Introduction

In this lecture we will start a new chapter and the name of the chapter is Fourier series expansion. We will see what Fourier series is, why we use Fourier series and what are different types of Fourier series expansion. First, let’s mention that; Fourier series, Fourier transform and their applications were all given by Joseph Fourier who was a French mathematician and physicist.

3-Definition

What is the use of Fourier series and what is Fourier series.

Fourier series expansion is used for periodic signals to expand them in terms of their harmonics which are sinusoidal and orthogonal to one another.

The first important thing which we can see is the use of Fourier series expansion only in the case of periodic signals.

There are two types of periodic signals.

- continuous time periodic signals and
- discrete-time periodic signals.

Based on these two types of signals (continuous-time and discrete-time) we have two types of Fourier series expansions.

- For continuous time the Fourier series expansion is known as continuous time Fourier series
- For discrete time the series is known as discrete time Fourier series.

4-Difference Between Fourier series and Fourier Transform

Fourier series expansion is used for periodic signals only.

what about non-periodic signals? If we talk about real life signals then we don't have the periodic signals generally we have the non-periodic signals. We know the real-life signals are non-periodic in nature.

So, for analysis of non-periodic signals, we need another tool which is known as Fourier transform

As you can estimate; there are two types of non-periodic signals the first one is continuous time non-periodic signals and for this we use the tool known as continuous time Fourier transform in the same way we have discrete-time Fourier transform. Both are for non-periodic signals.

This is the difference between the Fourier series and Fourier transform

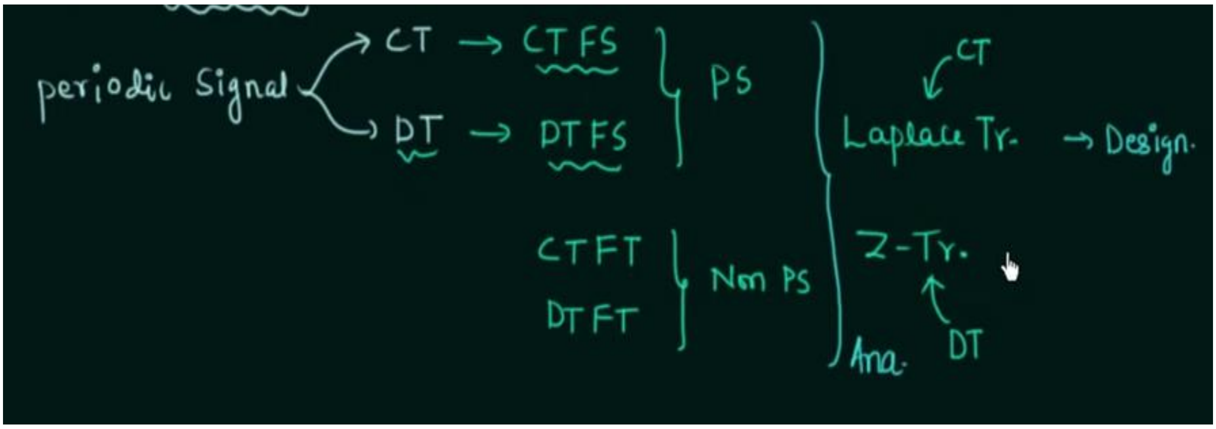
How about the use of the Laplace transform.

- Laplace transform is used for designing purposes.
- Fourier series and Fourier transform are used for analysis purposes.

Laplace transform is used for designing purposes. We obtain the transfer function using the Laplace transform and by using the different methods available, we can easily check the stability of the system and by using the results, we can design our system.

How about is Z transform.

- Laplace transform is for continuous time and
- Z transform is for discrete time



5-Periodic Signals

Periodic signals are those signals in which there is reputation of a particular structure from minus infinity to infinity or you can see the periodic signal is a signal in which the signal repeats itself after a particular interval of time and this particular interval of time is known as time period of the signal

so if we have a periodic signal x(t+T) it is a periodic signal, then we can see that plus T or minus T is equal to x(t). This is the condition for a signal to be periodic.

