Functions

Murat Osmanoglu

Relations

- For a cartesian product set $A \times B = \{(x, y) | x \in A \ \wedge \ y \in B\}$, a binary relation from A to B is a subset of $A \times B$, i.e. $R \subseteq A \times B$

- if $(\mathrm{a}, \mathrm{b}) \in R$, then a is said to be related to b by R, i.e $\mathrm{a}Rb$

- Let A be the set of students and B be the set of courses

  A = {Ahmet, Efe, Buse, Pelin, . . .}
  B = {Math, Physics, Discrete, Algorithms, . . .}

  Let R be the relation such that if student a is taking course b, $(\mathrm{a}, \mathrm{b}) \in R$.

  (Ahmet, Physics) $\in R$, (Efe, Discrete) $\notin R$

Functions as Relations

$$R \subseteq A \times B$$

domain codomain

R(A) : the image of R, $R(A) = \{y \in B | (x, y) \in R, \exists x \in A\}$

Function is a relation that satisfies two conditions :

- for every element x of the domain, there is an element y in the codomain such that (x,y) is an element of the relation

  Let $R \subseteq A \times B$ be the relation, $\forall x[(x \in A) \rightarrow (\exists y \in B \;\; s.t. (x, y) \in R)]$

- for every element x of the domain, there is only one element y of the codomain such that (x,y) is an element of the relation

  Let $R \subseteq A \times B$ be the relation, $\forall x[((x, y_1) \in R \wedge (x, y_2) \in R) \rightarrow (y_1 = y_2)]$

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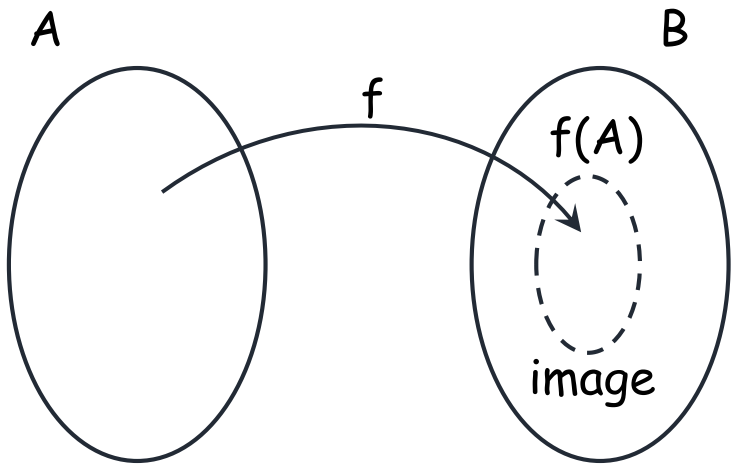

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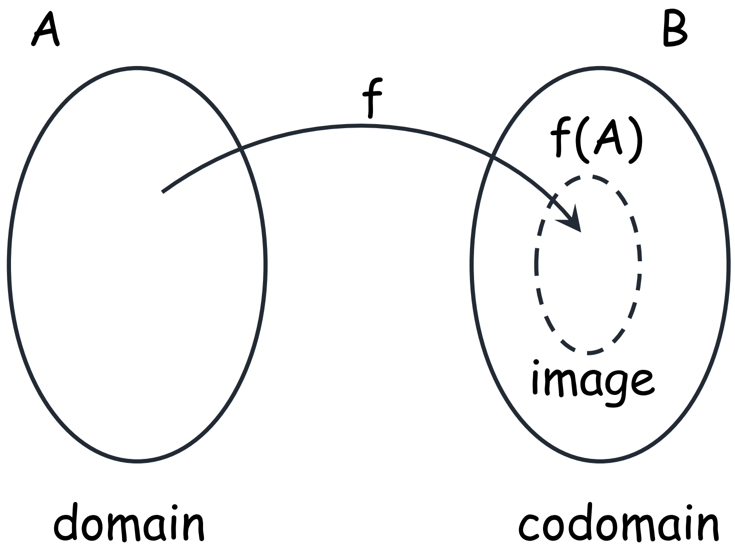

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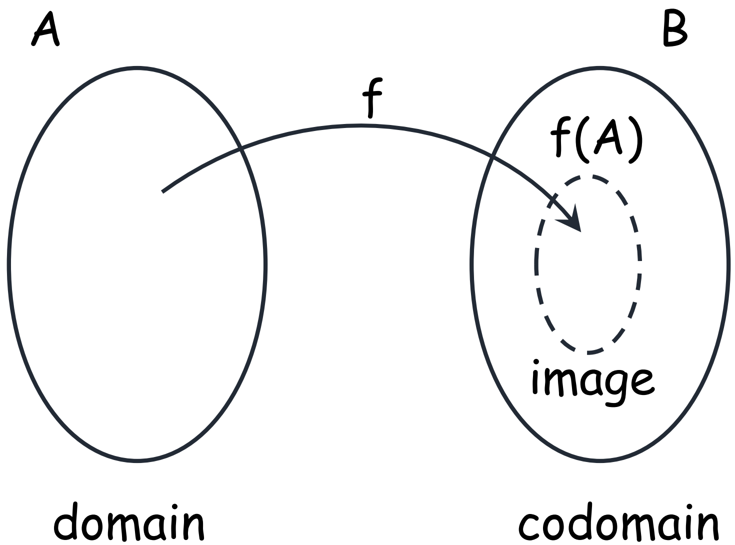

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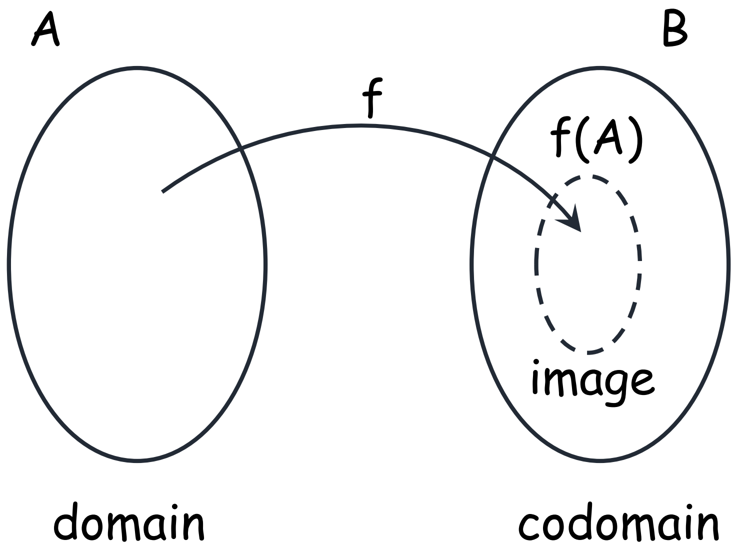

