}([(p_1, q_1), \ldots, (p_n, q_n)])$ and let M be linear time invariant and

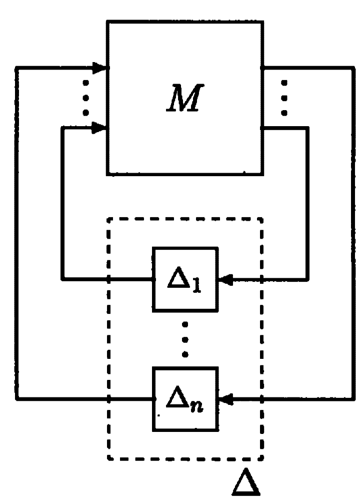

Figure 2. System with structured uncertainty.

$$M = \begin{bmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & & \vdots \\ M_{n1} & \cdots & M_{nn} \end{bmatrix} \tag{25}$$

where M_{ij} has p_j inputs and q_i outputs. The mth row of M_{ij} is denoted by $(M_{ij})_m$. Define the vectors of positive integers $\bar{p} := (p_1, \ldots, p_n)$ and $\bar{q} := (q_1, \ldots, q_n)$.

The set $\mathbb{K}$ is defined as

$$\mathbb{K} := \{(k_1, \ldots, k_n) : k_i \in \mathbb{N} \text{ and } 1 \le k_i \le q_i\} \tag{26}$$

For each $k = (k_1, \ldots, k_n) \in \mathbb{K}$, the matrix

$$M_k := \begin{bmatrix} (M_{11})_{k_1} & \cdots & (M_{1n})_{k_1} \\ \vdots & & \vdots \\ (M_{n1})_{k_n} & \cdots & (M_{nn})_{k_n} \end{bmatrix} \tag{27}$$

is defined. The robust stability of the system is determined by this matrix. Similarly,

$$\hat{M}_k := \begin{bmatrix} \|(M_{11})_{k_1}\|_{\mathcal{A}} & \cdots & \|(M_{1n})_{k_1}\|_{\mathcal{A}} \\ \vdots & & \vdots \\ \|(M_{n1})_{k_n}\|_{\mathcal{A}} & \cdots & \|(M_{nn})_{k_n}\|_{\mathcal{A}} \end{bmatrix} \tag{28}$$

Given $R = \mathrm{diag}(r_1, \ldots, r_n) \in \mathcal{R}$, then for the vector $\bar{p}$, the matrix $R_{\bar{p}}$ is defined as $R_{\bar{p}} := \mathrm{diag}(r_1, \ldots, r_1, \ldots, r_n, \ldots, r_n)$, where each r_i is repeated p_i times. Clearly, $R_{\bar{p}}$ depends on both R and $\bar{p}$. The matrix $R_{\bar{q}}$ is similarly defined.

Theorem 1: *The system of figure 2 is stable for all $\Delta \in \mathcal{D}([(p_1, q_1), \ldots, (p_n, q_n)])$ if and only if one of the following holds:*

(1) *for all $k \in \mathbb{K}$, $\rho(\hat{M}_k) < 1$;*
(2) *for any $k \in \mathbb{K}$, the inequalities $x \le \hat{M}_k x$ have no non-zero solution $x \in \mathbb{R}^n$ which satisfies $x \ge 0$;*
(3) $\inf_{R \in \mathcal{R}} \|R_{\bar{q}}^{-1} M R_{\bar{p}}\|_{\mathcal{A}} < 1$.

The proof is given by Khammash and Pearson (1991, 1993). The three conditions are equivalent.

4. Structured uncertainty for FWL controller implementation

Theorem 1 provides an easily computed test for ensuring closed loop stability and performance for the discrete time control system subjected to perturbations in the digital controller due to the FWL implementation. Only closed loop stability is considered here.

4.1. *FWL perturbation bound*

Let the controller parameterization

$$K \stackrel{s}{=} \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right] \tag{29}$$

be perturbed to K_Δ where

$$K_\Delta \stackrel{s}{=} \left[\begin{array}{c|c} A_k + \Delta_A & B_k + \Delta_B \\ \hline C_k + \Delta_C & D_k + \Delta_D \end{array}\right] \tag{30}$$

From (13), let each element of Δ be bounded by δ. Bounds on each perturbation block Δ_A, Δ_B, Δ_C, Δ_D can be obtained from the following lemma.

