Discrete Mathematics (có hướng dẫn)

Discrete Mathematics (Trường Đại học FPT)

Name:..

Class:..

Discrete Mathematics and Its Applications

Exercise Book

Trần Trọng Huỳnh - 2020

Chapter 1: The Foundations: Logic and Proofs

1.1 Propositional Logic

1. Which of these sentences are propositions? What are the truth values of those that are propositions?

a) Boston is the capital of Massachusetts.

b) Miami is the capital of Florida.

c) $2 + 3 = 5$.

d) $5 + 7 = 10$.

e) $x + 2 = 11$.

f) Answer this question.

a) This is a true proposition.

b) This is a false proposition (Tallahassee is the capital).

c) This is a true proposition.

d) This is a false proposition.

e) This is not a proposition (it contains a variable; the truth value depends on the value assigned to x).

f) This is not a proposition, since it does not assert anything.

2. What is the negation of each of these propositions?

a) Mei has an MP3 player.

b) There is no pollution in New Jersey.

c) $2 + 1 = 3$.

d) The summer in Maine is hot and sunny.

a) Mei does not have an MP3 player.

b) There is pollution in New Jersey.

c) 2+1#3

d) The summer in Maine is not hot or not sunny.

3. Let p and q be the propositions

p : I bought a lottery ticket this week.

q : I won the million dollar jackpot.

Express each of these propositions as an English sentence.

a) $\neg p$

b) $p \vee q$

c) $p \rightarrow q$

d) $p \wedge q$

e) $p \leftrightarrow q$

f) $\neg p \rightarrow \neg q$

g) $\neg p \wedge \neg q$

h) $\neg p \vee (p \wedge q)$

4. Let p and q be the propositions

p : It is below freezing.

q : It is snowing.

Write these propositions using p and q and logical connectives (including negations).

a) It is below freezing and snowing.

b) It is below freezing but not snowing.

c) It is not below freezing and it is not snowing.

d) It is either snowing or below freezing (or both)

$a)\ p \wedge q$

$b)\ p \wedge \neg q$

$c)\ \neg p \wedge \neg q$

$d)\ p \vee q$

5. Let p and q be the propositions

p : You drive over 65 miles per hour.

q : You get a speeding ticket.

Write these propositions using p and q and logical connectives (including negations).

a) You do not drive over 65 miles per hour.

b) You drive over 65 miles per hour, but you do not get a speeding ticket.

c) You will get a speeding ticket if you drive over 65 miles per hour.

d) If you do not drive over 65 miles per hour, then you will not get a speeding ticket.

e) Driving over 65 miles per hour is sufficient for getting a speeding ticket.

f) You get a speeding ticket, but you do not drive over 65 miles per hour.

g) Whenever you get a speeding ticket, you are driving over 65 miles per hour.

$a)\ \neg p$

$b)\ p \wedge \neg q$

$c)\ p \rightarrow q$

$d)\ \neg p \rightarrow \neg q$

$e)\ p \rightarrow q$

$f)\ \neg p \wedge q$

$g)\ q \rightarrow p$

6. Determine whether these biconditionals are true or false.

a) 2 + 2 = 4 if and only if 1 + 1 = 2. T

b) 1 + 1 = 2 if and only if 2 + 3 = 4. F

c) 1 + 1 = 3 if and only if monkeys can fly. T

d) 0 > 1 if and only if 2 > 1. F

7. Determine whether each of these conditional statements is true or false.

a) If 1 + 1 = 2, then 2 + 2 = 5. F

b) If 1 + 1 = 3, then 2 + 2 = 4. T

c) If 1 + 1 = 3, then 2 + 2 = 5. T

d) If monkeys can fly, then 1 + 1 = 3. T

8. Determine whether each of these conditional statements is true or false.

a) If 1 + 1 = 3, then unicorns exist. T

b) If 1 + 1 = 3, then dogs can fly. T

c) If 1 + 1 = 2, then dogs can fly. F

d) If 2 + 2 = 4, then 1 + 2 = 3. T

9. Write each of these statements in the form "if p, then q"

a) It is necessary to wash the boss's car to get promoted.

If I get promoted, then I wash the boss's car.

It is necessary to q to p ⇔ q is necessary for p ⇔ if p, then q

b) Winds from the south imply a spring thaw.

If Winds are from the south, then a spring thaw.

c) A sufficient condition for the warranty to be good is that you bought the computer less than a year ago.

If you bought the computer less than a year ago, then the warranty is good.

d) Willy gets caught whenever he cheats.

If Willy cheats, then he gets caught

e) You can access the website only if you pay a subscription fee.

If You can access the website, then you pay a subscription fee

f) Getting elected follows from knowing the right people.

If you know the right people, then you get elected

g) Carol gets seasick whenever she is on a boat.

If Carol is on a boat, then she gets seasick.

10. How many rows appear in a truth table for each of these compound propositions?

a) $p \to \neg p$

b) $(p \vee \neg r) \wedge (q \vee \neg s)$

c) $q \vee p \vee \neg s \vee \neg r \vee \neg t \vee u$

d) $(p \wedge r \wedge t) \leftrightarrow (q \wedge t)$

e) $(q \to \neg p) \vee (\neg p \to \neg q)$

f) $(p \vee \neg t) \wedge (p \vee \neg s)$

g) $(p \to r) \vee (\neg s \to \neg t) \vee (\neg u \to v)$

h) $(p \wedge r \wedge s) \vee (q \wedge t) \vee (r \wedge \neg t)$

A truth table will need 2^n rows if there are n variables.

11. Construct a truth table for each of these compound propositions.

a) $p \wedge \neg p$

b) $p \vee \neg p$

c) $(p \vee \neg q) \to q$

d) $(p \vee q) \to (p \wedge q)$

e) $(p \to q) \leftrightarrow (\neg q \to \neg p)$

f) $(p \to q) \to (q \to p)$

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \to \neg p$	$(p \to q) \leftrightarrow (\neg q \to \neg p)$

1	1	0	0	1	1	1
1	0	0	1	0	0	1
0	1	1	0	1	1	1
0	0	1	1	1	1	1

12. Construct a truth table for each of these compound propositions.

a) $p \rightarrow \neg p$ b) $p \leftrightarrow \neg p$ c) $p \oplus (p \vee q)$

d) $(p \wedge q) \rightarrow (p \vee q)$ e) $(q \rightarrow \neg p) \leftrightarrow (p \leftrightarrow q)$ f) $(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$

p	q	$\neg q$	$p \leftrightarrow q$	$p \leftrightarrow \neg q$	$(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$
1	1	0	1	0	1
1	0	1	0	1	1
0	1	0	0	1	1
0	0	1	1	0	1

13. What is the value of x after each of these statements is encountered in a computer program, if x = 1 before the statement is reached?

a) if x + 2 = 3 then x := x + 1 $x = 2$

b) if (x + 1 = 3) OR (2x + 2 = 3) then x := x + 1 $x = 1$

c) if (2x + 3 = 5) AND (3x + 4 = 7) then x := x + 1 $x = 2$

d) if (x + 1 = 2) XOR (x + 2 = 3) then x := x + 1 $x = 1$

14. Find the bitwise OR, bitwise AND, and bitwise XOR of each of these pairs of bit strings.

a) 101 1110, 010 0001 b) 1111 0000, 1010 1010

c) 00 0111 0001, 10 0100 1000 d) 11 1111 1111, 00 0000 0000

101 1110

010 0001

OR 111 1111

AND 000 0000

XOR 111 1111

a) bitwise *OR* = 111 1111; bitwise *AND*= 000 0000; bitwise *XOR* = 111 1111

b) bitwise *OR* = 1111 1010; bitwise *AND*= 1010 0000; bitwise *XOR* = 0101 1010

c) bitwise *OR*= 10 0111 1001; bitwise *AND*= 00 0100 0000; bitwise *XOR*= 10 0011 1001

d) bitwise *OR*= 11 1111 1111; bitwise *AND*= 00 0000 0000; bitwise *XOR*= 11 1111 1111

15. Evaluate each of these expressions.

a) 1 1000 ∧ (0 1011 ∨ 1 1011) = 1 1000 ∧ 1 1011 = 1 1000

$$\begin{array}{rr} & 0\,1011 \\ \text{OR} & 1\,1011 \\ \hline & 1\,1011 \\ \text{AND} & 1\,1000 \\ \hline & 1\,1000 \end{array}$$

b) (0 1111 ∧ 1 0101) ∨ 0 1000

c) (0 1010 ⊕ 1 1011) ⊕ 0 1000

d) (1 1011 ∨ 0 1010) ∧ (1 0001 ∨ 1 1011)

1.2-Propositional Equivalences

1. Show that each of these conditional statements is a tautology by using truth tables.

a) $(p \land q) \to p$ b) $p \to (p \lor q)$ c) $\neg p \to (p \to q)$

d) $(p \land q) \to (p \to q)$ e) $\neg(p \to q) \to p$ f) $\neg(p \to q) \to \neg q$

2. Show that each of these conditional statements is a tautology by using truth tables.

a) $[\neg p \land (p \lor q)] \to q$ b) $[(p \to q) \land (q \to r)] \to (p \to r)$

c) $[p \land (p \to q)] \to q$ d) $[(p \lor q) \land (p \to r) \land (q \to r)] \to r$

p	q	r	$p \lor q$	$p \to r$	$q \to r$	$[(p \lor q) \land (p \to r) \land (q \to r)]$	$[(p \lor q) \land (p \to r) \land (q \to r)] \to r$
1	1	1	1	1	1	1	1
1	1	0	1	0	0	0	1
1	0	1	1	1	1	1	1
1	0	0	1	0	1	0	1
0	1	1	1	1	1	1	1
0	1	0	1	1	0	0	1
0	0	1	0	1	1	0	1
0	0	0	0	1	1	0	1

$$[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$$
$$\equiv (p \vee q) \wedge (\neg p \vee r) \wedge (\neg q \vee r) \to r$$
$$\equiv (p \vee q) \wedge \left[(\neg p \wedge \neg q) \vee r\right] \to r$$
$$\equiv (p \vee q) \wedge \left[\neg(p \vee q) \vee r\right] \to r$$
$$\equiv \left[(p \vee q) \wedge r\right] \to r$$

3. Determine whether $(\neg p \land (p \to q)) \to \neg q$ is a tautology.

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg p \land (p \to q)$	$(\neg p \land (p \to q)) \to \neg q$
1	1	0	0	1	0	1
1	0	0	1	0	0	1
0	1	1	0	1	1	0
0	0	1	1	1	1	1

4. Determine whether $(\neg q \land (p \to q)) \to \neg p$ is a tautology.

p	q	¬p	¬q	p→q	¬q ∧ (p → q)	(¬q ∧ (p → q)) → ¬p
1	1	0	0	1	0	1
1	0	0	1	0	0	1
0	1	1	0	1	0	1
0	0	1	1	1	1	1

5. Show that (p ∨ q) ∧ (¬p ∨ r) → (q ∨ r) is a tautology.

p	q	r	¬p	p ∨ q	¬p ∨ r	(p ∨ q) ∧ (¬p ∨ r)	q ∨ r	(p ∨ q) ∧ (¬p ∨ r) → (q ∨ r)
1	1	1	0	1	1	1	1	1
1	1	0	0	1	0	0	1	1
1	0	1	0	1	1	1	1	1
1	0	0	0	1	0	0	0	1
0	1	1	1	1	1	1	1	1
0	1	0	1	1	1	1	1	1
0	0	1	1	0	1	0	1	1
0	0	0	1	0	1	0	0	1

6. Show that (p → q) → r and p → (q → r) are not logically equivalent.

p	q	r	p→q	q → r	(p → q) → r	p → (q → r)
1	1	1	1	1	1	1
1	1	0	1	0	0	0
1	0	1	0	1	1	1
1	0	0	0	1	1	1
0	1	1	1	1	1	1
0	1	0	1	0	0	1
0	0	1	1	1	1	1
0	0	0	1	1	0	1

7. Show that (p ∧ q) → r and (p → r) ∧ (q → r) are not logically equivalent.

8. Show that (p → q) → (r → s) and (p → r) → (q → s) are not logically equivalent.

The **dual** of a compound proposition that contains only the logical operators ∨, ∧, and ¬ is the compound proposition obtained by replacing each $\vee$ by $\wedge$, each $\wedge$ by $\vee$, each **T** by **F**, and each **F** by **T**.

9. Find the dual of each of these compound propositions.

a) $p \vee \neg q$ b) $p \wedge (q \vee (r \wedge T))$ c) $(p \wedge \neg q) \vee (q \wedge F)$

a) $p \wedge \neg q$ b) $p \vee (q \wedge (r \vee F))$ c) $(p \vee \neg q) \wedge (q \vee T)$

1.4 Predicates and Quantifier

1. Let P(x) denote the statement "$x \leq 4$". What are these truth values?

a) P(0) b) P(4) c) P(6)

2. Let P(x) be the statement "the word x contains the letter a." What are these truth values?

a) P (orange) b) P (lemon) c) P (true) d) P (false)

3. Let Q(x, y) denote the statement "x is the capital of y". What are these truth values?

a) Q(Denver, Colorado) b) Q(Detroit, Michigan)

c) Q(Massachusetts, Boston) d) Q(New York, New York)

a) This is true.

b) This is false, since Lansing, not Detroit, is the capital.

c) This is false (but Q(Boston, Massachusetts) is true).

d) This is false, since Albany, not New York, is the capital.

4. State the value of x after the statement if P(x) then x := 1 is executed, where P(x) is the statement "x > 1", if the value of x when this statement is reached is

a) x = 0. b) x = 1. c) x = 2.

5. Let P(x) be the statement "x spends more than five hours every weekday in class," where the domain for x consists of all students. Express each of these

a) $\exists x P(x)$ b) $\forall x P(x)$ c) $\exists x \neg P(x)$ d) $\forall x \neg P(x)$

a) There is a student who spends more than five hours every weekday in class.

b) Every student spends more than five hours every weekday in class.

c) There is a student who does not spend more than five hours every weekday in class.

d) No student spends more than five hours every weekday in class (Or, equivalently, every student spends less than or equal to five hours every weekday in class.)

6. Translate these statements into English, where C(x) is "x is a comedian" and F(x) is "x is funny" and the domain consists of all people.

a) $\forall x(C(x) \rightarrow F(x))$ b) $\forall x(C(x) \wedge F(x))$

c) $\exists x(C(x) \rightarrow F(x))$ d) $\exists x(C(x) \wedge F(x))$

a) Every comedian is funny

b) Every person is a funny comedian

c) There exists a person such that if she/he is a comedian, then she/he is funny

d) There exists a funny comedian or Some comedians are funny or Some funny people are comedians.

7. Let P(x) be the statement "$x = x^2$". If the domain consists of the integers, what are these truth values?

a) $P(0)$ b) $P(1)$ c) $P(2)$

d) $P(-1)$ e) $\exists xP(x)$ f) $\forall xP(x)$

a) T b) T c) F

d) F e) T f) F

8. Let Q(x) be the statement "$x + 1 > 2x$." If the domain consists of all integers, what are these truth values?

a) $Q(0)$ b) $Q(-1)$ c) $Q(1)$

d) $\exists xQ(x)$ e) $\forall xQ(x)$ f) $\exists x\neg Q(x)$

a) T b) T c) F

d) T e) F f) T

9. Determine the truth value of each of these statements if the domain consists of all integers.

a) $\forall n(n + 1 > n)$ b) $\exists n(2n = 3n)$ c) $\exists n(n = -n)$ d) $\forall n(3n \leq 4n)$

10. Determine the truth value of each of these statements if the domain consists of all real numbers.

a) $\exists x(x^3 = -1)$ b) $\exists x(x^4 < x^2)$ c) $\forall x((-x)^2 = x^2)$ d) $\forall x(2x > x)$

11. Determine the truth value of each of these statements if the domain for all variables consists of all integers.

a) $\forall n(n^2 \geq 0)$ b) $\exists n(n^2 = 2)$ c) $\forall n(n^2 \geq n)$ d) $\exists n(n^2 < 0)$

12. Suppose that the domain of the propositional function P(x) consists of the integers 0, 1, 2, 3, and 4. Write out each of these propositions using disjunctions, conjunctions, and negations.

a) $\exists x P(x)$ b) $\forall x P(x)$ c) $\exists x \neg P(x)$

d) $\forall x \neg P(x)$ e) $\neg\exists x P(x)$ f) $\neg\forall x P(x)$

$a)\ P(0) \vee P(1) \vee P(2) \vee P(3) \vee P(4)$

$b)\ P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge P(4)$

$c)\ \neg P(0) \vee \neg P(1) \vee \neg P(2) \vee \neg P(3) \vee \neg P(4)$

$d)\ \neg P(0) \wedge \neg P(1) \wedge \neg P(2) \wedge \neg P(3) \wedge \neg P(4)$

$e)\ \neg\left(P(0) \vee P(1) \vee P(2) \vee P(3) \vee P(4)\right)$

$f)\ \neg\left(P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge P(4)\right)$

13. Suppose that the domain of the propositional function P(x) consists of the integers 1, 2, 3, 4, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

a) $\exists x P(x)$ b) $\forall x P(x)$ c) $\neg\exists x P(x)$ d) $\neg\forall x P(x)$

1.5 Rules of Inference

1. Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?

If Socrates is human, then Socrates is mortal.

Socrates is human.

∴Socrates is mortal.

This is modus ponens. The first statement is $p \to q$, where p is "Socrates is human" and q is "Socrates is mortal." The second statement is p. The third is q. Modus ponens is valid.

2. Use rules of inference to show that the hypotheses "Randy works hard", "If Randy works hard, then he is a dull boy", and "If Randy is a dull boy, then he will not get the job" imply the conclusion "Randy will not get the job".

Let w be the proposition "Randy works hard," let d be the proposition "Randy is a dull boy," and let j be the proposition "Randy will get the job." We are given premises w, $w \to d$, and $d \to \neg j$. We want to conclude $\neg j$.

Step	Reason
1. w	Hypothesis
2. $w \to d$	Hypothesis
3. d	Modus ponens using (2) and (3)
4. $d \to \neg j$	Hypothesis
5. $\neg j$	Modus ponens using (3) and (4)

3. For each of these collections of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.

a) "If I take the day off, it either rains or snows". "I took Tuesday off or I took Thursday off". "It was sunny on Tuesday". "It did not snow on Thursday".

Because It was sunny on Tuesday so we assume that it did not rain and snow on Tuesday. Then we conclude that I did not take Tuesday off (Modus tollen). We now apply disjunctive syllogism to the disjunction in light of this conclusion, we see that I took Thursday off. Now use modus ponens on the universal instantiation of the given conditional statement applied to Thursday; we conclude that it rained or snowed on Thursday. One more application of disjunctive syllogism tells us that it rained on Thursday.

b) “If I eat spicy foods, then I have strange dreams”. “I have strange dreams if there is thunder while I sleep”. “I did not have strange dreams”.

Using modus tollens we conclude two things-that I did not eat spicy food and that it did not thunder. Therefore by the conjunction rule of inference, we conclude "I did not eat spicy food and it did not thunder."

c) “I am either clever or lucky”. “I am not lucky”. “If I am lucky, then I will win the lottery”

By disjunctive syllogism from the first two hypotheses we conclude that I am clever.

d) “Every computer science major has a personal computer”. “Ralph does not have a personal computer”. “Ann has a personal computer”.

Modus tollens tells us that Ralph is not a CS major. There are no conclusions to be drawn about Ann.

e) “What is good for corporations is good for the United States”. “What is good for the United States is good for you”. “What is good for corporations is for you to buy lots of stuff”.

The first two conditional statements can be phrased as "If x is good for corporations, then x is good for the U.S." and "If x is good for the U.S., then x is good for you." If we now apply universal instantiation with x being "for you to buy lots of stuff," then we can conclude using modus ponens twice that for you to buy lots of stuff is good for the U.S. and is good for you.

f) “All rodents gnaw their food”. “Mice are rodents”. “Rabbits do not gnaw their food”. “Bats are not rodents.

The given conditional statement is "For all *x,* if *x* is a rodent, then *x* gnaws its food." We can form the universal instantiation of this with *x* being a mouse, a rabbit, and a bat. Then modus ponens allows us to conclude that mice gnaw their food; and modus tollens allows us to conclude that rabbits are not rodents. We can conclude nothing about bats.

4. Determine whether each of the following arguments is valid or not valid.

a) All parrots like fruit. My pet bird is not a parrot. Therefore, my pet bird does not like fruit.

This is invalid. After applying universal instantiation, it contains the fallacy of denying the hypothesis.

b) Everyone who eats granola every day is healthy. Linda is not healthy. Therefore, Linda does not eat granola every day.

This is valid by universal instantiation and modus tollens.

c) No man is an island. Manhattan is an island. Therefore, Manhattan is not a man.

This is valid by universal instantiation and modus tollens.

d) A convertible car is fun to drive. Isaac's car is not a convertible. Therefore, Isaac's car is not fun to drive.

This is invalid. After applying universal instantiation, it contains the fallacy of denying the hypothesis

e) If Mai knows French, Mai is smart. But Mai doesn't know French. So, she is not smart.

This is invalid. After applying universal instantiation, it contains the fallacy of denying the hypothesis

f) Lin can't go fishing if she doesn't have a bike. Last week, Lin went fishing with her friends. Therefore, she has got a bike.

This is valid by universal instantiation and modus tollens.

Chapter 2: Basic Structures: Sets, Functions, Sequences, and Sums

2.1 Sets

1. List the members of these sets.

a) {x | x is a real number such that $x^2 = 1$}

b) {x | x is a positive integer less than 12}

c) {x | x is the square of an integer and x < 100}

d) {x | x is an integer such that $x^2 = 2$}

2. For each of the following sets, determine whether 2 is an element of that set.

a) {x ∈ R | x is an integer greater than 1} b) {x ∈ R | x is the square of an integer}

c) {2,{2}} d) {{2},{{2}}} e) {{2},{2,{2}}} f) {{{2}}}

3. Determine whether each of these statements is true or false.

a) 0 ∈ ∅ b) ∅ ∈ {0} c) {0} ⊂ ∅ d) ∅ ⊂ {0}

e) {0} ∈ {0} f) {0} ⊂ {0} g) {∅} ⊆ {∅}

4. Determine whether each of these statements is true or false.

a) x ∈ {x} b) {x} ⊆ {x} c) {x} ∈ {x}

d) {x} ∈ {{x}} e) ∅ ⊆ {x} f) ∅ ∈ {x}

5. What is the cardinality of each of these sets?

a) {a} b) {{a}} c) {a, {a}} d) {a, {a}, {a, {a}}}

6. What is the cardinality of each of these sets?

a) ∅ b) {∅} c) {∅, {∅}} d) {∅, {∅}, {∅, {∅}}}

7. Find the power set of each of these sets, where a and b are distinct elements.

a) {a} b) {a, b} c) {∅, {∅}}

a) $P\left(\{a\}\right)=\{\varnothing,\{a\}\}$

b) $P\left(\{a,b\}\right)=\{\varnothing,\{a\},\{b\},\{a,b\}\}$

c) $P\left(\{\varnothing,\{\varnothing\}\}\right)=\{\varnothing,\{\varnothing\},\{\{\varnothing\}\},\{\varnothing,\{\varnothing\}\}\}$

8. How many elements does each of these sets have where a and b are distinct elements?

a) P({a, b, {a, b}}) 8 b) P({∅, a, {a}, {{a}}}) 16 c) P(P(∅)) 2

$S=\{a,b,\{a,b\}\}\Rightarrow |S|=3\Rightarrow P(S)=2^3=8$

$P(S)=\{\varnothing,\{a\},\{b\},\{\{a,b\}\},\{a,b\},\{a,\{a,b\}\},\{b,\{a,b\}\},S\}$

9. Find A^2 and A^3 if

a) A = {1, 3} b) A = {1, a }

$A^2 = A \times A = \{(1,1),(1,3),(3,1),(3,3)\}$

$A^3 = A \times A \times A = \{(1,1,1),(1,1,3),(1,3,1),(1,3,3),(3,1,1),(3,1,3),(3,3,1),(3,3,3)\}$

10. Let $A = \{1,2,3\}$ and $B = \{1, a\}$. What is the cardinality of each of these sets?

$a)\ A \times B$ 6 $b)\ A^2$ 9 $c)\ P(B)$ 4

$d)\ P(B \times A)$ 64 $e)\ A \cup B$ 4

11. Find the truth set of each of these predicates where the domain is the set of integers.

a) P(x): $x^2 < 3$ b) Q(x): $x^2 > x$ c) R(x): $2x + 1 = 0$

$\{-1,0,1\}$ $\mathbb{Z} \setminus \{0,1\}$ $\varnothing$

2.2 Set operations

1. Let A = {1, 2, 3, 4, 5} and B = {0, 3, 6}. Find

a) A ∪ B b) A ∩ B c) A − B d) B − A.

$a) A \cup B = \{0,1,2,3,4,5,6\}$

$b) A \cap B = \{3\}$

$c) A - B = \{1, 2, 4, 5\}$

$d) B - A = \{0,6\}$

2. Let A = {a, b, c, d, e} and B = {a, b, c, d, e, f, g, h}. Find

a) A ∪ B b) A ∩ B c) A − B d) B − A.