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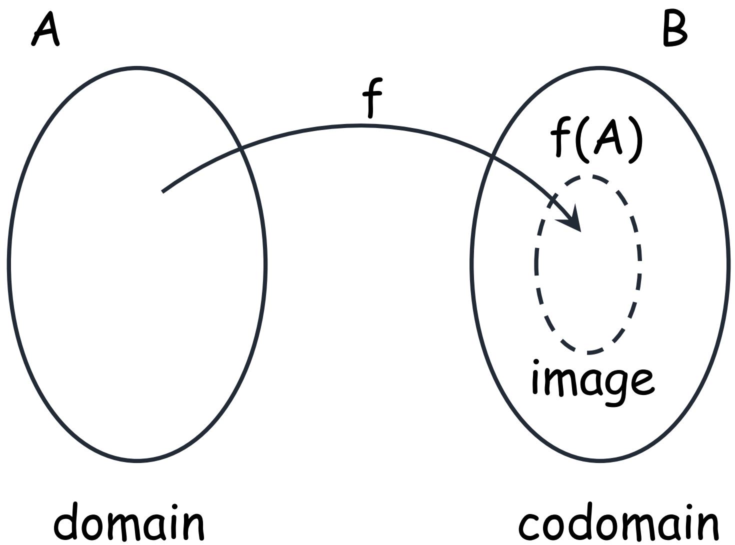

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$m^n = |B|^{|A|}$ functions

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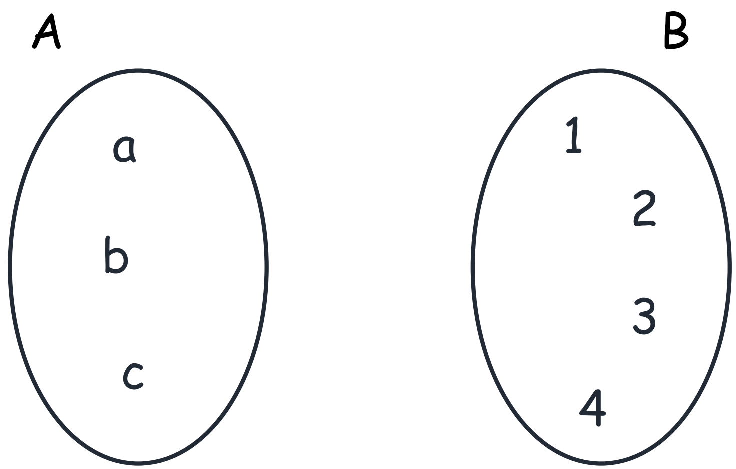

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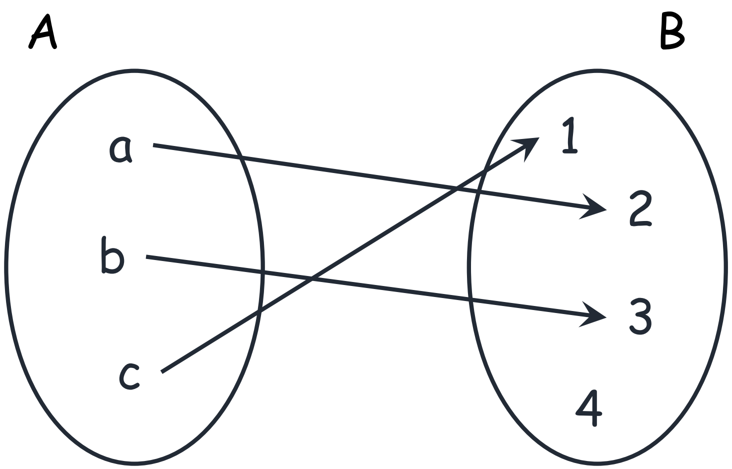

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for $x_1 = 1$ and $x_2 = -1$, $x_1 \neq x_2$ but $f(x_1) = f(x_2)$

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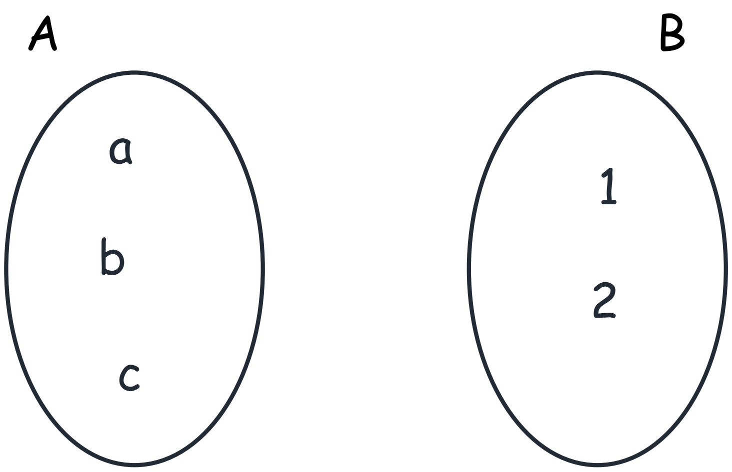

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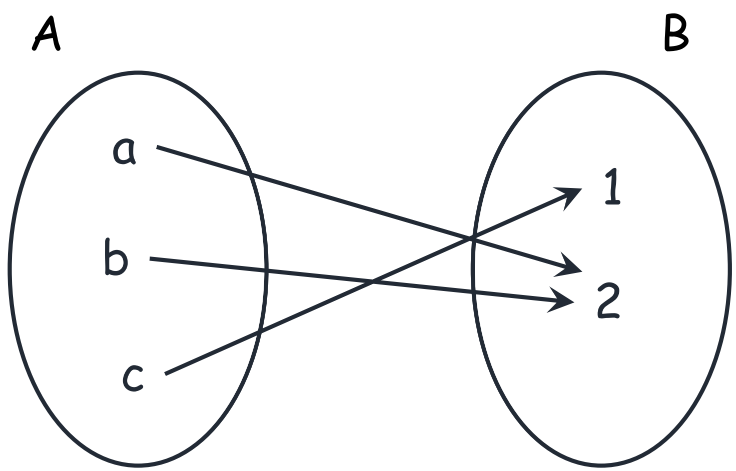

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for $5 \in \mathbb{Z}$, there is no integer $x \in \mathbb{Z}$ such that f(x) = 5.