Lemma 1: *Consider a time-invariant MIMO perturbation gain $\Delta \in \mathbb{R}^{p\times q}$ which maps l_∞^q to l_∞^p where $d \in l_\infty^p$ and $e \in l_\infty^q$ and $d(k) = \Delta e(k)$. Let*

$$\Delta = \begin{bmatrix} \Delta_{1,1} & \cdots & \Delta_{1,q} \\ \vdots & & \vdots \\ \Delta_{p,1} & \cdots & \Delta_{p,q} \end{bmatrix} \tag{31}$$

and

$$|\Delta_{i,j}| \le \delta \quad \forall i, j \tag{32}$$

Then, $\|\Delta\|_{\mathcal{A}} \le q\delta$.

Proof: From (23)

$$\|\Delta\|_{\mathcal{A}} = \max_i \sum_{j=1}^{q} |\Delta_{i,j}| \tag{33}$$

so, from (32), $\|\Delta\|_{\mathcal{A}} \le q\delta$. □

Thus the bounds on each uncertainty block are

$$\left.\begin{array}{ll} \|\Delta_A\|_{\mathcal{A}} \le n_k\delta, & \|\Delta_B\|_{\mathcal{A}} \le n_y\delta \\ \|\Delta_C\|_{\mathcal{A}} \le n_k\delta, & \|\Delta_D\|_{\mathcal{A}} \le n_y\delta \end{array}\right\} \tag{34}$$

4.2. *ℓ_∞ stability of FWL implementation*

We can use Theorem 1 to compute a sufficiency test on the stability of the closed loop system with a FWL implementation. Note that Theorem 1 becomes a sufficient condition because the perturbations on Δ are restricted to being linear and time invariant.

Theorem 2: *The system configured as in figure 3 is stable if one of the following holds:*

(1) *for all $k \in \mathbb{K}$, $\rho(Q\hat{M}_k) < 1/\delta$;*
(2) *for any $k \in \mathbb{K}$, the inequalities $x \le \delta Q\hat{M}_k x$ has no non-zero solution $x \in \mathbb{R}^n$ which satisfies $x \ge 0$;*
(3) $\inf_{R \in \mathcal{R}} \|R_{\bar{q}}^{-1} Q M R_{\bar{p}}\|_{\mathcal{A}} < 1/\delta$,

where $Q = \mathrm{diag}(n_k, \ldots, n_k, n_y, \ldots, n_y, n_k, \ldots, n_k, n_y, \ldots, n_y)$, with both n_k repeated n_k times and both n_y repeated n_y times, and where M is given by (38)–(42).

Proof: This follows directly from Theorem 1, from (34), and from the homogeneous property of matrix norms and the homogeneous property of the spectral

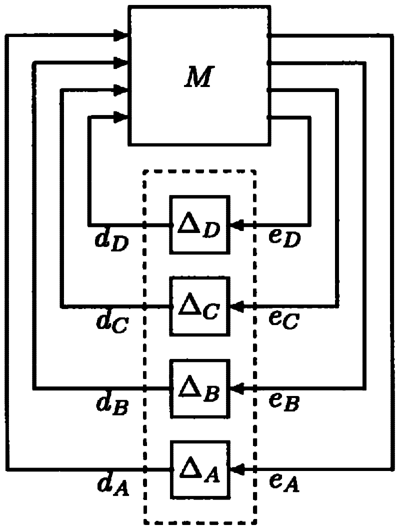

Figure 3. Structured uncertainty for FWL controller.

radius function. The loss of the necessity condition arises because Δ_A, Δ_B, Δ_C and Δ_D are linear time-invariant perturbation gains, subsets of $\Delta^{p\times q}$ defined by (24). □

From this theorem, a lower bound on μ_0 is obtained.

Corollary 1: *μ_ℓ is a lower bound on μ_0 where*

$$\mu_\ell = \left[\max_{k\in\mathbb{K}} \rho(Q\hat{M}_k)\right]^{-1} \tag{35}$$

(*or an equivalent from Theorem* 2).

