

Exam preparation cheetzheet:  
*Stuff from*

MNF130V2020

May 22, 2020

# 1 Chapter 1

**Propositional logic:** Logical  $\wedge, \vee, \oplus$  are trivial.

**Conditional statements (implication):**  $p \rightarrow q$ , if  $p$  then  $q \equiv p$  only if  $q \equiv p$  is a sufficient condition for  $q$ .

In other words,  $q$  is a necessary condition for  $p$ .  $p \rightarrow q$  is false then  $p$  is true and  $q$  is false and otherwise true.

$\neg(p \rightarrow q) \equiv p \wedge (\neg q)$ ,  $p \rightarrow q$  is equivalent to its contrapositive  $\neg q \rightarrow \neg p$ , but **not** to its **converse**  $q \rightarrow p$  or its inverse  $\neg p \rightarrow \neg q$ .

TABLE 6 Logical Equivalences.	
Equivalence	Name
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

TABLE 7 Logical Equivalences Involving Conditional Statements.
$p \rightarrow q \equiv \neg p \vee q$ $p \rightarrow q \equiv \neg q \rightarrow \neg p$ $p \vee q \equiv \neg p \rightarrow q$ $p \wedge q \equiv \neg(p \rightarrow \neg q)$ $\neg(p \rightarrow q) \equiv p \wedge \neg q$ $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$ $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$ $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$ $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.
$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$ $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$ $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

**Biconditional statements:**  $p \leftrightarrow q$  or expanded to  $(p \rightarrow q) \wedge (q \rightarrow p)$ .

**De Morgan:**  $\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$ ;  $\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$  Propositional logic can be represented by gates, creating combinational circuits which can represent **any** logical expression.

**Quantifiers:**

$\forall x(P(x) \rightarrow Q(x)) \equiv$  for all  $x$ , if  $P(x)$  then  $Q(x)$

$\exists x(P(x) \wedge Q(x)) \equiv$  there exists an  $x$  such that  $P(x)$  and  $Q(x)$

TABLE 2 De Morgan's Laws for Quantifiers.			
Negation	Equivalent Statement	When Is Negation True?	When False?
$\neg \exists x P(x)$	$\forall x \neg P(x)$	For every $x$ , $P(x)$ is false.	There is an $x$ for which $P(x)$ is true.
$\neg \forall x P(x)$	$\exists x \neg P(x)$	There is an $x$ for which $P(x)$ is false.	$P(x)$ is true for every $x$ .

TABLE 1 Quantifications of Two Variables.		
Statement	When True?	When False?
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair $x, y$ .	There is a pair $x, y$ for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every $x$ there is a $y$ for which $P(x, y)$ is true.	There is an $x$ such that $P(x, y)$ is false for every $y$ .
$\exists x \forall y P(x, y)$	There is an $x$ for which $P(x, y)$ is true for every $y$ .	For every $x$ there is a $y$ for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair $x, y$ for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair $x, y$ .

$P(x), Q(x)$  are propositional functions and there is always a **domain** or **universe of discourse**, either implicit or explicitly stated, over which the variable ranges.

**Negations of quantified propositions:**  $\neg \forall x P(x) \equiv \exists x \neg P(x)$ ;  $\neg \exists P(x) \equiv \forall x \neg P(x)$ .

**Theorem:** A proposition that can be proved; **lemma:** a simple theorem, commonly used as part of a greater picture to prove other theorems; **proof:** A demonstration that a proposition is true, **collorary:** A proposition that can be proved as a consequence of a theorem that has just been proved. A collorary can be seen as “Side effects” of the proved theorem.

A **valid** argument is an argument using correct rules of inference based on tautologies (something that will always give the **true** conclusion in **any** given scenario. I. E. a tautology is something that is always true for all possible combinations.)

An **invalid** argument can be referred to as a **fallacy**, such as affirming the conclusion, denying the hypothesis, begging the question or circular reasoning. They can lead to false conclusions.

**Some rules of inference:**

- $[p \wedge (p \rightarrow q)]$  Modus Ponens
- $[\neg q \wedge (p \rightarrow q)]$  Modus Tollens
- $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$  Hypothetical syllogism (Transitivity)
- $[(p \vee q) \wedge (\neg p)] \rightarrow q$  Disjunctive syllogism
- $\{P(a) \wedge \forall x [P(x) \rightarrow Q(x)]\} \rightarrow Q(a)$  Universal modus ponens
- $\{\neg Q(a) \wedge \forall x [P(x) \rightarrow Q(x)]\} \rightarrow \neg P(a)$  Universal modus tollens
- $(\forall x P(x)) \rightarrow P(c)$  Universal instantiation
- $(P(c) \text{ arbitrary } c) \rightarrow \forall x P(x)$  Universal generalization
- $(\exists x P(x)) \rightarrow (P(c) \text{ for some } c)$  Existential instantiation
- $(P(c) \text{ for some element } c) \rightarrow \exists x P(x)$  Existential generalization.

