Relations

Murat Osmanoglu

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  (Ahmet, Physics) $\in R$, (Efe, Discrete) $\notin R$

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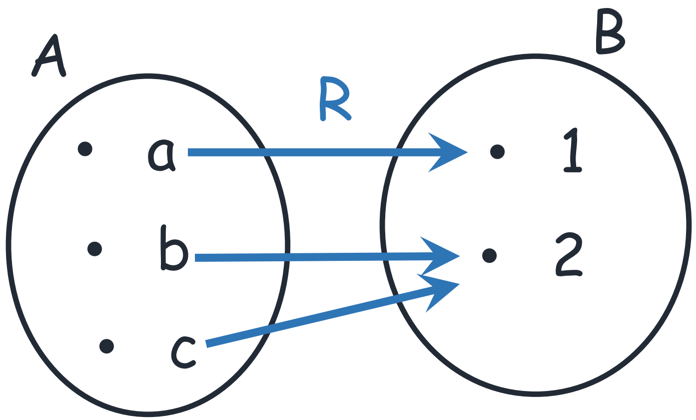

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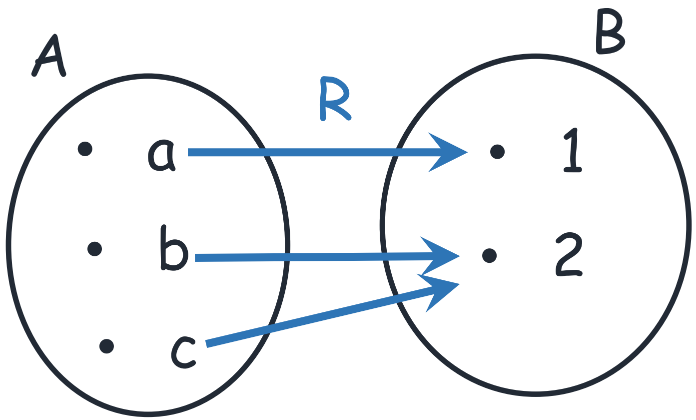

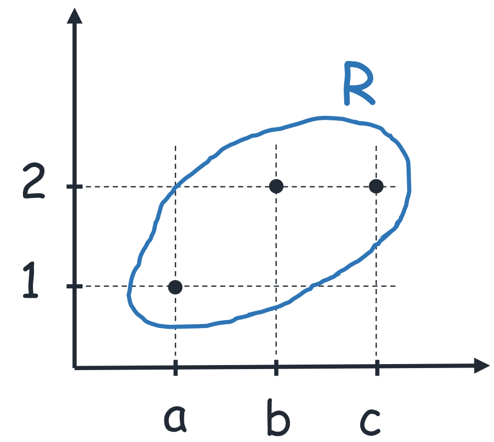

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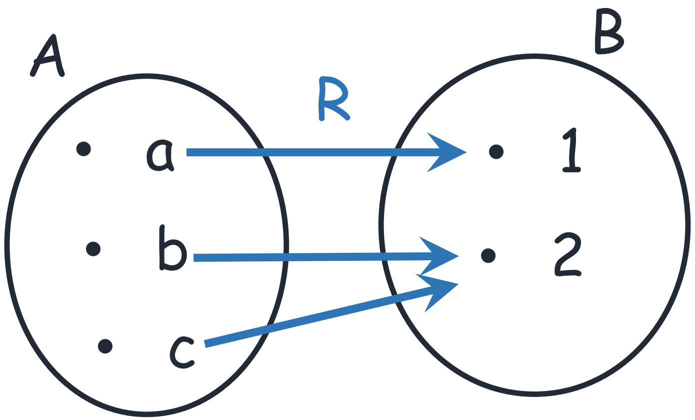

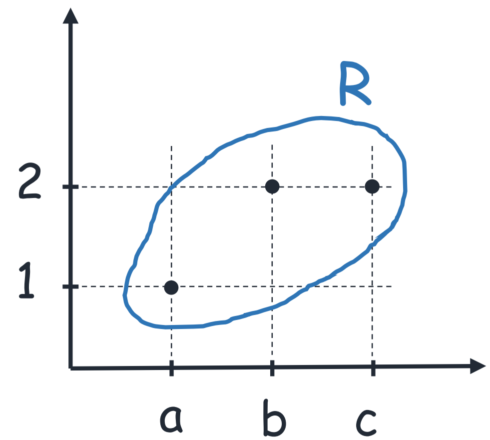

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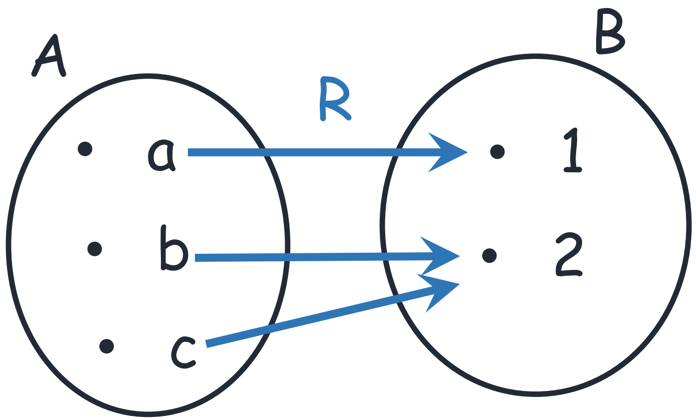

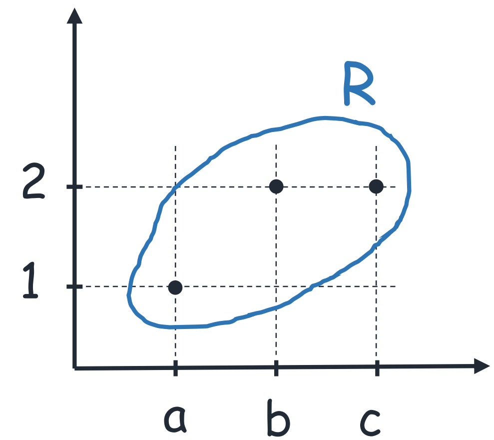

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- the number of relations that can be defined from A to B :

$$2^{|A||B|}$$

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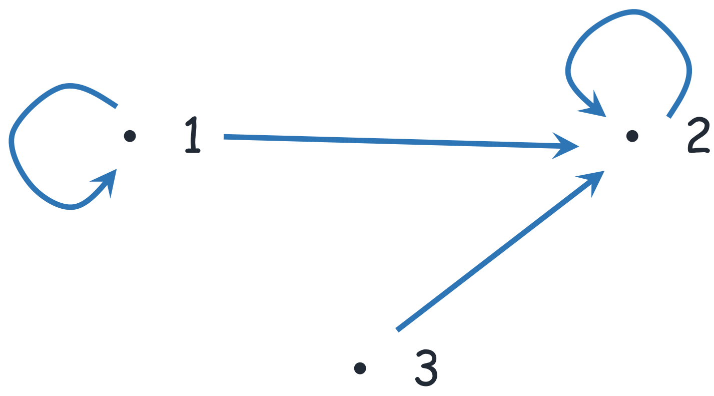

