Cumulative Link Model

when $J = 2$:

$$\begin{aligned} P(h\ wins) &= P(Y_{hi}^* > 0) = P(Y_h - Y_i > 0) \\ &= P(-Z + (\mu_h - \mu_i) > 0) = P(Z < \mu_h - \mu_i) \\ &= \frac{1}{1 + e^{-(\mu_h - \mu_i)}} \\ &= \frac{e^{\mu_h}}{e^{\mu_h} + e^{\mu_i}} \\ &= \frac{\gamma_h}{\gamma_h + \gamma_i} \quad \text{(Bradley-Terry Model)} \end{aligned}$$

when $J = 3$:

$$\begin{aligned} P(h\ wins) &= P(Y_{hi}^* > \alpha_2) = P(Y_h - Y_i > \alpha_2) \\ &= P(-Z + (\mu_h - \mu_i) > \alpha_2) = P(Z < \mu_h - \mu_i - \alpha_2) \\ &= \frac{1}{1 + e^{-(\mu_h - \mu_i - \alpha_2)}} \\ &= \frac{e^{\mu_h}}{e^{\mu_h} + e^{\mu_i} e^{\alpha_2}} \\ &= \frac{\gamma_h}{\gamma_h + \theta\gamma_i} \\ P(i\ wins) &= P(Y_i - Y_h > \alpha_2) \\ &= P(Z - (\mu_h - \mu_i) > \alpha_2) = P(Z > \alpha_2 + (\mu_h - \mu_i)) \\ &= 1 - \frac{1}{1 + e^{-(\alpha_2 + (\mu_h - \mu_i))}} = \frac{e^{\mu_i}}{e^{\mu_i} + e^{\mu_h} e^{\alpha_2}} \\ &= \frac{\gamma_i}{\gamma_i + \theta\gamma_h} \quad \text{(Rao-Kupper Model)} \end{aligned}$$

Adjacent Categories Logit Model

when $J = 2$:

$$\begin{aligned} P(h\ wins) &= P(Y_{hi} = 2) \\ &= \frac{P(Y_{hi} = 2)}{P(Y_{hi} = 2) + P(Y_{hi} = 1)} \\ &= \frac{1}{1 + P(Y_{hi} = 1)/P(Y_{hi} = 2)} \\ &= \frac{1}{1 + e^{\mu_i - \mu_h}} \\ &= \frac{e^{\mu_h}}{e^{\mu_h} + e^{\mu_i}} \\ &= \frac{\gamma_h}{\gamma_h + \gamma_i} \quad \text{(Bradley-Terry Model)} \end{aligned}$$

when $J = 3$:
since:

$$\begin{aligned} P(Y_{hi} = 1)/P(Y_{hi} = 2) = exp(-\alpha_2 - (\mu_h - \mu_i)) \\ P(Y_{hi} = 2)/P(Y_{hi} = 3) = exp(\alpha_2 - (\mu_h - \mu_i)) \end{aligned}$$

Then

$$\begin{aligned} P(Y_{hi} = 1) : P(Y_{hi} = 2) : P(Y_{hi} = 3) &= e^{2\mu_i} : e^{\alpha_2 + \mu_i + \mu_h} : e^{2\mu_h} \\ &= \gamma_i : \theta\sqrt{\gamma_i\gamma_h} : \gamma_h \quad \text{(Davidson Model)} \end{aligned}$$

