Math Capstone Project

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Part I:

Given $m\ddot{x} = -\gamma\dot{x} - Kx$ (1.1) , we can set $p = \dot{x}$ (1.2), which further gives $\dot{p} = \ddot{x}$ (1.3). We then substituted these new notations into (1.1) and obtained $\dot{p} = (-\gamma p - Kx)/m$ (1.4) after some simple algebraic operations.

Given the initial position of the Brownie motion particle, $x(0) = x_0$ and its initial velocity, $p(0) = p_0$, we are interested to approximate the positions of the particle at different values of t. By applying the Taylor expansion, we have,

$$
\begin{aligned}
x(h) &= x(0) + \dot{x}(0)h \\
&= x(0) + p(0)h && \textit{(by 1.2)} \\
&= x_0 + p_0 h && \textit{(by IVP condition)} \\
p(h) &= p(0) + \dot{p}(0)h \\
&= p(0) + ((-\gamma p(0) - Kx(0))/m)h && \textit{(by 1.4)} \\
&= p_0 + h(-\gamma p_0 - Kx_0)/m && \textit{(by IVP condition)} \\
x(2h) &= x(h) + \dot{x}(h)h \\
&= x(h) + p(h)h && \textit{(by 1.2)} \\
&= x(h) + h(p_0 + h(-\gamma p_0 - Kx_0)/m) && \textit{(by substitution of } p(h)\textit{)}
\end{aligned}
$$

From the above procedure, we moved from time 0 to time t. WLOG, by seting $t_{n+1} = t_n + h$, we can further generalised such a procedure to,

$$
\begin{aligned}
x(t_n) &= x(t_{n-1}) + \dot{x}(t_{n-1})h \\
&= x(t_{n-1}) + p(t_{n-1})h
\end{aligned}
$$

[*Notice that* $x(t_{n-1})$ *and* $p(t_{n-1})$ *were already obtained from the previous iteration.*]

$$\begin{aligned} p(t_n) &= p(t_{n-1}) + \dot{p}(t_{n-1})h \\ &= p(t_{n-1}) + h(-\gamma p(t_{n-1}) - Kx(t_{n-1}))/m \\ x(t_{n+1}) &= x(t_n) + \dot{x}(t_n)h \\ &= x(t_n) + p(t_n)h \end{aligned}$$

We can observe that in the final expression of $x(t_{n+1})$, we have already calculated the $x(t_n)$ and $p(t_n)$ from the previous steps. That makes our idea of approximating $x(t)$ for all t using Python's iterative function practicable.

Forward Euler Method in Python

```python
# Reference:
# https://blog.csdn.net/weixin_42376039/article/details/86485817
# https://www.epythonguru.com/2020/07/second-order-differential-equation.html
# https://apmonitor.com/pdc/index.php/Main/SolveDifferentialEquations

import matplotlib.pyplot as plt
# Ouput images with higher resolutions
import matplotlib_inline
matplotlib_inline.backend_inline.set_matplotlib_formats('svg')
from matplotlib.offsetbox import AnchoredText
from scipy.integrate import odeint
import numpy as np

# Initialized the parameters
h = 0.001 # time step used
interval_len = 10 # the length of the interval
# total num of observed time stamps N = interval_len / h
# And I will chop off the decimal places after N in cases N is a non-integer

m = 1 # mass
K = 2 # spring constant
gamma = 10 # frictional coefficient

# x_n := x(t_n)
# dx_n := the first derivative of x(t_(n-1))
x_0 = 0 # initial position of the particle
p_0 = 100 # initial velocity of the particle

# initialize the values for iterations
x_n = x_0
p_n = p_0

N_ls = [0] # The i list, (which are the index of ti, the observed timestamp)
x_n_ls = [x_n] # contains the trajectory of X at different time ti, i.e. X(t)

# N = truncate_to_int(interval_len/h) + 1
for N in range(1, int(interval_len/h)+1):
    x_n = x_n + p_n*h
    p_n = p_n + h*(-gamma*p_n - K*x_n)/m
    N_ls.append(N)
    x_n_ls.append(x_n)
```

```
# change the plot's labels in #
# i (the ith observation) to ti (observed timestamps) #
time_axis = [i * h for i in N_ls]

## Notice: for the illustration purpose ##
## points are plotted with stepsize = 1000 ##
## to contrast its discreteness with the continuous line ##
# list[start:stop:step]
# set the step size for ploting
step = 1000 # (constrast the discrete points with the continous line)
plt.plot(time_axis[0:len(time_axis):step], x_n_ls[0:len(time_axis):step],
         color='red', marker = ".", linestyle='None',
         markersize = 10, label='Numerical')
```

