Appendix B to "Distributionally Robust Pricing with Asymmetric Information"

Proof of Lemma 1. The dual problem of $\inf_{F\in\mathscr{F}} \mathsf{P}(V\geq p)$ is

$$\begin{aligned}
\sup_{\alpha_0,\alpha_1,\alpha_2,\beta} \quad & \alpha_0+\alpha_2\frac{1+s}{2}\sigma^2+\beta\frac{1-s}{2}\sigma^2 \\
\text{s.t.} \quad & \alpha_0-\alpha_1(\mu-v)+\beta(\mu-v)^2\leq \mathtt{I}_{\{v\geq p\}}, \quad \text{if } 0\leq v\leq \mu, \\
& \alpha_0+\alpha_1(v-\mu)+\alpha_2(v-\mu)^2\leq \mathtt{I}_{\{v\geq p\}}, \quad \text{if } v\geq \mu.
\end{aligned} \tag{OB.1}$$

We let $m_1=\mathrm{E}[(V-\mu)^{+2}]$ and $m_2=\mathrm{E}[(\mu-V)^{+2}]$. Recall $s=\frac{\mathrm{E}[(V-\mu)^{+2}]-\mathrm{E}[(\mu-V)^{+2}]}{\sigma^2}$ and the fact that $\mathrm{E}[(V-\mu)^{+2}]+\mathrm{E}[(\mu-V)^{+2}]=\sigma^2$, then $m_1=\frac{(1+s)}{2}\sigma^2$ and $m_2=\frac{(1-s)}{2}\sigma^2$. Note $\mathsf{P}(V\geq p)=\mathrm{E}[\mathtt{I}_{\{V\geq p\}}]$, where $\mathtt{I}$ is indicator function. Then $\mathscr{F}$ can be rewritten as

$$\mathscr{F}=\{F(v)\mid v\geq 0, \quad \mathrm{E}[(V-\mu)^{+}-(\mu-V)^{+}]=0, \quad \mathrm{E}[(V-\mu)^{+2}]=m_1, \quad \mathrm{E}[(\mu-V)^{+2}]=m_2\}.$$

Then the dual problem is as follows

$$\begin{aligned}
\sup_{\alpha_0,\alpha_1,\alpha_2,\beta} \quad & \alpha_0+\alpha_2 m_1+\beta m_2 \\
\text{s.t.} \quad & \alpha_0+\alpha_1((v-\mu)^{+}-(\mu-v)^{+})+\beta(\mu-v)^{+2}+\alpha_2(v-\mu)^{+2}\leq \mathtt{I}_{\{v\geq p\}}, \quad \text{if } v\geq 0,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\sup_{\alpha_0,\alpha_1,\alpha_2,\beta} \quad & \alpha_0+\alpha_2 m_1+\beta m_2 \\
\text{s.t.} \quad & \alpha_0-\alpha_1(\mu-v)+\beta(\mu-v)^2\leq \mathtt{I}_{\{v\geq p\}}, \quad \text{if } 0\leq v\leq \mu, \\
& \alpha_0+\alpha_1(v-\mu)+\alpha_2(v-\mu)^2\leq \mathtt{I}_{\{v\geq p\}}, \quad \text{if } v\geq \mu.
\end{aligned} \tag{OB.2}$$

Denote $x=\mu-v$ and $x=v-\mu$ in the first and second constraints in problem (OB.2), respectively. The problem (OB.2) can be rewritten as follows:

$$\begin{aligned}
\sup_{\alpha_0,\alpha_1,\alpha_2,\beta} \quad & \alpha_0+\alpha_2 m_1+\beta m_2 \\
\text{s.t.} \quad & \alpha_0-\alpha_1 x+\beta x^2\leq \mathtt{I}_{\{x\leq \mu-p\}}, \quad \text{if } 0\leq x\leq \mu, \\
& \alpha_0+\alpha_1 x+\alpha_2 x^2\leq \mathtt{I}_{\{x\geq p-\mu\}}, \quad \text{if } x\geq 0.
\end{aligned} \tag{OB.3}$$

We denote $g_1(x)=\alpha_0-\alpha_1 x+\beta x^2$, and $g_2(x)=\alpha_0+\alpha_1 x+\alpha_2 x^2$. The corresponding optimal primal and dual solutions should satisfy the following complementary slackness conditions:

$$f(\mu - x)(g_1(x) - \mathtt{I}_{\{x \le \mu - p\}}) = 0, \quad \text{if} \quad 0 \le x \le \mu, \tag{OB.4}$$

$$f(x + \mu)(g_2(x) - \mathtt{I}_{\{x \ge p - \mu\}}) = 0, \quad \text{if} \quad x \ge 0, \tag{OB.5}$$

where $f(x)$ is the probability at x of the optimal distribution in the primal problem.

Since $m_1 = \frac{(1+s)}{2}\sigma^2$ and $m_2 = \frac{(1-s)}{2}\sigma^2$, then $\mu > 0$, $\sigma > 0$, $s \in \left(\frac{\sigma^2 - \mu^2}{\sigma^2 + \mu^2}, 1\right)$ implies $m_1 > 0$, $m_2 > 0$ and $\frac{m_2}{m_1}(m_1 + m_2) < \mu^2$. Now we will consider the problem (OB.3) by two cases: $p \le \mu$ and $p \ge \mu$.

Case 1: $p \le \mu$. In this case, the first constraint in problem (OB.3) is equivalent to $g_1(x) \le 1$ for $x \in [0, \mu - p]$ and $g_1(x) \le 0$ for $x \in (\mu - p, \mu]$. It can be divided into the following 5 scenarios, which is illustrated in Figure OB.1.

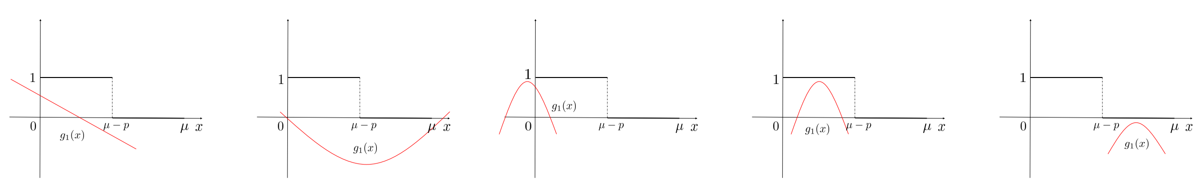
Scenario 1.1: $g_1(x)$ is a straight line, i.e., $\beta = 0$. Moreover, $g_1(0) = \alpha_0 \le 1$, $g_1(\mu - p) = \alpha_0 - \alpha_1(\mu - p) \le 0$ and $g_1(\mu) = \alpha_0 - \alpha_1 \mu \le 0$ due to $g_1(x)$ lies below 1 for $x \in [0, \mu - p]$ and 0 for $x \in (\mu - p, \mu]$.

Scenario 1.2: $g_1(x)$ is a quadratic function with $\beta > 0$. Moreover, $g_1(0) = \alpha_0 \le 1$, $g_1(\mu - p) = \alpha_0 - \alpha_1(\mu - p) + \beta(\mu - p)^2 \le 0$ and $g_1(\mu) = \alpha_0 - \alpha_1 \mu + \beta \mu^2 \le 0$ due to $g_1(x)$ lies below 1 for $x \in [0, \mu - p]$ and 0 for $x \in (\mu - p, \mu]$.

Scenario 1.3: $g_1(x)$ is a quadratic function with $\beta < 0$ and $\frac{\alpha_1}{2\beta} \le 0$. Moreover, $g_1(0) = \alpha_0 \le 1$, $g_1(\mu - p) = \alpha_0 - \alpha_1(\mu - p) + \beta(\mu - p)^2 \le 0$ due to $g_1(x)$ lies below 1 for $x \in [0, \mu - p]$ and 0 for $x \in (\mu - p, \mu]$.

Scenario 1.4: $g_1(x)$ is a quadratic function with $\beta < 0$ and $0 < \frac{\alpha_1}{2\beta} \le \mu - p$. Moreover, $g_1(0) = \alpha_0 \le 1$, $g_1(\mu - p) = \alpha_0 - \alpha_1(\mu - p) + \beta(\mu - p)^2 \le 0$ and $g_1(\frac{\alpha_1}{2\beta}) = \frac{4\beta\alpha_0 - \alpha_1^2}{4\beta} \le 1$ due to $g_1(x)$ lies below 1 for $x \in [0, \mu - p]$ and 0 for $x \in (\mu - p, \mu]$.

Scenario 1.5: $g_1(x)$ is a quadratic function with $\beta < 0$ and $\frac{\alpha_1}{2\beta} > \mu - p$. Moreover, $g_1(0) = \alpha_0 \le 1$, $g_1(\frac{\alpha_1}{2\beta}) = \frac{4\beta\alpha_0 - \alpha_1^2}{4\beta} \le 0$ due to $g_1(x)$ lies below 1 for $x \in [0, \mu - p]$ and 0 for $x \in (\mu - p, \mu]$.



(a) Scenario 1.1 (b) Scenario 1.2 (c) Scenario 1.3 (d) Scenario 1.4 (e) Scenario 1.5

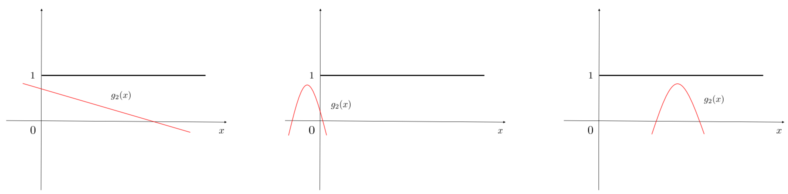
Figure OB.1 Graphical illustration of functions satisfying the first constraint of the problem (OB.3)

And the second constraint is equivalent to $g_2(x) \le 1$ for $x \in [0, +\infty)$, which is divided into three scenarios shown in Figure OB.2.

Scenario 2.1: $g_2(x)$ is a straight line, i.e., $\alpha_2 = 0$. Then $g_2(0) = \alpha_0 \leq 1$ and $\alpha_1 \leq 0$ due to $g_1(x)$ lies below 1 for $x \geq 0$.

Scenario 2.2: $g_2(x)$ is a quadratic function with $\alpha_2 < 0$ and $\frac{-\alpha_1}{2\alpha_2} < 0$. And $g_2(0) = \alpha_0 \leq 1$ due to $g_1(x)$ lies below 1 for $x \geq 0$.

Scenario 2.3: $g_2(x)$ is a quadratic function with $\alpha_2 < 0$ and $\frac{-\alpha_1}{2\alpha_2} \geq 0$. Moreover, $g_2(0) = \alpha_0 \leq 1$ and $g(\frac{-\alpha_1}{2\alpha_2}) = \frac{4\alpha_0\alpha_2 - \alpha_1^2}{4\alpha_2} \leq 1$ due to $g_1(x)$ lies below 1 for $x \geq 0$.



(a) Scenario 2.1 (b) Scenario 2.2 (c) Scenario 2.3

Figure OB.2 Graphical illustration of functions satisfying the second constraint of the problem (OB.3)

Since we have five scenarios for the first constraint and three scenarios for the second, then there are 15 candidate cases. However, we show that the optimal solution of problem (OB.3) only belongs to four cases below.