Proof: This follows immediately from Theorem 2, (15) and (16). □

4.3. *Closed loop transfer functions*

In order to evaluate μ_ℓ, the closed loop transfer function, M, is required. For a discrete-time linear state space plant $G(z) = C_g(zI - A_g)^{-1}B_g + D_g$ and discrete-time linear state space controller $K(z) = C_k(zI - A_k)^{-1}B_k + D_k$, with the uncertainty resulting from the FWL expressed as MIMO additive uncertainty on each of the matrices, the resulting closed loop transfer functions for the interconnected system shown in figure 4 are

$$\begin{bmatrix} zx_g \\ zx_k \end{bmatrix} = A\begin{bmatrix} x_g \\ x_k \end{bmatrix} + B\begin{bmatrix} d_A \\ d_B \\ d_C \\ d_D \end{bmatrix} \tag{36}$$

$$\begin{bmatrix} e_A \\ e_B \\ e_C \\ e_D \end{bmatrix} = C\begin{bmatrix} x_g \\ x_k \end{bmatrix} + D\begin{bmatrix} d_A \\ d_B \\ d_C \\ d_D \end{bmatrix} \tag{37}$$

that is

$$M \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{38}$$

where

$$A = \begin{bmatrix} A_g + B_g\tilde{E}^{-1}D_kC_g & B_g\tilde{E}^{-1}C_k \\ B_kE^{-1}C_g & A_k + B_kE^{-1}D_gC_k \end{bmatrix} \tag{39}$$

$$B = \begin{bmatrix} 0 & 0 & B_g\tilde{E}^{-1} & B_g\tilde{E}^{-1} \\ I & I & B_kE^{-1}D_g & B_kE^{-1}D_g \end{bmatrix} \tag{40}$$

$$C = \begin{bmatrix} 0 & I \\ E^{-1}C_g & E^{-1}D_gC_k \\ 0 & I \\ E^{-1}C_g & E^{-1}D_gC_k \end{bmatrix} \tag{41}$$

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & E^{-1}D_g & E^{-1}D_g \\ 0 & 0 & 0 & 0 \\ 0 & 0 & E^{-1}D_g & E^{-1}D_g \end{bmatrix} \tag{42}$$

and where $E = I - D_gD_k$, $\tilde{E} = I - D_kD_g$.

5. Optimal realization problem

In this section, the problem of determining the structure that minimizes the required word length is addressed. From (17), the problem to be solved is

$$\max_{K(T)\in\mathcal{K}} \mu_\ell(K) \tag{43}$$

5.1. *Closed loop realization transfer functions*

Let

$$\left.\begin{aligned} B_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ I & I & 0 & 0 \end{bmatrix} \\ B_2 &= \begin{bmatrix} 0 & 0 & B_g\tilde{E}^{-1} & B_g\tilde{E}^{-1} \\ 0 & 0 & B_kE^{-1}D_g & B_kE^{-1}D_g \end{bmatrix} \end{aligned}\right\} \tag{44}$$

$$C_1 = \begin{bmatrix} 0 & I \\ 0 & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 \\ E^{-1}C_g & E^{-1}D_gC_k \\ 0 & 0 \\ E^{-1}C_g & E^{-1}D_gC_k \end{bmatrix} \tag{45}$$

