CALCULUS OF VARIATIONS, HAMILTON'S PRINCIPLE AND THE EULER-LAGRANGE EQUATIONS

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Abstract. A foray into variational calculus and functionals. The mantelpiece of the subject, the Euler-Lagrange equation, is derived and applied to several canonical examples, namely Hamilton's principle. Hamilton's principle, expressed as the principle of least Action, is also derived, whose importance and power is demonstrated on examples in classical mechanics, and discussed in the context of general relativity and quantum field theory.

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1. Motivation and History

Variational calculus and the closely related discipline of functional analysis are incredibly beautiful and rich fields which are nearly as old as calculus itself. Without a doubt, every domain of physics from classical mechanics to quantum field theory has not only been embroidered, but built upon the results of the calculus of variations. The numerous applications in physics has made variational calculus a renowned subject, but bringing calculus to a higher degree of abstraction is what took analysis into exaltation in the 20th century.

Most historians agree that the onset of variational calculus was brought on by the Brachistochrone problem. *Brachis* - shortest, and *chronos* - time, together describe a path along which a particle falls under gravity in the shortest possible time. The problem was to find such a path passing between two points in $\mathbb{R}^2$. Johann Bernoulli (1667 - 1748) posed this problem in the 1696 issue of *Acta Eruditorum*, one of the earliest scientific journals in Europe. He encouraged his brother Jacob and many others in the scientific community to take the Brachistochrone challenge, especially Isaac Newton. It was first solved independently by Jacob and Johann Bernoulli, and Gottfried Leibniz. Newton, hearing about the problem a month after its prompt, wrote up his solution in the early hours of the morning and sent out his result, which was published anonymously. Johann Bernoulli, having been familiar with Newton's handwriting and distinguished problem solving style, recognized the authorship of the anonymous submission and remarked "*I recognize the paw of the lion*" [1], [2].

The methods applied to the Brachistochrone problem as it was solved in Newton's time were all based on geometry and early principles of calculus [3]. David Hilbert (1862 - 1943) appreciated the need for more sophistication to solve problems dealing equations that depend on arbitrary functions; specifically, integral equations that describe the variation of functionals. Hilbert suggested to his contemporaries in the 1900 lecture to the International Congress of Mathematicians that the subject of variational calculus, which encompasses the study of variational problems like the Brachistochrone, and other more complicated problems, should be advanced. Leading analysts in the 20th century like Weierstraß and Jacobi would continue this work, and develop the field of functional analysis from variational methods. Today, the results of variational calculus and functional analysis have proven their might in the arenas of mechanics and field theory time and time again. Their uses extend to very modern applications in mathematics and physics which have led to many fundamental and groundbreaking achievements.

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2. Functionals

Definition 1. A functional $J : X \to \mathbb{C}$ is a continuous mapping from a function space X to complex numbers.

A functional is a so-called "function of functions." They take an argument of a whole function and spit out a number. Functionals are also linear operators. Many linear operators have integral representations, which will be especially useful in the case of functionals; through a definite integral representation, we automatically gain the map from a function space to $\mathbb{R}$ or $\mathbb{C}$ in the evaluation step in the definite integral. Here's a simple example of a functional I can define:

$$\begin{aligned} J[f(x)] &= \int_0^1 [f(x)]dx \\ \to J[1] &= 1 \\ J[x] &= \frac{1}{2} \\ J[x - x^2] &= \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \end{aligned}$$

Notice that the argument of a functional is in square brackets [ ]; this is some common notation to distinguish a functional from a function. A functional can also depend explicitly on the variable x and even the first derivative of f, $\frac{df}{dx} = f'$. For example,

$$J[x, f(x), f'(x)] = \int_{-1}^1 [f'(x)]^2 dx$$

A natural question arises: Can one perform calculus on this mapping? In other words, since the functional is a continuous map, can I measure infinitessmal changes in the codomain if I vary the arguments of the functional by an infinitessimal amount? The answer is yes. The piece of machinery that does this is the functional derivative.

Definition 2. Functional Derivative
$F[f(x)] : X \to \mathbb{R}$, $\epsilon > 0$.

$$\frac{\delta F}{\delta f(x)} = \lim_{\epsilon \to 0} \frac{F[f(x) + \epsilon\eta(x)] - F[f(x)]}{\epsilon}$$

If $f(x) \in H \subseteq X$, then $\eta(x) \in H$ is such that $f(x) + \epsilon\eta(x) \in H \ \forall \epsilon > 0$.

$f(x) + \epsilon\eta(x)$ is called the *variation* of f. The arbitrary function $\eta(x)$, whose strength is controlled with ϵ, allows for the function f to be continuously moved around in the function space X. Sometimes I will denote it $\hat{f} = f + \epsilon\eta$ [2], [8].