First, if the second constraint of the optimal solution belongs to Scenario 2.1, we will show that the first constraint must be Scenario 1.2 or 1.3. Note the second constraint of any feasible solution in the Scenario 2.1 must satisfy $\alpha_2 = 0$, $\alpha_1 \leq 0$ and $\alpha_0 \leq 1$. We first show that the optimal solution must satisfy $\alpha_2 = 0$, $\alpha_1 = 0$, $\alpha_0 = 1$. On the one hand, if $\alpha_0 < 1$, then any feasible solution must satisfy $g_2(x) - 1 \leq g_2(0) - 1 = \alpha_0 - 1 < 0$ for $x \geq 0$ due to $\alpha_1 \leq 0$. According to the complementary slackness condition (OB.5), if an optimal dual solution satisfies $\alpha_0 < 1$, he corresponding optimal primal distribution must satisfy $f(x) = 0$, for $x \geq \mu$, which implies that the distribution's mean is less than μ. In other words, it is not a distribution that lies in the feasible set $\mathscr{F}$. On the other hand, if $\alpha_0 = 1$ and $\alpha_1 < 0$, then any feasible dual solution should satisfy $g_2(0) - 1 = \alpha_0 - 1 = 0$ and $g_2(x) - 1 = \alpha_0 + \alpha_1 x - 1 < \alpha_0 - 1 = 0$ for $x > 0$. According to the complementary slackness condition (OB.5), the corresponding optimal primal distribution should satisfy $f(\mu) \geq 0$ and $f(x) = 0$ for $x > \mu$, i.e., $f(\mu) = 1$, $f(x) = 0$, $x \neq \mu$, whose standard deviation is 0. In other words, it is not a distribution that lies in the feasible set $\mathscr{F}$. Therefore, the optimal solution must satisfy $\alpha_2 = 0$, $\alpha_1 = 0$, $\alpha_0 = 1$. Now we will show that the first constraint must be Scenario 1.3. If the first constraint of the optimal solution is in Scenario 1.1, recall that $g_1(\mu - p) = \alpha_0 - \alpha_1(\mu - p) \leq 0$, which contradicts $\alpha_1 = 0$, $\alpha_0 = 1$. If the first constraint of optimal solution is in Scenario 2.1, recall that $\beta > 0$ in Scenario 1.2 and $\alpha_1 = 0$, $\alpha_0 = 1$ in Scenario 2.1, then $g_1(x) = 1 + \beta x^2$ is minimized at $x = 0$ and

$g_1(0) = \alpha_0 = 1$. Therefore, $g_1(x) > g_1(0) = 1$, which is contradicted by the constraint $g_1(x) \leq 1$ in Scenario 1.2. Note that if the first constraint of the optimal solution is in Scenario 1.4 or 1.5, we have $\alpha_1 < 0$. So we only need to consider Scenario 1.3 (Case 1.1).

Second, we show that the second constraint of the optimal dual solution cannot be in Scenario 2.2. Otherwise, recall that $\alpha_2 < 0$ and $\frac{-\alpha_1}{2\alpha_2} < 0$. We have $g_2(x) - 1 < g_2(0) - 1 \leq 0$ for $x > 0$ due to $g_2(x)$ decreasing for $x > 0$. According to the complementary slackness condition (OB.5), the corresponding primal distribution satisfies $f(x) = 0$ for $x > \mu$, i.e., $f(\mu) = 1$ and $f(x) = 0$ for $x \neq \mu$, which is not in the feasible set $\mathscr{F}$ (note $\sigma > 0$).

Third, if the second constraint of the optimal solution belongs to Scenario 2.3, note that we have $\alpha_1 \geq 0$ in Scenario 2.3, which contradicts the constraint $\alpha_1 < 0$ in Scenario 1.4 and Scenario 1.5. So we only need to consider Scenario 1.1 (Case 1.2), Scenario 1.2 (Case 1.3) and Scenario 1.3 (Case 1.4).

In summary, we should solve the following four cases to derive the optimal dual solution.

Case 1.1: The first and second constraints of the optimal dual solution should satisfy Scenario 1.3 and Scenario 2.1, respectively. Recall that $\alpha_2 = 0$, $\alpha_1 = 0$, $\alpha_0 = 1$ in Scenario 1.1, problem (OB.3) can be rewritten as $\sup_{\beta \leq -\frac{1}{(\mu-p)^2}} \beta m_2 + 1$. Since the objective function $\beta m_2 + 1$ is increasing in β due to $m_2 > 0$, the $\beta^* = -\frac{1}{(\mu-p)^2}$. Therefore, the optimal value for the problem above is $1 - \frac{m_2}{(\mu-p)^2}$.

Case 1.2: The first and second constraints of the optimal dual solution should satisfy Scenario 1.1 and Scenario 2.3, respectively. Recall that the constraints in Scenario 1.1 and Scenario 2.3, problem (OB.3) is equivalent to:

$$\begin{aligned} &\sup_{\alpha_0,\alpha_1,\alpha_2,\beta} \quad \alpha_2 m_1 + \beta m_2 + \alpha_0 \\ &\qquad \text{s.t. } 4\alpha_2(\alpha_0 - 1) \geq \alpha_1^2, \alpha_0 - \alpha_1(\mu - p) + \beta(\mu - p)^2 \leq 0, \alpha_0 \leq 1, \alpha_1 \geq 0, \alpha_2 \leq 0, \beta < 0. \end{aligned} \tag{OB.6}$$

First, if $\alpha_0 = 1$, then $\alpha_1 = 0$ due to $4\alpha_2(\alpha_0 - 1) \geq \alpha_1^2$ and any $\alpha_2 \leq 0$ is feasible for F. Thus the problem above is rewritten as follows:

$$\sup_{\beta,\alpha_2} \quad \alpha_2 m_1 + \beta m_2 + 1 \qquad \text{s.t. } \beta \leq -\frac{1}{(\mu - p)^2}, \alpha_2 \leq 0.$$

Note that the objective value is increasing in α_2 and β. Therefore, the problem above is maximized at $\alpha_2 = 0$ and $\beta = -\frac{1}{(\mu-p)^2}$, and the corresponding objective value is $1 - \frac{m_2}{(\mu-p)^2}$, which is as the same as that in Case 1.1.

Second, if $\alpha_0 < 1$, note the objective function of problem (OB.6) is increasing in β and the constraint $\alpha_0 - \alpha_1(\mu - p) + \beta(\mu - p)^2 \leq 0$, i.e., $\beta \leq \frac{\alpha_1(\mu-p)-\alpha_0}{(\mu-p)^2}$. Then $\beta^* \to 0$ when $\frac{\alpha_1(\mu-p)-\alpha_0}{(\mu-p)^2} \geq 0$

and $\beta^* = \frac{\alpha_1(\mu-p)-\alpha_0}{(\mu-p)^2}$ when $\frac{\alpha_1(\mu-p)-\alpha_0}{(\mu-p)^2} \leq 0$.

If $\frac{\alpha_1(\mu-p)-\alpha_0}{(\mu-p)^2} \geq 0$, we can rewrite problem (OB.6) into:

$$\sup_{\alpha_0,\alpha_1,\alpha_2} \alpha_2 m_1 + \alpha_0 \qquad \text{s.t. } 4\alpha_2(\alpha_0 - 1) \geq \alpha_1^2, \alpha_0 - \alpha_1(\mu - p) \leq 0, \alpha_0 < 1, \alpha_1 \geq 0, \alpha_2 \leq 0.$$

Note that the objective value is increasing in α_2, and the constraint $4\alpha_2(\alpha_0 - 1) \geq \alpha_1^2$ and $\alpha_0 < 1$. We have $\alpha_2^* = \frac{\alpha_1^2}{4(\alpha_0-1)}$. Then the problem above can be rewritten as follows:

$$\sup_{\alpha_0,\alpha_1} \frac{\alpha_1^2}{4(\alpha_0 - 1)} m_1 + \alpha_0 \qquad \text{s.t. } \alpha_0 < 1, \alpha_1 \geq 0, \alpha_0 - \alpha_1(\mu - p) \leq 0.$$

Since $\frac{\alpha_1^2}{4(\alpha_0-1)} m_1 + \alpha_0$ is decreasing in α_1 and the constraint $\alpha_0 - \alpha_1(\mu - p) \leq 0$, i.e., $\alpha_1 \geq \frac{\alpha_0}{\mu-p}$. Then $\alpha_1^* = \frac{\alpha_0}{\mu-p}$, and the problem above is equivalent to

$$\sup_{0 \leq \alpha_0 < 1} \frac{\alpha_0^2}{4(\mu - p)^2(\alpha_0 - 1)} m_1 + \alpha_0 \tag{OB.7}$$

The optimal solution for problem (OB.7) is attained by the first-order condition $\frac{\partial \frac{\alpha_0^2}{4(\mu-p)^2(\alpha_0-1)} m_1 + \alpha_0}{\partial \alpha_0} = 0$, i.e., $\alpha_0^* = 1 - \sqrt{\frac{m_1}{m_1 + 4(\mu-p)^2}} \in [0, 1]$ and the corresponding objective value is $1 - \frac{2}{1+\sqrt{1+\frac{4(\mu-p)^2}{m_1}}}$.

If $\frac{\alpha_1(\mu-p)-\alpha_0}{(\mu-p)^2} \leq 0$, we can rewrite problem (OB.6) into:

$$\begin{aligned} &\sup_{\alpha_0,\alpha_1,\alpha_2} \alpha_2 m_1 + \frac{\alpha_1(\mu - p) - \alpha_0}{(\mu - p)^2} m_2 + \alpha_0 \\ &\qquad \text{s.t. } \alpha_1(\mu - p) - \alpha_0 \leq 0, 4\alpha_2(\alpha_0 - 1) \geq \alpha_1^2, 0 \leq \alpha_0 < 1, \alpha_2 \leq 0, \alpha_1 \geq 0. \end{aligned}$$

Since $\alpha_2 m_1 + \frac{\alpha_1(\mu-p)-\alpha_0}{(\mu-p)^2} m_2 + \alpha_0$ is increasing in α_2 and the constraint $4\alpha_2(\alpha_0 - 1) \geq \alpha_1^2$, i.e., $\alpha_2 \leq \frac{\alpha_1^2}{4(\alpha_0-1)}$. Then $\alpha_2^* = \frac{\alpha_1^2}{4(\alpha_0-1)}$, and the problem above is equivalent to

$$\begin{aligned} &\sup_{\alpha_0,\alpha_1} \frac{\alpha_1^2}{4(\alpha_0 - 1)} m_1 + \frac{\alpha_1(\mu - p) - \alpha_0}{(\mu - p)^2} m_2 + \alpha_0 \\ &\qquad \text{s.t. } \alpha_1(\mu - p) - \alpha_0 \leq 0, 0 \leq \alpha_0 < 1, \alpha_1 \geq 0. \end{aligned} \tag{OB.8}$$

Let $H(\alpha_0, \alpha_1) = \frac{\alpha_1^2}{4(\alpha_0-1)} m_1 + \frac{\alpha_1(\mu-p)-\alpha_0}{(\mu-p)^2} m_2 + \alpha_0$. Consider the first-order conditions:

$$\frac{\partial H(\alpha_0, \alpha_1)}{\partial \alpha_0} = -\frac{m_1}{4}\left(\frac{\alpha_1}{\alpha_0 - 1}\right)^2 - \frac{m_2}{(\mu - p)^2} + 1 = 0, \quad \frac{\partial H(\alpha_0, \alpha_1)}{\partial \alpha_1} = \frac{m_1}{2}\frac{\alpha_1}{\alpha_0 - 1} + \frac{m_2}{\mu - p} = 0.$$

Solving the above two equations, we have $\frac{\alpha_1}{2(\alpha_0-1)}=-\sqrt{\frac{1-\frac{m_2}{(\mu-p)^2}}{m_1}}=-\frac{1}{\mu-p}\frac{m_2}{m_1}$, which implies $(\mu-p)^2=\frac{(m_1+m_2)m_2}{m_1}$. Hence, if and only if $(\mu-p)^2=\frac{(m_1+m_2)m_2}{m_1}$, the optimal solution can be given by $\frac{\alpha_1}{2(\alpha_0-1)}=-\frac{1}{\mu-p}\frac{m_2}{m_1}$, where the objective function can be simplified to a constant $\frac{m_2}{m_1(\mu-p)^2}(1-\alpha_0)+\alpha_0(1-\frac{m_2}{(\mu-p)^2})=\frac{m_2}{m_1+m_2}=1-\frac{m_2}{(\mu-p)^2}$. Otherwise, i.e., $(\mu-p)^2\neq\frac{(m_1+m_2)m_2}{m_1}$, the first-order conditions can not be satisfied and the optimal solution will be attained at boundaries, i.e., $\alpha_1(\mu-p)=\alpha_0$ or $\alpha_0=0$. If $\alpha_0=0$, the constraint of problem (OB.8) becomes $\alpha_1\leq 0$ and $\alpha_1\geq 0$, which means that $\alpha_1=0$ and the objective value is 0; If $\alpha_1(\mu-p)=\alpha_0$, problem (OB.8) equals to problem (OB.7), and the optimal objective value is $1-\frac{2}{1+\sqrt{1+\frac{4(\mu-p)^2}{m_1}}}$.