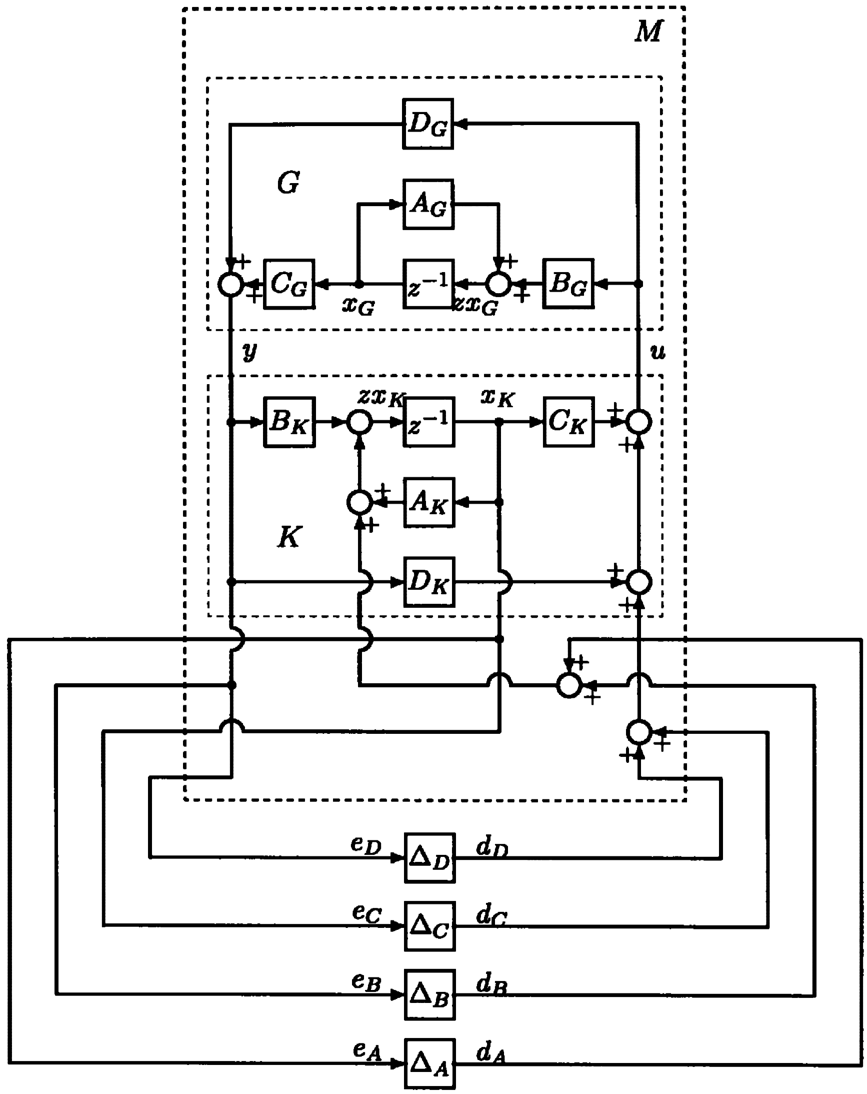

Figure 4. Structured uncertainty for FWL controller implementation.

$$I_2 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad \text{and} \quad T_I = \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix} \tag{46}$$

then, from (38)–(42), for some realization $(T^{-1}A_kT, T^{-1}B_k, C_kT, D_k)$, the closed loop system is given by

$$M_T(T) \stackrel{s}{=} \left[\begin{array}{c|c} T_I^{-1}AT_I & [I_2 \quad T_I^{-1}] \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ \hline [C_1 \quad C_2] \begin{bmatrix} I_2 \\ T_I \end{bmatrix} & D \end{array} \right] \tag{47}$$

This gives

$$M_T(z,T) = [C_1 \quad C_2] \begin{bmatrix} I_2 \\ T_I \end{bmatrix} (zI - T_I^{-1}AT_I)^{-1} \times [I_2 \quad T_I^{-1}] \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + D \tag{48}$$

which simplifies to

$$M_T(z,T) = [C_1 \quad C_2] \begin{bmatrix} T_I^{-1} \\ I_2 \end{bmatrix} (zI - A)^{-1} \times [T_I \quad I_2] \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + D \tag{49}$$

Clearly, as expected, the eigenvalues of M_T are unaffected by the parameterization.

5.2. *Optimal FWL realization*

From Corollary 1 and from Theorem 2, the problem given by (43) is equivalent to finding υ and hence maximal $]_\ell$ and an optimal $K(T)$ where

$$\upsilon = \min_{\substack{T \in \mathbb{R}^{n_k \times n_k} \\ \det(T) \neq 0}} \max_{k \in \mathbb{K}} \rho(Q\hat{M}_k(T)) \tag{50}$$

and maximal $\mu_{\ell} = 1/v$. The problem given by (50) is not convex, so local minima can be found by means of non-linear programming.

From (49), the system transition matrix, A, is independent of the parameterization T. This is an important point in terms of the efficiency of the computations required to solve (50), in that the impulse response kernel, $[I, \ A, \ A^2, \ \ldots]$, is constant and needs only to be solved once. The main computational load is then to solve $n_y^2 n_k^2$ (the number of elements in $\mathbb{K}$) fourth order spectral radius problems.

6. Application examples

The proposed approach is illustrated with two application examples, a PID controller for a steel rolling mill problem, and an H_∞-optimal controller for a fluid power system. The results for both problems are compared with results using the eigenvalue sensitivity based measures μ_1 and μ_2 defined in the appendix.

6.1. *A steel rolling mill PID controller*

A study of the FWL PID controller implementation for a steel rolling mill appears in Istepanian *et al.* (1998 b), and is based on a system described by Hori (1996). The torsional vibration model of the system is actually a distributed parameter system, and using modal analysis can be modelled as a multi-inertia system connected by springs. The simplest model considering up to the first mode is a two-mass model shown in figure 5. The motor and load moments of inertia are represented by J_M and J_L respectively; the motor and load coefficients of friction by B_M and M_L respectively, and the spring constant by K_S.

