Appendix A to "Distributionally Robust Pricing with Asymmetric Information"

Due to the limit of pages, we put Appendix B on the website:

https://anonymous.4open.science/r/DRO-Pricing-Asymmetry-Appendix/.

Proof of Lemma 1. Since the length of the proof is more than 10 pages, we put it in Appendix B. □

Proof of Lemma 2. We will solve the problem $\sup_{F\in\mathscr{F}} \mathsf{P}(V \geq p)$. As in the proof of Lemma 1, the dual problem can be written as follows:

$$\begin{aligned}
\inf_{\alpha_0,\alpha_1,\alpha_2,\beta} \quad & \alpha_0+\alpha_2 m_1+\beta m_2 \\
\text{s.t.} \quad & \alpha_0-\alpha_1 x+\beta x^2 \leq \mathtt{I}_{\{x\leq \mu-p\}}, \quad \text{if } 0\leq x\leq \mu, \\
& \alpha_0+\alpha_1 x+\alpha_2 x^2 \leq \mathtt{I}_{\{x\geq p-\mu\}}, \quad \text{if } x\geq 0.
\end{aligned} \tag{OA.1}$$

We denote $g_1(x)=\alpha_0-\alpha_1 x+\beta x^2$, and $g_2(x)=\alpha_0+\alpha_1 x+\alpha_2 x^2$. Now we will consider the problem (OA.1) in the following two cases: $p\leq\mu$ and $p\geq\mu$.

Case 1: $p\leq\mu$. In this case, the first constraint in problem (OA.1) is equivalent to $g_1(x)\geq 1$ for $x\in[0,\mu-p]$ and $g_1(x)\leq 0$ for $x\in(\mu-p,\mu]$. It can be divided into the following 5 scenarios, which are illustrated in Figure OA.1.

Scenario 1.1: $g_1(x)$ is a quadratic function or a straight line with $\beta\leq 0$. Moreover, $g_1(0)=\alpha_0\geq 1$, $g_1(\mu-p)=\alpha_0-\alpha_1(\mu-p)+\beta(\mu-p)^2\geq 1$ and $g_1(\mu)=\alpha_0-\alpha_1\mu+\beta\mu^2\geq 0$ due to $g_1(x)$ lies above 1 for $x\in[0,\mu-p]$ and 0 for $x\in(\mu-p,\mu]$.

Scenario 1.2: $g_1(x)$ is a quadratic function with $\beta>0$ and $\frac{\alpha_1}{2\beta}\leq 0$. Moreover, $g_1(0)=\alpha_0\geq 1$ due to $g_1(x)$ lies above 1 for $x\in[0,\mu-p]$ and 0 for $x\in(\mu-p,\mu]$.

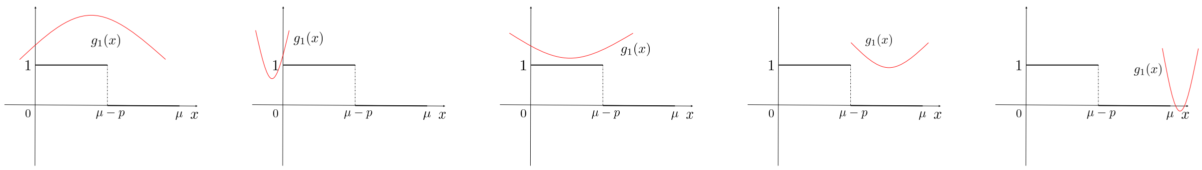
Scenario 1.3: $g_1(x)$ is a quadratic function with $\beta>0$ and $0<\frac{\alpha_1}{2\beta}\leq\mu-p$. Moreover, $g_1(0)=\alpha_0\geq 1$ and $g_1(\frac{\alpha_1}{2\beta})=\frac{4\beta\alpha_0-\alpha_1^2}{4\beta}\geq 1$ due to $g_1(x)$ lies above 1 for $x\in[0,\mu-p]$ and 0 for $x\in(\mu-p,\mu]$.

Scenario 1.4: $g_1(x)$ is a quadratic function with $\beta>0$ and $\mu-p<\frac{\alpha_1}{2\beta}\leq\mu$. Moreover, $g_1(0)=\alpha_0\geq 1$, $g_1(\mu-p)=\alpha_0-\alpha_1(\mu-p)+\beta(\mu-p)^2\geq 1$ and $g_1(\frac{\alpha_1}{2\beta})=\frac{4\beta\alpha_0-\alpha_1^2}{4\beta}\geq 0$ due to $g_1(x)$ lies above 1 for $x\in[0,\mu-p]$ and 0 for $x\in(\mu-p,\mu]$.

Scenario 1.5: $g_1(x)$ is a quadratic function with $\beta>0$ and $\frac{\alpha_1}{2\beta}>\mu$. Moreover, $g_1(\mu-p)=\alpha_0-\alpha_1(\mu-p)+\beta(\mu-p)^2\geq 1$ and $g_1(\mu)=\alpha_0-\alpha_1\mu+\beta\mu^2\geq 0$ due to $g_1(x)$ lies above 1 for $x\in[0,\mu-p]$ and 0 for $x\in(\mu-p,\mu]$.

The second constraint is equivalent to $g_2(x)\geq 1$ for $x\in[0,+\infty)$, which is divided into three scenarios shown in Figure OA.2.

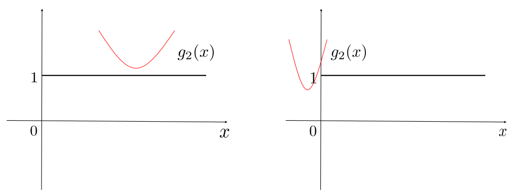
Scenario 2.1: $g_2(x)$ is a quadratic function with $\alpha_2<0$ and $\frac{-\alpha_1}{2\alpha_2}<0$. And $g(\frac{-\alpha_1}{2\alpha_2})=\frac{4\alpha_0\alpha_2-\alpha_1^2}{4\alpha_2}\geq 1$ due to $g_1(x)$ lies above 1 for $x\geq 0$.



(a) Scenario 1.1 (b) Scenario 1.2 (c) Scenario 1.3 (d) Scenario 1.4 (e) Scenario 1.5

Figure OA.1 Graphical illustration of functions satisfying the first constraint of the problem (OA.1)

Scenario 2.2: $g_2(x)$ is a quadratic function or a straight line with $\alpha_2 \geq 0$ and $\frac{-\alpha_1}{2\alpha_2} \leq 0$. And $g_2(0) = \alpha_0 \leq 1$ due to $g_1(x)$ lies above 1 for $x \geq 0$.



(a) Scenario 2.1 (b) Scenario 2.2

Figure OA.2 Graphical illustration of functions satisfying the second constraint of the problem (OA.1)

Since we have five scenarios for the first constraint and two scenarios for the second one, then there are 10 candidate cases. However, we show that the optimal solution of problem (OA.1) only belongs to two cases below.

Note that $\alpha_0 = 1$, $\alpha_1 = 0$, $\alpha_2 = 0$ and $\beta = 0$ is a feasible solution with objective value 1 and we must have $\alpha_0 \geq 1$ and $\alpha_2 \geq 0$ in all cases. Then $\alpha_2 \geq 0$ and $\beta \geq 0$ cannot hold simultaneously (except $\alpha_2 = \beta = 0$) otherwise the objective value will be greater than 1. Therefore, we only need to consider $\beta < 0$ and the optimal solution only belongs to the following two cases.

Case 1.1: The first and second constraints of the optimal dual solution should satisfy Scenario 1.2 and Scenario 2.1, respectively. The optimal solution is $\alpha_0 = \frac{\mu^2}{p(2\mu-p)}$, $\alpha_1 = 0$, $\alpha_2 = 0$ and $\beta = -\frac{\alpha_0}{\mu^2}$ with the optimal value $\frac{\mu^2}{p(2\mu-p)}(1 - \frac{m_2}{\mu^2})$.

Case 1.2: The first and second constraints of the optimal dual solution should satisfy Scenario 1.1 and Scenario 2.2, respectively. The optimal solution is $\alpha_0 = 1 + g(\mu - p)$, $\alpha_1 = \frac{gp(2\mu-p)-(\mu-p)}{\mu p}$, $\alpha_2 = \frac{\alpha_1^2}{4(\alpha_0-1)}$, $\beta = \frac{gp-1}{\mu p}$, where $g = \frac{1}{2\mu p}\sqrt{\frac{m_1(\mu-p)}{(\mu-p)+\frac{m_2}{\mu}+\frac{m_1(2\mu-p)^2}{4(\mu-p)\mu^2}}}$. And the optimal objective value of problem (OA.1) is $1 + \frac{m_1}{2\mu^2} - \frac{m_1+m_2}{\mu p} + \frac{\mu-p}{\mu p}\sqrt{m_1}\sqrt{1 + \frac{m_2}{\mu(\mu-p)} + \frac{m_1(2\mu-p)^2}{4(\mu-p)^2\mu^2}}$.

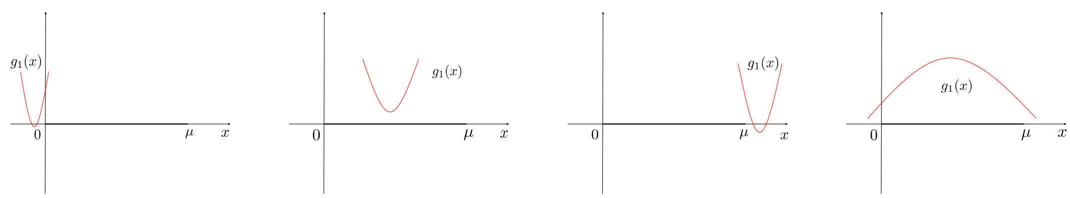
Case 2: $p > \mu$. In this case, the first constraint in problem (OA.1) is equivalent to $g_1(x) \geq 0$ for $x \in [0, \mu]$. It can be divided into the following 4 scenarios, which are illustrated in Figure OA.3.

Scenario 1.1: $g_1(x)$ is a quadratic function or a straight line with $\beta \geq 0$. Moreover, $g_1(0) = \alpha_0 \geq 0$ due to $g_1(x)$ lies above 0 for $x \in [0, \mu]$.

Scenario 1.2: $g_1(x)$ is a quadratic function with $\beta > 0$ and $0 \leq \frac{\alpha_1}{2\beta} \leq \mu$. Moreover, $g_1(\frac{\alpha_1}{2\beta}) = \frac{4\beta\alpha_0 - \alpha_1^2}{4\beta} \geq 0$ due to $g_1(x)$ lies above 0 for $x \in [0, \mu]$.

Scenario 1.3: $g_1(x)$ is a quadratic function with $\beta > 0$ and $\frac{\alpha_1}{2\beta} \geq \mu$. Moreover, $g_1(\mu) = \alpha_0 - \alpha_1\mu + \beta(\mu)^2 \geq 0$ due to $g_1(x)$ lies above 0 for $x \in [0, \mu]$.

Scenario 1.4: $g_1(x)$ is a quadratic function with $\beta < 0$. Moreover, $g_1(0) = \alpha_0 \geq 0$ and $g_1(\mu) = \alpha_0 - \alpha_1\mu + \beta\mu^2 \geq 0$ due to $g_1(x)$ lies above 0 for $x \in [0, \mu]$.



(a) Scenario 1.1 (b) Scenario 1.2 (c) Scenario 1.3 (d) Scenario 1.4

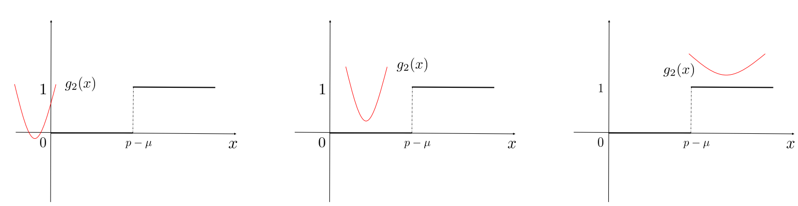
Figure OA.3 Graphical illustration of functions satisfying the first constraint of the problem (OA.1)

The second constraint is equivalent to $g_2(x) \geq 0$ for $x \in [0, p-\mu]$ and $g_2(x) \geq 1$ for $x \in [p-\mu, +\infty)$, which is divided into three scenarios shown in Figure OA.4.

Scenario 2.1: $g_2(x)$ is a quadratic function or straight line with $\alpha_2 \geq 0$ and $\frac{-\alpha_1}{2\alpha_2} < 0$. And $\alpha_0 \geq 0$, $g_2(p-\mu) = \alpha_0 + \alpha_1(p-\mu) + \alpha_2(p-\mu)^2 \geq 1$ due to $g_2(x) \geq 0$ for $x \in [0, p-\mu]$ and $g_2(x) \geq 1$ for $x \in [p-\mu, +\infty)$.

Scenario 2.2: $g_2(x)$ is a quadratic function or a straight line with $\alpha_2 \geq 0$ and $0 < \frac{-\alpha_1}{2\alpha_2} \leq p-\mu$. And $g_2(-\frac{\alpha_1}{2\alpha_2}) = \frac{4\alpha_2\alpha_0 - \alpha_1^2}{4\alpha_2} \geq 0$, $g_2(p-\mu) = \alpha_0 + \alpha_1(p-\mu) + \alpha_2(p-\mu)^2 \geq 1$ due to $g_2(x) \geq 0$ for $x \in [0, p-\mu]$ and $g_2(x) \geq 1$ for $x \in [p-\mu, +\infty)$.