Case 1.3: The first and second constraints of the optimal dual solution satisfy Scenario 1.2 and Scenario 2.3, respectively. Recall the constraints in Scenario 1.2 and Scenario 2.3, problem (OB.3) is equivalent to:

$$\begin{aligned}\sup_{\alpha_0,\alpha_1,\alpha_2,\beta}\ & \alpha_2 m_1+\beta m_2+\alpha_0\\ \text{s.t. } & 4\alpha_2(\alpha_0-1)\geq\alpha_1^2, \alpha_0-\alpha_1\mu+\beta\mu^2\leq 0,\\ & \beta(\mu-p)^2-\alpha_1(\mu-p)+\alpha_0\leq 0,\alpha_0\leq 1,\alpha_2<0,\alpha_1\geq 0,\beta>0.\end{aligned}\tag{OB.9}$$

First, if $\alpha_0=1$, then $\alpha_1=0$ due to $4\alpha_2(\alpha_0-1)\geq\alpha_1^2$. Thus $\alpha_0-\alpha_1\mu+\beta\mu^2=1+\beta\mu^2\leq 0$, which is contradicted with $\beta>0$.

Second, if $\alpha_0<1$, then $\alpha_2\leq\frac{\alpha_1^2}{4(\alpha_0-1)}$. Note that the objective value is increasing in α_2 due to $m_1>0$, then $\alpha_2^*=\frac{\alpha_1^2}{4(\alpha_0-1)}$. Moreover, $\beta\leq\frac{\alpha_1\mu-\alpha_0}{\mu^2}$ due to $\alpha_0-\alpha_1\mu+\beta\mu^2\leq 0$ and $\beta\leq\frac{\alpha_1(\mu-p)-\alpha_0}{(\mu-p)^2}$ due to $\beta(\mu-p)^2-\alpha_1(\mu-p)+\alpha_0\leq 0$. Since the objective value is increasing in β, then $\beta^*=\frac{\alpha_1\mu-\alpha_0}{\mu^2}$ when $\frac{\alpha_1\mu-\alpha_0}{\mu^2}\leq\frac{\alpha_1(\mu-p)-\alpha_0}{(\mu-p)^2}$, and $\beta^*=\frac{\alpha_1(\mu-p)-\alpha_0}{(\mu-p)^2}$ when $\frac{\alpha_1\mu-\alpha_0}{\mu^2}\geq\frac{\alpha_1(\mu-p)-\alpha_0}{(\mu-p)^2}$.

If $\frac{\alpha_1\mu-\alpha_0}{\mu^2}\leq\frac{\alpha_1(\mu-p)-\alpha_0}{(\mu-p)^2}$, then problem (OB.9) is rewritten as follows:

$$\begin{aligned}\sup_{\alpha_1,\alpha_0}\ & \frac{\alpha_1^2}{4(\alpha_0-1)}m_1+\frac{\alpha_1\mu-\alpha_0}{\mu^2}m_2+\alpha_0\\ \text{s.t. } & \frac{\alpha_1\mu-\alpha_0}{\mu^2}\leq\frac{\alpha_1(\mu-p)-\alpha_0}{(\mu-p)^2},\alpha_1(\mu-p)-\alpha_0\geq 0,\alpha_0<1,\alpha_1\geq 0.\end{aligned}$$

Let $J(\alpha_0,\alpha_1)=\frac{\alpha_1^2}{4(\alpha_0-1)}m_1+\frac{\alpha_1\mu-\alpha_0}{\mu^2}m_2+\alpha_0$, and consider the first-order conditions:

$$\frac{\partial J(\alpha_0,\alpha_1)}{\partial\alpha_0}=-\frac{\alpha_1^2}{4(\alpha_0-1)^2}m_1-\frac{1}{\mu^2}m_2+1=0,\quad \frac{\partial J(\alpha_0,\alpha_1)}{\partial\alpha_1}=\frac{\alpha_1}{2(\alpha_0-1)}m_1+\frac{1}{\mu}m_2=0.$$

Solving the above two equations, we have $\frac{\alpha_1}{2(\alpha_0-1)}=-\sqrt{\frac{1-\frac{m_2}{\mu^2}}{m_1}}=-\frac{1}{\mu}\frac{m_2}{m_1}$, which implies that $\mu^2=$

$\frac{(m_1+m_2)m_2}{m_1}$ and contradicts the assumption $\mu^2 > \frac{(m_1+m_2)m_2}{m_1}$. Thus, the first-order conditions can not be satisfied, and the optimal value is achieved at boundaries. First, we consider the boundary $\alpha_1 = 0$, then $J(\alpha_0,\alpha_1) = \alpha_0(1-\frac{m_2}{\mu^2})$ is increasing in α_0 due to $\mu > m_2$. Note $\alpha_1(\mu-p)-\alpha_0 = -\alpha_0 \geq 0$, i.e., $\alpha_0 \leq 0$. Thus the optimal objective value is 0, which is achieved at $\alpha_0 = 0$ and $\alpha_1 = 0$. Second, we consider the boundary $\alpha_1(\mu-p)-\alpha_0 = 0$, thus $\alpha_1\mu-\alpha_0 \geq 0$ due to $\alpha_1 \geq 0$. However, according to $\frac{\alpha_1\mu-\alpha_0}{\mu^2} \leq \frac{\alpha_1(\mu-p)-\alpha_0}{(\mu-p)^2}$, we have $\alpha_1\mu-\alpha_0 \leq 0$, which is a contradiction. Lastly, we consider the boundary $\frac{\alpha_1\mu-\alpha_0}{\mu^2} = \frac{\alpha_1(\mu-p)-\alpha_0}{(\mu-p)^2}$, i.e., $\mu(\mu-p)\alpha_1 = \alpha_0(2\mu-p) \geq 0$, then we have $J(\alpha_0,\alpha_1) = \frac{\alpha_0^2(2\mu-p)^2 m_1}{4\mu^2(\mu-p)^2(\alpha_0-1)} + \alpha_0(1+\frac{m_2}{\mu(\mu-p)}) + \alpha_0 \equiv L(\alpha_0)$. Taking the derivative of $L(\alpha_0)$, we have $L'(\alpha_0) = \frac{(1-\alpha_0)^2(m_1(2\mu-p)^2+4\mu(\mu-p)(m_2+\mu(\mu-p)))-m_1(2\mu-p)^2}{4\mu^2(\mu-p)^2(\alpha_0-1)^2}$, whose denominator is positive and numerator is decreasing in α_0. Let $L'(\alpha_0^*) = 0$, we have $\alpha_0^* = 1-\frac{\sqrt{1+a}}{\sqrt{1+b}}$ $(0 < \alpha_0^* < 1$ is feasible), and $L(\alpha_0)$ is maximized at α_0^*, where

$$a = \frac{m_2^2}{(2\mu-p)^2 m_1 - m_2^2}, \quad b = \frac{(m_2+2\mu(\mu-p))^2}{(2\mu-p)^2 m_1 - m_2^2}.$$

And the corresponding optimal objective value is $(\frac{\sqrt{1+b}-\sqrt{1+a}}{\sqrt{b}-\sqrt{a}})^2$. In sum, the problem (OB.9) is maximized at the boundary $\frac{\alpha_1\mu-\alpha_0}{\mu^2} = \frac{\alpha_1(\mu-p)-\alpha_0}{(\mu-p)^2}$, and the corresponding optimal objective value is $(\frac{\sqrt{1+b}-\sqrt{1+a}}{\sqrt{b}-\sqrt{a}})^2$.

If $\frac{\alpha_1\mu-\alpha_0}{\mu^2} \geq \frac{\alpha_1(\mu-p)-\alpha_0}{(\mu-p)^2}$, then problem (OB.9) is rewritten as follows:

$$\begin{aligned} \sup_{\alpha_1,\alpha_0} \quad & \frac{\alpha_1^2}{4(\alpha_0-1)} m_1 + \frac{\alpha_1(\mu-p)-\alpha_0}{(\mu-p)^2} m_2 + \alpha_0 \\ \text{s.t.} \quad & \frac{\alpha_1\mu-\alpha_0}{\mu^2} \geq \frac{\alpha_1(\mu-p)-\alpha_0}{(\mu-p)^2}, \alpha_1(\mu-p)-\alpha_0 \geq 0, \alpha_0 < 1, \alpha_1 \geq 0. \end{aligned}$$

The objective function is as the same as the objective function $H(\alpha_0,\alpha_1)$ in problem (OB.8). Recall that we have shown the first-order conditions $\frac{\partial H(\alpha_0,\alpha_1)}{\partial \alpha_0} = 0$ and $\frac{\partial H(\alpha_0,\alpha_1)}{\partial \alpha_1} = 0$ cannot be satisfied. Then we only need to consider the boundaries. First, we consider $\alpha_1(\mu-p)-\alpha_0 = 0$, then $0 \leq \alpha_0 \leq 1$ is feasible due to $\frac{\alpha_1\mu-\alpha_0}{\mu^2} \geq \frac{\alpha_1(\mu-p)-\alpha_0}{\mu^2} = 0 = \frac{\alpha_1(\mu-p)-\alpha_0}{(\mu-p)^2}$ and $\alpha_1 \geq 0$. Thus, the problem above is simplified to problem (OB.7). Second, we consider $\alpha_1 = 0$, then $-\alpha_0 = \alpha_1(\mu-p)-\alpha_0 \geq 0$, i.e., $\alpha_0 \leq 0$, and $\frac{\alpha_1\mu-\alpha_0}{\mu^2} - \frac{\alpha_1(\mu-p)-\alpha_0}{(\mu-p)^2} = \alpha_0(\frac{1}{(\mu-p)^2} - \frac{1}{\mu^2}) \geq 0$, which implies $\alpha_0 \geq 0$. Thus $\alpha_0^* = 0$ and the corresponding objective value is 0. Last, we consider $\frac{\alpha_1\mu-\alpha_0}{\mu^2} = \frac{\alpha_1(\mu-p)-\alpha_0}{(\mu-p)^2}$, which is as the same as the last case in the previous paragraph.