The state equations of the system are

$$\begin{bmatrix} \dot{\omega}_M \\ \dot{\theta}_S \\ \dot{\omega}_L \end{bmatrix} = \begin{bmatrix} -B_M/J_M & -K_S/J_M & 0 \\ 1 & 0 & 1 \\ 0 & K_S/J_L & -B_L/J_L \end{bmatrix} \begin{bmatrix} \omega_M \\ \theta_S \\ \omega_L \end{bmatrix} + \begin{bmatrix} 1/J_L \\ 0 \\ 0 \end{bmatrix} T_M + \begin{bmatrix} 0 \\ 0 \\ 1/J_L \end{bmatrix} T_L \qquad (51)$$

where the state variables are motor speed, ω_M, the torsional angle, θ_S, and the load speed, ω_L. The control input is the motor torque, T_M, and the measured output is the motor speed, ω_M. The main disturbance to the system is the load torque, T_L. Neglecting friction, the continuous time linearized ideal two-mass inertia model $G(s)$ of the system is given by

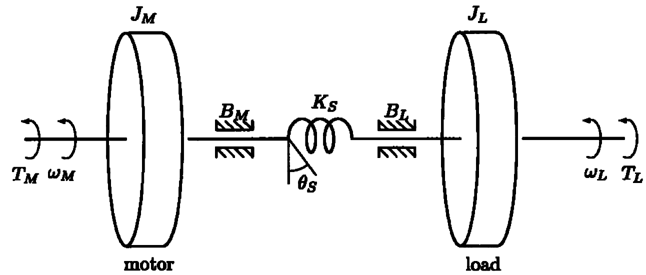

Figure 5. Two-inertia system model.

$$G(s) \overset{s}{=} \left[\begin{array}{ccc|c} 0 & -9763.7203 & 0 & 249.0300 \\ 1 & 0 & -1 & 0 \\ 0 & 13\,424.9859 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \end{array}\right] \qquad (52)$$

A PID vibration suppression and disturbance rejection controller, $K(s)$, is designed as

$$K(s) = \frac{0.00269s}{0.001s+1} - 0.435 - \frac{14.26}{s} \qquad (53)$$

Using a bilinear transform

$$s = \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}} \qquad (54)$$

where h is the sampling period of 0.001 s, the digital PID controller is

$$K(z) = -\frac{0.01426}{z-1} - \frac{1.1956}{z-0.3333} + 1.3512 \qquad (55)$$

The initial realization $K_0(z)$ is set to

$$K_0(z) \overset{s}{=} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 0.3333 & -1 \\ \hline 0.014\,26 & 1.1956 & 1.3512 \end{array}\right] \qquad (56)$$

The discretized plant $G(z)$ is given by (see (57)).

With the realization given by (56), the ℓ_1-based stability measure was calculated to be $\mu_{\ell} = 2.101$.

$$G(z) \overset{s}{=} \left[\begin{array}{ccc|c} 0.995\,127\,566\,229\,59 & -9.726\,029\,353\,189\,39 & 0.004\,872\,433\,770\,41 & 0.248\,625\,226\,382\,91 \\ 0.000\,996\,139\,694\,13 & 0.988\,428\,034\,426\,50 & -0.000\,996\,139\,694\,13 & 0.000\,124\,274\,574\,09 \\ 0.006\,699\,531\,803\,10 & 13.373\,161\,348\,093\,25 & 0.993\,300\,468\,196\,90 & 0.000\,556\,558\,354\,31 \\ \hline 1 & 0 & 0 & 0 \end{array}\right] \qquad (57)$$

Realization	μ_ℓ $(\times 10^{-3})$	$\hat{B}_{s\ell}^{\min}$	μ_1 $(\times 10^{-3})$	$\hat{B}_{s1}^{\min}$	μ_2 $(\times 10^{-3})$	$\hat{B}_{s2}^{\min}$	$B_s^{\min}$
I_2	2.101	8	1.948	9	1.077	9	6
T_1	5.358	7	8.929	6	4.895	7	3
T_2	7.488	7	5.277	7	4.896	7	3
T_ℓ	8.157	6	6.706	7	4.749	7	3
T_{bal}	7.571	7	5.272	7	4.888	7	3

Table 1. Comparative stability measures for different realizations of the steel rolling mill PID controller.

A Nelder Mead modified simplex search was used to find solutions to (50). The best solution produced a value of $\mu_\ell = 8.157$ with minimizing transformation matrix T_ℓ

$$T_\ell = \begin{bmatrix} -5.667\,740\,664\,683\,51 & -5.664\,816\,585\,856\,03 \\ -0.524\,845\,081\,594\,91 & 0.929\,242\,950\,433\,30 \end{bmatrix} \tag{58}$$

and an optimal realization of (see (59)).