Scenario 2.3: $g_2(x)$ is a quadratic function or a straight line with $\alpha_2 \geq 0$ and $\frac{-\alpha_1}{2\alpha_2} > p-\mu$. And $g_2(-\frac{\alpha_1}{2\alpha_2}) = \frac{4\alpha_2\alpha_0 - \alpha_1^2}{4\alpha_2} \geq 1$ due to $g_2(x) \geq 0$ for $x \in [0, p-\mu]$ and $g_2(x) \geq 1$ for $x \in [p-\mu, +\infty)$.



(a) Scenario 2.1 (b) Scenario 2.2 (c) Scenario 2.3

Figure OA.4 Graphical illustration of functions satisfying the second constraint of the problem (OA.1)

Since we have five scenarios for the first constraint and three scenarios for the second one, then there are 12 candidate cases. We next show the optimal solution of problem (OA.1) only belongs to six cases.

First, if the first constraint of the optimal solution belongs to Scenario 1.1, we will show that the second constraint must be scenario 2.2. Otherwise, we have $\alpha_0 \geq 1$, $\alpha_2 \geq 0$ and $\beta \geq 0$ with the objective value greater than 1, which cannot be the optimal solution of problem (OA.1).

Second, if the first constraint of the optimal solution belongs to Scenario 1.2 or Scenario 1.3, the second constraint must be Scenario 2.1 because Scenario 1.2 or Scenario 1.3 implies $\alpha_1 \geq 0$ but Scenario 2.2 or Scenario 2.3 implies $\alpha_1 < 0$.

Therefore, the optimal solution only belongs to the following six cases.

Case 1.1: The first and second constraints of the optimal dual solution satisfy Scenario 1.1 and Scenario 2.2, respectively. The optimal solution is $\alpha_0 = 0$, $\alpha_1 = 0$, $\alpha_2 = \frac{1}{(p-\mu)^2}$, $\beta = 0$ and the optimal objective value is $\frac{m_1}{(p-\mu)^2}$.

Case 1.2: The first and second constraints of the optimal dual solution satisfy Scenario 1.2 and Scenario 2.1, respectively. The optimal solution is $\alpha_0 = \frac{1}{\sqrt{1+\frac{4(p-\mu)^2}{m_2}}}$, $\alpha_1 = \frac{1-\alpha_0}{(p-\mu)}$, $\alpha = \frac{1-\alpha_0-\alpha_1(p-\mu)}{(p-\mu)^2}$, $\beta = \frac{(1-\alpha_0)^2}{4(p-\mu)^2\alpha_0}$ and the corresponding objective value is $\frac{1+\frac{m_2}{4(p-\mu)^2}(\sqrt{1+\frac{4(p-\mu)^2}{m_2}}-1)^2}{\sqrt{1+\frac{4(p-\mu)^2}{m_2}}}$.

Case 1.3: The first and second constraints of the optimal dual solution satisfy Scenario 1.3 and Scenario 2.1, respectively. The optimal objective value is $\frac{\mu}{p}$ or $\frac{m_2+\mu^2}{2\mu p-\mu^2}$, where the former solution is $\alpha_0 = \frac{\mu-\beta\mu^2(p-\mu)}{p}$, $\alpha_1 = \frac{1-\alpha_0}{p-\mu}$, $\alpha_2 = 0$, $\beta = 0$ and the latter is $\alpha_0 = \mu^2 p$, $\alpha_1 = \frac{1-\alpha_0}{p-\mu}$, $\alpha_2 = 0$, $\beta = \frac{1}{2\mu p-\mu^2}$.

Case 1.4: The first and second constraints of the optimal dual solution satisfy Scenario 1.4 and Scenario 2.1, respectively. The optimal solution is $\alpha_0 = 0$, $\alpha_1 = 0$, $\alpha_2 = \frac{1}{(p-\mu)^2}$, $\beta = 0$ and the optimal objective value is $\frac{m_1}{(p-\mu)^2}$.

Case 1.5: The first and second constraints of the optimal dual solution satisfy Scenario 1.4 and Scenario 2.2, respectively. The optimal objective value is $\frac{\frac{m_1}{\mu^2}(\mu^2-m_2-\frac{m_2^2}{m_1})}{(1-\frac{m_2}{\mu^2})(p-\mu)^2-2(p-\mu)\frac{m_2}{\mu}+m_1}$ and the optimal solution is $\alpha_0 = (1-\sqrt{\alpha_2}(p-\mu))^2$, $\alpha_1 = \frac{1-\alpha_0-\alpha_2(p-\mu)^2}{p-\mu}$, $\alpha_2 = \left(\frac{(p-\mu)(1-\frac{m_2}{\mu^2}-\frac{m_2}{\mu(p-\mu)})}{m_1-\frac{2(p-\mu)m_2}{\mu}+(p-\mu)^2-\frac{m_2(p-\mu)^2}{\mu^2}}\right)^2$, $\beta = \frac{\alpha_1\mu-\alpha_0}{\mu^2}$.

Case 1.6: The first and second constraints of the optimal dual solution satisfy Scenario 1.4 and Scenario 2.3, respectively. The optimal objective value is $1-\frac{m_2}{\mu^2}$ and the optimal solution is $\alpha_0 = 1$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\beta = -\frac{1}{\mu^2}$.

In summary, from these cases above, the optimal objective value of problem (OA.1) is $\min\{1, \frac{\mu^2}{p(2\mu-p)}(1-\frac{m_2}{\mu^2}), 1+\frac{m_1}{2\mu^2}-\frac{m_1+m_2}{\mu p}+\frac{\mu-p}{\mu p}\sqrt{m_1}\sqrt{1+\frac{m_2}{\mu(\mu-p)}+\frac{m_1(2\mu-p)^2}{4(\mu-p)^2\mu^2}}\}$ when $p \leq \mu$ and $\min\{1-\frac{m_2}{\mu^2}, \frac{m_1}{(p-\mu)^2}, \frac{\frac{m_1}{\mu^2}(\mu^2-m_2-\frac{m_2^2}{m_1})}{(1-\frac{m_2}{\mu^2})(p-\mu)^2-2(p-\mu)\frac{m_2}{\mu}+m_1}, \frac{1+\frac{m_2}{4(p-\mu)^2}(\sqrt{1+\frac{4(p-\mu)^2}{m_2}}-1)^2}{\sqrt{1+\frac{4(p-\mu)^2}{m_2}}}, \frac{\mu}{p}, \frac{m_2+\mu^2}{2\mu p-\mu^2}\}$ when $p \geq \mu$. Therefore, the following probability inequality holds:

$$\mathsf{P}(V\ge p)\le\begin{cases}1 & \text{if } 0\le p\le \mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)},\\ 1+\frac{m_1}{2\mu^2}-\frac{m_1+m_2}{\mu p}+\frac{\mu-p}{\mu p}\sqrt{m_1}\sqrt{1+\frac{m_2}{\mu(\mu-p)}+\frac{m_1(2\mu-p)^2}{4(\mu-p)^2\mu^2}} & \text{if } \mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)}\le p\le\mu,\\ 1-\frac{m_2}{\mu^2} & \text{if } \mu\le p\le\frac{\mu}{1-\frac{m_2}{\mu^2}},\\ \frac{1+\frac{m_2}{4(p-\mu)^2}(\sqrt{1+\frac{4(p-\mu)^2}{m_2}}-1)^2}{\sqrt{1+\frac{4(p-\mu)^2}{m_2}}} & \text{if } \frac{\mu}{1-\frac{m_2}{\mu^2}}\le p\le \mu+\sqrt{\frac{m_1}{m_2}(m_1+m_2)},\\ \frac{m_1}{(p-\mu)^2} & \text{if } \mu+\sqrt{\frac{m_1}{m_2}(m_1+m_2)}\le p\le\frac{m_1\mu}{m_2},\\ \frac{\frac{m_1}{\mu^2}(\mu^2-m_2-\frac{m_2^2}{m_1})}{(1-\frac{m_2}{\mu^2})(p-\mu)^2-2(p-\mu)\frac{m_2}{\mu}+m_1} & \text{if } p\ge\frac{m_1\mu}{m_2}.\end{cases} \tag{OA.2}$$

And in each segment, the upper bound can be approached by a three-point distribution based on the complementary slackness, which is summarized in Table 3. We obtain Lemma 2 by replacing $m_1=\frac{(1+s)}{2}\sigma^2$ and $m_2=\frac{(1-s)}{2}\sigma^2$. $\square$

Proof of Corollary 1. We remove "$v\ge 0$" in the first constraint of problem (OB.1), then obtain the dual problem as follows:

$$\begin{aligned}\sup_{\alpha_0,\alpha_1,\alpha_2,\beta}\quad & \alpha_0+\alpha_2 m_1+\beta m_2\\ \text{s.t.}\quad & \alpha_0-\alpha_1(\mu-v)+\beta(\mu-v)^2\le \mathtt{I}_{\{v\ge p\}},\quad \text{if } v\le\mu,\\ & \alpha_0+\alpha_1(v-\mu)+\alpha_2(v-\mu)^2\le \mathtt{I}_{\{v\ge p\}},\quad \text{if } v\ge\mu.\end{aligned} \tag{OA.3}$$

The strong duality holds when $\mu>0$, $\sigma>0$ and $s\in(-1,1)$ since the moment vector of the primal problem is an interior point of the set of feasible moment vectors based on theorem 2.2 of Bertsimas and Popescu (2005). Denote $x=\mu-v$ and $x=v-\mu$ in the first and second constraints in problem (OA.3), respectively. The problem (OA.3) can be rewritten as follows:

$$\begin{aligned}\sup_{\alpha_0,\alpha_1,\alpha_2,\beta}\quad & \alpha_0+\alpha_2 m_1+\beta m_2\\ \text{s.t.}\quad & \alpha_0-\alpha_1 x+\beta x^2\le \mathtt{I}_{\{x\le\mu-p\}},\quad \text{if } x\ge 0,\\ & \alpha_0+\alpha_1 x+\alpha_2 x^2\le \mathtt{I}_{\{x\ge p-\mu\}},\quad \text{if } x\ge 0.\end{aligned} \tag{OA.4}$$

We denote $g_1(x)=\alpha_0-\alpha_1x+\beta x^2$, $g_2(x)=\alpha_0+\alpha_1x+\alpha_2x^2$, and $f(x)$ is the probability at x of the optimal distribution in the primal problem. Moreover, the corresponding optimal primal and dual solutions should satisfy the complementary slackness conditions. Note that the only difference between problem (OB.3) and problem (OA.4) is the first constraint, where $0\le x\le\mu$ for the former and $x\ge 0$ for the latter. Thus, we can solve problem (OA.4) by a similar method of solving problem (OB.3). Similarly, we will consider the problem (OA.4) by two cases: $p\le\mu$ and $p\ge\mu$.

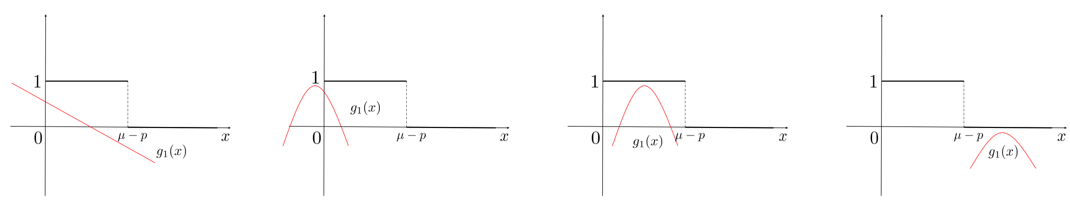
Case 1: $p \le \mu$. In this case, the first constraint in problem (OA.4) is equivalent to $g_1(x) \le 1$ for $x \in [0, \mu - p]$ and $g_1(x) \le 0$ for $x \in (\mu - p, +\infty)$. It can be divided into the following 4 scenarios, which are illustrated in Figure OA.5.

Scenario 1.1: $g_1(x)$ is a straight line, i.e., $\beta = 0$. Moreover, $g_1(0) = \alpha_0 \le 1$, $g_1(\mu - p) = \alpha_0 - \alpha_1(\mu - p) \le 0$ and $g_1(\mu) = \alpha_0 - \alpha_1\mu \le 0$ due to $g_1(x)$ lies below 1 for $x \in [0, \mu - p]$ and 0 for $x \in (\mu - p, +\infty)$.

Scenario 1.2: $g_1(x)$ is a quadratic function with $\beta < 0$ and $\frac{\alpha_1}{2\beta} \le 0$. Moreover, $g_1(0) = \alpha_0 \le 1$, $g_1(\mu - p) = \alpha_0 - \alpha_1(\mu - p) + \beta(\mu - p)^2 \le 0$ due to $g_1(x)$ lies below 1 for $x \in [0, \mu - p]$ and 0 for $x \in (\mu - p, +\infty)$.

Scenario 1.3: $g_1(x)$ is a quadratic function with $\beta < 0$ and $0 < \frac{\alpha_1}{2\beta} \le \mu - p$. Moreover, $g_1(0) = \alpha_0 \le 1$, $g_1(\mu - p) = \alpha_0 - \alpha_1(\mu - p) + \beta(\mu - p)^2 \le 0$ and $g_1(\frac{\alpha_1}{2\beta}) = \frac{4\beta\alpha_0 - \alpha_1^2}{4\beta} \le 1$ due to $g_1(x)$ lies below 1 for $x \in [0, \mu - p]$ and 0 for $x \in (\mu - p, +\infty)$.