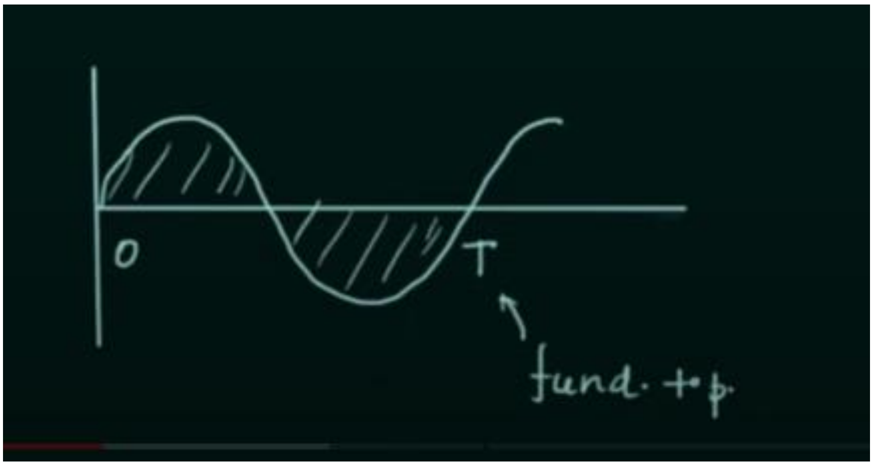
The Fourier series expansion is used for periodic signals to expand them in terms of their harmonics.

Here; we have a new term harmonic and to understand the harmonic you should understand the frequency.

6-Frequency

First we know the basic definition of frequency defined as the number of cycles per second it is number of cycles per second and we can easily calculate the frequency if we have the waveform.

Here is a quick plot of a sinusoidal waveform:



You can see one cycle is completed from zero to T. From 0 to T is the interval after which the signal repeats itself.

This T here is the fundamental time period of the sine function.

We can easily calculate the frequency. It is equal to number of cycles per second. (1/T).

For example, our signal in figure is taking T seconds for one cycle. So in 1 seconds in we will have 1 over T cycles. Therefore the frequency is going to be 1 over T cycles per second or simply Hertz. This is the definition of frequency.

We have mentioned that, our real-life signals are not periodic in nature. For example I am speaking right now. If we convert my voice to the electrical signal by the help of the microphone, we can easily see that voice signal is not a periodic signal. It depends on the word I am speaking and based on different words we have different spikes in the waveform. For example the waveform will look like this:

now you cannot use this formula to calculate the frequency in this case because you cannot find out the cycles here.

Therefore we need to define the frequency in more general way:

The frequency is defined as the rate of change

In this example voice signal, you can see that; in some intervals, the rate of change is too high and in some intervals change is small. In our previous sinusoidal signal there was only one frequency but here in voice signal we have multiple frequencies. We can say that it our voice signal carrying some information but we don't know. But we already know everything about the sinusoidal waveform either cosine or sine.

So for our voice signal, there are multiple frequencies available. When we have multiple frequencies we can express the signal in terms of a **fundamental component plus the harmonics.**

7-Harmonics

Now, we now know the meaning of frequency and now we will understand the meaning of harmonics. for this I will take one example in this example signal X(t) is a periodic signal and it is expressed as the sum of original signal and the harmonics:

$$x(t) = 2\sin\omega t + \sin 2\omega t + 7\sin 3\omega t + \cdots$$

Here; our signal X(t) is a periodic signal and it is expressed as the sum of the original signal and the harmonics. **2sinωt** (2 times sine Omega T), as you see in the first term is also known as the first harmonic. Here frequency is **ω (o**mega). It is the fundamental frequency or more specifically it is the fundamental angular frequency. **Angular frequency is equal to 2πf.**

If you examine the second term you will find the frequency is 2ω. So here the frequency is 2 times the fundamental frequency.

Whenever you have the frequency has integral multiple of the fundamental frequency we call the term harmonics.

If you see the third term you will find the frequency is 3ω. So again the frequency is integer multiple of the fundamental frequency. This one here is also harmonic in the same way. We can have other harmonics present in this signal and depending on the integer here we have even and odd harmonics. Here you can see 2ω and 2 is even. Therefore we can see 3ω, which is odd.

If you focus on the coefficient you will find here the coefficient is 1 but here the coefficient is 7 7 is greater than 1 so we can see that the effect of 3^{rd} harmonic is more as compared to the effect of 2nd harmonic and if the same pattern is followed for the other harmonics also we can see that odd harmonics are more dominant in this signal and even harmonics are less dominant. It depends on signal to signal in some signals odd harmonics are more dominant in some signals even harmonics are more dominant.