This optimal realization guarantees stability for a maximum permissible perturbation of $\delta = 0.008\,157$, compared to an optimal maximum permissible perturbation of $\delta = 0.008\,929$ obtained using the method of Istepanian *et al.* (1998 b). When implemented with a fractional 3-bit length implementation, the controller

$$K_{3\text{-bit}}(z) \stackrel{s}{=} \left[\begin{array}{cc|c} 0.75 & 0.375 & 0.75 \\ 0.25 & 0.625 & -0.625 \\ \hline -0.75 & 1. & 1.375 \end{array}\right] \tag{60}$$

is stabilizing.

Table 1 shows a comparison of the measures μ_ℓ, μ_1 and μ_2 for five different PID controller realizations. The measures μ_1 and μ_2 are defined in the appendix. The original controller realization is given by (56), i.e. $T_0 = I_2$. The ℓ_1-optimal realization is from transformation matrix T_ℓ given by (58). A μ_1-optimal realization is obtained using the method of Istepanian *et al.* (1998 b) with transformation matrix

$$T_1 = \begin{bmatrix} 11.995\,662\,094\,715\,59 & -6.253\,955\,559\,400\,68 \\ 1.506\,184\,242\,403\,66 & 0.594\,004\,412\,187\,48 \end{bmatrix} \tag{61}$$

A μ_2 sub-optimal realization is obtained using a method based on Li (1998) and Whidborne *et al.* (1999) with transformation matrix

$$T_2 = \mathrm{diag}\,(8.374\,140\,386\,271\,04, \quad 0.914\,549\,142\,377\,83) \tag{62}$$

Finally, a balanced realization of $K(z)$ (Laub *et al.* 1987) is obtained with transformation matrix

$$T_{\mathrm{bal}} = \mathrm{diag}\,(8.998\,580\,855\,475\,10, \quad 0.938\,963\,653\,790\,65) \tag{63}$$

The corresponding fractional bit-lengths for each realization and measure are also shown along with the actual minimal fractional bit-length $B_s^{\min}$ for closed loop stability for each realization. These results indicate that the ℓ_1 framework provides a tractable stability measure of similar suitability to that based on μ_1 and μ_2.

6.2. *A fluid power speed controller*

Fluid power systems have traditionally been the preferred choice in applications where high power is required to be smoothly delivered in a compact form. This is mainly due to the high power to weight ratio as well as high inherent stiffness of the transmission fluid. Despite their success, it has become difficult to satisfy increasing demands for high accuracy in low cost high power applications where huge pressure variations and non-linearities are invariably present. Inability to meet these demands efficiently in open loop has led to the consideration of feedback control. Unfortunately, model uncertainties caused by supply pressure variations and the heavy non-linearities caused by both fluid and component characteristics make the application of traditional linear feedback control challenging, especially where precise control is required over a range of operating conditions.

A robust controller has been designed by Njabeleke *et al.* (1997) for a fluid power speed control system consisting of a Moog–Donzelli servovalve controlled axial piston motor driving a load simulated by a hydraulic pump. The pump is used to simulate both

$$K_\ell(z) \stackrel{s}{=} \left[\begin{array}{cc|c} 0.759\,437\,654\,167\,62 & 0.425\,917\,802\,878\,39 & 0.800\,263\,62 \\ 0.240\,686\,519\,890\,17 & 0.573\,862\,345\,832\,38 & -0.624\,148\,48 \\ \hline -0.708\,326\,761\,433\,26 & 1.030\,222\,587\,023\,75 & 1.3512 \end{array}\right] \tag{59}$$

load disturbances at high speeds and reciprocating loads (when the motor drives a mechanism with linear movement) at low to medium speeds. These are typical applications in manufacturing. The controller has been designed using the H_∞ loop shaping design procedure of McFarlane and Glover (1992).

The plant model is given by (see (64)) and the continuous time H_∞-optimal controller is (see (65)).

The controller was discretized using the bilinear transform (54). A sampling period of 0.002 s (500 Hz) was selected to reflect the system bandwidth at high speed. The controller was implemented as a ratio of two polynomials using a difference equation. Unfortunately, the bandwidth falls by over a factor of 10 as the operating speed is reduced and this sampling rate leads to poor numerical conditioning as the closed loop poles are pushed close to the unit circle, causing instability (both in simulation and on an experimental rig) because the controller coefficients are truncated.