J=4

Cumulative Link Model

$$
\begin{aligned}
P(Y_{hi}=4) &= P(Y_h - Y_i > \alpha_3) \\
&= P(-Z + (\mu_h - \mu_i) > \alpha_3) = P(Z < (\mu_h - \mu_i) - \alpha_3) \\
&= \frac{1}{1+e^{-(\mu_h-\mu_i-\alpha_3)}} \\
&= \frac{e^{\mu_h}}{e^{\mu_h} + e^{\alpha_3}e^{\mu_i}} \\
&= \frac{\gamma_h}{\gamma_h + \theta\gamma_i} \\
P(Y_{hi}=3) &= P(Y_h - Yi > 0) - P(Y_h - Y_i > \alpha_3) \\
&= \frac{e^{\mu_h}}{e^{\mu_h} + e^{\mu_i}} - \frac{e^{\mu_h}}{e^{\mu_h} + e^{\alpha_3}e^{\mu_i}} \\
&= \frac{\gamma_h}{\gamma_h + \gamma_i} - \frac{\gamma_h}{\gamma_h + \theta\gamma_i} \\
P(Y_{hi}=1) &= P(Y_h - Y_i < -\alpha_3) \\
&= P(-Z + (\mu_h - \mu_i) < -\alpha_3) = P(Z > (\mu_h - \mu_i) + \alpha_3) \\
&= 1 - \frac{1}{1+e^{-(\mu_h-\mu_i+\alpha_3)}} \\
&= \frac{e^{\mu_i}}{e^{\mu_i} + e^{\alpha_3}e^{\mu_h}} \\
&= \frac{\gamma_i}{\gamma_i + \theta\gamma_h} \\
P(Y_{hi}=2) &= P(Y_h - Y_i < 0) - P(Y_h - Y_i < -\alpha_3) \\
&= \frac{e^{\mu_i}}{e^{\mu_i} + e^{\mu_h}} - \frac{e^{\mu_i}}{e^{\mu_i} + e^{\alpha_3}e^{\mu_h}} \\
&= \frac{\gamma_i}{\gamma_i + \gamma_h} - \frac{\gamma_i}{\gamma_i + \theta\gamma_h}
\end{aligned}
$$

Then its log-likelihood function is:

$$
\begin{aligned}
\ell(\boldsymbol{\gamma}, \theta) &= \sum_{i=1}^{m}\sum_{j=1}^{m}\{W_{ij(s)}ln(\frac{\gamma_i}{\gamma_i + \theta\gamma_j}) + W_{ij(w)}ln(\frac{\gamma_i}{\gamma_i + \gamma_j} - \frac{\gamma_i}{\gamma_i + \theta\gamma_j})\} \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m}\{W_{ij(s)}ln(\frac{\gamma_i}{\gamma_i + \theta\gamma_j}) + W_{ij(w)}ln(\frac{(\theta-1)\gamma_i\gamma_j}{(\gamma_i+\gamma_j)(\gamma_i+\theta\gamma_j)})\}
\end{aligned}
$$

where $W_{ij(s)}$ and $W_{ij(w)}$ denote the number of responses of stronger and weaker preferences for player i.
Then its minorizing function is (without constant terms):

$$
Q_k(\boldsymbol{\gamma}, \theta) = \sum_{i=1}^{m}\sum_{j=1}^{m}\{W_{ij(s)}(ln\gamma_i - \frac{\gamma_i + \theta\gamma_j}{\gamma_i^{(k)} + \theta^{(k)}\gamma_j^{(k)}}) + W_{ij(w)}(ln(\theta-1) + (ln\gamma_i - \frac{\gamma_i+\gamma_j}{\gamma_i^{(k)}+\gamma_j^{(k)}}) + (ln\gamma_j - \frac{\gamma_i + \theta\gamma_j}{\gamma_i^{(k)} + \theta^{(k)}\gamma_j^{(k)}}))\}
$$