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$\forall a \in A,\ \ f(a) = \mathrm{a}$, the preimage of a is itself

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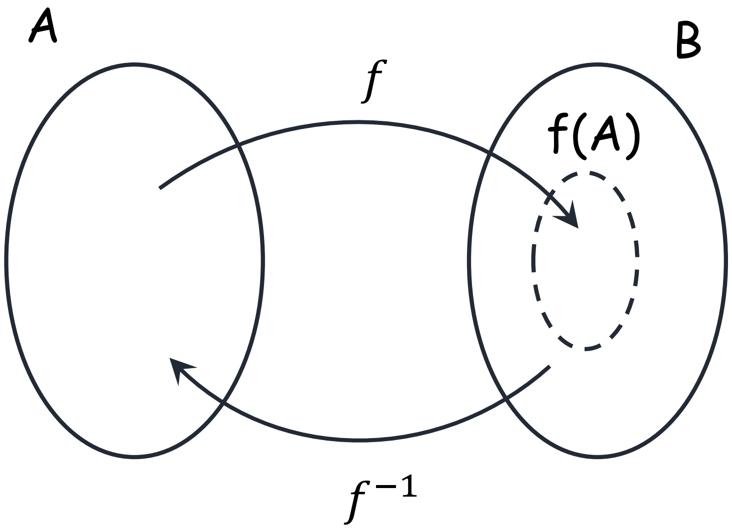

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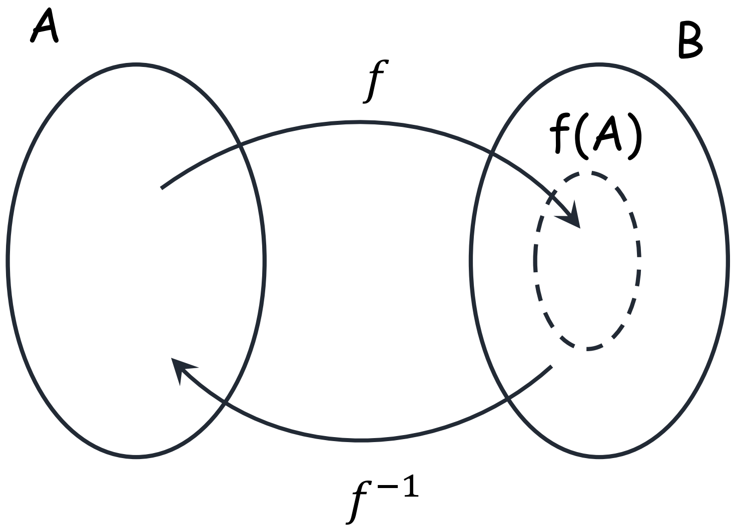

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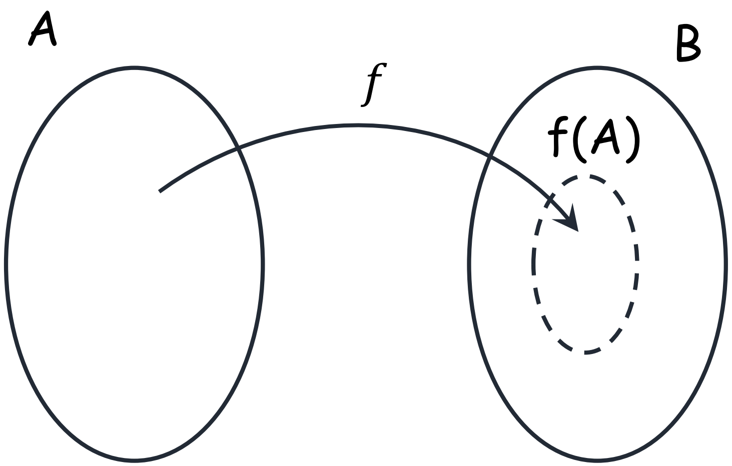

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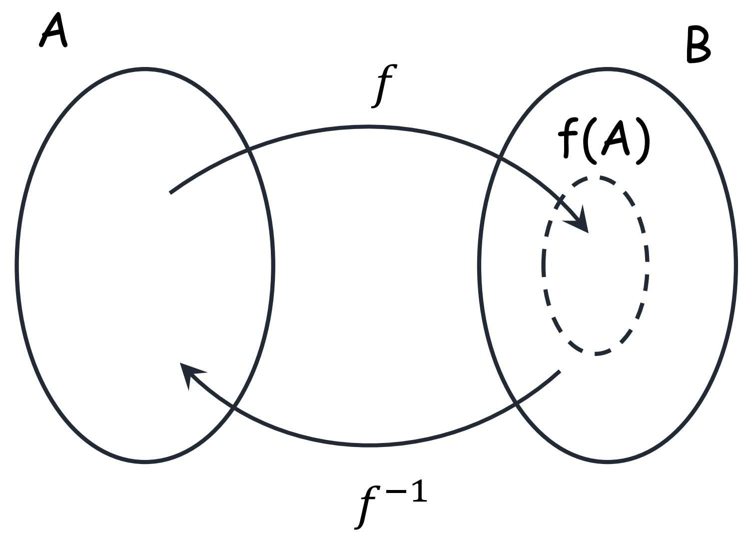

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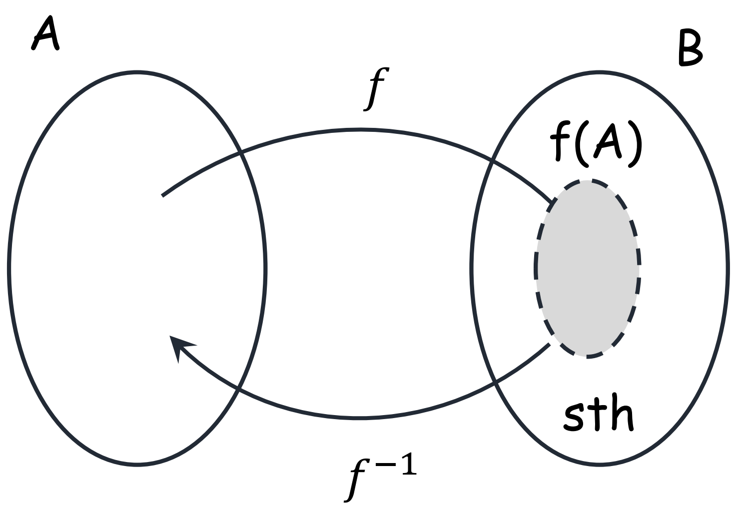

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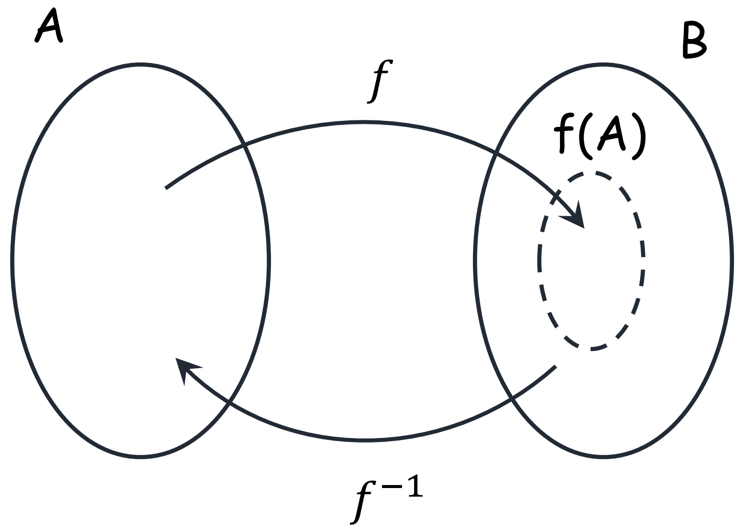

