Compulsory Assignment 3

MNF130V2020

Due on: Friday 13 March 2020, 14:00

- Submit your assignment by leaving it in the box marked **MNF130** at the **reception of the Department of Informatics (Datablokken, 4th floor)**. The box will be available from Monday 9 March.
- If you cannot deliver the assignment in person, only use a **UiB Pullprint scanner** and **submit on Mitt**. Assignments sent by email, or from camera pictures instead of scans, or otherwise illegible, will not count as a valid submission and will not be graded.
- Write your answers **one-sided** (don't use both sides of a page), and start a new page for every exercise.
- Write **your name** on every page.
- You may write your answers in English or Norwegian.
- The assignment covers the entire syllabus covered during the lectures so far.
- The assignment is scored on **30 points**. Hence you need to score **at least 10.5 points to pass**.

1 Propositional and predicate logic (8 points)

a) Fill out the truth table:

p	q	r	$p \to q$	$q \to r$	$(p \to q) \land (q \to r)$	$p \to r$	$[(p \to q) \land (q \to r)] \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

b) Let p, q, r be propositions. Is the compound proposition $[(p \to q) \land (q \to r)] \to (p \to r)$ a tautology? Explain why/why not.

The compound proposition is a tautology, because it is true for all cases in the truth table.
By definition, a tautology is an assertion that is true in every possible case/interpretation.

□

c) Let p, q, r be propositions. Use basic logical equivalences to prove that the compound propositions $(p \land \neg q) \to r$ and $p \to (q \lor r)$ are logically equivalent.

Write as disjunction.
$(p \land \neg q) \to r \equiv \neg(p \land \neg q) \lor r$
De Morgan.
$\neg(p \land \neg q) \lor r \equiv (\neg p \lor q) \lor r$
Assiociative laws.
$(\neg p \lor q) \lor r \equiv \neg p \lor (q \lor r)$
Logical equivalence for conditional statements.
$\neg p \lor (p \lor r) \equiv p \to (q \lor r)$

□

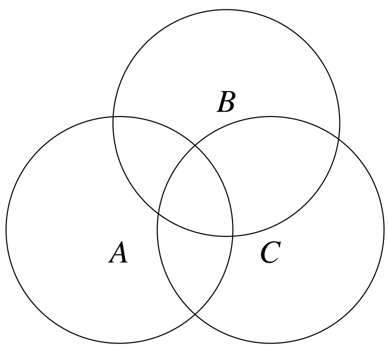
d) What are the truth values of the statements $\forall n \exists m (n + m = 0)$ and $\exists n \forall m (n < m^2)$ if the domain for all variables consists of $\mathbb{Z}$, the set of all integers? Explain your answer.

- The first statement, $\forall n \exists m (n + m = 0)$ is **true**, because the domain of integers contains a negative/positive counterpart for every value. For the particular statement, it means that for all n, there exists a counterpart such that the sum will be 0. This works because there only has to exist a single m for any n value.
- The second statement, $\exists n \forall m (n < m^2)$ is **true**, because there has to exist a value of $\mathbb{Z}$ such that all values in $\mathbb{Z}$ squared are greater that this number. You can pick n to be a negative number, and all m values will be strictly greater. So the statement is true.

□

2 Set theory and functions (8 points)

a) Let A, B, C be sets. Draw a Venn diagram and color the region $(A-C)\cap(C-B)$. Prove (using set identities) or disprove (give a counterexample) that $(A-C)\cap(C-B)=\emptyset$.



Let $x \in (A-C) \wedge x \in (C-B)$.
For the first part of the intersection, x is in A and not in C, for the second part of the intersection, x is in C and not in B.

Expanding this, we get:
$x \in C \wedge \neg(x \in C)$

Which is **F** by the negation law.

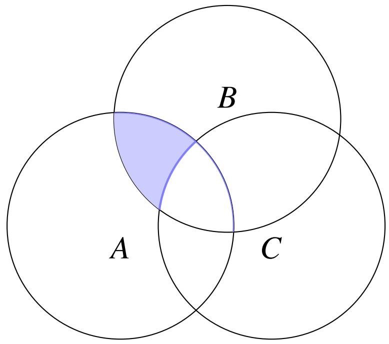
The proposition now looks like:
$x \in A \wedge \neg(x \in B) \wedge F$.

