Permutations and Combinations

Rosen Section 6.3

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MNF130V2020 – Week 13

Permutations

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- If A is a *set* of n objects, a permutation of A is an *element* of the n-fold Cartesian product $A^n = A \times A \times \cdots \times A$, that is, an *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$, with $a_i \in A$ for $i = 1, 2, \ldots, n$.

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Example

Let $A = \{a, b, c\}$. The permutations of A are:

$$\begin{array}{cc}
(a, b, c) & (a, c, b) \\
(b, a, c) & (b, c, a) \\
(c, a, b) & (c, b, a)
\end{array}$$

Theorem
If n is a positive integer, then there are $n!$ permutations of a set with n distinct elements.

Proof (direct, see also slides Section 2.3).

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- The number of permutations is the same as the number of ordered n-tuples that can be made with n distinct element.
- By the product rule for counting, to make an ordered n-tuple, we have:

$$\begin{aligned}
&n \text{ choices for the 1st element} \\
&\times(n-1) \text{ choices for the 2nd element} \\
&\times(n-2) \text{ choices for the 3rd element} \\
&\times \ldots \\
&\times 2 \text{ choices for the } (n-1)\text{th element} \\
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- This makes $n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1$ ways of making an ordered n-tuple.

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  - By the induction hypothesis, there are $m!$ permutations of the set $A - \{a\}$.
  - Each permutation $(a_1, a_2, \ldots, a_m)$ of $A - \{a\}$ can be combined with a to form a permutation $(a, a_1, a_2, \ldots, a_m)$ of A.

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- By the principle of mathematical induction, there are $n!$ permutations of a set with n distinct elements, for all integers $n \geq 1$.

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Example

Let $A = \{a, b, c\}$. The 2-permutations of A are:

$$\begin{array}{cc} (a, b) & (b, a) \\ (a, c) & (c, a) \\ (b, c) & (c, b) \end{array}$$

Theorem

If n, r are positive integers with $1 \leq r \leq n$, then there are

$$P(n,r) = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

r-permutations of a set with n distinct elements.

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- To make an ordered r-tuple, we have n choices for the 1st element, $n-1$ choices for the 2nd element, $n-2$ choices for the 3rd element, and so on, until there are $\big(n-(r-1)\big) = (n-r+1)$ choices for the $r$the element.

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- The number of ways to award the medals is the number of ordered 3-permutations of a set of 8 elements.
- Hence there are $P(8,3) = 8 \cdot 7 \cdot 6 = 336$ ways to award the medals.

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Theorem

If n, r *are integers with* $0 \leq r \leq n$*, then there are*

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r*-combinations of a set with* n *distinct elements.*

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- Let A be an arbitrary set with n elements.
- There are $P(n, r)$ r-permutations of A.

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- There are $P(n, r)$ r-permutations of A.
- To obtain the set of r-permutations of A, we can take first an r-combination (subset of r elements) of A, and then consider all r-permutations of this subset.

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- By the product rule, $P(n, r) = C(n, r) \cdot P(r, r)$.
- Hence

$$C(n, r) = \frac{P(n, r)}{P(r, r)} = \frac{n!}{(n-r)!r!}$$

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Corollary

If n, r are integers with $0 \leq r \leq n$, then $C(n, r) = C(n, n - r)$.

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Proof.

- This follows immediately from the formula

$$C(n, r) = \frac{n!}{(n-r)!r!} = C(n, n-r)$$

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- Because the order of the cards does not matter, each poker hand is a subset with 5 elements of a set with 52 elements.
- Hence the total number of poker hands is

$$C(52,5) = \frac{52!}{5!47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2} = 13 \cdot 17 \cdot 10 \cdot 49 \cdot 24$$
$$= 2,598,960$$

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- For the first player, the number of possible hands is the number of ways to select 13 cards from 52, or $C(52, 13)$.
- For the second player, for each hand of the first player, 39 cards remain, and the number of possible hands is $C(39, 13)$.

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- For the second player, for each hand of the first player, 39 cards remain, and the number of possible hands is $C(39, 13)$.
- For the third player, for each hand of the first and second players, 26 cards remain, and the number of possible hands is $C(26, 13)$.

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- For the third player, for each hand of the first and second players, 26 cards remain, and the number of possible hands is $C(26, 13)$.
- For the fourth player, for each hand of the first, second, and third players, 13 cards remain and that is their only possible hand.

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- For the third player, for each hand of the first and second players, 26 cards remain, and the number of possible hands is $C(26, 13)$.
- For the fourth player, for each hand of the first, second, and third players, 13 cards remain and that is their only possible hand.
- By the product rule, the total number of ways to divide a standard deck of 52 cards over 4 players is

$$C(52,13) \cdot C(39,13) \cdot C(26,13) \cdot 1 = \frac{52!}{13!39!} \frac{39!}{13!26!} \frac{26!}{13!13!} = \frac{52!}{(13!)^4}$$

$$= 5.3645 \times 10^{28}$$

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- Post questions on the discussion forum and participate in the discussion:
  `https://mitt.uib.no/courses/21678/discussion_topics/156554`