Functions as Relations

$$R \subseteq A \times B$$

$$\begin{array}{cc} \swarrow & \searrow \\ \text{domain} & \text{codomain} \end{array}$$

R(A) : the image of R, $R(A) = \{y \in B | (x, y) \in R, \exists x \in A\}$

Function is a relation that satisfies two conditions :

- for every element x of the domain, there is an element y in the codomain such that (x,y) is an element of the relation

  Let $R \subseteq A \times B$ be the relation, $\forall x[(x \in A) \rightarrow (\exists y \in B \;\; s.t. (x, y) \in R)]$

- for every element x of the domain, there is only one element y of the codomain such that (x,y) is an element of the relation

  Let $R \subseteq A \times B$ be the relation, $\forall x[((x, y_1) \in R \wedge (x, y_2) \in R) \rightarrow (y_1 = y_2)]$

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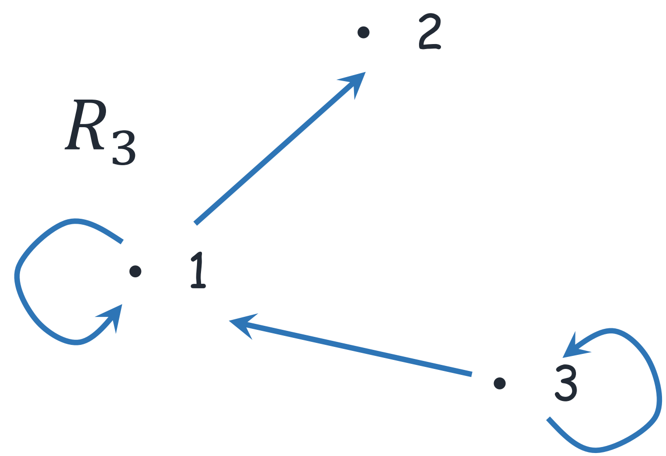
R_2	1	2	3
1	1	0	0
2	0	1	0
3	0	1	1

R_3



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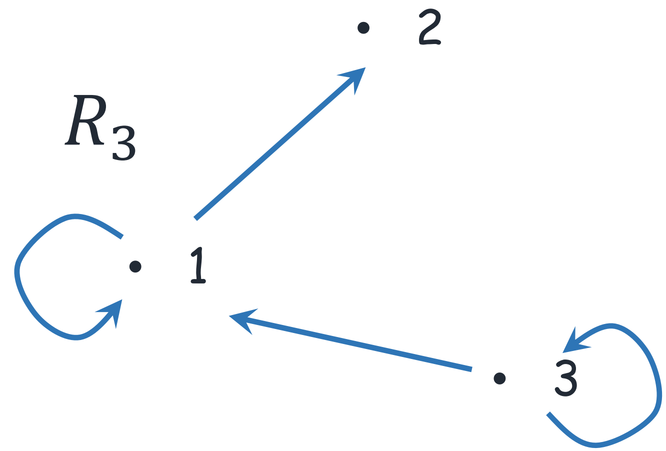

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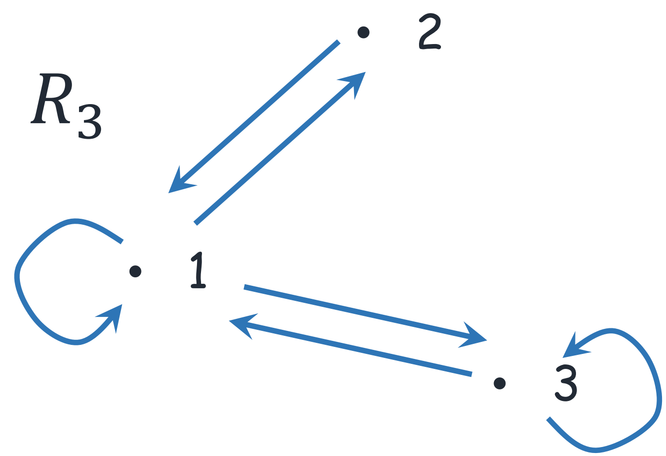

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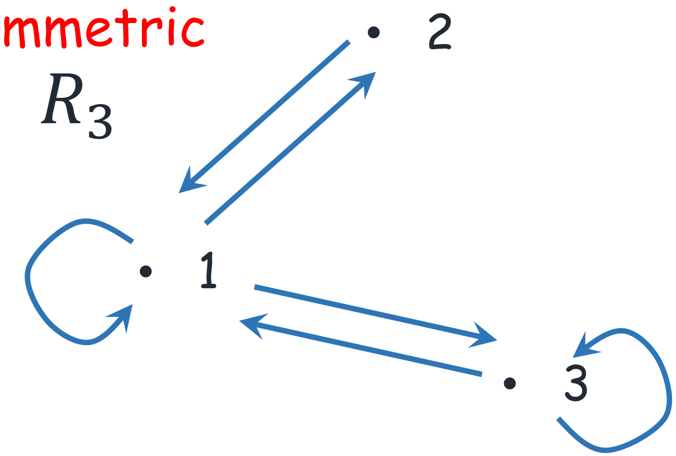

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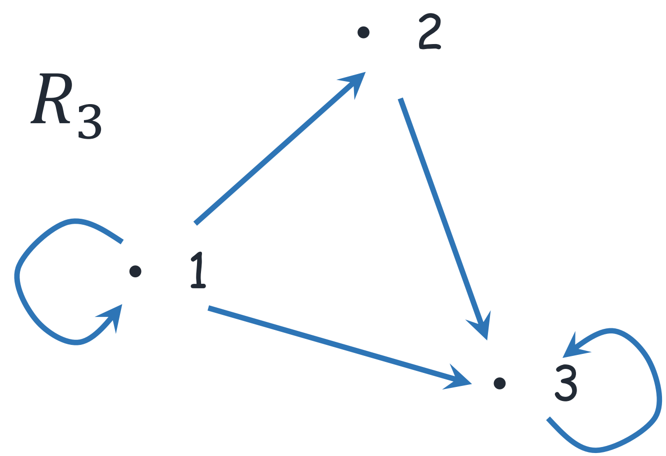

