Polynomial and exponential bounds for Ramsey numbers

Pham Hy Hieu, Le Viet Hai

Abstract. In the 19^{th} century, Frank Ramsey introduced the definition of Ramsey's number followed by many forms of theorems in Ramsey theory. These theorems are usually stated in the form: *For $t, k, n \in \mathbb{N}$ and sufficiently large $m \in \mathbb{N}$, if the k-tuples of the set M of cardinality m are t-colored then there exists $M' \subset M$ of cardinality n with all k-tuples of M' have the same color.* In this article we first introduce some well-known bounds for Ramsey's number of Erdos and Szekeres, then establish our two results about the lower bound of Ramsey numbers by imitating the method of them.

1 Introduction:

The purest definition of Ramsey numbers involves the 2-coloring of complete graphs.

Definition 1.1. ***(Ramsey's numbers)***
For $m, n \in \mathbb{N}^*$, $R(m, n) = R$ defines the minimum positive integer such that if we color every edge of K_R by two colors red and blue then there is either a mono-chromatic K_m or a mono-chromatic K_n.

An equivalent definition of Ramsey numbers involves the *independent sets*, which are subgraphs induced by a graph in which there is no edge and the *cliques*, which are complete subgraphs. In this perspective, $R(m, n)$ is the minimum positive integer R such that a graph with R vertices contains either a $\overline{K_m}$, which is an independent set of size m or a K_n, which is a clique of size n.

In generalization, the Ramsey number is defined as followed:

Definition 1.2. ***(General Ramsey's numbers)***
Consider two graphs G_1 and G_2. Ramsey number $R(G_1, G_2)$ is the minimum positive integer R such that any graph with R vertices contains either a subgraph isomorphic to G_1 or a subgraph isomorphic to $\overline{G_2}$ or equivalently, if every edge of K_R is colored by two color then there is either a subgraph of the first colored, say RED, isomorphic to G_1 or of the second color, say BLUE, isomorphic to G_2.

Obviously, the definition of simple Ramsey numbers is obtained by consider $(G_1, G_2) = (K_m, K_n)$. To define a closed formula for $R(G_1, G_2)$ is impossible due to the flexibility of graphs G_1 and G_2, but even a closed formula for $R(m, n)$, or $R(K_m, K_n)$, where G_1 and G_2 are so well defined, is a dream of human. Very few values of $R(m, n)$ is known and most are calculated in simple manners. However, there are bounds for $R(m, n)$ that are being improved significantly in recent decades. An old lower bound for Ramsey numbers appears in the polynomial form

Theorem 1.1. *For $k, l \geq 2$ we have*

$$R(k, l) \geq 2kl - 3k - 2l + 6$$

We will slightly improve this bound for $k, l \geq 5$. Also, we will set a lower bound for $R(k,l)$ in exponential function by imitating the amazingly simple probabilistic method of Erdos.

2 Upper and lower bounds for basic Ramsey numbers:

First we establish an upper bound for Ramsey numbers. This bound was first proved by Paul Erdos using the following result.

Theorem 2.1. *For $m, n \in \mathbb{N}$ and $m, n \geq 2$, we have:*

$$R(m,n) \leq R(m-1,n) + R(m,n-1)$$

Proof. Consider a complete graph K_s where $s = R(m-1,n) + R(m,n-1)$, obviously every vertex of K_s has the degree of $s-1$. Assume that in a 2-color of such graph, there is no mono-chromatic K_m of the first color, say RED, we will establish a mono-chromatic K_n of the second color, say BLUE. Indeed, since every vertex of K_s has degree $s-1$, there is either $R(m-1,n)$ red edges or $R(m,n-1)$ blue edges incident from a vertex.

If there are $R(m-1,n)$ red edges from a vertex, say A, consider these $R(m-1,n)$ vertices that are directly connected to A. By the definition of $R(m-1,n)$, if we 2-color the edges connect these vertices, there is either a red K_{m-1} or a blue K_n (where *"red K_{m-1}"* implies the K_{m-1} whose every edge is colored red). If there is a red K_{m-1}, along with the red edges from A to each vertex of this K_{m-1}, it establish a red K_m, which is contradicted to our assumption. Hence, there is a blue K_n and we are done.

