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Modified leaky competing accumulator model of decision making with multiple alternatives: the Lie-algebraic approach

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In this communication, based upon the stochastic Gompertz law of population growth, we have reformulated the Leaky Competing Accumulator (LCA) model with multiple alternatives such that the positive-definiteness of evidence accumulation is automatically satisfied. By exploiting the Lie symmetry of the backward Kolmogorov equation (or Fokker–Planck equation) assoicated with the modified model and applying the Wei–Norman theorem, we have succeeded in deriving the *N*-dimensional joint probability density function (p.d.f.) and marginal p.d.f. for each alternative in closed form. With this joint p.d.f., a likelihood function can be constructed and thus model-fitting procedures become feasible. We have also demonstrated that the calibration of model parameters based upon the Monte Carlo simulated time series is indeed both efficient and accurate. Moreover, it should be noted that the proposed Lie-algebraic approach can also be applied to tackle the modified LCA model with time-varying parameters.

Among the current models of decision making, the *Leaky Competing Accumulator* (LCA) model[1–6] has become fairly popular recently because it has been shown to account for a variety of behavioural datasets (mostly) related to two alternatives. In accordance with the model, evidence accumulation continues until an accumulator reaches a certain threshold level of activation, and a decision is made. Mathematically, evidence accumulation in this N-alternative model is described by the Ito stochastic differential equations (s.d.e.'s):

$$\begin{aligned} dx_i &= \left(I_i - \kappa x_i - \beta \sum_{j \neq i} x_j \right) dt + \xi \, dW_i \\ &= \left\{ I_i - (\kappa - \beta) x_i - \beta N \left(\frac{1}{N} \sum_{j=1}^{N} x_j \right) \right\} dt + \xi \, dW_i \end{aligned} \tag{1}$$

for $x_i \geqslant 0$ and $i = 1, 2, 3, \ldots, N$. Here dx_i is the change in activation of accumulator i, I_i is the input, dt is the time step size, ξ refers to the noise and dW_i denotes a standard Weiner process. In addition, κx_i and $\beta \sum_{j \neq i} x_j$ quantify the loss of activation of accumulator i due to leakage (sometimes called decay) and inhibition by the other accumulators, respectively. Unfortunately, the LCA model does not have a known likelihood function[7], and the only methods available to fit the model to data are simulation-based, i.e. the LCA model has to generate simulated data for each proposed set of parameters in order to calculate any measure of fit. Hence, model-fitting procedures are extremely slow, and a thorough investigation of model-fitting procedures, recoverability and identifiability of the LCA model has not been performed for multi-alternative cases[3]. Moreover, recent research has found that the LCA model suffers from an instability problem in parameter recovery studies so that inferences made directly on the estimated parameter values are unreliable and of little meaning when applied to real data[3–6].

Beyond question, in addition to the complicated couplings among the set of s.d.e.'s, a major hurdle in deriving a closed-form N-dimensional joint probability density function (p.d.f.) is the restriction that evidence accumulation is positive definite, i.e. $\{x_i \geqslant 0\}$. It has been observed that under reasonable parameter ranges (i.e. when the inputs $\{I_i\}$ are not too small) the effect of neglecting the restriction is insignificant[1,8–11] , so it is justifiable to

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drop the constraint. The resultant model is sometimes called the Linear LCA model[12]. The special case of two alternatives can then be modelled by an Ornstein-Ulenbeck (OU) process:

$$dx = \{(I_1 - I_2) - (\kappa - \beta)x\}dt + \sqrt{2}\xi\, dW \tag{2}$$

where $x \equiv x_1 - x_2$. The OU process is a well-known process and its properties can be easily found in the literature (see, e.g.[13]). Thus, most of current research work on decision making has focused on choices between two alternatives. It is obvious that in this OU process the leakage parameter κ and inhibition parameter β cannot be calibrated separately for the OU process depends upon their difference only. Likewise, in order that the current models of decision making are of relevance to real life decisions, they must be applicable to understanding decisions among more than two alternatives. Accordingly, it is the aim of this communication to propose a new reformulation of the LCA model such that not only the positive-definiteness of evidence accumulation can be fulfilled automatically but also the N-dimensional joint p.d.f. can be derived in closed form.