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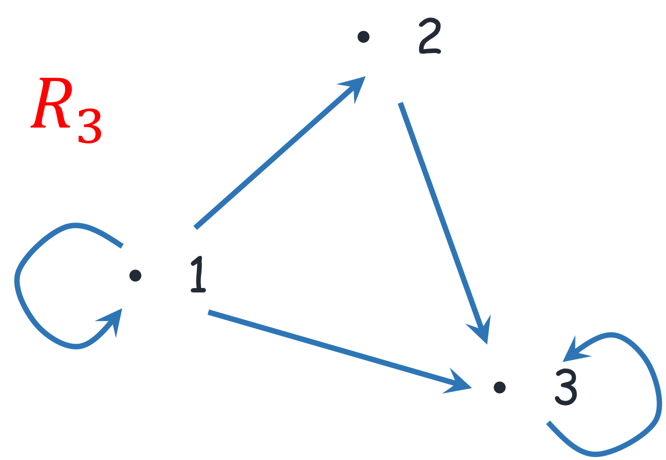

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- add each of them the pairs (1, 1), . . ., (n, n) to make them reflexive

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How many symmetric relations can be defined on a set A of n elements?

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How many symmetric relations can be defined on a set A of n elements?

- A = {1, 2, . . ., n}
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- How many subsets of the set {a,b,c,d} with the cardinality 3 contains the element a ?

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>
> {a, . , .} $\binom{4}{2} = 6$

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$$\left(2^n . 2^{\frac{n^2-n}{2}}\right) = 2^{(n^2+n)/2}$$

Operations

Operations

Union : Given $R, S \subseteq A \times B$,

$$\mathrm{T} = R \cup S = \{(x, y) | (x, y) \in R \ \vee \ (x, y) \in S\}$$

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$$\mathrm{T} = \bar{R} = \{(x, y) | (x, y) \notin R \ \}$$

Inverse : Given $R \subseteq A \times B$,

$$\mathrm{T} = R^{-1} = \{(y, x) \in B \times A | (x, y) \in R \ \}$$

Operations

Union : Given $R, S \subseteq A \times B$,

$$T = R \cup S = \{(x, y) | (x, y) \in R \ \vee \ (x, y) \in S\}$$

Intersection : Given $R, S \subseteq A \times B$,

$$T = R \cap S = \{(x, y) | (x, y) \in R \ \wedge \ (x, y) \in S\}$$

Complement : Given $R \subseteq A \times B$,

$$T = \bar{R} = \{(x, y) | (x, y) \notin R \ \}$$

Inverse : Given $R \subseteq A \times B$,

$$T = R^{-1} = \{(y, x) \in B \times A | (x, y) \in R \ \}$$

Composition : Given $R \subseteq A \times B$ and $S \subseteq B \times C$

$$T = S \circ R = \{(x, z) | (x, y) \in R \ \wedge \ (y, z) \in S\}$$

Operations

R	1	2
a	1	0
b	0	1
c	1	0

S	u	v
1	0	0
2	1	1

Operations

R	1	2
a	1	0
b	0	1
c	1	0

S∘R	u	v
a		
b		
c		

S	u	v
1	0	0
2	1	1

Operations

R	1	2
a	1	0
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S∘R	u	v
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Operations

R	1	2
a	1	0
b	0	1
c	1	0

$S \circ R$	u	v
a	0	0
b	1	1
c		

S	u	v
1	0	0
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Operations

R	1	2
a	1	0
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$S \circ R$	u	v
a	0	0
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S	u	v
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Operations

R	1	2
a	1	0
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c	1	0

S∘R	u	v
a	0	0
b	1	1
c	0	0

S	u	v
1	0	0
2	1	1

S^{-1}	1	2
u	0	1
v	0	1

Operations

- $A = \{ 1, 2, 3 \}$, $R = \{ (1, 1), (2, 1), (3, 2) \}$

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1			
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R	1	2	3
1	1	0	0
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2			
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R	1	2	3
1	1	0	0
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1	1	0	0
2	1	0	0
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1	1	0	
2			
3			

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1	1	0	0
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1	1	0	0
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3	0	1	0

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1	1	0	0
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1	1	0	0
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3	0	1	0

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  $R^3 = R^2 \circ R = \{(1,1), (2,1), (3,2)\}$

Operations

- A = { 1, 2, 3 },   R = { (1, 1), (2, 1), (3, 2) }

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- $R^2 = R \circ R = \{(1,1),(2,1),(3,2)\}$
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- The relation R on a set A is transitive if and only if $R^n \subseteq R$ for some $n \in Z^+$

Equivalence Relations

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Definition : A relation R on a set A is called an equivalence relation if it's reflexive, symmetric, and transitive. If $(a, b) \in R$, then a and b are called equivalent, i.e. a~b.

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  - $[(a, b) \in R \land (b, c) \in R] \rightarrow [a - b \in \mathbb{Z} \land b - c \in \mathbb{Z}]$
    $\rightarrow [a - c \in \mathbb{Z}] \rightarrow [(a, c) \in R]$ (transitive)

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  thus $(b, a) \in R_3$ (symmetric)

- if $(a, b) \in R_3 \wedge (b, c) \in R_3$,

  either $a = b$ and $b = c$, then $a = c$

  or $a = b$ and $b[1..3] = c[1..3]$, then $a[1..3] = c[1..3]$

  or $a[1..3] = b[1..3]$ and $b = c$, then $a[1..3] = c[1..3]$

  or $a[1..3] = b[1..3]$ and $b[1..3] = c[1..3]$, then $a[1..3] = c[1..3]$

  (transitive)

Equivalence Relations

- Let R_n be a relation on the set of strings built with $\{0,1\}$.

  For any two strings s and t,

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- $[\varepsilon]_{R_3} \cup [0]_{R_3} \cup \cdots \cup [111]_{R_3}$

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- $[\varepsilon]_{R_3} \cup [0]_{R_3} \cup \cdots \cup [111]_{R_3} = S$, the set of all strings

Equivalence Relations

- A given set S can be decomposed into disjoint subsets A_i. For a family of sets $A = \{A_i | i \in I\}$ such that $A_i \cap A_j \neq \emptyset$, a given set S can be written as

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Partial Order

Definition : A relation R on a set A is called a partial order if it's reflexive, antisymmetric, and transitive. A set S together with a partial order R is called partially ordered set or poset, (S, R)

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- Consider the poset $(\mathbb{Z}^+, '|')$.
  - Since $3|9$, 3 and 9 are comparable.

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Definition : The elements a and b of a poset (S, R) are called comparable if either aRb or bRa.