You should know about the harmonic is that:

whenever you have different frequency components along with the fundamental frequency component, we say harmonics are present in the signal and this is what we want to perform the analysis

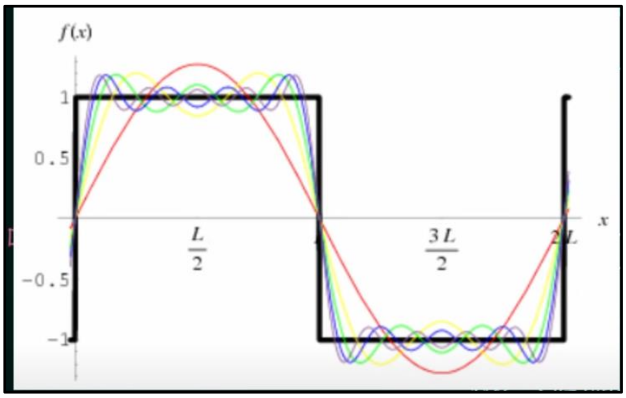
We can have the same result from the waveform of the signal using the Fourier series. So

- We want to analyze the signal
- For analyzing the signal we will analyze the harmonics. (present for example here we got the information that the 3rd harmonic is more dominant as compared to the second harmonic)
- So we want this expansion of this signal.
- And we can use the Fourier series expansion.

IMPORTANT NOTE: Fourier series expansion is only for the periodic signals. when we have non periodic signals like this we will use the Fourier transform

8-Harmonics of the square wave

Now; first we will first focus on Fourier series expansion.



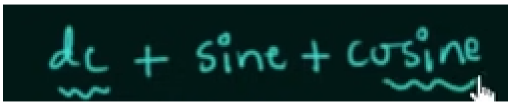
Here; you can see the square waveform and in this square waveform. You can see different harmonics present here. There is fundamental signal and if you try to visualize the waveform from minus infinity to infinity you will find it is like sine waveform. So when you obtain the Fourier series expansion of this particular square wave you will find all sine terms in the expansion and all the different harmonics will be present in the expansion. You can see in the waveform will be obtained in mathematical form using the Fourier series. There are other harmonics also but they are not very dominant so there is no need to write down all the harmonics.

So I hope you know have the clear understanding of the use of Fourier series and the term periodicity and harmonics.

Now we will understand the last point written in the definition statement the harmonics are sinusoidal and orthogonal to one another.