Case 1.4: The first and second constraints of the optimal dual solution should satisfy Scenario 1.3 and Scenario 2.3, respectively. Recall the constraints in Scenario 1.3 and Scenario 2.3, problem (OB.3) is equivalent to:

$$\sup_{\alpha_0,\alpha_1,\alpha_2,\beta} \quad \alpha_2 m_1 + \alpha_0$$
$$\text{s.t. } 4\alpha_2(\alpha_0 - 1) \geq \alpha_1^2, \alpha_0 - \alpha_1(\mu - p) \leq 0, \alpha_0 \leq 1, \alpha_1 \geq 0, \alpha_2 \leq 0.$$

Note we have solved the problem above in Case 1.2 (the case $\frac{\alpha_1(\mu-p)-\alpha_0}{(\mu-p)^2} \geq 0$), then the optimal objective value is $1 - \frac{2}{1+\sqrt{1+\frac{4(\mu-p)}{m_1}}}$ in this case.

In summary, from the four cases above, the optimal value of problem (OB.3) is $\max\{0, 1 - \frac{m_2}{(\mu-p)^2}, (\frac{\sqrt{1+b}-\sqrt{1+a}}{\sqrt{b}-\sqrt{a}})^2, 1 - \frac{2}{1+\sqrt{1+\frac{4(\mu-p)^2}{m_1}}}\}$ for $0 \leq p \leq \mu$. We denote P^* as the optimal value for the primal problem, D^* as the optimal value for the dual problem.

When $0 \leq p \leq \mu - \sqrt{\frac{m_2}{m_1}(m_1+m_2)}$, the objective value $1 - \frac{m_2}{(\mu-p)^2}$ is asymptotically achievable by a series of three-point distributions:

$$V = \begin{cases} p - \epsilon & \text{with prob. } \frac{m_2}{(\mu-p+\epsilon)^2}, \\ \mu & \text{with prob. } 1 - \frac{m_2}{(\mu-p+\epsilon)^2} - \frac{m_2^2}{m_1(\mu-p+\epsilon)^2}, \\ \mu + \frac{m_1}{m_2}(\mu - p + \epsilon) & \text{with prob. } \frac{m_2^2}{m_1(\mu-p+\epsilon)^2}, \end{cases}$$

with $\epsilon \searrow 0$.

Then $1 - \frac{m_2}{(\mu-p)^2} \geq P^* = D^* = \max\{0, 1 - \frac{m_2}{(\mu-p)^2}, (\frac{\sqrt{1+b}-\sqrt{1+a}}{\sqrt{b}-\sqrt{a}})^2, 1 - \frac{2}{1+\sqrt{1+\frac{4(\mu-p)^2}{m_1}}}\} \geq 1 - \frac{m_2}{(\mu-p)^2}$. Therefore, we obtain $P^* = D^* = 1 - \frac{m_2}{(\mu-p)^2}$ in this case, which can be achieved by the dual solution from Case 1.1 with $\beta = -\frac{m_2}{(\mu-p)^2}$, α_2, $\alpha_1 = 0$, $\alpha_0 = 1$, and the asymptotic distributions for the primal problem above.

When $\mu - \sqrt{\frac{m_2}{m_1}(m_1+m_2)} < p \leq \mu$, the objective value $(\frac{\sqrt{1+b}-\sqrt{1+a}}{\sqrt{b}-\sqrt{a}})^2$ can be asymptotically achievable by a series of three-point distributions:

$$V = \begin{cases} 0 & \text{with prob. } 1 - (\frac{\sqrt{1+b_\epsilon}-\sqrt{1+a_\epsilon}}{\sqrt{b_\epsilon}-\sqrt{a_\epsilon}})^2(\frac{\mu}{p-\epsilon}(1 + \frac{2\sqrt{1+a_\epsilon}(\mu-p+\epsilon)}{(2\mu-p+\epsilon)(\sqrt{1+b_\epsilon}-\sqrt{1+a_\epsilon})}) - 1), \\ p - \epsilon & \text{with prob. } \frac{\mu}{p-\epsilon}((\frac{\sqrt{1+b_\epsilon}-\sqrt{1+a_\epsilon}}{\sqrt{b_\epsilon}-\sqrt{a_\epsilon}})^2 + \frac{2\sqrt{1+a_\epsilon}(\mu-p+\epsilon)(\sqrt{1+b_\epsilon}-\sqrt{1+a_\epsilon})}{(2\mu-p+\epsilon)(\sqrt{b_\epsilon}-\sqrt{a_\epsilon})^2}), \\ \frac{2\sqrt{1+a_\epsilon}\mu(\mu-p+\epsilon)}{(\sqrt{1+b_\epsilon}-\sqrt{1+a_\epsilon})(2\mu-p+\epsilon)} + \mu & \text{with prob. } (\frac{\sqrt{1+b_\epsilon}-\sqrt{1+a_\epsilon}}{\sqrt{b_\epsilon}-\sqrt{a_\epsilon}})^2, \end{cases}$$

with $\epsilon \searrow 0$, where $a_\epsilon = \frac{m_2^2}{(2\mu-p+\epsilon)^2 m_1 - m_2^2}, b_\epsilon = \frac{(m_2+2\mu(\mu-p+\epsilon))^2}{(2\mu-p+\epsilon)^2 m_1 - m_2^2}$. Similarly, by the strong duality, $(\frac{\sqrt{1+b}-\sqrt{1+a}}{\sqrt{b}-\sqrt{a}})^2 \geq P^* = D^* = \max\{0, 1 - \frac{m_2}{(\mu-p)^2}, (\frac{\sqrt{1+b}-\sqrt{1+a}}{\sqrt{b}-\sqrt{a}})^2, 1 - \frac{2}{1+\sqrt{1+\frac{4(\mu-p)^2}{m_1}}}\} \geq (\frac{\sqrt{1+b}-\sqrt{1+a}}{\sqrt{b}-\sqrt{a}})^2$. Therefore, we get $P^* = D^* = (\frac{\sqrt{1+b}-\sqrt{1+a}}{\sqrt{b}-\sqrt{a}})^2$ in this case, which can be achieved by the dual solution from Case 1.3 with $\alpha_0 = 1 - \frac{\sqrt{1+a}}{\sqrt{1+b}}$, $\alpha_1 = \frac{\alpha_0}{\mu-p}$, $\beta = \frac{\alpha_1(\mu-p)-\alpha_0}{(\mu-p)^2}$, $\alpha_2 = \frac{\alpha_1^2}{4(\alpha_0-1)}$.

Therefore, when $p \leq \mu$, the optimal objective value of program (OB.3) is:

$$\begin{cases} 1-\frac{m_2}{(\mu-p)^2} & \text{if } 0\le p\le \mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)}, \\ (\frac{\sqrt{1+b}-\sqrt{1+a}}{\sqrt{b}-\sqrt{a}})^2 & \text{if } p>\mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)}. \end{cases}$$

Case 2: $p \ge \mu$. In this case, the first constraint in problem (OB.3) is equivalent to $g_1(x)\le 0$ for $x\in[0,\mu]$. It can be divided into the following 5 scenarios, which are illustrated in Figure OB.3. Recall that $g_1(x)=\alpha_0-\alpha_1 x+\beta x^2$, and $g_2(x)=\alpha_0+\alpha_1 x+\alpha_2 x^2$.

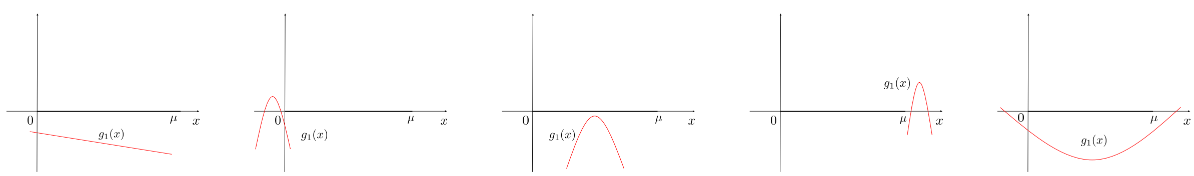
Scenario 1.1: $g_1(x)$ is a straight line, i.e., $\beta=0$. Moreover, $g_1(0)=\alpha_0\le 0$ and $g_1(\mu)=\alpha_0-\alpha_1\mu\le 0$.

Scenario 1.2: $g_1(x)$ is a quadratic function with $\beta<0$ and $\frac{\alpha_1}{2\beta}\le 0$. Moreover, $g_1(0)=\alpha_0\le 0$.

Scenario 1.3: $g_1(x)$ is a quadratic function with $\beta<0$ and $0<\frac{\alpha_1}{2\beta}<\mu$. Moreover, $g_1(\frac{\alpha_1}{2\beta})=\frac{4\beta\alpha_0-\alpha_1^2}{4\beta}\le 0$.

Scenario 1.4: $g_1(x)$ is a quadratic function with $\beta<0$ and $\frac{\alpha_1}{2\beta}\le\mu$. Moreover, $g_1(\mu)=\alpha_0-\alpha_1\mu+\beta\mu^2\le 0$.

Scenario 1.5: $g_1(x)$ is a quadratic function with $\beta>0$. Moreover, $g_1(0)=\alpha_0\le 1$ and $g_1(\mu)=\alpha_0-\alpha_1\mu+\beta\mu^2\le 0$.



(a) Scenario 1.1 (b) Scenario 1.2 (c) Scenario 1.3 (d) Scenario 1.4 (e) Scenario 1.5

Figure OB.3 Graphical illustration of functions satisfying the first constraint of the problem (OB.3)

The second constraint is equivalent to $g_2(x)\le 0$ for $x\in[0,p-\mu)$ and $g_2(x)\le 1$ for $x\in[p-\mu,+\infty)$, which is divided into the 4 scenarios shown in Figure OB.4.

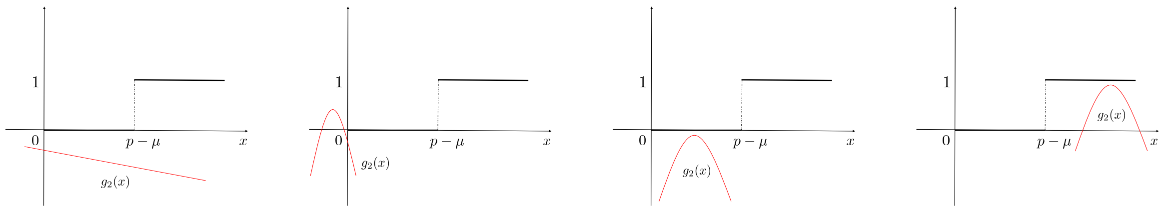
Scenario 2.1: $g_2(x)$ is a straight line, i.e., $\alpha_2=0$. Moreover, $g_2(0)=\alpha_0\le 0$ and $\alpha_1\le 0$ due to $g_2(x)\le 1$ for $x\in[p-\mu,+\infty)$.

Scenario 2.2: $g_2(x)$ is a quadratic function with $\alpha_2<0$ and $-\frac{\alpha_1}{2\alpha_2}\le 0$. Moreover, $g_2(0)=\alpha_0\le 0$.

Scenario 2.3: $g_2(x)$ is a quadratic function with $\alpha_2<0$ and $0<-\frac{\alpha_1}{2\alpha_2}<p-\mu$. Moreover, $g_2(-\frac{\alpha_1}{2\alpha_2})=\frac{4\alpha_2\alpha_0-\alpha_1^2}{4\alpha_2}\le 0$.

Scenario 2.4: $g_2(x)$ is a quadratic function with $\alpha_2<0$ and $-\frac{\alpha_1}{2\alpha_2}\ge p-\mu$. Moreover, $g_2(p-\mu)=\alpha_0+\alpha_1(p-\mu)+\alpha_2(p-\mu)^2\le 0$ and $g_2(-\frac{\alpha_1}{2\alpha_2})=\frac{4\alpha_2\alpha_0-\alpha_1^2}{4\alpha_2}\le 1$.