If there are $R(m,n-1)$ blue edges from a vertex, apply the same argument and we obtain a blue K_n, which complete the proofs. □

Corollary 2.1.1. *For $m, n \in \mathbb{N}^*$ we have:*

$$R(m,n) \leq \binom{m+n-2}{m-1}$$

Proof. We use the method of induction on the sum $m+n$. The base cases $R(2,2)$ is trivial. Now assume that our result is true for $m+n=s$, consider cases when $m'+n'=s+1$, according to **Theorem 2.1.** we have

$$\begin{aligned}
R(m',n') &\leq R(m'-1,n') + R(m',n'-1) \\
&\leq \binom{(m'-1)+n'-2}{(m'-1)-1} + \binom{m'+(n'-1)-2}{m'-1} \text{ (since } (m'+n'-1=s)\text{)} \\
&= \binom{m'+n'-3}{m'-2} + \binom{m'+n'-3}{m'-1} \\
&= \binom{m'+n'-2}{m'-1} \text{ (by Pascal's identity)}
\end{aligned}$$

Thus the induction is completed. □

This upper bound is very weak, for instance $R(4,4) = 18$ but $\binom{6}{3} = 20$. Note that even a unit of improvement of the bound is precious since the algorithm of finding the graphs takes significantly huge time. We have a slightly sharper bound from the proof of the theorem above.

Corollary 2.1.2. *If both $R(m-1,n)$ and $R(m,n-1)$ are even then*

$$R(m,n) \leq R(m-1,n) + R(m,n-1) - 1$$

Now establish the lower bound for Ramsey numbers. We use the following lemmas:

Lemma 2.1. *For $k, l \geq 3$ we have*

$$R(k,l) \geq \max\{R(k-1,l) + 2l - 3; R(k,l-1) + 2k - 3\}$$

The proof of the lemma above based on the construction of a counter-example. If applying the *Lemma of hand shakes* we can obtain a stronger result in case k is odd.

Lemma 2.2. *For $k \geq 2$ and $l \geq 5$ we have*

$$R(2k-1,l) \geq \max\{4R(k-1,l) - 3; 4R(k,l-1) - 3\}$$

These two lemmas are included in [1]. Now we come to prove another lemma by the method of construction.

Lemma 2.3. *For $k \geq 3$ we have*

$$R(3,k) \geq 3(k-1)$$

Proof. We first introduce the notation F_n^k. This one denotes the graph of n vertices placed on a circle where each vertex is connected to the vertices whose distances to it do not exceed k. For instance, F_5^1 is the graph of vertices A, B, C, D, E placed on a circle clock-wise and the edges AB, BC, CD, DE, EA while F_5^2 is the complete graph K_5.

Now for the proof. We will cite that F_{3k-4}^{k-2} as a counter-example. Let v_i where $i = 1, 2, \cdots, 3k-4$ be the vertices of a F_{3k-4}^{k-2}. We define the *length* of the edge $\{i, j\}$ the distance between vertices i and j on the cycle. By definition, the vertex v_i is adjacent to $k-2$ nearest vertices in both directions. Therefore, in F_{3k-4}^{k-2}, the $2(k-2)$ lines that are adjacent to v_1 have length $1, 2, ..., k-2$. Moreover, there are $3k-4-2(k-2)-1 = k-1$ consecutive vertices of F_{3k-4}^{k-2} not adjacent to v_1. Obviously, the length of the lines joining these consecutive vertices do not exceed $k-2$.

First we prove that there is no K_k in F_{3k-4}^{k-2}. Assume that there is a K_k in F_{3k-4}^{k-2} then because of the symmetry, we can assume that v_1 is a vertex of this K_k. Thus, the other $k-1$ vertices of this K_k are among the $2(k-2)$ vertices of F_{3k-4}^{k-2} that are adjacent to v_1. Since we need $k-1$ vertices and in each direction of v_1, there is only $k-2$ vertices, by the *Principle of Pigeon-Holes*, there must be vertices on both directions of v_1 in this K_k. However, this poses a contradiction to the definition of F_{3k-4}^{k-2} since the line joining the furthest vertices on both sides of v_1 has the length at least $k-1$. Hence, there is no K_k in F_{3k-4}^{k-2}.

Now we prove that there is no K_3 in $\overline{F_{3k-4}^{k-2}}$. Indeed, since in $\overline{F_{3k-4}^{k-2}}$, each vertex, for instance v_1 only is adjacent to $k-1$ consecutive points whose lines joining them to v_1 has the length more than $k-2$. However, the lines joining these points are all in F_{3k-4}^{k-2}, since the length of every line of them does not exceed $k-2$, and hence, there is no edge joins the two vertices which are both adjacent to v_1, and vice versa, for every v_i. Therefore, there is no K_3 in $\overline{F_{3k-4}^{k-2}}$.

Thus, there is neither K_k in F_{3k-4}^{k-2} nor K_3 in $\overline{F_{3k-4}^{k-2}}$. From this we deduce that $R(3,k) > 3k-4$ or $R(3,k) \geq 3(k-1)$. $\square$

With **Lemma 2.3.** we can establish a polynomial lower bound for $R(k,l)$, which is presented in the following theorem.