Scenario 1.4: $g_1(x)$ is a quadratic function with $\beta < 0$ and $\frac{\alpha_1}{2\beta} > \mu - p$. Moreover, $g_1(0) = \alpha_0 \le 1$, $g_1(\frac{\alpha_1}{2\beta}) = \frac{4\beta\alpha_0 - \alpha_1^2}{4\beta} \le 0$ due to $g_1(x)$ lies below 1 for $x \in [0, \mu - p]$ and 0 for $x \in (\mu - p, +\infty)$.



(a) Scenario 1.1 (b) Scenario 1.2 (c) Scenario 1.3 (d) Scenario 1.4

Figure OA.5 Graphical illustration of functions satisfying the first constraint of the problem (OA.4)

Note that analysis of the second constraints is exactly as the same as the proof of Lemma 1. Moreover, from Figure OA.5 and Figure OB.1, the constraints for $g_1(x)$ in the two problems in each scenario are exactly the same except that problem (OB.3) has an additional scenario 1.2. Therefore, as in the analysis of problem (OB.3), we can show that the optimal solution of problem (OA.4) only belongs to the following three cases.

Case 1.1: The first and second constraints of the optimal dual solution should satisfy Scenario 1.2 and Scenario 2.1, respectively. The optimal solution is $\alpha_2^* = 0$, $\alpha_1^* = 0$, $\alpha_0^* = 1$ and $\beta^* = -\frac{1}{(\mu-p)^2}$. And the optimal objective value for the problem above is $1 - \frac{m_2}{(\mu-p)^2}$.

Case 1.2: The first and second constraints of the optimal dual solution should satisfy Scenario 1.1 and Scenario 2.3, respectively. The optimal solution is $\beta^* = 0$, $\alpha_2^* = \frac{\alpha_1^2}{4(\alpha_0-1)}$, $\alpha_1^* = \frac{\alpha_0}{\mu-p}$, $\alpha_0^* = 1 - \sqrt{\frac{m_1}{m_1+4(\mu-p)^2}} \in [0,1]$ and the corresponding objective value is $1 - \frac{2}{1+\sqrt{1+\frac{4(\mu-p)^2}{m_1}}}$.

Case 1.3: The first and second constraints of the optimal dual solution should satisfy Scenario 1.3 and Scenario 2.3, respectively. The optimal objective value is $1 - \frac{2}{1+\sqrt{1+\frac{4(\mu-p)^2}{m_1}}}$ in this case.

In summary, from the three cases above, the optimal value of problem (OA.4) is $\max\{1-\frac{2}{1+\sqrt{1+\frac{4(\mu-p)^2}{m_1}}}, 1-\frac{m_2}{(\mu-p)^2}\}$, which is $1-\frac{m_2}{(\mu-p)^2}$ when $p\le\mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)}$ and $1-\frac{2}{1+\sqrt{1+\frac{4(\mu-p)^2}{m_1}}}$ when $\mu\le p\le\mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)}$. Moreover, based on complementary slackness, we can derive a series of optimal primal distributions.

Case 2: $p\ge\mu$. Note that $1-\frac{2}{1+\sqrt{1+\frac{4(\mu-p)^2}{m_1}}}=0$ when $p=\mu$. Therefore, the optimal value is also 0 for all $p\ge\mu$ due to $\inf_{F\in\mathscr{F}}\mathsf{P}(V\ge p)$ decreasing in p.

The optimal primal distribution and dual solution are summarized in Table 4, and we obtain Corollary 1 by replacing $m_1=\frac{(1+s)}{2}\sigma^2$ and $m_2=\frac{(1-s)}{2}\sigma^2$. $\square$

Proof of Proposition 1. Recall $m_1=\frac{1+s}{2}\sigma^2$ and $m_2=\frac{1-s}{2}\sigma^2$, based on Lemma 1, we can derive the proposition immediately. $\square$

Proof of Lemma 3. (i) First, note $l_1(p)=(p-c)(1-\frac{m_2}{(\mu-p)^2})$ is defined in $p\in[c,\mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)}]$. If $\mu-c<\sqrt{\frac{m_2}{m_1}(m_1+m_2)}$, $l_1(p)$ does not exist. Otherwise, $\frac{\mathrm{d}^2l_1(p)}{\mathrm{d}p^2}=-\frac{2m_2(p+2\mu-3c)}{(\mu-p)^4}\le 0$, which implies $l_1(p)$ is concave. Note $\frac{\mathrm{d}l_1(p)}{\mathrm{d}p}=\frac{m_2(2c-\mu-p)+(\mu-p)^3}{(\mu-p)^3}$, then we have $\frac{\mathrm{d}l_1(p)}{\mathrm{d}p}\Big|_{p=c}=\frac{(\mu-c)^2-m_2}{(\mu-c)^2}\ge\frac{\frac{m_2}{m_1}(m_1+m_2)-m_2}{(\mu-c)^2}=\frac{m_2^2}{m_1(\mu-c)^2}>0$, where the first inequality is due to $\mu-c\ge\sqrt{\frac{m_2}{m_1}(m_1+m_2)}$. Since $\frac{\mathrm{d}l_1(p)}{\mathrm{d}p}\big|_{p=\mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)}}=\frac{m_2\left[2c-2\mu+\sqrt{\frac{m_2}{m_1}(m_1+m_2)}\right]+\left(\sqrt{\frac{m_2}{m_1}(m_1+m_2)}\right)^3}{\left(\sqrt{\frac{m_2}{m_1}(m_1+m_2)}\right)^3}=\frac{m_2\left[2c-2\mu+\frac{2m_1+m_2}{m_1}\sqrt{\frac{m_2}{m_1}(m_1+m_2)}\right]}{\left(\sqrt{\frac{m_2}{m_1}(m_1+m_2)}\right)^3}$. Thus if $\mu-c\le(\frac{m_2}{2m_1}+1)\sqrt{\frac{m_2}{m_1}(m_1+m_2)}$, $\frac{\mathrm{d}l_1(p)}{\mathrm{d}p}\big|_{p=\mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)}}\ge 0$, otherwise, $\frac{\mathrm{d}l_1(p)}{\mathrm{d}p}\big|_{p=\mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)}}<0$. Hence, $l_1(p)$ is increasing for $\mu-c\le(\frac{m_2}{2m_1}+1)\sqrt{\frac{m_2}{m_1}(m_1+m_2)}$ and is unimodal otherwise. When $l_1(p)$ is unimodal, the unique $p_1^*=\arg\max_{p\in[c,\mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)}]}l_1(p)$ is given by $\frac{\mathrm{d}l_1(p)}{\mathrm{d}p}=\frac{m_2(2c-\mu-p)+(\mu-p)^3}{(\mu-p)^3}=0$, i.e., $m_2(2c-\mu-p)+(\mu-p)^3=0$. By Cardano's solution for a cubic function, the unique solution for $m_2(2c-\mu-p)+(\mu-p)^3=0$ is $p_1^*=\mu-\frac{\sqrt[3]{2}m_2}{\sqrt[3]{\sqrt{108m_2^3+(54m_2c-54m_2\mu)^2}+54m_2c-54m_2\mu}}+\frac{\sqrt[3]{\sqrt{108m_2^3+(54m_2c-54m_2\mu)^2}+54m_2c-54m_2\mu}}{3\sqrt[3]{2}}$.

Second, note $l_2(p)=(p-c)(\frac{\sqrt{1+b}-\sqrt{1+a}}{\sqrt{b}-\sqrt{a}})^2$ is defined in $p\in[\mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)},\mu]$, and $a=\frac{m_2^2}{(2\mu-p)^2m_1-m_2^2}$, $b=\frac{(m_2+2\mu(\mu-p))^2}{(2\mu-p)^2m_1-m_2^2}$. We first show $\frac{\sqrt{1+b}-\sqrt{1+a}}{\sqrt{b}-\sqrt{a}}$ is decreasing in p. Taking the first-order differential of $\frac{\sqrt{1+b}-\sqrt{1+a}}{\sqrt{b}-\sqrt{a}}$, $\frac{\mathrm{d}\frac{\sqrt{1+b}-\sqrt{1+a}}{\sqrt{b}-\sqrt{a}}}{\mathrm{d}p}=\frac{\sqrt{1+a}\sqrt{1+b}-\sqrt{ab}-1}{2(\sqrt{b}-\sqrt{a})^2}(\frac{a'(p)}{\sqrt{a}\sqrt{a+1}}+\frac{b'(p)}{\sqrt{b}\sqrt{b+1}})$, where $a'(p)=\frac{2m_1m_2^2(2\mu-p)}{(m_1(2\mu-p)^2-m_2^2)^2}\ge 0$ and $b'(p)=-\frac{2(m_2+2\mu(\mu-p))}{(m_1(2\mu-p)^2-m_2^2)^2}[m_1(m_2-2\mu^2)(2\mu-p)+2m_2^2\mu]\le 0$. Note $\sqrt{1+a}\sqrt{1+b}-\sqrt{ab}-1\ge 0$ since $\sqrt{1+a}\sqrt{1+b}\ge\sqrt{ab}+1$ due to Cauchy inequality. Then it is equivalent to show $(\frac{a'(p)}{\sqrt{a}\sqrt{a+1}}+\frac{b'(p)}{\sqrt{b}\sqrt{b+1}})=\frac{a'(p)\sqrt{b}\sqrt{b+1}+b'(p)\sqrt{a}\sqrt{a+1}}{\sqrt{a}\sqrt{a+1}\sqrt{b}\sqrt{b+1}}\le 0$. Since $a'(p)\sqrt{b}\sqrt{b+1}\le b'(p)\sqrt{a}\sqrt{a+1}$ is equivalent to $(a'(p))^2b(b+1)-(b'(p))^2a(a+1)=-\frac{(m_2+2\mu(\mu-p))^2}{(m_1(2\mu-p)^2-m_2^2)^4((2\mu-p)^2m_1-m_2^2)^2}4\mu^2m_1(\mu^2-\frac{m_2^2}{m_1}-m_2)[m_1(\mu^2-\frac{m_2^2}{m_1}-m_2)+2m_1\mu(\mu-p)+m_1(\mu-p)^2+m_1m_2]\le 0$, where the inequality is due to $\mu^2-\frac{m_2^2}{m_1}-m_2>0$ and $p\le\mu$. Therefore, we have $\frac{\sqrt{1+b}-\sqrt{1+a}}{\sqrt{b}-\sqrt{a}}$ is decreasing in p.