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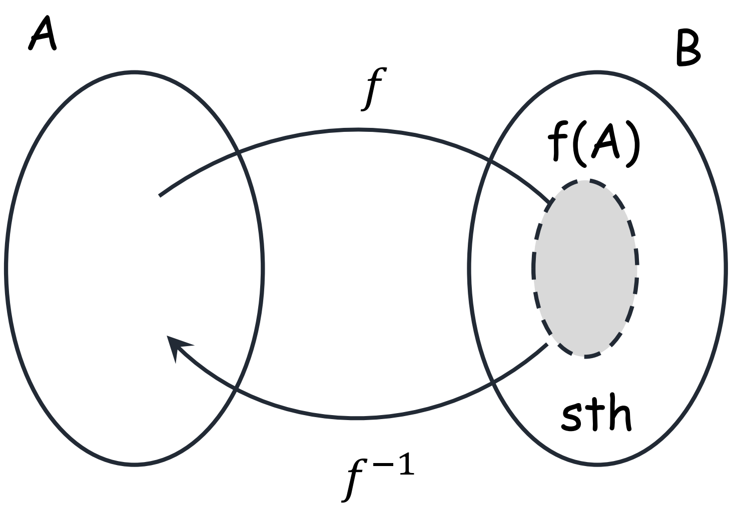

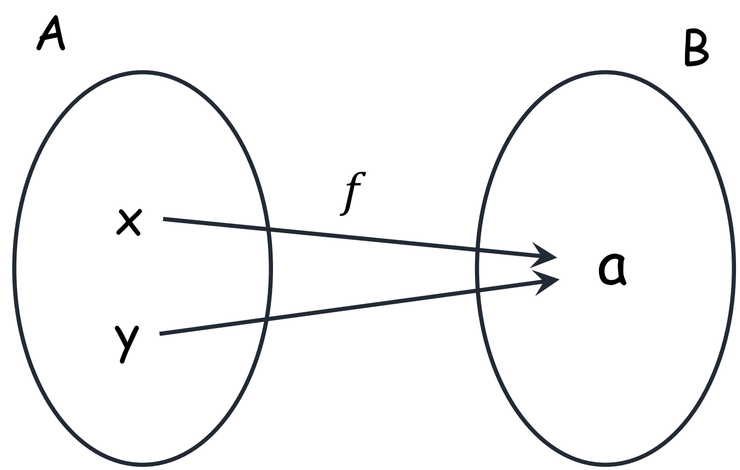

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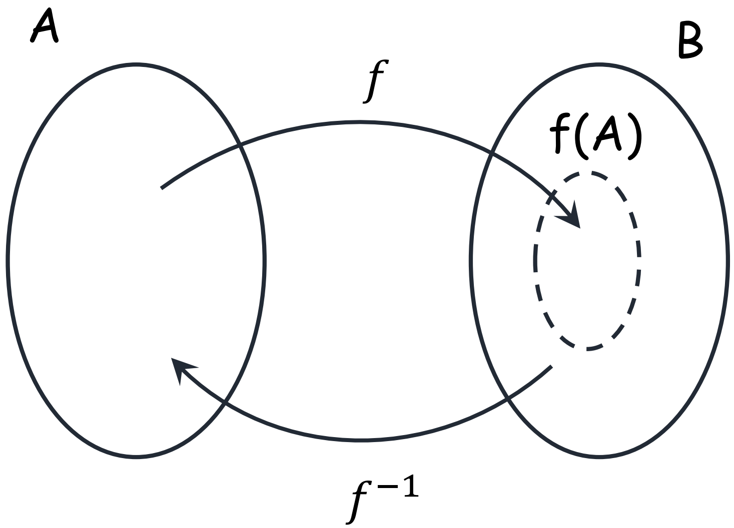

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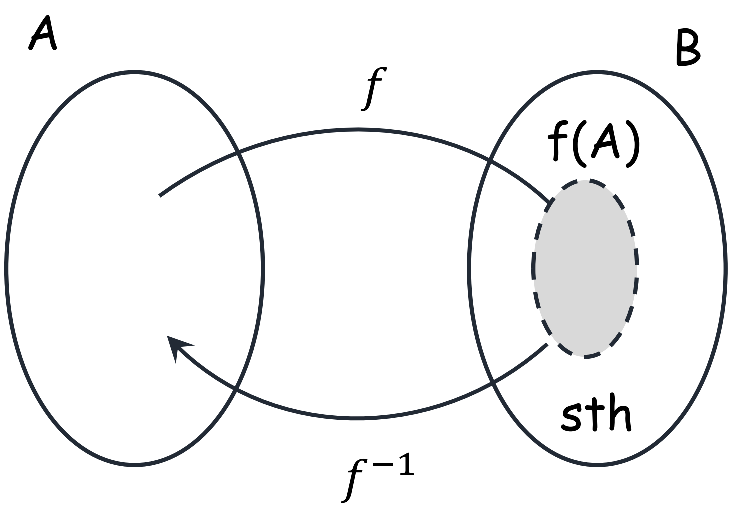

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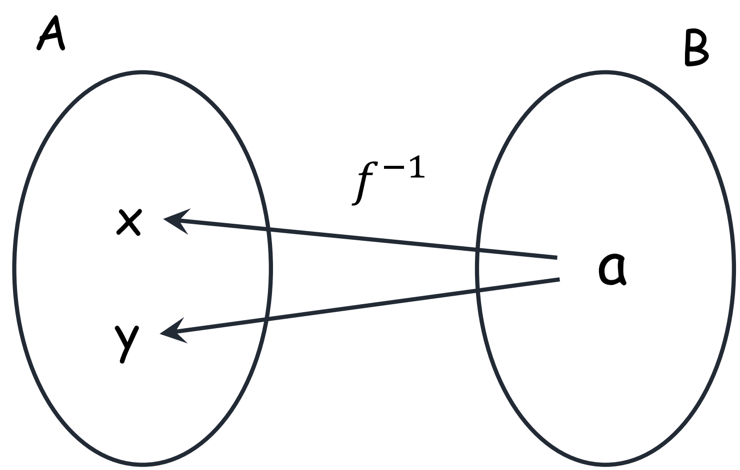