First of all, by drawing a similarity between evidence accumulation and population growth, we propose to model the evidence accumulation by a generalization of the stochastic version of the Gompertz law of population growth. If $y(t)$ is the size of the cell at time t, the Gompertz law models the cell growth by the equation[14–17]:

$$\frac{dy}{y} = \left(A_1 - A_2 \ln y\right)dt \quad \text{for} \quad A_2 > 0, \tag{3}$$

where A_1, the intrinsic growth rate of the cell, is a parameter related to the initial mitosis rate and A_2, the growth deceleration factor, is related to the antiangiogenic processes. However, it should be stressed that quite often discrepancies exist between clinical data and theoretical predictions, due to more or less intense environmental fluctuations. Thus, a better model is needed to reflect the external randomness that affects the cell growth behaviour. The simplest stochastic version of the Gompertz law can be derived via assuming that the growth deceleration factor A_2 does not change while the variability of environmental conditions induces fluctuations in the intrinsic growth rate A_1[18]. By assuming that the intrinsic growth rate varies in time according to

$$\theta(t) = A_1 + \sigma\varepsilon(t), \tag{4}$$

where A_1 is the constant mean value of $\theta(t)$, σ is the diffusion coefficient, and $\varepsilon(t)$ is a Gaussian white noise process, the proposed stochastic version of the Gompertz law is defined by the s.d.e.:

$$\frac{dy}{y} = \left(A_1 - A_2 \ln y\right)dt + \sigma\, dW, \tag{5}$$

where dW denotes the standard Wiener process. By Ito's lemma, this s.d.e. implies that the exponent $x \equiv \ln y$ follows the OU process:

$$dx = \left\{\left(A_1 - \frac{1}{2}\sigma^2\right) - A_2 x\right\}dt + \sigma\, dW \tag{6}$$

with the long term mean $\left\{A_1 - (1/2)\sigma^2\right\}/A_2$. It is obvious that the positive-definiteness of the size y of the cell is automatically fulfilled. This stochastic Gompertz model has been popularly applied to model tumour cell growth and similate the effects of a therapy recently[19–23]. In addition, this model is commonly known as the Schwartz model for modelling the mean-reverting stochastic behaviour of commodity prices in finance[24]. Beyond question, one can readily recognise that Eq. (6) is identical to the s.d.e. describing the evidence accumulation of each alternative in the absence of inhibition by the other accumulators. As a consequence, the complementary approach of letting x_i in Eq. (1) represent the logarithm of evidence accumulation y_i of accumulator i rather than the evidence accumulation presents a simple way to deal with the positive-definiteness of evidence accumulation. By Ito's lemma, the corresponding set of s.d.e.'s which govern the evidence accumulation of each accumulator are thus given by

$$\begin{aligned}\frac{dy_i}{y_i} &= \left(\tilde{I}_i - \kappa \ln y_i - \beta \sum_{j\neq i} \ln y_j\right)dt + \xi\, dW_i \\ &= \left\{\tilde{I}_i - (\kappa - \beta)\ln y_i - \beta N\left(\frac{1}{N}\sum_{j=1}^{N} \ln y_j\right)\right\}dt + \xi\, dW_i\end{aligned} \tag{7}$$

for $\tilde{I}_i \equiv I_i + (1/2)\xi^2$ and $i = 1, 2, 3, \ldots, N$.

Next, we propose a new method, namely the Lie-algebraic approach, to tackle the modified LCA model with multiple alternatives. By exploiting the Lie symmetry of the backward Kolmogorov equation (or Fokker–Planck equation) assoicated with the model and applying the Wei–Norman theorem (see Appendix A;[25]), we have succeeded in deriving the N-dimensional joint p.d.f. and marginal p.d.f. for each alternative in closed form. Then, a likelihood function can be constructed and model-fitting procedures become feasible. More importantly, the instability problem in parameter recovery is completely solved.