Let $x=\frac{\sqrt{1+b}-\sqrt{1+a}}{\sqrt{b}-\sqrt{a}}$, then $p=\frac{2\sqrt{m_1}x\mu-m_2+x^2\mu^2-\mu^2}{\sqrt{m_1}x+x^2\mu-\mu}$. And $p\in[\mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)},\mu]$ is equivalent to $x\in[\frac{m_2}{\sqrt{m_1}\mu},\sqrt{\frac{m_2}{m_1+m_2}}]$. Thus we have $l_2(x)=(\frac{2\sqrt{m_1}x\mu-m_2+x^2\mu^2-\mu^2}{\sqrt{m_1}x+x^2\mu-\mu}-c)x^2$. Taking the first-order differential of $l_2(x)$, we have $\frac{\mathrm{d}l_2(x)}{\mathrm{d}x}=\frac{x}{(\sqrt{m_1}x+x^2\mu-\mu)^2}[2\mu^2(\mu-c)x^4+\sqrt{m_1}\mu(5\mu-4c)x^3+2(2m_1\mu-m_1c+2c\mu^2-2\mu^3)x^2+(4\sqrt{m_1}c\mu-\sqrt{m_1}m_2-7\sqrt{m_1}\mu^2)x+2m_2\mu-2c\mu^2+2\mu^3]$. Let $G(x)=2\mu^2(\mu-c)x^4+\sqrt{m_1}\mu(5\mu-4c)x^3+2(2m_1\mu-m_1c+2c\mu^2-2\mu^3)x^2+(4\sqrt{m_1}c\mu-\sqrt{m_1}m_2-7\sqrt{m_1}\mu^2)x+2m_2\mu-2c\mu^2+2\mu^3$. Then we will show that $G(x)$ is decreasing in x. Note $\frac{\mathrm{d}^2G(x)}{\mathrm{d}x^2}=24\mu^2(\mu-c)x^2+6\sqrt{m_1}\mu(5\mu-4c)x+4(2m_1\mu-m_1c+2c\mu^2-2\mu^3)$ is a quadratic function with $24\mu^2(\mu-c)>0$ and $6\sqrt{m_1}\mu(5\mu-4c)>0$, which implies $\frac{\mathrm{d}^2G(x)}{\mathrm{d}x^2}$ is increasing in $x\in[\frac{m_2}{\sqrt{m_1}\mu},\sqrt{\frac{m_2}{m_1+m_2}}]$. Hence, $\frac{\mathrm{d}G(x)}{\mathrm{d}x}$ is convex. We have $\frac{\mathrm{d}G(x)}{\mathrm{d}x}=8\mu^2(\mu-c)x^3+3\sqrt{m_1}\mu(5\mu-4c)x^2+4(2m_1\mu-m_1c+2c\mu^2-2\mu^3)x+4\sqrt{m_1}c\mu-\sqrt{m_1}m_2-7\sqrt{m_1}\mu^2$. Then $\frac{\mathrm{d}G(x)}{\mathrm{d}x}|_{x=\frac{m_2}{\sqrt{m_1}\mu}}=\frac{-1}{\mu\sqrt{m_1}}(\mu^2-\frac{m_2^2}{m_1}-m_2)(4m_1(\mu-c)+3m_1\mu+8m_2(\mu-c))<0$, where the inequality is due to $\mu^2-\frac{m_2^2}{m_1}-m_2>0$. Note $\frac{\mathrm{d}G(x)}{\mathrm{d}x}|_{x=\sqrt{\frac{m_2}{m_1+m_2}}}$ is a linear function of c with coefficient $-8\mu^2x^3-12\sqrt{m_1}\mu x^2-4m_1x+4\sqrt{m_1}\mu+8\mu^2x=4[(2\frac{m_1}{m_1+m_2}\sqrt{\frac{m_2}{m_1+m_2}})\mu^2+\mu(\sqrt{m_1}-3\sqrt{m_1}\frac{m_2}{m_1+m_2})-m_1\sqrt{\frac{m_2}{m_1+m_2}}]\equiv K(\mu)$. $K(\mu)$ is a quadratic function with $-\frac{\sqrt{m_1}-3\sqrt{m_1}\frac{m_2}{m_1+m_2}}{4\frac{m_1}{m_1+m_2}\sqrt{\frac{m_2}{m_1+m_2}})}\le\sqrt{(m_1+m_2)\frac{m_2}{m_1}}<\mu$. Thus $K(\mu)\ge K(\sqrt{(m_1+m_2)\frac{m_2}{m_1}})=0$ and $\frac{\mathrm{d}G(x)}{\mathrm{d}x}|_{x=\sqrt{\frac{m_2}{m_1+m_2}}}$ is increasing in c. When $c=\mu$, $\frac{\mathrm{d}G(x)}{\mathrm{d}x}|_{x=\sqrt{\frac{m_2}{m_1+m_2}}}=\frac{\sqrt{m_1}}{m_1+m_2}(-\sqrt{m_1}\mu+\sqrt{m_2(m_1+m_2)})(3\sqrt{m_1}\mu-\sqrt{m_2(m_1+m_2)})\le 0$. Therefore, $\frac{\mathrm{d}G(x)}{\mathrm{d}x}\le 0$ for $x\in[\frac{m_2}{\sqrt{m_1}\mu},\sqrt{\frac{m_2}{m_1+m_2}}]$, which implies $G(x)$ is decreasing in x. Then $l_2(x)$ as well as $l_2(p)$ is either monotonous or unimodal. We further explore the conditions for $l_2(x)$ being monotonous or unimodal. We have $\frac{\mathrm{d}l_2(x)}{\mathrm{d}x}|_{x=\frac{m_2}{\sqrt{m_1}\mu}}\ge 0\Leftrightarrow G(\frac{m_2}{\sqrt{m_1}\mu})\ge 0\Leftrightarrow\mu-c\ge\frac{m_2\mu}{2(\mu^2-\frac{m_2}{m_1}(m_1+m_2))}$, and $\frac{\mathrm{d}l_2(x)}{\mathrm{d}x}|_{x=\sqrt{\frac{m_2}{m_1+m_2}}}\ge 0\Leftrightarrow G(\sqrt{\frac{m_2}{m_1+m_2}})\ge 0\Leftrightarrow\mu-c\ge\frac{\sqrt{\frac{m_1m_2}{m_1+m_2}}\left(\left(m_2+3\mu^2\right)(m_1+m_2)-\mu^2m_2\right)-2m_2\mu(2m_1+m_2)}{-4\mu\sqrt{\frac{m_1m_2}{m_1+m_2}}m_1+\frac{2m_1^2}{m_1+m_2}\left(\mu^2+m_2+\frac{m_2^2}{m_1}\right)}$. Note that $\frac{\sqrt{\frac{m_1m_2}{m_1+m_2}}\left(\left(m_2+3\mu^2\right)(m_1+m_2)-\mu^2m_2\right)-2m_2\mu(2m_1+m_2)}{-4\mu\sqrt{\frac{m_1m_2}{m_1+m_2}}m_1+\frac{2m_1^2}{m_1+m_2}\left(\mu^2+m_2+\frac{m_2^2}{m_1}\right)}>\frac{m_2\mu}{2(\mu^2-\frac{m_2}{m_1}(m_1+m_2))}$. If $\mu-c\le\frac{\mu m_2}{2(\mu^2-\frac{m_2}{m_1}(m_1+m_2))}$, $l_2(x)$ is decreasing in x, which implies $l_2(p)$ is increasing in p due to x is decreasing in p; if $\frac{\mu m_2}{2(\mu^2-\frac{m_2}{m_1}(m_1+m_2))}\le\mu-c\le\frac{\sqrt{\frac{m_1m_2}{m_1+m_2}}\left(\left(m_2+3\mu^2\right)(m_1+m_2)-\mu^2m_2\right)-2m_2\mu(2m_1+m_2)}{-4\mu\sqrt{\frac{m_1m_2}{m_1+m_2}}m_1+\frac{2m_1^2}{m_1+m_2}\left(\mu^2+m_2+\frac{m_2^2}{m_1}\right)}$, $l_2(x)$ is unimodal and is maximized at x^*, which is uniquely given by $\frac{\mathrm{d}l_2(x)}{\mathrm{d}x}=0$, in other words, $2\mu^2(\mu-c)x^4+\sqrt{m_1}\mu(5\mu-4c)x^3+2(2m_1\mu-m_1c+2c\mu^2-2\mu^3)x^2+(4\sqrt{m_1}c\mu-\sqrt{m_1}m_2-7\sqrt{m_1}\mu^2)x+2m_2\mu-2c\mu^2+2\mu^3=0$. And $l_2(p)$ is also unimodal and maximized at $p_2^*=\frac{2\sqrt{m_1}x^*\mu-m_2+x^{*2}\mu^2-\mu^2}{\sqrt{m_1}x^*+x^{*2}\mu-\mu}$; otherwise, $l_2(x)$ is increasing in x, which implies $l_2(p)$ is decreasing in p.

Lastly, note $l_3(p)=(p-c)\frac{(p-\mu)^2[1-\frac{m_2}{\mu^2}-\frac{m_2}{\mu(p-\mu)}]^2}{m_1-\frac{2m_2}{\mu}(p-\mu)+(p-\mu)^2(1-\frac{m_2}{\mu^2})}$ is defined in $p\in[\mu,\mu+\frac{m_2}{\mu(1-\frac{m_2}{\mu^2})}]$. Let $y=p-\mu\in[0,\frac{m_2}{\mu(1-\frac{m_2}{\mu^2})}]$. Then we will show $l_3(y)$ as well as $l_3(p)$ is unimodal or monotonous. We have $\frac{\mathrm{d}l_3(y)}{\mathrm{d}y}=\frac{(\frac{m_2}{\mu}-(1-\frac{m_2}{\mu^2})y)H(y)}{[m_1-\frac{m_2}{\mu}y+y^2(1-\frac{m_2}{\mu^2})]^2}$, where $H(y)=-(1-\frac{m_2}{\mu^2})^2y^3+3\frac{m_2}{\mu}(1-\frac{m_2}{\mu^2})y^2-3m_1(1-\frac{m_2}{\mu^2})y+2(\mu-c)\frac{m_2^2}{\mu^2}+\frac{m_1m_2}{\mu}-2(1-\frac{m_2}{\mu^2})(\mu-c)m_1$. Then we have $\frac{\mathrm{d}H(y)}{\mathrm{d}y}=-3(1-\frac{m_2}{\mu^2})^2y^2+6y\frac{m_2}{\mu}(1-\frac{m_2}{\mu^2})-3m_1(1-$

$\frac{m_2}{\mu^2})$, whose discriminant is $36\frac{m_1}{\mu^2}(1-\frac{m_2}{\mu^2})^2[\frac{m_2}{m_1}(m_1+m_2)-\mu^2]<0$ due to $\mu^2>\frac{m_2}{m_1}(m_1+m_2)$. Thus $\frac{\mathrm{d}H(y)}{\mathrm{d}y}<0$ and $H(y)$ is decreasing in y. Therefore, $l_3(y)$ as well as $l_3(p)$ is either monotonous or unimodal. Since $H(0)\geq 0$ when $\mu-c\leq\frac{\mu m_2}{2(\mu^2-\frac{(m_1+m_2)m_2}{m_1})}$ and $H(\frac{m_2}{\mu(1-\frac{m_2}{\mu^2})})\leq 0$ due to $\mu^2>\frac{m_2}{m_1}(m_1+m_2)$. Hence, if $\mu-c\geq\frac{\mu m_2}{2(\mu^2-\frac{(m_1+m_2)m_2}{m_1})}$, $H(y)\leq 0$, which means $l_3(y)$ as well as $l_3(p)$ decreases; if $\mu-c\leq\frac{\mu m_2}{2(\mu^2-\frac{(m_1+m_2)m_2}{m_1})}$, $l_3(y)$ is unimodal and is maximized at y^*, which is uniquely given by $H(y)=0$, and $l_3(p)$ is also unimodal and maximized at $p_3^*=y^*+\mu$.

(ii) Since we have proved $l_i(p)$ is either monotonous or unimodal, then the global maximum of $L(p)$ should be derived from $\{p_1^*,\mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)},p_2^*,\mu,p_3^*\}$, i.e., the possible local maximums in each $l_i(p)$. Then we will prove that the global maximum of $L(p)$ can not be $\mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)}$ or μ.

First, we show $L(p)$ can not be maximized at $p=\mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)}$ by contradiction. We suppose $L(p)$ is maximized at $\mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)}$, then $l_1(\mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)})=l_2(\mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)})=(\mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)}-c)\frac{m_2}{m_1+m_2}$ is the maximum in both $l_1(p)$ and $l_2(p)$. Recall that $l_i(p)$ is either monotonous or unimodal, then $l_1(p)$ must increase in p or do not exist and $l_2(p)$ must decrease in p, i.e., $\mu-c\leq(\frac{m_2}{2m_1}+1)\sqrt{\frac{m_2}{m_1}(m_1+m_2)}$ and $\mu-c\geq\frac{\sqrt{\frac{m_1m_2}{m_1+m_2}}\Big(\big(m_2+3\mu^2\big)(m_1+m_2)-\mu^2m_2\Big)-2m_2\mu(2m_1+m_2)}{-4\mu\sqrt{\frac{m_1m_2}{m_1+m_2}}m_1+\frac{2m_1^2}{m_1+m_2}\left(\mu^2+m_2+\frac{m_2^2}{m_1}\right)}$ based on the proof of Lemma 3 (i). However, $(\frac{m_2}{2m_1}+1)\sqrt{\frac{m_2}{m_1}(m_1+m_2)}<\frac{\sqrt{\frac{m_1m_2}{m_1+m_2}}\Big(\big(m_2+3\mu^2\big)(m_1+m_2)-\mu^2m_2\Big)-2m_2\mu(2m_1+m_2)}{-4\mu\sqrt{\frac{m_1m_2}{m_1+m_2}}m_1+\frac{2m_1^2}{m_1+m_2}\left(\mu^2+m_2+\frac{m_2^2}{m_1}\right)}$ due to $\mu^2>\frac{m_2}{m_1}(m_1+m_2)$, which is a contradiction.

Second, we will show that if $L(p)$ is maximized at $p=\mu$, then $p_3^*=\mu$, in other words, $L(p)$ is maximized at $p=p_3^*$. Suppose $L(p)$ is maximized at $p=\mu$, then $l_2(\mu)=l_3(\mu)=(\mu-c)\frac{m_2^2}{m_1\mu^2}$ is the maximum in both $l_2(p)$ and $l_3(p)$. Recall that $l_i(p)$ is either monotonous or unimodal, then $l_2(p)$ must increase in p or do not exist and $l_3(p)$ must decrease in p, i.e. $\mu-c\leq\frac{\mu m_2}{2(\mu^2-\frac{m_2}{m_1}(m_1+m_2))}$ and $\mu-c\geq\frac{\mu m_2}{2(\mu^2-\frac{(m_1+m_2)m_2}{m_1})}$ based on the proof of Lemma 3 (i), which implies $\mu-c=\frac{\mu m_2}{2(\mu^2-\frac{(m_1+m_2)m_2}{m_1})}$. When $\mu-c=\frac{\mu m_2}{2(\mu^2-\frac{(m_1+m_2)m_2}{m_1})}$, the solution for $H(y)=0$ is 0, i.e., $p_3^*=\mu$.

In summary, p_1^*, p_2^* and p_3^* are the unique solutions of the equations in Table 5, respectively. Thus, they can be solved by Cardano's formula with close forms. □

Proof of Proposition 2. Recall $m_1=\frac{1+s}{2}\sigma^2$ and $m_2=\frac{1-s}{2}\sigma^2$, we have $(\frac{m_2}{2m_1}+1)\sqrt{\frac{m_2}{m_1}(m_1+m_2)}=\sigma\theta_1$, $\frac{\mu m_2}{2(\mu^2-\frac{m_2}{m_1}(m_1+m_2))}=\sigma\theta_2$ and $\frac{\sqrt{\frac{m_1m_2}{m_1+m_2}}\Big(\big(m_2+3\mu^2\big)(m_1+m_2)-\mu^2m_2\Big)-2m_2\mu(2m_1+m_2)}{-4\mu\sqrt{\frac{m_1m_2}{m_1+m_2}}m_1+\frac{2m_1^2}{m_1+m_2}\left(\mu^2+m_2+\frac{m_2^2}{m_1}\right)}=\sigma\theta_3$. Moreover, $\max\{\sigma\theta_1,\sigma\theta_2\}<\sigma\theta_3$. By Lemma 3, we have $p^*\in\{p_1^*,p_2^*,p_3^*\}$. We will consider the following four cases.

