CS101 Advanced Engineering Mathematics (I)

工程數學(一)

[Guidelines]

- All the homework in this course will involve solving advanced engineering mathematics problems (differential equations in particular) by hand and computer.
- While discussion with other classmates is allowed, you MUST work independently to generate your own solutions to the problems.
- Python programming will be used for plotting solutions. You should reference the Python Tutorial (課程講義) for detail information.
- For each homework, you must submit a ***written report*** **(書面報告)**.

[General Instructions]

To get a good grading in homework assignments, you are advised to do the following:

- Do not copy other classmate's works! (請遵守學術倫理，嚴禁抄襲)
- Provide correct answers in details. (詳細推導過程與標明正確答案)
- Prepare your written reports in good quality (使用 Template 檔並書寫工整).
- Meet the deadline! Late homework will **not** be collected. (按時繳交，逾時不候)

指導教授：張元翔

Homework Assignment 1

Review of Calculus & First-Order Differential Equations

Deadline: 11 / 11 / 2022 (星期五)

(請繳交書面報告至電學 603 計算機視覺研究室)

[Instructions]

The ***Gradient Descent method*** & ***Newton's method*** are useful for solving the optimization problems. In this homework assignment, your goal is to design the Python program that applies the two methods.

The iteration equation using the ***Gradient Descent method*** can be defined as:

$$x_{n+1} = x_n - \alpha f'(x_n)$$

where x_n is the nth iteration and α is the learning rate.

The iteration equation using the ***Newton's method*** can be defined as:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Please reference the Python Tutorial (課程講義) for details.

[Problems]

1. Suppose the function can be defined as: $y = f(x) = x^2 - 2x + 3$, please answer the following:
   (a) Design the Python program using the ***Gradient Descent*** method and print the values of $(x_n, y_n), n = 0...10$ (i.e., the first 10 iterations) given the initial guess $x_0 = 0$ and $\alpha = 0.1$. (5%)
   (b) Design the Python program using the ***Newton's method*** and print the values of $(x_n, y_n), n = 0...10$ (i.e., the first 10 iterations) given the initial guess $x_0 = 0$. (5%)
   (c) Plot the function and show $(x_n, y_n), n = 0...10$ in the ***same figure*** (For plotting, x is in the range of –0.5 ~ 1.5). As a result, you will generate two plots for the two methods. (10%)

Note: The figures must be carefully ***labeled***, ***titled***, and with your own ***copyright*** for full credits.

[Instructions]

Calculus is useful for solving optimization problems. In this homework assignment, our goal is to learn the ***method of least squares***, also known as ***Linear Regression*** (線性迴歸).

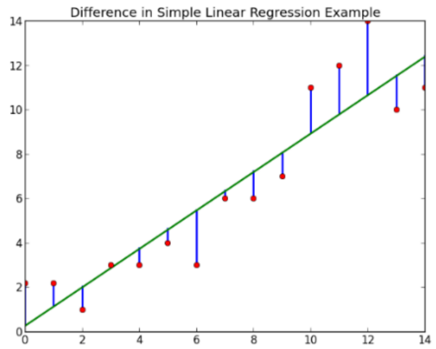
[Problems]

2. 給定一組資料點 (x_i, y_i), $i = 1 \ldots n$，**最小平方法** (Method of Least Squares) 的目的是找一直線 $y = ax + b$，使得每一點至直線的垂直距離總和(又稱為**平方誤差和** Sum of Square Errors) 可以達到最小值：

$$\varepsilon = \sum_{i=1}^{n} \left[y_i - ax_i - b \right]^2$$

假設給定一組資料點 (2, 1)、(3, 2)、(4, 3)、(5, 2)，試利用最小平方法求得最佳之直線方程式 (10%，請列出手寫推導過程即可)

【提示】 分別**設微分為 0**，即 $\dfrac{\partial \varepsilon}{\partial a} = 0$ 與 $\dfrac{\partial \varepsilon}{\partial b} = 0$。



[Instructions]

Direction fields are particularly useful for solving first-order differential equations when analytic solutions can't be found. To plot a direction field for a first-order differential equation with Python programming, the equation must be in **normal form**. A first-order differential equation is in normal form if it is expressed as:

$$\frac{dy}{dx} = f(x, y)$$

Please reference the Python Tutorial (課程講義) for details.