$a) A \cup B = B$

$b) A \cap B = A$

$c) A - B = \varnothing$

$d) B - A = \{f, g, h\}$

3. Find the sets A and B if A − B = {1, 5, 7, 8}, B − A = {2, 10}, and A ∩ B = {3, 6, 9}.

$A - B = \{1, 5, 7, 8\} \Rightarrow 1, 5, 7, 8 \in A$

$B - A = \{2,10\} \Rightarrow 2,10 \in B$

$A \cap B = \{3, 6, 9\}$

$A = \{1,3,5,6,7,8,9\}$

$B = \{2,3,6,9,10\}$

4. Let A and B be sets. Show that

a) (A ∩ B) ⊆ A b) A ⊆ (A ∪ B) c) A − B ⊆ A

d) A ∩ (B − A) = ∅ e) A ∪ (B − A) = A ∪ B f) A ⊕ B = (A ∪ B) − (A ∩ B).

$A\cup(B-A)=A\cup B$

$+\forall x\in A\cup(B-A)\Rightarrow x\in A\vee x\in B-A$

$\Rightarrow\left[\begin{array}{l}x\in A\\ x\in B\wedge x\notin A\end{array}\right.\Rightarrow x\in A\cup B\Rightarrow A\cup(B-A)\subset A\cup B$

$+\forall x\in A\cup B\Rightarrow\left[\begin{array}{l}x\in A\\ x\in B\end{array}\right.\Rightarrow\left[\begin{array}{l}x\in A\\ x\in B\wedge x\in A\\ x\in B\wedge x\notin A\end{array}\right.\Rightarrow\left[\begin{array}{l}x\in A\\ x\in A\cap B\\ x\in B-A\end{array}\right.$

$\Rightarrow x\in A\cup(B-A)\Rightarrow A\cup B\subset A\cup(B-A)$

5. Suppose that the universal set is U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. Express each of these sets with bit strings where the i^{th} bit in the string is 1 if i is in the set and 0 otherwise.

a) {3, 4, 5} ="00 1110 0000"

b) {1, 3, 6, 10} ="10 1001 0001"

c) {2, 3, 4, 7, 8, 9} ="01 1100 1110"

2.3 Functions

1. Why is *f* not a function from R to R if

a) $f(x)=\frac{1}{x}$ F b) $f(x)=\sqrt{x}$ F c) $f(x)=\pm\sqrt{x^2+1}$ F

2. Determine whether *f* is a function from Z to R if

a) $f(n)=\pm n$ F b) $f(n)=\sqrt{n^2+1}$ T c) $f(n)=\frac{1}{n^2-4}$ F

3. Find these values

a) $\lceil 1.1\rceil$ b) $\lceil -0.1\rceil$ c) $\lceil 4\rceil$ d) $\lfloor 3.2\rfloor$

e) $\lfloor -5.2\rfloor$ e) $\lfloor 2\rfloor$ f) $\left\lfloor \frac{1}{2}+\left\lceil\frac{2}{3}\right\rceil\right\rfloor$

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4. Determine whether each of these functions from {a, b, c, d} to itself is one-to-one (onto)

a) f (a) = b, f (b) = a, f (c) = c, f (d) = d onto and one – to – one

b) f (a) = b, f (b) = b, f (c) = d, f (d) = c not onto but not one – to – one

c) f (a) = d, f (b) = b, f (c) = c, f (d) = d not onto but not one – to – one

5. Determine whether each of these functions from Z to Z is one-to-one (onto)

a) f(n) = n − 1 b) f(n) = n^2 + 1 c) f(n) = n^3 d) $f(n)=\left\lceil \frac{n}{2} \right\rceil$

a) onto and one – to – one b) not onto but not one – to – one

c) not onto and one – to – one d) onto but not one – to – one

$c)\ f(n)=n^3$

$+\forall a,b\in\mathbb{Z}: a\neq b\Rightarrow a^3\neq b^3\Rightarrow f(a)\neq f(b)\Rightarrow f \text{ is one to one}$

$+2\in\mathbb{Z}, \nexists n\in\mathbb{Z}: n^3=2\Rightarrow f(n)\neq 2\Rightarrow f \text{ is not onto}$

$d)\ f(1)=\left\lceil \frac{1}{2}\right\rceil=1,\ f(2)=\left\lceil \frac{2}{2}\right\rceil=1\Rightarrow f$ is not one to one

$\forall m\in\mathbb{Z},\ n=2m: f(n)=\left\lceil \frac{2m}{2}\right\rceil=\lceil m\rceil=m\Rightarrow f$ is onto

6. Determine whether f : Z × Z → Z is onto if

a) f(m, n) = 2m − n b) f(m, n) = $m^2 - n^2$ c) f(m, n) = m + n + 1

onto not onto onto

với $p=2$, ta tìm $m,n\in\mathbb{Z}: f(m,n)=2$

$\Leftrightarrow m^2-n^2=2$

$\Leftrightarrow (m-n)(m+n)=2$

vì $m-n\in\mathbb{Z},\ m+n\in\mathbb{Z}$ nên ta có

$$\begin{cases} m+n=2\\ m-n=1\end{cases}\vee\begin{cases} m+n=1\\ m-n=2\end{cases}\vee\begin{cases} m+n=-1\\ m-n=-2\end{cases}\vee\begin{cases} m+n=-2\\ m-n=-1\end{cases}$$

Giải các hpt trên ta suy ra $\nexists (m,n)\in\mathbb{Z}\times\mathbb{Z}: f(m,n)=2$

d) f (m, n) = |m| − |n| e) f (m, n) = m^2 − 4 f) f (m, n) = m + n

onto not onto onto

$\forall p\geq 0, \exists m=2p, \exists n=p: f(m,\ n)=|m|-|n|=p$

$\forall p<0, \exists m=p, \exists n=2p: f(m,\ n)=|m|-|n|=|p|-|2p|=-|p|=p$

7. Determine whether each of these functions is a bijection from R to R.

a) f (x) = −3x + 4 bijection b) f (x) = $-3x^2$ + 7 not bijection

c) f (x) = (x + 1)/(x + 2) not bijection d) f (x) = x^5 + 1 bijection

$+f$ is one – to – one because

$$\forall x_1, x_2 \in \mathbb{R}: f(x_1) = f(x_2)$$

$$\Leftrightarrow x_1^5 + 1 = x_2^5 + 1$$

$$\Leftrightarrow x_1^5 = x_2^5$$

$$\Leftrightarrow x_1 = x_2$$

$+f$ is onto because $\forall y \in \mathbb{R}, \exists x = \sqrt[5]{y-1} : f(x) = f\left(\sqrt[5]{y-1}\right) = y - 1 + 1 = y$

$f : X \to Y$, $A \subset X$, $B \subset Y$

$f(A) = \left\{ f(x) \middle| x \in S \right\} \subset Y$

$y \in f(A) \Leftrightarrow \exists x \in A : y = f(x)$

$f^{-1}(B) = \left\{ x \in X \middle| f(x) \in B \right\} \subset X$

$x \in f^{-1}(B) \Leftrightarrow f(x) \in B$

8. Let S = {−1, 0, 2, 4, 7}. Find f(S) if

a) f(x) = 1 b) f(x) = 2x + 1 c) $f(x) = \left\lceil \frac{x}{5} \right\rceil$

$f(S) = \{1\}$ $f(S) = \{-1, 1, 5, 9, 15\}$ $f(S) = \{0, 1, 2\}$

9. Let f be the function from R to R defined by f(x) = x^2. Find

a) $f^{-1}(\{1\})$ b) $f^{-1}(\{x \mid 0 < x < 1\})$ c) $f^{-1}(\{x \mid x > 4\})$

$f^{-1}\left(\{1\}\right) = \left\{ x \in \mathbb{R} \middle| f(x) = 1 \right\} = \left\{ x \in \mathbb{R} \middle| x^2 = 1 \right\} = \{-1, 1\}$

$f^{-1}\left(\left\{ x \in \mathbb{R} \middle| 0 < x < 1 \right\}\right) = \left\{ x \in \mathbb{R} \middle| 0 < f(x) < 1 \right\} = \left\{ x \in \mathbb{R} \middle| 0 < x^2 < 1 \right\} = (-1, 1) \setminus \{0\}$

$f^{-1}\left(\left\{ x \in \mathbb{R} \middle| x > 4 \right\}\right) = \left\{ x \in \mathbb{R} \middle| f(x) > 4 \right\} = \left\{ x \in \mathbb{R} \middle| x^2 > 4 \right\} = (-\infty, -2) \cup (2, \infty)$

2.4 Sequences and Summations

1. Find these terms of the sequence $\{a_n\}$, where $a_n = 2(-3)^n + 5n$.

a) a_0 b) a_1 c) a_4 d) a_5

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2. What is the term a_8 of the sequence $\{a_n\}$ if an equals

a) 2n−1? b) 7? c) $1 + (-1)^n$? d) $-(-2)^n$?

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3. Find the first five terms of the sequence defined by each of these recurrence relations and initial conditions.

a) $a_n = 6a_{n-1}$, $a_0 = 2$ b) $a_n = a_{n-1}^2$, $a_1 = 2$ c) $a_n = a_{n-1} + 3a_{n-2}$, $a_0 = 1$, $a_1 = 2$

$a_2 = a_1 + 3a_0 = 2 + 3.1 = 5$
$a_3 = a_2 + 3a_1 = 5 + 3.2 = 11$

4. Find the solution to each of these recurrence relations and initial conditions.

a) $a_n = -a_{n-1}$, $a_0 = 5$ b) $a_n = a_{n-1} + 3$, $a_0 = 1$ c) $a_n = a_{n-1} - n$, $a_0 = 4$

d) $a_n = 2na_{n-1}$, $a_0 = 3$ e) $a_n = 5a_{n-1} - 6a_{n-2}$, $a_0 = 2$, $a_1 = -1$

a) $a_n = -a_{n-1} = -(-a_{n-2}) = (-1)^2 a_{n-2} = (-1)^2(-a_{n-3}) = (-1)^3 a_{n-3} = ... = (-1)^n a_0 = (-1)^n.5$

b) $a_n = a_{n-1} + 3 = (a_{n-2} + 3) + 3 = a_{n-2} + 2.3 = (a_{n-3} + 3) + 2.3 = a_{n-3} + 3.3 = ... = a_0 + n.3 = 1 + 3n$

c) $a_n = a_{n-1} - n$ -a = (-1)a

$$\left.\begin{array}{l} a_n - a_{n-1} = -n \\ a_{n-1} - a_{n-2} = -(n-1) \\ a_{n-2} - a_{n-3} = -(n-2) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_2 - a_1 = -2 \\ a_1 - a_0 = -1 \end{array}\right\} \Rightarrow a_n = (a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + ... + (a_2 - a_1) + (a_1 - a_0) + a_0$$

$$\Rightarrow a_n = -[n + (n-1) + ... + 2 + 1] + 4 = 4 - \frac{n(n+1)}{2}$$

$$d)\ a_n = 2na_{n-1} = 2n[2(n-1)a_{n-2}] = 2^2 n(n-1)a_{n-2} = 2^2 n(n-1)[2(n-2)a_{n-3}]$$
$$= 2^3 n(n-1)(n-2)a_{n-3} = ... = 2^n n(n-1)(n-2)....1.a_0 = 3.2^n.n!$$

$e)\ a_n = 5a_{n-1} - 6a_{n-2}$

$$\Leftrightarrow a_n - 3a_{n-1} = 2a_{n-1} - 6a_{n-2} = 2(a_{n-1} - 3a_{n-2}) = 2[2(a_{n-2} - 3a_{n-3})] = 2^2(a_{n-2} - 3a_{n-3}) = ... = 2^{n-1}(a_1 - 3a_0) = -7.2^{n-1}$$

$a_n = 5a_{n-1} - 6a_{n-2}$

$$\Leftrightarrow a_n - 2a_{n-1} = 3a_{n-1} - 6a_{n-2} = 3(a_{n-1} - 2a_{n-2}) = 3[3(a_{n-2} - 2a_{n-3})] = 3^2(a_{n-2} - 2a_{n-3}) = ... = 3^{n-1}(a_1 - 2a_0) = -5.3^{n-1}$$

Ta có hệ phương trình $\begin{cases} a_n - 3a_{n-1} = -7.2^{n-1} \\ a_n - 2a_{n-1} = -5.3^{n-1} \end{cases} \Leftrightarrow a_n = -5.3^n + 7.2^n$

5. What are the values of these sums?

a) $\sum_{k=1}^{5}(k+1)$ b) $\sum_{j=1}^{4}(j+2)^4$ c) $\sum_{i=0}^{2}\sum_{j=1}^{3}(2i-3j)$ d) $\sum_{i=1}^{3}\sum_{j=2}^{4} ij$

$$\sum_{i=0}^{2}\sum_{j=1}^{3}(2i-3j)=\sum_{i=0}^{2}\left[(2i-3)+(2i-6)+(2i-9)\right]=\sum_{i=0}^{2}(6i-18)$$

$$=(6.0-18)+(6.1-18)+(6.2-18)=-36$$

$$\sum_{i=1}^{3}\sum_{j=2}^{4}ij=\sum_{i=1}^{3}(2i+3i+4i)=\sum_{i=1}^{3}9i=9(1+2+3)=54$$

6. What are the values of these sums, where S = {1, 3, 5, 7}?

a) $\sum_{j\in S}\left(j+\frac{1}{j}\right)=\left(1+\frac{1}{1}\right)+\left(3+\frac{1}{3}\right)+\left(5+\frac{1}{5}\right)+\left(7+\frac{1}{7}\right)=\ldots$ b) $\sum_{j\in S}j^2$ c) $\sum_{j\in S}2=\sum_{j\in S}(2+0.j)$

7. What are the values of the following products?

a) $\prod_{i=0}^{10}i=0.1.2....10=0$ b) $\prod_{i=1}^{100}(-1)^i=(-1)^1.(-1)^2....(-1)^{100}=(-1)^{1+2+...+100}=(-1)^{5050}=1$

c) $\prod_{i=0}^{4}i!=0!1!2!3!4!=1.1.2.6.24=288$ $\prod_{i=1}^{n}x_i=x_1.x_2....x_n$

Chapter 3: The Fundamentals: Algorithms, the Integers, and Matrices

3.1 Algorithms

1. List all the steps used by Algorithm "max" to find the maximum of the list 1, 8, 12, 9, 11, 2, 14, 5, 10, 4.

max : = 1 ;

i : = 2, max < 8 $\Rightarrow$ max : = 8 ;

i : = 3, max < 12 $\Rightarrow$ max : = 12 ;

i : = 4 ; i : = 5 ; i : = 6 $\Rightarrow$ max : = 12 ;

i : = 7 , max < 14 $\Rightarrow$ max : = 14

i : = 8 ; i : = 9 ; i : = 10 ; i : = 11 $\Rightarrow$ a_i < max $\Rightarrow$ max : = 14

2. Devise an algorithm that finds the sum of all the integers in a list.

procedure *Sum* (a_1 , a_2 , ... , a_n: integers)

Sum:= a_1

for i := 2 to n

Sum:= *Sum* + a_i

return *Sum*

{*Sum* is the sum of all the elements in the list}

3. List all the steps used to search for 9 in the sequence 1, 3, 4, 5, 6, 8, 9, 11 using

a) a linear search b) a binary search.

$a)\ n=8,\ x=9$

$i:=1,(1\le 8\ and\ 9\ne 1)\Rightarrow i:=i+1=2$

$i:=2,(2\le 8\ and\ 9\ne 3)\Rightarrow i:=i+1=3$

$i:=3,(3\le 8\ and\ 9\ne 4)\Rightarrow i:=i+1=4$

$i:=4,(4\le 8\ and\ 9\ne 5)\Rightarrow i:=i+1=5$

$i:=5,(5\le 8\ and\ 9\ne 6)\Rightarrow i:=i+1=6$

$i:=6,(6\le 8\ and\ 9\ne 8)\Rightarrow i:=i+1=7$

$i:=7,(7\le 8\ and\ 9\ne 9)\Rightarrow\ false$

$7\le 8\Rightarrow location=7$

$b)\ n=8,\ x=9$

$i:=1<j:=8\ ,\ m=\left\lfloor \frac{i+j}{2}\right\rfloor=4\ ,\ a_4=5$

$9>5\Rightarrow i:=m+1=4+1=5$

$i:=5<j:=8,\ m=\left\lfloor \frac{i+j}{2}\right\rfloor=6\ ,\ a_6=8$

$9>8\Rightarrow i:=m+1=6+1=7$

$9>9(False)\Rightarrow\ j:=7$

$x=a_7\Rightarrow location:=7$

4. Describe an algorithm that inserts an integer x in the appropriate position into the list a_1, a_2, . . . , a_n of integers that are in increasing order.

Procedure insert ($x, a_1, a_2, ..., a_n$: integers) $(a_1 \le a_2 \le .... \le a_n)$

$a_{n+1} := x + 1$

$i := 1$

while $x > a_i$

$i := i + 1$

for $j := 1$ **to** $n - i$

$a_{n-j+1} = a_{n-j}$

$a_i := x$

5. Use the bubble sort to sort 3, 1, 5, 7, 4, showing the lists obtained at each step.

3	1	1	1
1	3	3	3
5	5	5	4
7	7	4	5
4	4	7	7

6. Consider the Linear search algorithm:

procedure linear search(x: integer, $a_1, a_2, ..., a_n$: distinct integers)

i := 1

while (i $\le$ n and x $\ne$ a_i)

i := i + 1

if i $\le$ n then location := i

else location := 0

return location

Given the sequence a_n: 3, 1, 5, 7, 4, 6. How many comparisons required for searching x = 7?

$n = 6,\ x = 7$

$i := 1, (1 \le 6 \textit{ and } 7 \ne 3) \Rightarrow i := i + 1 = 2$

$i := 2, (2 \le 6 \textit{ and } 7 \ne 1) \Rightarrow i := i + 1 = 3$

$i := 3, (3 \le 6 \textit{ and } 7 \ne 5) \Rightarrow i := i + 1 = 4$

$i := 4, (4 \le 6 \textit{ and } 7 \ne 7) \Rightarrow \textit{ false}$

$4 \le 6 \Rightarrow location = 4$

There are 9 comparisons ($\le$, $\ne$) required.

3.2 The Growth of Functions

1. Determine whether each of these functions is O(x).

a) f(x) = 10

$f(x) \le 10x\ ,\ \forall x \ge 1$

$\Rightarrow f(x) = O(x),\ C = 10,\ k = 1$

b) f(x) = 3x + 7

$f(x) \le 3x + 7x = 10x\ ,\ \forall x \ge 1$

$\Rightarrow f(x) = O(x),\ C = 10,\ k = 1$

c) f(x) = x^2 + x + 1

$\forall C > 0, \forall k > 0, \exists x_0 = C + k > k:$

$f(x_0) = (C+k)^2 + C + k + 1 > C(C+k) = Cx_0$

d) f(x) = 5logx

$f(x) \le 5x\ ,\ \forall x > 1$

$\Rightarrow f(x) = O(x),\ C = 5,\ k = 1$

$f(x) = O(g(x)) \Leftrightarrow \exists C > 0, \exists k, \forall x > k: |f(x)| \le C|g(x)|$

$f(x) \ne O(g(x)) \Leftrightarrow \forall C > 0, \forall k, \exists x > k: |f(x)| > C|g(x)|$

2. Determine whether each of these functions is O(x^2).

a) f(x) = 17x + 11

$f(x) = 17x + 11 \le 17x^2 + 11x^2 = 28x^2\ ,\ \forall x \ge 1$

$\Rightarrow f(x) = O(x^2),\ C = 28,\ k = 1$

b) f(x) = x^2 + 1000

$f(x) \le x^2 + x^2 = 2x^2\ ,\ \forall x \ge 10\sqrt{10}$

$\Rightarrow f(x) = O(x^2),\ C = 2,\ k = 10\sqrt{10}$

c) f(x) = xlogx

$f(x) = x\log x \le x^2\ ,\ \forall x > 1$

$\Rightarrow f(x) = O(x^2),\ C = 1,\ k = 1$

3. Find the least integer n such that f(x) is O(x^n) for each of these functions.

a) f(x) = $2x^3 + x^2\log x$

$f(x) \le 2x^3 + x^2x = 3x^3,\ \forall x > 1$

$\Rightarrow f(x) = O(x^3),\ C = 3,\ k = 1$

$\Rightarrow n = 3$

b) f(x) = $3x^3 + (\log x)^4$

$f(x) = 3x^3 + (\log x)^4 \le 3x^3 + x^2 \le 3x^3 + x^3 = 4x^3,\ \forall x > 1$

$\Rightarrow f(x) = O(x^3),\ C = 4,\ k = 1$

$\Rightarrow n = 3$

c) f(x) = $(x^4 + x^2 + 1)/(x^3 + 1)$

$$\frac{x^4 + x^2 + 1}{x^3 + 1} \le \frac{x^4 + x^4 + x^4}{x^3} \le \frac{3x^4}{x^3} = 3x\ ,\ x > 1$$

$\Rightarrow f(x) \le 3x\ ,\ \forall x > 1$

$\Rightarrow f(x) = O(x),\ C = 3,\ k = 1$

$\Rightarrow n = 1$

d) f(x) = $(x^4 + 5\log x)/(x^4 + 1)$

$$\frac{x^4 + 5\log x}{x^4 + 1} \le \frac{x^4 + 5x^4}{x^4} = 6\ ,\ x > 1$$

$\Rightarrow f(x) \le 6\ ,\ \forall x > 1$

$\Rightarrow f(x) = O(1),\ C = 6,\ k = 1$

$\Rightarrow n = 0$

4. Determine whether x^3 is $O(g(x))$ for each of these functions $g(x)$.

a) $g(x) = x^2$ F

b) $g(x) = x^3$ T

c) $g(x) = x^2 + x^3$ T

d) $g(x) = x^2 + x^4$ T

e) $g(x) = 3^x$ T

f) $g(x) = x^3/2$ T

$x^3 < x^4 < x^2 + x^4$, $x > 1$

$\Rightarrow x^3 = O(g(x))$, $C = 1$, $k = 1$

5. Arrange the functions $\sqrt{n}$,1000logn, nlogn, 2n!, 2^n, 3^n, and $n^2/1{,}000{,}000$ in a list so that each function is big-O of the next function.

$$1000\log n < \sqrt{n} < n\log n < \frac{n^2}{1{,}000{,}000} < 2^n < 3^n < 2n! \quad \text{(when } n \text{ is big enough)}$$

6. Give as good a big-O estimate as possible for each of these functions.

a) $(n^2 + 8)(n + 1)$

$n^2 + 8 = O(n^2)$, $n + 1 = O(n)$

$\Rightarrow (n^2 + 8)(n + 1) = O(n^2.n) = O(n^3)$

b) $(n\log n + n^2)(n^3 + 2)$

$n\log n + n^2 = O(n^2)$, $n^3 + 2 = O(n^3)$

$\Rightarrow (n\log n + n^2)(n^3 + 2) = O(n^2.n^3) = O(n^5)$

c) $(n! + 2^n)(n^3 + \log(n^2 + 1))$

$2^n < n!$, $n > 4$

$\Rightarrow n! + 2^n < 2n!$, $n > 4$

$\Rightarrow n! + 2^n = O(n!)$, $C = 2$, $k = 4$

$\log(n^2 + 1) < n^2 + 1 < n^3$, $n > 4$

$\Rightarrow n^3 + \log(n^2 + 1) < n^3 + n^3 = 2n^3$, $n > 4$

$\Rightarrow n^3 + \log(n^2 + 1) = O(n^3)$, $C = 2$, $k = 4$

$n! + 2^n = O(n!)$, $n^3 + \log(n^2 + 1) = O(n^3)$

$\Rightarrow (n! + 2^n)(n^3 + \log(n^2 + 1)) = O(n!n^3)$

3.3 Complexity of Algorithms

1. Consider the algorithm:

procedure *giaithuat* $(a_1, a_2, \ldots, a_n :$ *integers*$)$

count:= 0

for i:= 1 to n do

if $a_i > 0$ then *count*: = *count* + 1

print(*count*)

Give the best big-O complexity for the algorithm above.

The Complexity of Algorithm is O(n)

2. Consider the algorithm:

procedure GT (n : positive integer)

F:=1

for i:= 1 to n do

F: = F * i

Print(F)

Give the best big-O complexity for the algorithm above.

The Complexity of Algorithm is O(n)

3. Consider the algorithm:

procedure max($a_1, a_2, \ldots\ldots, a_n$: reals)

max:= a_1

for i:=2 to n

if max < a_i then max:= a_i

Give the best big-O complexity for the algorithm above.

The Complexity of Algorithm is O(n)

3.4 The Integers and Division

1. Does 17 divide each of these numbers?

a) 68 b) 84 c) 357 d) 1001

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2. What are the quotient and remainder when

a) 19 is divided by 7? b) −111 is divided by 11? c) 789 is divided by 23?

d) 1001 is divided by 13? e) 0 is divided by 19? f) 3 is divided by 5?

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3. Suppose that a and b are integers, a ≡ 4 (mod 13), and b ≡ 9 (mod 13). Find the integer c with $0 \le c \le 12$ such that

a) c ≡ 9a (mod 13). b) c ≡ 11b (mod 13). c) c ≡ a + b (mod 13).

d) c ≡ 2a + 3b (mod 13). e) $c \equiv a^2 + b^2$ (mod 13). f) $c \equiv a^3 - b^3$ (mod 13).