(i) When $\frac{\mu-c}{\sigma}\leq\min\{\theta_1,\theta_2\}$, i.e., $\mu-c\leq\min\{\sigma\theta_1,\sigma\theta_2\}$. From the proof of Lemma 3, we know $l_1(p)$ is increasing for $\mu-c\leq\sigma\theta_1$, $l_2(p)$ is increasing for $\mu-c\leq\sigma\theta_2$, and $l_3(p)$ is unimodal for $\mu-c\leq\sigma\theta_1$. Thus there is only one local maximal point p_3^* and $p^*=p_3^*$.

(ii) When $\min\{\theta_1,\theta_2\} < \frac{\mu-c}{\sigma} \le \max\{\theta_1,\theta_2\}$, there are two scenarios. If $\theta_1 \ge \theta_2$, i.e. $\sigma\theta_2 < \mu - c \le \sigma\theta_1$. From the proof of Lemma 3, $l_1(p)$ increases for $\mu - c \le \sigma\theta_1$, $l_2(p)$ is unimodal for $\sigma\theta_2 < \mu - c \le \sigma\theta_1$ and $l_3(p)$ decreases for $\mu - c > \sigma\theta_2$. Thus $L(p)$ is unimodal and $p^* = p_2^*$. If $\theta_1 < \theta_2$, i.e. $\sigma\theta_1 < \mu - c \le \sigma\theta_2$. From the proof of Lemma 3, $l_1(p)$ is unimodal for $\mu - c > \sigma\theta_1$, $l_2(p)$ is increasing for $\mu - c \le \sigma\theta_2$ and $l_3(p)$ is unimodal for $\mu - c \le \sigma\theta_3$. Moreover, $\sigma\theta_2 < \sigma\theta_3$. Thus $L(p)$ is bimodal and $p^* = \underset{p\in\{p_1^*,p_3^*\}}{\arg\max}\{L(p_1^*), L(p_3^*)\}$.

(iii) When $\max\{\theta_1,\theta_2\} < \frac{\mu-c}{\sigma} \le \theta_3$, i.e. $\max\{\sigma\theta_1,\sigma\theta_2\} < \mu - c \le \sigma\theta_3$. From the proof of Lemma 3, $l_1(p)$ is unimodal for $\mu - c > \sigma\theta_1$, $l_2(p)$ is unimodal for $\sigma\theta_2 < \mu - c \le \sigma\theta_3$ and $l_3(p)$ decreases for $\mu - c > \sigma\theta_2$. Thus $L(p)$ is bimodal and $p^* = \underset{p\in\{p_1^*,p_2^*\}}{\arg\max}\{L(p_1^*), L(p_2^*)\}$.

(iv) When $\frac{\mu-c}{\sigma} > \theta_3$, i.e. $\mu - c > \sigma\theta_3$. From the proof of Lemma 3, $l_1(p)$ is unimodal for $\mu - c > \sigma\theta_1$ while $l_2(p)$ and $l_3(p)$ decreases for $\mu - c > \sigma\theta_2$. Due to $\max\{\sigma\theta_1,\sigma\theta_2\} < \sigma\theta_3$, $L(p)$ is unimodal and $p^* = p_1^*$ for $\mu - c > \sigma\theta_3$. □

Proof of Corollary 2. We first consider $\theta_1 \ge \theta_2$. Based on the proof of Lemma 3, if $\frac{\mu-c}{\sigma} \le \min\{\theta_1,\theta_2\} = \theta_2$, $L(p)$ is unimodal; if $\theta_2 = \min\{\theta_1,\theta_2\} < \frac{\mu-c}{\sigma} \le \max\{\theta_1,\theta_2\} = \theta_1$, $L(p)$ is unimodal; if $\theta_1 = \max\{\theta_1,\theta_2\} < \frac{\mu-c}{\sigma} \le \theta_3$, $L(p)$ is bimodal; if $\frac{\mu-c}{\sigma} > \theta_3$, $L(p)$ is unimodal. Thus $L(p)$ is unimodal for $\frac{\mu-c}{\sigma} < \theta_1$ or $\frac{\mu-c}{\sigma} > \theta_3$, and $L(p)$ is bimodal otherwise.

We next consider $\theta_1 < \theta_2$. Based on the proof of Lemma 3, if $\frac{\mu-c}{\sigma} \le \min\{\theta_1,\theta_2\} = \theta_1$, $L(p)$ is unimodal; if $\sigma\theta_1 = \min\{\theta_1,\theta_2\} < \frac{\mu-c}{\sigma} \le \max\{\theta_1,\theta_2\} = \sigma\theta_2$, $L(p)$ is bimodal; if $\theta_2 = \max\{\theta_1,\theta_2\} < \frac{\mu-c}{\sigma} \le \theta_3$, $L(p)$ is bimodal; if $\frac{\mu-c}{\sigma} > \theta_3$, $L(p)$ is unimodal. Thus $L(p)$ is unimodal for $\frac{\mu-c}{\sigma} < \theta_1$ or $\frac{\mu-c}{\sigma} > \theta_3$, and $L(p)$ is bimodal otherwise. □

Proof of Corollary 3. We first prove p_i^* for $i = 1,2,3$ increases in c in each segment.

First, we prove p_1^* is increasing in c. We have $l_1'(p|c) = 1 - \frac{m2(\mu+p)}{(\mu-p)^3} + \frac{2m_2}{(\mu-p)^3}c$, which is increasing in c. Let $c_1 \ge c_2$, we will prove $p_1^*(c_1) \le p_1^*(c_2)$ by contradiction. Suppose $p_1^*(c_1) > p_1^*(c_2)$. Recall that $l_1'(p|c)$ is decreasing in p, then $l_1'(p_1^*(c_2)|c_2) > l_1'(p_1^*(c_1)|c_2) \ge l_1'(p_1^*(c_1)|c_1)$, where the first inequality is due to $p_1^*(c_1) > p_1^*(c_2)$, and the second inequality is due to $l_1'(p|c)$ is increasing in c. Thus it contradicts to $l_1'(p_1^*(c_1)|c_1) = l_1'(p_1^*(c_2)|c_2) = 0$.

Second, we prove p_2^* is increasing in c by contradiction. Suppose p_2^* is decreasing in c, recall that p_2^* is decreasing in x^*, where $G(x^*) = A_1 + (-2\mu^2 x^{*4} - 4\sqrt{m_1}\mu x^{*3} - 2m_1 x^{*2} + 4\mu^2 x^{*2} + 4\sqrt{m_1}\mu x^* - 2\mu^2)c = A_1 - 2(\sqrt{m_1}x^* + \mu x^{*2} - \mu)^2 c = 0$ and A_1 is independent on c. Then $G(x)$ is decreasing in c and x^* is increasing in c. Let $c_1 \ge c_2$, then we have $x^*(c_1) > x^*(c_2)$. Then $G(x^*(c_2)|c_2) > G(x^*(c_1)|c_2) \ge G(x^*(c_1)|c_1)$, where the first inequality is due to $x^*(c_1) > x^*(c_2)$ and the second inequality is due to $G(x^*|c)$ is decreasing in c. Thus it contradicts to $G(x^*(c_2)|c_2) = G(x^*(c_1)|c_1) = 0$.

Third, we prove p_3^* is increasing in c by contradiction. Suppose p_3^* is decreasing in c, recall that $y^* = p_3^* - \mu$, $H(y^*) = A_2 + 2((1 - \frac{m_2}{\mu^2})m_1 - \frac{m_2^2}{\mu^2})c$, $y^* \ge 0$ and A_2 is is independent on c. Note $H(y)$ is decreasing in y, then y^* is decreasing in c, and $H(y^*)$ is increasing in c due to $\mu^2 - m_2 - \frac{m_2^2}{m_1} > 0$ and $(1-\frac{m_2}{\mu^2})m_1 - \frac{m_2^2}{\mu^2} > 0$. Let $c_1 \geq c_2$, then we have $y^*(c_1) < y^*(c_2)$. Then $H(y^*(c_2)|c_2) < H(y^*(c_1)|c_2) \leq H(y^*(c_1)|c_1)$, where the first inequality is due to $y^*(c_1) < y^*(c_2)$ and the second inequality is due to $H(y^*|c)$ is increasing in c. Thus it contradicts to $H(y^*(c_2)|c_2) = H(y^*(c_1)|c_1) = 0$.

Last, we prove p^* will be first p_1^*, then p_2^* and last p_3^* when c increases, which completes the proof due to p_i^* is increasing in c. By Table 2, $\inf\limits_{F\in\mathscr{F}} \mathsf{P}(V \geq p)$ equals to $1 - \frac{m_2}{(\mu-p)^2}$ when $0 \leq p \leq \mu - \sqrt{\frac{m_2}{m_1}(m_1+m_2)}$, $(\frac{\sqrt{1+b}-\sqrt{1+a}}{\sqrt{b}-\sqrt{a}})^2$ when $\mu - \sqrt{\frac{m_2}{m_1}(m_1+m_2)} \leq p \leq \mu$ and $\frac{\left[\frac{m_2}{\mu}-(p-\mu)\left(1-\frac{m_2}{\mu^2}\right)\right]^2}{m_1 - \frac{2m_2}{\mu}(p_3-\mu)+(p_3-\mu)^2\left(1-\frac{m_2}{\mu^2}\right)}$ when $\mu \leq p \leq \mu + \frac{m_2}{\mu(1-\frac{m_2}{\mu^2})}$, which are denoted by $P_1(p)$, $P_2(p)$, $P_3(p)$, respectively. Let $p_1^{*'} = \arg\sup\limits_{p\in[0,\underline{p}]} (p-c)P_1(p) = \arg\sup\limits_{p\in[0,\underline{p}]} l_1(p)$, $p_2^{*'} = \arg\sup\limits_{p\in[\underline{p},\mu]} (p-c)P_2(p) = \arg\sup\limits_{p\in[\underline{p},\mu]} l_2(p)$, $p_3^{*'} = \arg\sup\limits_{p\in[\mu,\bar{p}]} (p-c)P_3(p) = \arg\sup\limits_{p\in[\mu,\bar{p}]} l_3(p)$, where $\underline{p} = \mu - \sqrt{\frac{m_2}{m_1}(m_1+m_2)}$ and $\bar{p} = \mu + \frac{m_2}{\mu(1-\frac{m_2}{\mu^2})}$. Thus, $l_1(p_1^{*'}) - l_2(p_2^{*'}) = \sup_{p_1\in[0,\underline{p}]} l_1(p_1) - \sup_{p_2\in[\underline{p},\mu]} l_2(p_2) = \sup_{p_1\in[0,\underline{p}],p_2\in[\underline{p},\mu]} \{p_1P_1(p_1) - p_2P_2(p_2) - c(P_1(p_1) - P_2(p_2))\}$ is decreasing in c because $p_1 \leq p_2$ and $\inf\limits_{F\in\mathscr{F}} \mathsf{P}(V \geq p)$ is decreasing in p. Similarly, $l_1(p_1^{*'}) - l_3(p_3^{*'})$ and $l_2(p_2^{*'}) - l_3(p_3^{*'})$ are also decreasing in c. Therefore, there exist $\underline{c}$ and $\bar{c}$, if $c \leq \underline{c}$, $l_1(p_1^{*'}) \geq l_2(p_2^{*'}) \geq l_3(p_3^{*'})$; if $\underline{c} \leq c \leq \bar{c}$, $l_2(p_2^{*'}) \geq l_1(p_1^{*'})$ and $l_2(p_2^{*'}) \geq l_3(p_3^{*'})$; otherwise, $l_3(p_1^{*'}) \geq l_2(p_2^{*'}) \geq l_1(p_3^{*'})$. Then we prove that p^* will be achieved first in the interval $[0,\underline{p}]$, then in the interval $[\underline{p},\mu]$, and last in the interval $[\mu,\bar{p}]$ when c increases. We thus get the results. □

Proof of Proposition 3. Denote $U(p) = (p-c)\sup\limits_{F\in\mathscr{F}} \mathsf{P}(V \geq p)$ and $p_u^* = \arg\sup\limits_{p} U(p)$. We will first prove $p_u^* > \mu$ by contradiction. Suppose $p_u^* \leq \mu$, then $U(p) \leq \mu - c$ due to $\mathsf{P}(V \geq p) \leq 1$. Recall $m_1 = \frac{1+s}{2}\sigma^2$ and $m_2 = \frac{1-s}{2}\sigma^2$, when $p = \frac{\mu}{1-\frac{m_2}{\mu^2}}$, $U(p) = (p-c)(1-\frac{m_2}{\mu^2}) = \mu - c(1-\frac{m_2}{\mu^2}) > \mu - c \geq U(p_u^*)$, which contradicts to the definition of p_u^*. Next, we will focus on $p > \mu$. By (OA.2), we should consider the following four cases.

(i) When $\mu < p \leq \frac{\mu}{1-\frac{m_2}{\mu^2}}$, $U(p) = (p-c)(1-\frac{m_2}{\mu^2})$, which is increasing in p.

