Equivalence Relations

Rosen Section 9.5

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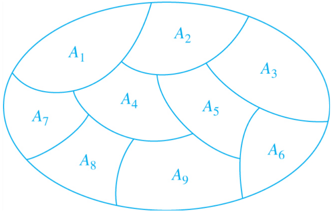
MNF130V2020 – Week 16

Partition of a set

We often want to divide a large set into a smaller number of **classes** such that items sharing a certain characteristic belong to the same class. If each item belongs to *exactly one* class, we obtain a **partition** of the set.

Example



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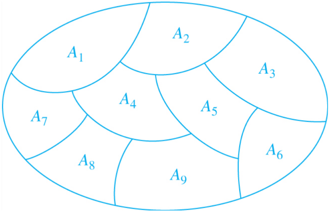

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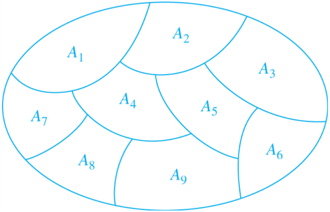

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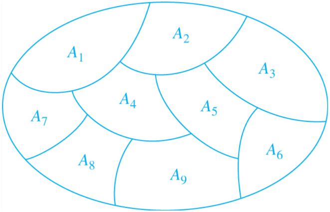
- Positive vs. negative real numbers.
- Books written by the same author.
- ...



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  - If $a \sim b$ and $b \sim c$ then $a \sim c$ for all a, b, c, because R is transitive.

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- <u>R is transitive</u>: if $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$ then there exist k, l such that $a - b = k \cdot m$ and $b - c = l \cdot m$. Hence $a - c = (k + l)m$ and $a \equiv c \pmod{m}$

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- $\underline{R \text{ is symmetric}}$: if b has the same author as a then a has the same author as b.
- $\underline{R \text{ is transitive}}$: if b has the same author as a and c has the same author as b, then c has the same author as a.

The graph of an equivalence relation

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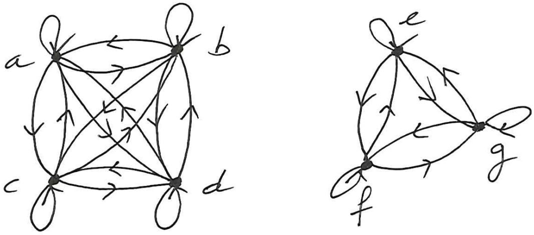

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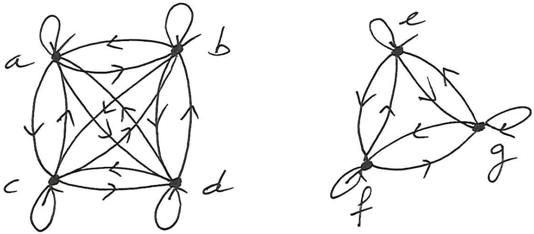
- These subsets are called **equivalence classes** and define a **partition** of A.



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- Any element of an equivalence class can be used as a representative, they are all equivalent.
- If only one relation is under consideration, we write $[a]$ instead of $[a]_R$.

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- If $a < 0$ or $a \geq m$, then there exists $0 \leq k < m$ such that $a \in [k]$, namely $k = a \bmod m$.
- Hence the set of all equivalence classes of R is $\{[0], [1], \ldots, [m-1]\}$.

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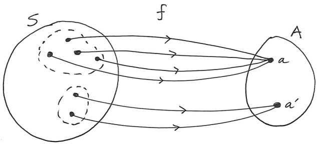

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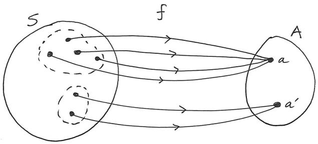

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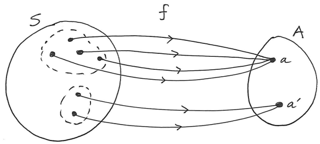

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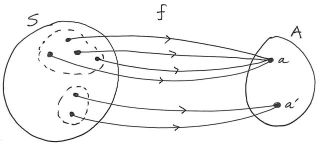

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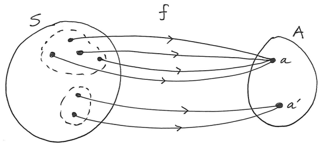

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Equivalence classes and partitions: formal treatment

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Let R be an equivalence relation on a set A. For all $a, b \in A$, the following statements are equivalent:

$$(i)\ a \sim b \qquad (ii)\ [a] = [b] \qquad (iii)\ [a] \cap [b] \neq \emptyset$$

Proof.