**TABLE 1 Rules of Inference.**

<i>Rule of Inference</i>	<i>Tautology</i>	<i>Name</i>
$\frac{p \quad p \rightarrow q}{\therefore q}$	$(p \wedge (p \rightarrow q)) \rightarrow q$	Modus ponens
$\frac{\neg q \quad p \rightarrow q}{\therefore \neg p}$	$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$	Modus tollens
$\frac{p \rightarrow q \quad q \rightarrow r}{\therefore p \rightarrow r}$	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$\frac{p \vee q \quad \neg p}{\therefore q}$	$((p \vee q) \wedge \neg p) \rightarrow q$	Disjunctive syllogism
$\frac{p}{\therefore p \vee q}$	$p \rightarrow (p \vee q)$	Addition
$\frac{p \wedge q}{\therefore p}$	$(p \wedge q) \rightarrow p$	Simplification
$\frac{p \quad q}{\therefore p \wedge q}$	$((p) \wedge (q)) \rightarrow (p \wedge q)$	Conjunction
$\frac{p \vee q \quad \neg p \vee r}{\therefore q \vee r}$	$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$	Resolution

## a Proofs

**Trivial proof:** A proof that  $p \rightarrow q$  just shows that  $q$  is true without using the hypothesis  $p$ .

**Vacuous proof:** A proof of  $p \rightarrow q$  that just shows that the hypothesis  $p$  is false.

**Direct proof:** A proof of  $p \rightarrow q$  that shows that the assumption of the hypothesis  $p$  implies the conclusion of  $q$ .

**Proof by contraposition:** A proof of  $p \rightarrow q$  that shows that the assumption of the negation of the conclusion  $q$  implies the negation of the hypothesis  $p$  (in other words, proof of contrapositive).

**Proof by contradiction:** A proof of  $p$  that shows that the assumption of the negation of  $p$  leads to a contradiction.

**Proof by cases:** A proof of  $(p_1 \vee p_2 \vee p_3 \dots p_n) \rightarrow q$  that shows that each conditional statement  $p_i \rightarrow q$  is true. Statements of the form  $p \leftrightarrow q$  require that both  $p \rightarrow q$  and  $q \rightarrow p$  be proved. It is sometimes necessary to give the two separate proofs (usually a direct proof or a proof by contraposition); other times a string of equivalences can be constructed starting with  $p$  and ending with  $q$ :  $p \leftrightarrow p_1 \leftrightarrow p_2 \dots \leftrightarrow p_n \leftrightarrow q$ .

To give a **constructive proof** of  $\exists x P(x)$  is to show how to find an element  $x$  that makes  $P(x)$  true. **Non-constructive existence proofs** are also possible, often using **proof by contradiction**.

One can **disprove** a universally quantified proposition  $\forall x P(x)$  simply by giving a **counter example**, i.e. an object  $x$  such that  $P(x)$  is **false**. One can, however, not prove it with such an example.

**Fermat's last theorem:** There are no positive integer solutions of  $x^n + y^n = z^n$  if  $n > 2$ .

An integer is **even** if it can be written as  $2k$  for some integer  $k$ ; an integer is **odd** if it can be written as  $2k + 1$  for some integer  $k$ . Every number is even or odd but not both. A number is **rational**, if it can be written as  $p/q$  with  $p$  being an integer and  $q$  strictly a non-zero integer.

## 2 Chapter 2

### a Sets

TABLE 1 Set Identities.	
Identity	Name
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{\overline{A}} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

**Empty set:** A set with no elements, commonly denoted as  $\emptyset$ . Do not confuse this with the set only containing the empty set. The difference is that the empty itself is empty, whereas the set containing the empty set has a single element.

**Subset:**  $A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$ , whereas a proper subset is  $A \subset B \equiv (A \subseteq B) \wedge (A \neq B)$ , in other words,  $B$  has at least one element different from the set  $A$ .

**Equality of sets:**  $A = B \equiv (A \subseteq B \wedge B \subseteq A) \equiv \forall x(x \in A \leftrightarrow x \in B)$ .

**Power set:**  $\mathcal{P}(A) = \{B | B \subseteq A\}$ , the set of all subsets of  $A$ . A set with  $n$  elements has  $2^n$  subsets.

**Cardinality:**  $|S|$ , the number of elements in  $S$ .

Some specific sets in regards to cardinality:  $\mathbb{R}$  is the set of real numbers, represented by either finite or infinite decimals;

$\mathbb{N}$  is the set of all natural numbers (eg.  $\{0, 1, 2, 3, 4, 5, \dots\}$ ),  $\mathbb{Z}$  is the set of integers  $\{\dots - 2, -1, 0, 1, 2, \dots\}$  and can also be denoted with only the positive or negative subset.  $\mathbb{Q}$  is the set of rational numbers, where  $\{p/q | p, q \in \mathbb{Z} \wedge q \neq 0\}$ ,  $\mathbb{Q}^+$  is the set of positive rational numbers and a subset of  $\mathbb{Q}$ .