Theorem 2.2. *For $5 \leq k \leq l$ we have*

$$R(k,l) \geq 2kl - 3k + 2l - 12$$

Proof. By continuously applying **Lemma 2.1., Lemma 2.2.** and **Lemma 2.3.**, we have

$$\begin{aligned}
R(k,l) &\geq R(l,k-1) + 2l - 3 \\
&\geq R(l,k-2) + 2(2l-3) \\
&\geq ... \\
&\geq R(l,k-i) + i(2l-3) \\
&\geq ... \\
&\geq R(l,5) + (k-5)(2l-3) \\
&\geq 4R(3,l-1) - 3 + (k-5)(2l-3) \\
&\geq 4 \cdot 3(l-2) - 3 + (k-5)(2l-3) \\
&= 2kl - 3k + 2l - 12
\end{aligned}$$

This lower bound is sharp for $R(3,3), R(3,4)$ and $R(4,4)$, but for greater k and l this bound is very weak. Sharper results for some cases have been found in [3] for only special cases.

Theorem 2.3. *For $k \geq 2$ we have*

$$R(k,k) \geq 2^{\frac{t}{2}}$$

Even sharper bound is found for particular case $(l,k) = (3,k)$ for $k \geq 2$.

Theorem 2.4. *For $k \geq 2$ we have*

$$c_1 \cdot \frac{k^2}{\log k} \leq R(3,k) \leq c_2 \cdot \frac{k^2}{\log k}$$

Through years, the constants c_1, c_2 are improved very slightly. This again shows the difficulty of the problem. However, we hope that the bound can be improved as followed.

Conjecture 2.1. *For $k \geq 3$ there is positive real number c such that*

$$R(3,k) \geq c \cdot \left(\frac{k^2}{\log k}\right)^2$$

Imitating the Sieve method, we can prove a more general result for $k,l \in \mathbb{N}^*$.

Theorem 2.5. *For $k,l \geq 2$ we have*

$$R(k,l) \geq \min\{2^{\frac{m-1}{2}}; 2^{\frac{n-1}{2}}\}$$

Proof. The proof for this theorem is not constructive. Put $R := R(k,l)$. We will use the method of probability. Let's consider the probability for a 2-coloring of K_R to consider a mono-chromatic K_k in red. Clearly, the number of ways of choosing k vertices is $\binom{R}{k}$. For each k vertices, we have $\binom{k}{2}$ edges and hence the probability of all of them to be red is $\frac{1}{2^{\binom{k}{2}}}$. Therefore the appropriate probability is

$$\frac{\binom{R}{k}}{2^{\binom{k}{2}}}$$

For the probability of K_R to have a mono-chromatic K_l in blue, similarly, we get

$$\frac{\binom{R}{l}}{2^{\binom{l}{2}}}$$

There exists a 2-coloring of K_R which does not satisfy the Ramsey condition if and only if the probability of K_R to have either a mono-chromatic red K_k or a mono-chromatic blue K_l is less then 1. Thus we need the following inequality to hold

$$\frac{\binom{R}{k}}{2^{\binom{k}{2}}} + \frac{\binom{R}{l}}{2^{\binom{l}{2}}} < 1$$

Consider $R = \lfloor 2^{\frac{k-1}{2}} \rfloor$ we have

$$\begin{aligned}
\frac{\binom{R}{k}}{2^{\binom{k}{2}}} &= 2^{-\frac{k(k-1)}{2}} \cdot \frac{R(R-1)\cdots(R-k+1)}{k!} \\
&\le 2^{-\frac{k(k-1)}{2}} \cdot \frac{R^k}{k!} \\
&\le \frac{1}{2} \cdot 2^{-\frac{k(k-1)}{2}} \cdot 2^{\frac{k(k-1)}{2}} \\
&= \frac{1}{2}
\end{aligned}$$

Similarly when $R = \lfloor 2^{\frac{l-1}{2}} \rfloor$ we have

$$\frac{\binom{R}{l}}{2^{\binom{l}{2}}} \le \frac{1}{2}$$

Therefore when $R < \min\{2^{\frac{m-1}{2}}; 2^{\frac{n-1}{2}}\}$ we have

$$\frac{\binom{R}{k}}{2^{\binom{k}{2}}} + \frac{\binom{R}{l}}{2^{\binom{l}{2}}} \le 2 \cdot \max\{\frac{\binom{\lfloor 2^{\frac{k-1}{2}} \rfloor}{k}}{2^{\binom{k}{2}}}; \frac{\binom{\lfloor 2^{\frac{l-1}{2}} \rfloor}{l}}{2^{\binom{l}{2}}}\} \le 1$$

Hence, even we can not point out exactly what is the counter-example, we are sure about its existence. Due to this, we need

$$R(k,l) \ge \min\{2^{\frac{m-1}{2}}; 2^{\frac{n-1}{2}}\}$$

The theorem is proved. $\square$

The bound in exponential function is established. However, we rise the following conjecture

Conjecture 2.2. *There exists positive real number C such that for $k, l \geq 2$ we have*

$$R(k,l) \geq C \cdot \min\{k \cdot 2^{\frac{k-1}{2}}; l \cdot 2^{\frac{l-1}{2}}\}$$

3 References:

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