- Consider the poset $(\mathbb{Z}^+, '|')$.
  - Since $3|9$, 3 and 9 are comparable.
  - Since $7\nmid 5$ or $5\nmid 7$, 5 and 7 are not comparable

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Definition : If every pair of elements in S are comparable, then R is called total order. (S, R) is called totally ordered set.

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- the poset $(\mathbb{Z}^+, '|')$ is not totally ordered set.

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- the poset $(\mathbb{Z}^+, '|')$ is not totally ordered set.
- the poset $(\mathbb{Z}^+, \leq)$ is a totally ordered set.

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- the poset $(\mathbb{Z}^+, '|')$ is not totally ordered set.
- the poset $(\mathbb{Z}^+, \leq)$ is a totally ordered set.

- For every $a, b \in \mathbb{Z}^+$, either $a \leq b$ or $b \leq a$. Thus, either $(a, b) \in (\leq)$ or $(b, a) \in (\leq)$

Partial Order

Definition : Consider a poset (S, R). An element a is called maximal if there is no $b \in S$ such that aRb. An element a is called minimal if there is no $b \in S$ such that bRa.

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- Consider the poset $(S, '|')$ where S = {2, 4, 5, 10, 12, 15, 20, 30}
  - maximal elements of $(S, '|')$ {12, 20, 30}
  - minimal elements of $(S, '|')$ {2, 5}

Definition : An element a is called the greatest element if bRa for all $b \in S$. An element a is called the least element if aRb for all $b \in S$.

Partial Order

Definition : Consider a poset (S, R). An element a is called maximal if there is no $b \in S$ such that aRb. An element a is called minimal if there is no $b \in S$ such that bRa.

- Consider the poset $(S, '|')$ where S = {2, 4, 5, 10, 12, 15, 20, 30}
  - maximal elements of $(S, '|')$ {12, 20, 30}
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Definition : An element a is called the greatest element if bRa for all $b \in S$. An element a is called the least element if aRb for all $b \in S$.

- Consider the power set of a given set S.

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    or $a < c$, and $c = e$ and $d \leq f$,
    or $a = c$ and $d \leq f$, and $c < e$,
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    either $a < c$, then $((c,d),(a,b)) \notin R$
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    or $a = c$ and $d \leq f$, and $c < e$, then $a < e$, $\big((a,b),(e,f)\big) \in R$
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  - for all $\big((a,b),(c,d)\big) \in R$ and $\big((c,d),(e,f)\big) \in R$, transitive
    either $a < c$ and $c < e$, then $a < e$, $\big((a,b),(e,f)\big) \in R$
    or $a < c$, and $c = e$ and $d \le f$, then $a < e$, $\big((a,b),(e,f)\big) \in R$
    or $a = c$ and $d \le f$, and $c < e$, then $a < e$, $\big((a,b),(e,f)\big) \in R$
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- a type of directed graph used to represent finite posets.

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- if $(x, y) \in (\leq)$, draw a line from x to y

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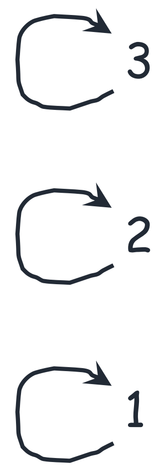

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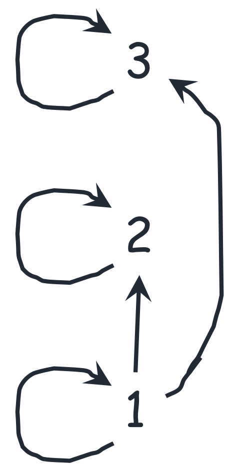

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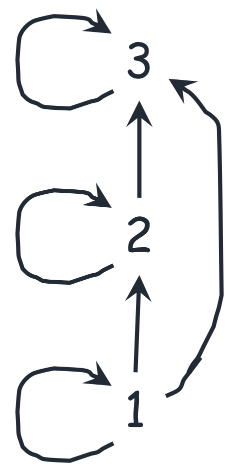

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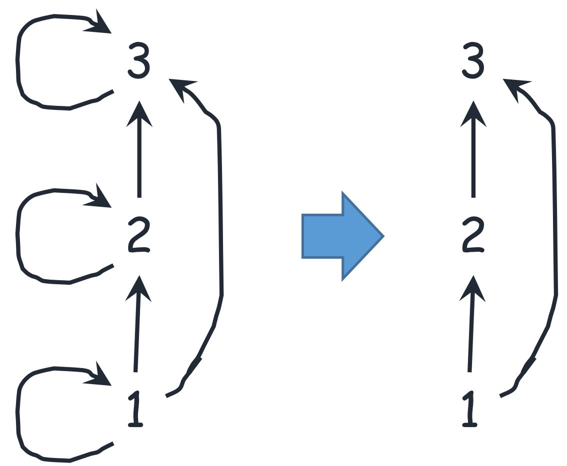

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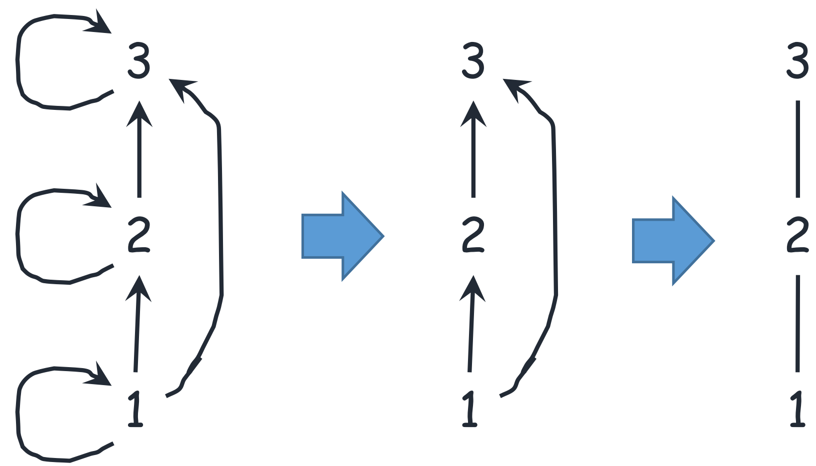

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2 3

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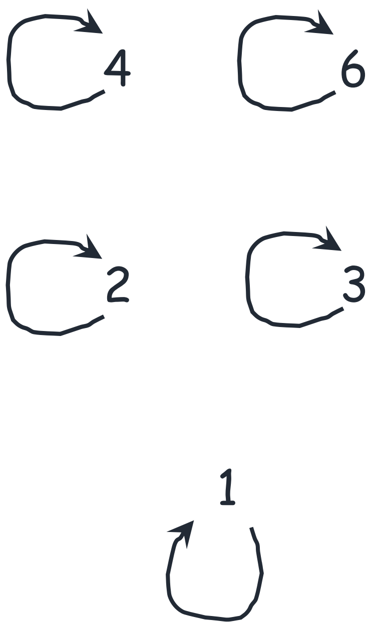