(ii) When $\frac{\mu}{1-\frac{m_2}{\mu^2}} \leq p \leq \mu + \sqrt{\frac{m_1}{m_2}(m_1+m_2)}$, $U(p) = (p-c)\frac{1+\frac{m_2}{4(p-\mu)^2}(\sqrt{1+\frac{4(p-\mu)^2}{m_2}}-1)^2}{\sqrt{1+\frac{4(p-\mu)^2}{m_2}}}$. Denote $x = \frac{4(p-\mu)^2}{m_2} \in [\frac{4\frac{m_2}{\mu^2}}{(1-\frac{m_2}{\mu^2})^2}, \frac{4m_1(m_1+m_2)}{m_2^2}]$, then $p = \mu + \frac{\sqrt{m_2}}{2}\sqrt{x}$ and $U(x) = U(p(x)) = (\mu - c + \frac{\sqrt{m_2}}{2}\sqrt{x})\frac{1+\frac{1}{x}(\sqrt{1+x}-1)^2}{\sqrt{1+x}}$. We have $\frac{\mathrm{d}U(x)}{\mathrm{d}x} = \frac{[\frac{\sqrt{m_2}\sqrt{x}}{2}+2(\mu-c)](\sqrt{1+x}-1)-x(\mu-c)}{x^2\sqrt{1+x}}$. Then $\frac{\mathrm{d}U(x)}{\mathrm{d}x} \geq 0$ if and only if $[\frac{\sqrt{m_2}\sqrt{x}}{2} + 2(\mu-c)](\sqrt{1+x}-1) \geq x(\mu-c)$, which is equivalent to $\frac{\sqrt{x}}{\sqrt{1+x}-1} - (\mu-c)\frac{2}{\sqrt{m_2}} \geq 0$. Note $t(x) = \frac{\sqrt{x}}{\sqrt{1+x}-1} - (\mu-c)\frac{2}{\sqrt{m_2}}$ is a decreasing function in $[\frac{4\frac{m_2}{\mu^2}}{(1-\frac{m_2}{\mu^2})^2}, \frac{4m_1(m_1+m_2)}{m_2^2}]$, then $U(x)$ is increasing when $t(\frac{4m_1(m_1+m_2)}{m_2^2}) = \frac{\sqrt{m_1+m_2}}{\sqrt{m_1}} - (\mu-c)\frac{2}{\sqrt{m_2}} \geq 0$, unimodal when $t(\frac{4m_1(m_1+m_2)}{m_2^2}) = \frac{\sqrt{m_1+m_2}}{\sqrt{m_1}} - (\mu-c)\frac{2}{\sqrt{m_2}} \leq 0$ and $t(\frac{4\frac{m_2}{\mu^2}}{(1-\frac{m_2}{\mu^2})^2}) = \frac{2c-\mu}{\sqrt{m_2}} \geq 0$, and decreasing when $t(\frac{4\frac{m_2}{\mu^2}}{(1-\frac{m_2}{\mu^2})^2}) \leq 0$. Note p is increasing in x, then we have the following statements: if $c \leq \frac{1}{2}\mu$, $U(p)$ is decreasing in p; if $\frac{1}{2}\mu < c \leq \mu - \frac{1}{2}\sqrt{\frac{m_2(m_1+m_2)}{m_1}}$, $U(p)$ is unimodal; if $\mu - \frac{1}{2}\sqrt{\frac{m_2(m_1+m_2)}{m_1}} < c \leq \mu$, $U(p)$ is increasing in p.

(iii) When $\mu+\sqrt{\frac{m_1}{m_2}(m_1+m_2)}\le p\le\mu+\frac{m_1\mu}{m_2}$, $U(p)=(p-c)\frac{m_1}{(p-\mu)^2}$. We have $\frac{\mathrm{d}U(p)}{\mathrm{d}p}=m_1\frac{2c-p-\mu}{(p-\mu)^3}\le 0$ due to $p>\mu\ge c$. Hence, $U(p)$ is decreasing in p.

(iv) When $p\ge\mu+\frac{m_1\mu}{m_2}$, $U(p)=(p-c)\frac{\frac{m_1}{\mu^2}(\mu^2-m_2-\frac{m_2^2}{m_1})}{(1-\frac{m_2}{\mu^2})(p-\mu)^2-2(p-\mu)\frac{m_2}{\mu}+m_1}$. We first prove $\frac{\frac{m_1}{\mu^2}(\mu^2-m_2-\frac{m_2^2}{m_1})}{(1-\frac{m_2}{\mu^2})(p-\mu)^2-2(p-\mu)\frac{m_2}{\mu}+m_1}\le\frac{m_1}{(p-\mu)^2}$, which is equivalent to $(\frac{m_2}{\mu}(p-\mu)-m_1)^2\ge 0$. Then $U(p)\le(p-c)\frac{m_1}{(p-\mu)^2}$, and the equality is attained by $p=\mu+\frac{m_1\mu}{m_2}$.

Based on the above analysis, we have $\frac{\mu}{1-\frac{m_2}{\mu^2}}\le p_u^*\le\mu+\sqrt{\frac{m_1}{m_2}(m_1+m_2)}$, i.e, p_u^* is given by case (ii). Moreover, if $c\le\frac{1}{2}\mu$, $p_u^*=\frac{\mu}{1-\frac{m_2}{\mu^2}}$; if $\frac{1}{2}\mu<c\le\mu-\frac{1}{2}\sqrt{\frac{m_2(m_1+m_2)}{m_1}}$, p_u^* is the unique solution of $\frac{\sqrt{m_2}(p_u^*+\mu-2c)}{\sqrt{4(p_u^*-\mu)^2+m_2}+\sqrt{m_2}}=(\mu-c)$ for $p_u^*\in[\frac{\mu}{1-\frac{m_2}{\mu^2}},\mu+\sqrt{\frac{m_1}{m_2}(m_1+m_2)}]$; if $\mu-\frac{1}{2}\sqrt{\frac{m_2(m_1+m_2)}{m_1}}<c\le\mu$, $p_u^*=\mu+\sqrt{\frac{m_1}{m_2}(m_1+m_2)}$. Plugging $p=p_u^*$ into $U(p)=(p-c)\frac{1+\frac{m_2}{4(p-\mu)^2}(\sqrt{1+\frac{4(p-\mu)^2}{m_2}}-1)^2}{\sqrt{1+\frac{4(p-\mu)^2}{m_2}}}$, and replace $m_1=\frac{(1+s)}{2}\sigma^2$ and $m_2=\frac{(1-s)}{2}\sigma^2$, we get the desired results. $\square$

Proof of Proposition 4. (i) Based on the proof of Theorem 1(b) of Chen et al. (2022), they derive the lower and upper bound of the probability involving mean and variance. Because we also involve semivariance as information and derive tight lower and upper bound, our probability will be tighter and thus the performance bound will be greater.

(ii) The optimal robust price of the MV model according to Chen et al. (2022) is $p_{MV}^*=\mu-k_{MV}^*\sigma$, where $k_{MV}^*=\sqrt[3]{\tau+\sqrt{\tau^2+1}}+\sqrt[3]{\tau-\sqrt{\tau^2+1}}\ge 0$ is the solution of $G_{MV}(k)=-k^3-3k+2\tau=0$ and $\tau=\frac{\mu-c}{\sigma}$.

When p^* is attained in $l_1(p)$, i.e., $p^*=p_1^*$. We denote $p^*=\mu-k^*\sigma$, which satisfies $l_1'(p)=\frac{m_2(2c-\mu-p)+(\mu-p)^3}{(\mu-p)^3}=0$ for $0\le p\le\mu-\sigma\sqrt{\frac{1-s}{1+s}}$. In other words, k^* is the unique solution of $H(k)=k^3-(1-s)\tau+\frac{1-s}{2}k=0$, $k\in[\sqrt{\frac{1-s}{1+s}},\tau]$. Due to $H'(k)=3k^2+\frac{1-s}{2}\ge 0$, $H(k)$ is increasing in k. Moreover, $H(k_{MV}^*)=k_{MV}^{*3}-(1-s)\tau+\frac{1-s}{2}k_{MV}^*=2\tau-3k_{MV}^*-(1-s)\tau+\frac{1-s}{2}k_{MV}^*$, where the second equality is due to $k_{MV}^{*3}+3k_{MV}^*-2\tau=0$. Therefore, $H(k_{MV}^*)\le 0$, which is equivalent to $s\le\frac{5\left(\sqrt[3]{\tau+\sqrt{\tau^2+1}}+\sqrt[3]{\tau-\sqrt{\tau^2+1}}\right)-2\tau}{2\tau-\sqrt[3]{\tau+\sqrt{\tau^2+1}}-\sqrt[3]{\tau-\sqrt{\tau^2+1}}}$, means that $p_{MV}^*\ge p^*$ due to $H(k_{MVS}^*)=0$. It is easy to see $\frac{5\left(\sqrt[3]{\tau+\sqrt{\tau^2+1}}+\sqrt[3]{\tau-\sqrt{\tau^2+1}}\right)-2\tau}{2\tau-\sqrt[3]{\tau+\sqrt{\tau^2+1}}-\sqrt[3]{\tau-\sqrt{\tau^2+1}}}<1$. Recall $\frac{\sigma^2-\mu^2}{\sigma^2+\mu^2}<s<1$, then $p_{MV}^*\ge p^*$ if $s\le\max\left\{\frac{\sigma^2-\mu^2}{\sigma^2+\mu^2},\frac{5\left(\sqrt[3]{\tau+\sqrt{\tau^2+1}}+\sqrt[3]{\tau-\sqrt{\tau^2+1}}\right)-2\tau}{2\tau-\sqrt[3]{\tau+\sqrt{\tau^2+1}}-\sqrt[3]{\tau-\sqrt{\tau^2+1}}}\right\}$, and $p_{MV}^*\le p^*$ otherwise.

When p^* is attained in $l_2(p)$, i.e., $p^*=p_2^*$, denote $p(x)=\frac{2\sqrt{m_1}x\mu-m_2+x^2\mu^2-\mu^2}{\sqrt{m_1}x+x^2\mu-\mu}$ and $k(x)=(\mu-p(x))/\sigma=(\mu-p(x))/\sqrt{m_1+m_2}=\frac{m_2-\sqrt{m_1}x\mu}{(\sqrt{m_1}x+x^2\mu-\mu)\sqrt{m_1+m_2}}$. Then $p^*=p(x^*)=\mu-k(x^*)\sigma$. Recall $p_{MV}^*=\mu-k_{MV}^*\sigma$. Therefore, to show $p^*\ge p_{MV}^*$ is equivalent to show $k^*\le k_{MV}^*$. Recall $p(x)$ is decreasing for $x\in[\frac{m_2}{\sqrt{m_1}\mu},\sqrt{\frac{m_2}{m_1+m_2}}]$ and $p^*=p(x^*)\in[\mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)},\mu]$, where x^* satisfies $G(x^*)=0$ and $x^*\in[\frac{m_2}{\sqrt{m_1}\mu},\sqrt{\frac{m_2}{m_1+m_2}}]$. Let $k_{MV}^*=k(x_{MV}^*)$, then x_{MV}^* satisfies $G_{MV}(k_{MV}^*)=G_{MV}(k(x_{MV}^*))=0$, which is equivalent to

$$\begin{aligned}Q_{MV}(x) =& 2(\mu-c)\left(m_1+m_2\right)\left(\sqrt{m_1}x+x^2\mu-\mu\right)^3-\left(m_2-\sqrt{m_1}x\mu\right)^3\\ &-3\left(m_1+m_2\right)\left(\sqrt{m_1}x+x^2\mu-\mu\right)^2\left(m_2-\sqrt{m_1}x\mu\right)=0.\end{aligned}$$

Moreover, since $p(x)$ is decreasing in x, then $k(x)$ is increasing in x. Hence, to show $k^*\le k^*_{MV}$ is equivalent to show $x^*\le x^*_{MV}$. Denote $K(x)=(m_1+m_2)(\sqrt{m_1}x+x^2\mu-\mu)$, and $K(x)<0$ due to $K(x)$ is increasing in $x\ge 0$ and $K(\sqrt{\frac{m_2}{m_1+m_2}})<0$ since $\mu^2>\frac{m_2}{m_1}(m_1+m_2)$. Recall that $G(x)$ is decreasing in x, then $G(x)>0$ for $x<x^*$ and $K(x)G(x)<0$ for $x<x^*$. Moreover, we have

$$\begin{aligned}K(x)G(x)-Q_{MV}(x)=&\,m_2(m_1+m_2)\left(-\mu+\sqrt{m_1}x+x^2\mu\right)^2\\ &+(m_2-\sqrt{m_1}\mu x)\left((-\mu+\sqrt{m_1}x+x^2\mu)(m_1+m_2)(2\mu x^2+\sqrt{m_1}x)+(\sqrt{m_1}x\mu-m_2)^2\right)\ge 0,\end{aligned}$$

where the first term is nonnegative obviously. And we have

$$\begin{aligned}&(-\mu+\sqrt{m_1}x+x^2\mu)(m_1+m_2)(2\mu x^2+\sqrt{m_1}x)+(\sqrt{m_1}x\mu-m_2)^2\\ =&(2\mu^2(m_1+m_2)x^4-2m_2\mu^2x^2)+(3\mu\sqrt{m_1}(m_1+m_2)x^3-3\sqrt{m_1}m_2\mu x)\\ &+(x^2(m_1m_2+m_1^2)-x\sqrt{m_1}m_1\mu)+(m_2^2-x^2m_1\mu^2)\le 0.\end{aligned}$$

Notice that each term in the second function is less than or equal to 0 because $\frac{m_2}{\sqrt{m_1}\mu}\le x\le\sqrt{\frac{m_2}{m_1+m_2}}$. Therefore, the second term is also nonnegative because $m_2-\sqrt{m_1}\mu x\le 0$. Then $Q_{MV}(x)\le K(x)G(x)<0$ for $x<x^*$, which means that any $x<x^*$ is not the unique solution of $Q_{MV}(x)=0$. Hence, $x^*_{MV}\ge x^*$.