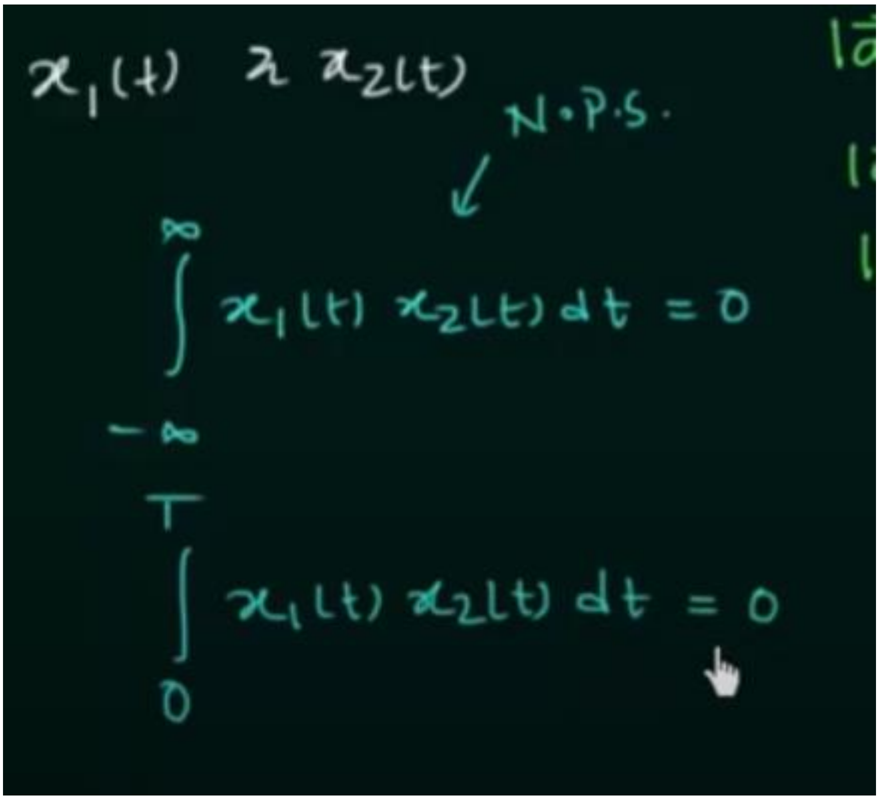
8.1-Sinusoidal and orthogonal

The harmonics are sinusoidal this means whenever you have the expansion of the given signal by the help of Fourier series you will have a DC value plus sine terms plus cosine terms.



So we have DC plus sinusoidal terms in the expansion. Sometimes you will have DC and cosine, sometimes you will have sine plus DC, and sometimes you will have DC sine and cosine.

The next important point is the harmonics are orthogonal to one another we already know the meaning of orthogonal which means perpendicular to each other.



The next point is Fourier series is also used to calculate the power and phase content of a particular harmonic present in the expansion. Like our case, you can see the harmonics present in the expansion and we can calculate the phase content and the power easily. Another important point is that; the power is equally distributed to the harmonics.

9-Types Of Fourier Series Expansion

Now we will talk about different types of Fourier series expansion:

1. Trigonometric Fourier series expansion
2. Complex exponential Fourier series expansion or simply the exponential Fourier series expansion
3. Polar or harmonic Fourier series expansion polar or harmonic Fourier series expansion

We will first study trigonometric Fourier series expansion. After this I will explain complex exponential Fourier series expansion. This is very important and we will focus more on complex exponential Fourier series expansion. The third one is polar or harmonic Fourier series. We will not focus on this type.

10-Existence of Fourier Series

We mentioned that Fourier series expansion is used for periodic signals but it does not mean whenever we have periodic signal the Fourier series expansion will exist.

The existence of Fourier series expansion depends on three conditions and the conditions are known as **Dirichlet conditions.** There are three conditions for the existence of Fourier series expansion even if the signal is periodic the three conditions must be satisfied

11-Dirichlet Conditions

We will start with the conditions for the existence of Fourier series.

We said that; we use Fourier series for the analysis of periodic signals. So;

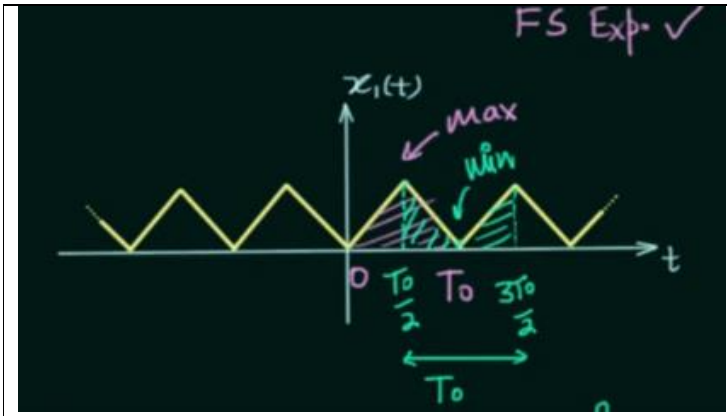
- If you have a periodic signal and you want to analyze it then you can use the tool known as Fourier series expansion.
- If you want to analyze the non-periodic signal then you can use another tool known as Fourier transform

But there are some conditions. You cannot say all periodic signals will have the Fourier series. And you also cannot say all non-periodic signals will have the Fourier transform. There are some conditions. These conditions were given by German mathematician Dirichlet.

Therefore these conditions are also known as Dirichlet conditions. It includes both Fourier Series expansion (for periodic signals) and Fourier Transform (for non-periodic signals)

Condition 1 : There must be finite number of minima and maxima in the function.