Maximization of $Q_k(\boldsymbol{\gamma}, \theta^{(k)})$ w.r.t. γ_i gives:

$$
\gamma_i = (W_{i(4)} + W_{i(3)} + W_{i(2)})\{\sum_{j=1}^{m}(\frac{w_{ij(s)} + w_{ij(w)}}{\gamma_i^{(k)} + \theta^{(k)}\gamma_j^{(k)}} + \frac{\theta(w_{ji(s)} + w_{ji(w)})}{\gamma_j^{(k)} + \theta^{(k)}\gamma_i^{(k)}} + \frac{w_{ij(w)} + w_{ji(w)}}{\gamma_i^{(k)} + \gamma_j^{(k)}})\}^{-1}
$$

$$
\text{where } (W_{i(4)}, W_{i(3)}, W_{i(2)}) = (\sum_{j=1}^{m} w_{ij(s)}, \sum_{j=1}^{m} w_{ij(w)}, \sum_{j=1}^{m} w_{ji(w)})
$$

Maximization of $Q_k(\boldsymbol{\gamma}^{(k)}, \theta)$ w.r.t. θ gives:

$$\theta = W_3\{\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\gamma_j(w_{ij(s)} + w_{ij(w)})}{\gamma_i^{(k)} + \theta^{(k)}\gamma_j^{(k)}}\}^{-1} + 1$$

$$\text{where } W_3 = \sum_{i=1}^{m}\sum_{j=1}^{m} w_{ij(w)}$$

Adjacent Categories Logit Model

since:

$$P(Y_{hi} = 1)/P(Y_{hi} = 2) = exp(-\alpha_3 - (\mu_h - \mu_i))$$
$$P(Y_{hi} = 2)/P(Y_{hi} = 3) = exp(0 - (\mu_h - \mu_i))$$
$$P(Y_{hi} = 3)/P(Y_{hi} = 4) = exp(\alpha_3 - (\mu_h - \mu_i))$$

Then:

$$\begin{aligned} &P(Y_{hi} = 1) : P(Y_{hi} = 2) : P(Y_{hi} = 3) : P(Y_{hi} = 4) \\ &= exp(3\mu_i) : exp(\alpha_3 + \mu_h + 2\mu_i) : exp(\alpha_3 + 2\mu_h + \mu_i) : exp(3\mu_h) \\ &= \gamma_i : \theta\gamma_h^{\frac{1}{3}}\gamma_i^{\frac{2}{3}} : \theta\gamma_h^{\frac{2}{3}}\gamma_i^{\frac{1}{3}} : \gamma_h \end{aligned}$$

Then its log-likelihood function is:

$$\ell(\boldsymbol{\gamma}, \theta) = \sum_{i=1}^{m}\sum_{j=1}^{m}\{W_{ij(s)}ln(\frac{\gamma_i}{\gamma_i + \gamma_j + \theta\gamma_i^{\frac{1}{3}}\gamma_j^{\frac{2}{3}} + \theta\gamma_i^{\frac{2}{3}}\gamma_j^{\frac{1}{3}}}) + W_{ij(w)}ln(\frac{\theta\gamma_i^{\frac{2}{3}}\gamma_j^{\frac{1}{3}}}{\gamma_i + \gamma_j + \theta\gamma_i^{\frac{1}{3}}\gamma_j^{\frac{2}{3}} + \theta\gamma_i^{\frac{2}{3}}\gamma_j^{\frac{1}{3}}})\}$$

where $W_{ij(s)}$ and $W_{ij(w)}$ denote the number of responses of the stronger and weaker preferences for player i. Then its minorizing function is (without constant terms):

$$Q_k^*(\boldsymbol{\gamma}, \theta) = \sum_{i=1}^{m}\sum_{j=1}^{m}\{W_{ij(s)}ln\gamma_i + W_{ij(w)}ln(\theta\gamma_i^{\frac{2}{3}}\gamma_j^{\frac{1}{3}}) - (W_{ij(s)} + W_{ij(w)})(\frac{\gamma_i + \gamma_j + \theta\gamma_i^{\frac{1}{3}}\gamma_j^{\frac{2}{3}} + \theta\gamma_i^{\frac{2}{3}}\gamma_j^{\frac{1}{3}}}{\gamma_i^{(k)} + \gamma_j^{(k)} + \theta^{(k)}\gamma_i^{(k)\frac{1}{3}}\gamma_j^{(k)\frac{2}{3}} + \theta^{(k)}\gamma_i^{(k)\frac{2}{3}}\gamma_j^{(k)\frac{1}{3}}})\}$$

