Discrete Probability

Rosen Section 7.1

Tom Michoel

MNF130V2020 – Week 13

Probability theory

- **Probability theory** is about the study of the probability or chance of certain events to occur.

Probability theory

- **Probability theory** is about the study of the probability or chance of certain events to occur.
- In **discrete probabilty**, the number of possible outcomes of an *experiment* is finite.

Probability theory

- **Probability theory** is about the study of the probability or chance of certain events to occur.
- In **discrete probabilty**, the number of possible outcomes of an *experiment* is finite.
- Then the probability of an event is the number of outcomes corresponding to that event, divided by the total number of outcomes.

Probability theory

- **Probability theory** is about the study of the probability or chance of certain events to occur.
- In **discrete probabilty**, the number of possible outcomes of an *experiment* is finite.
- Then the probability of an event is the number of outcomes corresponding to that event, divided by the total number of outcomes.

Example

Roll a dice. What is the probability to obtain a ⚅?

Probability theory

- **Probability theory** is about the study of the probability or chance of certain events to occur.
- In **discrete probabilty**, the number of possible outcomes of an *experiment* is finite.
- Then the probability of an event is the number of outcomes corresponding to that event, divided by the total number of outcomes.

Example

Roll a dice. What is the probability to obtain a ⚅?

- There are 6 possible outcomes when a dice is rolled.

Probability theory

- **Probability theory** is about the study of the probability or chance of certain events to occur.
- In **discrete probabilty**, the number of possible outcomes of an *experiment* is finite.
- Then the probability of an event is the number of outcomes corresponding to that event, divided by the total number of outcomes.

Example

Roll a dice. What is the probability to obtain a ⚅?

- There are 6 possible outcomes when a dice is rolled.
- One outcome corresponds to having a ⚅.

Probability theory

- **Probability theory** is about the study of the probability or chance of certain events to occur.
- In **discrete probabilty**, the number of possible outcomes of an *experiment* is finite.
- Then the probability of an event is the number of outcomes corresponding to that event, divided by the total number of outcomes.

Example

Roll a dice. What is the probability to obtain a ⚅?

- There are 6 possible outcomes when a dice is rolled.
- One outcome corresponds to having a ⚅.
- Hence the probability is $\frac{1}{6}$

Example

Roll two dice. What is the probability that the sum of the numbers on the two dice is 10?

Example

Roll two dice. What is the probability that the sum of the numbers on the two dice is 10?

- By the product rule, there are $6 \cdot 6 = 36$ possible outcomes when two dice are rolled.

Example

Roll two dice. What is the probability that the sum of the numbers on the two dice is 10?

- By the product rule, there are $6 \cdot 6 = 36$ possible outcomes when two dice are rolled.
- There are 3 successful outcomes where the sum of the numbers is 10: (4,6), (5,5), (6,4).

Example

Roll two dice. What is the probability that the sum of the numbers on the two dice is 10?

- By the product rule, there are $6 \cdot 6 = 36$ possible outcomes when two dice are rolled.
- There are 3 successful outcomes where the sum of the numbers is 10: (4,6), (5,5), (6,4).
- Hence the probability that the two dice sum up to 10 is $\frac{3}{36} = \frac{1}{12}$.

Finite probability

Definition

- An **experiment** is a procedure that yields one of a given set if possible *outcomes*.

Finite probability

Definition

- An **experiment** is a procedure that yields one of a given set if possible *outcomes*.
- The **sample space** of the experiments is the *set* of possible outcomes.

Finite probability

Definition

- An **experiment** is a procedure that yields one of a given set if possible *outcomes*.
- The **sample space** of the experiments is the *set* of possible outcomes.
- An **event** is a *subset* of the sample space.

Finite probability

Definition

- An **experiment** is a procedure that yields one of a given set if possible *outcomes*.
- The **sample space** of the experiments is the *set* of possible outcomes.
- An **event** is a *subset* of the sample space.
- If S is a finite nonempty sample space of equally likely outcomes, and E is an event (a subset of S), the the **probability** of E is $p(E) = \frac{|E|}{|S|}$.