**Set operations:**  $A \times B = \{(a, b) | a \in A \wedge b \in B\}$  (**Cartesian Product**);  $\overline{A}$  is the set of elements in the universe which are **not** in  $A$  (**complement**);  $A \cap B = \{x | x \in A \wedge x \in B\}$  (**intersection**);  $A \cup B = \{x | x \in A \vee x \in B\}$  (**union**);  $A - B = A \cap \overline{B}$  (**difference**);  $A \oplus B = (A - B) \cup (B - A)$ , (**symmetric difference/xor**)

**Inclusion-exclusion (simple case):**  $|A \cup B| = |A| + |B| - |A \cap B|$

**De Morgan's laws for sets:**  $\overline{A \cap B} = \overline{A} \cup \overline{B}$ ;  $\overline{A \cup B} = \overline{A} \cap \overline{B}$

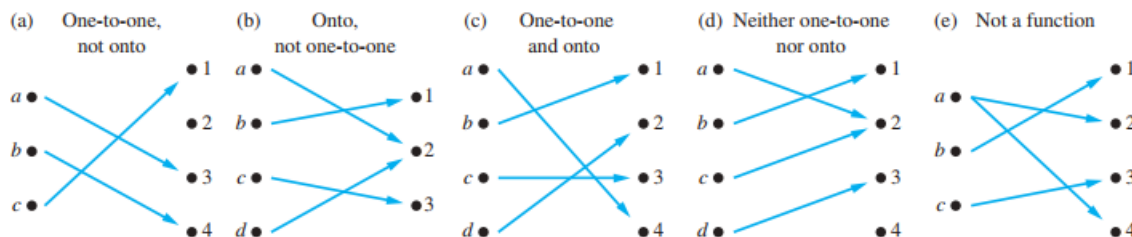
A **function**  $f$  from  $A$  (**the domain**) to  $B$  (**the co-domain**) is an assignment of a unique element of  $B$  to each element of  $A$ . Write  $f : A \rightarrow B$ . Write  $f(a) = b$  if  $b$  is assigned to  $a$ . **Range** of  $f$  is  $\{f(a) | a \in A\}$ ;  $f$  is **onto/surjective**  $\equiv \text{range}(f) = B$ ;  $f$  is **one-to-one/injective**  $\equiv \forall a_1 \forall a_2 [f(a_1) = f(a_2) \rightarrow a_1 = a_2]$

**EXAMPLE 11** Use set builder notation and logical equivalences to establish the first De Morgan law  $\overline{A \cap B} = \overline{A} \cup \overline{B}$ .

*Solution:* We can prove this identity with the following steps.

$$\begin{aligned}
 \overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement} \\
 &= \{x \mid \neg(x \in (A \cap B))\} && \text{by definition of does not belong symbol} \\
 &= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{by definition of intersection} \\
 &= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by the first De Morgan law for logical equivalences} \\
 &= \{x \mid x \notin A \vee x \notin B\} && \text{by definition of does not belong symbol} \\
 &= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{by definition of complement} \\
 &= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by definition of union} \\
 &= \overline{A} \cup \overline{B} && \text{by meaning of set builder notation}
 \end{aligned}$$

If  $f$  is one-to-one **and** onto, it is **bijective** and the **inverse** function  $f^{-1} : B \rightarrow A$  is defined by  $f^{-1}(y) = x \equiv f(x) = y$ .



If  $f : B \rightarrow C$  and  $g : A \rightarrow B$ , then the **composition**  $f \circ g$  is the function from  $A$  to  $C$  defined by  $f \circ g(x) = f(g(x))$ .

**Rounding functions:**  $\lfloor x \rfloor$  is the largest integer less than or equal to  $x$  **floor function**;  $\lceil x \rceil$  is the smallest integer greater than or equal to  $x$  **the ceiling function**.

**Summation notation:**

$$\sum_{n=1}^n a_i = a_1 + a_2 + a_3 + \dots + a_n$$

**Sum of first  $n$  positive integers:**

$$\sum_{j=1}^n j = 1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

**Sum of squares of first  $n$  positive integers:**

$$\sum_{j=1}^n j^2 = 1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

**TABLE 2** Some Useful Summation Formulae.

<i>Sum</i>	<i>Closed Form</i>
$\sum_{k=0}^n ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^n k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^n k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^n k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k,  x  < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1},  x  < 1$	$\frac{1}{(1-x)^2}$

**Sum of geometric progression:** (*I don't think we did this in the course*)



Two sets are said to have the **same cardinality** if there is a **bijection** between them. We can say that  $|A| \leq |B|$  if there is a one-to-one function from  $A$  to  $B$ .