then use the inequality:

$$-\gamma_i^{\frac{1}{3}}\gamma_j^{\frac{2}{3}} \geq -\frac{1}{3}\gamma_i(\frac{\gamma_j^{(k)}}{\gamma_i^{(k)}})^{\frac{2}{3}} - \frac{2}{3}\gamma_j(\frac{\gamma_i^{(k)}}{\gamma_j^{(k)}})^{\frac{1}{3}}$$

$$-\gamma_i^{\frac{2}{3}}\gamma_j^{\frac{1}{3}} \geq -\frac{2}{3}\gamma_i(\frac{\gamma_j^{(k)}}{\gamma_i^{(k)}})^{\frac{1}{3}} - \frac{1}{3}\gamma_j(\frac{\gamma_i^{(k)}}{\gamma_j^{(k)}})^{\frac{2}{3}}$$

then we have the minorizer of Q_k^*:

$$\begin{aligned} Q_k(\boldsymbol{\gamma}, \theta) = \sum_{i=1}^{m}\sum_{j=1}^{m}\{&W_{ij(s)}ln\gamma_i + W_{ij(w)}ln(\theta\gamma_i^{\frac{2}{3}}\gamma_j^{\frac{1}{3}}) \\ &- (W_{ij(s)} + W_{ij(w)})(\frac{\gamma_i + \gamma_j + \frac{\theta}{3}[\gamma_i((\frac{\gamma_j^{(k)}}{\gamma_i^{(k)}})^{\frac{2}{3}} + 2(\frac{\gamma_j^{(k)}}{\gamma_i^{(k)}})^{\frac{1}{3}}) + \gamma_j(2(\frac{\gamma_i^{(k)}}{\gamma_j^{(k)}})^{\frac{1}{3}} + (\frac{\gamma_i^{(k)}}{\gamma_j^{(k)}})^{\frac{2}{3}})]}{\gamma_i^{(k)} + \gamma_j^{(k)} + \theta^{(k)}\gamma_i^{(k)\frac{1}{3}}\gamma_j^{(k)\frac{2}{3}} + \theta^{(k)}\gamma_i^{(k)\frac{2}{3}}\gamma_j^{(k)\frac{1}{3}}})\} \end{aligned}$$

Maximization of $Q_k(\boldsymbol{\gamma}, \theta^{(k)})$ w.r.t. γ_i gives:

$$\gamma_i = (W_{i(4)} + \frac{2}{3}W_{i(3)} + \frac{1}{3}W_{i(2)})\{\sum_{j=1}^{m}\frac{(w_{ij(s)} + w_{ij(w)} + w_{ji(s)} + w_{ji(w)})(\frac{\theta}{3}((\frac{\gamma_j^{(k)}}{\gamma_i^{(k)}})^{\frac{2}{3}} + 2(\frac{\gamma_j^{(k)}}{\gamma_i^{(k)}})^{\frac{1}{3}}) + 1)}{\gamma_i^{(k)} + \gamma_j^{(k)} + \theta^{(k)}\gamma_i^{(k)\frac{1}{3}}\gamma_j^{(k)\frac{2}{3}} + \theta^{(k)}\gamma_i^{(k)\frac{2}{3}}\gamma_j^{(k)\frac{1}{3}}}\}^{-1}$$