Since one of the principal aims of the calculus of variations is to find extrema, we can now do so. A functional J is extremal at f when

$$\frac{\delta J[f(x)]}{\delta f} = 0.$$

As an example, consider the boundary-value problem in Figure 1. The function $q(t)$

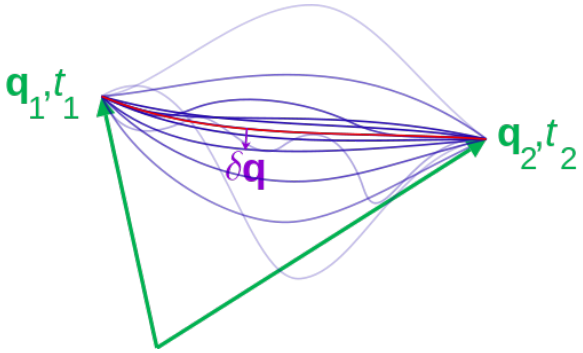

FIGURE 1. The variation of a function passing between two boundary points [11].

is varied (blue) away from the "true" function in red. $q(t)$ takes on boundary values $q(t_1) = q_1$ and $q(t_2) = q_2$. Notice that the variations of q must also pass through the boundary ($\hat{q}(t_1) = q_1$, $\hat{q}(t_2) = q_2$); this is guaranteed by requiring that the variation $\hat{q}$ always remains in the same subspace as q as stated in Definition 2. This is all we need to jump right in to the following major theorem.

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3. The Euler-Lagrange Equations

Theorem 3. *Euler-Lagrange*
Define a functional $J : C^2[x_1, x_2] \to \mathbb{R}$, with $J[f(x)] = \int_{x_1}^{x_2} F[x, f(x), f'(x)]dx$ such that F has continuous second order partial derivatives with respect to x, $f(x)$, and $f'(x)$. Consider the subset S of $C^2[x_1, x_2]$ containing functions that are well-defined on the boundary:

$$S = \{f \in C^2[x_1, x_2] : f(x_1) = y_1, f(x_2) = y_2\}$$

. If $f(x) \in S$, and if $\frac{\delta J[f]}{\delta f} = 0$, then

$$\frac{d}{dx}\left(\frac{\partial F}{\partial f'}\right) - \frac{\partial F}{\partial f} = 0.$$

Proof. The proof is done directly by evaluating $\frac{\delta J[f]}{\delta f}$ and supposing J is extremal at f.

$$\frac{\delta J[f]}{\delta f} = \lim_{\epsilon \to 0} \frac{J[f + \epsilon\eta] - J[f]}{\epsilon} = \lim_{\epsilon \to 0} \frac{J[\hat{f}] - J[f]}{\epsilon}$$

Now, since

$$J[\hat{f}] = \int_{x_1}^{x_2} F[x, \hat{f}(x), \hat{f}'(x)]dx$$

and

$$J[f] = \int_{x_1}^{x_2} F[x, f(x), f'(x)]dx,$$

it is useful to first simplify $F[x, \hat{f}(x), \hat{f}'(x)]$ by expanding it about the point (x, f, f') in a Talor expansion. By Taylor's theorem,

$$\begin{aligned} F[x, \hat{f}(x), \hat{f}'(x)] &= F[x, f(x) + \epsilon\eta(x), f'(x) + \epsilon\eta'(x)] \\ &= F[x, f(x), f'(x)] + \epsilon\left(\eta\frac{\partial F}{\partial f} + \eta'\frac{\partial F}{\partial f'}\right) + O(\epsilon^2). \end{aligned}$$

After taking the limit, the terms in the power series of order ϵ^2 will vanish. Now rewrite $\frac{\delta J[f]}{\delta f}$ with the expanded functional.

$$\frac{\delta J[f]}{\delta f} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int_{x_1}^{x_2}\left(F[x, f(x), f'(x)] + \epsilon\left(\eta\frac{\partial F}{\partial f} + \eta'\frac{\partial F}{\partial f'}\right) + O(\epsilon^2) - F[x, f(x), f'(x)]\right)dx\right]$$

$$\frac{\delta J[f]}{\delta f} = \int_{x_1}^{x_2}\left(\eta\frac{\partial F}{\partial f} + \eta'\frac{\partial F}{\partial f'}\right)dx$$

Integrating the second term in the integrand by parts is a nice trick that leaves us with no derivatives of η:

$$\int_{x_1}^{x_2}\eta'\frac{\partial F}{\partial f'}dx = \eta\frac{\partial F}{\partial f'}\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2}\eta\left[\frac{d}{dx}\frac{\partial F}{\partial f'}\right]dx$$

Since $f \in S$, $\hat{f} \in S \to \eta(x_1) = \eta(x_2) = 0$. Hence,

$$\int_{x_1}^{x_2}\eta'\frac{\partial F}{\partial f'}dx = -\int_{x_1}^{x_2}\eta\left[\frac{d}{dx}\frac{\partial F}{\partial f'}\right]dx$$