$e)\ c \equiv a^2 + b^2 (\mathrm{mod}\,13) \equiv 4^2 + 9^2 (\mathrm{mod}\,13) \equiv 97\,(\mathrm{mod}\,13) \Rightarrow c = 6$

$f)\ c \equiv a^3 - b^3 (\mathrm{mod}\,13) \equiv 4^3 - 9^3 (\mathrm{mod}\,13) \equiv -665\,(\mathrm{mod}\,13) \Rightarrow c = 11\ (-665 = -52.13 + 11)$

4. Evaluate these quantities.

a) 13 mod 3 b) −97 mod 11 c) 155 mod 19 d) −221 mod 23

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5. Find a **div** m and a **mod** m when

a) a = −111, m = 99. b) a = −9999, m = 101.

c) a = 10299, m = 999. d) a = 123456, m = 1001.

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6. Decide whether each of these integers is congruent to 5 modulo 17.

a) 80 b) 103 c) −29 d) −122

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7. Find each of these values.

a) $(992 \text{ mod } 32)^3$ mod 15 b) $(34 \text{ mod } 17)^2$ mod 11

c) $(193 \text{ mod } 23)^2$ mod 31 d) $(893 \text{ mod } 79)^4$ mod 26

$a)\ (992 \bmod 32)^3 \bmod 15 = (0)^3 \bmod 15 = 0$

$b)\ (34 \bmod 17)^2 \bmod 11 = 0^2 (\bmod 11) = 0$

$c)\ (193 \bmod 23)^2 \bmod 31 = 9^2 \bmod 31 = 19$

$d)\ (893 \bmod 79)^4 \bmod 26 = 24^4 \bmod 26 = 576^2 \bmod 26 = 4^2 \bmod 26 = 16$

8. Convert the decimal expansion of each of these integers to a binary expansion.

a) 23 b) 45 c) 241 d) 1025

$241 = 2.120 + 1$ $15 = 2.7 + 1$

$120 = 2.60 + 0$ $7 = 2.3 + 1$

$60 = 2.30 + 0$ $3 = 2.1 + 1$

$30 = 2.15 + 0$ $1 = 1.0 + 1$

$$241 = (1111\ 0001)_2$$

9. Convert the binary expansion of each of these integers to a decimal expansion.

a) (1 1011)2 b) (10 1011 0101)2

c) (11 1011 1110)2 d) (111 1100 0001 1111)

$(1\ 1011)_2 = 1.2^4 + 1.2^3 + 0.2^2 + 1.2 + 1 = 27$

10. Convert the octal expansion of each of these integers to a binary expansion.

a) $(572)_8$ b) $(1604)_8$ c) $(423)_8$ d) $(2417)_8$

$a)\ (572)_8 = (1\,0111\,1010)_2\ \left((5)_8 = (101)_2, (7)_8 = (111)_2, (2)_8 = (010)_2\right)$

$b)\ (1604)_8 = (11\,1000\,0100)_2$

$c)\ (423)_8 = (1\,0001\,0011)_2$

$d)\ (2417)_8 = (101\,0000\,1111)_2$

11. Convert each of the following expansions to **decimal expansion**.

a) $(1021)_3$ b) $(325)_7$ c) $(A3)_{12}$ d) $(401)_5$ e) $(12B7)_{13}$

$(A3)_{12} = 10.12 + 3 = 123$

$(12B7)_{13} = 1.13^3 + 2.13^2 + 11.13 + 7 = 2685$

12. Convert 69 to

a) a binary expansion b) a base 6 expansion c) a base 9 expansion

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13. Suppose a ***mod*** 3 = 2 and b ***mod*** 6 = 4, find ab ***mod*** 3.

* Cách 1

$a \bmod 3 = 2 \Rightarrow a \equiv 2 \pmod 3$

$b \bmod 6 = 4 \Rightarrow b \equiv 4 \pmod 6 \Rightarrow b - 4 ⋮ 6 \Rightarrow b - 4 ⋮ 3$

$\Rightarrow b \equiv 4 \pmod 3 \equiv 1 \pmod 3$

$\Rightarrow ab \equiv 2.1 \pmod 3 = 2 \pmod 3$

$\Rightarrow ab \bmod 3 = 2$

* Cách 2

$a \bmod 3 = 2 \Rightarrow a = 3k + 2\ (k \in \mathbb{Z})$

$b \bmod 6 = 4 \Rightarrow b = 6m + 4\ (m \in \mathbb{Z})$

$ab = (3k+2)(6m+4) = 18km + 12k + 12m + 8 = 3(6km + 4k + 4m + 2) + 2$

$\Rightarrow ab \bmod 3 = 2$

3.5 Primes and Greatest Common Divisors

1. Determine whether each of these integers is prime.

a) 21 b) 29 c) 71 d) 97

e) 111 f) 143 g) 93 h) 101

$\sqrt{97} \approx 9.84$, The only primes not exceeding $\sqrt{97}$ are 2, 3, 5, 7. Because 97 is not divisible by 2, 3, 5, 7 so it is prime.

2. Find the prime factorization of each of these integers.

a) 39 b) 81 c) 101

d) 143 e) 289 f) 899 = 29.31

3. Find the prime factorization of 10!

$$10! = 2.3.4.5.6.7.8.9.10 = 2.3.2^2.5.(2.3).7.2^3.3^2.(2.5) = 2^8.3^4.5^2.7$$

4. Which positive integers less than 12 are relatively prime to 12? 1, 5, 7, 11

5. Which positive integers less than 30 are relatively prime to 30? 1, 7, 11, 13, 17, 19, 23, 29

6. Find these values of the Euler φ-function.

a) φ(4) b) φ(10) c) φ(13)

$$\begin{aligned} \varphi : \mathbb{N}^* &\to \mathbb{N}^* \\ n &\mapsto \varphi(n) \end{aligned}$$

$\varphi(n)$ = the number of positive integers less than or equal to n that are **relatively prime** to n.

$$n = p_1^{\alpha_1}.p_2^{\alpha_2}...p_k^{\alpha_k}$$

$$\varphi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right).....\left(1-\frac{1}{p_k}\right)$$

if p is the prime, then $\varphi(p) = p\left(1-\frac{1}{p}\right) = p-1$

$10 = 2.5$

$$\varphi(10) = 10\left(1-\frac{1}{2}\right)\left(1-\frac{1}{5}\right) = 4$$

$100 = 4.25 = 2^2.5^2$

$$\varphi(100) = 100\left(1-\frac{1}{2}\right)\left(1-\frac{1}{5}\right) = 40$$

7. What are the greatest common divisors of these pairs of integers?

a) 37 · 53 · 73, 211 · 35 · 59 b) 11 · 13 · 17, 29 · 37 · 55 · 73

c) 2331, 2317 d) 313 · 517, 212 · 721

$a)\ 37.53.73 = 37.53.73$

$211.35.59 = 5.7.59.211$

$\gcd(37.53.73,\ 211.35.59) = 1$

$b)\ 29.37.55.73 = 5.11.29.37.73$

$\gcd(11.13.17,\ 29.37.55.73) = 11$

$c)\ 2331 = 3^2.7.37$

$2317 = 7.331$

$\gcd(2331, 2317) = 7$

3.6 Integers and Algorithms

1. Suppose ***pseudo-random numbers*** are produced by using: $x_{n+1} = (3x_n + 11) \bmod 13$.

If $x_3 = 5$, find x_2 and x_4. $x_{n+1} = (ax_n + c) \bmod m \Leftrightarrow ax_n + c \equiv x_{n+1} (\bmod m) \Leftrightarrow ax_n + c - x_{n+1} \vdots m$

$x_4 = (3x_3 + 11) \bmod 13$

$= (3.5 + 11) \bmod 13 = 0$

$x_3 = (3x_2 + 11) \bmod 13$

So, $5 = (3x_2 + 11) \bmod 13$

$\Leftrightarrow 13 \mid (3x_2 + 11 - 5)$

$\Leftrightarrow 13 \mid (3x_2 + 6)$ (*)

Note that x_2 is in $\{0,..,12\}$ ➔ $x_2 = 11$ is the solution of (*).

2. Suppose pseudo-random numbers are produced by using: $x_{n+1} = (2x_n + 7) \bmod 9$.

a) If $x_0 = 1$, find x_2 and x_3

b) If $x_3 = 3$, find x_2 and x_4.

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3. Using the function $f(x) = (x + 10) \bmod 26$ $\left(f(x) = (x + p) \bmod 26\right)$ to encrypt messages.

Answer each of these questions.

a) Encrypt the message STOP

b) Decrypt the message LEI

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

x	18	19	14	15
f(x) = (x+10) mod 26	2	3	24	25

2	3	24	25
C	D	Y	Z

STOP has been encrypted to CDYZ

We will **decrypt** the message LEI using the inverse function $f^{-1}(x) = (x - 10) \bmod 26$.

Encrypted form	L	E	I
x	11	4	8
$f^{-1}(x) = (x - 10) \bmod 26$	1	20	24
Original message	B	U	Y

4. Which memory locations are assigned by the **hashing function** $h(k) = k \bmod 101$ to the records of insurance company customers with these Social Security Numbers?

a) 104578690

b) 501338753

a/ h(104578690) = 104578690 mod 101 = 58.

The memory location 58 is assigned to the customer with the Social Security number 104578690.

b/ h(501338753) = 501338753 mod 101 = 3.

The memory location 3 is assigned to the customer with the Social Security number 501338753.

5. Use the **Euclidean algorithm** to find

a) gcd(14, 28) b) gcd(8, 28) c) gcd(100, 101) d) gcd(28,35)

35 = 28.1 + 7

28 = 7.4 + 0

gcd(28,35) = 7

e) lcm(7, 28) f) lcm(12, 28) g) lcm(100, 101) h) lcm(28,3)

28 = 12.2 + 4

12 = 4.3 + 0

gcd (12, 28) = 4 $\Rightarrow$ lcm(12,28) = 12.28/gcd(12,28) = 12.28/4 = 84

$$a = bq + r \quad (0 < r < b)$$
$$(a,b) = (b,r)$$
$$ab = (a,b).[a,b]$$

Chapter 4: Induction and Recursion

4.1 Mathematical Induction

1. Let P(n) be the statement that $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$ for the positive integer n.

a) What is the statement P(1)?

b) Show that P(1) is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step, identifying where you use the inductive hypothesis.

a) When n = 1 we have that P(1) is the statement $1^2 = 1.2.3/6$.

b) Both sides of P(1) shown in part (a) equal 1.

c) The inductive hypothesis is the statement that

$$1^2 + 2^2 + 3^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6}$$

d) For the inductive step, we want to show for each $k \geq 1$ that P(k) implies P(k +1). In other words, we want to show that assuming the inductive hypothesis (see part (c)) we can show

$$1^2 + 2^2 + 3^2 + ... + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

e) We have

$$1^2 + 2^2 + 3^2 + ... + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)\left[k(2k+1) + 6(k+1)\right]}{6} = \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

2. Let P(n) be the statement that $1 + \frac{1}{4} + \frac{1}{9} + ... + \frac{1}{n^2} < 2 - \frac{1}{n}$ where n is an integer greater than 1

a) What is the statement P(2)?

b) Show that P(2) is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step.

a) When n = 2 we have that P(2) is the statement $1+\frac{1}{4}<2-\frac{1}{2}$

b) This is true because 5 / 4 is less than 6 / 4.

c) The inductive hypothesis is the statement that

$$1+\frac{1}{4}+\frac{1}{9}+....+\frac{1}{k^2}<2-\frac{1}{k}$$

d) For the inductive step, we want to show for each $k \geq 2$ that P(k) implies P($k+1$). In other words, we want to show that assuming the inductive hypothesis (see part (c)) we can show

$$1+\frac{1}{4}+\frac{1}{9}+....+\frac{1}{(k+1)^2}<2-\frac{1}{k+1}$$

e) We have

$$1+\frac{1}{4}+\frac{1}{9}+....+\frac{1}{(k+1)^2}=1+\frac{1}{4}+\frac{1}{9}+....+\frac{1}{k^2}+\frac{1}{(k+1)^2}<2-\frac{1}{k}+\frac{1}{(k+1)^2}$$

but $\frac{1}{(k+1)^2}<\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$, $\forall k \geq 2$

$\Rightarrow 2-\frac{1}{k}+\frac{1}{(k+1)^2}<2-\frac{1}{k}+\frac{1}{k}-\frac{1}{k+1}=2-\frac{1}{k+1}$

$\Rightarrow 1+\frac{1}{4}+\frac{1}{9}+....+\frac{1}{(k+1)^2}<2-\frac{1}{k+1}$

3. Prove the statement "6 divides n^3 - n for all integers n ≥ 0".

Let P(n) be the statement that "6 divides n^3 - n"

When n = 0, The statement P(0) is true because $0^3-0=0 \vdots 6$

We assume that P(k) is true, that is $k^3-k \vdots 6$, $\forall k>0$

We must show that P(k + 1) is true, that is $(k+1)^3-(k+1) \vdots 6$, $\forall k>0$

We have $(k+1)^3-(k+1)=k^3+3k^2+3k+1-k-1=k^3-k+3k(k+1)$

But $k^3-k \vdots 6$ (inductive hypothesis) and $3k(k+1) \vdots 6$ $[k(k+1) \vdots 2]$

$\Rightarrow (k+1)^3-(k+1) \vdots 6$

4. Prove that $3^n < n!$ if n is an integer greater than 6.

Let P(n) be the statement that "$3^n < n!$"

When n = 7, The statement P(7) is true because $3^7 < 7!$

We assume that P(k) is true, that is $3^k < k!$, $\forall k > 6$

We must show that P(k + 1) is true, that is $3^{k+1} < (k+1)!$, $\forall k > 6$

We have $3^{k+1} = 3.3^k < 3.k! < (k+1).k! < (k+1)!$

5. Prove that $2^n > n^2$ if n is an integer greater than 4.

Let P(n) be the statement that "$2^n > n^2$"

When n = 5, The statement P(5) is true because $2^5 > 5^2$

We assume that P(k) is true, that is $2^k > k^2$, $\forall k > 5$

We must show that P(k + 1) is true, that is $2^{k+1} > (k+1)^2$, $\forall k > 5$

We have $2^{k+1} = 2.2^k > 2k^2 > (k+1)^2$, $\forall k > 5$

6. Prove that for every positive integer n then $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} > 2\sqrt{n+1} - 2$

Let P(n) be the statement that "$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} > 2\sqrt{n+1} - 2$"

When n = 1, The statement P(1) is true because $1 > 2\sqrt{1+1} - 2$

We assume that P(k) is true, that is $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{k}} > 2\sqrt{k+1} - 2$, $\forall k > 1$

We must show that P(k + 1) is true, that is $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{k+1}} > 2\sqrt{k+2} - 2$, $\forall k > 1$

We have

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{k+1}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > 2\sqrt{k+1} - 2 + \frac{1}{\sqrt{k+1}}$$

We will prove that $2\sqrt{k+1} - 2 + \frac{1}{\sqrt{k+1}} > 2\sqrt{k+2} - 2$, $\forall k > 1$ (*)

$$(*) \Leftrightarrow 2\sqrt{k+1} + \frac{1}{\sqrt{k+1}} = \frac{2(k+1)+1}{\sqrt{k+1}} > 2\sqrt{k+2}$$

$$\Leftrightarrow 2k + 3 > 2\sqrt{(k+1)(k+2)} = 2\sqrt{k^2 + 3k + 2}$$

$$\Leftrightarrow (2k+3)^2 > 4(k^2+3k+2)$$
$$\Leftrightarrow 4k^2+12k+9 > 4k^2+12k+8$$
$$\Leftrightarrow 9 > 8$$

7. Prove that $\ln n < \sum_{i=1}^{n} \frac{1}{i}$ whenever n is a positive integer.

Let P(n) be the statement that "$\ln n < \sum_{i=1}^{n} \frac{1}{i}$"

When n = 1, The statement P(1) is true because $0 < 1$

We assume that P(k) is true, that is $\ln k < \sum_{i=1}^{k} \frac{1}{i} = 1 + \frac{1}{2} + ... + \frac{1}{k}$, $\forall k > 1$

We must show that P(k + 1) is true, that is $\ln(k+1) < \sum_{i=1}^{k+1} \frac{1}{i} = 1 + \frac{1}{2} + ... + \frac{1}{k+1}$, $\forall k > 1$

We have

$$\ln(k+1) = \ln(k+1) - \ln k + \ln k - \ln(k-1) + \ldots + \ln 3 - \ln 2 + \ln 2 - \ln 1$$
$$= \sum_{i=1}^{k} \left[\ln(i+1) - \ln i\right] = \sum_{i=1}^{k} \ln\left(\frac{i+1}{i}\right) = \sum_{i=1}^{k} \ln\left(1 + \frac{1}{i}\right)$$

but $\ln\left(1 + \frac{1}{i}\right) < \frac{1}{i}$, $\forall i \geq 1$

$$\Rightarrow \ln(k+1) < \sum_{i=1}^{k} \frac{1}{i} < \sum_{i=1}^{k+1} \frac{1}{i}$$

4.2 Strong Induction

1. Let P(n) be the statement that a postage of n cents can be formed using just 3-cent stamps and 5-cent stamps. The parts of this exercise outline a strong induction proof that P(n) is true for n ≥ 8

a) Show that the statements P(8), P(9), and P(10) are true, completing the basis step of the proof.

b) What is the inductive hypothesis of the proof?

c) What do you need to prove in the inductive step?

d) Complete the inductive step for k ≥ 10.

a) P(8) is true, because we can form 8 cents of postage with one 3-cent stamp and one 5-cent stamp. P(9) is true, because we can form 9 cents of postage with three 3-cent stamps. P(10) is true, because we can form 10 cents of postage with two 5-cent stamps.

b) The inductive hypothesis is the statement that using just 3-cent and 5-cent stamps we can form j cents postage for all j with $8 \le j \le k$, where we assume that $k \ge 10$.

c) In the inductive step we must show, assuming the inductive hypothesis, that we can form k + 1 cents postage using just 3-cent and 5-cent stamps.

d) We want to form k + 1 cents of postage. Since $k \ge 10$, we know that P(k - 2) is true, that is, that we can form k - 2 cents of postage. Put one more 3-cent stamp on the envelope, and we have formed k + 1 cents of postage, as desired.

2. Let P (n) be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that P(n) is true for n ≥ 18

a) Show statements P(18), P(19), P(20), and P(21) are true, completing the basis step of the proof.

b) What is the inductive hypothesis of the proof?

c) What do you need to prove in the inductive step?

d) Complete the inductive step for k ≥ 21.

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4.3 Recursive Definitions and Structural Induction

1. Find f(1), f(2), f(3), and f(4) if f(n) is defined recursively by f(0) = 1 and for n = 0, 1, . . .

a) f(n + 1) = f(n) + 2 b) f(n + 1) = 3f(n)

c) $f(n + 1) = 2^{f(n)}$ d) $f(n + 1) = [f(n)]^2 + f(n) + 1$

$f(1) = 2^{f(0)} = 2^1 = 2$, $f(2) = 2^{f(1)} = 2^2 = 4$

$f(3) = 2^{f(2)} = 2^4 = 16$, $f(4) = 2^{f(3)} = 2^{16} = 65,636$

2. Find f (2), f (3), f (4), and f (5) if f is defined recursively by f(0) = −1, f(1) = 2, and for n = 1, 2, . . .

a) f(n + 1) = f(n) + 3f(n − 1) b) $f(n + 1) = [f(n)]^2 f(n - 1)$

c) $f(n + 1) = 3[f(n)]^2 - 4[f(n - 1)]^2$ d) f(n + 1) = f(n − 1)/f(n)

$f(2) = 3[f(1)]^2 - 4[f(0)]^2 = 3(2)^2 - 4(-1)^2 = 8$

$f(3) = 3[f(2)]^2 - 4[f(1)]^2 = 3(8)^2 - 4(2)^2 = 176$

$f(4) = 3[f(3)]^2 - 4[f(2)]^2 = 3(176)^2 - 4(8)^2 = 92,672$

$f(5) = 3[f(4)]^2 - 4[f(3)]^2 = 3(92,672)^2 - 4(176)^2 = 25,764,174,848$

3. Find f (2), f (3), f (4), and f (5) if f is defined recursively by f (0) = f (1) = 1 and for n = 1, 2, . . .

a) f(n + 1) = f(n) − f(n − 1)

b) f(n + 1) = f(n)f(n − 1)

c) f(n + 1) = $[f(n)]^2 + [f(n – 1)]^3$

d) f(n + 1) = f(n)/f(n − 1)

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4. Determine whether each of these proposed definitions is a valid recursive definition of a function f from the set of nonnegative integers to the set of integers. If f is well defined, find a formula for f(n) when n is a nonnegative integer and prove that your formula is valid.

a) f (0) = 0, f(n) = 2f(n − 2) for n ≥ 1

b) f (0) = 1, f(n) = f(n − 1) − 1 for n ≥ 1

c) f (0) = 2, f (1) = 3, f(n) = f(n − 1) − 1 for n ≥ 2

d) f (0) = 1, f (1) = 2, f (n) = 2f(n − 2) for n ≥ 2

e) f (0) = 1, f(n) = 3f(n − 1) if n is odd and n ≥ 1 and f(n) = 9f(n − 2) if n is even and n ≥ 2

a) This is not valid, since letting n = 1 we would have f(l) = 2f(-1), but f(-1) is not defined.

b) This is valid.

$$f(n) = f(n-1) - 1 = \left[f(n-2) - 1\right] - 1 = f(n-2) - 2 = \left[f(n-3) - 1\right] - 2$$
$$= f(n-3) - 3 = ... = f(1) - (n-1) = f(0) - n = 1 - n$$
$$\Rightarrow f(n) = 1 - n\ ,\ \forall n \geq 1$$

c) This is valid.

$$f(n) = f(n-1) - 1 = \left[f(n-2) - 1\right] - 1 = f(n-2) - 2 = \left[f(n-3) - 1\right] - 2$$
$$= f(n-3) - 3 = ... = f(1) - (n-1) = 3 - n + 1 = 4 - n$$
$$\Rightarrow f(n) = 4 - n\ ,\ \forall n \geq 1$$

d) This is valid.

$$f(2) = 2f(0) = 2\ ,\ f(3) = 2f(1) = 2.2 = 4\ ,\ f(4) = 2f(2) = 2.2 = 4$$
$$f(5) = 2f(3) = 2.4 = 8\ ,\ f(6) = 2f(4) = 2.4 = 8\ , ...$$

1,2,2,4,4,8,8,.........

$$\Rightarrow f(n) = 2^{\left\lfloor \frac{n+1}{2} \right\rfloor}, \forall n \geq 0$$

e) This is valid.

$$n = 1 \Rightarrow f(1) = 3f(0) = 3.1 = 3$$
$$n = 2 \Rightarrow f(2) = 9f(0) = 9.1 = 9 = 3^2$$

$n=3 \Rightarrow f(3)=3f(2)=3.3^2=3^3$

$n=4 \Rightarrow f(4)=9f(2)=9.3^2=3^4$

....

$\Rightarrow f(n)=3^n,\ n\geq 0$

5. Give a recursive definition of the sequence $\{a_n\}$, n = 1, 2, 3, . . . if

a) $a_n = 6n$ b) $a_n = 2n + 1$ c) $a_n = 10n$ d) $a_n = 5$

e) $a_n = 4n - 2$ f) $a_n = 1 + (-1)^n$ g) $a_n = n(n + 1)$ h) $a_n = n^2$

$f)\ a_1=1+(-1)^1=0$

$a_{n+1}=1+(-1)^{n+1}$

$\Rightarrow a_{n+1}=a_n+2(-1)^{n+1},\ \forall n\geq 1$

$g)\ a_1=1(1+1)=2$

$a_{n+1}=(n+1)(n+2)$

$\Rightarrow a_{n+1}=a_n+2(n+1),\ \forall n\geq 1$

6. Let F be the function such that F(n) is the sum of the first n positive integers. Give a recursive definition of F(n).

$F(n)=\sum_{k=1}^{n} k=1+2+...+n$

$F(1)=1$

$F(n+1)=\sum_{k=1}^{n+1} k=1+2+...+(n+1)=1+2+...+n+(n+1)=F(n)+n+1$

7. Give a **recursive definition** of each of these sets.

a) A = {2, 5, 8, 11, 14, …}

$2\in A$

$if\ x\in A\ then\ x+3\in A$

b) B = {…, -5, -1, 3, 7, 11, …}

$3\in B$

$if\ x\in B\ then\ x+4\in B$ or $x-4\in B$

c) C = {3, 12, 48, 192, 768, …}

$3\in C$

$if\ x\in C\ then\ 4x\in C$

d) D = {1, 2, 4, 7, 11, 16, …}

$1\in D$

$if\ x\in D\ then\ x+i(x)\in D$

$i(x)$: vị trí của phần tử x trong D

8. The reversal of a string is the string consisting of the symbols of the string in reverse order.

The reversal of the string w is denoted by w^R. Find the reversal of the following bit strings.

a) 0101 b) 1 1011 c) 1000 1001 0111

$(1000\ 1001\ 0111)^R=1110\ 1001\ 0001$

9. When does a string belong to the set A of bit strings defined recursively by

λ ∈ A, 0x1 ∈ A if x ∈ A, where λ is the empty string?