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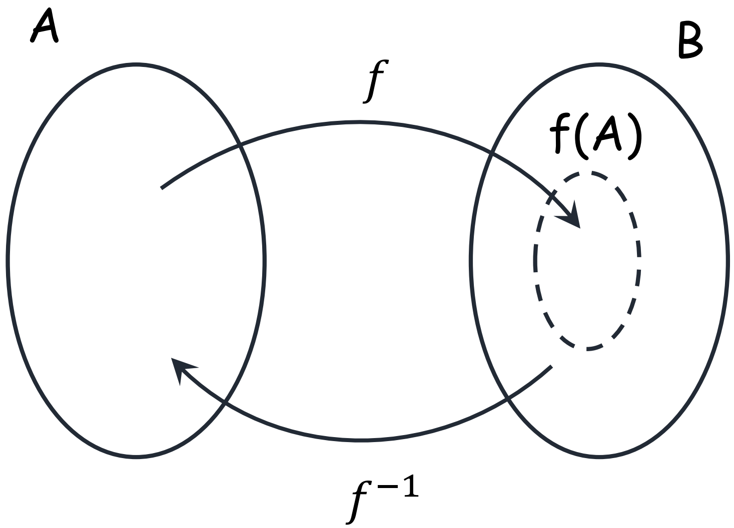

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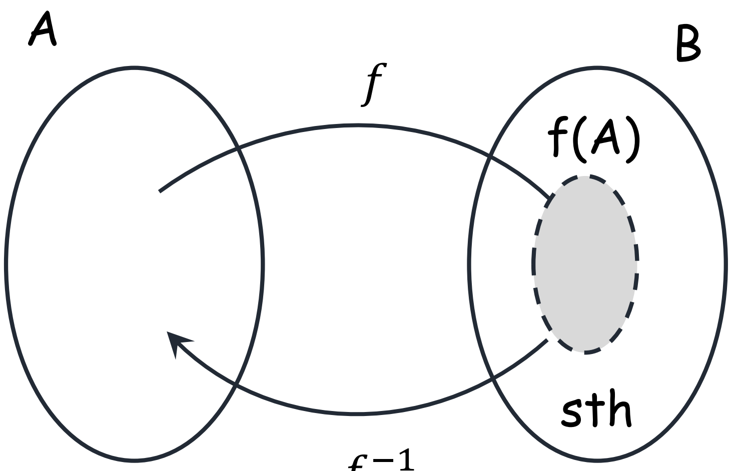

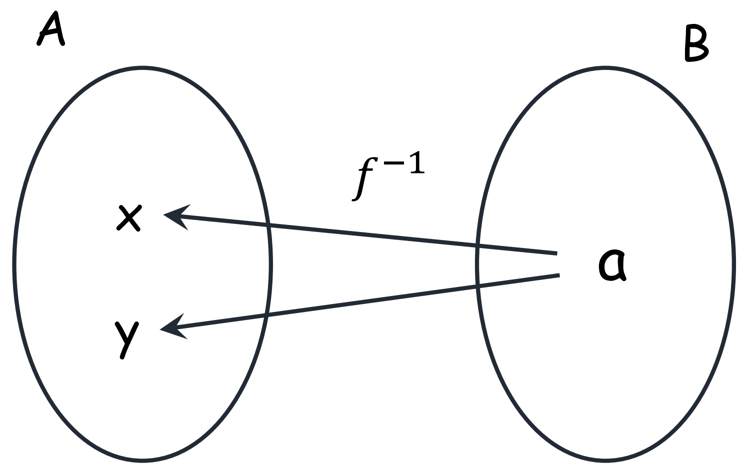

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$$\forall y \in \mathbb{Z}, f(x) = \mathrm{y}$$

Inverse

- If a function both one-to-one and onto, it is called bijection.

  If f is a bijection, then f^{-1} can be defined, i.e. f is invertible

- $f: \mathbb{Z} \to \mathbb{Z}$, defined as $f(x) = x + 1$, f is invertible ?

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$$\leftrightarrow x = y - 1 \in \mathbb{Z} \text{ (onto)}$$

$$f^{-1}(x) = x - 1$$

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- $f: \mathbb{Z} \rightarrow \mathbb{Z}$, defined as $f(x) = 2x + 1$, f is invertible ?

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  $\forall x_1, x_2 \in \mathbb{Z},\ \ f(x_1) = f(x_2)$

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$$\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}\ f(x) = \mathrm{y}$$

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$$\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}\ f(x) = y \leftrightarrow 2x + 1 = y$$
$$\leftrightarrow x = \frac{y-1}{2}$$

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$$\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}\ f(x) = y \leftrightarrow 2x + 1 = y$$

$$\leftrightarrow x = \frac{y-1}{2}$$

but for some $y \in \mathbb{Z}$, $x = \frac{y-1}{2} \notin \mathbb{Z}$ (not onto)

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- $f\colon \mathbb{Z} \to \mathbb{N}$, defined as $f(x) = \begin{cases} 2x-1 & if \;\; x > 0 \\ -2x & if \;\; x \le 0 \end{cases}$, f is invertible ?

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$\forall y \in \mathbb{N}, \exists x \in \mathbb{Z}$, if $\mathrm{y} = 2k, \exists k \in \mathbb{Z}$,

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$$\leftrightarrow x = -\frac{y}{2} = -k \in \mathbb{Z}$$

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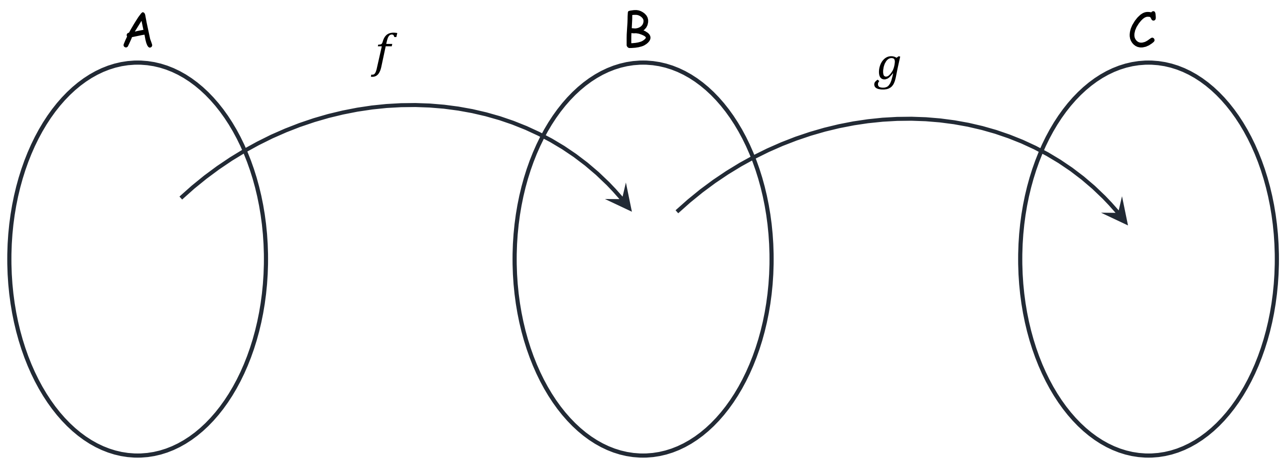
$$\text{then } f(x) = y \leftrightarrow 2x - 1 = y$$
$$\leftrightarrow x = \frac{y+1}{2} = k + 1 \in \mathbb{Z}$$
(onto)