Since we have 5 scenarios for the first constraint and 4 scenarios for the second one, then there are 20 candidate cases. However, we will show that the optimal solution of problem (OB.3) can only possibly belong to 2 cases.



(a) Scenario 2.1 (b) Scenario 2.2 (c) Scenario 2.3 (d) Scenario 2.4

Figure OB.4 Graphical illustration of functions satisfying the second constraint of the problem (OB.3)

First, we show that the positive optimal value must be derived in Scenario 1.5. Recall that all scenarios in the second constraint have $\alpha_0 \le 0$, $\alpha_2 \le 0$ and that all scenarios except Scenario 1.5 in the first constraint require $\beta \le 0$. Therefore, any feasible solution in Scenarios 1.1, 1.2, 1.3 and 1.4 will have the objective value $\alpha_0 + \alpha_2 m_1 + \beta m_2 \le 0$. Note that $(\alpha_1, \alpha_2, \beta, \alpha_0) = \mathbf{0}$ is a feasible solution and the corresponding objective value is 0. Therefore, we only need to consider Scenario 1.5 in the first constraint to derive the optimal solution in the following analysis.

Second, we will show the second constraint of the optimal dual problem can not be in Scenario 2.1 when the first constraint satisfies Scenario 1.5. Recall that we have $\alpha_1 \le 0$, $\alpha_0 \le 0$ in Scenario 2.1 and $\beta > 0$, $g_1(\mu) = \alpha_0 - \alpha_1\mu + \beta\mu^2 \le 0$ in Scenario 1.5. On the one hand, according to the complementary slackness condition (OB.5), if an optimal dual solution satisfies $\alpha_1 < 0$, we must have a corresponding optimal primal distribution satisfy $f(x) = 0$, for $x > \mu$, which implies the distribution's mean is strictly less than μ otherwise $\sigma = 0$. In other words, it is not a distribution that lies in the feasible set $\mathscr{F}$. On the other hand, if an optimal dual solution satisfies $\alpha_0 < 0$ and $\alpha_1 = 0$, we must have the corresponding optimal primal distribution satisfying $f(x) = 0$ for $x \ge \mu$, which is also not in $\mathscr{F}$. Finally, if an optimal dual solution satisfies $\alpha_0 = 0$ and $\alpha_1 = 0$, the constraint $g_1(\mu) = \beta\mu^2 \le 0$ in Scenario 1.5 contradict to $\beta > 0$ in Scenario 1.5. In sum, the second constraint of the optimal dual problem cannot be in Scenario 2.1.

Third, we will prove that the second constraint of the optimal dual problem cannot be in Scenario 2.2 when the first constraint is in Scenario 1.5. Recall the constraints in Scenario 2.2 and we have $g_2(x) < g_2(0) \le 0$ for $x \in (0, +\infty)$. According to the complementary slackness condition (OB.5), the corresponding optimal primal distribution must satisfy $f(x) = 0$ for $x > \mu$, which implies that the distribution's mean is strictly less than μ otherwise $\sigma = 0$, which is not a distribution that lies in the feasible set $\mathscr{F}$. Thus the optimal solution cannot be in Scenario 2.2.

In summary, we should solve the following 2 cases to derive the optimal dual solution.

Case 2.1: The optimal solution satisfies Scenario 1.5 and Scenario 2.3. Recall the constraints illustrated before, problem (OB.3) is equivalent to:

$$\begin{aligned}\sup_{\alpha_0,\alpha_1,\alpha_2,\beta} \quad & \alpha_2 m_1 + \beta m_2 + \alpha_0 \\ \text{s.t. } & \alpha_1 + 2\alpha_2(p-\mu) \le 0, 4\alpha_2\alpha_0 \ge \alpha_1^2, \alpha_0 - \alpha_1\mu + \beta\mu^2 \le 0, \\ & \alpha_2 < 0, \alpha_1 \ge 0, \alpha_0 \le 0, \beta \ge 0.\end{aligned}$$

Note the objective value $\alpha_2 m_1 + \beta m_2 + \alpha_0$ is increasing in β, and the constraint $\alpha_0 - \alpha_1\mu + \beta\mu^2 \le 0$, i.e., $\beta \le \frac{\alpha_1\mu - \alpha_0}{\mu^2}$. Thus, it is optimal to let $\beta = \frac{\alpha_1\mu-\alpha_0}{\mu^2}$. Thus problem above is equivalent to:

$$\begin{aligned}\sup_{\alpha_0,\alpha_1,\alpha_2} \quad & \alpha_2 m_1 + \frac{m_2}{\mu}\alpha_1 + \alpha_0(1 - \frac{m_2}{\mu^2}) \\ \text{s.t. } & \alpha_1 + 2\alpha_2(p-\mu) \le 0, 4\alpha_2\alpha_0 \ge \alpha_1^2, \alpha_2 < 0, \alpha_1 \ge 0, \alpha_0 \le 0.\end{aligned}$$

Recall that $m_2 + \frac{m_2^2}{m_1} = \frac{m_2}{m_1}(m_1+m_2) < \mu^2$, $m_1 > 0$ and $m_2 > 0$, then $m_2 + \frac{m_2^2}{m_1} = \frac{m_2}{m_1}(m_1+m_2) < \mu^2$, which implies that $m_2 < \mu^2$. Hence, the objective value $\alpha_2 m_1 + \frac{m_2}{\mu}\alpha_1 + \alpha_0(1-\frac{m_2}{\mu^2})$ is increasing in α_0. Note the constraints $4\alpha_2\alpha_0 \ge \alpha_1^2$, $\alpha_2 < 0$ and $\alpha_0 \le 0$. Thus, it is optimal to let $\alpha_0 = \frac{\alpha_1^2}{4\alpha_2}$. Then the problem above is simplified to:

$$\sup_{\alpha_1,\alpha_2} \alpha_2 m_1 + \frac{m_2}{\mu}\alpha_1 + \frac{\alpha_1^2}{4\alpha_2}(1-\frac{m_2}{\mu^2}) \qquad \text{s.t. } \alpha_1 + 2\alpha_2(p-\mu) \le 0, \alpha_2 < 0, \alpha_1 \ge 0.$$

Let $J(\alpha_1,\alpha_2) = \alpha_2 m_1 + \frac{m_2}{\mu}\alpha_1 + \frac{\alpha_1^2}{4\alpha_2}(1-\frac{m_2}{\mu^2})$, and consider the first-order conditions:

$$\frac{\partial J(\alpha_1,\alpha_2)}{\partial \alpha_1} = \frac{m_2}{\mu} + \frac{\alpha_1}{2\alpha_2}(1-\frac{m_2}{\mu^2}) = 0, \quad \frac{\partial J(\alpha_1,\alpha_2)}{\partial \alpha_2} = m_1 - \frac{\alpha_1^2}{4\alpha_2^2}(1-\frac{m_2}{\mu^2}) = 0.$$

Solving the two equations above, we have $\frac{\alpha_1}{\alpha_2} = -2\sqrt{\frac{m_1}{1-\frac{m_2}{\mu^2}}} = -\frac{2m_2}{\mu(1-\frac{m_2}{\mu^2})}$, which implies that $\mu^2 = \frac{m_2}{m_1}(m_1+m_2)$ and contradicts our assumption $\mu^2 > \frac{m_2}{m_1}(m_1+m_2)$. Thus, the first-order conditions cannot be satisfied and the optimal value is achieved at boundaries. First, if $\alpha_1 = 0$, the problem becomes maximizing $\alpha_2 m_1$ for $\alpha_2 < 0$, and the optimal objective value is less than 0. Second, if $\alpha_1 + 2\alpha_2(p-\mu) = 0$, the problem becomes maximizing $\alpha_2(m_1 - 2\frac{m_2}{\mu}(p-\mu) + (p-\mu)^2(1-\frac{m_2}{\mu^2}))$ for $\alpha_2 < 0$. The discriminant of the quadratic function $m_1 - 2\frac{m_2}{\mu}x + (1-\frac{m_2}{\mu^2})x^2$ is $4\frac{m_2^2}{\mu^2} - 4m_1(1-\frac{m_2}{\mu^2}) = 4\frac{m_2^2 - m_1\mu^2 + m_1 m_2}{\mu^2} < 0$ due to $\mu^2 > \frac{m_2}{m_1}(m_1+m_2)$. Note $1 - \frac{m_2}{\mu^2} > 0$ since $\mu^2 > \frac{m_2}{m_1}(m_1+m_2) > m_2$. Hence, $m_1 - 2\frac{m_2}{\mu}x + (1-\frac{m_2}{\mu^2})x^2 > 0$ for any x. Thus the objective value decreases in α_2 and the optimal value is less than 0.

Case 2.2: The optimal solution satisfies Scenario 1.5 and Scenario 2.4. Recall that the constraints illustrated before, problem (OB.3) is equivalent to:

$$\begin{aligned}\sup_{\alpha_0,\alpha_1,\alpha_2,\beta}\quad & \alpha_2 m_1+\beta m_2+\alpha_0,\\ \text{s.t. } & \alpha_1+2\alpha_2(p-\mu)\geq 0, 4\alpha_2(\alpha_0-1)\geq \alpha_1^2,\\ & \alpha_2(p-\mu)^2+\alpha_1(p-\mu)+\alpha_0\leq 0, \alpha_0-\alpha_1\mu+\beta\mu^2\leq 0,\\ & \alpha_2<0, \alpha_1\geq 0, \alpha_0\leq 0, \beta\geq 0.\end{aligned}$$

Note that the objective value $\alpha_2 m_1+\beta m_2+\alpha_0$ is increasing in β, and the constraint $\alpha_0-\alpha_1\mu+\beta\mu^2\leq 0$, i.e., $\beta\leq\frac{\alpha_1\mu-\alpha_0}{\mu^2}$. Thus it is optimal to let $\beta=\frac{\alpha_1\mu-\alpha_0}{\mu^2}$. Then problem above is equivalent to:

$$\begin{aligned}\sup_{\alpha_0,\alpha_1,\alpha_2}\ & \alpha_2 m_1+\frac{m_2}{\mu}\alpha_1+\alpha_0(1-\frac{m_2}{\mu^2})\\ \text{s.t. } & 4\alpha_2(\alpha_0-1)\geq\alpha_1^2, \alpha_2(p-\mu)^2+\alpha_1(p-\mu)+\alpha_0\leq 0,\\ & \alpha_1+2\alpha_2(p-\mu)\geq 0, \alpha_2<0, \alpha_1\geq 0, \alpha_0\leq 0.\end{aligned}$$