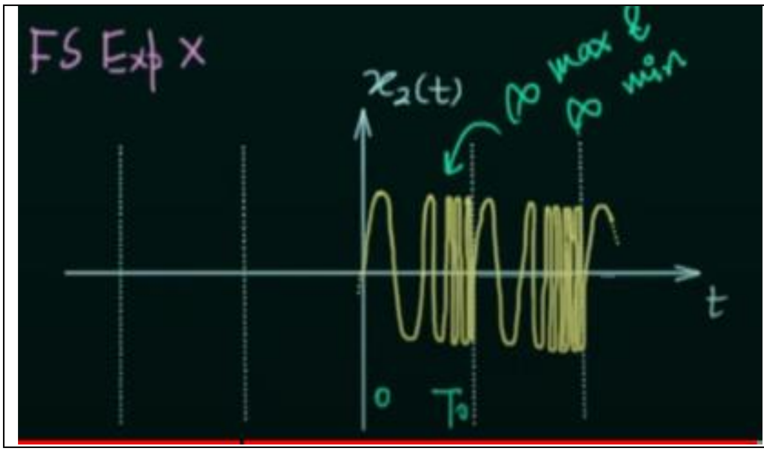
For a periodic signal to have Fourier series expansion, the signal should have a finite number of maxima and finite number of minima over the range of time period.



(Maxima is the highest amplitude or peak and minima is the lowest amplitude or peak value.)

- In the case of periodic signals, the number of maxima and minima over the range of time period should be finite.
- Here, signal x1(t) is given. From the waveform, we can easily find out the fundamental time period of the signal from 0 (zero) to T. You can see there is only one Maxima and there is only one Maxima. Hence the Fourier series expansion for this signal can exist.

Now we will move to signal X2(t). In this second figure, it can be seen that the maxima and minima are not constant, and the time period has some inconsistency, hence the signal is discontinuous or infinite.



- **Here you can see the signal is periodic. The same type of a structure is repeated after some interval of time but here you can see there are infinite Maximas and infinite minimas**
- **So we don't have the finite number of maxima and minima. Therefore, the Fourier series expansion for signal x2(t) will not exist.**

Condition 2: The periodic signal should have a finite number of discontinuities over the range of time period.

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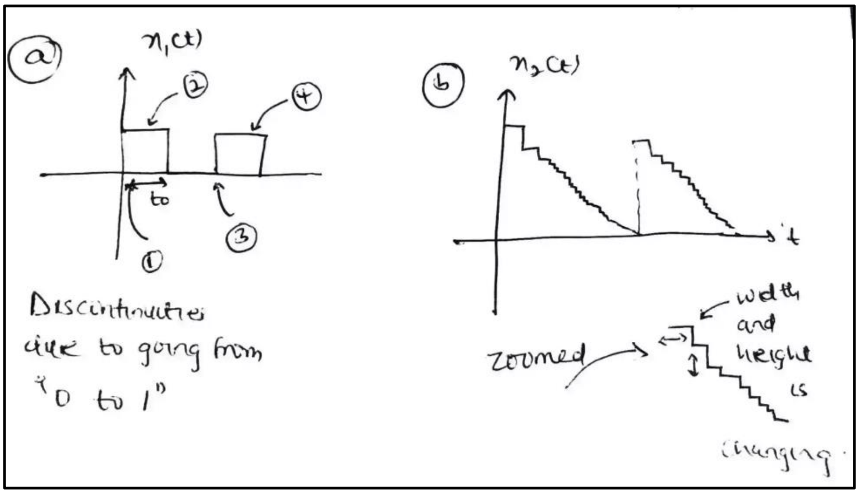
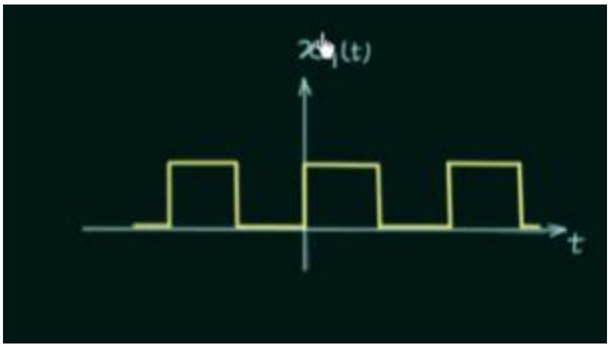


In the first figure, there are finite discontinuities while in the second figure, there are infinite discontinuities as the width and height is changing randomly and the discontinuities present can tend to infinity. We had another problem with the second figure. The signal has infinite maxima and infinite minima.