This paper is organized as follows. In second section the Lie-algebraic approach is applied to tackle the problem of the modified LCA model with N alternatives. Both the N-dimensional joint p.d.f. and marginal p.d.f. for

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each alternative are obtained in closed form. In "Numerical analysis" section some calibrated results based upon the Monte Carlo simulated time series are discussed and the final section presents the conclusion.

Lie-algebraic approach

To derive the joint p.d.f. $P(\{x_i\},t;\{x_{i0}\},t_0)$ of the stochastic variables $\{x_1,x_2,x_3,\ldots,x_N\}$ described by Eq. (1) without the restriction that $x_i \geqslant 0$ for $i=1,2,3,\ldots,N$, we need to solve the associated multi-dimenstional backward Kolmogorov equation:

$$\begin{aligned}-\frac{\partial P(\{x_i\},t;\{x_{i0}\},t_0)}{\partial t_0} &= \sum_{i=1}^{N}\left\{\frac{1}{2}\xi^2\frac{\partial^2 P(\{x_i\},t;\{x_{i0}\},t_0)}{\partial x_{i0}^2}\right.\\ &\left.+\left(I_i-\kappa x_{i0}-\beta\sum_{j\neq i}x_{j0}\right)\frac{\partial P(\{x_i\},t;\{x_{i0}\},t_0)}{\partial x_{i0}}\right\}\\ &=\left(\sum_{i=1}^{6}a_i\hat{L}_i\right)P(\{x_i\},t;\{x_{i0}\},t_0)\end{aligned} \tag{8}$$

subject to the condition $P(\{x_i\},t;\{x_{i0}\},t_0\to t)=\prod_{i=1}^{N}\delta(x_i-x_{i0})$, where

$$\begin{aligned}&\hat{L}_1=\sum_{i=1}^{N}\Delta I_i\frac{\partial}{\partial x_{i0}},\quad \hat{L}_2=\sum_{i=1}^{N}\frac{\partial}{\partial x_{i0}},\quad \hat{L}_3=\sum_{i=1}^{N}x_{i0}\frac{\partial}{\partial x_{i0}}\\ &\hat{L}_4=\sum_{j=1}^{N}x_{j0}\sum_{i=1}^{N}\frac{\partial}{\partial x_{i0}},\quad \hat{L}_5=\sum_{i=1}^{N}\frac{\partial^2}{\partial x_{i0}^2},\quad \hat{L}_6=\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_{i0}\partial x_{j0}}\\ &\bar{I}=\frac{1}{N}\sum_{i=1}^{N}I_i,\quad \Delta I_i=I_i-\bar{I},\quad a_1=1,\quad a_2=\bar{I}\\ &a_3=-(\kappa-\beta),\quad a_4=-\beta,\quad a_5=\frac{1}{2}\xi^2,\quad a_6=0.\end{aligned} \tag{9}$$

Introducing the backward time $\tau\equiv t-t_0$, Eq. (8) can be rewritten as

$$\frac{\partial P(\{x_i\};\{x_{i0}\},\tau)}{\partial\tau}=\left(\sum_{i=1}^{6}a_i\hat{L}_i\right)P(\{x_i\};\{x_{i0}\},\tau) \tag{10}$$

with $P(\{x_i\};\{x_{i0}\},0)=\prod_{i=1}^{N}\delta(x_i-x_{i0})$. It is not difficult to show that the operators $\left\{\hat{L}_i\right\}$ are the generators of a closed Lie algebra defined by the non-vanishing commutation relations:

$$\begin{aligned}&\left[\hat{L}_1,\hat{L}_3\right]\equiv\hat{L}_1\hat{L}_3-\hat{L}_3\hat{L}_1=\hat{L}_1\\ &\left[\hat{L}_2,\hat{L}_3\right]=\frac{1}{N}\left[\hat{L}_2,\hat{L}_4\right]=\hat{L}_2,\quad \left[\hat{L}_3,\hat{L}_5\right]=-2\hat{L}_5\\ &\left[\hat{L}_3,\hat{L}_6\right]=\left[\hat{L}_4,\hat{L}_5\right]=\frac{1}{N}\left[\hat{L}_4,\hat{L}_6\right]=-2\hat{L}_6.\end{aligned} \tag{11}$$