Note that $\alpha_2 m_1+\frac{m_2}{\mu}\alpha_1+\alpha_0(1-\frac{m_2}{\mu^2})$ is increasing in α_1, and the constraints $4\alpha_2(\alpha_0-1)\geq\alpha_1^2$ and $\alpha_2(p-\mu)^2+\alpha_1(p-\mu)+\alpha_0\leq 0$. Hence, it is optimal to let $\alpha_1=\min\{\frac{-\alpha_0-\alpha_2(p-\mu)^2}{p-\mu}, \sqrt{4\alpha_2(\alpha_0-1)}\}$. Thus, the problem above is divided into two cases $\alpha_1=\frac{-\alpha_0-\alpha_2(p-\mu)^2}{p-\mu}$ and $\alpha_1=\sqrt{4\alpha_2(\alpha_0-1)}$:

$$\begin{aligned}\sup_{\alpha_0,\alpha_2}\quad & \alpha_2 m_1+\frac{m_2}{\mu}\frac{-\alpha_0-\alpha_2(p-\mu)^2}{p-\mu}+\alpha_0\left(1-\frac{m_2}{\mu^2}\right)\\ \text{s.t. } & [\alpha_0+\alpha_2(p-\mu)^2]^2\leq 4\alpha_2(p-\mu)^2(\alpha_0-1), \alpha_2(p-\mu)^2\geq\alpha_0, \alpha_0\leq 0, \alpha_2<0,\end{aligned}\tag{OB.10}$$

and

$$\begin{aligned}\sup_{\alpha_0,\alpha_2}\quad & \alpha_2 m_1+\frac{m_2}{\mu}\sqrt{4\alpha_2(\alpha_0-1)}+\alpha_0\left(1-\frac{m_2}{\mu^2}\right)\\ \text{s.t. } & [\alpha_0+\alpha_2(p-\mu)^2]^2\geq 4\alpha_2(p-\mu)^2(\alpha_0-1), \alpha_2(p-\mu)^2\geq\alpha_0, \alpha_0\leq 0, \alpha_2<0.\end{aligned}\tag{OB.11}$$

For problem (OB.10), note $[\alpha_0+\alpha_2(p-\mu)^2]^2\leq 4\alpha_2(p-\mu)^2(\alpha_0-1)$ is equivalent to $[\alpha_0-\alpha_2(p-\mu)^2]^2\leq -4\alpha_2(p-\mu)^2$, then we must have $\alpha_2(p-\mu)^2-\alpha_0\leq 2(p-\mu)\sqrt{-\alpha_2}$ because of the constraint $\alpha_2(p-\mu)^2\geq\alpha_0$. Then problem (OB.10) can be rewritten as follows:

$$\begin{aligned}\sup_{\alpha_0,\alpha_2}\quad & \alpha_2(m_1-\frac{m_2}{\mu}(p-\mu))+\alpha_0\left[1-\frac{m_2}{\mu^2}-\frac{m_2}{\mu(p-\mu)}\right]\\ \text{s.t. } & \alpha_2(p-\mu)^2-2(p-\mu)\sqrt{-\alpha_2}\leq\alpha_0\leq\alpha_2(p-\mu)^2, \alpha_2<0.\end{aligned}$$

If $1-\frac{m_2}{\mu^2}-\frac{m_2}{\mu(p-\mu)}>0$, the objective value increases in α_0, thus it is optimal to let $\alpha_0=\alpha_2(p-\mu)^2$. Then the problem above is equivalent to maximizing $\alpha_2(m_1-2\frac{m_2}{\mu}(p-\mu)+(p-\mu)^2(1-\frac{m_2}{\mu^2}))$ for $\alpha_2<0$, which is as the same as Case 2.1. If $1-\frac{m_2}{\mu^2}-\frac{m_2}{\mu(p-\mu)}<0$, i.e., $p\geq\mu+\frac{m_2}{\mu(1-\frac{m_2}{\mu^2})}$, the objective value decreases in α_0, thus it is optimal to let $\alpha_0=\alpha_2(p-\mu)^2-2(p-\mu)\sqrt{-\alpha_2}$. Then the problem above is equivalent to maximizing $-(\sqrt{-\alpha_2})^2[m_1-2\frac{m_2}{\mu}(p-\mu)+(1-\frac{m_2}{\mu^2})(p-\mu)^2]-2\sqrt{-\alpha_2}(p-\mu)[1-\frac{m_2}{\mu^2}-\frac{m_2}{\mu(p-\mu)}]$ for $\alpha_2<0$. Therefore, it is a quadratic function about $\sqrt{-\alpha_2}$, and the optimal value is $\frac{(p-\mu)^2[1-\frac{m_2}{\mu^2}-\frac{m_2}{\mu(p-\mu)}]^2}{m_1-2\frac{m_2}{\mu}(p-\mu)+(1-\frac{m_2}{\mu^2})(p-\mu)^2}$, which is achieved by $\sqrt{-\alpha_2}=\frac{-(p-\mu)[1-\frac{m_2}{\mu^2}-\frac{m_2}{\mu(p-\mu)}]}{m_1-2\frac{m_2}{\mu}(p-\mu)+(1-\frac{m_2}{\mu^2})(p-\mu)^2}$.

For problem (OB.11), note that $[\alpha_0+\alpha_2(p-\mu)^2]^2\geq 4\alpha_2(p-\mu)^2(\alpha_0-1)$ is equivalent to $[\alpha_0-\alpha_2(p-\mu)^2]^2\geq -4\alpha_2(p-\mu)^2$, then we must have $\alpha_2(p-\mu)^2-\alpha_0\geq 2(p-\mu)\sqrt{-\alpha_2}$ due to the constraint $\alpha_2(p-\mu)^2\geq\alpha_0$. Then problem (OB.11) is equivalent to:

$$\begin{aligned}\sup_{\alpha_0,\alpha_2}\quad & \alpha_2 m_1+\frac{m_2}{\mu}\sqrt{4\alpha_2(\alpha_0-1)}+\alpha_0(1-\frac{m_2}{\mu^2})\\ \text{s.t. } & \alpha_2(p-\mu)^2-2(p-\mu)\sqrt{-\alpha_2}\geq\alpha_0,\alpha_0\leq 0,\alpha_2<0.\end{aligned}$$

Let $J(\alpha_0,\alpha_2)=\alpha_2 m_1+2\frac{m_2}{\mu}\sqrt{-\alpha_2(1-\alpha_0)}+\alpha_0(1-\frac{m_2}{\mu^2})$, and consider the first-order conditions:

$$\frac{\partial J(\alpha_0,\alpha_2)}{\partial\alpha_0}=1-\frac{m_2}{\mu^2}-\frac{m_2}{\mu}\sqrt{\frac{-\alpha_2}{1-\alpha_0}}=0,\quad \frac{\partial J(\alpha_0,\alpha_2)}{\partial\alpha_2}=m_1-\frac{m_2}{\mu}\sqrt{\frac{-\alpha_2}{1-\alpha_0}}=0.$$

Solving the two equations above, we have $\sqrt{\frac{-\alpha_2}{1-\alpha_0}}=(1-\frac{m_2}{\mu^2})\frac{\mu}{m_2}=\frac{m_2}{\mu m_1}$, i.e., $\mu^2=\frac{m_2}{m_1}(m_1+m_2)$. This contradicts our assumption $\mu^2>\frac{m_2}{m_1}(m_1+m_2)$. Thus, the first-order conditions cannot be satisfied and the optimal objective value is achieved at boundaries. First, if $\alpha_0=0$, the constraint $\alpha_2(p-\mu)^2-2(p-\mu)\sqrt{-\alpha_2}\geq\alpha_0$ is equivalent to $(p-\mu)\sqrt{-\alpha_2}[-\sqrt{-\alpha_2}(p-\mu)-2]\geq 0$, which is not possible due to $p\geq\mu$. Second, if $\alpha_2(p-\mu)^2-\alpha_0=2(p-\mu)\sqrt{-\alpha_2}$, then the problem is as the same as problem (OB.10).

Based on analysis of Cases 2.1 and 2.2, when $p\geq\mu$, the objective value is $\max\{\frac{(p-\mu)^2[1-\frac{m_2}{\mu^2}-\frac{m_2}{\mu(p-\mu)}]^2}{m_1-2\frac{m_2}{\mu}(p-\mu)+(1-\frac{m_2}{\mu^2})(p-\mu)^2},0\}$. Since $\frac{(p-\mu)^2[1-\frac{m_2}{\mu^2}-\frac{m_2}{\mu(p-\mu)}]^2}{m_1-2\frac{m_2}{\mu}(p-\mu)+(1-\frac{m_2}{\mu^2})(p-\mu)^2}\geq 0$ if and only if $p\leq\mu+\frac{m_2}{\mu(1-\frac{m_2}{\mu^2})}$, the optimal objective value is:

$$\begin{cases}\frac{\left[\frac{m_2}{\mu}-(p-\mu)\left(1-\frac{m_2}{\mu^2}\right)\right]^2}{m_1-\frac{2m_2}{\mu}(p-\mu)+(p-\mu)^2\left(1-\frac{m_2}{\mu^2}\right)} & \text{if } \mu\leq p\leq\mu+\frac{m_2}{\mu(1-\frac{m_2}{\mu^2})},\\ 0 & \text{if } p\geq\mu+\frac{m_2}{\mu(1-\frac{m_2}{\mu^2})}.\end{cases}$$

Moreover, when $p=\mu$, the optimal objective value $\frac{m^2}{m_1\mu^2}$ can be obtained asymptotically by the distributions in Case 1; when $\mu<p\leq\mu+\frac{m_2}{\mu(1-\frac{m_2}{\mu^2})}$, the optimal objective value $\frac{(\frac{m_2}{\mu}-(p-\mu)(1-\frac{m_2}{\mu^2}))^2}{m_1-\frac{m_2}{\mu}(p-\mu)+(p-\mu)^2(1-\frac{m_2}{\mu^2})}$

is asymptotically achievable by a series of three-point distributions:

$$V = \begin{cases} 0 & \text{with prob. } \frac{m_2}{\mu^2}, \\ p-\epsilon & \text{with prob. } 1-\frac{m_2}{\mu^2}-\frac{[\frac{m_2}{\mu}-(p-\epsilon-\mu)(1-\frac{m_2}{\mu^2})]^2}{m_1-\frac{2m_2}{\mu}(p-\epsilon-\mu)+(p-\epsilon-\mu)^2(1-\frac{m_2}{\mu^2})} \equiv p_2, \\ \frac{\mu-(p-\epsilon)p_2}{p_3} & \text{with prob. } \frac{[\frac{m_2}{\mu}-(p-\epsilon-\mu)(1-\frac{m_2}{\mu^2})]^2}{m_1-\frac{2m_2}{\mu}(p-\epsilon-\mu)+(p-\epsilon-\mu)^2(1-\frac{m_2}{\mu^2})} \equiv p_3, \end{cases}$$

with $\epsilon \searrow 0$. It is also achievable by the dual solution $\alpha_2 = -\frac{-(p-\mu)^2[1-\frac{m_2}{\mu^2}-\frac{m_2}{\mu(p-\mu)}]^2}{(m_1-2\frac{m_2}{\mu}(p-\mu)+(1-\frac{m_2}{\mu^2})(p-\mu)^2)^2}$, $\alpha_0 = \alpha_2(p-\mu)^2 - 2(p-\mu)\sqrt{-\alpha_2}$, $\beta = \frac{\alpha_1\mu-\alpha_0}{\mu^2}$, $\alpha_1 = \sqrt{4\alpha_2(\alpha_0-1)}$.

The primal and dual results are summarized in Table 2, and we obtain Lemma 1 by replacing $m_1 = \frac{(1+s)}{2}\sigma^2$ and $m_2 = \frac{(1-s)}{2}\sigma^2$. $\square$

Proof of Lemma 4. We consider two cases: $p \le \mu$ and $p > \mu$, respectively.

