Relations

Rosen Section 9.1, 9.3

Tom Michoel

MNF130V2020 – Week 14

Binary relations

Definition

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Example
$\leq$ is a relation on $\mathbb{R}$. Formally, we can write

$$R_{\leq} = \{(a, b) \mid a, b \in \mathbb{R}, a \leq b\}$$

Relations vs. functions

- A *function* f from A to B assigns *exactly* one element of B to each element of A. The *graph* of f is the set $\{(a, f(a)) \mid a \in A\} \subset A \times B$. Hence **every function defines a binary relation**.

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- Relations are a generalization of graphs of functions: **not every binary relation defines a function**.
- Main difference between relations and (graphs of) functions: relations can be **one-to-many** (assigning more than one element of B to an element of A.)

  Example: $R_{\leq} = \{(a, b) \mid a, b \in \mathbb{R}, a \leq b\}$

Representing relations using directed graphs

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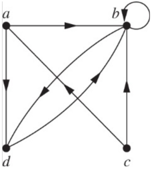
A **directed graph**, or **digraph**, $\mathcal{G} = (V, E)$ consists of a set of *vertices* (or *nodes*) V together with a set E of ordered pairs of elements of V called *edges* (or *arcs*).



$$R = \{(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)\}$$

Figure 1: A directed graph and the relation is represents

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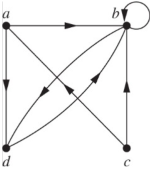

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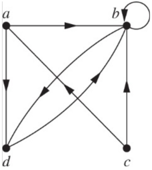
- The vertex a is called the *initial* vertex of the edge (a, b) and the vertex b is called the *terminal* vertex of this edge. An edge of the form (a, a) is called a *loop*.



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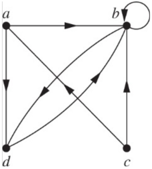
- The vertex a is called the *initial* vertex of the edge (a, b) and the vertex b is called the *terminal* vertex of this edge. An edge of the form (a, a) is called a *loop*.
- A relation R on a set A can be represented by the digraph $\mathcal{G} = (A, R)$



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Relations on a set: reflexive

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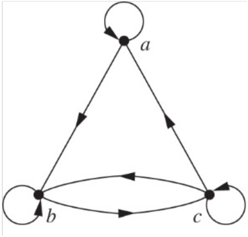

Figure 2: A reflexive relation

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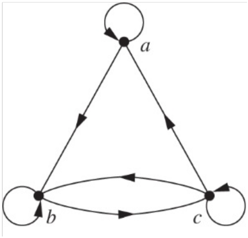
- A relation R on a set A is **reflexive** if $(a, a) \in R$ for all $a \in A$.
- A relation is reflexive if and only if its digraph representation has a loop at every vertex.



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- A relation R on a set A is **symmetric** if $(a, b) \in R \rightarrow (b, a) \in R$, for all $a, b \in A$.

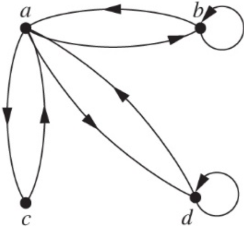

Figure 3: A symmetric relation

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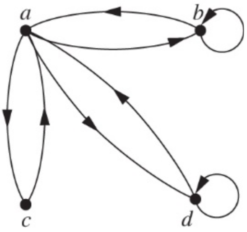
- A relation R on a set A is **symmetric** if $(a, b) \in R \to (b, a) \in R$, for all $a, b \in A$.

- A relation is symmetric if and only if for every edge between distinct vertices in its digraph there is also an edge in the opposite direction.



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Relations on a set: antisymmetric

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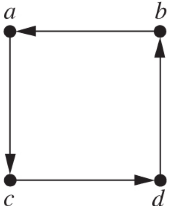

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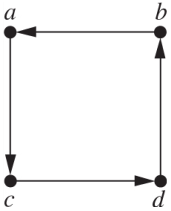
- A relation is antisymmetric if and only if there are never two edges in opposite direction between two distinct vertices.



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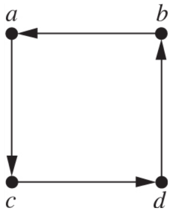
- A relation is antisymmetric if and only if there are never two edges in opposite direction between two distinct vertices.
- Relations on a set can be both symmetric and antisymmetric.
  Example: Let $R = \{(a,a) \mid a \in A\}$.



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Relations on a set: transitive

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- A relation R on a set A is **transitive** if $\big((a,b) \in R \wedge (b,c) \in R\big) \rightarrow (a,c) \in R$, for all $a, b, c \in A$.

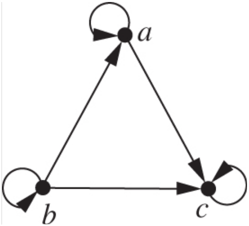

Figure 5: A transitive relation

Relations on a set: transitive

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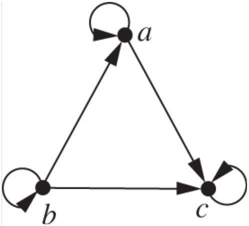
- A relation is transitive if and only if whenever there is an edge from vertex x to y and an edge from vertex y to z, there is an edge from x to z (completing triangles).