The original discrete controller realization was (to four significant digits) (see (66)). With this realization, the ℓ_1-based stability measure was calculated to be $\mu_\ell = 48.178 \times 10^{-6}$. With the discrete controller realization in the observable companion form, the stability measure is $\mu_\ell = 4.7439 \times 10^{-9}$. Similarly, with the discrete controller realization in the controllable companion form, the stability measure is $\mu_\ell = 2.3835 \times 10^{-9}$.

A Nelder Mead modified simplex search was used to find solutions to (50). The best solution produced a value of $\upsilon = 4.0623 \times 10^3$ with (to four significant digits) (see (67)). For this optimal realization, the stability measure is $\mu_\ell = 0.2462 \times 10^{-3}$. This is an improvement by five orders of magnitude on the stability measure for the controllable and observable companion realizations.

Table 2 shows a comparison of the measures μ_ℓ, μ_1 and μ_2 for five different controller realizations calculated using similar methods as for table 1.

Although μ_1 provides a lower optimal estimate of fractional bit-length $B_{s1}^{\min}$ than $B_{s\ell}^{\min}$, the actual fractional bit-length $B_s^{\min}$ is smaller for the realization T_ℓ than for T_1. These results again indicate that the ℓ_1 framework provides a tractable stability measure, μ_ℓ, of similar suitability to μ_1 or μ_2.

$$G(s) \stackrel{s}{=} \left[\begin{array}{cccc|c} 0 & 0 & 80.0437378 & -80.0437378 & 0 \\ 0 & -1.014\times10^{4} & 1349.6368 & 1282.155 & 2.7833 \\ -2044.3703 & 1.3496\times10^{6} & -1.3496\times10^{6} & 0 & -3.4981\times10^{-6} \\ 2044.3703 & 1.2822\times10^{6} & 0 & -1.28454\times10^{6} & -9.8088\times10^{4} \\ \hline & 0 & 0 & 0 & 0 \end{array}\right] \quad (64)$$

$$K(s) \stackrel{s}{=} \left[\begin{array}{cccc|c} 0 & 0.0004844 & 0.009798 & -0.4226 & 2.781\times10^{-10} \\ 0 & -2000 & 0.06998 & -7.506 & -0.01216 \\ 0 & -0.02 & -0.6615 & 1.37 & 0.01623 \\ 0 & 8.364 & -3.082 & -15.09 & 7.679 \\ \hline 1 & 0.0004844 & 0.009798 & -0.4226 & 2.781\times10^{-10} \end{array}\right] \quad (65)$$

$$K(z) \stackrel{s}{=} \left[\begin{array}{cccc|c} 1 & -1.999\times10^{-6} & 0.00002215 & -0.0008326 & -0.003197 \\ 0 & -0.3334 & 0.0000618 & -0.004929 & -0.02298 \\ 0 & -5.801\times10^{-6} & 0.9987 & 0.002697 & 0.02658 \\ 0 & 0.005493 & -0.006067 & 0.9702 & 7.565 \\ \hline 10.002 & -2.001\times10^{-6} & 0.00002217 & -0.0008335 & -0.0032 \end{array}\right] \quad (66)$$

$$K(z) \stackrel{s}{=} \left[\begin{array}{cccc|c} 1 & -0.00008615 & 0.0004425 & 0.0001094 & -0.007195 \\ 0.04796 & -0.3337 & 0.1863 & 0.4083 & -0.02601 \\ 0.0009544 & -0.0007019 & 0.9701 & -0.009106 & -0.2859 \\ -0.00002970 & -0.0009021 & 0.002686 & 0.9992 & 0.01625 \\ \hline 0.02498 & -0.00001743 & 0.02185 & 0.007117 & -0.003200 \end{array}\right] \quad (67)$$

Realization	μ_{ℓ} ($\times 10^{-6}$)	$\hat{B}_{s\ell}^{\min}$	μ_1 ($\times 10^{-6}$)	$\hat{B}_{s1}^{\min}$	μ_2 ($\times 10^{-6}$)	$\hat{B}_{s2}^{\min}$	$B_s^{\min}$
I_2	48.178	14	41.872	14	13.118	16	11
T_1	204.497	12	616.821	10	273.202	11	8
T_2	167.620	12	516.496	10	244.206	11	9
T_{ℓ}	246.17	11	243.605	12	99.105	13	7
T_{bal}	207.404	12	538.973	10	273.154	11	8

Table 2. Comparative stability measures for different realizations of the fluid power speed controller.