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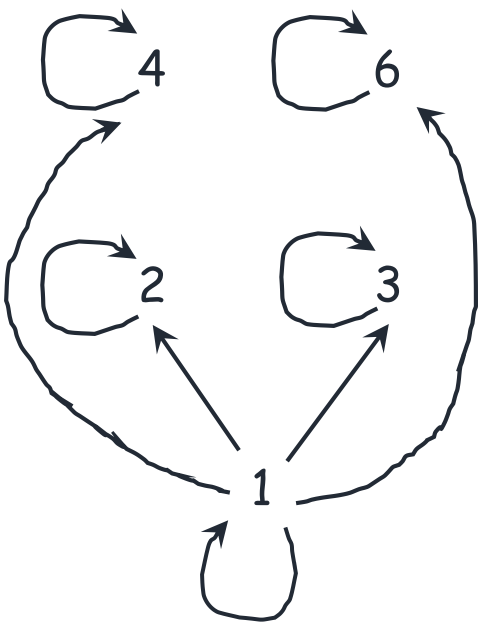

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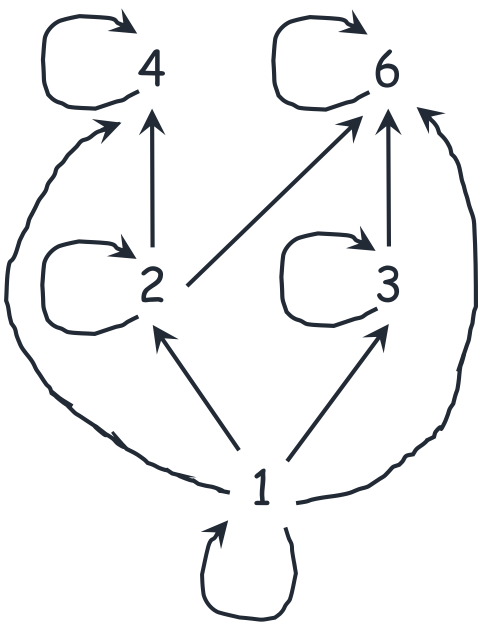

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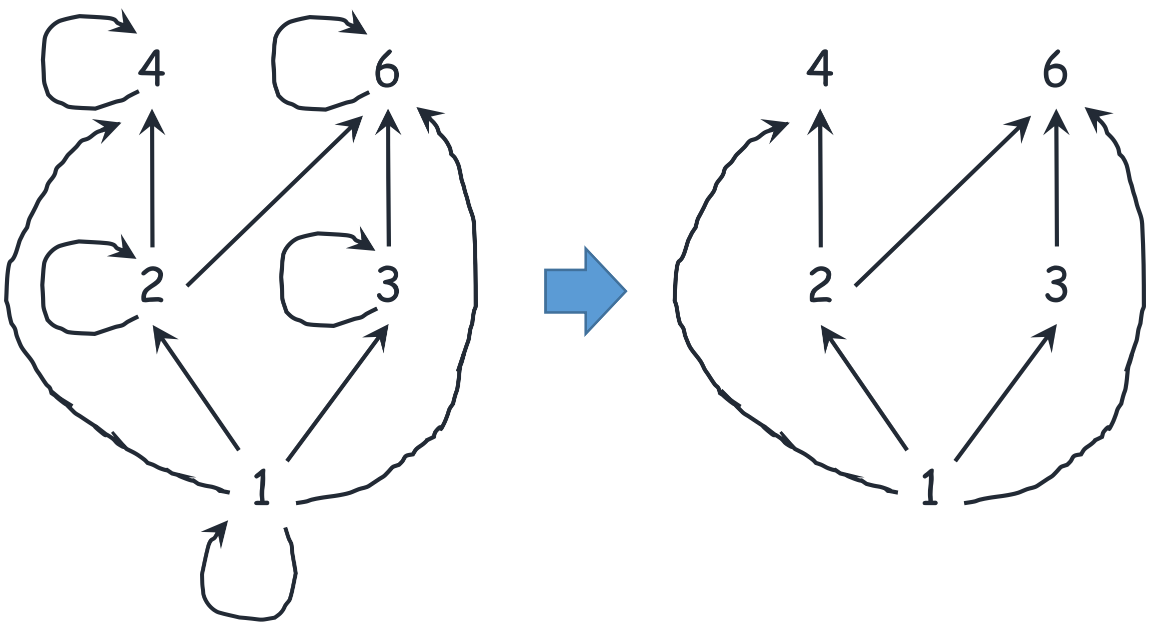

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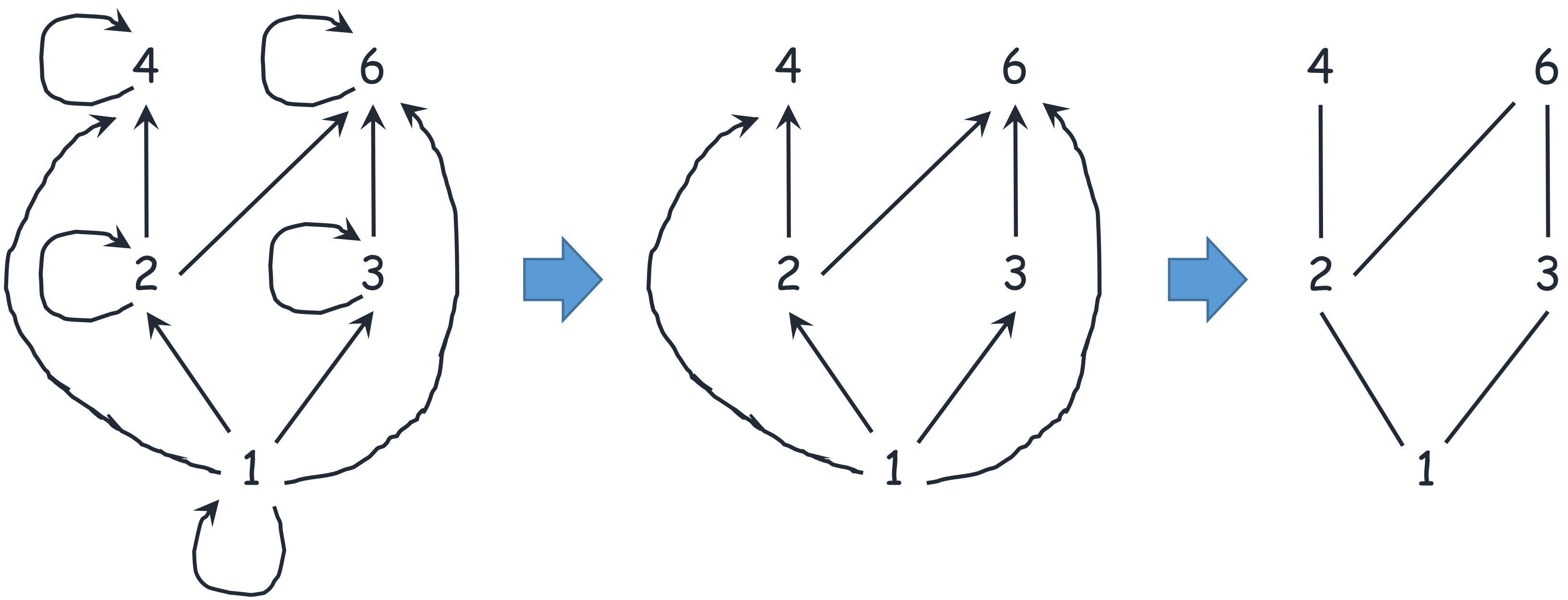