Finite probability

Definition

- An **experiment** is a procedure that yields one of a given set if possible *outcomes*.
- The **sample space** of the experiments is the *set* of possible outcomes.
- An **event** is a *subset* of the sample space.
- If S is a finite nonempty sample space of equally likely outcomes, and E is an event (a subset of S), the the **probability** of E is $p(E) = \frac{|E|}{|S|}$.
- Note that, because $E \subseteq S$, $0 \leq p(E) \leq 1$.

Example

Roll two dice. What is the probability that the sum of the numbers on the two dice is 10?

Example

Roll two dice. What is the probability that the sum of the numbers on the two dice is 10?

- The **experiment** is the rolling of two dice.

Example

Roll two dice. What is the probability that the sum of the numbers on the two dice is 10?

- The **experiment** is the rolling of two dice.
- The **sample space** is

$$S = D \times D, \quad \text{where} \quad D = \{1, 2, 3, 4, 5, 6\}$$

Example

Roll two dice. What is the probability that the sum of the numbers on the two dice is 10?

- The **experiment** is the rolling of two dice.
- The **sample space** is

$$S = D \times D, \quad \text{where} \quad D = \{1, 2, 3, 4, 5, 6\}$$

- The **event** is

$$E = \{(4, 6), (5, 5), (6, 4)\} \subset S$$

Example

Roll two dice. What is the probability that the sum of the numbers on the two dice is 10?

- The **experiment** is the rolling of two dice.
- The **sample space** is

$$S = D \times D, \quad \text{where} \quad D = \{1, 2, 3, 4, 5, 6\}$$

- The **event** is

$$E = \{(4, 6), (5, 5), (6, 4)\} \subset S$$

- The **probability** of E is

$$p(E) = \frac{|E|}{|S|} = \frac{3}{36} = \frac{1}{12}$$

Probabilities of complements of events

Theorem

Let E be an event in a sample space S. The probability of the event $\bar{E} = S - E$, the **complementary event** *of E, is*

$$p(\bar{E}) = 1 - p(E)$$

Proof.

$\square$

Probabilities of complements of events

Theorem

Let E be an event in a sample space S. The probability of the event $\bar{E} = S - E$, the **complementary event** *of E, is*

$$p(\bar{E}) = 1 - p(E)$$

Proof.

- Because $\bar{E} = S - E$, $|\bar{E}| = |S| - |E|$.

$\square$

Probabilities of complements of events

Theorem

Let E be an event in a sample space S. The probability of the event $\bar{E} = S - E$, the **complementary event** *of E, is*

$$p(\bar{E}) = 1 - p(E)$$

Proof.

- Because $\bar{E} = S - E$, $|\bar{E}| = |S| - |E|$.
- Hence

$$p(\bar{E}) = \frac{|\bar{E}|}{|S|} = \frac{|S| - |E|}{|S|} = 1 - \frac{|E|}{|S|} = 1 - p(E)$$

$\square$

Sometimes it is easier to compute the probability of a complementary event than the event of interest.

Example

A sequence of 10 bits is randomly generated. What is the probability that at least one of the bits is 0?

Sometimes it is easier to compute the probability of a complementary event than the event of interest.

Example

A sequence of 10 bits is randomly generated. What is the probability that at least one of the bits is 0?

- E is the event that at least one bit is 0, that is, the subset of all 10-bit strings that contain at least one 0.

Sometimes it is easier to compute the probability of a complementary event than the event of interest.

Example

A sequence of 10 bits is randomly generated. What is the probability that at least one of the bits is 0?

- E is the event that at least one bit is 0, that is, the subset of all 10-bit strings that contain at least one 0.
- Then $\bar{E}$ is the event that *none* of the bits are 0, that is, $\bar{E} = \{(1111111111)_2\}$.

Sometimes it is easier to compute the probability of a complementary event than the event of interest.

Example

A sequence of 10 bits is randomly generated. What is the probability that at least one of the bits is 0?

- E is the event that at least one bit is 0, that is, the subset of all 10-bit strings that contain at least one 0.
- Then $\bar{E}$ is the event that *none* of the bits are 0, that is, $\bar{E} = \{(1111111111)_2\}$.
- It follows that

$$p(E) = 1 - p(\bar{E}) = 1 - \frac{1}{2^{10}} = 1 - \frac{1}{1024} = \frac{1023}{1024} \approx 0.999$$

Probabilities of unions of events

Theorem

Let E_1 and E_2 be events in a sample space S. Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2) \tag{1}$$

Remark

Eq. (1) is an application of the inclusion-exclusion rule (see Section 6.1) for probabilities.