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Examples

Relation	Refl	Symm	Antisymm	Trans

Table 1: Example relations on $\mathbb{R}$

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$R_4 = \{(a, b) \mid a = b\}$	✓	✓	✓	✓
$R_5 = \{(a, b) \mid b = a + 1\}$	–	–	✓	–

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$R_5 = \{(a, b) \mid b = a + 1\}$	–	–	✓	–
$R_6 = \{(a, b) \mid a \equiv b \pmod 3\}$	✓	✓	–	✓

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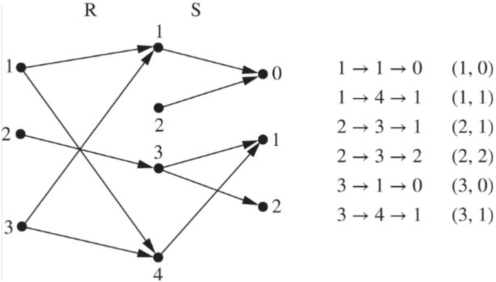
Composite relations

Definition

The **composite** $S \circ R$ of a relation R from A to B and a relation S from B to C is the relation consisting of ordered pairs (a, c), where



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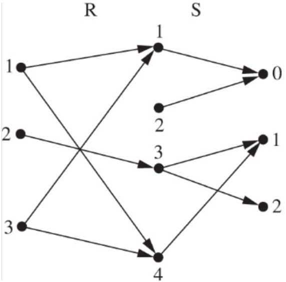

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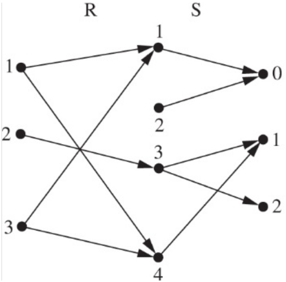

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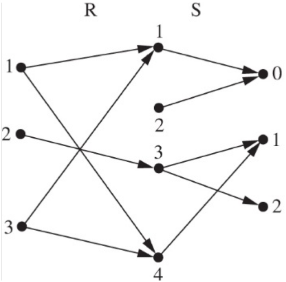
- $a \in A$, $c \in C$
- there exists $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.
- In one line:
  $S \circ R = \{(a, c) \mid a \in A, c \in C, \exists b \in B\big((a, b) \in R \land (b, c) \in S\big)\}$



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Composing the parent relation with itself

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The **powers** R^n, $n = 1, 2, 3, \ldots$, of a relation R on a set A are defined recursively by

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- Then $(a, c) \in R^2$ if and only if $\exists b$ such that $(a, b) \in R$ and $(b, c) \in R$, that is $b = a + 1$ and $c = b + 1$, or $c = a + 2$.

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- Hence $R^2 = \{(a, b) \mid a, b \in \mathbb{N}, b - a = 2\}$.
- More generally, $R^n = \{(a, b) \mid a, b \in \mathbb{N}, b - a = n\}$ for all n.

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A relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n \geq 1$.

Proof.

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- Let $p =$ "*R is transitive*" and $Q(n) = R^n \subseteq R$. To prove the equivalence $p \leftrightarrow \forall n Q(n)$, we first prove $p \rightarrow \forall n Q(n)$ and then $\forall n Q(n) \rightarrow p$.

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    - Take $(a, c) \in R^{k+1}$ arbitrarily.
    - Because $R^{k+1} = R^k \circ R$, there must exist $b \in R$ such that $(a, b) \in R \wedge (b, c) \in R^k$.

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    - Because $R^{k+1} = R^k \circ R$, there must exist $b \in R$ such that $(a, b) \in R \wedge (b, c) \in R^k$.
    - By the inductive hypothesis $R^k \subseteq R$ and hence $(b, c) \in R$.

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  - Hence p is true.

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Representing relations using matrices

A relation between finite sets $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_m\}$ can also be represented using the $m \times n$ **zero-one** or **Boolean matrix** $\mathbf{M}_R = (M_{ij})$ where

$$M_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{otherwise} \end{cases}$$

Example

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$$\mathbf{M}_R = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Matrix properties for reflexive, symmetric and antisymmetric relations

If R is a relation on a set A (that is, $A = B$), then R is:

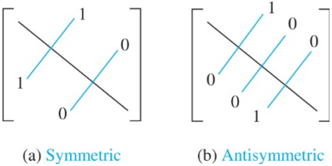

Figure 7: Zero-one matrices for symmetric and antisymmetric relations

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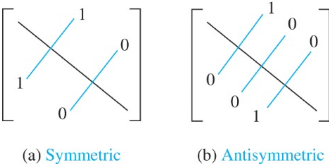

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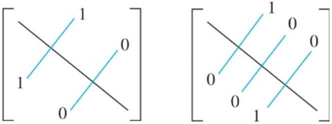

(a) Symmetric (b) Antisymmetric

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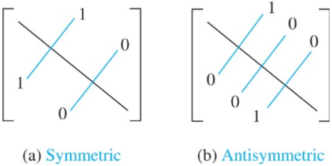
- **symmetric** if and only if $(\mathbf{M}_R)^t = \mathbf{M}_R$.
- **antisymmetric** if $M_{ij} = 0 \vee M_{ji} = 0$ for all $i \neq j$.



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