A set is *countable* if it is finite or there is a **bijection** from the positive integers to the set. **In other words**, if the elements of the set can be listed (e.g.  $a_1, a_2, \dots$ ). Sets of the latter type are called *countably infinite* and the **cardinality of these sets are denoted with  $\aleph_0$** . The empty set, the integers and the rational numbers **are countable**. The union of a countable number of countable sets is countable.

**Schroder-Bernstein theorem:** If  $|A| \leq |B|$  and  $|B| \leq |A|$  then it must be that  $|A| = |B|$ . This can be explained as if there is a one-to-one function from  $A$  to  $B$  and a one-to-one function from  $B$  to  $A$ , then there is a one-to-one and onto function from  $A$  to  $B$ .

**Matrix Multiplication:** The  $(i, j)^{th}$  entry of  $\mathbf{AB}$  is  $\sum_{t=1}^k a_{it}b_{tj}$  for  $1 \leq i \leq m$  and  $1 \leq j \leq n$ , where  $\mathbf{A}$  is an  $m \times k$  matrix and  $\mathbf{B}$  is a  $k \times n$  matrix.

**Identity matrix**  $I_n$  with 1's on the main diagonal and 0's elsewhere is the multiplicative identity.

Cardinality arguments can be used to show that some functions are **uncomputable**.

Matrix addition (+), Boolean meet ( $\wedge$ ) and join ( $\vee$ ) are done entry-wise; Boolean matrix product ( $\odot$ ) is like matrix multiplication using boolean operators.

**Transpose:**  $\mathbf{A}^t$  is the matrix whose  $(i, j)^{th}$  entry is  $a_{ij}$  (the  $(j, i)^{th}$  entry of  $\mathbf{A}$ );

$\mathbf{A}$  is **symmetric** if  $\mathbf{A}^t = \mathbf{A}$ ;

### 3 Chapter 3

**Algorithm** are commonly expressed in **pseudo-code** when not directly implemented in a domain specific area.

**Keywords for algorithms:** {input, output, definiteness, correctness, finiteness, effectiveness, generality}.

**Greedy algorithms:** Will examine and pick the best choice at a given step. Not always the best.

**Brute forcing:** Specifically in discrete mathematics, this refers to examining the entire space of solutions and then determine the best one (very inefficient, sometimes necessary). Not explained in this course: **dynamic programming, probabilistic algorithms, divide-and-conquer.**

**Halting problem:** The unsolvable computing problem whether a program will halt given input. (Alan Turing for reference...)

**Big-O:** Half of inf102 is just this:

$f(x) = O(g(x))$  means  $\exists C \exists k \forall x (x > k \rightarrow |f(x)| \leq C|g(x)|)$ . Big-O of a sum is the largest (fastest growing) of the functions in the sum. Big-O of a product is the product of the big-O's of the factors. If  $f$  is  $O(g)$ , the  $g$  is  $\Omega(f)$  "big-omega". If  $f$  is both big-O and big-Omega of  $g$ , then  $f$  is  $\theta(g)$  "big-theta".

**Little-O:** We say that  $f(x)$  is  $o(g(x))$  if  $\lim_{x \rightarrow \infty} f(x)/g(x) = 0$ .

**Powers grow faster than logs:**  $(\log n)^c$  is  $O(n^d)$  but not the other way around, where  $c$  and  $d$  are positive numbers. If  $f_1(x)$  is  $O(g_1(x))$  and  $f_2(x)$  is  $O(g_2(x))$ , then  $(f_1 + f_2)(x)$  is  $O(\max(g_1(x), g_2(x)))$  and  $(f_1 f_2)(x)$  is  $O(g_1(x) g_2(x))$ .  $\log n!$  is  $O(n \log n)$ .

**Time complexity:** Binary search =  $O(\log n)$  (cut half of possibilities at each step), linear search  $O(n)$  (all input is examined exactly once), both have **space complexity (in terms of computer memory)  $O(1)$**  without taking the input into account. Bubble sort and insertion sort have  $O(n^2)$ .

Matrix multiplication has standard algorithm time complexity of  $O(m_1 m_2 m_3)$  if the matrices have dimensions  $m_1 \times m_2$  and  $m_2 \times m_3$ .

Efficient algorithms can reduce the complexity of multiplying two  $n \times x$  matrices from  $O(n^3)$  to  $O(n^{\sqrt{7}})$  Important complexity classes include polynomials  $n^b$ , exponential ( $b^n$  for  $b > 1$ ) and factorial ( $n!$ ).

A problem that can be solved by an algorithm with polynomial worst-case time complexity is called **tractable**; otherwise **intractable**.