The formal solution of the backward Kolmogorov equation is given by

$$P(\{x_i\};\{x_{i0}\},\tau)=\exp\left\{\tau\left(\sum_{i=1}^{6}a_i\hat{L}_i\right)\right\}P(\{x_i\};\{x_{i0}\},0)\,. \tag{12}$$

In accordance with the Wei–Norman theorem[25], the exponential operator can be disentangled into the product form:

$$U(\tau)\equiv\exp\left\{\tau\left(\sum_{i=1}^{6}a_i\hat{L}_i\right)\right\}=\prod_{i=1}^{6}\exp\left\{b_i(\tau)\hat{L}_i\right\} \tag{13}$$

where the functions $\{b_i(\tau)\}$ are determined by solving a set of six coupled nonlinear ordinary differential equations with the conditions: $b_i(0)=0$ for all i (see Appendix A). After some simple algebra we obtain

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$$
\begin{aligned}
b_1(\tau) &= \frac{1}{\kappa-\beta}\left\{e^{(\kappa-\beta)\tau}-1\right\}\\
b_2(\tau) &= \frac{\bar{I}}{\kappa+\beta(N-1)}\left\{e^{[\kappa+\beta(N-1)]\tau}-1\right\}\\
b_3(\tau) &= -(\kappa-\beta)\tau, \qquad b_4(t) = -\beta\tau\\
b_5(\tau) &= \frac{\xi^2}{4(\kappa-\beta)}\left\{1-e^{-2(\kappa-\beta)\tau}\right\}\\
b_6(\tau) &= \frac{\xi^2}{4N}\left\{\frac{1-e^{-2[\kappa+\beta(N-1)]\tau}}{\kappa+\beta(N-1)}-\frac{1-e^{-2(\kappa-\beta)\tau}}{\kappa-\beta}\right\}.
\end{aligned}
\tag{14}
$$

The corresponding closed-form joint p.d.f. is then given by

$$
P(\{x_i\};\{x_{i0}\},\tau) = \frac{1}{\sqrt{(4\pi)^N \det(\mathbf{\Omega})}}\exp\left\{-\frac{1}{4}\sum_{i,j=1}^{N}(X_{i0}-x_i)\left(\mathbf{\Omega}^{-1}\right)_{ij}\left(X_{j0}-x_j\right)\right\} \tag{15}
$$

where

$$
X_{i0} = \left[x_{i0} + b_2(\tau)e^{Nb_4(\tau)} + b_1(\tau)\Delta I_i + \frac{e^{Nb_4(\tau)}-1}{N}\sum_{j=1}^{N}x_{j0}\right]e^{b_3(\tau)}. \tag{16}
$$

Here the $N\times N$ matrix $\mathbf{\Omega}(\tau)$ is defined by its elements as follows:

$$
\Omega_{ij}(\tau) = b_5(\tau)\delta_{ij} + b_6(\tau), \tag{17}
$$

and $\mathbf{\Omega}^{-1}(\tau)$ is its inverse.

Moreover, the marginal p.d.f. $p_i(x_i;\{x_{i0}\},\tau)$ for the stochastic variable x_i can be obtained from the joint p.d.f. by integrating out the other $N-1$ variables:

$$
p_i(x_i;\{x_{i0}\},\tau) = \int_{-\infty}^{\infty}dx_1\cdots\int_{-\infty}^{\infty}dx_{i-1}\int_{-\infty}^{\infty}dx_{i+1}\cdots\int_{-\infty}^{\infty}dx_N P(\{x_i\};\{x_{i0}\},\tau). \tag{18}
$$