To set up a valid criterion to verify our numerical approximation, we will also derive the analytical solutions for (1.1). Moving terms in (1.1) gives us $m\ddot{x} + \gamma\dot{x} + Kx = 0$ (1.5). Since m, γ and K are all constants, we recognize (1.5) as a homogeneous, linear, second order differential equation. The characteristic equation of (1.5), hence, is $mw^2 + rw + K = 0$. Solving this quadratic equation gives $w = (-\gamma \pm \sqrt{\gamma^2 - 4mK})/2m$ so we will further separate our discussion into three cases.

<u>Case 1:</u> When $r^2 - 4mK > 0$, then the roots of the characteristic equation, w_1 and w_2 are real and distinct. The general solution of (1.5) is, hence, $x(t) = c_1 e^{w_1 t} + c_2 e^{w_2 t}$ (1.6). Given $x(0) = x_0$, we obtained $c_1 + c_2 = x_0$ (1.7). Given $\dot{x}(0) = p(0) = p_0$, we first took first derivative w.r.t t on both sides of (1.6) and attained $\dot{x} = c_1 w_1 e^{w_1 t} + c_2 w_2 e^{w_2 t}$. Further evaluating $\dot{x}$ at point 0 gives, $\dot{x}(0) = c_1 w_1 + c_2 w_2 = p_0$ (1.8). By equating (1.7) and (1.8) and using the Cramer's rule, we obtained,

$$\begin{cases} c_1 + c_2 = x_0 \\ c_1 w_1 + c_2 w_2 = p_0 \end{cases} \Rightarrow \begin{cases} c_1 = (x_0 w_2 - p_0)/(w_2 - w_1) \\ c_2 = (p_0 - w_1 x_0)/(w_2 - w_1) \end{cases}$$

The general solution of (1.5) under IVP condition is, therefore, $x(t) = (x_0 w_2 - p_0)/(w_2 - w_1) e^{w_1 t} + (p_0 - w_1 x_0)/(w_2 - w_1) e^{w_2 t}$, where $w_{1,2} = (-\gamma \pm \sqrt{\gamma^2 - 4mK})/(2m)$.

<u>Case 2:</u> When $\gamma^2 - 4mK = 0$, then the roots of the characteristic equation, w_1, w_2 are real and but repeated (i.e.$w_1 = w_2 = w$). The general solution of (1.5) is, hence, $x(t) = c_1 e^{wt} + c_2 t e^{wt}$ (1.9). Given $x(0) = x_0$, we obtained $c_1 = x_0$ (2.0). Given $\dot{x}(0) = p(0) = p_0$, we first took first derivative w.r.t t on both sides of (1.9) and attained $\dot{x} = c_1 w e^{wt} + c_2(wte^{wt} + e^{wt})$. Further evaluating $\dot{x}$ at point 0 gives, $\dot{x}(0) = c_1 w + c_2 = p_0$ (2.1). By equating (2.0) and (2.1), we obtained $c_2 = p_0 - x_0 w$. The general solution of (1.9), hence, is $x(t) = x_0 e^{wt} + (p_0 - x_0 w) t e^{wt}$, where $w = (-\gamma + \sqrt{\gamma^2 - 4mK})/(2m)$.