**P=NP problem:** The class **P** is the class of tractable problems. The class **NP** consists of the problems for which it is possible to check solutions (**not FIND solutions**) in polynomial time. This means that  $P \subseteq NP$  yet the **P=NP** problem is unsolved because it has not been shown whether **P=NP**.

## 4 Chapter 4

**Divisibility:**  $a|b$  means  $a \neq 0 \wedge \exists c(a \cdot c = b)$  ( $a$  is a **divisor** or **factor** of  $b$  such that  $b$  is a multiple of  $a$ ).

**Base  $b$  representations:**  $(a_{n-1}a_{n-2}\dots a_2a_1a_0)_b = a_{n-1}b^{n-1} + \dots + a_2b^2 + a_1b + a_0$ .

To convert from base 10 to base  $b$ , continually divide by  $b$  and record remainders as  $a_0, a_1, a_2, \dots$  ( $b = 8$  is **octal**,  $b = 16$  is **hexadecimal**, using A-F for 10-15). Convert from binary to octal by grouping bits by **threes**, from the right, to hexadecimal by grouping by fours; because  $2^3 = 8$  and  $2^4 = 16$ .

**Addition:** of two **binary numerals** each of  $n$  bits  $((a_{n-1}a_{n-2}\dots a_2a_1a_0)_2)$  requires  $O(n)$  bit operations.

**Multiplication:** requires  $O(n^2)$  bit operations if done naively,  $O(n^{1.585})$  steps by more sophisticated algorithms.

**Division "algorithm":**

$$\forall a \forall d > 0 \exists q \exists r (a = dq + r \wedge 0 \leq r < d)$$

where  $q$  is the quotient and  $r$  is the remainder; we write  $a \bmod d$  for the remainder.

**Example of the division "algorithm":**

$$-18 = 5 \cdot (-4) + 2 \rightarrow -15 \bmod 5 = 2$$

**Congruent modulo  $m$ :**  $a \equiv b \pmod{m} \leftrightarrow m|a - b \leftrightarrow a \bmod m = b \bmod m$

One can do arithmetic in  $\mathbb{Z}_m = \{0, 1, \dots, m-1\}$  by working modulo  $m$ . There are fast algorithms for computing  $b^n \bmod m$ , based on successive squaring.

Integer  $n > 1$  is said to be **prime**  $\leftrightarrow$  its only factors are 1 and itself; otherwise it is referred to as a **composite**.

There are infinitely many **primes**, but it is not known whether there are infinitely many twin primes (**primes that differ by 2**), or whether every even positive integer greater than 2 is the sum of two primes (**Goldbach's conjecture**) or whether there are infinitely many **Mersenne primes**  $\rightarrow$  **primes of the form**  $2^p - 1$ .

**Naive test for primeness and test for prime factorization:** To find prime factorization of  $n$ , successively divide it by all primes less than  $\sqrt{n}$ ; if none is found, then  **$n$  is prime**. If a prime factor  $p$  is found, then continue the process to find the prime factorization of the remaining factor, namely  $n/p$ ; this time the trial divisions can start with  $p$ . Continue until a prime factor remains.

**Prime number theorem** states that there are approximately  $n/\ln(n)$  primes less than or equal to  $n$ .

**Fundamental theorem of arithmetic:** Every integer greater than 1 can be written as a product of one or more primes, and the product is unique except for the order of the factors. (A proof based on fact that if a prime divides a product of integers, then it divides at least one of those integers.)

**Euclidean algorithm for greatest common divisor:**  $\gcd(x, y) = \gcd(y, x \bmod y)$  if  $y \neq 0$ ;  $\gcd(x, 0) = x$ .

Using extended Euclidean algorithm or working backwards, one can find **Bezout coefficients** and write  $\gcd(a, b) = sa + tb$ .

Two integers are **relatively prime** if their greatest common divisor ( $\gcd$ ) is 1. The integers  $a_1, a_2, \dots, a_n$  are **pairwise relatively prime**  $\leftrightarrow \gcd(a_i, a_j) = 1$  whenever  $1 \leq i < j \leq n$ .

**Chinese remainder theorem:** If  $m_1, m_2, \dots, m_n$  are pairwise relatively prime, then the system  $\forall i (x \equiv a_i \pmod{m_i})$  has unique solution modulo  $m_1 m_2 \dots m_n$ . An example of application of this: Handling very large integers on a computer.

**Fermat's little theorem:**  $a^{p-1} \equiv 1 \pmod{p}$  if  $p$  prime and does not divide  $a$ . The converse is not true; for example  $2^{340} \equiv 1 \pmod{341}$  so  $341 (= 11 \cdot 31)$  is referred to as a **pseudo prime**;

If  $a$  and  $b$  are positive integers, then there exist integers  $s$  and  $t$  such that  $as + bt = \gcd(a, b)$  **linear combination**. This theorem allows one to compute the **multiplicative inverse**  $\bar{a}$  of  $a$  modulo  $b$  (i.e.  $\bar{a}a \equiv 1 \pmod{b}$ ) as long as  $a$  and  $b$  are relatively prime, which enables one to solve **linear congruences**  $ax \equiv c \pmod{b}$ .