It should be noted that the marginal p.d.f. $p_i(x_i;\{x_{i0}\},\tau)$ satisfies the backward Kolmogorov equation subject to the condition $p_i(x_i;\{x_{i0}\},0)=\delta(x_i-x_{i0})$. Thus, a more efficient way to derive $p_i(x_i;\{x_{i0}\},\tau)$ is to solve the backward Kolmogorov equation with the condition $p_i(x_i;\{x_{i0}\},0)=\delta(x_i-x_{i0})$ directly as follows:

$$
\begin{aligned}
p_i(x_i;\{x_{i0}\},\tau) &= \prod_{k=1}^{6}\exp\left\{b_k(\tau)\hat{L}_k\right\}\delta(x_i-x_{i0})\\
&= \frac{1}{\sqrt{4\pi\left[b_5(\tau)+b_6(\tau)\right]}}\exp\left\{-\frac{(X_{i0}-x_i)^2}{4[b_5(\tau)+b_6(\tau)]}\right\}.
\end{aligned}
\tag{19}
$$

Consequently, the closed-form joint p.d.f. $P(\{x_i\};\{x_{i0}\},\tau)$ enables us to construct a likelihood function for the modified LCA model with multiple alternatives so that maximum-likelihood analyses can be performed to calibrate the parameters of the model.

As a final remark, it should be noted that the joint p.d.f. of the stochastic variables $\{x_i\}$ can also be derived from solving the associated multi-dimensional Fokker–Planck equation. The details are shown in the Appendix B.

Numerical analysis

The Monte Carlo method based upon the strong order 1.5 Taylor scheme[26] is employed to generate the time series of the LCA model. That is, the simulation of evidence accumulation of the ith accumulator is performed in accordance with the following discretized version of the s.d.e. given in Eq. (1):

$$
\begin{aligned}
x_i^{t+\Delta t} = x_i^t + a_i\Delta t + \xi\,\Delta W_i - \frac{1}{2}\left\{(\kappa-\beta)a_i + \beta\sum_{j=1}^{N}a_j\right\}(\Delta t)^2\\
-\xi\left\{(\kappa-\beta)\Delta Z_i + \beta\sum_{j=1}^{N}\Delta Z_j\right\},
\end{aligned}
\tag{20}
$$

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N	Data points	Average elapsed time of calibration per time series (s)
2	20,000	1.3783
3	20,000	1.7902
10	20,000	11.592

Table 1. The average elapsed time of calibration from 128 time series.

	κ	β	I_1	I_2	ξ
Exact value	4	1	0.9	1.1	0.25
Calibrated value	4.01	0.97	0.894	1.095	0.24998
Standard error	0.10	0.10	0.026	0.026	0.00089
z-score	39.7	9.7	33.9	41.4	280.1

Table 2. Calibrated results for the case of two alternatives.

	κ	β	I_1	I_2	I_3	ξ
Exact value	4	1	0.9	1.1	0.98	0.25
Calibrated value	3.963	0.998	0.894	1.094	0.978	0.24991
Standard error	0.088	0.051	0.023	0.023	0.023	0.00073
z-score	45.0	19.6	38.8	47.2	42.9	342.5

Table 3. Calibrated results for the case of three alternatives.

where

$$\begin{aligned} a_i &= \left\{ I_i - (\kappa - \beta)x_i - \beta \sum_{j=1}^{N} x_j \right\} \\ \Delta W_i &= U_{i,1}\sqrt{\Delta t} \\ \Delta Z_i &= \frac{1}{2}(\Delta t)^{3/2}\left(U_{i,1} + \frac{1}{\sqrt{3}} U_{i,2} \right) \end{aligned} \tag{21}$$

and x_i^t denotes the logarithm of evidence accumulation of the ith accumulator at time t. Here $U_{i,1}$ and $U_{i,2}$ are uncorrelated random numbers drawn from a normal distribution with zero mean and unit variance whilst ΔZ_i is normally distributed with zero mean, variance $E((\Delta Z_i)^2) = \frac{1}{3}(\Delta t)^3$ and covariance $E(\Delta Z_i \Delta W_i) = \frac{1}{2}(\Delta t)^2$. In order to simulate that the initial value of evidence accumulation is sufficiently close to zero, $x_i^{t=0}$ is always set equal to -5 in this study. For illustration, we have examined three different cases, namely the two-, three- and ten-alternative case, in each of which 128 simulated time series are generated. For each time series there are 20, 000 data points with $\Delta t = 0.01$.