$01\in A,\ 0011\in A,\ 000111\in A, ...,\ \underbrace{00....00}_{n}\underbrace{11....11}_{n}\in A$

4.4 Recursive Algorithms

1. Give a recursive algorithm for computing *nx* whenever *n* is a positive integer and *x* is an integer, using just addition. $nx = (n-1)x + x$

procedure *product* (*n:* positive integer, *x:* integer)

if $n = 1$ then return x

else return *product* (n -1, x) + x

2. Consider an **recursive algorithm** to compute the n^{th} Fibonacci number:

procedure Fibo(n : positive integer)

if n = 1 return 1

else if n = 2 return 1

else return Fibo(n – 1) + Fibo(n – 2)

How many additions (+) are used to find Fibo(6) by the algorithm above

Fibo(6) = Fibo(5) + Fibo(4)

Fibo(5) = Fibo(4) + Fibo(3)

Fibo(4) = Fibo(3) + Fibo(2)

Fibo(3) = Fibo(2) + Fibo(1)

3. Give a recursive algorithm for finding the sum of the first n odd positive integers.

1 + 3 + 5 +…..+ (2n – 1)

procedure *sum of odds* (*n:* positive integer)

if n = 1 then return 1

else return *sum of odds* (n - 1) + 2n - 1

4. Consider the following algorithm:

procedure tinh (a: real number; n: positive integer)

if n = 1 return a

else return a.tinh(a, n-1).

a) What is the output if inputs are: n = 4, a = 2.5? Explain your answer.

b) Show that the algorithm computes a^n using Mathematical Induction.

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5. Consider the following algorithm:

procedure F(a_1, a_2, a_3, ...,a_n : integers) if $n = 1$ return 0

else return $a_n + F(a_1, a_2, a_3, ..., a_{n-1})$

Find

a) $F(1,3,5)$ b) $F(1,2,3,5,9)$

a) $F(1,3,5) = 5 + F(1,3) = 5 + 3 + F(1) = 8 + F(1) = 8 + 0 = 8$

b) $F(1,2,3,5,9) = 9 + F(1,2,3,5) = 9 + 5 + F(1,2,3) = 9 + 5 + 3 + F(1,2)$

$= 9 + 5 + 3 + 2 + F(1) = 19 + F(1) = 19 + 0 = 19$

Chapter 5-7: Counting

5.1 The Basics of Counting

1. How many different bit strings of length seven are there?

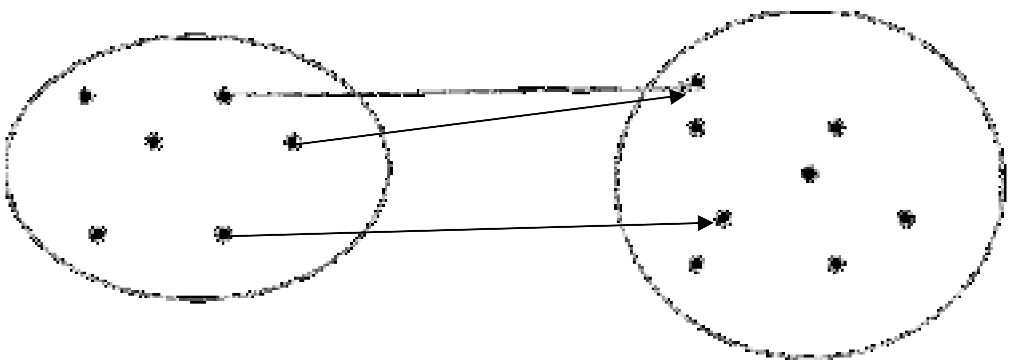
6.1 The Basics of Counting – page **406 -** Example 4

2. How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits (and no sequences of letters are prohibited, even if they are obscene)?

6.1 The Basics of Counting – page **407 -** Example 5

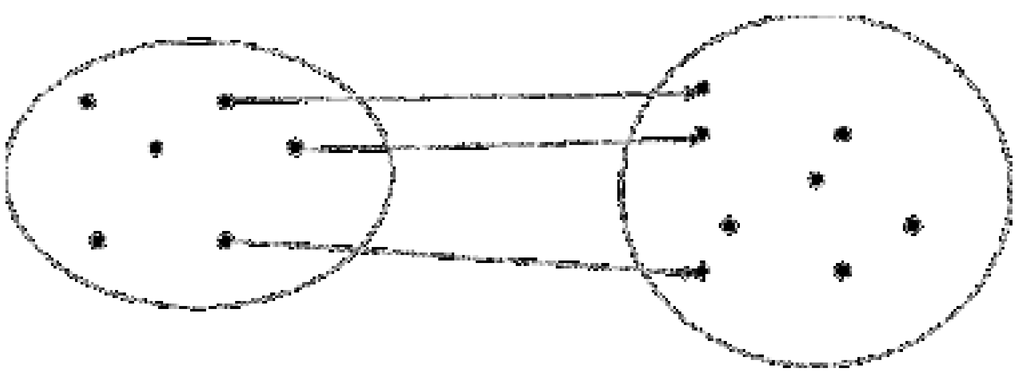
3. **Counting Functions**: How many functions are there from a set with m elements to a set with n elements?

6.1 The Basics of Counting – page **407 -** Example 6



4. **Counting One-to-One Functions**: How many one-to-one functions are there from a set with m elements to one with n elements?

6.1 The Basics of Counting – page **407 -** Example 7



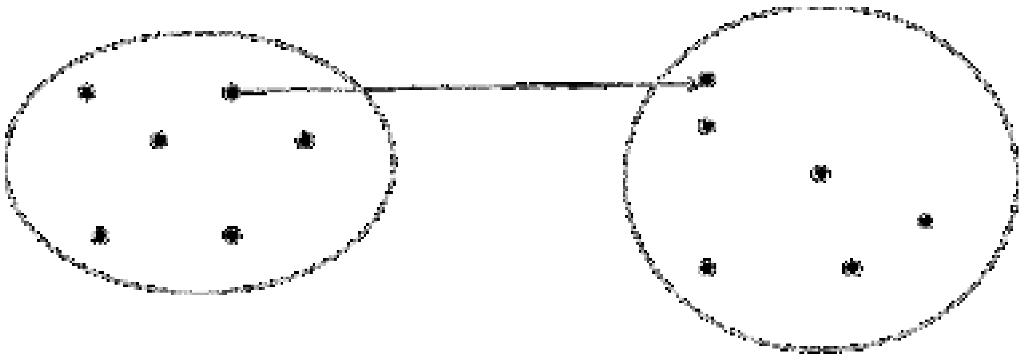
5. **Counting bijection Functions**: How many one-to-one functions are there from a set with m elements to one with n elements?

when $m \neq n$ there are no one-to-one functions from a set with m elements to a set with n elements. Therefore $m = n$.

$f:\{a_1,a_2,...,a_n\}\to\{b_1,b_2,...,b_n\}$

f is bijection function $\Leftrightarrow \forall b_i \in\{b_1,b_2,...,b_n\}, \exists! a_j \in\{a_1,a_2,...,a_n\}: f(a_j)=b_i \ (i,j=1,2,...,n)$

So, there are $n!=1.2.3.....(n-1).n$ one-to-one functions from a set with m elements to a set with n elements.



6. Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

6.1 The Basics of Counting – page **411 -** Example 16

7. How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

6.1 The Basics of Counting – page **413 -** Example 18

8. How many bit strings of length four do not have two consecutive 1s?

6.1 The Basics of Counting – page **415 -** Example 22

9. How many bit strings of length ten both begin and end with a 1? 2^8

10. How many positive integers between 5 and 31

a) are divisible by 3?

$D_3^5=\{m\in\mathbb{N}^*|m\vdots 3\wedge m\le 5\}\Rightarrow |D_3^5|=\lfloor 5/3\rfloor=1$

$D_3^{31}=\{m\in\mathbb{N}^*|m\vdots 3\wedge m< 31\}\Rightarrow |D_3^{31}|=\lfloor 31/3\rfloor=10$

$D_3^{5<m<31}=\{m\in\mathbb{N}^*|m\vdots 3\wedge 5<m< 31\}\Rightarrow |D_3^{5<m<31}|=10-1=9$

b) are divisible by 4?

$D_4^5=\{m\in\mathbb{N}^*|m\vdots 4\wedge m\le 5\}\Rightarrow |D_4^5|=\lfloor 5/4\rfloor=1$

$D_4^{31}=\{m\in\mathbb{N}^*|m\vdots 4\wedge m< 31\}\Rightarrow |D_4^{31}|=\lfloor 31/4\rfloor=7$

$D_4^{5<m<31}=\{m\in\mathbb{N}^*|m\vdots 4\wedge 5<m< 31\}\Rightarrow |D_4^{5<m<31}|=7-1=6$

c) are divisible by 3 and by 4?

$D_{3,4}^{5<m<31} = \{m \in \mathbb{N}^* | m \vdots 3 \wedge m \vdots 4 \wedge 5 < m < 31\} = \{m \in \mathbb{N}^* | m \vdots 12 \wedge 5 < m < 31\}$

$\Rightarrow |D_{3,4}^{5<m<31}| = \lfloor 31/12 \rfloor = 2$

11. How many positive integers between 50 and 100

a) are divisible by 7? b) are divisible by 11? c) are divisible by both 7 and 11?

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12. How many positive integers less than 1000 $D = \{1, 2, \ldots\ldots, 999\}$

a) are divisible by 7?

$D_7 = \{m \in \mathbb{N}^* | m \vdots 7 \wedge m \le 999\} \Rightarrow |D_7| = \lfloor 999/7 \rfloor = 142$

b) are divisible by 7 but not by 11?

c) are divisible by both 7 and 11?

$D_{11} = \{m \in \mathbb{N}^* | m \vdots 11 \wedge m \le 999\} \Rightarrow |D_{11}| = \lfloor 999/11 \rfloor = 90$

$D_7 \cap D_{11} = \{m \in \mathbb{N}^* | m \vdots 7 \wedge m \vdots 11 \wedge m \le 999\} = \{m \in \mathbb{N}^* | m \vdots 77 \wedge m \le 999\}$

$\Rightarrow |D_7 \cap D_{11}| = \lfloor 999/77 \rfloor = 12$

We have $\begin{cases} (D_7 \cap D_{11}) \cup (D_7 \cap \overline{D_{11}}) = D_7 \\ (D_7 \cap D_{11}) \cap (D_7 \cap \overline{D_{11}}) = \varnothing \end{cases} \Rightarrow |D_7 \cap D_{11}| + |D_7 \cap \overline{D_{11}}| = |D_7|$

$\Rightarrow |D_7 \cap \overline{D_{11}}| = |D_7| - |D_7 \cap D_{11}| = 142 - 12 = 130$

d) are divisible by either 7 or 11?

$|D_7 \cup D_{11}| = |D_7| + |D_{11}| - |D_7 \cap D_{11}| = 142 + 90 - 12 = 220$

e) are divisible by exactly one of 7 and 11?

$E = \{m \in \mathbb{N}^* | [(m \vdots 11 \wedge m \not\vdots 7) \vee (m \not\vdots 11 \wedge m \vdots 7)] \wedge m \le 999\}$

We have $\begin{cases} (D_7 \cap D_{11}) \cup (\overline{D_7} \cap D_{11}) = D_{11} \\ (D_7 \cap D_{11}) \cap (\overline{D_7} \cap D_{11}) = \varnothing \end{cases} \Rightarrow |D_7 \cap D_{11}| + |\overline{D_7} \cap D_{11}| = |D_{11}|$

$\Rightarrow |\overline{D_7} \cap D_{11}| = |D_{11}| - |D_7 \cap D_{11}| = 90 - 12 = 78$

$\Rightarrow |E| = |D_7 \cap \overline{D_{11}}| + |\overline{D_7} \cap D_{11}| = 78 + 130 = 208$

f) are divisible by neither 7 nor 11?

$\begin{cases} \overline{D_7} \cap \overline{D_{11}} = \overline{D_7 \cup D_{11}} \\ (\overline{D_7 \cup D_{11}}) \cup (D_7 \cup D_{11}) \end{cases} \Rightarrow |\overline{D_7 \cup D_{11}}| = |D| - |D_7 \cup D_{11}| = 999 - 220 = 779$

g) have distinct digits?

Let P be the set of positive integers less than 1000 and have distinct digits

P_i be the set of positive integers less than 1000 and have i distinct digits (i = 1, 2, 3)

$P_1 = \{m \in D | 1 \le m \le 9\} \Rightarrow |P_1| = 9$

$P_2 = \{m \in D | 10 \le m \le 99,\ m = \overline{ab}\}$; $a, b \in \{0,1,2,,...,9\}$

$\Rightarrow |P_2| = 9.9 = 81$

$P_3 = \{m \in D | 100 \le m \le 999,\ m = \overline{abc}\}$; $a, b, c \in \{0,1,2,,...,9\}$

$\Rightarrow |P_3| = 9.9.8 = 648$

We have $P = P_1 \cup P_2 \cup P_3$ so $|P| = |P_1 \cup P_2 \cup P_3| = |P_1| + |P_2| + |P_3| = 9 + 81 + 648 = 738$

h) have distinct digits and are even?

Let Q be the set of positive integers less than 1000 and have distinct digits and are even

Q_i be the set of positive integers less than 1000 and have i distinct digits and are odd

(i = 1, 2, 3)

$Q_1 = \{m \in D | 1 \le m \le 9 \wedge m \not\vdots\, 2\} \Rightarrow |Q_1| = 5$

$Q_2 = \{m \in D | 10 \le m \le 99 \wedge m = \overline{ab} \not\vdots\, 2\}$

$\Rightarrow |Q_2| = 8.5 = 40$

$Q_3 = \{m \in D | 100 \le m \le 999 \wedge m = \overline{abc} \not\vdots\, 2\}$

$\Rightarrow |Q_3| = 8.8.5 = 320$

$|Q_1 \cup Q_2 \cup Q_3| = |Q_1| + |Q_2| + |Q_3| = 5 + 40 + 320 = 365$

$P = Q \cup (Q_1 \cup Q_2 \cup Q_3) \Rightarrow |Q| = |P| - |Q_1 \cup Q_2 \cup Q_3| = 738 - 365 = 373$

13. How many positive integers between 100 and 999 inclusive

a) are divisible by 7?

b) are odd?

c) have the same three decimal digits?

d) are not divisible by 4?

e) are divisible by 3 or 4?

f) are not divisible by either 3 or 4?

g) are divisible by 3 but not by 4?

h) are divisible by 3 and 4?

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14. How many one-to-one functions are there from a set with five elements to sets with the following number of elements?

a) 4 b) 5 c) 6 d) 7

0 5.4.3.2.1 = 120 6.5.4.3.2 = 720 7.6.5.4.3 = 2520

15. How many bit strings of length seven either begin with two 0s or end with three 1s?

There are $2^5 = 32$ bit strings of length seven begin with two 0s (00_ _ _ _ _)

There are $2^4 = 16$ bit strings of length seven end with three 1s (_ _ _ _ 111)

There are $2^2 = 4$ bit strings of length seven begin with two 0s and end with three 1s (00_ _ 111)

Therefore, there are 32 + 16 – 4 = 44 bit strings of length seven either begin with two 0s or end with three 1s.

16. How many bit strings of length 10 either begin with three 0s or end with two 0s?

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17. How many bit strings of length 8 begin with 11 or end with 00?

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18. Let B be the set {a, b, c}. How many functions are there from B^2 to B?

$$B = \{a, b, c\}$$

$$B^2 = B \times B = \{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)\}$$

$$f : B^2 \to B$$

$$(x, y) \mapsto f(x, y)$$

$$(x, y) \in \{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)\}$$

$$f(x, y) \in \{a, b, c\}$$

There are 3^9 functions are there from B^2 to B.

19. How many bit strings with length 10 that has exactly three 1s and end with 0?

Let A be the set of bit strings with length 10 that has exactly three 1s and end with 0

$$A = \{x = x_1x_2....x_9 0, x \text{ has exactly three } 1s \mid x_i \in \{0,1\}, i = 1,2,...9\}$$

$$|A| = C_9^3 = \frac{9!}{3!(9-3)!} = 84$$

20. A game consisting of flipping a coin ends when the player gets two heads in a row, two tails in a row or flips the coin four times. In how many ways can the game end?

{NN, SS, SNN, NSS, SNSS, NSNN, NSNS, SNSN}

There are 8 ways can the game end.

7.1 Recurrence Relations

1. a) Find a recurrence relation for the number of permutations of a set with n elements.

b) Use this recurrence relation to find the number of permutations of a set with n elements using iteration.

$A_n = \{x_1, x_2, ..., x_n\}$

$A_{n-1} = \{x_1, x_2, ..., x_{n-1}\}$

$A_3 = \{a, b, c\}$

$abc, acb, bac, bca, cab, cba$

$A_1 = \{a, b\}$, $A_2 = \{b, c\}, A_3 = \{a, c\}$

$ab \to abc, acb , cab$

$ba \to cba, bca, bac$

a) Each permutation of a set with n elements is an ordered arrangement of n elements of that set. Let a_n be the number of permutations of a set with n elements. Each permutation of a set with n elements (which we will call a permutation of length n) will be generated from a permutation of length n – 1 by concatenating an element remaining from the original set when remove n – 1 elements to create permutations of length n – 1. There are n ways to fit the remaining element into each permutation of length n – 1. Hence there are n ways of generating permutations of length n words permutations of length n – 1. We have recurrence relation $a_1 = 1$, $a_n = n a_{n-1}$, $n \geq 2$

b) $a_n = n a_{n-1} = n(n-1) a_{n-2} = n(n-1)(n-2) a_{n-3} = ... = n(n-1)(n-2)....2 a_1$

$= n(n-1)(n-2)....21 = n!$

2. a) Find a recurrence relation for the number of bit strings of length n that do not contain three consecutive 0s.

b) What are the initial conditions?

c) How many bit strings of length seven do not contain three consecutive 0s?

a) Let a_n be the number of bit strings of length n that do not contain three consecutive 0's. In order to construct a bit string of length n of this type we could start with 1 and follow with a string of length n - 1 not containing three consecutive 0's, or we could start with a 01 and follow with a string of length n - 2 not containing three consecutive 0's, or we could start with a 001 and follow with a string of length n - 2 not containing three consecutive 0's. These three cases are mutually exclusive and exhaust the possibilities for how the string might start, since it cannot start 000. From this analysis we can immediately write down the recurrence relation, valid for all $n \geq 3$:

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$

			Number of bit strings of length n with no three consecutive 0s:
Start with a 1	Any bit string of length $n-1$ with no three consecutive 0s	1_ _ _ _ _ _ _	a_{n-1}
Start with a 01	Any bit string of length $n-2$ with no three consecutive 0s	01_ _ _ _ _ _	a_{n-2}
Start with a 001	Any bit string of length $n-3$ with no three consecutive 0s	001_ _ _ _ _	a_{n-3}

b) The initial conditions are $a_0 = 1$, $a_1 = 2$ ("1" ; "0") and $a_2 = 4$ ("00" ; "01" ; "11" ; "10") since all strings of length less than 3 satisfy the conditions (recall that the empty string has length 0).

c) $a_3 = a_2 + a_1 + a_0 = 1 + 2 + 4 = 7$

$a_4 = a_3 + a_2 + a_1 = 2 + 4 + 7 = 13$

$a_5 = a_4 + a_3 + a_2 = 4 + 7 + 13 = 24$

$a_6 = a_5 + a_4 + a_3 = 7 + 13 + 24 = 44$

$a_7 = a_6 + a_5 + a_4 = 13 + 24 + 44 = 81$

3. What is the solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$

$$a_n = a_{n-1} + 2a_{n-2}$$

$$\Leftrightarrow a_n + a_{n-1} = 2(a_{n-1} + a_{n-2}) = 2\left[2(a_{n-2} + a_{n-3})\right] = 2^2(a_{n-2} + a_{n-3}) = ... = 2^{n-1}(a_1 + a_0) = 9.2^{n-1}$$

$$a_n = a_{n-1} + 2a_{n-2}$$

$$\Leftrightarrow a_n - 2a_{n-1} = -(a_{n-1} - 2a_{n-2}) = (-1)\left[(-1)(a_{n-2} - 2a_{n-3})\right] = (-1)^2(a_{n-2} + a_{n-3}) = ... = (-1)^{n-1}(a_1 - 2a_0)$$

$$= 3.(-1)^{n-1}$$

Ta có hệ phương trình $\begin{cases} a_n + a_{n-1} = 9.2^{n-1} \\ a_n - 2a_{n-1} = 3.(-1)^{n-1} \end{cases} \Leftrightarrow a_n = 3.2^n - (-1)^n \text{ , } n \geq 1$

7.3 Divide-and-Conquer Algorithms and recurrence Relations

1. How many comparisons are needed for a binary search in a set of 64 elements?

Let $f(n)$ be the number of comparisons needed in a binary search of a list of n elements.

We know that f satisfies the divide-and-conquer recurrence relation $f(n) = f(n/2) + 2$

$(a = 1, b = 2, c = 2)$

We have $f(n) = f(1) + ck \quad (k = \log n)$. Because there are 2 comparisons are needed for a list with one element so $f(1) = 2$ and $k = \log n = 6$, therefore $f(64) = f(1) + ck = 2 + 2.6 = 14$

2. Suppose that f(n) = f(n/3) + 1 when n is a positive integer divisible by 3, and f(1) = 1. Find

a) f(3) b) f(27) c) f(729)

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3. Suppose that f(n) = 2f(n/2) + 3 when n is an even positive integer, and f(1) = 5. Find

a) f(2) b) f(8) c) f(64) d) f(1024).

$f(n) = 2f(n/2) + 3 \quad (a = 2, b = 2, c = 3)$

$$f(n) = a^k\left[f(1) + \frac{c}{a-1}\right] - \frac{c}{a-1} \quad \text{if } n = b^k \ (k \in \mathbb{N}^*)$$

$$\Rightarrow f(n) = 2^k\left[5 + \frac{3}{2-1}\right] - \frac{3}{2-1} = 8.2^k - 3 = 2^{k+3} - 3$$

$n = 1024 = 2^{10} \Rightarrow k = 10$. Therefore $f(1024) = 2^{10+3} - 3 = 8189$

4. Suppose that f(n) = f(n/5) + 3n when n is a positive integer divisible by 5, and f(1) = 4. Find

a) f(5) b) f(125) c) f(3125)

$f(n) = f(n/5) + 3n \quad (a = 1, b = 5, g(n) = 3n)$

$$\Rightarrow f(n) = f(1) + \sum_{j=0}^{k-1} g\left(\frac{n}{b^j}\right) = 4 + \sum_{j=0}^{k-1}\frac{3n}{5^j} = 4 + 3n\sum_{j=0}^{k-1}\frac{1}{5^j} \quad \text{if } n = b^k \ (k \in \mathbb{N}^*)$$

$n = 3125 = 5^5 \Rightarrow k = 5$

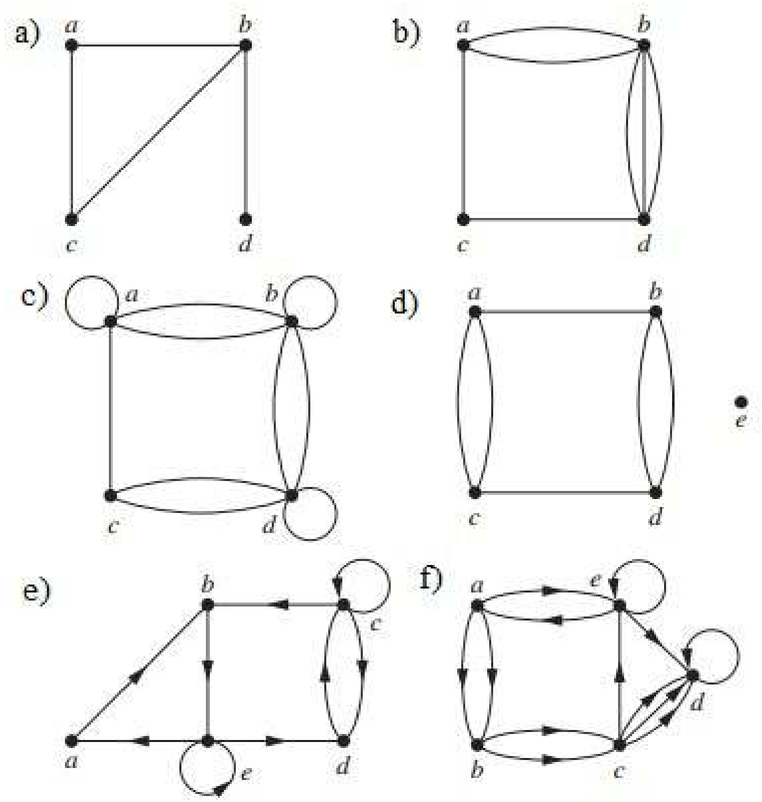
$$\Rightarrow f(3125) = 4 + 3.3125\sum_{j=0}^{4}\frac{1}{5^j} = 4 + 3.3125\left(\frac{1}{5^0} + \frac{1}{5^1} + \dots + \frac{1}{5^4}\right) = 11719$$

$f(n) = af\left(\frac{n}{b}\right) + g(n) \quad (b \mid n)$	
$a \neq 1, n = b^k, g(n) \neq c$	$f(n) = a^k f(1) + \sum_{j=0}^{k-1} a^j g\left(\frac{n}{b^j}\right)$
$a = 1, n = b^k, g(n) \neq c$	$f(n) = f(1) + \sum_{j=0}^{k-1} g\left(\frac{n}{b^j}\right)$
$a \neq 1, n = b^k, g(n) = c$	$f(n) = a^k\left[f(1) + \frac{c}{a-1}\right] - \frac{c}{a-1} \quad (k = \log_b n)$
$a = 1, n = b^k, g(n) = c$	$f(n) = f(1) + ck \quad (k = \log_b n)$

Chapter 9: Graphs

9.1 Graphs and Graph Models

1. Determine whether the graph shown has directed or undirected edges, whether it has multiple edges, and whether it has one or more loops



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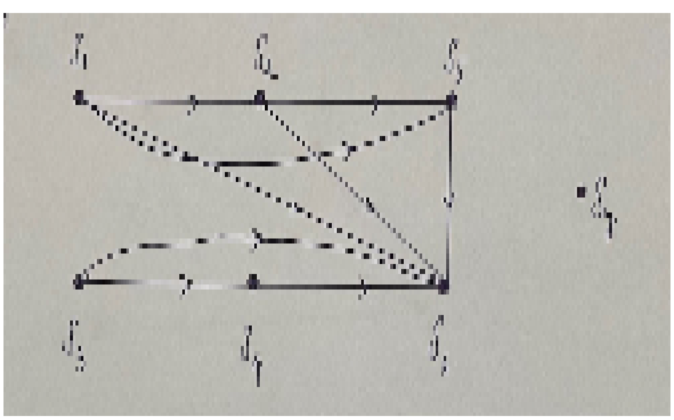
2. Construct a precedence graph for the following program:

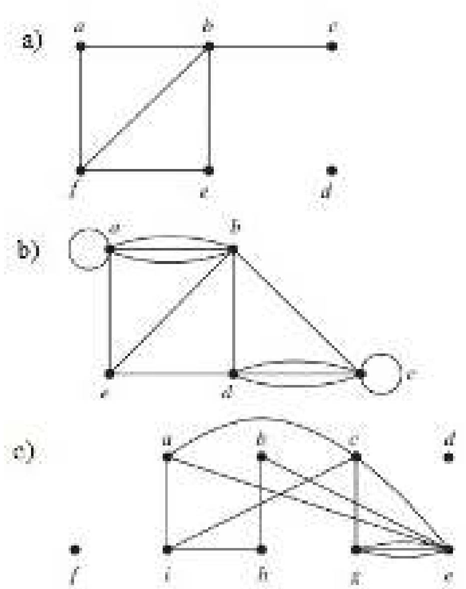
S_1: x := 0 S_2: x := x + 1 S_3: y := 2

S_4: z := y S_5: x := x + 2 S_6: y := x + z S_7: z := 4



9.2 Graph Terminology and Special Types of Graphs

1. find the number of vertices, the number of edges, and the degree of each vertex in the given undirected graph. Identify all isolated and pendant vertices.