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- We will show that (i) implies (ii), (ii) implies (iii) and (iii) implies (i), and therefore all three statements are equivalent.

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Proof ($[a] = [b] \to [a] \cap [b] \neq \emptyset$).

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- In other words $\{A_1, A_2, \ldots, A_n\}$ is a partition of A if and only if

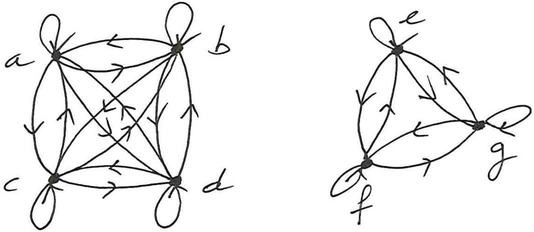
$$\begin{aligned} A_i &\neq \emptyset \;\text{ for } i = 1, \ldots, n \\ A_i \cap A_j &= \emptyset \;\text{ for all } i \neq j \\ \bigcup_{i=1}^{n} A_i &= S \end{aligned}$$

Theorem

- *Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S.*
- *Conversely, given a partition $\{A_1, \ldots, A_n\}$ of S, there exists an equivalence relation R that has the sets A_i as its equivalence classes.*

Proof (1st part, nonconstructive).

□



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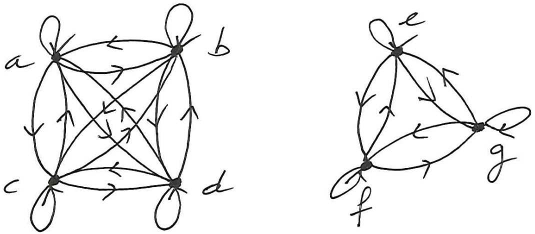

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- Let R be an equivalence relation.
- Then for all a, $[a]_R \neq \emptyset$ because $a \in [a]_R$.

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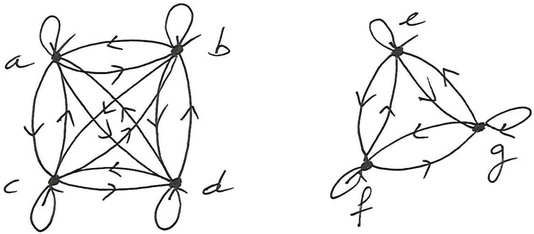

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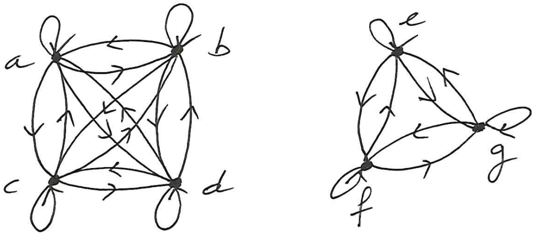

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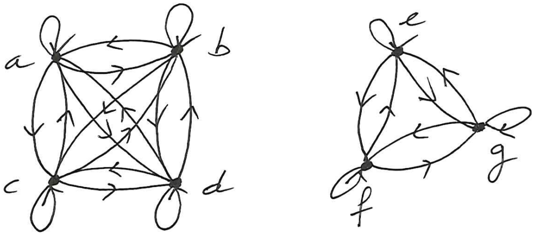

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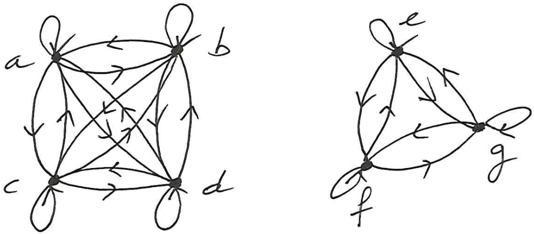
- Hence the equivalence classes of R form a partition of S.

□



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Algorithm

procedure PARTITION(S: set, R: equivalence relation)

$n := 0$

while S is nonempty **do**

$n := n + 1$

Take $a \in S$ arbitrary.

$A_n := [a]_R$

$S := S \setminus [a]_R$

return $\{A_1, \dots, A_n\}$

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  - R is transitive
- Note $a \sim b$ if and only if a, b belong to the same A_i.

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