Case 1: $p \le \mu$. We first show $\inf_{F\in\mathscr{F}_s} \mathsf{P}(V \ge p)$ is given by the following problem

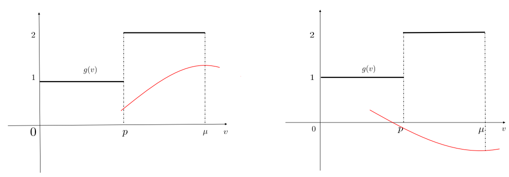
$$\begin{aligned} \underline{\mathsf{P}}_{s1} = &\inf_F \ \mathsf{P}(p \le V < \mu) + \mathsf{P}(V \le \mu) \\ &\text{s.t.} \ \ v \ge 0, \quad \mathrm{E}[1]=1, \quad \mathrm{E}[(V-\mu)^2\mathtt{I}_{\{0\le V\le\mu\}}] = \tfrac{\sigma^2}{2}, \quad \mathsf{P}(V\le\mu) = \mathsf{P}(V\ge\mu). \end{aligned} \tag{OB.12}$$

We first prove $\underline{\mathsf{P}}_{s1} \ge \inf_{F\in\mathscr{F}_s} \mathsf{P}(V\ge p)$. Suppose F^* is an optimal distribution for problem (OB.12), and $f^*(x)$ is its probability density function. We construct a symmetric distribution F with $f(x) = f^*(x)$ for $x \le \mu$ and $f(x) = f(2\mu - x)$ for $x > \mu$. Then $\mathrm{E}_F[1] = \mathsf{P}_F(V\le\mu) + \mathsf{P}_F(V>\mu) = \mathsf{P}_{F^*}(V \le \mu) + \mathsf{P}_{F^*}(V<\mu) = \mathsf{P}_{F^*}(V\le\mu) + \mathsf{P}_{F^*}(V>\mu) = 1$, where the third equality is due to $\mathsf{P}_{F^*}(V\le\mu) = \mathsf{P}_{F^*}(V\ge\mu)$. On the one hand, $\underline{\mathsf{P}}_{s1} = \mathsf{P}_{F^*}(p\le V<\mu) + \mathsf{P}_{F^*}(V\le\mu) = \mathsf{P}_F(p\le V<\mu) + \mathsf{P}_F(V\le\mu)$. On the other hand, F is symmetric, i.e., $F\in\mathscr{F}_s$. Hence, $\inf_{F\in\mathscr{F}_s}\mathsf{P}(V\ge p) \le \mathsf{P}_F(V\ge p) = \mathsf{P}_F(p\le V< \mu) + \mathsf{P}_F(V\le\mu) = \underline{\mathsf{P}}_{s1}$. We next show $\underline{\mathsf{P}}_{s1} \le \inf_{F\in\mathscr{F}_s}\mathsf{P}(V\ge p)$. Suppose $F^* = \arg\inf_{F\in\mathscr{F}_s}\mathsf{P}(V\le p)$, then it is a feasible solution for problem (OB.12) due to $\mathrm{E}[(V-\mu)^2\mathtt{I}_{\{0\le V\le\mu\}}] = \frac{\sigma^2}{2}$ holds for any symmetric distribution. Thus, $\underline{\mathsf{P}}_{s1} \le \mathsf{P}_{F^*}(p\le V<\mu) + \mathsf{P}_{F^*}(V\le\mu) = \mathsf{P}_{F^*}(p\le V<\mu) + \mathsf{P}_{F^*}(V\ge\mu) = \mathsf{P}_{F^*}(V\ge p) = \inf_{F\in\mathscr{F}_s}\mathsf{P}(V\le p)$, where the first equality is due to F^* is symmetric.

As in Lemma 1, theorem 2.2 of Bertsimas and Popescu (2005) shows that if the moment vector of the primal problem is an interior point of the set of feasible moment vectors, then strong duality holds in a moment problem, which is achieved because $\mu > \sigma > 0$. Then the optimal value of the dual problem (OB.13) is equivalent to that of $\inf_{F\in\mathscr{F}_s}\mathsf{P}(V\ge p)$.

$$\begin{aligned} \sup_{\alpha_0,\alpha_1} \quad & \alpha_0 + \frac{\sigma^2}{2}\alpha_2 \\ \text{s.t.} \quad & \alpha_0 + \alpha_1 + \alpha_2(v-\mu)^2 \leq 1 + \mathtt{I}_{\{v \geq p\}}, \quad \text{if } 0 \leq v \leq \mu, \\ & \alpha_0 \leq 1, \quad \alpha_0 \leq \alpha_1. \end{aligned} \tag{OB.13}$$

Note that the first and third constraints are equivalent to $\alpha_0 \leq \alpha_1 \leq 1 + \mathtt{I}_{\{v \geq p\}} - \alpha_2(v-\mu)^2 - \alpha_0$ and the objective function is independent of α_1, then the two constraint are equivalent to $\alpha_0 \leq 1 + \mathtt{I}_{\{v \geq p\}} - \alpha_2(v-\mu)^2 - \alpha_0$, i.e., $2\alpha_0 + \alpha_2(v-\mu)^2 \leq 1 + \mathtt{I}_{\{v \geq p\}}$. Note $1 + \mathtt{I}_{\{v \geq p\}}$ is 1 when $v < p$ and 2 when $p \leq v \leq \mu$, and $g(v) = 2\alpha_0 + \alpha_2(v-\mu)^2$ is monotonous in v for $v \in [0,\mu]$. Then it can be divided into the following 2 scenarios, which are illustrated in Figure OB.5.



(a) Scenario 1.1 (b) Scenario 1.2

Figure OB.5 Graphical illustration of functions satisfying the constraints of problem (OB.13)

Scenario 1.1: $g(v)$ is a quadratic or linear function with $\alpha_2 \leq 0$ and increasing in $[0,\mu]$. Moreover, $g(p) = 2\alpha_0 + \alpha_2(\mu - p)^2 \leq 1$, $g(\mu) = 2\alpha_0 \leq 2$ due to $g(v)$ liying below 1 for $v \in [0,p]$ and 2 for $v \in [p,\mu]$. Then the problem (OB.13) is equivalent to

$$\sup_{\alpha_0,\alpha_2} \quad \alpha_0 + \frac{\sigma^2}{2}\alpha_2 \qquad \text{s.t.} \quad g(p) = 2\alpha_0 + \alpha_2(\mu-p)^2 \leq 1, \quad g(\mu) = 2\alpha_0 \leq 2, \quad \alpha_2 \leq 0.$$

The constraints are equivalent to $\alpha_2 \leq 0$, $\alpha_2 \leq \frac{1-2\alpha_0}{(\mu-p)^2}$ and $\alpha_0 \leq 1$. Then the problem is equivalent to

$$\sup_{\alpha_0,\alpha_2} \quad \alpha_0 + \frac{\sigma^2}{2}\alpha_2 \qquad \text{s.t.} \quad \alpha_0 \leq 1, \quad \alpha_2 \leq \min\left\{0, \frac{1-2\alpha_0}{(\mu-p)^2}\right\}.$$

When $\alpha_0 \leq \frac{1}{2}$, the constraints are equivalent to $\alpha_0 \leq \frac{1}{2}$ and $\alpha_2 \leq 0$, which induces the optimal objective value to be $\frac{1}{2}$ with $\alpha_0 = \frac{1}{2}$ and $\alpha_2 = 0$. When $\frac{1}{2} \leq \alpha_0 \leq 1$, the constraints are equivalent to $\frac{1}{2} \leq \alpha_0 \leq 1$ and $\alpha_2 \leq \frac{1-2\alpha_0}{(\mu-p)^2}$, thus it is optimal to let $\alpha_2 = \frac{1-2\alpha_0}{(\mu-p)^2}$ since the objective function is increasing in α_2. Then the objective function becomes $\frac{\sigma^2}{2(\mu-p)^2} + \alpha_0(1 - \frac{\sigma^2}{(\mu-p)^2})$. Therefore, it is easy to prove that the optimal objective value is $1 - \frac{\sigma^2}{2(\mu-p)^2}$ with $\alpha_0 = 1$ for $p \leq \mu - \sigma$, and is $\frac{1}{2}$ with $\alpha_0 = \frac{1}{2}$ for $\mu - \sigma \leq p \leq \mu$.

Scenario 1.2: $g(v)$ is a quadratic or linear function with $\alpha_2 \geq 0$ and decreasing in $[0,\mu]$. Moreover, $g(0) = 2\alpha_0 + \alpha_2\mu^2 \leq 1$ due to $g(v)$ lies below 1 for $v \in [0,p]$ and 2 for $v \in [p,\mu]$. Then the problem (OB.13) is equivalent to

$$\sup_{\alpha_0,\alpha_2} \quad \alpha_0 + \frac{\sigma^2}{2}\alpha_2 \quad \text{s.t.} \quad g(0) = 2\alpha_0 + \alpha_2\mu^2 \leq 1, \quad \alpha_2 \geq 0.$$

The constraints are equivalent to $0 \leq \alpha_2 \leq \frac{1-2\alpha_0}{\mu^2}$ and $\alpha_0 \leq 1$, where the optimal solution is $\alpha_0 = \frac{1}{2}$ and $\alpha_2 = 0$, and the corresponding objective value is $\frac{1}{2}$.

In summary, the optimal objective value is $\max\{1 - \frac{\sigma^2}{2(\mu-p)^2}, \frac{1}{2}\}$ for $p \leq \mu$, which is $1 - \frac{\sigma^2}{2(\mu-p)^2}$ for $p \leq \mu - \sigma$, and $\frac{1}{2}$ for $\mu - \sigma \leq p \leq \mu$. By the complementary slackness, we can derive the corresponding primal distributions as follows.

When $p \leq \mu - \sigma$, the optimal objective value $1 - \frac{\sigma^2}{2(\mu-p)^2}$ is asymptotically achievable by a series of three-point distributions:

$$V = \begin{cases} p - \epsilon & \text{with prob. } \frac{\sigma^2}{2(\mu-p+\epsilon)^2}, \\ \mu & \text{with prob. } 1 - \frac{\sigma^2}{(\mu-p+\epsilon)^2}, \\ 2\mu - p + \epsilon & \text{with prob. } \frac{\sigma^2}{2(\mu-p+\epsilon)^2}. \end{cases}$$

with $\epsilon \searrow 0$. And the optimal value can be achieved by the dual solution with $\alpha_0 = 1$, $\alpha_1 = 1$, $\alpha_2 = \frac{-1}{(\mu-p)^2}$.

When $\mu - \sigma \leq p \leq \mu$, the optimal objective value $\frac{1}{2}$ is asymptotically achievable by a series of two-point distributions:

$$V = \begin{cases} \mu - \sigma \text{ with prob. } \frac{1}{2}, \\ \mu + \sigma \text{ with prob. } \frac{1}{2}. \end{cases}$$

with $\epsilon \searrow 0$. And the optimal value can be achieved by the dual solution with $\alpha_0 = 1 - \sqrt{\frac{\sigma^2}{\sigma^2 + 8(\mu-p)^2}}$, $\alpha_1 = \frac{\alpha_0}{\mu - p}$, $\alpha_2 = \frac{\alpha_1^2}{4(\alpha_0 - 1)}$.