Maximization of $Q_k(\boldsymbol{\gamma}^{(k)}, \theta)$ w.r.t. θ gives:

$$\theta = W_3\{\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{(w_{ij(s)} + w_{ij(w)})[\gamma_i^{(k)\frac{1}{3}}\gamma_j^{(k)\frac{2}{3}} + \gamma_i^{(k)\frac{2}{3}}\gamma_j^{(k)\frac{1}{3}}]}{\gamma_i^{(k)} + \gamma_j^{(k)} + \theta^{(k)}\gamma_i^{(k)\frac{1}{3}}\gamma_j^{(k)\frac{2}{3}} + \theta^{(k)}\gamma_i^{(k)\frac{2}{3}}\gamma_j^{(k)\frac{1}{3}}}\}^{-1}$$

J=5

Cumulative Link Model

$Y_{ih} = 1$:

$$\begin{aligned}P(Y_h - Y_i > \alpha_4) &= P(\mu_h - \mu_i - Z > \alpha_4) = P(Z < \mu_h - \mu_i - \alpha_4)\\ &= \frac{1}{1+e^{-(\mu_h-\mu_i-\alpha_4)}} = \frac{e^{\mu_h}}{e^{\mu_h}+e^{\mu_i}e^{\alpha_4}}\\ &= \frac{\gamma_h}{\gamma_h+\theta_2\gamma_i}\end{aligned}$$

$Y_{ih} = 2$:

$$\begin{aligned}P(\alpha_3 < Y_h - Y_i < \alpha_4) &= P(Y_h - Y_i > \alpha_3) - P(Y_h - Y_i > \alpha_4)\\ &= \frac{e^{\mu_h}}{e^{\mu_h}+e^{\mu_i}e^{\alpha_3}} - \frac{e^{\mu_h}}{e^{\mu_h}+e^{\mu_i}e^{\alpha_4}}\\ &= \frac{e^{\mu_h}e^{\mu_i}(e^{\alpha_4}-e^{\alpha_3})}{(e^{\mu_h}+e^{\mu_i}e^{\alpha_3})(e^{\mu_h}+e^{\mu_i}e^{\alpha_4})}\\ &= \frac{\gamma_h\gamma_i(\theta_2-\theta_1)}{(\gamma_h+\theta_1\gamma_i)(\gamma_h+\theta_2\gamma_i)}\end{aligned}$$

$Y_{ih} = 3$:

$$\begin{aligned}P(\alpha_2 < Y_h - Y_i < \alpha_3) &= P(Y_h - Y_i > \alpha_2) - P(Y_h - Y_i > \alpha_3)\\ &= \frac{e^{\mu_h}e^{\mu_i}(e^{\alpha_3}-e^{-\alpha_3})}{(e^{\mu_h}+e^{\mu_i}e^{\alpha_3})(e^{\mu_h}+e^{\mu_i}e^{-\alpha_3})}\\ &= \frac{\gamma_h\gamma_i(\theta_1^2-1)}{(\theta_1\gamma_h+\gamma_i)(\gamma_h+\theta_1\gamma_i)}\end{aligned}$$

$Y_{ih} = 4$:

$$\begin{aligned}P(\alpha_1 < Y_h - Y_i < \alpha_2) &= P(Y_h - Y_i > \alpha_1) - P(Y_h - Y_i > \alpha_2)\\ &= \frac{e^{\mu_h}e^{\mu_i}(e^{-\alpha_3}-e^{-\alpha_4})}{(e^{\mu_h}+e^{\mu_i}e^{-\alpha_3})+(e^{\mu_h}+e^{\mu_i}e^{-\alpha_4})}\\ &= \frac{e^{\mu_h}e^{\mu_i}(e^{\alpha_4}-e^{\alpha_3})}{(e^{\mu_i}+e^{\mu_h}e^{-\alpha_3})+(e^{\mu_i}+e^{\mu_h}e^{-\alpha_4})}\\ &= \frac{\gamma_h\gamma_i(\theta_2-\theta_1)}{(\gamma_i+\theta_1\gamma_h)(\gamma_i+\theta_2\gamma_h)}\end{aligned}$$