When $p^*\ge\mu$, we have $p^*\ge p^*_{MV}$ due to $p^*_{MV}\le\mu$. $\square$

Proof of Proposition 5. When $s\nearrow 1$, we have $p^*\to\mu$ and $L(p^*)\to\mu-c$ based on Proposition 1 and Proposition 2. It indicates $1=\inf_{F\in\mathscr{F}}\mathsf{P}(V\ge\mu)\le\sup_{F\in\mathscr{F}}\mathsf{P}(V\ge\mu)\le 1$. Then we have $1=\inf_{F\in\mathscr{F}}\mathsf{P}(V\ge\mu)\le\inf_{F\in\mathscr{F}}\mathsf{P}(V\ge p^*_{MV})\le\sup_{F\in\mathscr{F}}\mathsf{P}(V\ge p^*_{MV})\le 1$ due to $p^*_{MV}\le\mu$. Therefore, $1=\inf_{F\in\mathscr{F}}\mathsf{P}(V\ge\mu)=\sup_{F\in\mathscr{F}}\mathsf{P}(V\ge\mu)=\inf_{F\in\mathscr{F}}\mathsf{P}(V\ge p^*_{MV})=\sup_{F\in\mathscr{F}}\mathsf{P}(V\ge p^*_{MV})$. Then $\inf_{F\in\mathscr{F}}\frac{\pi(p^*;F)}{\pi(p^*_{MV};F)}=\frac{p^*-c}{p^*_{MV}-c}=\frac{\mu-c}{p^*_{MV}-c}\ge 1$ due to $p^*_{MV}\le\mu$.

When $s\searrow\frac{\sigma^2-\mu^2}{\sigma^2+\mu^2}$, recall that the feasible distribution converges to a unique distribution

$$\begin{cases}0 & \text{with prob. } \frac{\sigma^2}{\sigma^2+\mu^2},\\ \frac{\sigma^2+\mu^2}{\mu} & \text{with prob. } \frac{\mu^2}{\sigma^2+\mu^2}.\end{cases}$$

Moreover, we have $p^*\to\frac{\sigma^2+\mu^2}{\mu}>\mu$ with $L(p^*)\to\mu-c\frac{\mu^2}{\sigma^2+\mu^2}$. Because $0<p^*_{MV}\le\mu$ and the distribution is unique, we have $\inf_{F\in\mathscr{F}}\mathsf{P}(V\ge\mu)=\sup_{F\in\mathscr{F}}\mathsf{P}(V\ge\mu)=\inf_{F\in\mathscr{F}}\mathsf{P}(V\ge p^*_{MV})=\sup_{F\in\mathscr{F}}\mathsf{P}(V\ge p^*_{MV})=$

$\frac{\mu^2}{\sigma^2+\mu^2}$. Therefore, $\inf_{F\in\mathscr{F}} \frac{\pi(p^*;F)}{\pi(p^*_{MV};F)} = \frac{p^*-c}{p^*_{MV}-c} = \frac{\frac{\sigma^2+\mu^2}{\mu}-c}{p^*_{MV}-c} \geq 1+\frac{\sigma^2}{\mu(\mu-c)}$. $\square$

Proof of Lemma 4. Due to the length of the proof, we put it in the Appendix B. $\square$

Proof of Proposition 7. Based on Proposition 6, when $p \leq \mu-\sigma$, $L_s(p) = (p-c)\big(1-\frac{\sigma^2}{2(\mu-p)^2}\big) = (p-c)\big(1-\frac{m_2}{(\mu-p)^2}\big)$. Then $\frac{\partial L_s(p)}{\partial p} = 1-\frac{\sigma^2}{2(\mu-p)^2}-\frac{\sigma^2(p-c)}{(\mu-p)^3}$, which is decreasing in p. Thus, $\frac{\partial L_s(p)}{\partial p} \geq \frac{\partial L_s(p)}{\partial p}|_{p=\mu-\sigma} = \frac{1}{2}-\frac{\mu-c-\sigma}{\sigma} \geq 0$ for $\mu-c\leq 3\sigma/2$. Hence, $L_s(p)$ is concave. If $\mu-c\leq 3\sigma/2$, $L_s(p)$ is increasing for $p\leq\mu-\sigma$. Otherwise, $L_s(p)$ is maximized at $p^*_{s_1}$, where $p^*_{s_1}$ is given by $\frac{\partial L_s(p)}{\partial p}=0$, which is equivalent to $\frac{\sigma^2}{2}(2c-\mu-p)+(\mu-p)^3=0$.

When $\mu-\sigma\leq p\leq\mu$, $L_s(p)=\frac{p-c}{2}$ and is maximized at μ.

When $\mu\leq p\leq\mu+\sigma$, $L_s(p)=\frac{1}{2}(p-c)\frac{\sigma^2-(p-\mu)^2}{\mu^2-(p-\mu)^2}$, then $\frac{\partial L_s(p)}{\partial p} = \frac{2c(p-\mu)(\mu^2-\sigma^2)+p^2(3\mu^2-4\mu p+p^2+\sigma^2)}{2p^2(p-2\mu)^2}$ and $\frac{\partial^2 L_s(p)}{\partial p^2} = \frac{(\mu^2-\sigma^2)(c(4\mu^2-6\mu p+3p^2)-p^3)}{p^3(2\mu-p)^3}$. We will show $\frac{\partial^2 L_s(p)}{\partial p^2}\leq 0$, i.e., $c(4\mu^2-6\mu p+3p^2)-p^3\equiv h(p)\leq 0$. Note $\frac{\partial h(p)}{\partial p}=2(-p^2-2\mu c+2pc)\leq 0$ for $p\geq c$. Then $h(\mu)=-\mu^2(\mu-c)\leq 0$ leads to $h(p)\leq 0$ for $\mu< p\leq\mu+\sigma$. Therefore, $L_s(p)$ is concave. Since $\frac{\partial L_s(p)}{\partial p}|_{p=\mu}=\frac{\sigma^2}{2\mu^2}>0$ and $\frac{\partial L_s(p)}{\partial p}|_{p=\mu+\sigma}=\frac{\sigma(-c-\mu-\sigma)}{\mu^2-\sigma^2}<0$, $L_s(p)$ is maximized at $p^*_{s_2}$, where $p^*_{s_2}$ is given by $\frac{\partial L_s(p)}{\partial p}=0$, which is equivalent to $p^2(3\mu-p)(\mu-p)+p^2\sigma^2-2c(\mu-p)(\mu^2-\sigma^2)=0$.

Based on Proposition 6, we have $L_s(\mu)=\frac{\mu-c}{2} > \lim_{p\searrow\mu} L_s(p) = \frac{\mu-c}{2}\frac{\sigma^2}{\mu^2}$. Therefore, if $\mu-c>3\sigma/2$, the optimal robust price is $p^*_s = \arg\max_{p\in\{\mu,p^*_{s_1},p^*_{s_2}\}} \{L_s(\mu), L_s(p^*_{s_1}), L_s(p^*_{s_2})\}$; otherwise, the optimal robust price is $p^*_s=\arg\max_{p\in\{\mu,p^*_{s_2}\}}\{L_s(\mu),L_s(p^*_{s_2})\}$. $\square$

Proof of Corollary 4. We fist prove $L_s(p)\geq L(p)$ for $p\leq\mu$. By Proposition 6 and Proposition 1, $L_s(p)=L(p)$ for $c\leq p\leq\max\{c,\mu-\sigma\}$ when $s=0$. When $\max\{c,\mu-\sigma\}\leq p\leq\mu$, it is equivalent to prove $\frac{1}{2}\geq\Big[\frac{\sqrt{(2\mu-p)^2\frac{\sigma^2}{2}-\frac{\sigma^4}{4}+(\frac{\sigma^2}{2}+2\mu(\mu-p)^2)^2}-\sqrt{\frac{(2\mu-p)^2\sigma^2}{2}}}{2\mu(\mu-p)}\Big]^2 = \inf_{F\in\mathscr{F}}\mathsf{P}(V\geq p)$. Recall that $\inf_{F\in\mathscr{F}}\mathsf{P}(V\geq p)$ is decreasing in p and $\inf_{F\in\mathscr{F}}\mathsf{P}(V\geq\mu-\sigma)=\frac{1}{2}$. Thus $\inf_{F\in\mathscr{F}}\mathsf{P}(V\geq p)\leq\frac{1}{2}$ for $p\geq\mu-\sigma$.

Then we prove $L_s(p)\geq L(p)$ for $p\geq\mu$. Note $\mu>\sigma$, then $\mu+\sigma>\mu+\frac{\sigma^2}{\mu(2-\frac{\sigma^2}{\mu^2})}$. For $\mu+\frac{\sigma^2}{\mu(2-\frac{\sigma^2}{\mu^2})}\leq p\leq\mu+\sigma$, $L_s(p)\geq 0=L(p)$ and $L_s(p)=L(p)=0$ for $p\geq\mu+\sigma$. Now we will prove $L_s(p)\geq L(p)$ for $\mu\leq p\leq\mu+\frac{\sigma^2}{\mu(2-\frac{\sigma^2}{\mu^2})}$, that is $\frac{1}{2}\frac{\sigma^2-(p-\mu)^2}{\mu^2-(p-\mu)^2}\geq\frac{[\frac{\sigma^2}{2\mu}-(1-\frac{\sigma^2}{2\mu^2})(p-\mu)]^2}{\frac{\sigma^2}{2}-\frac{\sigma^2}{\mu}(p-\mu)+(1-\frac{\sigma^2}{2\mu^2})(p-\mu)^2}$. Let $x=\frac{\sigma}{\mu}$, $y=\frac{p-\mu}{\mu}$, then $\frac{1}{2}\frac{\sigma^2-(p-\mu)^2}{\mu^2-(p-\mu)^2}=\frac{1}{2}\frac{x^2-y^2}{1-y^2}$ and $\frac{[\frac{\sigma^2}{2\mu}-(1-\frac{\sigma^2}{2\mu^2})(p-\mu)]^2}{\frac{\sigma^2}{2}-\frac{\sigma^2}{\mu}(p-\mu)+(1-\frac{\sigma^2}{2\mu^2})(p-\mu)^2} = \frac{[\frac{x^2}{2}-(1-\frac{x^2}{2})y]^2}{\frac{x^2}{2}-x^2y+(1-\frac{x^2}{2})y^2}$. Moreover, we have $0\leq y\leq\frac{x^2}{2-x^2}$ and $0<x<1$ due to $\mu\leq p\leq\mu+\frac{\sigma^2}{\mu(2-\frac{\sigma^2}{\mu^2})}$ and $\mu>\sigma$. Note $\frac{x^2}{2-x^2}$ is increasing in x, then $y<1$ due to $0<x<1$. Note that $\frac{1}{2}\frac{x^2-y^2}{1-y^2}\geq\frac{[\frac{x^2}{2}-(1-\frac{x^2}{2})y]^2}{\frac{x^2}{2}-x^2y+(1-\frac{x^2}{2})y^2}$ is equivalent to $(1-x^2)y[(2-x^2)y^3-2x^2y^2-(4-x^2)y+4]\geq 0$. It is equivalent to show $(2-x^2)y^3-2x^2y^2-(4-x^2)y+4\geq 0$ due to $0<x<1$ and $y\geq 0$. We have $(2-x^2)y^3-2x^2y^2-(4-x^2)y+4>(2-x^2)y^3-2y^2-(4-x^2)y+4=(1-y)[-(2-x^2)y(1+y)+2(1+y)+2(1-y)]\geq(1-y)[-(2-x^2)(1+y)+2(1+y)+2(1-y)]=(1-y)[x^2(1+y)+2(1-y)]\geq 0$, where the first inequality is due to $0<x<1$, the second and third inequalities are due to $0\leq y\leq 1$. Then we get the desired results. $\square$

Proof of Proposition 8. We first prove the profit of the upper bound increases before p_u^*.

For the first segment, obviously $U_1(p)$ will increase in p.