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2. Find the sum of the degrees of the vertices of each graph in Exercises 1 and verify that it equals twice the number of edges in the graph.

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3. Can a simple graph exist with 15 vertices each of degree five?

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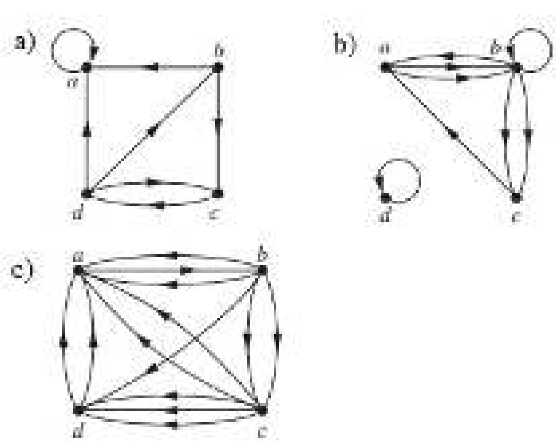
4. Determine the number of vertices and edges and find the in-degree and out-degree of each vertex for the given directed multigraph



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5. Draw these graphs.

a) K_7	b) $K_{1,8}$	c) $K_{4,4}$
d) C_7	e) W_7	f) Q_4

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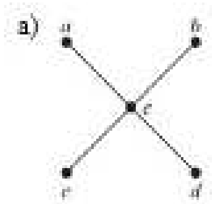
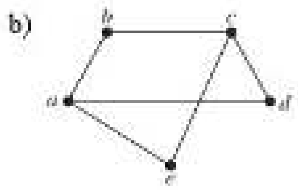
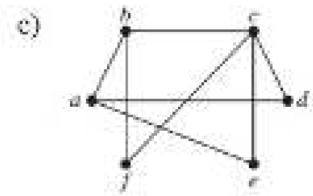
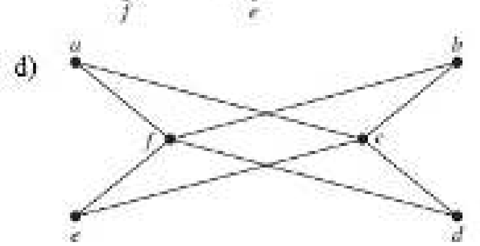
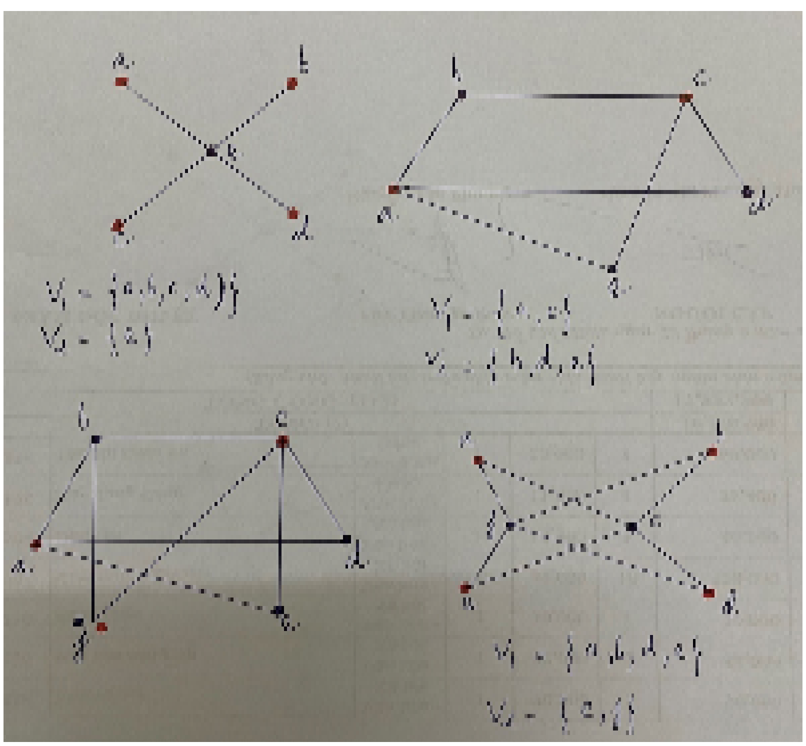
6. Determine whether the graph is bipartite











7. For which values of n are these graphs bipartite?

a) K_n b) C_n c) W_n d) Q_n

$n = 2$ n is an even $\nexists n$ $\forall n \geq 1$

8. How many vertices and how many edges do these graphs have?

a) K_n b) C_n c) W_n d) $K_{m,n}$ e) Q_n

a) K_n has n vertices. It has $C(n, 2) = n(n - 1)/2$ edges.

b) C_n has n vertices, n edges.

c) W_n is the same as C_n with an extra vertex and n extra edges incident to that vertex. Therefore, it has $n + 1$ vertices and $n + n = 2n$ edges.

d) By definition $K_{m,n}$ has $m + n$ vertices. Since it has one edge for each choice of a vertex in the one part and a vertex in the other part, it has mn edges.

e) Since the vertices of Q_n are the bit strings of length n, there are 2^n vertices. Each vertex has degree n, since there are n strings that differ from any given string in exactly one bit (any one of the n different bits can be changed). Thus the sum of the degrees is $n2^n$. Since this must equal twice the number of edges (by the handshaking theorem), we know that there are $n2^n /2 = n2^{n-1}$ edges.

9. Find the degree sequences for each of the graphs in Exercises 6.

a) 1, 1, 1, 1, 4 b) 3, 2, 3, 2, 2, c) 3, 3, 4, 2, 2, 2 d) 2, 2, 4, 2, 2, 4

10. Find the degree sequence of each of the following graphs.

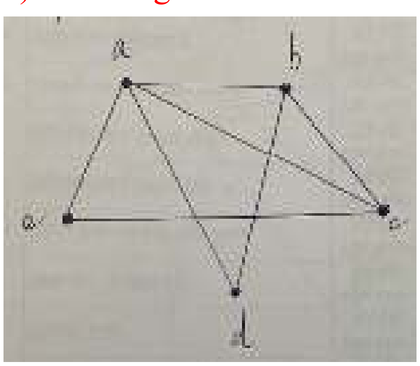
a) K_4 b) C_4 c) W_4 d) $K_{2,3}$ e) Q_3

a) 3, 3, 3, 3 b) 2, 2, 2, 2 c) 4, 3, 3, 3, 3

d) 3, 3, 2, 2, 2. e) 3, 3, 3, 3, 3, 3, 3, 3

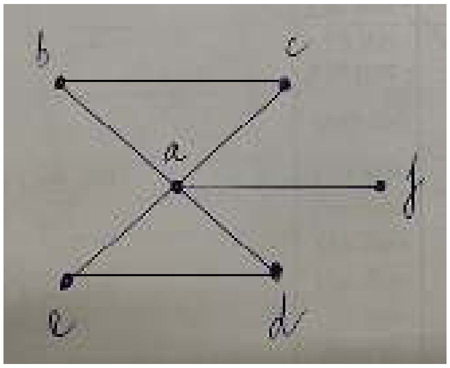
11. How many edges does a graph have if its degree sequence is 4, 3, 3, 2, 2? Draw such a graph.

The number of edges is half the sum of the degrees. Therefore this graph has $(4 + 3 + 3 + 2 + 2)/2 = 7$ edges



12. How many edges does a graph have if its degree sequence is 5, 2, 2, 2, 2, 1? Draw such a graph.

The number of edges is half the sum of the degrees (Theorem 1). Therefore this graph has $(5 + 2 + 2 + 2 + 2 + 1)/2 = 7$ edges

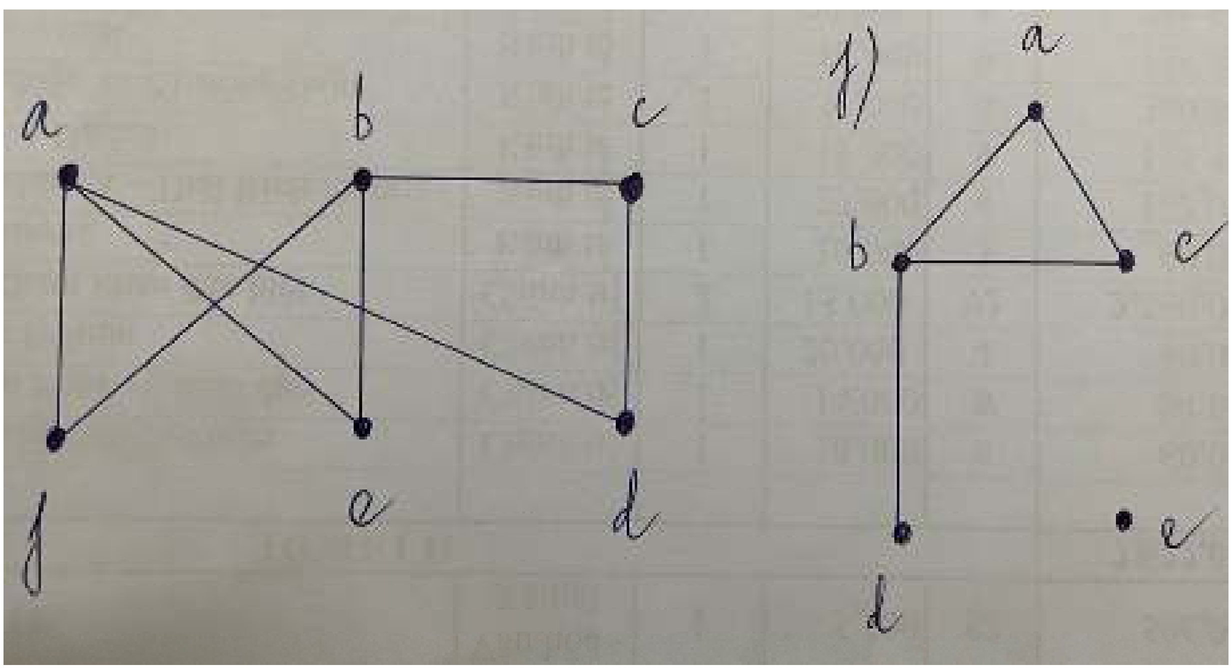


A sequence $d_1, d_2, \ldots, d_n$ is called **graphic** if it is the degree sequence of a simple graph.

13. Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence.

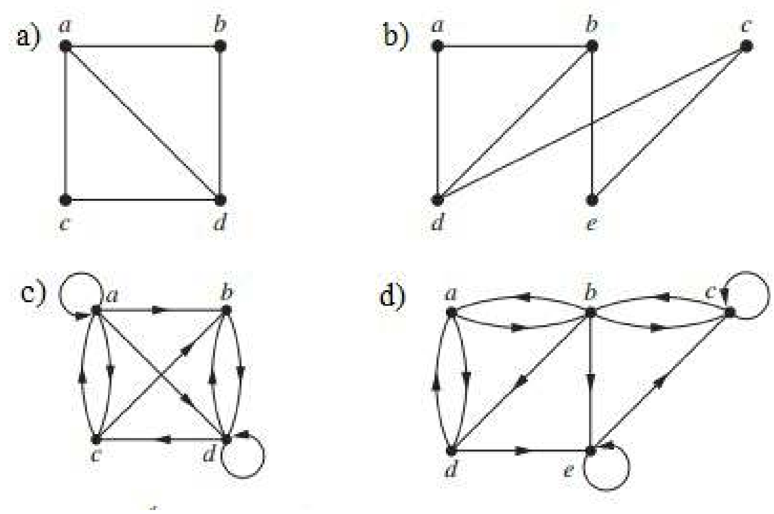
a) 5, 4, 3, 2, 1, 0 (no) b) 6, 5, 4, 3, 2, 1 (no) c) 4, 4, 3, 2, 1 (no)

d) 3, 3, 3, 2, 2, 2 (no) e) 3, 3, 2, 2, 2, 2 (yes) f) 3, 2, 2, 1, 0 (yes)



9.3 Representing Graphs and Graph Isomorphism

1. use an adjacency list to represent the given graph.



c)

Initial vertex	Terminal vertices
a	a, b, c, d
b	d
c	a, b
d	b, c, d

d)

Initial vertex	Terminal vertices
a	b, d
b	a, c, d, e
c	b, c
d	a, e
e	e

2. Represent the graph in Exercise 1 with an adjacency matrix.

c) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

d) $B = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

3. Represent each of these graphs with an adjacency matrix.

a) K_4 b) $K_{1,4}$ c) $K_{2,3}$ d) C_4

e) W_4 f) Q_3

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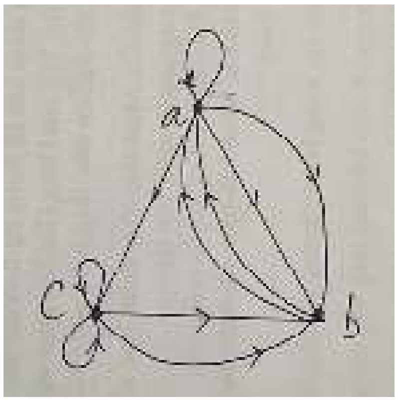
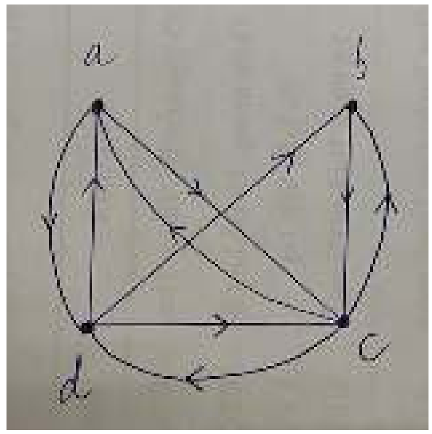
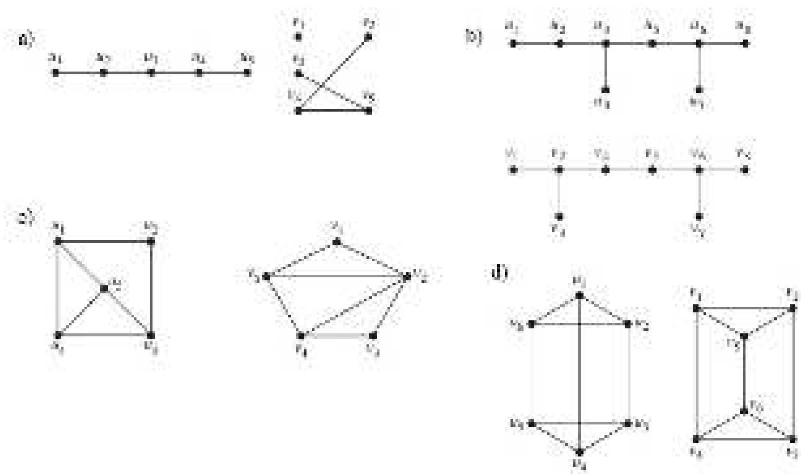
4. Draw a graph with the given adjacency matrix

a) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}$ c) $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$





5. Determine whether the given pair of graphs is isomorphic



a) These two graphs are isomorphic, one isomorphism is $f(u_1) = v_1$, $f(u_2) = v_2$, $f(u_3) = v_4$, $f(u_4) = v_5$, $f(u_5) = v_3$

b) These graphs are not isomorphic. In the first graph the vertices of degree 3 are adjacent to a common vertex. This is not true of the second graph.

c) These graphs are not isomorphic. In the second graph there is deg(v_2) = 4 while the first graph has no vertices of degree 4.

d) These two graphs are isomorphic, one isomorphism is $f(u_1) = v_5$, $f(u_2) = v_2$, $f(u_3) = v_3$, $f(u_4) = v_6$, $f(u_5) = v_4$, $f(u_6) = v_1$

6. Are the simple graphs with the following adjacency matrices isomorphic?

a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

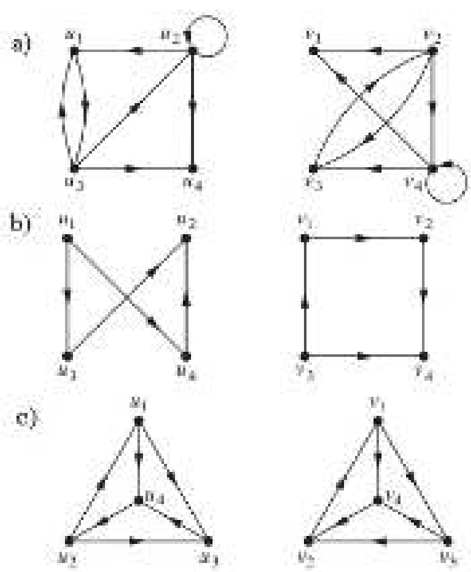
c) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

a) Both graphs consist of 2 sides of a triangle; they are clearly isomorphic.

b) The graphs are not isomorphic, since the first has 4 edges and the second has 5 edges.

c) The graphs are not isomorphic, since the first has 4 edges and the second has 3 edges.

7. Determine whether the given pair of directed graphs are isomorphic.



$u_1 - v_3$, $u_2 - v_4$, $u_3 - v_2$, $u_4 - v_1$

a) These two graphs are isomorphic, one isomorphism is $f(u_1) = v_3$, $f(u_2) = v_4$, $f(u_3) = v_2$, $f(u_4) = v_1$

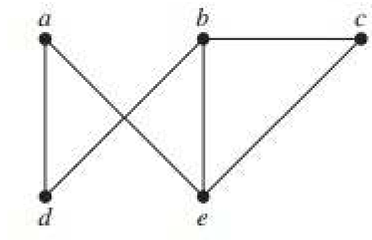
b) These graphs are not isomorphic

c) These two graphs are isomorphic, one isomorphism is $f(u_1) = v_3$, $f(u_2) = v_1$, $f(u_3) = v_4$, $f(u_4) = v_2$

9.4 Connectivity

1. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

a) a, e, b, c, b b) a, e, a, d, b, c, a c) e, b, a, d, b, e d) c, b, d, a, e, c



a) This is a path of length 4, but it is not simple, since edge {*b,* c} is used twice. It is not a circuit, since it ends at a different vertex from the one at which it began.

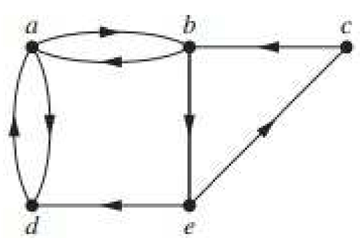
b) This is not a path, since there is no edge from *c* to *a.*

c) This is not a path, since there is no edge from *b* to *a.*

d) This is a path of length 5 (it has 5 edges in it). It is simple, since no edge is repeated. It is a circuit since it ends at the same vertex at which it began.

2. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

a) a, b, e, c, b b) a, d, a, d, a c) a, d, b, e, a d) a, b, e, c, b, d, a



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3. Find the number of paths of length n between two different vertices in K_4 if n is

a) 2 b) 3 c) 4 d) 5

Without loss of generality, we assume that the vertices are called 1, 2, 3, 4, and the path is to run from 1 to 2 . A path of length *n* is determined by choosing the *n* - 1 intermediate vertices. Each vertex in the path must differ from the one immediately preceding it.

a) A path of length 2 requires the choice of 1 intermediate vertex, which must be different from both of the ends. Vertices 3 and 4 are the only ones available. Therefore the answer is 2.

b) Let the path be denoted 1, *x, y,* 2 . If $x = 2$, then there are 3 choices for *y.* If $x = 3$, then there are 2 choices for *y;* similarly if $x = 4$. Therefore there are $3 + 2 + 2 = 7$ possibilities in all.

c) Let the path be denoted 1, *x, y, z,* 2. If $x = 3$, then by part (b) there are 7 choices for *y* and *z.* Similarly, if $x = 4$. If $x = 2$, then *y* and *z* can be any two distinct members of {1,3,4}, and there are 6 ways to choose them. Therefore there are $7 + 7 + 6 = 20$ possibilities in all.

d) Let the path be denoted 1, *w*, *x*, *y*, *z*, 2. If $w = 3$, then by part (c) there are 20 choices for *x, y,* and *z.* Similarly, if $w = 4$. If $w = 2$, then *x* must be different from 2, and there are 3 choices for *x.* For each of these there are by part (b) 7 choices for *y* and *z.* This gives a total of 21 possibilities in this case. Therefore, the answer is $20 + 20 + 21 = 61$.

$$A=\begin{bmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{bmatrix}\Rightarrow A^2=\begin{bmatrix}3&2&2&2\\2&3&2&2\\2&2&3&2\\2&2&2&3\end{bmatrix}\Rightarrow A^3=\begin{bmatrix}6&7&7&7\\7&6&7&7\\7&7&6&7\\7&7&7&6\end{bmatrix}\Rightarrow A^4=\begin{bmatrix}21&20&20&20\\20&21&20&20\\20&20&21&20\\20&20&20&21\end{bmatrix}\Rightarrow A^5=\begin{bmatrix}60&61&61&61\\61&60&61&61\\61&61&60&61\\61&61&61&60\end{bmatrix}$$

4. Find the number of paths between c and d in the graph of length

a) 2 b) 3 c) 4 d) 5 e) 6 f) 7

b) c, x, y, d

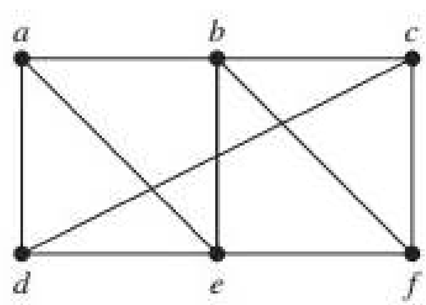
x ∈ {b, d, f}

x = b → c, b, y, d → 3 đường

x = d → c, d, y, d → 3 đường

x = f → c, f, y, d → 2 đường

c-b-a-d, c-b-e-d, c-b-c-d, c-d-c-d, c-d-a-d, c-d-e-d, c-f-c-d, c-f-e-d



c) c, x, y, z, d

x ∈ {b, d, f}

x = b → c, b, y, z, d → 4 đường

x = d → c, d, y, z, d → 2 đường

x = f → c, f, y, z, d → 4 đường

d) c, x, y, z, u, d

x ∈ {b, d, f}

x = b → c, b, y, z, u, d

→ y ∈ {a, c, e, f}

y = a → c, b, a, z, u, d → 7 đường

y = c → c, b, c, z, u, d → 8 đường

y = e → c, b, e, z, u, d → 9 đường

y = f → c, b, f, z, u, d → 4 đường

x = d → c, d, y, z, u, d

→ y ∈ {a, c, e,}

y = a → c, d, a, z, u, d → 7 đường

y = c → c, d, c, z, u, d → 8 đường

y = e → c, d, e, z, u, d → 9 đường

x = f → c, f, y, z, u, d

→ y ∈ {b, c, e,}

y = a → c, f, b, z, u, d → 4 đường

y = c → c, f, c, z, u, d → 8 đường

y = e → c, f, e, z, u, d → 9 đường

	a	b	c	d	e	f	
	0	1	0	1	1	0	a
	1	0	1	0	1	1	b
M =	0	1	0	1	0	1	c
	1	0	1	0	1	0	d
	1	1	0	1	0	1	e
	0	1	1	0	1	0	f