<u>Case 3:</u> When $\gamma^2 - 4mK < 0$, then the roots of the characteristic equation, w_1 and w_2 are complex and it is in the form of $w_{1,2} = \lambda \pm \mu i$. The general solution

of (1.5) is, hence, $x(t) = c_1 e^{\lambda t} cos(\mu t) + c_2 e^{\lambda t} sin(\mu t)$(2.2). Given $x(0) = x_0$, we obtained $c_1 = x_0$ (2.3). Given $\dot{x}(0) = p(0) = p_0$, we first took first derivative w.r.t t on both sides of (2.2) and attained
$\dot{x} = c_1 e^{\lambda t}[-\mu \sin(\mu t) + \lambda \cos(\mu t)] + c_2 e^{\lambda t}[\mu \cos(\mu t) + \lambda \sin(\mu t)]$. Further evaluating $\dot{x}$ at point 0 gives, $\dot{x}(0) = c_1\lambda + c_2\mu = p_0$ (2.4). By equating (2.3) and (2.4), we obtained $c_2 = (p_0 - x_0\lambda)/\mu$. The general solution of (1.9), hence, is $x(t) = x_0 e^{\lambda t} cos(\mu t) + (p_0 - x_0\lambda) e^{\lambda t} sin(\mu t)/\mu$, where
$w_{1,2} = (-\gamma \pm \sqrt{\gamma^2 - 4mK})/(2m) = \lambda \pm \mu i$.

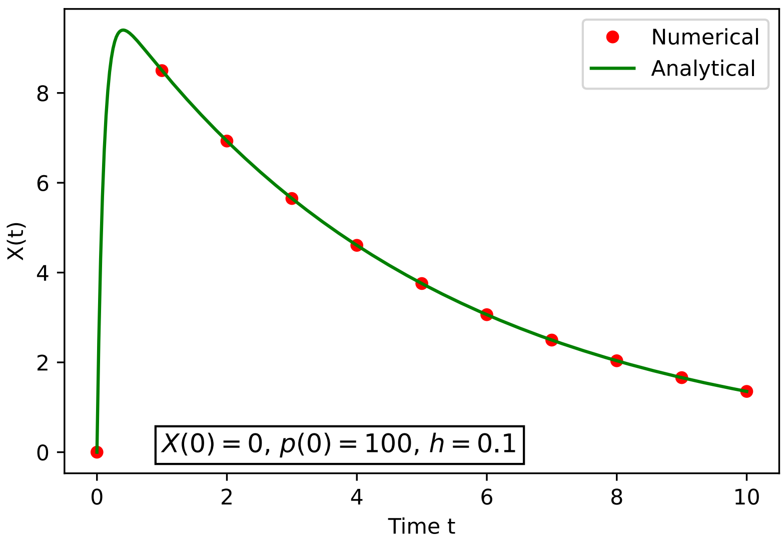


Figure 1: $m = 1$, $K = 2$, $\gamma = 10$ (Overdamped)

An Analytical Approach to solve ODEs in Python

```
## We will plot the analytical ODE sol in a continous line ##
## to represent the ground truth ##
# odeint() can define more than one 1st order differential equations
    # And we can solve even higher order ODEs by using multiple 1st order ODEs.
        def derivatives(initial_values, time_interval):
            x_0 = initial_values[0]
            dxdt_0 = initial_values[1]
            sec_dxdt_0 = -gamma/m*dxdt_0-K/m*x_0
            return(dxdt_0, sec_dxdt_0)
        initial_values = [x_0, p_0]
        time_interval = np.linspace(0, interval_len, N_boosted)
        ODE_sol = odeint(derivatives, initial_values, time_interval)
        ODE_sol = ODE_sol[:,0]
```