Therefore Fourier series expansion of the signal in second figure will not exist.

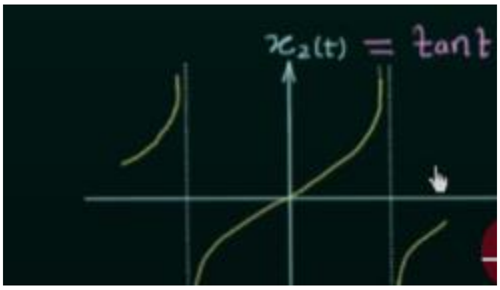
Condition 3: Periodic signal should be absolutely integrable over the range of time period.

The meaning of absolutely integrable is very simple. Let's discuss our example signal x1(t)



When you integrate absolute value of signal x1(t) you should get something finite. This means you should get something which is less than infinity. Now there is one thing which may confuse you we already know. For periodic signals there are only two possibilities: The first possibility is that it can be power signal and the second possibility is that it can be neither energy nor power signal.

It can never be an energy signal. We already know energy signals are absolutely integrable signals. So why we are having this word absolutely integrable associated with the periodic signal in condition number 3. This is because we are talking about the range of time period. In case of energy signals we integrated the signal from minus infinity to infinity. In this case the result should be finite or less than infinity. Here, in Dirichlet condition number 3 we are not integrating from minus infinity to infinity but we are integrating over the time period T. Because we already know when we integrate the periodic signal from minus infinity to infinity we will get infinite area. The integration will give you area and the extension of periodic signal is from minus infinity to infinity. But when you integrate it in one time period T then the result may or may not be infinity. When it goes to o infinity we can say that condition number three is violated because the signal is not absolutely integrable over the range of time period and T. When it is less than infinity this means when it is finite then we can see that condition number three is satisfied.



Let's let us try to understand this by the help of two examples. In first example we have signal x1(t) and here when you calculate the integration over one time period from zero to T naught you will find it is finite. Our integration will be over the time period: From 0 to to T .

The result will be T/2 which is finite. Therefore Fourier series expansion for signal x1(t) will exist.

Now we will move to the second example in which we have x2(t). This is the waveform for the tan function and we already know when T is equal to PI /2 (pi over 2) value of signal X2(t) will approach to infinity. So when you find out the **integral over the time period** you will find it is equal to infinity. Therefore the given signal is not absolutely integrable over the range of time period and therefore the Fourier series expansion is not possible for this .

So I hope all the three conditions are clear to you. Let's continue types of Fourier Series:

1. Trigonometric Fourier series expansion
2. Complex exponential Fourier series expansion or simply the exponential Fourier series expansion
3. Polar or harmonic Fourier series expansion polar or harmonic Fourier series expansion

12-Trigonometric Fourier series expansion

Now, we are going to discuss the first type which is the trigonometric Fourier series expansion. Iwill first give you the formulas and then we will understand different terms involved in the formula and once we are done with the basics we will move to the examples

I have mentioned that the Fourier series expansion is used only for periodic signals we cannot have the Fourier series expansion of a non-periodic signal. The same thing you will see in the formulas involved in the trigonometric Fourier series expansion. Let's say there is a periodic signal X(t) and in case of trigonometric Fourier series expansion we can represent this periodic signal as

Sum of DC or average value of signal X(t) plus all the cosine terms plus all the sine terms.

So our task in the trigonometric Fourier series is to obtain the DC or average value of signal X(t) which we already know how to calculate. Then we will obtain the cosine terms and then we will obtain the sine terms. Their sum will give us the Fourier series expansion of the given periodic signal now this is the representation in mathematical form:

$$\boxed{x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t}$$

In this equation a_0 is the DC or average value of signal X(t) and we already know how to calculate the DC or average value of a periodic signal. You simply need to calculate the total area in one time period and then divide it by one time period.

$$a_0 = \frac{1}{T_0} \int_{T_0} x(t)\, dt$$

Now we will consider a_n which is the Fourier coefficient multiplied to cosine terms.

$$a_n = \frac{2}{T_0} \int_{T_0} x(t) \cdot \cos n\omega_0 t\, dt$$

We will do the same thing for the sine terms for this we have another b_n

$$b_n = \frac{2}{T_0} \int_{T_0} x(t) \cdot \sin n\omega_0 t\, dt$$

All three coefficients a_0, a_n, and b_n are Fourier coefficients of trigonometric expansion.

will focus on different coefficients in this expansion first we will talk about a_0 is simply the DC or average value of the given periodic signal. Here we are having the fundamental time period T_0.