	3	1	2	1	2	2
	1	4	1	3	2	2
$M^2 =$	2	1	3	0	3	1
	1	3	0	3	1	2
	2	2	3	1	4	1
	2	2	1	2	1	3

	4	9	4	7	7	5
	9	6	9	4	10	7
$M^3 =$	4	9	2	8	4	7
	7	4	8	2	9	4
	7	10	4	9	6	9
	5	7	7	4	9	4

	23	20	21	15	25	20
	20	35	17	28	26	25
$M^4 =$	21	17	24	10	28	15
	15	28	10	24	17	21
	25	26	28	17	35	20
	20	25	15	21	20	23

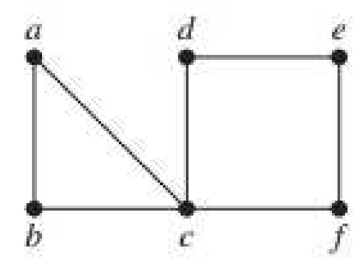
	60	89	55	69	78	66
	89	88	88	63	108	78
$M^5 =$	55	88	42	73	63	69
	69	63	73	42	88	55
	78	108	63	88	88	89
	66	78	69	55	89	60

	236	259	224	193	284	222
	259	363	229	285	318	284
$M^6 =$	224	229	230	160	285	193
	193	285	160	230	229	224
	284	318	285	229	363	259
	222	284	193	224	259	236

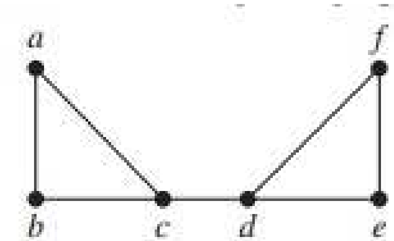
	736	966	674	744	910	767
	966	1090	932	806	1191	910
$M^7 =$	674	932	582	739	806	744
	744	806	739	582	932	674
	910	1191	806	932	1090	966
	767	910	744	674	966	736

5. Find all the cut vertices of the given graph

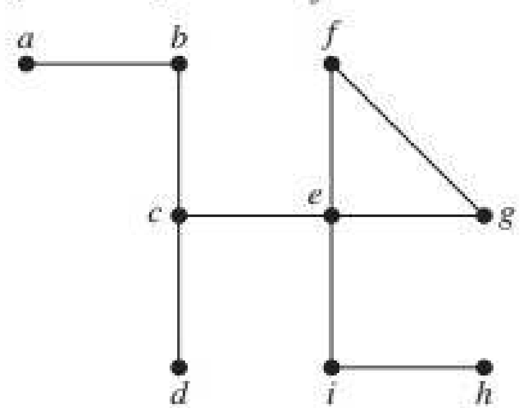
a)



b)



c)



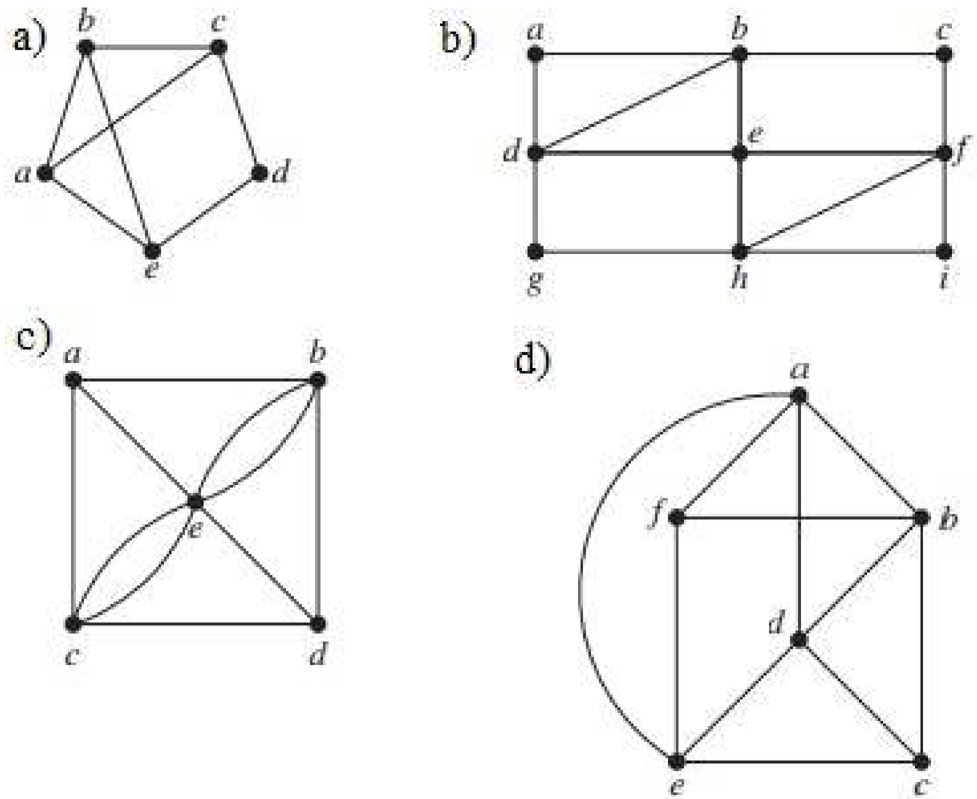
a) vertex c is a cut vertex.

b) vertices c, d are two cut vertices.

c) vertices b, c, e, i are four cut vertices.

9.5 Euler and Hamilton Paths

1. Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists.



a) Since there are four vertices of odd degree (*a, b,* c, and e), this graph has not Euler circuit.

b) Since the vertices of the graph have even degree, the graph has an Euler circuit.

a, b, d, g, h, i, f, c, b, e, h, f, e, d, a

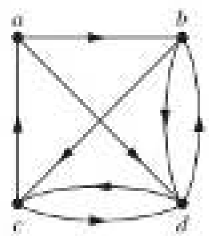
c) Since there are two vertices of odd degree (*a, d*), this graph has not Euler circuit.

d) Since there are two vertices of odd degree (*c, f*), this graph has not Euler circuit.

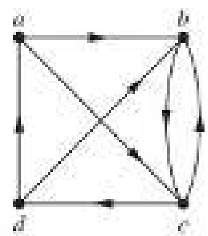
2. Determine whether the directed graph shown has an Euler circuit. Construct an Euler circuit if one exists.

A directed multigraph having no isolated vertices has an Euler circuit if and only if the graph is weakly connected and the in-degree and out-degree of each vertex are equal.

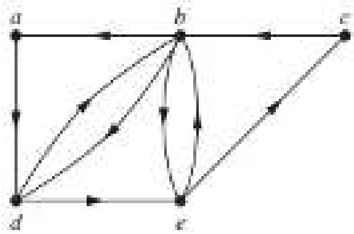
a)



b)



c)



a), b) Since $\deg^-(a) \neq \deg^+(a)$ $(1 \neq 2)$, this graph has not Euler circuit.

c) Since the graph is weakly connected and the in-degree and out-degree of each vertex are equal, it has an Euler circuit (a, d, e, c, b, e, b, d, b, a).

3. For which values of n do these graphs have an Euler circuit?

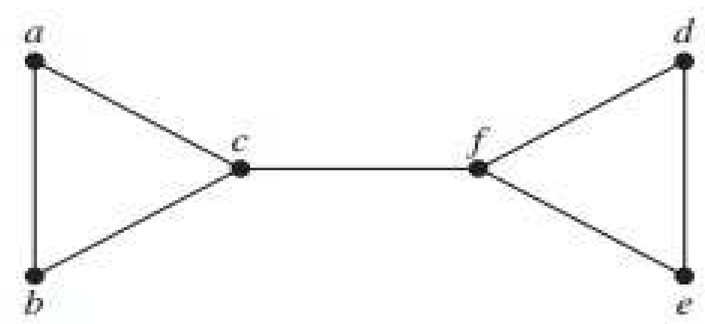
a) K_n	b) C_n	c) W_n	d) Q_n
n is an odd and $n > 1$	$\forall n \geq 3$	$\nexists n$	n is an even

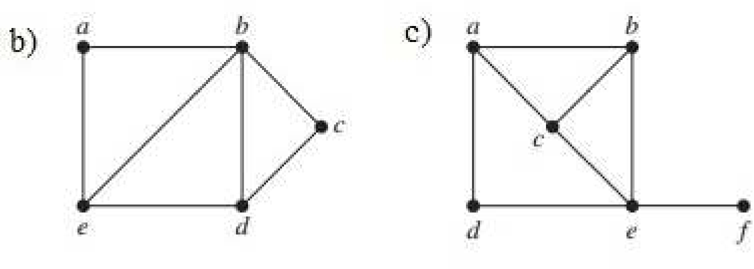
4. For which values of m and n does the complete bipartite graph $K_{m,n}$ have an

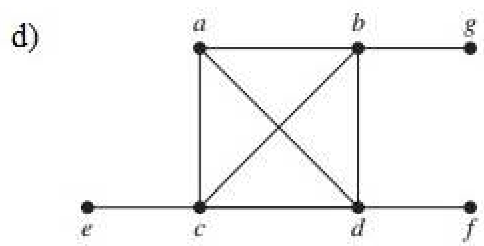
a) Euler circuit?	b) Euler path?
m, n is an even	$m = 2$ and n is an odd or m is an odd and $n = 2$

5. Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit.

a)







a) There is no Hamilton circuit.

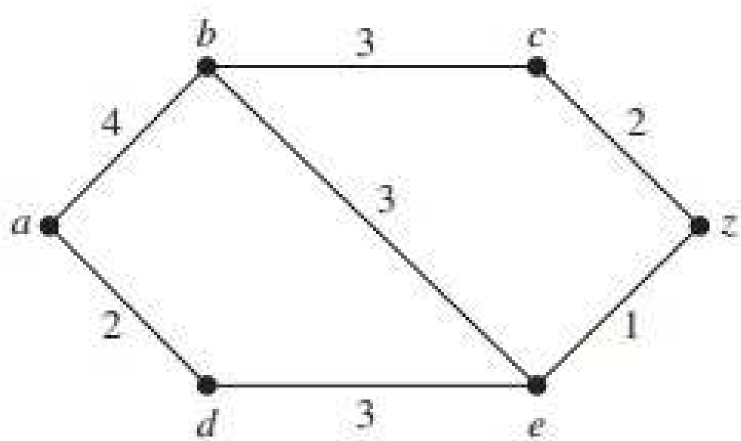
b) It is clear that a, b, c, d, e, a is a Hamilton circuit.

c) There is no Hamilton circuit.

d) There is no Hamilton circuit.

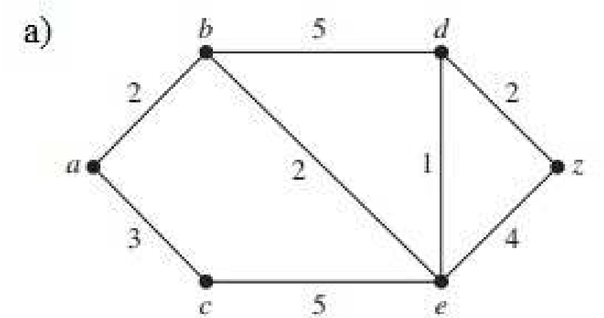
9.6 Shortest-Path Problems

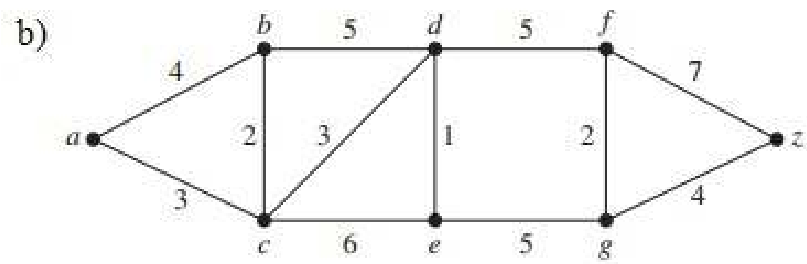
1. What is the length of a shortest path between a and z in the weighted graph shown in the Figure?



A shortest path between a and z is a, d, e, z and the length of the shortest path from a to z is 6.

2. Find a shortest path between a and z in the given weighted graph.





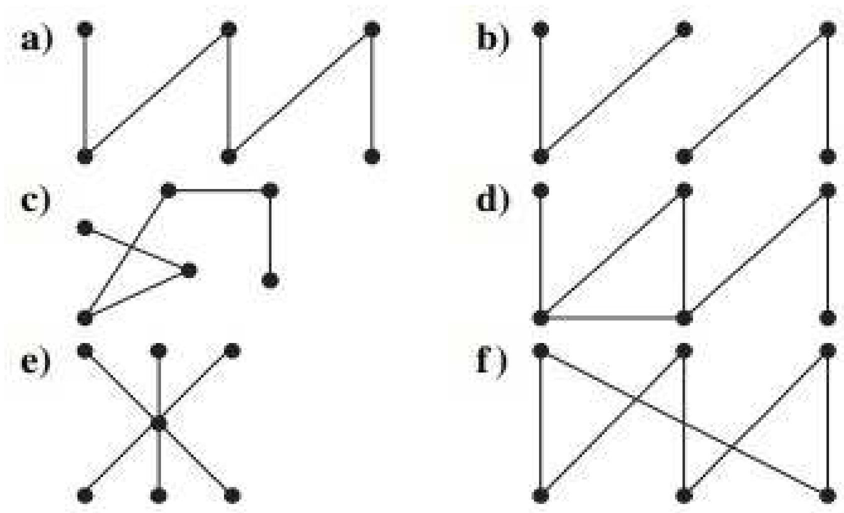
a) A shortest path between a and z is a, b, e, d, z and the length of the shortest path from a to z is 7.

b) A shortest path between a and z is a, c, d, e, g, z and the length of the shortest path from a to z is 16.

Chapter 10. Trees

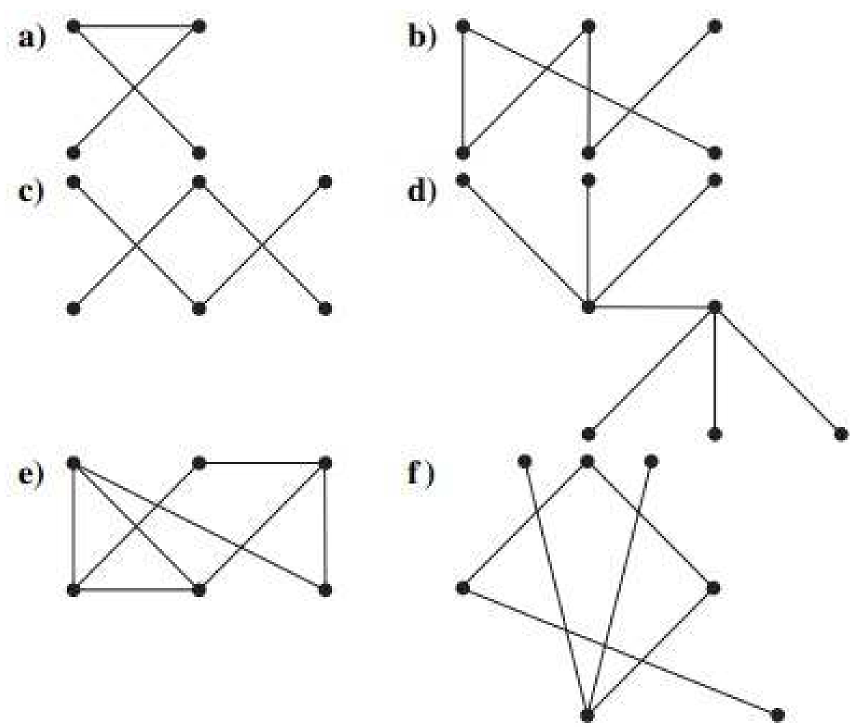
10.1 Introduction to Trees

1. Which of these graphs are trees?



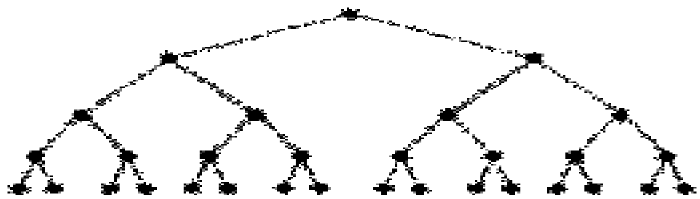
a), c), e)

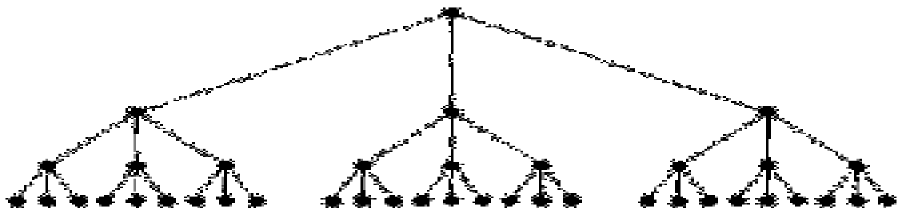
2. Which of these graphs are trees?



a), b), d), f)

3. Construct a complete binary tree of height 4 and a complete 3-ary tree of height 3.





4. Which complete bipartite graphs $K_{m,n}$, where m and n are positive integers, are trees?

m = 1 or n = 1

5. How many edges does a tree with 10,000 vertices have? e = 10000 – 1 = 9999

6. How many vertices does a full 5-ary tree with 100 internal vertices have?

m = 5 , i = 100 => n = mi + 1 = 501

7. How many edges does a full binary tree with 1000 internal vertices have?

m = 2 , i = 1000 => n = mi + 1 = 2001 => e = 2001 – 1 = 2000

8. How many leaves does a full 3-ary tree with 100 vertices have?

m = 3 , n = 100 => i = (n – 1)/m = (100 – 1)/3 = 33

n = l + i => l = n – i = 100 – 33 = 67

10.2 Applications of Trees

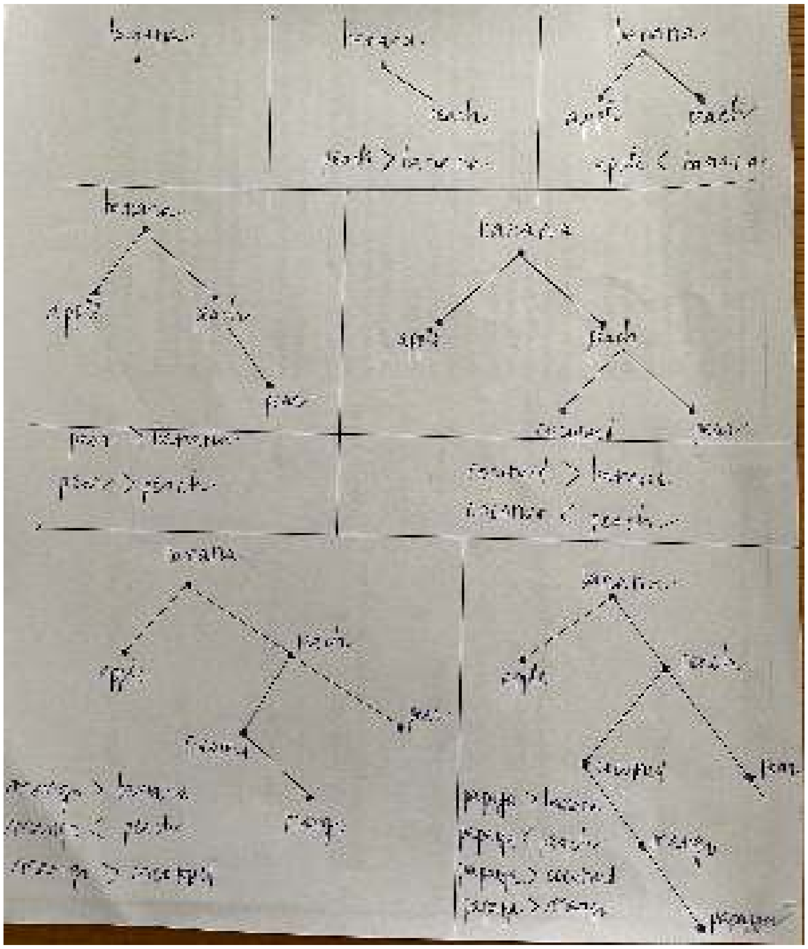
1. Form a binary search tree for the words *mathematics, physics, geography, zoology, meteorology, geology, psychology* and *chemistry* (using alphabetical order).

11.2. **Applications of Trees** – page **794 -** Example 1

2. Use Huffman coding to encode the following symbols with the frequencies listed: A: 0.08, B: 0.10, C: 0.12, D: 0.15, E: 0.20, F: 0.35. What is the average number of bits used to encode a character?

11.2. **Applications of Trees** – page **800 -** Example 5

3. Build a binary search tree for the words *banana, peach, apple, pear, coconut, mango* and *papaya* using alphabetical order.



4. Build a binary search tree for the words *oenology, phrenology, campanology, ornithology, ichthyology, limnology, alchemy* and *astrology* using alphabetical order.

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5. How many comparisons are needed to locate or to add each of these words in the search tree for Exercise 3, starting fresh each time?

a) pear b) banana c) kumquat d) orange

a) To find *pear,* we compare it with the root *(banana),* then with the right child of the root *(peach),* and finally with the right child of that vertex *(pear).* Thus 3 comparisons are needed.

b) Only 1 comparison is needed, since the item being searched for is the root.

c) We fail to locate *kumquat* by comparing it successively to *banana, peach, coconut,* and *mango.* Once we determine that *kumquat* should be in the left subtree of *mango,* and find no vertices there. we know that *kumquat* is not in the tree. Thus 4 comparisons were used.

d) This one is similar to part (c), except that 5 comparisons are used. We compare *orange* successively to *banana, peach, coconut, mango,* and *papaya.*

6. How many comparisons are needed to locate or to add each of the words in the search tree for Exercise 4, starting fresh each time?

a) palmistry b) etymology c) paleontology d) glaciology

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7. Which of these codes are prefix codes?

a) a: 11, e: 00, t: 10, s: 01

b) a: 0, e: 1, t: 01, s: 001

c) a: 101, e: 11, t: 001, s: 011, n: 010

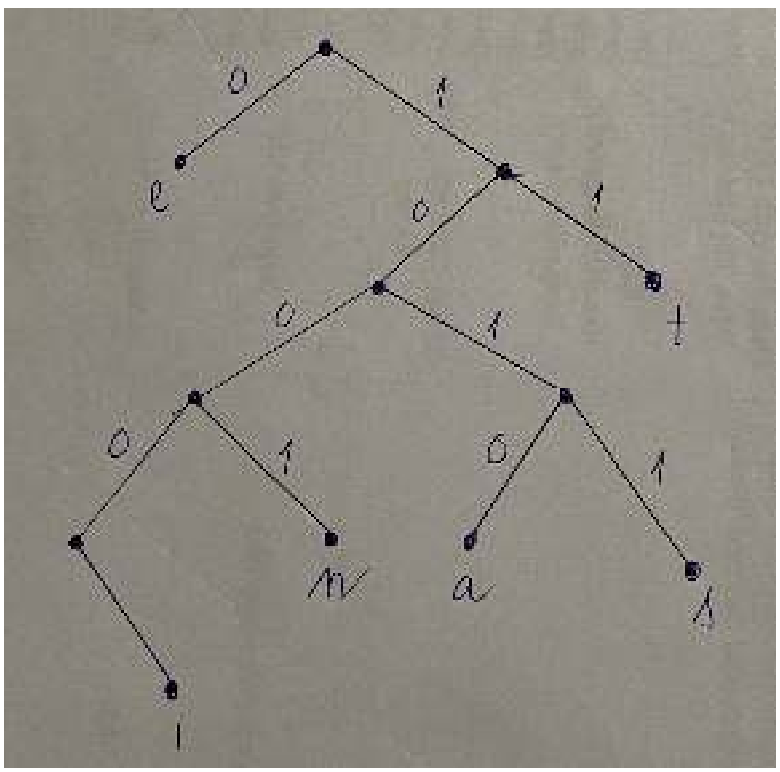
d) a: 010, e: 11, t: 011, s: 1011, n: 1001, i: 10101

