Math Capstone Project

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After setting up the basic framework, we now introduce the noise to the the current system for a more realistic simulation. The governing equation, hence, turns to be $\gamma\dot{x} = -Kx + \xi$ (3.1), where ξ represents the noise. In Physics, we regard these noises as constant kicks. These kicks guarantee the particle will never stop at a fixed point, even at its equilibrium state. We assume the noise in our model has zero mean, i.e., $<\xi>= 0$, considering noise's mean reversion property in many real-life cases. We expect noises not to be a self-perpetuating force in favor of any specific direction but instead as a random disturbance to the system. In our model, noises oscillate with positive and negative signs and are supposed to be a neutral force in the sense that they overall cancel out each other's effect with a zero mean. We additionally play the universe's role for now and leave further explanations to this article's next section. At this moment, we directly assume a specified autocorrelation function to the noise to ease the discussion. That is

$$R_{\xi}(t,t') =< \xi(t)\xi(t') >= \Gamma\delta(t-t') \tag{3.2}$$

where $\Gamma = 2\gamma k_B T$ with $k_B T$ being the thermal energy.

The particle now moves consistently due to the introduction of noise. Abandoning the world with a still particle, we are now interested in studying the distribution of this particle's trajectory under its equilibrium state. But to achieve our goal, firstly, we need to find the particle's trajectory's expression. Since we can regard $\int_{(i-1)h}^{ih} \dot{x}(t)\,\mathrm{d}t$ as Δdistance = velocity $\times$ Δtime, we write,

$$\begin{aligned}
x_i &= x_{i-1} + \int_{(i-1)h}^{ih} \dot{x}(t)\,\mathrm{d}t \\
&= x_{i-1} + \int_{(i-1)h}^{ih} (-\frac{k}{r}x + \frac{\xi(t)}{r})\,\mathrm{d}t && \textit{(by substitutions of (3.1))} \\
&= x_{i-1} - \frac{k}{r}\int_{(i-1)h}^{ih} x(t)\,\mathrm{d}t + \frac{1}{r}\int_{(i-1)h}^{ih} \xi(t)\,\mathrm{d}t
\end{aligned}$$

For the second term in the above step, we can again solve it by applying Forward Euler Method. $x(t)$ is a solution of the ODE. It is, hence, differentiable.

Otherwise, it will be meaningless to discuss the ODE when $\dot{x}$ does not exist and do so other higher order derivative terms, e.g., $\ddot{x}$ as a consequence. $x(t)$'s differentiability implies $x(t)$ is a continuous function, which satisfies the condition of using the first fundamental theorem of calculus. We let $F = \int_0^x x(t)\,\mathrm{d}t$. The theorem states that F, an indefinite integral is also an antiderivative of $x(t)$, i.e. $F'(t) = x(t)$. Along with Newton–Leibniz axiom, we use Talyor expansion to further simplify the integral to,

$$\int_{(i-1)h}^{ih} x(t)\,\mathrm{d}t = F(ih) - F((i-1)h) = F'((i-1))h = x(i-1)h$$

Introducing a new notation $\zeta_i = \int_{(i-1)h}^{ih} \xi(t)\,\mathrm{d}t$ and defining ζ_i on the interval i finally gives the core equation we used to perform the iteration task in Python,

$$x_i = x_{i-1} - \frac{k}{r} x_{i-1} h + \frac{\zeta_i}{r}$$