With the closed-form joint p.d.f., maximum likelihood analyses are then applied to calibrate the model parameters and check whether the actual values can be recovered. The global maximum of the log-likelihood function is determined by the Nelder–Mead simplex algorithm, and the implementation is performed by means of the "'*fminsearch*" function of the MATLAB. To ensure convergence, we iterate the estimation until the discrepancy between the guess and estimated value is smaller than 10^{-6} in magnitude. Table 1 tabulates the average elapsed time for the maximum likelihood estimation per time series for the three illustrative cases. In fact, the calibration can be completed within a minute even for a large number of alternatives. The calibration is carried out using a 4.7 GHz Intel Core i7-10700K PC.

In Table 2 the input model parameters and the calibrated values (based upon 128 simulated time series) are presented for the case of two alternatives. The corresponding standard errors and z-scores of the calibrated values are tabulated, too. It is obvious that the calibrated values are in good agreement with the exact values. Tables 3 and 4 present the same set of informations for the three- and ten-alternative case respectively, and the same observation can be made. As a result, it is beyond question that the calibration of parameters is both effcient and accurate.

Conclusion

Based upon the stochastic Gompertz law of population growth, we have reformulated the LCA model with multiple alternatives such that the positive-definiteness of evidence accumulation is automatically satisfied. By exploiting the Lie symmetry of the backward Kolmogorov equation (or Fokker–Planck equation) assoicated with the modified model and applying the Wei–Norman theorem, we have also succeeded in deriving the

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	κ	θ	I_1	I_2	I_3	I_4	I_5
Exact value	4	1	0.9	1.1	0.95	1.2	0.5
Calibrated value	4.08	0.990	0.898	1.103	0.954	1.199	0.484
Standard error	0.17	0.020	0.056	0.057	0.056	0.058	0.062
z-score	24.3	48.9	16.1	19.4	17.1	20.6	7.9
	I_6	I_7	I_8	I_9	I_{10}	ξ	
Exact value	0.75	0.8	1	1	1	0.25	
Calibrated value	0.741	0.796	1.006	1.008	1.005	0.25001	
Standard error	0.057	0.056	0.056	0.056	0.056	0.00040	
z-score	13.0	14.1	17.9	18.0	17.9	631.6	

Table 4. Calibrated results for the case of ten alternatives.

N-dimensional joint p.d.f. and marginal p.d.f. for each alternative in closed form. With the joint p.d.f., a likelihood function can be constructed and thus model-fitting procedures become feasible and efficient. We have also demonstrated that the calibration of model parameters based upon the Monte Carlo simulated time series is indeed both efficient and accurate. Moreover, it should be noted that the proposed Lie-algebraic approach can also be applied to tackle the modified LCA model with time-varying parameters.