A **primitive root** modulo a prime  $p$  is an integer  $r$  in  $\mathbb{Z}_p$  such that every nonzero element of  $\mathbb{Z}_p$  is a power of  $r$ .

**Discrete logarithms:**  $\log_r a = e$  modulo  $p$  if  $r^e \bmod p = a$  and  $1 \leq e \leq p-1$

A common **hashing function:**  $h(k) = k \bmod m$ , where  $k$  is the key.

**Check digits** for error-correcting codes like UPCs, involve modular arithmetic (?????????)

**Pseudorandom numbers:** can be generated by the **linear congruential method:**  $x_{n+1} = (ax_n + c) \bmod m$ , where

$x_0$  is arbitrarily chosen **seed**. Then  $\{x_n/m\}$  will be rather randomly distributed numbers between 0 and 1.

**Shift cipher:**  $f(p) = (p + k) \bmod 26$  [ $A \leftrightarrow 0, B \leftrightarrow 1, \dots$ ]. Caesar cipher used  $k = 3$ .

**Affine cipher:** Uses  $f(p) = (ap + b) \bmod 26$  with  $\gcd(a, 26) = 1$ .

**RSA public key encryption system:** An integer  $M$  representing the plaintext is translated into an integer  $C$  representing the ciphertext using the function  $C = M^e \bmod n$ , where  $n$  is a public number that is the product of two large (100-digit or so) primes and  $e$  is a public number *relatively* prime to  $(p - 1)(q - 1)$ ; the primes  $p$  and  $q$  are kept secret. Decryption is accomplished via  $M = C^d \bmod n$ , where  $d$  is an inverse of  $e$  modulo  $(p - 1)(q - 1)$ . It is infeasible to compute  $d$  without knowing  $p$  and  $q$ , which are infeasible to compute from  $n$ .

Similar methods can be used for **key exchange protocols, digital signatures, signing stuff in general**.

## 5 Chapter 5

**The well-ordering property:** Every non-empty set of nonnegative integers has a "least element".

**The principle of mathematical induction:** Let  $P(n)$  be a propositional function in which the domain (the universe of discourse) is the set of positive integers. Then if one can show that  $P(1)$  is true (through **Base case/Base step**) and that for every positive integer  $k$ , the conditional statement  $P(k) \rightarrow P(k+1)$  is true (**inductive step**), then one has proved that  $\forall n P(n)$ . The hypothesis  $P(k)$  in a proof of the inductive step is called the **inductive hypothesis**.

More generally, the induction can start at any integer, and there could potentially be several base cases.

### *Template for Proofs by Mathematical Induction*

1. Express the statement that is to be proved in the form "for all  $n \geq b$ ,  $P(n)$ " for a fixed integer  $b$ . For statements of the form " $P(n)$  for all positive integers  $n$ ," let  $b = 1$ , and for statements of the form " $P(n)$  for all nonnegative integers  $n$ ," let  $b = 0$ . For some statements of the form  $P(n)$ , such as inequalities, you may need to determine the appropriate value of  $b$  by checking the truth values of  $P(n)$  for small values of  $n$ , as is done in Example 6.
2. Write out the words "Basis Step." Then show that  $P(b)$  is true, taking care that the correct value of  $b$  is used. This completes the first part of the proof.
3. Write out the words "Inductive Step" and state, and clearly identify, the inductive hypothesis, in the form "Assume that  $P(k)$  is true for an arbitrary fixed integer  $k \geq b$ ."
4. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what  $P(k+1)$  says.
5. Prove the statement  $P(k+1)$  making use of the assumption  $P(k)$ . (Generally, this is the most difficult part of a mathematical induction proof. Decide on the most promising proof strategy and look ahead to see how to use the induction hypothesis to build your proof of the inductive step. Also, be sure that your proof is valid for all integers  $k$  with  $k \geq b$ , taking care that the proof works for small values of  $k$ , including  $k = b$ .)
6. Clearly identify the conclusion of the inductive step, such as by saying "This completes the inductive step."
7. After completing the basis step and the inductive step, state the conclusion, namely, "By mathematical induction,  $P(n)$  is true for all integers  $n$  with  $n \geq b$ ".

**Strong induction:** Let  $P(n)$  be a propositional function in which the domain (again, **universe of discourse**) is the set of positive integers. Then if one can show that  $P(1)$  is true, and that for every positive integer  $k$  the conditional statement  $[P(1) \wedge P(2) \wedge \dots \wedge P(k)] \rightarrow P(k+1)$  is true (**inductive step**), then one has proved  $\forall n P(n)$ . The hypothesis  $\forall j \leq k P(j)$  in a proof of the inductive step is called the (**strong**) **inductive hypothesis**. Again, the induction can start at any integer, and there can be several base cases.