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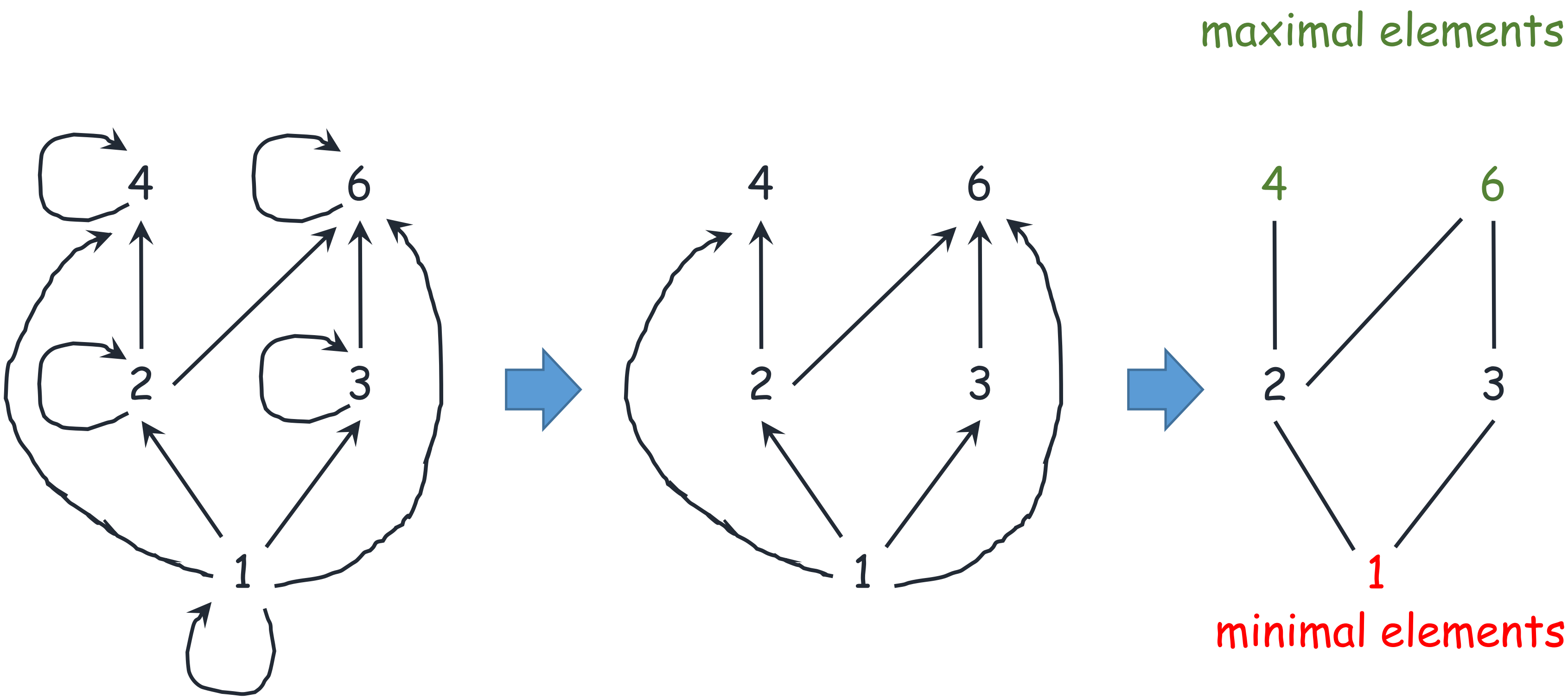

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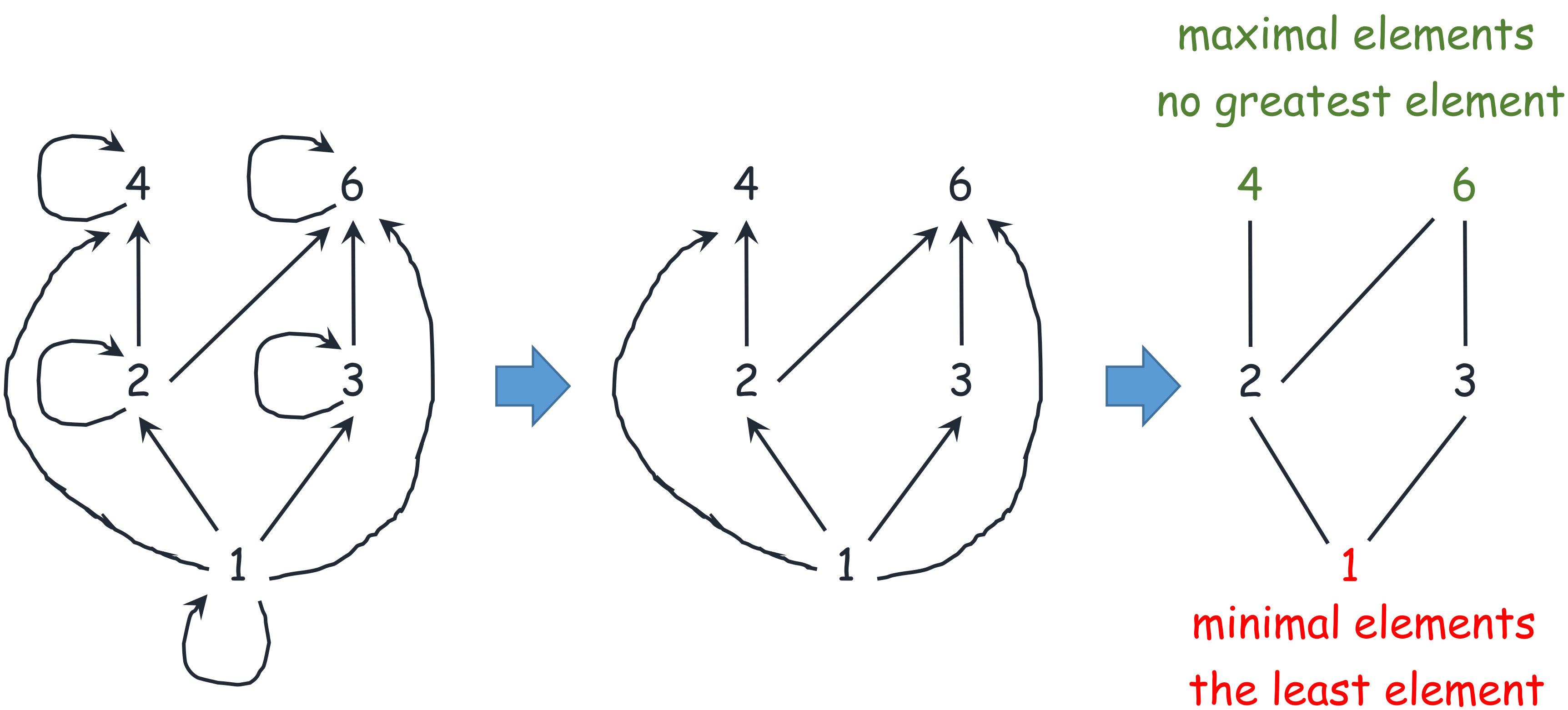