7. Discussion and conclusions

The results of this paper show that the new methodology based on structured ℓ_1 uncertainty provides a practical means of determining the minimum bit length for stable FWL implementation. The methodology can be simply extended to consider FWL performance.

Like all the current approaches, the method is conservative, but the examples demonstrate that it is not excessively so and that it provides a less conservative stability measure than μ_1 or μ_2 for certain controller realizations. Whilst the calculation of μ_{ℓ} requires more computation than μ_1 or μ_2, the computational effort is not demanding. For the fluid power speed controller problem, for example, the main effort is in solving 25 fourth order spectral radius problems at each iteration. In this paper, the `eig` function from MATLAB (MatLab 1992) was used, but since the spectral radius is required, a more efficient method could be devised. In addition, there is a certain amount of sparseness and repetition in M that could be exploited.

The optimization problem using μ_{ℓ} is non-convex and requires non-linear programming for the solution, hence it would not be practical for very high order controllers. Note that the problem using μ_1 is also non-convex (Istepanian *et al.* 1999), and the proposed method using μ_2 by Li (1998) provides sub-optimal solutions (Whidborne *et al.* 1999).

The main advantage of the proposed approach over eigenvalue sensitivity methods is that the method can be simply extended to include plant uncertainty and to ensure robust performance, by adding extra uncertainty blocks representing plant uncertainty and one additional block to convert from a robust performance to a robust FWL stability problem (Dahleh and Khammash 1993, Khammash and Pearson 1993). Future work will address this issue. In addition, in on-going work, the authors are utilizing evolutionary algorithms to address the relevant sparseness issues (Whidborne 1999).

Acknowledgments

The authors thank Prof. Y. Hori and Y. Chun of the University of Tokyo for information on the steel rolling mill problem, and Dr. I. Njabaleke formerly of the University of Bath for information on the fluid power system. The authors are also grateful to an anonymous reviewer for suggestions.

Appendix. FWL stability measures

Let the discrete-time linear state space plant, $G(z)$, be strictly proper, i.e. $D_g = 0$ in (1). Then, from (5), the closed loop system transition matrix $\bar{A}$ is given by

$$\bar{A} = \begin{bmatrix} A_k & B_k C_g \\ B_g C_k & A_g + B_g D_k C_g \end{bmatrix} \tag{68}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & A_g \end{bmatrix} + \begin{bmatrix} I_{n_k} & 0 \\ 0 & B_g \end{bmatrix} \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \begin{bmatrix} I_{n_g} & 0 \\ 0 & C_g \end{bmatrix} \tag{69}$$

$$= M_0 + M_1 K M_2 = \bar{A}(K) \tag{70}$$

where K is the controller parameterization matrix given by (2).

A lower bound on $\mu_0(K)$ is defined as (Istepanian *et al.* 1998 a)

$$\mu_1(K) := \min_{m \in \{1,\dots,n_k+n_g\}} \frac{1 - |\lambda_m|}{\sum_{j=1}^{n_k+n_u} \sum_{i=1}^{n_k+n_y} \left| \frac{\partial \lambda_m}{\partial p_{i,j}} \right|} \tag{71}$$

where $\{\lambda_m\}$: $(m = 1, \dots, n_k + n_g)$ represents the set of all eigenvalues of $\bar{A}(K)$. The term $\partial\lambda_m/\partial K$ can be easily computed (Istepanian *et al.* 1998 a, Li 1998), hence $\mu_1(K)$ is a tractable stability measure of K with FWL considerations. Based on $\mu_1(K)$, one can compute integer $\hat{B}_{s1}^{\min}$ as a super estimation of $B_s^{\min}$, the minimal word-length that can guarantee the closed loop system stability when subject to FWL perturbations.

In Li (1998), another stability robustness measure with FWL considerations is given as

$$\mu_2(K) := \min_{m \in \{1,\dots,n_k+n_g\}} \frac{1 - |\lambda_m|}{\sqrt{N \sum_{j=1}^{n_k+n_u} \sum_{i=1}^{n_k+n_y} \left| \frac{\partial \lambda_i}{\partial p_j} \right|^2}} \tag{72}$$

where $N = (n_k + n_u) \times (n_k + n_y)$. The measure μ_2 is also a lower bound of $\mu_0(K)$. Similarly, based on $\mu_2(K)$, one can obtain $\hat{B}_{s2}^{\min}$ as a estimation of $B_s^{\min}$. It can be easily seen from (71) and (72) that $\mu_2(K) \le \mu_1(K) \le \mu_0(K)$.

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