Case 2: $p > \mu$. We first show $\inf_{F \in \mathscr{F}_s} \mathsf{P}(V \geq p)$ is given by the following problem

$$\begin{aligned} \underline{\mathsf{P}}_{s2} = & \inf_F \ \mathsf{P}(V \geq p) \\ & \text{s.t.} \ \ v \geq 0, \quad \mathrm{E}[1] = 1, \quad \mathrm{E}[(V-\mu)^2 \mathtt{I}_{\{\mu \leq V \leq 2\mu\}}] = \frac{\sigma^2}{2}, \quad \mathsf{P}(V \leq \mu) = \mathsf{P}(V \geq \mu). \end{aligned} \tag{OB.14}$$

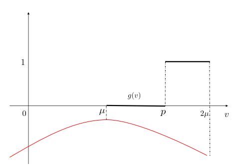
We first prove $\underline{\mathsf{P}}_{s2} \geq \inf_{F \in \mathscr{F}_s} \mathsf{P}(V \geq p)$. Suppose that F^* is an optimal distribution for problem (OB.14), and $f^*(x)$ is its probability density function. Then we construct a symmetric distribution F with $f(x) = f^*(x)$ for $x \geq \mu$ and $f(x) = f(2\mu - x)$ for $x < \mu$. Then $\mathrm{E}_F[1] = \mathsf{P}_F(V < \mu) + \mathsf{P}_F(V \geq \mu) =$

$\mathsf{P}_{F^*}(V > \mu) + \mathsf{P}_{F^*}(V \geq \mu) = \mathsf{P}_{F^*}(V < \mu) + \mathsf{P}_{F^*}(V \geq \mu) = 1$, where the third equality is due to $\mathsf{P}_{F^*}(V \leq \mu) = \mathsf{P}_{F^*}(V \geq \mu)$. On the one hand, $\underline{\mathsf{P}}_{s2} = \mathsf{P}_{F^*}(V \geq p) = \mathsf{P}_F(V \geq p)$ due to $f(x) = f^*(x)$ for $x \geq \mu$ and $p \geq \mu$. On the other hand, F is symmetric, i.e., $F \in \mathscr{F}_s$. Hence, $\inf_{F \in \mathscr{F}_s} \mathsf{P}(V \geq p) \leq \mathsf{P}_F(V \geq p) = \underline{\mathsf{P}}_{s2}$. We next show $\underline{\mathsf{P}}_{s2} \leq \inf_{F \in \mathscr{F}_s} \mathsf{P}(V \geq p)$. Suppose $F^* = \arg \inf_{F \in \mathscr{F}_s} \mathsf{P}(V \leq p)$, then it is a feasible solution for problem (OB.14) due to $\mathrm{E}[(V-\mu)^2 \mathtt{I}_{\{\mu \leq V \leq 2\mu\}}] = \frac{\sigma^2}{2}$ holds for any symmetric distribution. Thus, $\underline{\mathsf{P}}_{s2} \leq \mathsf{P}_{F^*}(V \geq p) = \inf_{F \in \mathscr{F}_s} \mathsf{P}(V \geq p)$.

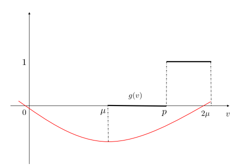
The dual problem of (OB.14) is

$$\begin{aligned} \sup_{\alpha_0, \alpha_1} \quad & \alpha_0 + \frac{\sigma^2}{2}\alpha_2 \\ \text{s.t.} \quad & \alpha_0 + \alpha_1 + \alpha_2(v-\mu)^2 \leq \mathtt{I}_{\{v \geq p\}}, \quad \text{if } \mu < v \leq 2\mu, \quad \alpha_0 \leq \alpha_1, \quad \alpha_0 \leq 0. \end{aligned} \tag{OB.15}$$

Note that the first and third constraints are equivalent to $\alpha_0 \leq \alpha_1 \leq \mathtt{I}_{\{v \geq p\}} - \alpha_2(v-\mu)^2 - \alpha_0$ and the objective function is independent of α_1, then the two constraints are equivalent to $\alpha_0 \leq \mathtt{I}_{\{v \geq p\}} - \alpha_2(v-\mu)^2 - \alpha_0$, i.e., $2\alpha_0 + \alpha_2(v-\mu)^2 \leq \mathtt{I}_{\{v \geq p\}}$. Note $\mathtt{I}_{\{v \geq p\}}$ is 0 when $V < p$ and 1 when $v \geq p$, and $g(v) = 2\alpha_0 + \alpha_2(v-\mu)^2$ is monotonous in v for $v \in (\mu, 2\mu]$. Then it can be divided into the following 2 scenarios, which are illustrated in Figure OB.6.



(a) Scenario 2.1



(b) Scenario 2.2

Figure OB.6 Graphical illustration of functions satisfying the constraints of problem (OB.15)

Scenario 2.1: $g(v)$ is a quadratic or linear function with $\alpha_2 \leq 0$ and decreasing in $(\mu, 2\mu]$. Moreover, $g(\mu) = 2\alpha_0 \leq 0$ due to $g(v)$ lies below 0 for $v \in (\mu, p]$ and 1 for $v \in [p, 2\mu]$. Then the problem (OB.15) is equivalent to $\sup_{\alpha_0, \alpha_2} \alpha_0 + \frac{\sigma^2}{2}\alpha_2 \quad \text{s.t.} \quad g(\mu) = 2\alpha_0 \leq 0, \ \alpha_2 \leq 0$. The constraints are equivalent to $\alpha_0 \leq 0$ and $\alpha_2 \leq 0$. Then the optimal objective value is 0 with $\alpha_0 = 0$ and $\alpha_2 = 0$.

Scenario 2.2: $g(v)$ is a quadratic or linear function with $\alpha_2 > 0$ and increasing in $(\mu, 2\mu]$. Moreover, $g(p) = 2\alpha_0 + \alpha_2(\mu - p)^2 \leq 0$, $g(2\mu) = 2\alpha_0 + \alpha_2\mu^2 \leq 1$ due to $g(v)$ lies below 0 for $v \in (\mu, p]$ and 1 for $v \in [p, 2\mu]$. Then the problem (OB.15) is equivalent to

$$\sup_{\alpha_0, \alpha_2} \quad \alpha_0 + \frac{\sigma^2}{2}\alpha_2 \quad \text{s.t.} \quad g(p) = 2\alpha_0 + \alpha_2(p-\mu)^2 \leq 0, \quad g(2\mu) = 2\alpha_0 + \alpha_2\mu^2 \leq 1, \quad \alpha_0 \leq 0, \quad \alpha_2 \geq 0,$$

which is equivalent to

$$\sup_{\alpha_0,\alpha_2} \quad \alpha_0 + \frac{\sigma^2}{2}\alpha_2 \quad \text{s.t.} \quad \alpha_0 \le -\frac{\alpha_2(p-\mu)^2}{2}, \quad \alpha_0 \le \frac{1}{2} - \frac{\mu^2\alpha_2}{2}, \quad \alpha_2 \ge 0.$$

Thus it is optimal to let $\alpha_0 = \min\{-\frac{\alpha_2(p-\mu)^2}{2}, \frac{1}{2} - \alpha_2\frac{\mu^2}{2}\}$ due to $\alpha_0 + \frac{\sigma^2}{2}\alpha_2$ is increasing in α_0. When $\alpha_2 \le \frac{1}{\mu^2-(p-\mu)^2}$, i.e., $-\frac{\alpha_2(p-\mu)^2}{2} \le \frac{1}{2} - \alpha_2\frac{\mu^2}{2}$, then $\alpha_0 = -\frac{\alpha_2(p-\mu)^2}{2}$. Thus, the optimal objective value is $\frac{1}{2}\frac{\sigma^2-(p-\mu)^2}{\mu^2-(p-\mu)^2}$ with $\alpha_2 = \frac{1}{\mu^2-(p-\mu)^2}$ for $\mu < p \le \mu+\sigma$ and 0 with $\alpha_2 = 0$ for $p > \mu+\sigma$. When $\alpha_2 \ge \frac{1}{\mu^2-(p-\mu)^2}$, i.e., $-\frac{\alpha_2(p-\mu)^2}{2} \ge \frac{1}{2} - \alpha_2\frac{\mu^2}{2}$, then $\alpha_0 = \frac{1}{2} - \alpha_2\frac{\mu^2}{2}$. Thus, the optimal objective value is $\frac{1}{2}\frac{\sigma^2-(p-\mu)^2}{\mu^2-(p-\mu)^2}$ with $\alpha_2 = \frac{1}{\mu^2-(p-\mu)^2}$ for $\mu < p \le \mu+\sigma$ and 0 with $\alpha_2 = 0$ for $p \ge \mu+\sigma$.

In sum, the optimal objective value is $\max\{\frac{1}{2}\frac{\sigma^2-(p-\mu)^2}{\mu^2-(p-\mu)^2}, 0\}$, which is $\frac{1}{2}\frac{\sigma^2-(p-\mu)^2}{\mu^2-(p-\mu)^2}$ for $\mu < p \le \mu+\sigma$ and 0 for $p \ge \mu+\sigma$. By the complementary slackness, we can derive the corresponding primal distributions as follows.

When $\mu < p \le \mu+\sigma$, the optimal objective value $\frac{1}{2}\frac{\sigma^2-(p-\mu)^2}{\mu^2-(p-\mu)^2}$ can be asymptotically achievable by a series of four-point distributions:

$$V = \begin{cases} 0 & \text{with prob. } \frac{1}{2}\frac{\sigma^2-(p-\mu-\epsilon)^2}{\mu^2-(p-\mu-\epsilon)^2}, \\ 2\mu - p + \epsilon & \text{with prob. } \frac{1}{2}\frac{\mu^2-\sigma^2}{\mu^2-(p-\mu-\epsilon)^2}, \\ p - \epsilon & \text{with prob. } \frac{1}{2}\frac{\mu^2-\sigma^2}{\mu^2-(p-\mu-\epsilon)^2}, \\ 2\mu & \text{with prob. } \frac{1}{2}\frac{\sigma^2-(p-\mu-\epsilon)^2}{\mu^2-(p-\mu-\epsilon)^2}. \end{cases}$$

with $\epsilon \searrow 0$. And the optimal value can be achieved by the dual solution with $\alpha_0 = \alpha_1 = -\frac{(p-\mu)^2}{\mu^2-(p-\mu)^2}$, $\alpha_2 = \frac{1}{\mu^2-(p-\mu)^2}$.

The results in the symmetric distribution are summarized in Table 6. $\square$

Proof of Proposition 10. (i) For an exponential distribution with parameter a, we have $\mu = 1/a$ and $\sigma = 1/a$. Moreover, $m_1 = \int_{\frac{1}{a}}^{+\infty}(x-\frac{1}{a})^2 a e^{-ax}dx = 2/a^2e$ and $m_2 = \int_0^{\frac{1}{a}}(x-\frac{1}{a})^2 a e^{-ax}dx = 1/a^2 - 2/a^2e$, which implies $s = 4/e - 1$. By the definitions of θ_i in Proposition 2, we have $\theta_1 = \frac{e+2}{4}\sqrt{\frac{e-2}{2}}$, $\theta_2 = \frac{e-2}{e(4-e)}$ and $\theta_3 = \frac{2}{\sqrt{e-2}}\frac{\sqrt{2}(e+1)-\sqrt{e-2}}{e-2+\sqrt{2(e-2)}}$. Therefore, based on the proof of Proposition 2, we can derive Table 8 immediately. In particular, if $c = 0$, then $p^* = p_3^* \approx 0.71/a$. The robust optimal profit is around $0.36/a$. For the exponential distribution, it is easy to see that the optimal price is $c + 1/a$ and the optimal profit is e^{-ac-1}/a. Thus if $c = 0$, the profit of our heuristic captures at least 96% of the optimal.

(ii) For a uniform distribution defined in $[0,a]$, we have $s = 0$, $\mu = \frac{a}{2}$ and $\sigma = \frac{a}{2\sqrt{3}}$. By the definitions of θ_i in Proposition 2, $\theta_1 = \frac{3}{2}$, $\theta_2 = \frac{\sqrt{3}}{8}$ and $\theta_3 = 7$. Therefore, based on the proof of

Proposition 2, we can derive Table 8 immediately. In particular, if $c = 0$, then $p^* = p_2^* \approx 0.37a$. The robust optimal profit is around $0.23a$. For the uniform distribution, it is easy to see that the optimal price is $\frac{a+c}{2}$ and the optimal profit is $\frac{(a-c)^2}{4a}$. Thus if $c = 0$, the profit of our heuristic captures at least 94% of the optimal. $\square$