Here; a_0 , a_n and b_n are all known as they are known as the Fourier coefficients and they are very important.

QUESTIONS and ANSWERS

1- What are periodic signals?

Those signals which repeat themselves in a fixed interval of time are called periodic signals. The continuous-time signal x(t) is periodic if and only if
x(t+T)= x(t).

Periodic signals are same in case of continuous time and discrete time signals.
In case of continuous time signal, x(t)=x(t+T), for all t>0
In case of discrete time signal, x(n)=x(n+N), for all n>0.

The periodic signals have actually a time period between t=-∞ and at t= + ∞. These signals have an infinite time period, that is periodic signals are actually continued forever. But this is not possible in case of real time signals.

2-What is a fundamental period?

The first time interval of a periodic signal after which it repeats itself is called a fundamental period. It should be noted that the fundamental period is the first positive value of frequency for which the signal repeats itself.

3-Is a constant signal periodic?

A constant signal is **NOT** periodic. It is because it does not repeat itself over in time. It is constant at any time, it is aperiodic.

4-What is the necessary and sufficient condition for a sum of a periodic continuous time signal to be periodic?

The necessary and sufficient condition for a sum of a periodic continuous time signal to be periodic is that the ratio of a period of the first signal to the period of other signals should be rational.

T/Ti = a rational number.

5- What is the fundamental period of the signal : e^{jwt}?

The complex exponential signal can be represented as

$e^{jwt}= e^{jwt+jwT}$

Hence, wt=2 π, T_0= 2π/w.

6- What is the period of the signal :e^{jw11t}?

From the definition of periodic signal, we express a periodic exponential signal as :

$e^{jw11t} = e^{jwt+jwT}$

Hence, $11wT_0=2\pi$, which gives the fundamental period as $2\pi/11$.

7-Is the sum of discrete time periodic signals periodic?

The sum of discrete time periodic signals always periodic because the period ratios N/N are always rational. For the continuous time, it depends on the ratio.

8-What is the period of the signal: 2cost/6?

Comparing the above signal with the standard form $A\cos 2\pi Ft$, where A is the amplitude and F is the frequency,

We get, $2\pi F=\frac{1}{6}$

So, $F= 1/12\pi$ Hence, $t= 12\pi$.

9-What is Fourier series?

The Fourier series is the representation of periodic signals in terms of complex exponentials, or equivalently in terms of sine and cosine waveform leads to Fourier series. In other words, Fourier series is a mathematical tool that allows representation of any periodic wave as a sum of harmonically related sinusoids.

10-What are the conditions called which are required for a signal to fulfil to be represented as Fourier series?

When the Dirichlet's conditions are satisfied, then only for a signal, the fourier series exist. Fourier series is of two types- trigonometric series and exponential series.

11- How is a trigonometric Fourier series represented?

$A_0 + \sum[a_n\cos(w_0t)+ b_n\sin(w_0t)]$

12-How is the exponential Fourier series represented?

The exponential Fourier series is represented as: $X(t)=\sum X_n e^{jnwt}$.

Here, the X(t) is the signal and Xn=1/T ∫x(t)e-jnwt.

13-Fourier series uses which domain representation of signals?

Fourier series uses frequency domain representation of signals.

14-What are the properties of continuous time Fourier series?

Linearity,

time shifting,

frequency shifting,

time reversal,

time scaling,

periodic convolution,

multiplication,

differentiation are some of the properties followed by continuous time fourier series.

15-If x(t) and y(t) are two periodic signals with coefficients Xn and Yn then the linearity is represented as?

ax(t) + by(t) = aXn + bYn,

x(t) and y(t) are two periodic signals with coefficients Xn and Yn.

This means; Fourier Series of two periodic signals x(t) and y(t) equals to sum of Fourier convolutions of these two signals.

16-Why does the signal change while time scaling?

The fourier coefficients have not changed but the representation has changed because of changes in fundamental frequency So the signal changes while time scaling because, the frequency changes.