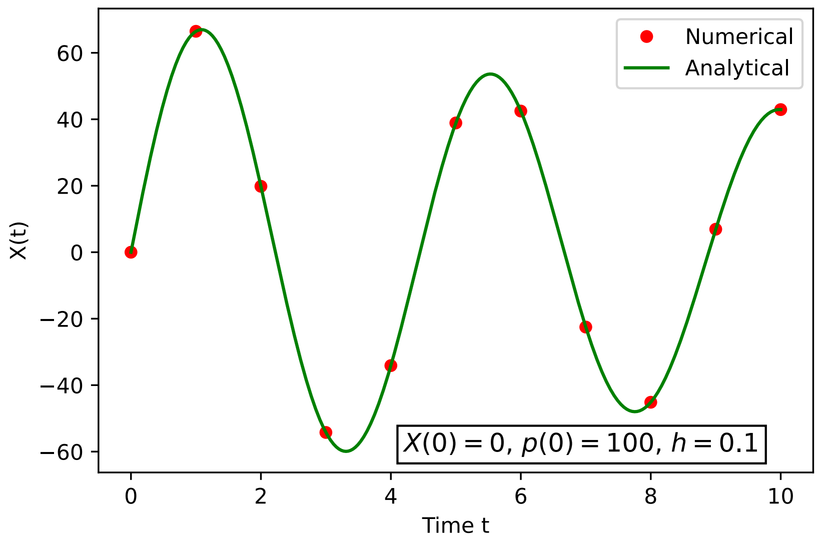


Figure 2: $m = 1,\ K = 2,\ \gamma = 10$ (Underdamped)

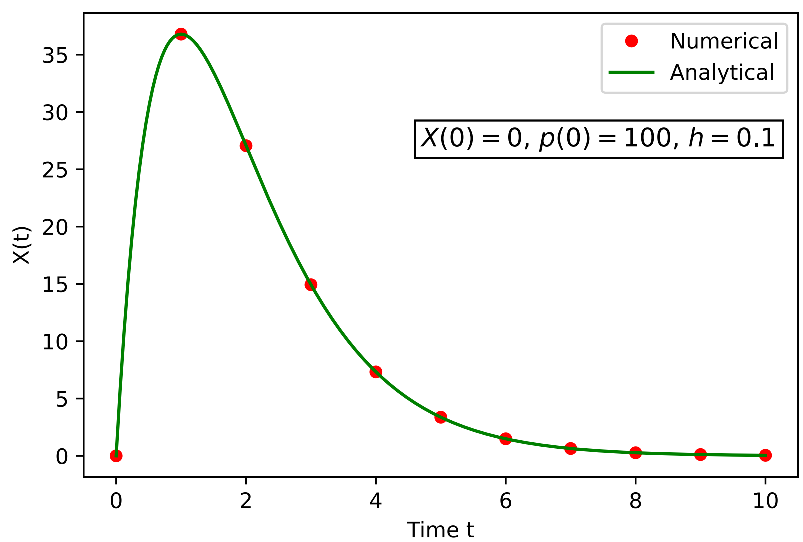
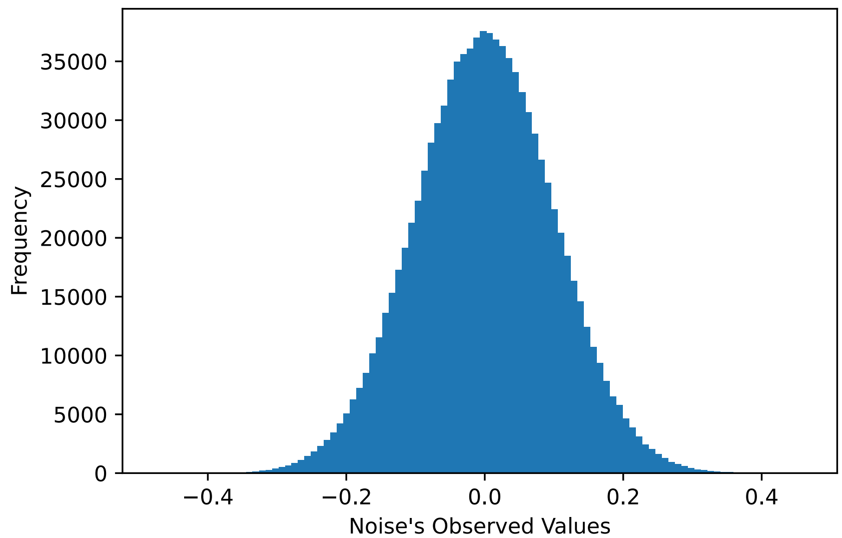
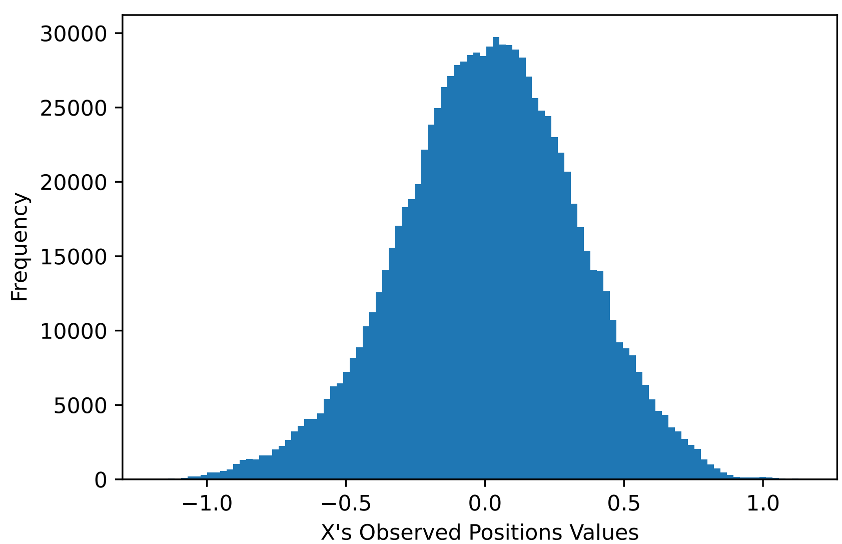