$Y_{ih} = 5$:

$$\begin{aligned}P(Y_h - Y_i < \alpha_2) &= P(Y_i - Y_h > \alpha_4)\\ &= \frac{e^{\mu_i}}{e^{\mu_i}+e^{\mu_h}e^{\alpha_4}}\\ &= \frac{\gamma_i}{\gamma_i+\theta_2\gamma_h}\end{aligned}$$

Then its log-likelihood function is:

$$\begin{aligned}&\ell(\boldsymbol{\gamma},\theta_1,\theta_2) =\\ &\sum_{h=1}^{m}\sum_{i=1}^{m}\{w_{ih(1)}ln(\frac{\gamma_h}{\gamma_h+\theta_2\gamma_i}) + w_{ih(5)}ln(\frac{\gamma_i}{\gamma_i+\theta_2\gamma_h}) + w_{ih(2)}ln(\frac{\gamma_h\gamma_i(\theta_2-\theta_1)}{(\gamma_h+\theta_2\gamma_i)(\gamma_h+\theta_1\gamma_i)})\\ &+ w_{ih(4)}ln(\frac{\gamma_h\gamma_i(\theta_2-\theta_1)}{(\gamma_i+\theta_1\gamma_h)(\gamma_i+\theta_2\gamma_h)}) + w_{ih(3)}ln(\frac{\gamma_h\gamma_i(\theta_1^2-1)}{(\theta_1\gamma_h+\gamma_i)(\gamma_h+\theta_1\gamma_i)})\}\end{aligned}$$

Obtain its minorizing function by using the inequality (9) twice:

$$Q_k(\boldsymbol{\gamma}, \boldsymbol{\theta_1}, \boldsymbol{\theta_2}) = \sum_{h=1}^{m}\sum_{i=1}^{m}\{w_{ih(1)}(ln\gamma_h - \frac{\gamma_h + \theta_2\gamma_i}{\gamma_h^{(k)} + \gamma_i^{(k)}\theta_2^{(k)}}) + w_{ih(5)}(ln\gamma_i - \frac{\gamma_i + \theta_2\gamma_h}{\gamma_i^{(k)} + \theta_2^{(k)}\gamma_h^{(k)}}) +$$
$$w_{ih(2)}(ln\gamma_h + ln\gamma_i + ln(\theta_2 - \theta_1) - \frac{\gamma_h + \theta_2\gamma_i}{\gamma_h^{(k)} + \theta_2^{(k)}\gamma_i^{(k)}} - \frac{\gamma_h + \theta_1\gamma_i}{\gamma_h^{(k)} + \theta_1^{(k)}\gamma_i^{(k)}}) +$$
$$w_{ih(4)}(ln\gamma_h + ln\gamma_i + ln(\theta_2 - \theta_1) - \frac{\gamma_i + \theta_2\gamma_h}{\gamma_i^{(k)} + \theta_2^{(k)}\gamma_h^{(k)}} - \frac{\gamma_i + \theta_1\gamma_h}{\gamma_i^{(k)} + \theta_1^{(k)}\gamma_h^{(k)}}) +$$
$$w_{ih(3)}(ln\gamma_h + ln\gamma_i + \frac{\theta_1^2 - 1}{\theta_1^{(k)2} - 1} - \frac{\theta_1\gamma_h + \gamma_i}{\theta_1^{(k)}\gamma_h^{(k)} + \gamma_i^{(k)}} - \frac{\gamma_h + \theta_1\gamma_i}{\gamma_h^{(k)} + \theta_1^{(k)}\gamma_i^{(k)}})\}$$