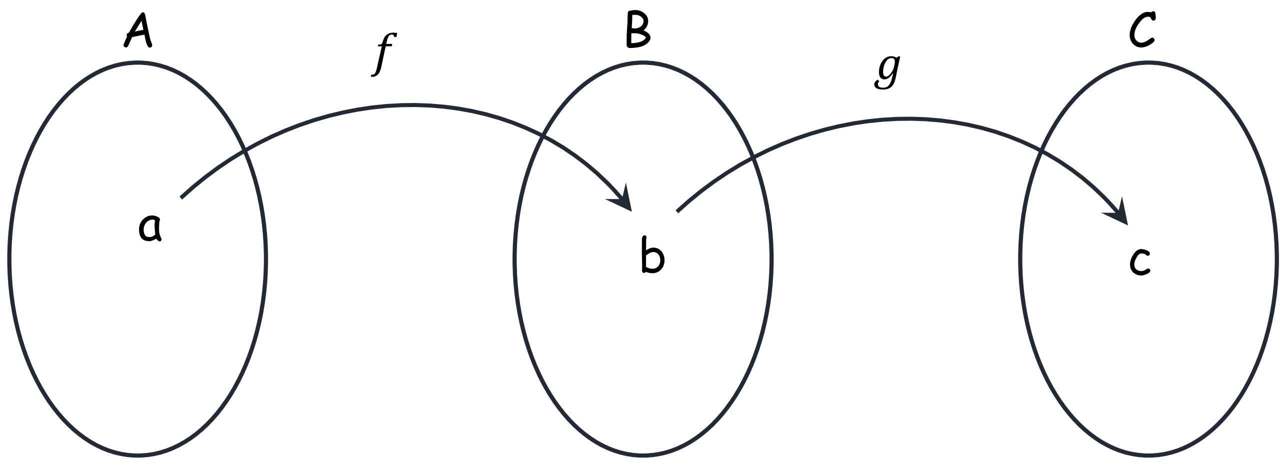
Composition



$f: A \to B$ and $g: B \to C$

$g \circ f: A \to C$

Composition

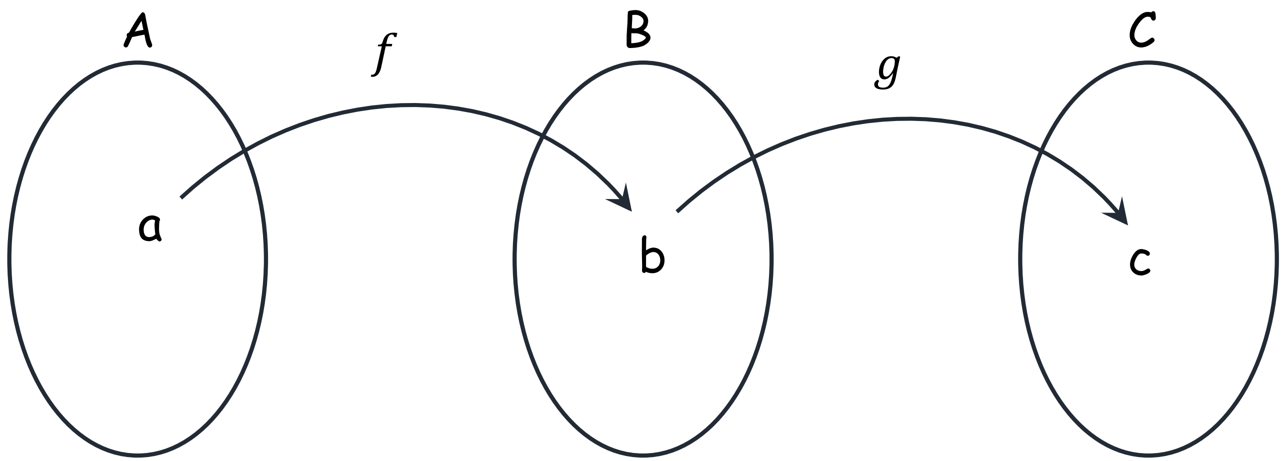


$f: A \to B$ and $g: B \to C$

$g \circ f: A \to C$

$f(a) = b$ and $\mathrm{g}(b) = c$

Composition



$f: A \rightarrow B$ and $g: B \rightarrow C$

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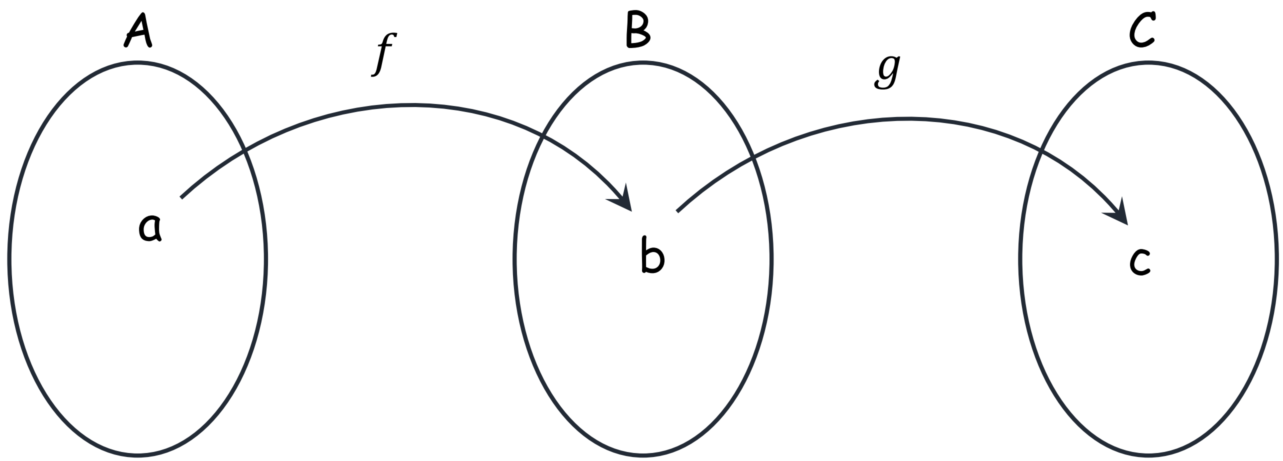
$g \circ f(a)$

