Stat 250: Probability and Statistics

Dr.Gültaç E.İNAN

Some Special Continuous Random Variables:

Normal Random Variable :

Standard Normal Random Variable:

***t* Random Variable:**

Normal Random Variable :

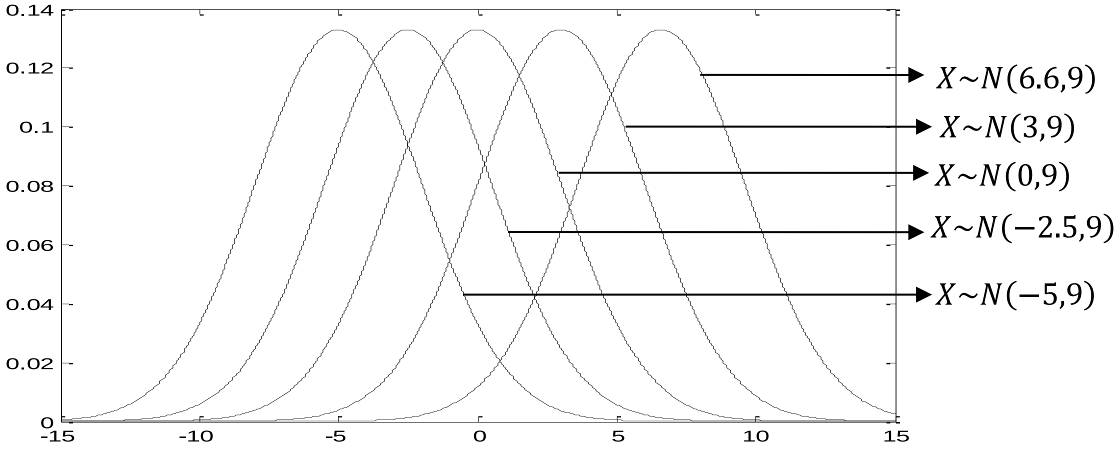
Normal distribution is a very important distribution used in both applied and theoretical statistics. The reason why the normal distribution has an important place in statistics is that many observation results give a bell-shaped distribution and most distributions approach normal distribution as the number of observations increases. The probability density function of X random variable is given by,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty, -\infty < \mu < \infty, \sigma^2 > 0$$

μ: average (parameter)
σ^2: variances (parameter)
$\mathrm{e} = 2.71825$
$\pi = 3.1416$

Notation: $X \sim N(\mu, \sigma^2)$

The probability density functions graphs of normal random variables with $(\sigma^2 = 9)$ variances but different expected values $\mu = -5, -2.5, 0, 3, 6.6$. **(Ozturk 2010).**



Properties of the Normal Random Variable

1.

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1$$

2. The normal distribution is symmetrical with respect to the mean. It means;

$$\int_{-\infty}^{\mu} f(x)\,dx = \int_{\mu}^{+\infty} f(x) = \frac{1}{2}$$

If , $\mu = 0$ $\sigma^2 = 1$, the normal random variable is called standard normal random variable.The standard normal random variable is denoted by the letter Z.

Notation: $Z \sim N(0,1)$

The probability density function of Z random variable is given by,

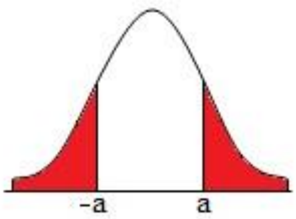
$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, -\infty < z < \infty$$

Transformation:

If $X \sim N(\mu, \sigma^2)$ is normal random variable, $Z = \frac{X-\mu}{\sigma}$ is a standard normal random variable. In this case;

$$P(X \leq x) = P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right)$$
$$= P(Z \leq z)$$

This is called standardization. From the symmetry property;



$$P(Z \leq -a) = P(Z \geq a)$$
$$= 1 - P(Z < a)$$

For the probabilities calculations, the Z table prepared with the values 0 to nearly 4 is used.

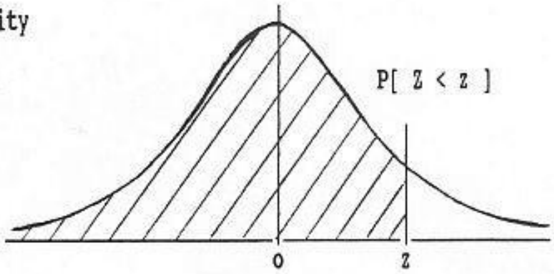
Z table is given as follows;

<u>STANDARD STATISTICAL TABLES</u>

1. Areas under the Normal Distribution

The table gives the cumulative probability up to the standardised normal value z
i.e.

$$P[\ Z < z\ ] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}Z^2)\ dZ$$



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5159	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7854
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8804	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9865	0.9868	0.9871	0.9874	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9980	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

z	3.00	3.10	3.20	3.30	3.40	3.50	3.60	3.70	3.80	3.90
P	0.9986	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000

Examples: Reading the Table

1.