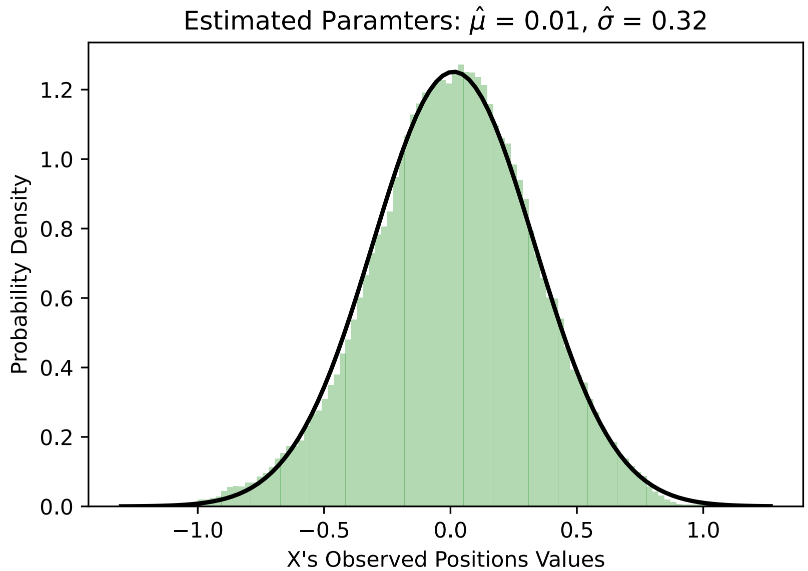
Figure 3: $m = 1,\ K = 1,\ \gamma = 2$



35000
30000
25000
20000
15000
10000
5000
0
Frequency
−0.4
−0.2
0.0
0.2
0.4
Noise's Observed Values