Composition



$f: A \to B$ and $g: B \to C$

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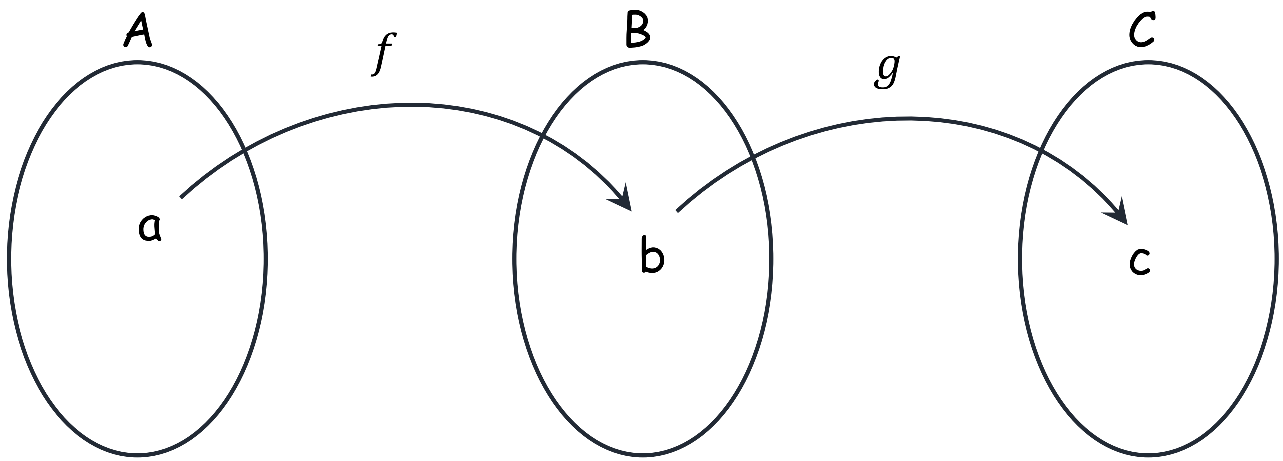
$f(a) = b$ and $\mathrm{g}(b) = c$

$g \circ f(a) = g\big(f(a)\big)$

Composition



$f: A \to B$ and $g: B \to C$

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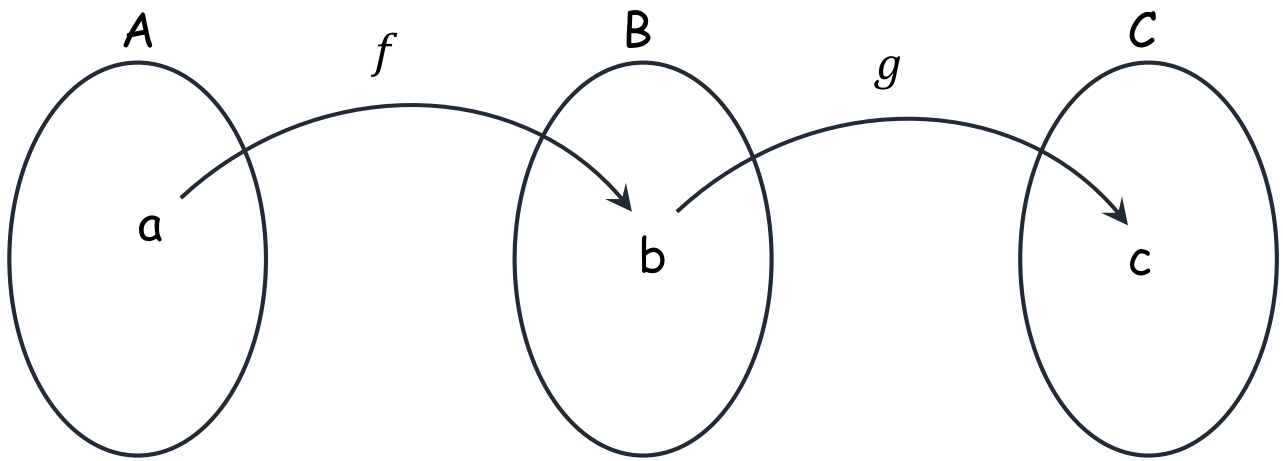
$f(a) = b$ and $\mathrm{g}(b) = c$

$g \circ f(a) = g\big(f(a)\big) = g(b)$

Composition



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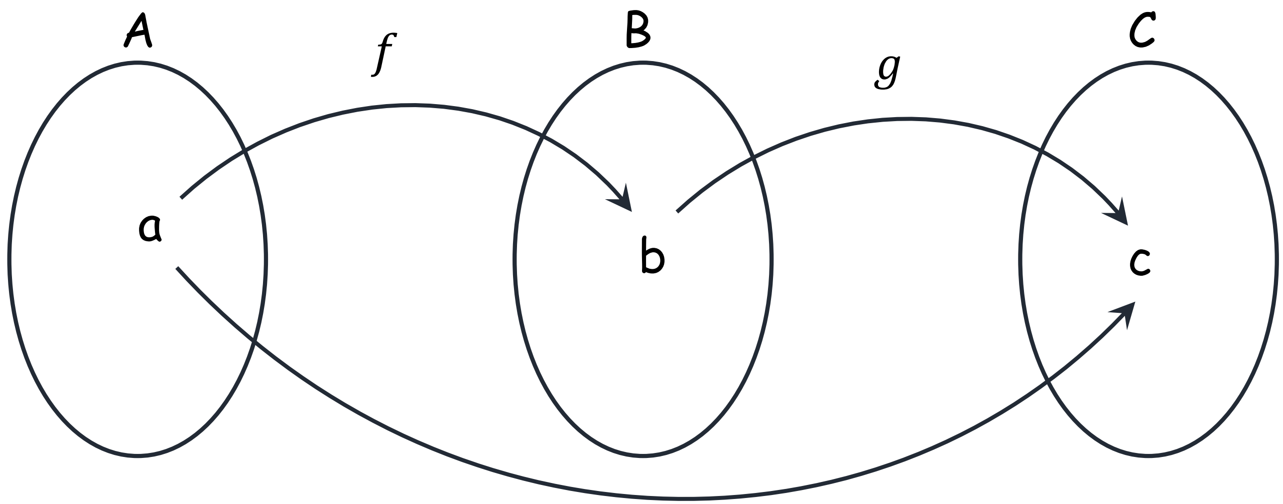

$g \circ f$

$f: A \to B$ and $g: B \to C$

$g \circ f: A \to C$

$f(a) = b$ and $\mathrm{g}(b) = c$

$g \circ f(a) = g\big(f(a)\big) = g(b) = c$

Composition

- $f, g: \mathbb{Z} \to \mathbb{Z}$,
  $f(x) = 3x + 1$ and $\mathrm{g}(x) = 2x - 1$