$P(Z < 1.5) = 0.9332 \rightarrow P(Z > 1.5) = 1 - 0.9332 = 0.0668$

$P(Z < 2.2) = 0.9861 \rightarrow P(Z > 2.2) = 1 - 0.9861 = 0.0139$

$P(Z < 0.42) = 0.6628 \rightarrow P(Z > 0.42) = 1 - 0.6628 = 0.3372$

$P(Z < 1.53) = 0.9370 \rightarrow P(Z > 1.53) = 1 - 0.9370 = 0.0630$

2. $P(Z < a) = 0.9878 \rightarrow a = 2.25$

3. $P(Z > a) = 0.0301 \rightarrow P(Z < a) = 0.9699 \rightarrow a = 1.88$

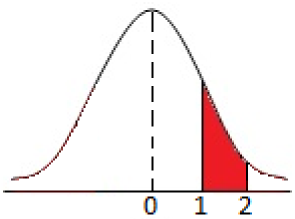
4. Student

*$P(Z > 1.36) =?$

$*\ P(Z < a) = 0.8264 \rightarrow a =?$

* $P(Z > a) = 0.0495 \rightarrow a =?$

5. $P(1 < Z < 2) =?$

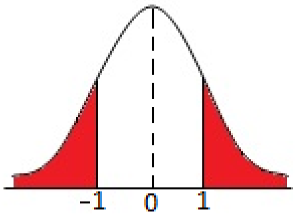


$P(1 < Z < 2) = P(0 < Z < 2) - P(0 < Z < 1)$

$= 0.4773 - 0.4413$

=0.036

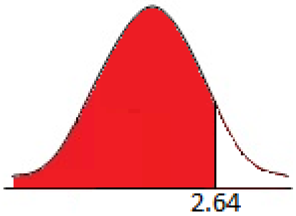
6. $P(Z < -1)$ =?



$P(Z < -1) = P(Z > 1) = 1 - P(Z < 1)$

=1-0.8413=0.1587

7. $P(Z > -2.64)$ =?



$P(Z > -2.64) = P(Z < 2.64)$=0.9959

8. $P(-1.32 < Z < 2.87) = P(Z < 2.87) - P(Z < -1.32)$

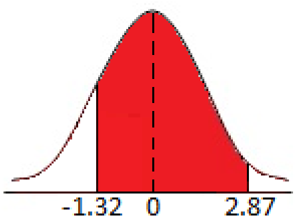
=$P(Z < 2.87) - P(Z > 1.32)$

= $P(Z < 2.87) - \left(1 - P(Z < 1.32)\right)$

= $P(Z < 1.32) + P(Z < 2.87) - 1$

=0.9066+0.9980-1

=0.9046



9. $P(-3 < Z < 3) = 2 * P(0 < Z < 3)$

=2*0.4986

=0.9972

Student:

10. $P(-1.32 < Z < -2.87) = ?$

11. $P(-1.32 < Z < 0) = ?$

12. $P(-1.32 < Z < a) = 0.1269 \rightarrow a = ?$

13. $P(Z < a) = 0.1269 \rightarrow a = ?$

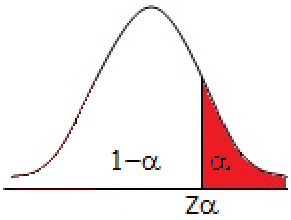
14. $P(0 < Z < a) = 0.4032 \rightarrow a = ?$

<u>Transformations from X to Z</u>

$X \sim N(\mu, \sigma^2)$ normal random variable$\rightarrow Z = \frac{X-\mu}{\sigma}$ standard normal random variable.

<u>Percentile:</u>

The $1-\alpha th$ percentile of a set of measurements **is the value** $(z_\alpha)$for which at most $1-\alpha\%$ of the measurements are less than that value.



$P(Z > z_\alpha) = \alpha$

$P(Z < z_\alpha) = 1-\alpha \rightarrow z_\alpha$: $1-\alpha th$ percentile

Example: $90th$ percentile means:

$P(Z < z_{0.10}) = 0.90 \rightarrow z_{0.10}$: $90th$ percentile

Examples: For Most used α Values.

1.

$P(Z < z_{0.05}) = 0.95$ **($95th$ percentile)**

$z_{0.05} = 1.645$ $\left(1.64 + \frac{1.65}{2} = \mathbf{1.645}\right)$

2. ($975th$ percentile)

$P(Z < z_{0.025}) = 0.975 \rightarrow \boldsymbol{z_{0.025}}$**=1.96**

3. ($90th$percentile)

$P(Z < z_{0.10}) = 0.90$

$$\boldsymbol{z_{0.10}} = \frac{(1.28 + 1.29)}{2} = \mathbf{1.285}$$

Examples:

X : Lifetime for a particular type of battery.It is known that;

$X \sim N(\mu = 35\ hours, \sigma^2 = 16\ hours)$

a. For a randomly selected battery, $P(X \geq 45)$ (Life time of the battery is more than 45 hours)

Solution:

$$P\left(\frac{X-35}{4} \geq \frac{45-35}{4}\right) = P(Z \geq 2.5) = 1 - P(Z < 2.5) = 1 - 0.9938 = 0.0062$$

b. $P(40 < X < 45) = P\left(\frac{40-35}{4} < Z < \frac{45-35}{4}\right) = P\left(\frac{5}{4} < X < \frac{10}{4}\right)$

$= P(1.25 < Z < 2.5)$

$= P(Z < 2.5)$- $P(Z < 1.25)$

=0.9938-0.8944

=0.0994

c.Student:

$P(X < 40)$ =?

d. 95.percentile of X=?

$P(X < x) = 0.05 \rightarrow x =?$

Solution:

$P(Z < a) = 0.95 \rightarrow a = z_{0.05} = 1.645$

$$a = 1.645 \rightarrow 1.645 = \frac{x-35}{4} \rightarrow x = 1.645 * 4 + 35 \rightarrow x = 41.58$$

e. 90.percentile of X=?

Example: Student

The amount of time it takes to assemble a computer is normally distributed, with a mean of 50 minutes and a standard deviation of 10 minutes. What is the probability that

a.

a computer is assembled in a time between 45 and 60 minutes?

b. Define some probabilities and calculate.

c. 95.percentile of X=?