8. Construct the binary tree with prefix codes representing these coding schemes.

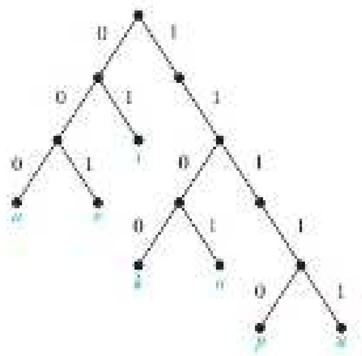
a) a: 11, e: 0, t: 101, s: 100

b) a: 1, e: 01, t: 001, s: 0001, n: 00001

c) a: 1010, e: 0, t: 11, s: 1011, n: 1001, i: 10001



9. What are the codes for a, e, i, k, o, p, and u if the coding scheme is represented by this tree?



a → 000 k → 1100 p → 11110

e → 001 o → 1101 u → 11111

i → 01

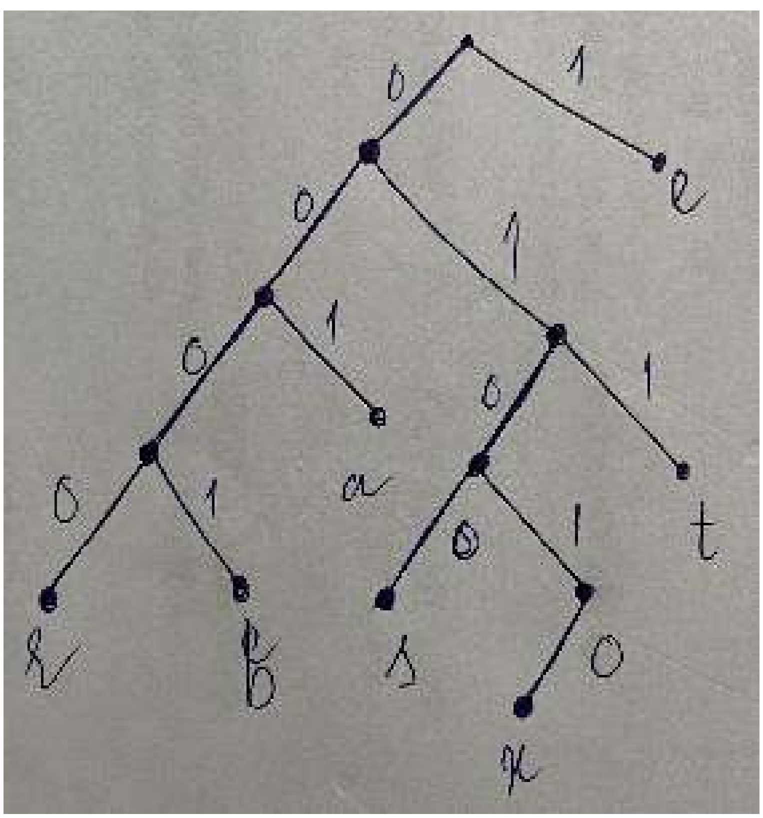
10. Given the coding scheme a: 001, b: 0001, e: 1, r: 0000, s: 0100, t: 011, x: 01010, find the word represented by

a) 011 1 0100 011 → test

b) 0001 1 1 0000 → beer

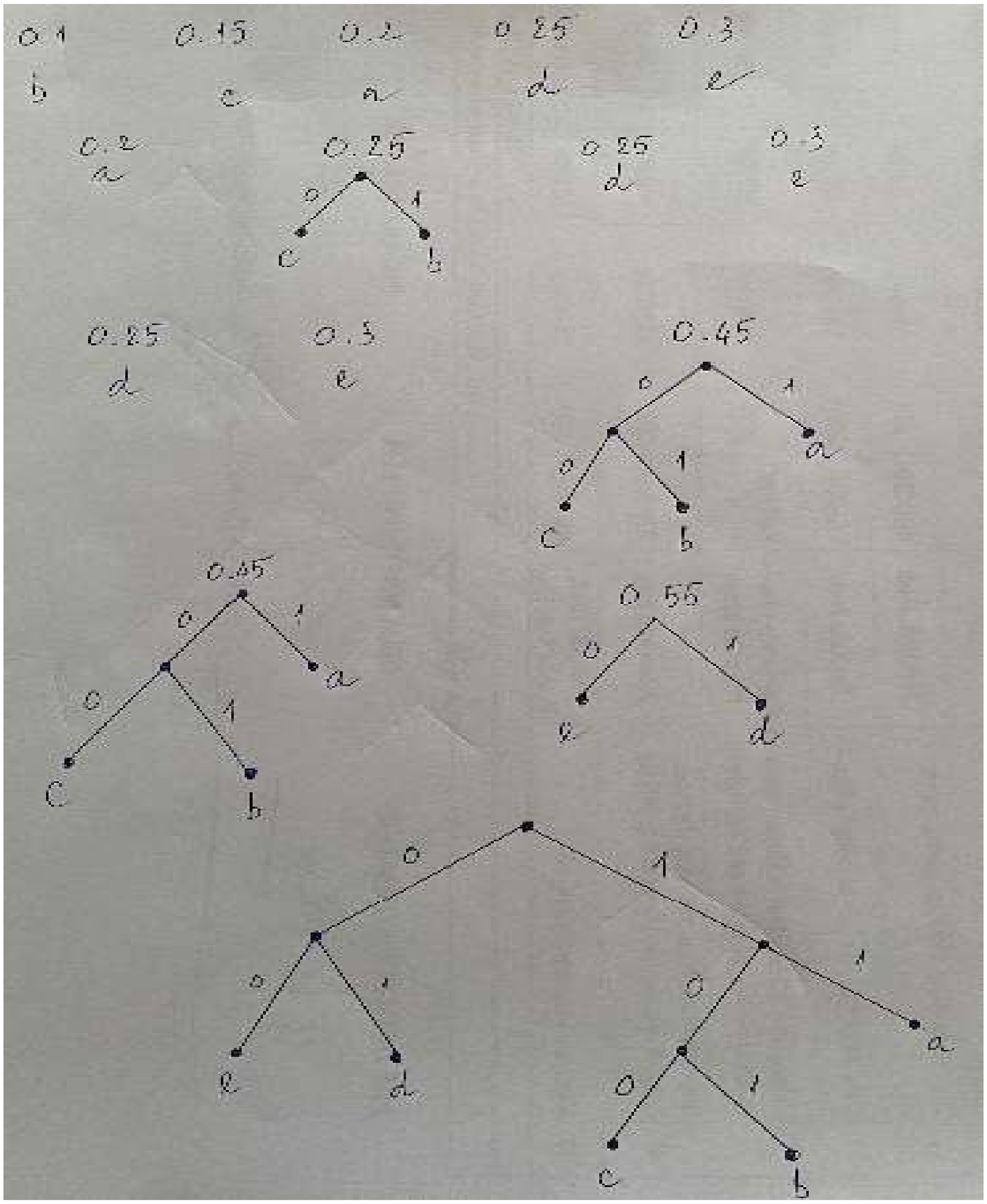
c) 0100 1 01010 → sex

d) 011 001 01010 → tax



11. Use Huffman coding to encode these symbols with given frequencies: a: 0.20, b: 0.10, c:

0.15, d: 0.25, e: 0.30. What is the average number of bits required to encode a character?



a → 11 c → 100 e → 00

b → 101 d → 01

We get 2 x 0.20 + 3 x 0.10 + 3 x 0.15 + 2 x 0.25 + 2 x 0.30 = 2.25.

Thus on the average, 2.25 bits are needed per character.

12. Use Huffman coding to encode these symbols with given frequencies: A: 0.10, B: 0.25, C: 0.05, D: 0.15, E: 0.30, F: 0.07, G: 0.08. What is the average number of bits required to encode a symbol?

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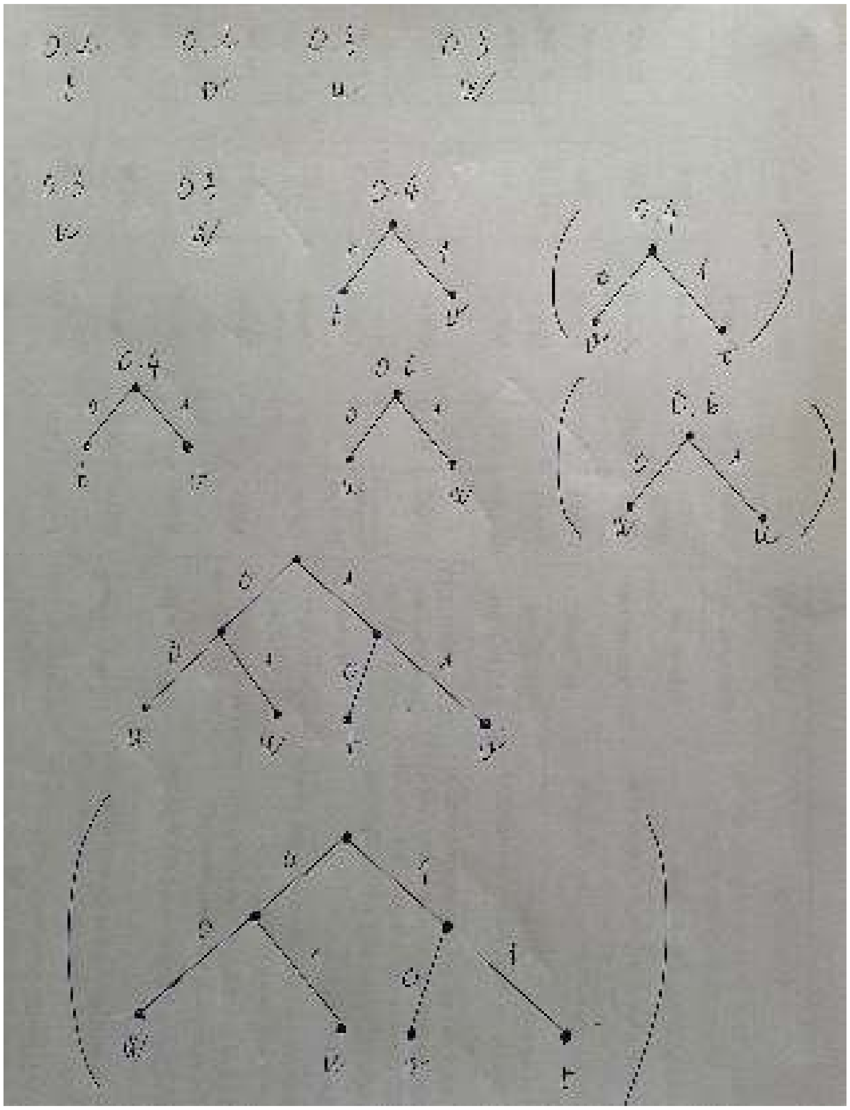
13. Construct two different Huffman codes for these symbols and frequencies: t: 0.2, u: 0.3, v: 0.2, w: 0.3.



10.3 Tree Traversal

1. What is the ordered rooted tree that represents the expression *((x + y)↑2) + ((x − 4)/3)* ?

11.3. **Tree Traversal** – page **816 -** Example 5

2. What is the prefix form for *((x + y) ↑ 2) + ((x − 4)/3)* ?

11.3. **Tree Traversal** – page **817 -** Example 6

3. What is the value of the prefix expression + − ∗ *2 3 5/↑ 2 3 4* ?

11.3. **Tree Traversal** – page **817 -** Example 7

4. What is the postfix form of the expression *((x + y) ↑ 2) + ((x − 4)/3)* ?

11.3. **Tree Traversal** – page **817 -** Example 8

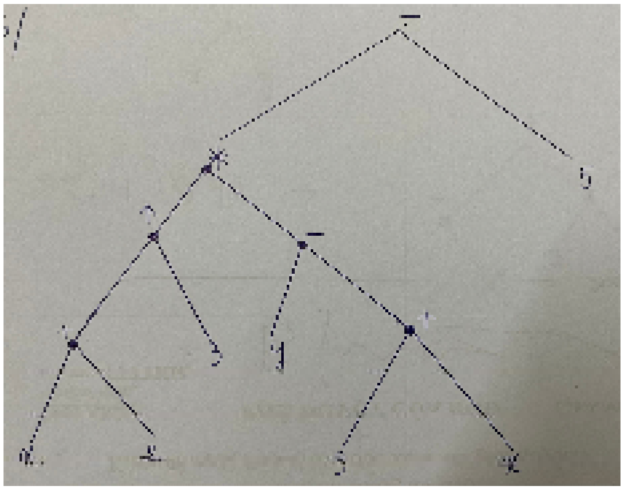
5. What is the value of the postfix expression $7\ 2\ 3 * -\ 4 \uparrow 9\ 3/+$?

11.3. **Tree Traversal** – page **818 -** Example 9

6. a) Represent the expression $((x + 2) \uparrow 3) * (y - (3 + x)) - 5$ using a binary tree.

Write this expression in

b) prefix notation c) postfix notation d) infix notation



$b)\ - * \uparrow x\ 2\ 3 - y + 3\ x\ 5$

$c)\ x\ 2 + 3 \uparrow y\ 3\ x + - * 5 -$

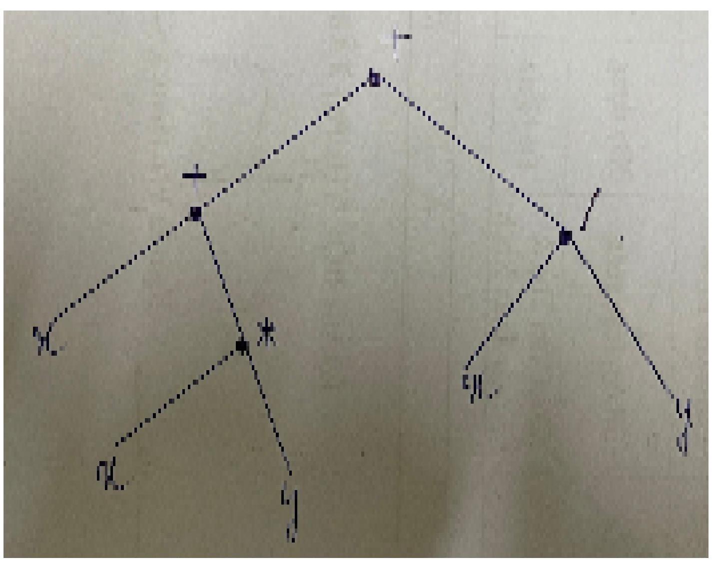
$d)\ (((x+2) \uparrow 3) * (y-(3+x))) - 5$

7. a) Represent the expressions $(x + xy) + (x/y)$ and $x + ((xy + x)/y)$ using binary trees.

Write these expressions in

b) prefix notation c) postfix notation d) infix notation

$(x + xy) + (x/y)$

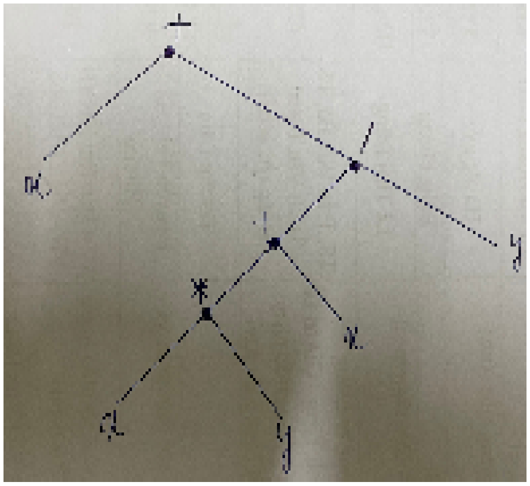


$b)\ ++x*x\ y\ /\ x\ y$

$c)\ x\ x\ y*+x\ y/+$

$d)\ ((x+(x*y))+(x/y))$

$x + ((xy + x)/y)$



$b)\ +x/+*x\ y\ x\ y$

$c)\ x\ x\ y*x+y/+$

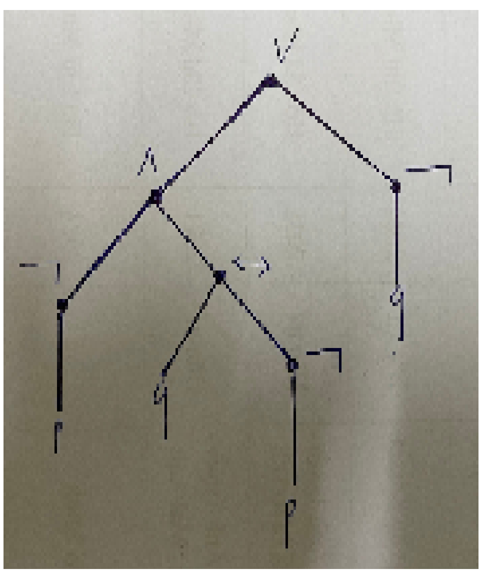
$d)\ (x+(((x*y)+x)/y))$

8. a) Represent the compound propositions ¬(p ∧ q) ↔ (¬p ∨ ¬q) and (¬p ∧ (q ↔ ¬p)) ∨ ¬q using ordered rooted trees.

Write these expressions in

b) prefix notation c) postfix notation d) infix notation

(¬p ∧ (q ↔ ¬p)) ∨ ¬q



$b)\ \vee \wedge \neg\ p \leftrightarrow q \neg\ p \neg\ q$

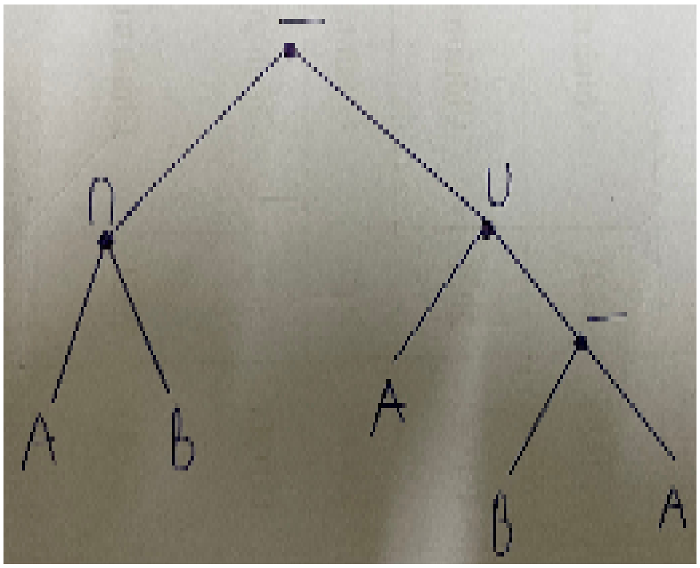
$c)\ \ p \neg\ q\ \ p \neg \leftrightarrow \wedge\ q \neg \vee$

$d)\ (((\neg p\ ) \wedge (q \leftrightarrow (\neg p))) \vee (\neg q\ ))$

9. a) Represent (A ∩ B) − (A ∪ (B − A)) using an ordered rooted tree.

Write this expression in

b) prefix notation c) postfix notation d) infix notation



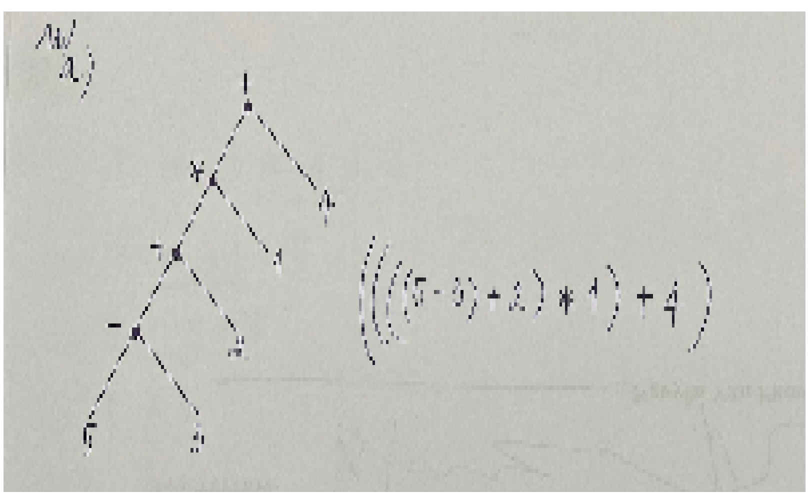
$b)\ -\cap\ A\ B \cup\ A - B\ \ A$

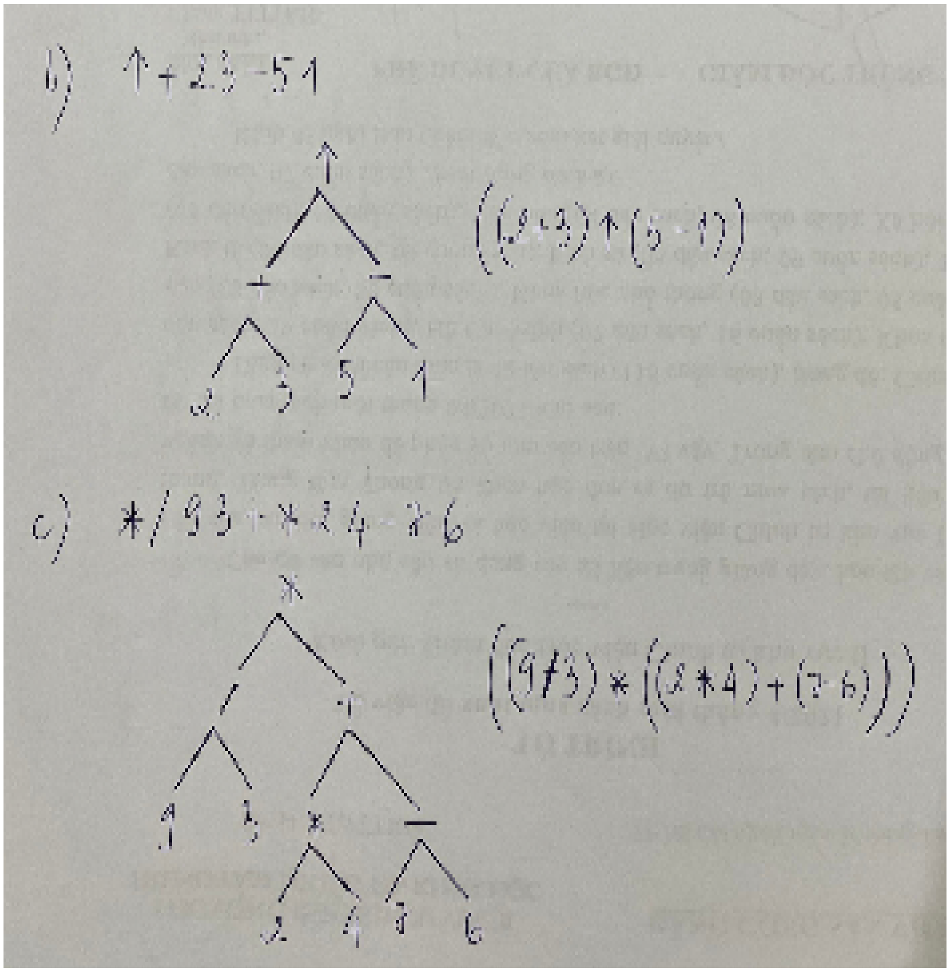
$c)\ \ A\ B \cap\ A\ \ B\ \ A - \cup -$

$d)\ ((A \cap B) - (A \cup (B - A)))$

10. Draw the ordered rooted tree corresponding to each of these arithmetic expressions written in prefix notation. Then write each expression using infix notation.

a) + ∗ + − 5 3 2 1 4 b) ↑ + 2 3 − 5 1 c) ∗ / 9 3 + ∗ 2 4 − 7 6





11. What is the value of each of these prefix expressions?

a) $- * 2 / 8\ 4\ 3$

b) $\uparrow - * 3\ 3 * 4\ 2\ 5$

c) $+ - \uparrow 3\ 2 \uparrow 2\ 3 / 6 - 4\ 2$

d) $* + 3 + 3 \uparrow 3 + 3\ 3\ 3$

b) $\uparrow - * 3\ 3 * 4\ 2\ 5$

$= \uparrow - * 3\ 3\ 8\ 5$

$= \uparrow - 9\ 8\ 5$

$= \uparrow 1\ 5$

$= 1$

c) $+ - \uparrow 3\ 2 \uparrow 2\ 3 / 6 - 4\ 2$

$= + - \uparrow 3\ 2 \uparrow 2\ 3 / 6\ 2$

$= + - \uparrow 3\ 2 \uparrow 2\ 3\ 3$

$= + - \uparrow 3\ 2\ 8\ 3$

$= + - 9\ 8\ 3$

$= + 1\ 3$

$= 4$

12. What is the value of each of these postfix expressions?

a) $5\ 2\ 1 - - 3\ 1\ 4 + + *$

b) $9\ 3 / 5 + 7\ 2 - *$

c) $3\ 2 * 2 \uparrow 5\ 3 - 8\ 4 / * -$

$c)\ 3\ 2 * 2 \uparrow 5\ 3 - 8\ 4 / * -$

$= 6\ 2 \uparrow 5\ 3 - 8\ 4 / * -$

$= 36\ 5\ 3 - 8\ 4 / * -$

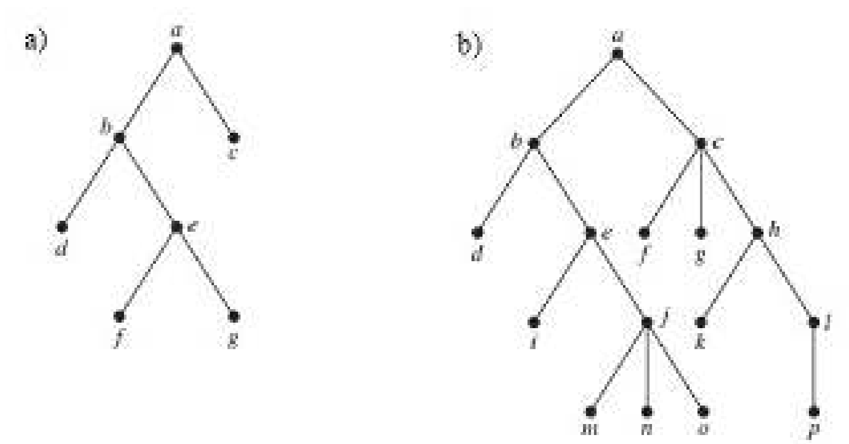
$= 36\ 2\ 8\ 4 / * -$

$= 36\ 2\ 2 * -$

$= 36\ 4 -$

$= 32$

13. Determine the order in which a preorder traversal visits the vertices of the given ordered rooted tree.



a) a b d e f g c

b) a b d e i j m n o c f g h k l p

14. In which order are the vertices of the ordered rooted tree in Exercise 13 visited using an inorder traversal?

a) d b f e g a c

b) d b i e m j n o a f c g k h p l

15. In which order are the vertices of the ordered rooted tree in Exercise 13 visited using a postorder traversal?

a) d f g e b c a

b) d i m n o j e b f g k p l h c a

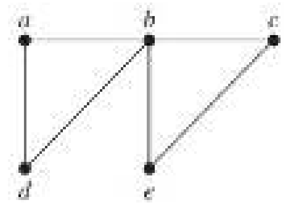
10.4 Spanning Trees

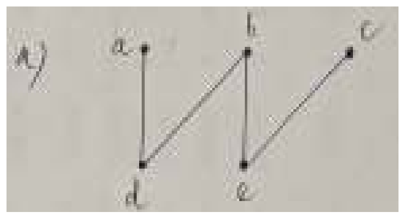
1. How many edges must be removed from a connected graph with n vertices and m edges to produce a spanning tree?