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  $f \circ g(x)$

Composition

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  $g \circ f(x) = g\big(f(x)\big)$

  $f \circ g(x)$

Composition

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$$1 \neq 1 + 1$$

Sequences

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- $a_n = \frac{1}{n}$ $a_1 = 1,\ a_2 = \frac{1}{2}, a_3 = \frac{1}{3}, . . .$
- $a_n = \frac{1}{3^n+2}$ $a_0 = \frac{1}{2}, a_1 = \frac{1}{5},\ a_2 = \frac{1}{11}, \ldots$

Sequences

Geometric Sequence :

Sequences

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$$a, ar, ar^2, . . ., ar^n, . . .$$

Sequences

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$$a, ar, ar^2, \ldots, ar^n, \ldots$$

initial term

common ratio

general term

Sequences

Geometric Sequence :

$$a\,,\,ar\,,\,ar^2,\,.\,.\,.,\,ar^n,\,.\,.\,.$$

initial term

common ratio

general term

$$a_n = (-1)^n$$

$$1, -1, 1, -1, \ldots$$

Sequences

Geometric Sequence :

$$a\,,\ ar\,,\ ar^2\,, \ldots,\ ar^n\,, \ldots$$

initial term

common ratio

general term

$$a_n = (-1)^n \qquad\qquad a_n = 2.3^n$$

$$1, -1, 1, -1, \ldots \qquad\qquad 2, 2.3, 2.9, 2.27, \ldots$$

Sequences

Geometric Sequence :

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

initial term

common ratio

general term

$$a_n = (-1)^n \qquad a_n = 2.3^n \qquad a_n = 3.(1/2)^n$$

$$1, -1, 1, -1, \ldots \qquad 2, 2.3, 2.9, 2.27, \ldots \qquad 3, 3/2, 3/4, 3/8, \ldots$$

Sequences

Arithmetic Sequence :

Sequences

Arithmetic Sequence :

$$a, a + d, a + 2d, . . ., a + n.d, . . .$$

Sequences

Arithmetic Sequence :

$$a, a + d, a + 2d, . . ., a + n.d, . . .$$

initial term

common difference

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Sequences

Arithmetic Sequence :

$$a,\, a + d,\, a + 2d,\, .\,.\,.,\, a + n.d,\, .\,.\,.$$

initial term

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$$a_n = 1 + n$$

$$1, 2, 3, 4, \ldots$$

Sequences

Arithmetic Sequence :

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$$a_n = 1 + n \qquad\qquad a_n = 2 - 4n$$

$$1, 2, 3, 4, \ldots \qquad\qquad 2, -2, -6, -10, \ldots$$

Sequences

Arithmetic Sequence :

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$$1, 2, 3, 4, \ldots$$

$$a_n = 2 - 4n$$

$$2, -2, -6, -10, \ldots$$

$$a_n = -1 + 8n$$

$$-1, 7, 15, 23, \ldots$$

Summations

- $\sum_{i=m}^{n} a_i$

Summations

- $\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_{n-1} + a_n$

'summation notation' represents the sum of the terms from a_m to a_n from the sequence $\{a_n\}$

Summations

- $\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_{n-1} + a_n$

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Summations

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Summations

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  $= (\mathrm{n} + 1) + (\mathrm{n} + 1) + \ldots + (n + 1)$

Summations

- $\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_{n-1} + a_n$

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- $$\begin{aligned} \sum_{i=1}^{n} i &= 1 + 2 + \ldots + \frac{n}{2} + \left(\frac{n}{2} + 1\right) + \ldots + (n-1) + n \\ &= (\mathrm{n} + 1) + (\mathrm{n} + 1) + \ldots + (n + 1) \\ &= \frac{n}{2} (\mathrm{n} + 1) \end{aligned}$$

Summations

- $a, a + d, a + 2d, . . ., a + n.d$

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  $\sum_{i=0}^{n}(a+id)$

Summations

- $a, a + d, a + 2d, \ldots, a + n.d$

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$$S_n = \sum_{i=0}^{n} ar^i$$

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$$S_n = \sum_{i=0}^{n} ar^i \rightarrow rS_n = r\sum_{i=0}^{n} ar^i$$

Summations

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$$rS_n = \sum_{i=1}^{n+1} ar^i$$

Summations

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$$rS_n = \sum_{i=1}^{n+1} ar^i = \sum_{i=1}^{n} ar^i + ar^{n+1}$$

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$$rS_n = \sum_{i=0}^{n} ar^i + ar^{n+1} - a$$

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Summations

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$$rS_n = S_n + ar^{n+1} - a \;\rightarrow\; S_n = \frac{ar^{n+1}-a}{r-1}$$

Recurrence Relations

- sometimes the elements of the sequence are defined recursively in terms of previous and the initial elements of the sequence

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Definition : an equation that express the general term of the sequence in terms of previous terms.

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Definition : an equation that express the general term of the sequence in terms of previous terms. A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

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$$a_3 = 225 = 3.(3.(3.5))$$

Recurrence Relations

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$$
\begin{aligned}
a_1 &= 15 = 3.5 \\
a_2 &= 75 = 3.(3.5) \\
a_3 &= 225 = 3.(3.(3.5)) \\
&\vdots \\
a_n &= 3^n 5
\end{aligned}
$$

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$$
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a_3 &= 225 = 3.(3.(3.5)) \\
&\vdots
\end{aligned}
$$

$a_n = 3^n 5$; the unique solution of the given recurrence relation

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  $a_2 = A.7^2 \rightarrow 98 = A.49 \rightarrow A = 2$

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$a_n = 3^n 5$; the unique solution of the given recurrence relation

- $a_{n+1} = d.a_n, \ a_0 = A$ where d is constant

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- solve the recurrence relation $a_{n+1} = 7.a_n$ where $n \geq 1$ and $a_2 = 98$

  $a_2 = A.7^2 \rightarrow 98 = A.49 \rightarrow A = 2$

  the solution is $a_n = 2.7^n$

Recurrence Relations

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$$3$$
$$1 + 2$$
$$2 + 1$$
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The solutions for the characteristic equation are called characteristic roots; r_1 and r_2

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the solution will be in the form of $b_n = c_1(\sqrt{2})^n + c_2(-\sqrt{2})^n$

$$b_1 = c_1(\sqrt{2})^1 + c_2(-\sqrt{2})^1 \rightarrow \quad 1 = \sqrt{2}c_1 - \sqrt{2}c_2$$

$$b_2 = c_1(\sqrt{2})^2 + c_2(-\sqrt{2})^2 \rightarrow \quad 2 = 2c_1 + 2c_2$$

$$b_n = \left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)(\sqrt{2})^n + \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)(-\sqrt{2})^n$$