17-What is the Laplace Transform of of δ(t). (unit impulse signal)

The Answer is : 1

Laplace transform, L{x(t)} = X(s) = ∫∞−∞x(t)e−stdt

L{δ(t)} = ∫∞−∞δ(t)e−stdt

[x(t)δ(t) = x(0)δ(t)]

= ∫∞−∞δ(t)dt

= 1.

18-Find the Laplace transform of u(t) (unit step signal)

The answer is 1/s

Laplace transform, $L\{x(t)\} = X(s) = \int x(t)e^{-st}\, dt$

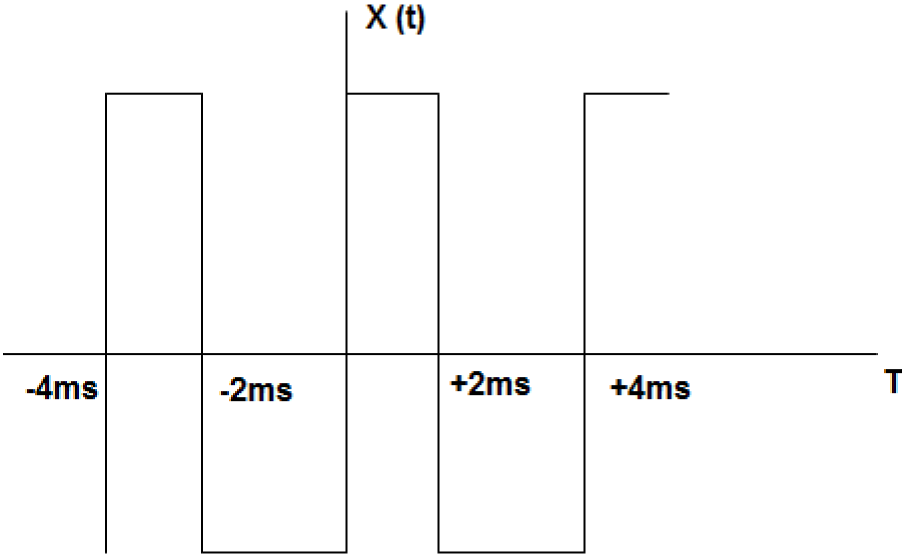
$L\{u(t)\} = \int u(t)e^{-st}\, dt = \quad 1/s \quad$ when $s>0$

19-What is the outcome of a periodic convolution of signals in case of continuous time fourier series?

This is a very important property of continuous time fourier series, it leads to the conclusion that the outcome of a periodic convolution is the multiplication of the signals in frequency domain representation.

$X(t)*y(t)=z(t) \leftrightarrow X_nY_n$

20-A periodic rectangular signal X (t) has the waveform as shown below. The frequency of the fifth harmonic of its spectrum is _______________



Periodic time = 4 ms = 4 × 10^{-3}

Fundamental frequency = 10^3/4 = 250 Hz

Frequency of the fifth harmonic = 250 × 5 = 1250 Hz.

21-What is the equivalent of unit impulse δ(at) ?

$\delta(at) = (1/a)\, \delta(at)$, $a>0$

22-What is the frequency domain?

frequency domain is defined as an analysis of a signal or a system with respect to its frequency. This concept has emerged from the transformations and the 'spectrum' concept.

23- What are the mathematical tools to convert a system from a time domain to frequency domain?

Fourier series,

Fourier transform,

Laplace transform,

z-transform are some tools to convert a system from a time domain to frequency domain analysis to make it simpler. In fact, the concept of frequency domain has emerged from these transformations. It was first given by Joseph Fourier.

24-**One of the main limitations of time domain analysis is the noise and frequency. Is it true?**

True, one of the main limitations of time domain analysis is the noise and frequency. This is because it is easier in the frequency domain to read it and detect it and solve it. Time domain analysis is much tedious and difficult to perform when it comes to lengthy solvable problems.

25-What are the types of symmetry shown by signals?

The types of symmetry shown by signals are- Even, odd, half-wave and quarter wave symmetry.

26-How is time shifting represented in case of periodic signal?

If x(t) is a periodic signal with coefficients Xn , then if a signal is shifted to t0, then the property says,

Xn = x(t-t0), →Yn = Xne-^{njwt0}