**Inductive/Recursive definitions (functions):** Is a definition of a function  $f$  with the set of nonnegative integers as its domain: Specification of  $f(0)$ , together with, for each  $n > 0$ , a rule for finding  $f(n)$  from values of  $f(k)$  for  $k < n$ .

**Example:**  $0! = 1$  and  $(n+1)! = (n+1) \cdot n!$  (**factorial function**)

**Inductive/Recursive definitions (sets):** Definition of a set  $S$ : A rule specifying one or more particular elements of  $S$ , together with a rule for obtaining more elements of  $S$  from those already in it. It is understood that  $S$  consists precisely of those elements that can be obtained by applying these two rules.

**Structural induction:** can be used to prove facts about recursively defined objects.

**Fibonacci numbers:**  $f_0, f_1, f_2, \dots : f_0 = 0, f_1 = 1, f_n = f_{n-1} + f_{n-2}$  for all  $n \geq 2$ .

**Lame's theorem:** The number of divisions used by the Euclidean algorithm to find  $\gcd(a, b)$  is  $O(\log b)$ .

An algorithm is **recursive** if it solves a problem by reducing it to an instance of the same problem with smaller input. It is **iterative** if it is based on the repeated use of operations in a loop.

There is an efficient recursive algorithm for computing **modular powers** ( $b^n \bmod m$ ), based on computing  $b^{\lfloor \frac{n}{2} \rfloor} \bmod m$ .

**Merge sort:** is an efficient recursive algorithm for sorting a list: break the list into two parts, recursively sort each half, and then merge them together in order. It has  $O(n \log n)$  time complexity in **all** cases.

A program segment  $S$  is **partially correct** with respect to **initial assertion**  $p$  and **final assertion**  $q$ , written  $p\{S\}q$ , if whenever  $p$  is true for the input values of  $S$  and  $S$  terminates,  $q$  is true for the output values of  $S$ .

A **loop invariant** for **while condition**  $S$  is an assertion  $p$  that remains true each time  $S$  is executed in the loop; i.e.  $(p \wedge \text{condition})\{S\}p$ . If  $p$  is true before the program segment is executed, then  $p$  and  $\neg \text{condition}$  are true after it terminates (if it terminates at all). In symbols,  $p\{\text{while condition } S\}(\neg \text{condition} \wedge p)$ .

## 6 Chapter 6

**Sum rule:** Given  $t$  mutually exclusive tasks, if task  $i$  can be done in  $n_i$  ways, then the number of ways to do exactly one of the tasks is  $n_1 + n_2 + \dots + n_t$ .

**Size of union of disjoint sets:**  $|A_1 \cup A_2 \cup \dots \cup A_n| = |A_1| + |A_2| + \dots + |A_n|$ . ?? Duplicates ??

**Two-set case of inclusion-exclusion:**  $|A \cup B| = |A| + |B| - |A \cap B|$ . **Product rule:** If a task consists of successively performing  $t$  tasks, and if task  $i$  can be done in  $n_i$  ways (after previous tasks have been completed), then the number of ways to do the task is  $n_1 \cdot n_2 \dots n_t$ .

A set with  $n$  elements has  $2^n$  subsets (equivalently, there are  $2^n$  bit strings of length  $n$ ).

**Tree diagrams:** Can be used to organize counting problems.

**Pigeonhole principle:** If more than  $k$  objects are placed in  $k$  boxes, then some box will have more than 1 object.

**Generalized version of the pigeonhole principle:** If  $N$  objects are placed in  $k$  boxes, then some box will have at least  $\lceil N/k \rceil$  objects.

**Ramsey number:**  $R(m, n)$  is the smallest number of people there must be at a party so that there exist either  $m$  mutual friends or  $n$  mutual enemies (assuming each pair of people are either friends or enemies).  $R(3, 3) = 6$ .

**r-permutation** of set with  $n$  objects, *ordered* arrangement of  $r$  of the objects from the set (no repetitions allowed); there are  $P(n, r) = n!/(n-r)!$  such permutations. **r-combination** of set with  $n$  objects, *unordered* selection (i.e. subset) of  $r$  of the objects from the set (no repetitions allowed); there are  $C(n, r) = n!/[r!(n-r)!]$  such combinations. Alternative notation is  $\binom{n}{r}$ , also called the binomial coefficient.

**Pascal's identity:** *not in curriculum*.

**Combinatorial identities:** *not in curriculum*.

Number of **r-permutations** of an  $n$ -set **with repetitions allowed** is  $n^r$ ; number of **r-combinations** of an  $n$ -set **with repetitions allowed** is  $C(n+r-1, r)$ . This latter value is the same as the **number of solutions in nonnegative integers** to  $x_1 + x_2 + \dots + x_n = r$ .