Received: 13 February 2021; Accepted: 10 May 2021
Published online: 25 May 2021

References

1. Usher, M. & McClelland, J.L. (2001). The time course of perceptual choice: The leaky, competing accumulator model. Psychological Review 108, 550–592.
2. Churchland, A.K. & Ditterich, J. (2012). New advances in understanding decisions among multiple alternatives. Current Opinion in Neurobiology 22, 920–926.
3. Miletić, S., Turner, B.M., Forstmann, B.U. & Van Maanen, L. (2017). Parameter recovery for the leaky competing accumulator model. Journal of Mathematical Psychology 76, 25–50.
4. Evans, N.J. (2019). A method, framework and tutorial for efficiently simulating models of decision-making. Behavior Research Methods 51, 2390–2404.
5. Evans, N.J., Holmes, W.R. & Trueblood, J.S. (2019). Response-time data provide critical constraints on dynamic models of multi-alternative, multi-attribute choice. Psychonomic Bulletin and Review 26, 901–933.
6. Evans, N.J., Dutilh, G., Wagenmakers, E.J. & van der Maas, H.L.J. (2020). Double responding: A new constraint for models of speeded decision making. Cognitive Psychology 121, 101292.
7. Turner, B.M. & Sederberg, P.B. (2014). A generalized, likelihood-free method for posterior estimation. Psychological Review 21, 227–250.
8. Brown, E. & Holmes, P. (2001). Modelling a simple choice task: Stochastic dynamics of mutually inhibitory neural groups. Stochastics and Dynamics 1, 159–191.
9. McMillen, T. & Holmes, P. (2006). The dynamics of choice among multiple alternatives. Journal of Mathematical Psychology 50, 30–57.
10. Usher, M. & McClelland, J.L. (2004). Loss aversion and inhibition in dynamical models of multialternative choice, Psychological Review 111, 757–769.
11. van Ravenzwaaij, D., van der Maas, H.L.J. & Wagenmakers, E.J. (2012). Optimal decision making in neural inhibition models. Psychological Review 119, 201–215.
12. Bogacz, R., Usher, M., Zhang, J. & McClelland, J.L. (2007). Extending a biologically inspired model of choice: multi-alternatives, nonlinearity and value-based multidimensional choice. Phil. Trans. R. Soc. B 362, 1655–1670.
13. Gardiner, G.W. (1985). Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences, 2nd edn. Springer, Berlin.
14. Bass, L., Green, H.S. & Boxenbaum, H. (1989). Gompertzian mortality derived from competition between cell-types: congenial, toxicologic and biometric determinants of longevity. J. Theor. Biol. 140(2), 263–278.
15. Qi, A.S., Zheng, X., Du, C.Y. & An, B.S. (1993). A cellular automaton model of cancerous growth. J. Theor. Biol. 161(1), 1–12.
16. Bassukas, I.D. (1994). Comparative Gompertzian analysis of alterations of tumor-growth patterns. Cancer Res. 54(16), 4385–4392.
17. Rygaard, K. & Spang-Thomsen, M. (1997). Quantitation and Gompertzian analysis of tumor growth. Breast Cancer: Res. Treat. 46(2–3), 303–312.
18. Ferrante, L., Bompadre, S., Possati, L. & Leone, L. (2000). Parameter estimation in a Gompertzian stochastic model for tumor growth. Biometrics 56(4), 1076–1081.
19. Albano, G. & Giorno, V. (2006). A stochastic model in tumor growth. J. Theor. Biol. 242(2), 329–336.
20. Lo, C.F. (2007). Stochastic Gompertz model of tumour cell growth. J. Theor. Biol. 248(2), 317–321.
21. Albano, G., Giorno, V., Roman-Roman, P. & Torres-Ruiz, F. (2011). Inferring the effect of therapy on tumors showing stochastic Gompertzian growth. J. Theor. Biol. 276(1), 67–77.
22. Moummou, E.K., Gutierrez, R. & Gutierrez-Sanchez, R. (2012). A stochastic Gompertz model with logarithmic therapy functions: parameter estimation. Applied Mathematics and Computation 219(8), 3729–3739.
23. Giorno V, Spina S (2013) A stochastic Gompertz model with jumps for an intermittent treatment in cancer growth. In: Moreno-Diaz R., Pichler, F., Quesada-Arencibia, A. (eds.) Computer Aided Systems Theory—EUROCAST 2013, vol. 8111, pp. 61-68. Springer, Berlin
24. Schwartz, E.S. (1997). The stochastic behavior of commodity prices: implications for valuation and hedging. Journal of Finance 52(3), 923–973.
25. Wei, J. & Norman, E. (1963). Lie algebraic solution of linear differential equations. Journal of Mathematical Physics 4, 575–581.
26. Kloeden, R.E. & Platen, E. (1992). Numerical Solution of Stochastic Differential Equations, Springer-Verlag: Berlin.

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Author contributions

C.-F.L. is responsible for proposing the modified LCA model and deriving the probability density functions. H.-Y.I. is responsible for performing the Monte Carlo simulation and the calibration of model parameters based upon the simulated time series.

Competing interests

The authors declare no competing interests.

Additional information

Supplementary Information The online version contains supplementary material available at https://doi.org/10.1038/s41598-021-90356-7.

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