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∅

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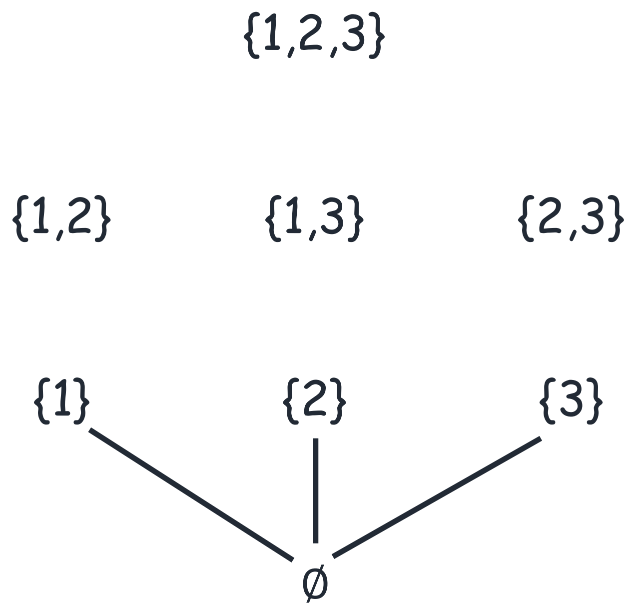

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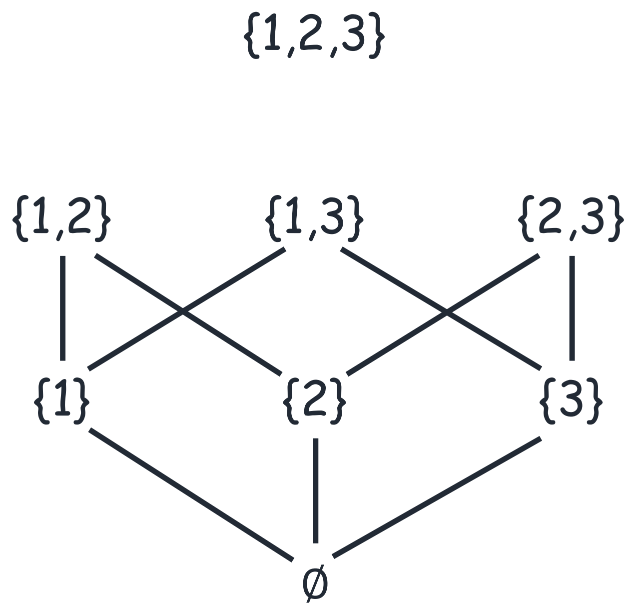

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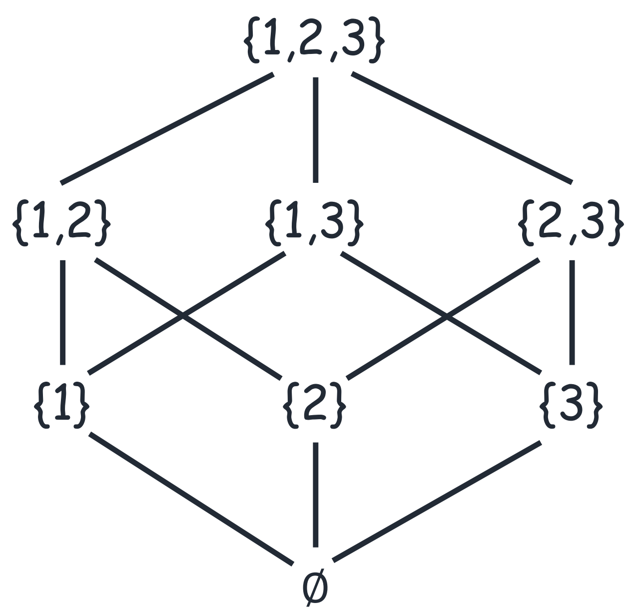

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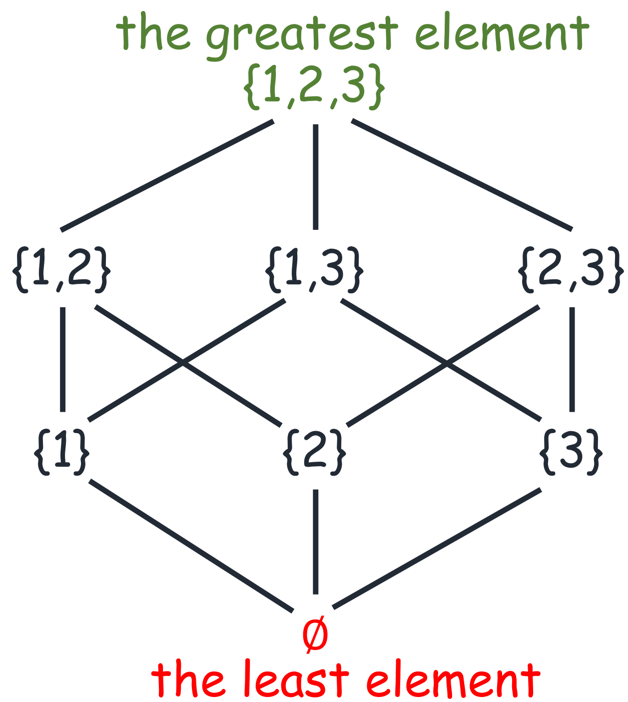