Proof.

□

Probabilities of unions of events

Theorem

Let E_1 and E_2 be events in a sample space S. Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2) \tag{1}$$

Remark

Eq. (1) is an application of the inclusion-exclusion rule (see Section 6.1) for probabilities.

Proof.

- The number of elements in the union of two sets is given by (see Section 2.2): $|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$.

□

Probabilities of unions of events

Theorem

Let E_1 and E_2 be events in a sample space S. Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2) \tag{1}$$

Remark

Eq. (1) is an application of the inclusion-exclusion rule (see Section 6.1) for probabilities.

Proof.

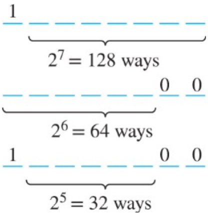
- The number of elements in the union of two sets is given by (see Section 2.2): $|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$.
- Hence

$$\begin{aligned} p(E_1 \cup E_2) &= \frac{|E_1 \cup E_2|}{|S|} = \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|S|} \\ &= \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} - \frac{|E_1 \cap E_2|}{|S|} = p(E_1) + p(E_2) - p(E_1 \cap E_2) \end{aligned}$$

$\square$

Example

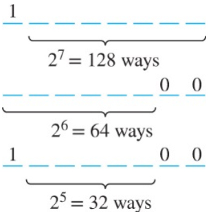
A sequence of 8 bits is randomly generated. What is the probability that the sequence starts with a 1 or ends with 00?



Example

A sequence of 8 bits is randomly generated. What is the probability that the sequence starts with a 1 or ends with 00?

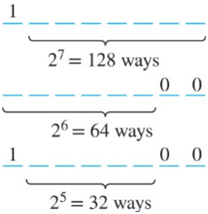
- E is the event that the sequence starts with a 1 or ends with 00, and can be written as $E = E_1 \cup E_2$:



Example

A sequence of 8 bits is randomly generated. What is the probability that the sequence starts with a 1 or ends with 00?

- E is the event that the sequence starts with a 1 or ends with 00, and can be written as $E = E_1 \cup E_2$:

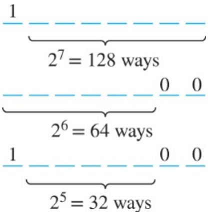


- E_1 is the event that the sequence starts with 1.

Example

A sequence of 8 bits is randomly generated. What is the probability that the sequence starts with a 1 or ends with 00?

- E is the event that the sequence starts with a 1 or ends with 00, and can be written as $E = E_1 \cup E_2$:

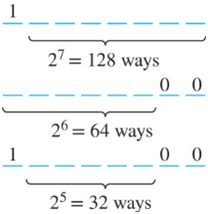


- E_1 is the event that the sequence starts with 1.
- Hence $|E_1|$ is the number of bit strings of length 8 starting with 1, that is $|E_1| = 2^7 = 128$.

Example

A sequence of 8 bits is randomly generated. What is the probability that the sequence starts with a 1 or ends with 00?

- E is the event that the sequence starts with a 1 or ends with 00, and can be written as $E = E_1 \cup E_2$:

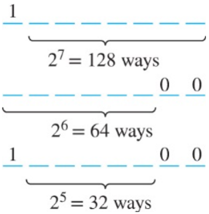


  - E_1 is the event that the sequence starts with 1.
  - Hence $|E_1|$ is the number of bit strings of length 8 starting with 1, that is $|E_1| = 2^7 = 128$.
  - E_2 is the event that the sequence ends with 00.

Example

A sequence of 8 bits is randomly generated. What is the probability that the sequence starts with a 1 or ends with 00?

- E is the event that the sequence starts with a 1 or ends with 00, and can be written as $E = E_1 \cup E_2$:

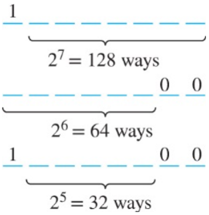


- E_1 is the event that the sequence starts with 1.
- Hence $|E_1|$ is the number of bit strings of length 8 starting with 1, that is $|E_1| = 2^7 = 128$.
- E_2 is the event that the sequence ends with 00.
- Hence $|E_2|$ is the number of bit strings of length 8 ending with 00, that is $|E_2| = 2^6 = 64$.