[Problems]

3. Following the aforementioned instructions, use the Python programming to obtain the direction field for each of the following differential equations (the interval *I* is given for (x, y) coordinates accordingly). The figures must be carefully ***labeled***, ***titled***, and with your own ***copyright*** for full credits. (20%)

(a) $\frac{dy}{dx} = x + y$; *I*: [-5 : 0.5 : 5, -5 : 0.5 : 5]

(b) $\frac{dy}{dx} = x - y$; *I*: [-5 : 0.5 : 5, -5 : 0.5 : 5]

(c) $\frac{dy}{dx} = xy$; *I*: [-5 : 0.5 : 5, -5 : 0.5 : 5]

(d) $\frac{dy}{dx} = \sin x \cos y$; *I*: [-5 : 0.5 : 5, -5 : 0.5 : 5]

[Instructions]

Python is useful for generating 2D plots using Numpy and Matplotlib. For example, the following Python source codes can be used to plot the function:

$$y = f(x) = e^{-x}\sin(2x)$$

Python Source Codes:

```
import numpy as np
import matplotlib.pyplot as plt

def f ( x ) :
    y = np.exp ( -x ) * np.sin ( 2 * x )
    return y

def main( ):
    x = np.linspace ( 0, 2 * np.pi, 100 )      # 於 [0, 2π] 產生 100 個點
    y = f ( x )
    plt.plot ( x, y )
    plt.xlabel ( ‘x’ )
    plt.ylabel ( ‘f(x)’ )
    plt.title ( ‘Plot of the Function f(x) = exp(-x)*sin(2x)’ )
    plt.text ( 4, -0.1, ‘Copyright@Chang’ )   # 請加上你的數位簽章
    plt.show ( )

main( )
```

Therefore, if an *explicit* solution can be found for a differential equation, you may plot the solution curve using the Python programming.

[Problems]

4. Solve the following initial value problems, and plot the solution curves. The interval *I* is given for the *x*-data in the plots. The figures must be carefully ***labeled***, ***titled***, and with your own ***copyright*** for full credits.

   注意：請先用手寫推導解題，再用 Python 畫圖，手寫推導過程與圖須放在同一頁面 (20% 每題 5 分)

   (a) $\dfrac{dy}{dx} = -xy,\ y(0) = 1,\quad I: [-3, 3]$

(b) $\dfrac{dy}{dx} = x\sqrt{1-y^2}$, $y(0)=1$, $I:[0,\, 2\pi]$

(c) $y' + (\tan x)y = \cos x$, $y(0)=1$, $I:[0,\, 4\pi]$

(d) $\dfrac{dy}{dx} = (x+y+1)^2$, $y(0)=-1$, $I:[0,\, 4\pi]$

[Instructions]

For many differential equations, *explicit* solutions may not be found. Instead, we may find the *implicit* solutions only. In these cases, we plot the 3D plot (surface plot) and level curves (contours) as the implicit solution curves. Therefore, if the implicit solution is defined by the relationship $f(x, y) = c$, you may plot the solution curves using Python programming.

The following Python source codes can be used to plot the function:

$$z = f(x, y) = x \cdot e^{-x^2 - y^2} = c$$

Python Source Codes:

```
# Calculate the Data
X = np.linspace ( -2, 2, 100 )        #  於 [-2, 2] 產生 100 個點
Y = np.linspace ( -2, 2, 100 )
x, y = np.meshgrid ( X, Y )           #  產生網狀格點
z = x * np.exp ( - x * x - y * y )    #  函數 z = f (x, y)

# 3D Plot (Surface Plot)
fig = plt.figure ( 1 )
ax = fig.gca ( projection = ‘3d’ )
ax.plot_surface ( x, y, z, cmap = cm.coolwarm, linewidth = 0, antialiased = False )
plt.xlabel ( 'x' )
plt.ylabel ( 'y' )
plt.title ( '3D Plot of f (x,y)' )

# Contour Plot
plt.figure ( 2 )
plt.contour ( x, y, z, 30 )      #  產生 30 個 Contours
plt.xlabel ( 'x' )
plt.ylabel ( 'y' )
plt.title ( 'Contour Plot of f (x,y)' )
plt.text ( 0.8, -1.8,    'Copyright@Chang' )

plt.show ( )
```

Please reference the Python Tutorial (課程講義) for details.

[Problems]

5. Solve the following initial value problems, and plot the solution curves. The interval I is given for the xy-data in the plots. The figures must be carefully ***labeled***, ***titled***, and with your own ***copyright*** for full credits.

   注意：請先用手寫推導解題，再用 Python 畫圖，手寫推導過程與圖須放在同一頁面 (20% 每題 5 分)

   (a) $2xy\ dx + (x^2 + 1)\ dy = 0$, $\quad I$: [-2 ~ 2, -2 ~ 2]

   (b) $(e^x + y)dx + (2 + x + ye^y)dy = 0$, $\quad I$: [0 ~ 2π, 0 ~ 2π]

   (c) $\cos x dx + \left(1 + \frac{2}{y}\right)\sin x dy = 0$, $\quad I$: [-2 ~ 2, -2 ~ 2]

   (d) $(\sin y - y\sin x)dx + (\cos x + x\cos y - y)dy = 0$, I : [-2π ~ 2π, -2π ~ 2π]