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Definition : Topological sorting of n elements from a poset (S, R) is $s_1 s_2 \ldots s_n$ such that there is no $(s_i, s_j) \in R$ where $j < i$

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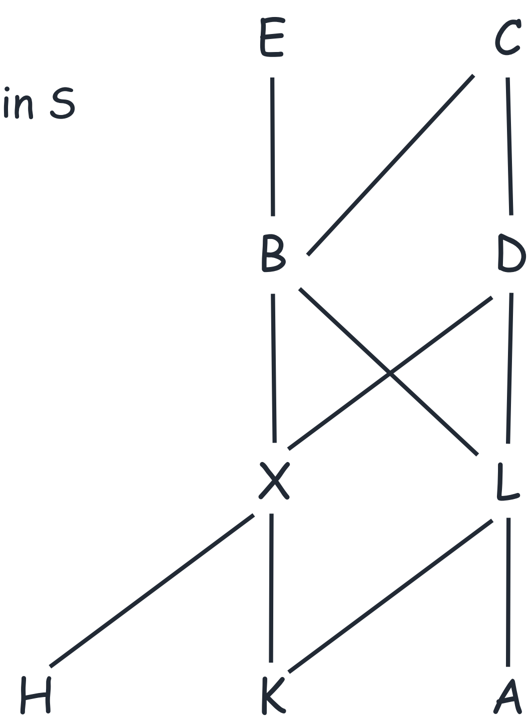

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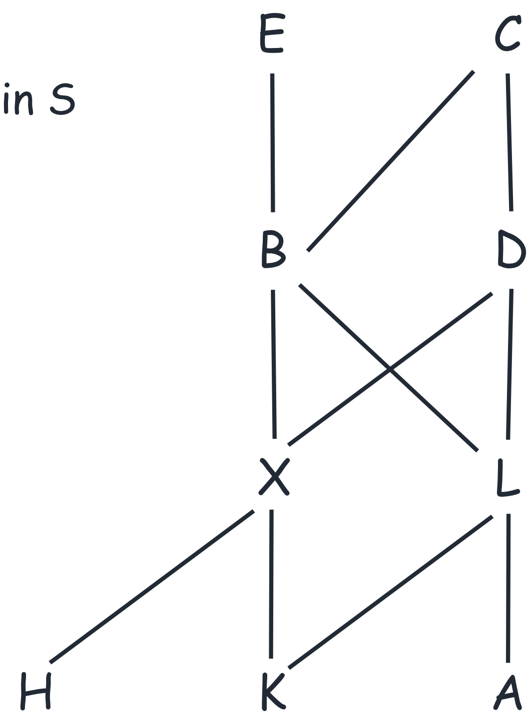

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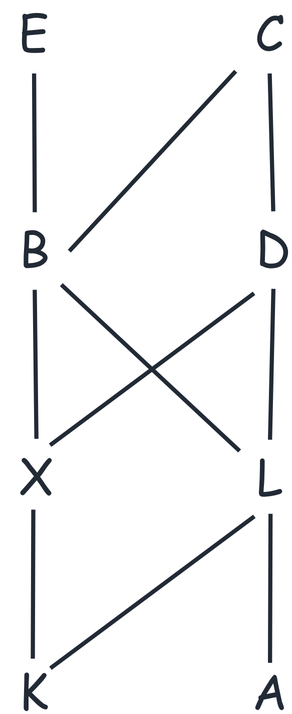

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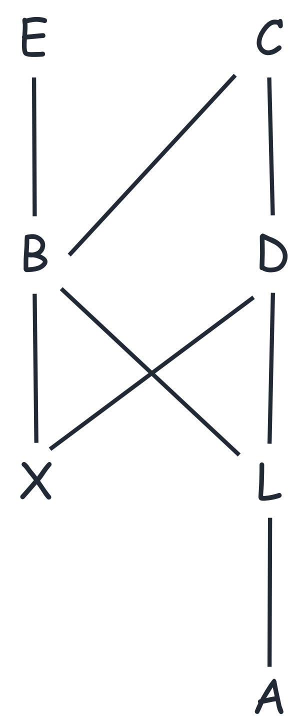


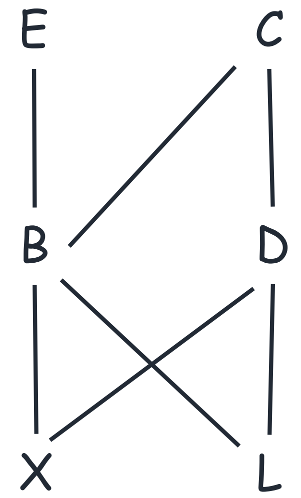
Q : H K

Topological Sorting

Definition : Topological sorting of n elements from a poset (S, R) is $s_1 s_2 \ldots s_n$ such that there is no $(s_i, s_j) \in R$ where $j < i$

input : a finite poset (S, R)
output : topological sorting of elements in S

initialize an empty queue Q
while $S \neq \emptyset$
    a = a minimal element of S
    S = S – {a}
    add a to Q
return Q



Q : H K A

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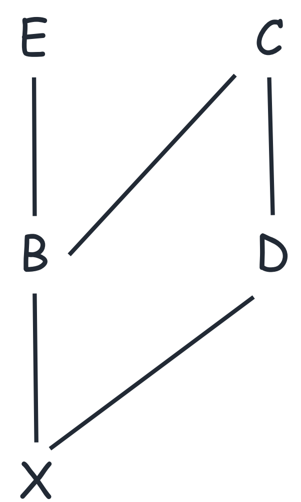

Q : H K A L

Topological Sorting

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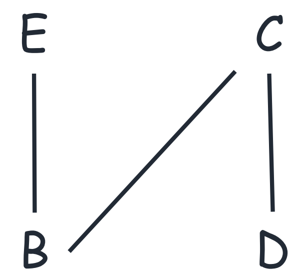

Q : H K A L X

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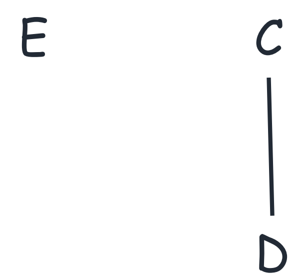

Q : H K A L X B

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Topological Sorting

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C

input : a finite poset (S, R)
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initialize an empty queue Q
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    S = S – {a}
    add a to Q
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