Example

A sequence of 8 bits is randomly generated. What is the probability that the sequence starts with a 1 or ends with 00?

- E is the event that the sequence starts with a 1 or ends with 00, and can be written as $E = E_1 \cup E_2$:



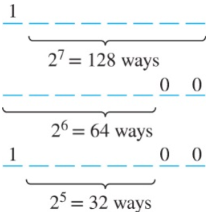
  - E_1 is the event that the sequence starts with 1.
  - Hence $|E_1|$ is the number of bit strings of length 8 starting with 1, that is $|E_1| = 2^7 = 128$.
  - E_2 is the event that the sequence ends with 00.
  - Hence $|E_2|$ is the number of bit strings of length 8 ending with 00, that is $|E_2| = 2^6 = 64$.

- $E_1 \cap E_2$ is the event that the sequence starts with 1 **and** ends with 00.

Example

A sequence of 8 bits is randomly generated. What is the probability that the sequence starts with a 1 or ends with 00?

- E is the event that the sequence starts with a 1 or ends with 00, and can be written as $E = E_1 \cup E_2$:



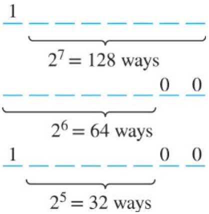
  - E_1 is the event that the sequence starts with 1.
  - Hence $|E_1|$ is the number of bit strings of length 8 starting with 1, that is $|E_1| = 2^7 = 128$.
  - E_2 is the event that the sequence ends with 00.
  - Hence $|E_2|$ is the number of bit strings of length 8 ending with 00, that is $|E_2| = 2^6 = 64$.

- $E_1 \cap E_2$ is the event that the sequence starts with 1 **and** ends with 00.
- Hence $|E_1 \cap E_2|$ is the number of bit strings of length 8 starting with 1 **and** ending with 00, that is $|E_1 \cap E_2| = 2^5 = 32$.

Example

A sequence of 8 bits is randomly generated. What is the probability that the sequence starts with a 1 or ends with 00?

- E is the event that the sequence starts with a 1 or ends with 00, and can be written as $E = E_1 \cup E_2$:



  - E_1 is the event that the sequence starts with 1.
  - Hence $|E_1|$ is the number of bit strings of length 8 starting with 1, that is $|E_1| = 2^7 = 128$.
  - E_2 is the event that the sequence ends with 00.
  - Hence $|E_2|$ is the number of bit strings of length 8 ending with 00, that is $|E_2| = 2^6 = 64$.
- $E_1 \cap E_2$ is the event that the sequence starts with 1 **and** ends with 00.
- Hence $|E_1 \cap E_2|$ is the number of bit strings of length 8 starting with 1 **and** ending with 00, that is $|E_1 \cap E_2| = 2^5 = 32$.
- Hence

$$p(E_1 \cup E_2) = \frac{2^7}{2^8} + \frac{2^6}{2^8} - \frac{2^5}{2^8} = \frac{2^5(2^2 + 2 - 1)}{2^8} = \frac{5}{8} = 0.625$$

What to do next?

What to do next?

- Read Rosen Section 7.1, , especially all the extra examples. Pay extra attention to Example 10 (the Monty Hall 3-door puzzle).

What to do next?

- Read Rosen Section 7.1, , especially all the extra examples. Pay extra attention to Example 10 (the Monty Hall 3-door puzzle).
- Solve exercises. Some recommended exercises are in the assignment for week 14:
  `https://mitt.uib.no/courses/21678/assignments/26392`

What to do next?

- Read Rosen Section 7.1, , especially all the extra examples. Pay extra attention to Example 10 (the Monty Hall 3-door puzzle).
- Solve exercises. Some recommended exercises are in the assignment for week 14:
  `https://mitt.uib.no/courses/21678/assignments/26392`
- Post questions on the discussion forum and participate in the discussion:
  `https://mitt.uib.no/courses/21678/discussion_topics/157824`