For the second segment, recall that we let $x=\frac{\sqrt{1+b}-\sqrt{1+a}}{\sqrt{b}-\sqrt{a}}$ when we analyze the lower bound of the second segment, then $p=\frac{2\sqrt{m_1}x\mu-m_2+x^2\mu^2-\mu^2}{\sqrt{m_1}x+x^2\mu-\mu}=\mu+\frac{\sqrt{m_1}x\mu-m_2}{\sqrt{m_1}x+x^2\mu-\mu}$. Based on the analysis in Lemma 3, we have x decreasing in p and $x\in[\frac{m_2}{\sqrt{m_1}\mu},\sqrt{\frac{m_2}{m_1+m_2}}]$. Then we denote the upper bound profit in the second segment as $U_2(p)$. Then we have

$$U_2(p)=\frac{2\sqrt{m_1}x\mu+x^2(\mu^2-m_1-m_2)-\mu^2}{\sqrt{m_1}x+x^2\mu-\mu}-c(\frac{2\sqrt{m_1}x\mu+x^2(\mu^2-m_1-m_2)-\mu^2}{2\sqrt{m_1}x\mu-m_2+x^2\mu^2-\mu^2}):=U_2(x).$$

Then $U_2(x)=\frac{2\sqrt{m_1}x\mu+x^2(\mu^2-m_1-m_2)-\mu^2}{\sqrt{m_1}x+x^2\mu-\mu}-c(\frac{2\sqrt{m_1}x\mu+x^2(\mu^2-m_1-m_2)-\mu^2}{2\sqrt{m_1}x\mu-m_2+x^2\mu^2-\mu^2})$. We first prove that $U_2(x)$ decreases when $c=0$. When $c=0$, $\frac{\mathrm{d}U_2(x)}{\mathrm{d}x}=\frac{1}{(\sqrt{m_1}x+x^2\mu-\mu)^2}(-m_1\sqrt{m_1}x^2+2m_1\mu x-\sqrt{m_1}(m_2x^2+\mu^2(x^2+1))+2m_2\mu x)$. We have $-m_1\sqrt{m_1}x^2+2m_1\mu x-\sqrt{m_1}(m_2x^2+\mu^2(x^2+1))+2m_2\mu x=-\sqrt{m_1}(m_1+m_2+\mu^2)x^2+2(m_1+m_2)\mu x-\sqrt{m_1}\mu^2<0$ for all x because the function is concave and the maximum is smaller than 0 due to $\mu^2>\frac{m_2}{m_1}(m_1+m_2)$. Then we will prove that $(\frac{2\sqrt{m_1}x\mu+x^2(\mu^2-m_1-m_2)-\mu^2}{2\sqrt{m_1}x\mu-m_2+x^2\mu^2-\mu^2})$ increases in x, which will finish the proof. $\frac{\partial}{\partial x}(\frac{2\sqrt{m_1}x\mu+x^2(\mu^2-m_1-m_2)-\mu^2}{2\sqrt{m_1}x\mu-m_2+x^2\mu^2-\mu^2})=-\frac{2((m_1+m_2)x-\sqrt{m_1}\mu)(\sqrt{m_1}\mu x-m_2)}{(2\sqrt{m_1}x\mu-m_2+x^2\mu^2-\mu^2)^2}$, where $\sqrt{m_1}\mu x-m_2\geq 0$ due to $x\geq\frac{m_2}{\sqrt{m_1}\mu}$ and $(m_1+m_2)x-\sqrt{m_1}\mu\leq 0$ due to $x\leq\sqrt{\frac{m_2}{m_1+m_2}}<\frac{\sqrt{m_1}\mu}{m_1+m_2}$. Therefore, $(\frac{2\sqrt{m_1}x\mu+x^2(\mu^2-m_1-m_2)-\mu^2}{2\sqrt{m_1}x\mu-m_2+x^2\mu^2-\mu^2})$ increases in x.

And combining with the previous analysis in Proposition 3, we have $U(p)$ increases in $[c,p_u^*]$ and then decreases in $[p_u^*,+\infty]$. If $p_h^*<p_l^*$, $\lambda L(p_l^*)+(1-\lambda)U(p_l^*)\geq\lambda L(p_h^*)+(1-\lambda)U(p_h^*)$ due to $L(p_l^*)\geq L(p_h^*)$ and $U(p_l^*)\geq U(p_h^*)$, which will lead to a contradiction. If $p_h^*>p_u^*\geq\mu$, $\lambda L(p_u^*)+(1-\lambda)U(p_u^*)\geq\lambda L(p_h^*)+(1-\lambda)U(p_h^*)$ due to $L(p_u^*)=L(p_h^*)=0$ and $U(p_u^*)\geq U(p_h^*)$, which will lead to a contradiction. Therefore, $p_l^*\leq p_h^*\leq p_u^*$.

Let $p^*(\lambda)=\arg\max\{\lambda L(p)+(1-\lambda)U(p)\}=\arg\max\{L(p)+\frac{(1-\lambda)}{\lambda}U(p)\}$ for $\lambda\in(0,1]$. We claim that $\lambda_1>\lambda_2$ and $p^*(\lambda_1)\leq p^*(\lambda_2)$. Otherwise, $p^*(\lambda_1)>p^*(\lambda_2)$. And $L(p^*(\lambda_1))+\frac{(1-\lambda_1)}{\lambda_1}U(p^*(\lambda_1))\geq L(p^*(\lambda_2))+\frac{(1-\lambda_1)}{\lambda_1}U(p^*(\lambda_2))$, $L(p^*(\lambda_2))+\frac{(1-\lambda_2)}{\lambda_2}U(p^*(\lambda_2))\geq L(p^*(\lambda_1))+\frac{(1-\lambda_2)}{\lambda_2}U(p^*(\lambda_1))$. By adding the left side and the right side of the two inequalities respectively, we have $(\frac{(1-\lambda_1)}{\lambda_1}-\frac{(1-\lambda_2)}{\lambda_2})(U(p^*(\lambda_1))-U(p^*(\lambda_2)))\geq 0$. Because $U(p)$ will increase in $[0,p_u^*]$, we have $U(p^*(\lambda_1))-U(p^*(\lambda_2))>0$. And $\frac{(1-\lambda_1)}{\lambda_1}-\frac{(1-\lambda_2)}{\lambda_2}<0$ due to $\lambda_1>\lambda_2$. It will lead to a contradiction.

And for $\lambda=0$, $p^*(0)=p_u^*\geq p^*(\lambda)$ if $\lambda>0$. Then we finish the proof. $\square$

Proof of Proposition 9. First, we consider the local optimal price in $p\in[c,\mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)}]$. Note $h_1(p)=\lambda(p-c)(1-\frac{m_2}{(\mu-p)^2})+(1-\lambda)(p-c)$ is the expected profit in the Hurwicz criterion. If $\mu-c<\sqrt{\frac{m_2}{m_1}(m_1+m_2)}$, $h_1(p)$ does not exist. Otherwise, $\frac{\mathrm{d}^2h_1(p)}{\mathrm{d}p^2}=-\lambda\frac{2m_2(p+2\mu-3c)}{(\mu-p)^4}\leq 0$, which implies that $h_1(p)$ is concave. Note $\frac{\mathrm{d}h_1(p)}{\mathrm{d}p}=\lambda\frac{m_2(2c-\mu-p)+(\mu-p)^3}{(\mu-p)^3}+(1-\lambda)$, then we have $\frac{\mathrm{d}h_1(p)}{\mathrm{d}p}|_{p=c}=\lambda\frac{(\mu-c)^2-m_2}{(\mu-c)^2}+(1-\lambda)\geq\lambda\frac{(\mu-c)^2-m_2}{(\mu-c)^2}\geq\lambda\frac{\frac{m_2}{m_1}(m_1+m_2)-m_2}{(\mu-c)^2}=\lambda\frac{m_2^2}{m_1(\mu-c)^2}>0$, where the second inequality is due to $\mu - c \geq \sqrt{\frac{m_2}{m_1}(m_1+m_2)}$. Since $\frac{\mathrm{d}h_1(p)}{\mathrm{d}p}|_{p=\mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)}} = \lambda \frac{m_2\left[2c-2\mu+\sqrt{\frac{m_2}{m_1}(m_1+m_2)}\right]+\left(\sqrt{\frac{m_2}{m_1}(m_1+m_2)}\right)^3}{\left(\sqrt{\frac{m_2}{m_1}(m_1+m_2)}\right)^3} + (1-\lambda) = \lambda \frac{m_2\left[2c-2\mu+\frac{2m_1+m_2}{m_1}\sqrt{\frac{m_2}{m_1}(m_1+m_2)}\right]}{\left(\sqrt{\frac{m_2}{m_1}(m_1+m_2)}\right)^3} + (1-\lambda)$. Thus if $\mu - c \leq \frac{1}{2}(\frac{m_1+m_2}{\lambda m_1}+1)\sqrt{\frac{m_2}{m_1}(m_1+m_2)}$, $\frac{\mathrm{d}h_1(p)}{\mathrm{d}p}|_{p=\mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)}} \geq 0$, otherwise, $\frac{\mathrm{d}h_1(p)}{\mathrm{d}p}|_{p=\mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)}} < 0$. Hence, $h_1(p)$ is increasing for $\mu - c \leq \frac{1}{2}(\frac{m_1+m_2}{\lambda m_1}+1)\sqrt{\frac{m_2}{m_1}(m_1+m_2)}$ and is unimodal otherwise. When $h_1(p)$ is unimodal, the unique $p^*_{1h} = \arg\max_{p\in[c,\mu-\sqrt{\frac{m_2}{m_1}(m_1+m_2)}]} h_1(p)$ is given by $\frac{\mathrm{d}h_1(p)}{\mathrm{d}p} = 0$, i.e., $\lambda m_2(2c-\mu-p)+(\mu-p)^3 = 0$. By Cardano's solution for a cubic function, we can derive the unique solution in a closed form.

Second, for the price candidates in $[\mu - \sqrt{\frac{m_2}{m_1}(m_1+m_2)}, \mu]$, we have $\frac{\mathrm{d}U_2(x)}{\mathrm{d}x} = \frac{1}{(\sqrt{m_1}x+x^2\mu-\mu)^2}(-m_1\sqrt{m_1}x^2 + 2m_1\mu x - \sqrt{m_1}(m_2x^2+\mu^2(x^2+1)) + 2m_2\mu x) + c\frac{2((m_1+m_2)x-\sqrt{m_1}\mu)(\sqrt{m_1}\mu x-m_2)}{(2\sqrt{m_1}x\mu-m_2+x^2\mu^2-\mu^2)^2}$ and $\frac{\mathrm{d}l_2(x)}{\mathrm{d}x} = \frac{x}{(\sqrt{m_1}x+x^2\mu-\mu)^2}[2\mu^2(\mu-c)x^4 + \sqrt{m_1}\mu(5\mu-4c)x^3 + 2(2m_1\mu - m_1c + 2c\mu^2 - 2\mu^3)x^2 + (4\sqrt{m_1}c\mu - \sqrt{m_1}m_2 - 7\sqrt{m_1}\mu^2)x + 2m_2\mu - 2c\mu^2 + 2\mu^3]$. Therefore, the first order condition of Hurwicz criterion is $\lambda\frac{\mathrm{d}l_2(x)}{\mathrm{d}x} + (1-\lambda)\frac{\mathrm{d}U_2(x)}{\mathrm{d}x} = 0$, where the numerator is a polynomial with ninth degree. The zeros of the polynomial will be the local extreme points.

Based on previous analysis, $G(x)$ of $\frac{\mathrm{d}l_2(x)}{\mathrm{d}x}$ is decreasing in x and the numerator of $\frac{\mathrm{d}U_2(x)}{\mathrm{d}x} \leq 0$. Therefore, if $G(x) < 0$, it indicates that the numerator of the Hurwicz criterion is less than 0, which cannot be the local extreme points. Thus if $\theta_2 \leq \frac{\mu-c}{\sigma} \leq \theta_3$, the smallest price candidate is greater than p^*_2; if $\frac{\mu-c}{\sigma} < \theta_2$, we have $G(x) \leq 0$ and then the Hurwicz profit is increasing in this segment with local optimal price μ.

Third, we focus on the local optimal price in $p \in [\mu, \frac{\mu}{1-\frac{m_2}{\mu^2}}]$. Note $h_3(p) = \lambda(p-c)\frac{(p-\mu)^2[1-\frac{m_2}{\mu^2}-\frac{m_2}{\mu(p-\mu)}]^2}{m_1-\frac{2m_2}{\mu}(p-\mu)+(p-\mu)^2(1-\frac{m_2}{\mu^2})} + (1-\lambda)(p-c)(1-\frac{m_2}{\mu^2})$ is the expected profit in Hurwicz criterion. Let $y = p - \mu \in [0, \frac{m_2}{\mu(1-\frac{m_2}{\mu^2})}]$. Then we will show $h_3(y)$ as well as $h_3(p)$ is unimodal or monotonous. We have $\frac{\mathrm{d}h_3(y)}{\mathrm{d}y} = \lambda\frac{(\frac{m_2}{\mu}-(1-\frac{m_2}{\mu^2})y)H(y)}{[m_1-\frac{2m_2}{\mu}y+y^2(1-\frac{m_2}{\mu^2})]^2} + (1-\lambda)(1-\frac{m_2}{\mu^2}) = \frac{\lambda(\frac{m_2}{\mu}-(1-\frac{m_2}{\mu^2})y)H(y)+(1-\lambda)(1-\frac{m_2}{\mu^2})[m_1-\frac{2m_2}{\mu}y+y^2(1-\frac{m_2}{\mu^2})]^2}{[m_1-\frac{2m_2}{\mu}y+y^2(1-\frac{m_2}{\mu^2})]^2}$, where $H(y) = -(1-\frac{m_2}{\mu^2})^2y^3 + 3\frac{m_2}{\mu}(1-\frac{m_2}{\mu^2})y^2 - 3m_1(1-\frac{m_2}{\mu^2})y + 2(\mu-c)\frac{m_2^2}{\mu^2} + \frac{m_1m_2}{\mu} - 2(1-\frac{m_2}{\mu^2})(\mu-c)m_1$. The numerator is a quartic function, which can be solved by Cardano's formula.

Last, when $p \geq \frac{m_2}{\mu(1-\frac{m_2}{\mu^2})}$, the expected profit is $(1-\lambda)U(p)$ and thus has the same shape as $U(p)$. Similarly, we can derive the closed-form candidates. $\square$

Proof of Proposition 10. Due to the limited number of pages, we put it into Appendix B. $\square$