Exam preparation crib sheet

MNF130V2020

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1 Chapter 1

Propositional logic:. Logical $\wedge, \vee, \oplus$ are trivial.
Conditional statements (implication): $p \rightarrow q$, if p then $q \equiv p$ only if $q \equiv p$ is a sufficient condition for q.
In other words, q is a necessary condition for p. $p \rightarrow q$ is false then p is true and q is false and otherwise true. $\neg(p \rightarrow q) \equiv p \wedge (\neg q)$, $p \rightarrow q$ is equivalent to its contrapositive $\neg q \rightarrow \neg p$, but **not** to its **converse** $q \rightarrow p$ **or** its inverse $\neg p \rightarrow \neg q$.
Biconditional statements: $p \leftrightarrow q$ or expanded to $(p \rightarrow q) \wedge (q \rightarrow p)$.
De Morgan: $\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$; $\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$ Propositional logic can be represented by gates, creating combinational circuits which can represent **any** logical expression.
Quantifiers:

$\forall x(P(x) \rightarrow Q(x)) \equiv$ *for all x, if P(x) then Q(x)*
$\exists x(P(x) \wedge Q(x)) \equiv$ *there exists an x such that P(x) and Q(x)*

P(x),Q(x) are propositional functions and there is always a **domain** or **universe of discourse**, either implicit or explicitly stated,over which the variable ranges.
Negations of quantified propositions: $\neg \forall x P(x) \equiv \exists x \neg P(x); \neg \exists P(x) \equiv \forall x \neg P(x)$.

Theorem: A proposition that can be proved; **lemma:** a simple theorem, commonly used as part of a greater picture to prove other theorems; **proof:** A demonstration that a proposition is true, **collorary:** A proposition that can be proved as a consequence of a theorem that has just been proved. A collorary can be seen as “Side effects“ of the prooved theorem.
A **valid** argument is an argument using correct rules of inference based on tautologies (something that will always give the **true** conclusion in **any** given scenario. I. E. a tautology is something that is always true for all possible combinations.)
An **invalid** argument can be referred to as a **fallacy**, such as affirming the conclusion, denying the hypothesis, begging the question or circular reasoning. They can lead to false conclusions.
Some rules of inference:

- $[p \wedge (p \rightarrow q)]$ Modus Ponens
- $[\neg q \wedge (p \rightarrow q)]$ Modus Tollens
- $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ Hypothetical syllogism (Transitivity)
- $[(p \vee q) \wedge (\neg p)] \rightarrow q$ Disjunctive syllogism
- $\{P(a) \wedge \forall x[P(x) \rightarrow Q(x)]\} \rightarrow Q(a)$ Universal modus ponens
- $\{\neg Q(a) \wedge \forall x[P(x) \rightarrow Q(x)]\} \rightarrow \neg P(a)$ Universal modus tollens
- $(\forall x P(x)) \rightarrow P(c)$ Universal instantiation
- $(P(c) arbitrary\ c) \rightarrow \forall x P(x)$ Universal generalization
- $(\exists x P(x)) \rightarrow (P(c)\ for\ some\ c)$ Existential instantiation
- $(P(c)\ for\ some\ element\ c) \rightarrow \exists x P(x)$ Existential generalization.

a Proofs

Trivial proof: A proof that $p \to q$ just shows that q is true witout using the hypothesis p.
Vacuous proof: A proof of $p \to q$ that just shows that the hypothesis p is false.
Direct proof: A proof of $p \to q$ that shows that the assumption of the hypothesis p implies the conclusion of q.
Proof by contraposition: A proof of $p \to q$ that shows that the assumption of the negation of the conclusion q implies the negation of the hypothesis p (in other words, proof of contrapositive).
Proof by contradiction: A proof of p that shows that the assumption of the negation of p leads to a contradiction.
Proof by cases: A proof of $(p_1 \vee p_2 \vee p_3 ... p_n) \to q$ that shows that each conditional statement $p_i \to q$ is true.
Statements of the form $p \leftrightarrow q$ require that both $p \to q$ and $q \to p$ be proved. It is sometimes necessary to give the two separate proof (usually a direct proof or a proof by contraposition); other times a string of equivalences can be constructed starting with p and ending with $q : p \leftrightarrow p_1 \leftrightarrow p_2 ... \leftrightarrow p_n \leftrightarrow q$.
To give a **constructive proof** of $\exists x P(x)$ is to show how to find an element x that makes $P(x)$ true. **Non-constructive existence proofs** are also possible, often using **proof by contradiction**.
One can **disprove** a universally quantified proposition $\forall x P(x)$ simply by giving a **counter example**, i.e. an object x such that $P(x)$ is **false**. One can, however, not proove it with such an example.
Fermat's last theorem: There are no positive integer solutions of $x^n + y^n = z^n \; if \; n > 2$.

An integer is **even** if it can be written as $2k$ for some integer k; an integer is **odd** if it can be written as $2k+1$ for some integer k. Every number is even or odd but not both. A number is **rational**, if it can be written as p/q with p being an integer and q strictly a non-zero integer.