With set identities, this can be written as:
$(A-C)\cap(C-B) \equiv (A\cap\overline{C})\cap(C\cap\overline{B}) \equiv (A\cap\overline{B})\cap(C\cap\overline{C})$

$(A\cap\overline{B})\cap(C\cap\overline{C}) \equiv (A\cap\overline{B})\cap\emptyset \equiv \emptyset$

This means that $(A-C)\cap(C-B)\subseteq\emptyset$. And thus the initial proposition is a subset of the empty set. □

b) Let A, B, C be sets. Draw a Venn diagram and color the region $(A-C)\cap(B-C)$. Prove (using set identities) or disprove (give a counterexample) that $(A-C)\cap(B-C)=\emptyset$.



Looking at the diagram, I will try to create a counterexample, because it looks like that's whats needed...

Let $A = B = \{1\}$ and $C = \emptyset$.

$A - C = A - \emptyset = A = B = \{1\}$.
$B - C = B - \emptyset = B = A = \{1\}$.

$(A - C) \cap (B - C) = A \cap B = A \cap A = A = B = \{1\}$.

$\{1\} \neq \emptyset$ and therefore we disproved the proposition by counterexample. □

c) Let $f(x) = x^2$ be a function from the set of real numbers to the set of real numbers. Is f one-to-one (injective)? Onto (surjective)? A one-to-one correspondence (bijective)? Explain why/why not.

- It is **not** injective for $\mathbb{R}$ because multiple inputs will map to the same output. A quick example would be 3 and -3.
- It is also **not** surjective because the negative co-domain $\mathbb{R}^-$ does not have pre-images mapping to them. No input x will map to a value $y < 0$. Thus is not surjective.
- It is also **not** bijective because it is neither injective or surjective. □

3 Number theory (6 points)

a) Let a be an integer that is not divisible by 3. Prove that $(a+1)(a+2)$ is divisible by 3.

b) Use the Euclidean algorithm to find $\gcd(252, 356)$.

$356/252 = 1$, the remainder is 104. 356 can be written as $252 \cdot 1 + 104$. Continue with gcd(252,104).

$252/104 = 2$, remainder 44. $252 = 104 * 2 + 44$. gcd(104,44).

$104/44 = 2$, remainder 16. $104 = 44 * 2 + 16$. gcd(44, 16).

$44/16 = 2$, remainder 12. $44 = 16 * 2 + 12$. gcd(16,12)

$16/12 = 1$, remainder 4. $16 = 12 * 1 + 4$. gcd(12,4)

$12/4 = 3$. Greater common denominator with 356 and 252 is **3**. $\square$

c) Find each of these values:

- $(177 \bmod 31 + 270 \bmod 31) \bmod 31$

  177 mod 31 = 22 and 270 mod 31 = 22.
  Simplify the equation as $(22+22)$ mod 31 = 44 mod 31 = **13**. $\square$

- $\left[5(99^2 \bmod 32)\right] \bmod 15$
  99^2 = 9801.

  9801 mod 32 = 9.

  9 * 5 = 45.

  45 mod 15 = **0**. $\square$

4 Induction (8 points)

Use mathematical induction to prove that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

for all integers $n \geq 1$.

Let $P(n)$ =

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

- Basis step, verify $P(1)$ (smallest n in this case is 1).

$$P(1) = \sum_{k=1}^{1} k^2 = \frac{1(1+1)(2\cdot 1+1)}{6} \Rightarrow 1^2 = \frac{6}{6} = 1$$

- Next, we assume P(m) is true for an arbitrary $m \geq 1$.
  To write the function in terms of m:

$$P(m) = \sum_{k=1}^{m} k^2 = \frac{m(m+1)(2m+1)}{6}$$

- $P(m+1)$ can be written as:

$$P(m+1) = \sum_{k=1}^{m} k^2 + (m+1)^2 = \frac{m(m+1)(2m+1)}{6} + (m+1)^2$$

$$= \frac{m(m+1)(2m+1)+6(m+1)^2}{6} = \frac{m(m+1)(2m+1)+6(m+1)(m+1)}{6}$$

$$= \frac{(m+1)(m(2m+1)+6(m+1))}{6} = \frac{(m+1)(m+2)(2(m+1)+1)}{6}$$

$$= \frac{(m+1)((m+1)+1)(2(m+1)+1)}{6}$$

Now we have shown that the basis is verified, and it will hold for $P(m)$ and some $P(m+1)$ where $m \geq 1$.

$\square$