The graph has *m* edges. The spanning tree has n-1 edges. Therefore we need to remove m-(n-1) edges

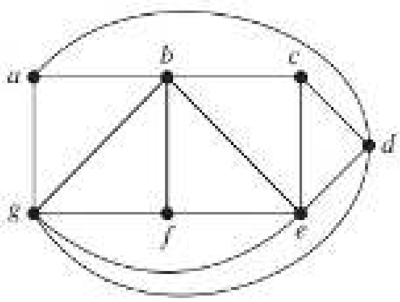
2. Find a spanning tree for the graph shown by removing edges in simple circuits.

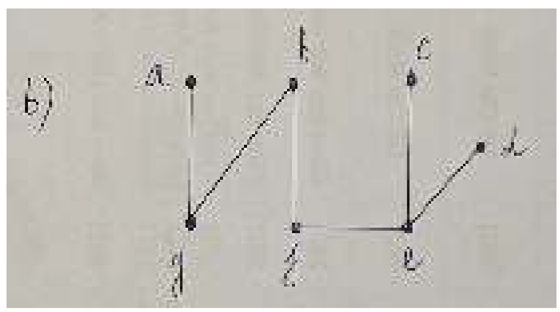
a.



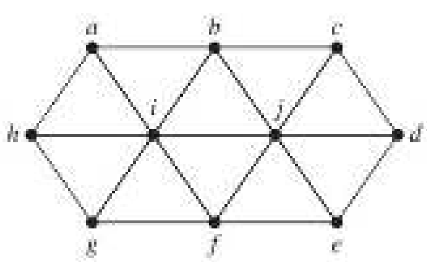


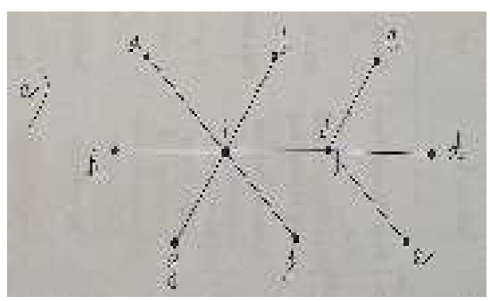
b.



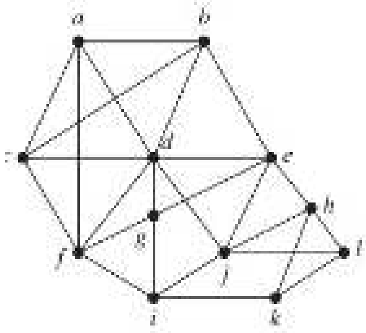


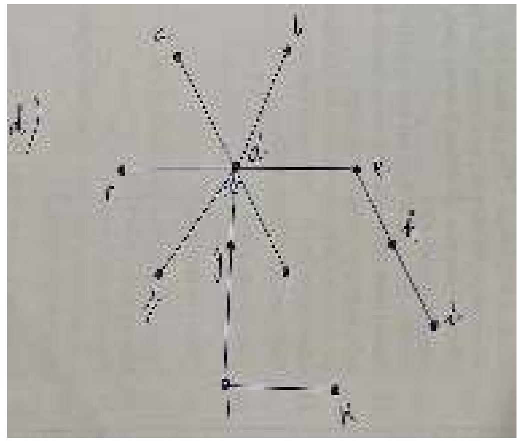
c.



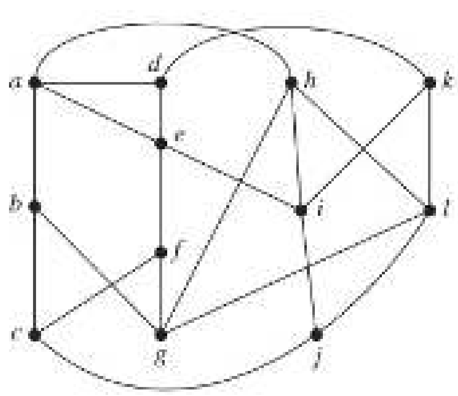


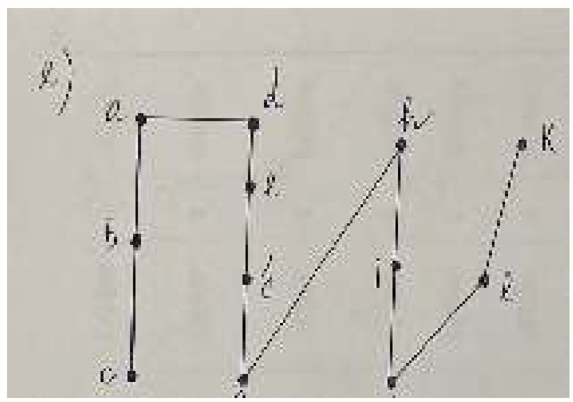
d.





e.

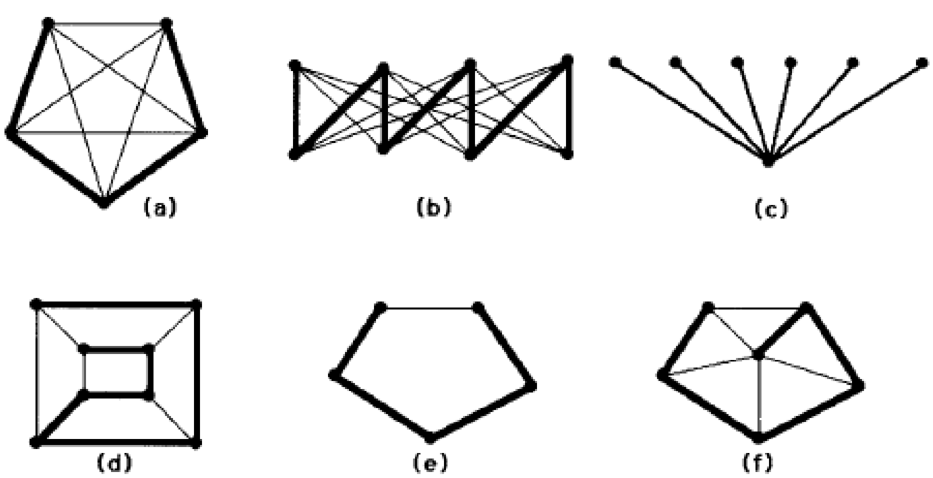




3. Find a spanning tree for each of these graphs.

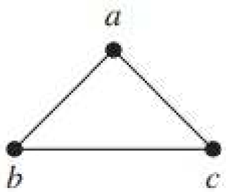
a) K_5 **b)** $K_{4,4}$ **c)** $K_{1,6}$

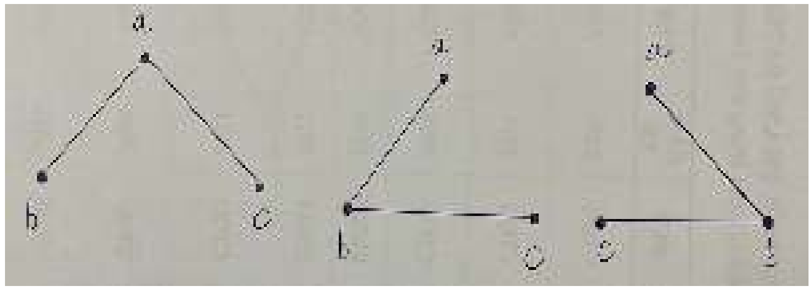
d) Q_3 **e)** C_5 **f)** W_5



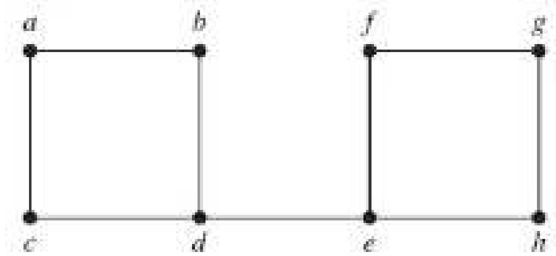
4. Draw all the spanning trees of the given simple graphs.

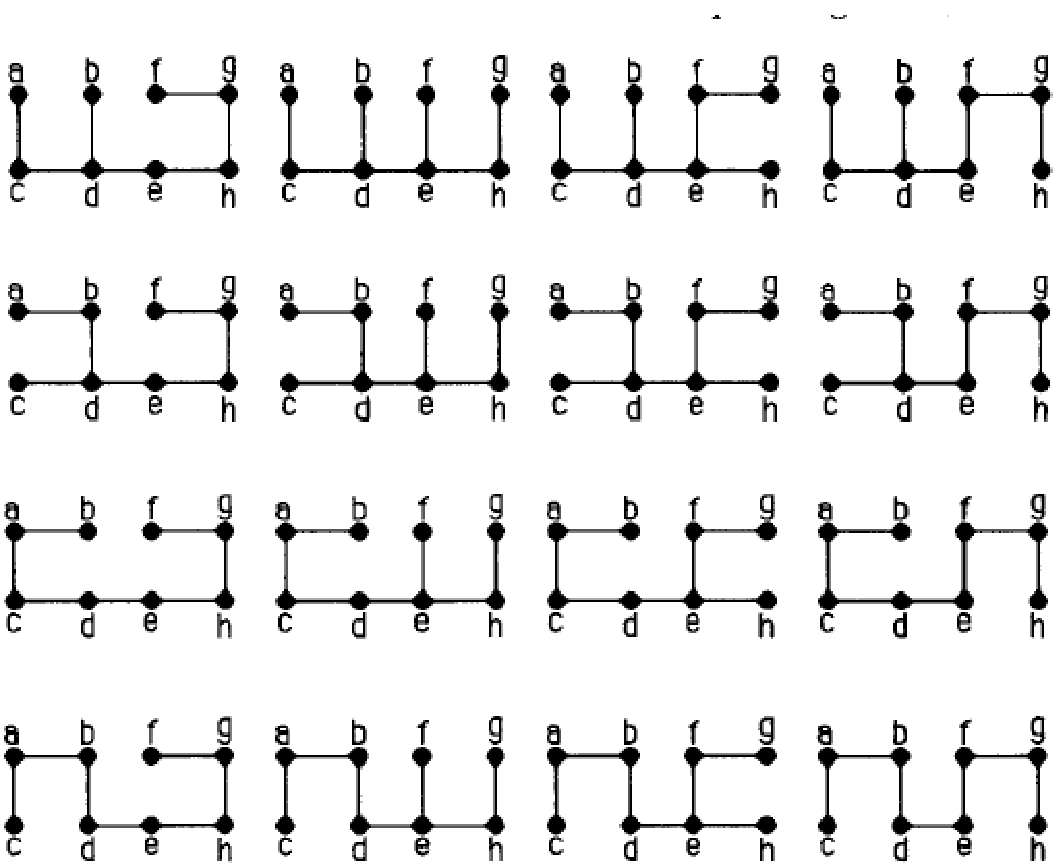
a.





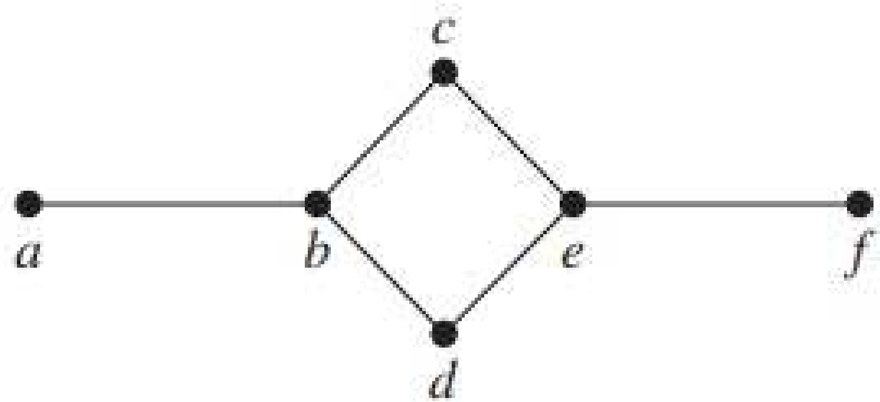
b.

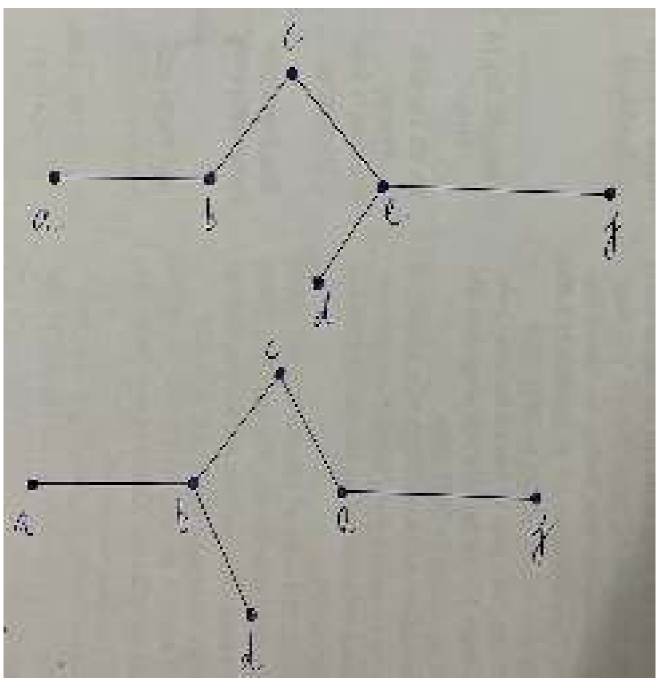


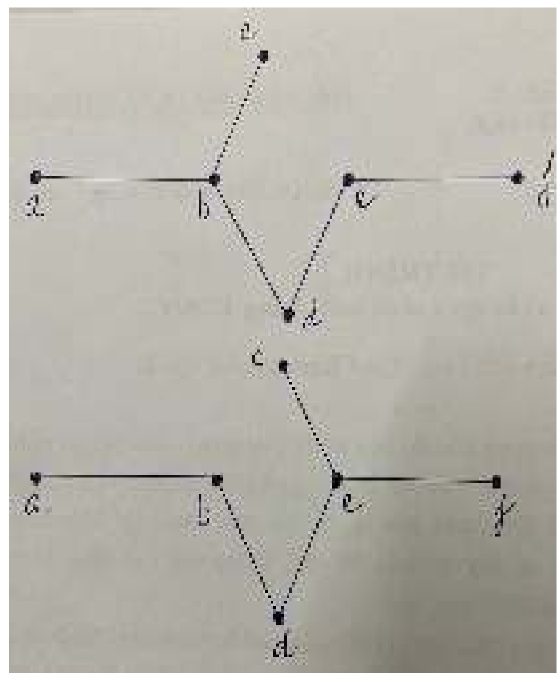


c

.



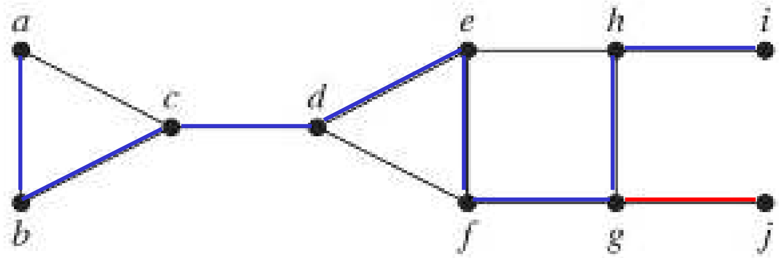




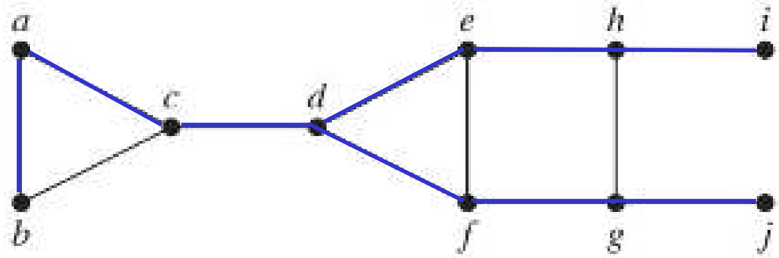
5. Use depth-first search and breadth-first search to produce a spanning tree for the given simple graph. Choose a as the root of this spanning tree and assume that the vertices are ordered alphabetically

a.

Depth-first search

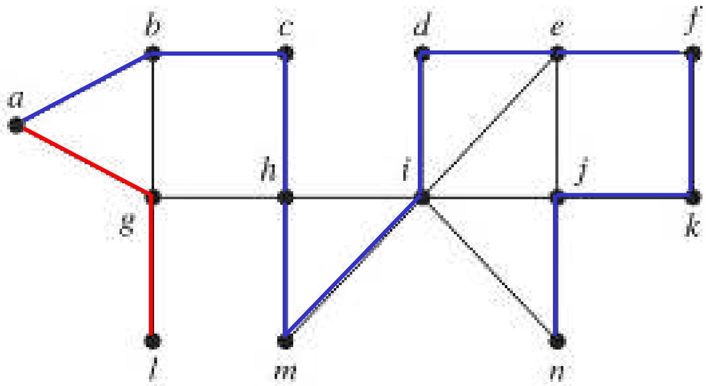


Breadth-first search

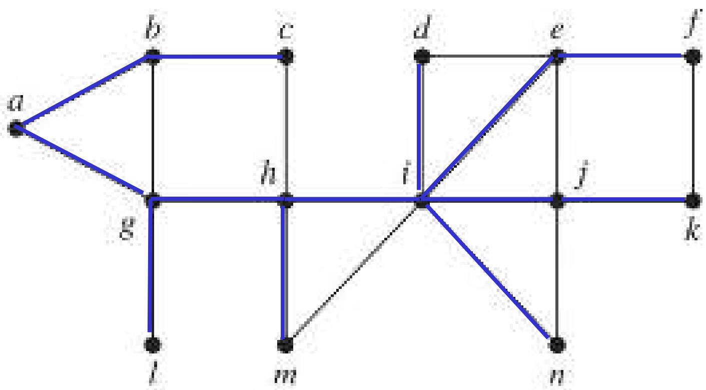


b.

Depth-first search

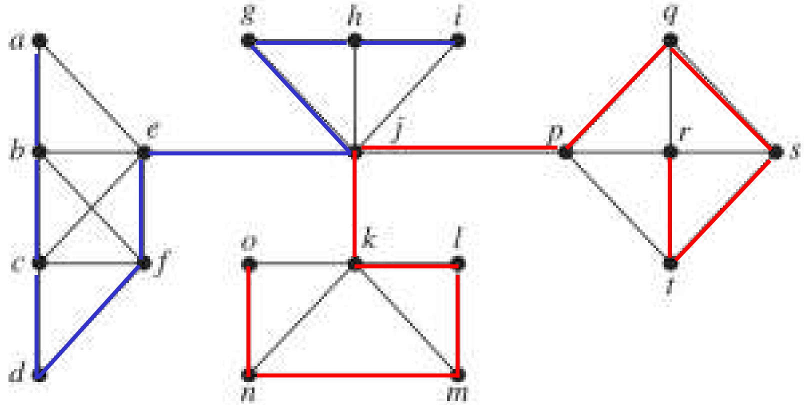


Breadth-first search

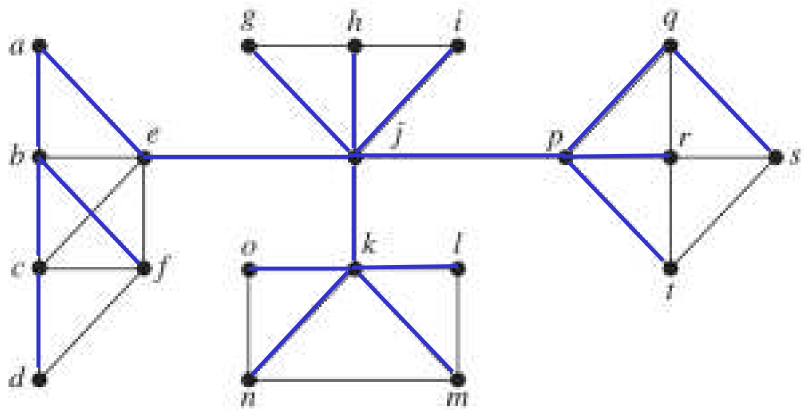


c.

Depth-first search



Breadth-first search



6. Use depth-first search and breadth-first search to find a spanning tree of each of these graphs.

a) W_6 ,starting at the vertex of degree 6 b) K_5

c) $K_{3,4}$, starting at a vertex of degree 3 d) Q_3

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7. Use backtracking to solve the n-queens problem for these values of n.

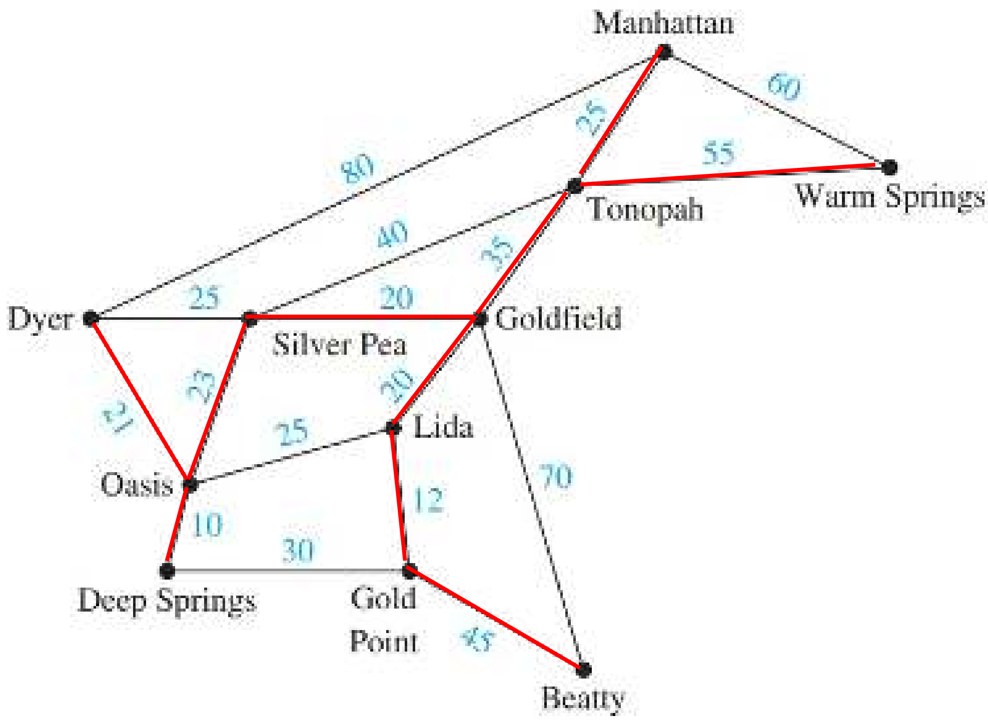
a) n = 3 b) n = 5 c) n = 6

8. Use backtracking to find a subset, if it exists, of the set {27, 24, 19, 14, 11, 8} with sum

a) 20 b) 41 c) 60

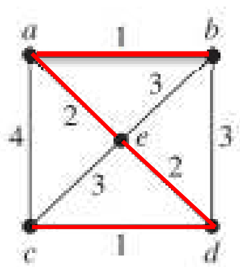
10.5 Minimum Spanning Trees

1. The roads represented by this graph are all unpaved. The lengths of the roads between pairs of towns are represented by edge weights. Which roads should be paved so that there is a path of paved roads between each pair of towns so that a minimum road length is paved? (Note: These towns are in Nevada.)



2. Use Prim's algorithm and Kruskal's algorithm to find a minimum spanning tree for the given weighted graph

a.



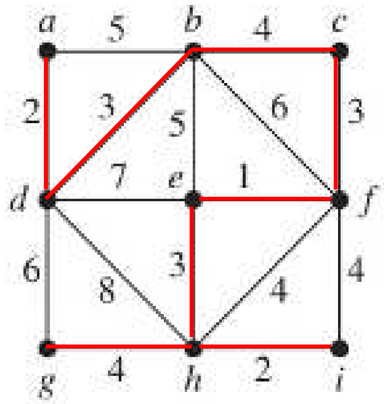
Prim's algorithm

{a , b} , {a , e} , {e , d} , {c , d}

Kruskal's algorithm

{a , b} , {c , d} , {a , e} , {e , d}

b.



Prim’s algorithm

{e , f} , {c , f} , {e , h} , {h , i} , {b , c} , {b , d} , {a , d} , {g , h}

Kruskal’s algorithm

{e , f} , {a , d} , {h , i} , {b , d} , {c , f} , {e , h} , {b , c} , {g , h}

c.

Prim’s algorithm

{a , b} , {a , e} , {b , c} , {c , d} , {d , h} , {e , f} , {g , h} , {e , i} , {i , j} , {i , m} , {m , n}

{n , o} , {o , p} , {l , p} , {k , l}

Kruskal’s algorithm

{a , b} , {a , e} , {c , d} , {d , h} , {b , c} , {e , f} , {g , h} , {e , i} , {l , p} , {m , n} , {n , o}

{i , j} , {i , m} , {k , l} , {o , p}