30000
25000
20000
15000
10000
5000
0
Frequency
−1.0
−0.5
0.0
0.5
1.0
X's Observed Positions Values




```
    plt.plot(time_interval,ODE_sol, linestyle ='-',
             color='green', label='Analytical' )
    plt.text(1, 0, '$X(0) = 0$, $p(0) = 100$, $h = 0.1$', fontsize = 12,
              bbox = dict(facecolor='none', edgecolor='black', pad = 3))
    plt.xlabel("Time t")
    plt.ylabel("X(t)")
    plt.legend()
    plt.savefig(r'C:\Users\alexi\Desktop\Plots\Overdamped.svg')
```

Introducing Noise to X's trajectory in Python

```
# Initialized the parameters
h = 0.001 # time step
interval_len = 1000 # the length of the interval

m = 2 # mass
K = 5 # spring constant
gamma = 10 # frictional coefficient

mu, sigma = 0, 0.1 # mean and standard deviation
noise_sample = np.random.normal(mu, sigma, N)
plt.hist(noise_sample, bins = 100)
plt.xlabel("Noise's Observed Values")
plt.ylabel("Frequency")
plt.show()

x_noise_i = x_0
x_noise_ls = [x_noise_i]
for i in range(1, int(interval_len/h)+1):
    x_noise_i = x_noise_i - K/gamma*x_noise_i*h + noise_sample[i-1]/gamma
    x_noise_ls.append(x_noise_i)
```

```
plt.hist(x_noise_ls, bins = 100)
plt.xlabel("X's Observed Positions Values")
plt.ylabel("Frequency")
plt.show()
```

$$x_i = x_{i-1} + \int_{(i-1)h}^{ih} \dot{x}(t)\,\mathrm{d}t$$

[we can regard $\int_{(i-1)h}^{ih} \dot{x}(t)\,\mathrm{d}t$ as distance = velocity × time]

$$\begin{aligned}
&= x_{i-1} + \int_{(i-1)h}^{ih} (-\frac{k}{r}x + \frac{\xi(t)}{r})\,\mathrm{d}t \\
&= x_{i-1} - \frac{k}{r}\int_{(i-1)h}^{ih} x(t)\,\mathrm{d}t + \frac{1}{r}\int_{(i-1)h}^{ih} \xi(t)\,\mathrm{d}t \\
&= x_{i-1} - \frac{k}{r}x_{i-1}h + \frac{\xi_i}{r}
\end{aligned}$$

Find the Best Fit Normal Distribution to $X(t)$

```
# Reference:
# https://stackoverflow.com/questions/20011122/fitting-a-normal-distribution-to-1d-data
import numpy as np
from scipy.stats import norm

## Find the best fitted normal distribution to X(t) ##
mu, std = norm.fit(x_noise_ls) # x_noise_ls: contains the trajectory of X(t)

## Plot the histogram of X(t) ##
# density: transfer the frequence on y-axis into probability density
# alpha: constrols the histogram's transparency
plt.hist(x_noise_ls, bins = 100, density=True, alpha=0.3, color='green')

xmin, xmax = plt.xlim() # set the boundary for x_axis
# larger N is, the higher the resolutions of (smoother) #
# the fitted line of the PDF values curve #
selected_pts_on_x_axis = np.linspace(xmin, xmax, 100)
# 100 is the number of equally spaced points, N

corresponding_pdf_values = norm.pdf(selected_pts_on_x_axis , mu, std)
plt.plot(selected_pts_on_x_axis, corresponding_pdf_values, 'k', linewidth=2)
# k:= (col = black)
title = "Estimated Paramters: $\hat{\mu}$ = %.2f, $\hat{\sigma}$ = %.2f" % (mu, std)
# %.2f: rounds up to 2 decimal places

plt.title(title)
plt.xlabel("X's Observed Positions Values")
plt.ylabel("Probability Density")
plt.savefig(r'C:\Users\alexi\Desktop\Plots\Bestfit.svg')
```