Maximization of $Q_k(\boldsymbol{\gamma}, \theta_1^{(k)}, \theta_2^{(k)})$ w.r.t. γ_i gives:

$$\gamma_i = (W_{i(5)} + W_{i(4)} + W_{i(3)} + W_{i(2)})\{\sum_{h=1}^{m}[\frac{w_{ih(5)}}{\gamma_i^{(k)} + \theta_2^{(k)}\gamma_h^{(k)}} + \frac{\theta_2^{(k)}w_{ih(1)}}{\gamma_h^{(k)} + \theta_2^{(k)}\gamma_i^{(k)}} + \frac{w_{ih(4)}}{\gamma_i^{(k)} + \theta_2^{(k)}\gamma_h^{(k)}} +$$
$$\frac{\theta_2^{(k)}w_{ih(2)}}{\gamma_h^{(k)} + \theta_2^{(k)}\gamma_i^{(k)}} + \frac{w_{ih(4)}}{\gamma_i^{(k)} + \theta_1^{(k)}\gamma_h^{(k)}} + \frac{\theta_1^{(k)}w_{ih(2)}}{\gamma_h^{(k)} + \theta_1^{(k)}\gamma_i^{(k)}} + \frac{w_{ih(3)}\theta_1^{(k)}}{\theta_1^{(k)}\gamma_i^{(k)} + \gamma_h^{(k)}} + \frac{w_{ih(3)}}{\gamma_i^{(k)} + \theta_1^{(k)}\gamma_h^{(k)}}]\}^{-1}$$

where $W_{i(n)} = \sum_{h=1}^{m} w_{ih(n)}$

Maximization of $Q_k(\boldsymbol{\gamma}^{(k)}, \theta_1^{(k)}, \theta_2)$ w.r.t. θ_2 gives:

$$\theta_2 = (W_2 + W_4)\{\sum_{h=1}^{m}\sum_{i=1}^{m}[\frac{2w_{ih(1)}\gamma_i^{(k)}}{\gamma_h^{(k)} + \theta_2^{(k)}\gamma_i^{(k)}} + \frac{2w_{ih(2)}\gamma_i^{(k)}}{\gamma_h^{(k)} + \theta_2^{(k)}\gamma_i^{(k)}}]\}^{-1} + \theta_1^{(k)}$$

where $W_n = \sum_{h=1}^{m}\sum_{i=1}^{m} w_{ih(n)}$

Maximization of $Q_k(\boldsymbol{\gamma}^{(k)}, \theta_1, \theta_2^{(k)})$ w.r.t. θ_1 gives:

$$\frac{W_2 + W_4}{\theta_2^{(k)} - \theta_1} + \frac{2\theta_1}{\theta_1^{(k)2} - 1} - C_k = 0$$

$$C_k = \sum_{h=1}^{m}\sum_{i=1}^{m} \frac{\gamma_i^{(k)}w_{ih(2)}}{\gamma_h^{(k)} + \theta_1^{(k)}\gamma_i^{(k)}} + \frac{\gamma_h^{(k)}w_{ih(4)}}{\gamma_i^{(k)} + \theta_1^{(k)}\gamma_h^{(k)}} + \frac{2\gamma_h^{(k)}w_{ih(3)}}{\theta_1^{(k)}\gamma_h^{(k)} + \gamma_i^{(k)}}$$

then we have

$$(W_2 + W_4)(\theta_1^{(k)2} - 1) + 2\theta_1(\theta_2^{(k)} - \theta_1) - C_k(\theta_1^{(k)2} - 1)(\theta_2^{(k)} - \theta_1) = 0$$
$$2\theta_1^2 - (2\theta_2^{(k)} + C_k(\theta_1^{(k)2} - 1))\theta_1 - (W_2 + W_4)(\theta_1^{(k)2} - 1) + C_k(\theta_1^{(k)2} - 1)\theta_2^{(k)} = 0$$
$$\theta_1 = \frac{1}{4}[2\theta_2^{(k)} + C_k(\theta_1^{(k)2} - 1) + \sqrt{(2\theta_2^{(k)} + C_k(\theta_1^{(k)2} - 1))^2 + 8(W_2 + W_4 - C_k\theta_2^{(k)})(\theta_1^{(k)2} - 1)]}$$