## 7 Chapter 7

If all outcomes are equally likely in a **sample space**  $S$  with  $n$  outcomes, then the **probability of an event**  $E$  is  $p(E) = |E|/n$ ; more generally, if  $p(s_i)$  is probability of  $i^{th}$  outcome  $s_i$ , then  $p(E) = \sum_{s_i \in E} p(s_i)$ .

**Probability distributions:** satisfy these conditions:  $0 \leq p(s) \leq 1$  for each  $s \in S$  and  $\sum_{s \in S} p(s) = 1$ .

For **complementary event**,  $p(\bar{E}) = 1 - p(E)$ ; for **union** of two events (either one or both happen),  $p(E \cup F) = p(E) + p(F) - p(E \cap F)$ ; for **independent events**,  $p(E \cap F) = p(E)p(F)$ .

The **conditional probability** of  $E$  given  $F$  (probability that  $E$  will happen after it is known that  $F$  happened) is  $p(E|F) = p(E \cap F)/p(F)$ .

**Bernoulli trials:** If only two outcomes are **success** and **failure**, with  $p(\text{success}) = p$  and  $p(\text{failure}) = q = 1 - p$ , then the **binomial distribution** applies, with probability of exactly  $k$  success in  $n$  trials being  $b(k; n, p) = C(n, k)p^k q^{n-k}$ .

### THEOREM 2

The probability of exactly  $k$  successes in  $n$  independent Bernoulli trials, with probability of success  $p$  and probability of failure  $q = 1 - p$ , is

$$C(n, k)p^k q^{n-k}.$$

**Bayes Theorem:** *Dont think this was in the curriculum.*



## 8 Chapter 9 (Chapter 8 not in curriculum)

**(Binary) Relations:** A binary relation  $R$  from  $A$  to  $B$ : subset of the **cartesian** set  $A \times B$ . Write  $aRb$  for  $(a, b) \in R$ ; relation **on**  $A$  is relation from  $A$  to  $A$ ; graph of a function from  $A$  to  $B$  is a relation such that  $\forall a \in A \exists$  exactly one pair  $(a, b)$  in the relation.

Relation  $R$  on a set  $A$  is **reflexive** if  $aRa$  for all  $a \in A$ .

**Irreflexive** if  $aRa$  for no  $a \in A$ .

**Symmetric** if  $aRb$  implies  $bRa$  for all  $a, b \in A$ .

**Asymmetric** if  $aRb$  implies that  $b$  is not related to  $a$ , for all  $a, b \in A$ .

**Antisymmetric** if  $aRb \wedge bRa$  implies  $a = b$  for all  $(a, b) \in A$ .

**Transitive** if  $aRb \wedge bRc$  implies  $aRc$  for all  $a, b, c \in A$ .

**Inverse** relation to  $R$  is given by  $bR^{-1}a \leftrightarrow aRB$ .

If  $R$  is a relation from  $A$  to  $B$  and  $S$  is a relation from  $B$  to  $C$ , then the **composite** is the relation  $S \circ R$  from  $A$  to  $C$  in which  $a$  is related to  $c \leftrightarrow$  there exists a  $b \in B$  such that  $aRb$  and  $bSc$ .  $R^n = R \circ R \circ \dots \circ R$ .

**N-ary relations:** on domains  $A_1, A_2, \dots, A_n$  is a subset of  $A_1 \times A_2 \times \dots \times A_n$ ; databases using the relational data model are just sets of **n-ary** relations in which each  $n$ -tuple is called a **record** made up of **fields** ( $n$  is the **degree**).

**A relation as matrix representation:** A relation  $R$  on  $A = \{a_1, a_2, \dots, a_n\}$  can be represented by an  $n \times n$  matrix  $M_R$  whose  $(i, j)^{th}$  entry is 1 if  $a_iRa_j$  and is 0 otherwise. Reflexivity, symmetry, antisymmetry, transitivity can easily be read off the matrix. Boolean products of the matrices give the matrix for the composite ( $M_{S \circ R} = M_R \odot M_S$ ).

A domain is a **primary key** if the corresponding field uniquely determines the record.

**Equivalence classes:** A relation  $R$  of set  $A$  is an **equivalence relation** if  $R$  is reflexive, symmetric and transitive. Usually, equivalence relations can be recognized by their definitions being of the form "two elements are related  $\leftrightarrow$  they have the same [something]".

For each  $a \in A$  the set of elements in  $A$  related to (i.e **equivalent to**)  $a$  is the **equivalence class** of  $a$ , denoted  $[a]$ .

**Canonical example:** Congruence modulo  $m$  with equivalence.

The set of equivalence classes partitions  $A$  into pairwise disjoint nonempty sets; conversely, every partition of  $A$  induces an equivalence relation by declaring two elements to be related if they are in the same set of the partition.