Returning now to $\frac{\delta J[f]}{\delta f}$, we obtain

$$\begin{aligned} \frac{\delta J[f]}{\delta f} &= \int_{x_1}^{x_2}\left(\eta\frac{\partial F}{\partial f} - \eta\left[\frac{d}{dx}\frac{\partial F}{\partial f'}\right]\right)dx \\ &= \int_{x_1}^{x_2}\eta\left(\frac{\partial F}{\partial f} - \frac{d}{dx}\frac{\partial F}{\partial f'}\right)dx = 0 \end{aligned} \tag{1}$$

Now, two lemmas are required to safely make the argument that the factor multiplying η in the integrand is zero, thus completing the proof. They are stated without proof in the interest of brevity.

Lemma 4. *Let $a, b \in \mathbb{R}$ such that $b > a$. $\exists\nu(x) \in C^2(\mathbb{R})$ such that $\nu(x) > 0\ \forall x \in (a, b)$ and $\nu(x) = 0\ \forall x \in \mathbb{R}\backslash(a, b)$.*

Lemma 5. *Suppose $\eta \in S = \{f \in C^2[x_1, x_2] : f(x_1) = y_1, f(x_2) = y_2\}$ and the inner product $\langle\eta, g\rangle = 0\ \forall\eta \in S$. If $g : [x_1, x_2] \to \mathbb{R}$ is continuous, then $g = 0$ on $[x_1, x_2]$.*

By Lemma 4, η can be chosen such that it is not zero on the interval. Then by Lemma 5, the integral in 1, viewed as the inner product

$$\left\langle\nu, \left(\frac{\partial F}{\partial f} - \frac{d}{dx}\frac{\partial F}{\partial f'}\right)\right\rangle,$$

must be zero on the interval $[x_1, x_2]$. Therefore,

$$\frac{\partial F}{\partial f} - \frac{d}{dx}\frac{\partial F}{\partial f'} = 0 \tag{2}$$

The above equality known as the Euler-Lagrange equation [1]. □

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4. Hamilton's Principle, or the Principle of Least (Extremized) Action

This principle is a purely mathematical result that relies on the definitions of the kinetic and potential energies as functionals, and the functional derivative.

Define the time-averaged potential energy as a functional, taking arguments of trajectories $x(t)$:

$$\overline{V}[x] = \frac{1}{\tau}\int_0^\tau V[x(t)]dt$$

The time-averaged kinetic energy:

$$\overline{T}[x] = \frac{1}{\tau}\int_0^\tau \frac{1}{2}m[x(t)]^{\frac{1}{2}}dt$$

Again, these functionals can be thought of as operations on trajectories $x(t)$ by integration. They map elements (functions) in function space to numbers in the real line. Taking the functional derivatives of both of these objects will lead to an interesting result.

For the following computations it is useful to introduce a particular form of the functional derivative. This form involves the Dirac-delta function, a function constructed by the following limit.

Definition 6. Dirac-delta

$$\delta(x'-x) = \lim_{a\to\infty}\begin{cases} 0 & x' < \left(x - \frac{1}{2a}\right) \\ a & \left(x - \frac{1}{2a}\right) \le x' \le \left(x + \frac{1}{2a}\right) \\ 0 & x' > \left(x + \frac{1}{2a}\right) \end{cases}$$

$\delta(x'-x)$ is a function of x'. It is constructed by the square function centered at x is stretched into an infinitely tall and infinitely narrow spike at x, yielding

$$\delta(x'-x) = \begin{cases} \infty & x' = x \\ 0 & x' \neq x \end{cases}$$

The Dirac-delta has many useful properties as a result of the way it is constructed. Namely, definition 6 guarantees the area under the curve to be unity;

$$\int_{-\infty}^{\infty}\delta(x'-x)dx = 1$$

Also, the Dirac-delta can be integrated with other functions of x to pick out their value at x'.

$$\int_{-\infty}^{\infty} \phi(x')\delta(x'-x)dx = \phi(x)$$

In this way, $\delta(x'-x)$ is a *perturbation* at x'. This is an incredibly useful function that is used to solve many problems in physics. With it, a new definition of the functional derivative is born, this time replacing the arbitrary variational function $\eta(x)$ with $\delta(x'-x)$ as a trial function;

$$\frac{\delta F}{\delta f(x')} = \lim_{\epsilon\to 0} \frac{F[f(x)+\epsilon\delta(x'-x)] - F[f(x)]}{\epsilon}.$$

This definition has a very strict application, however. $\delta(x-x')$ being a perturbation of $f(x)$ at x', $F[f(x)]$ should converge exactly to its Taylor expansion at $f(x')$. Most of the time this is perfectly O.K., since many classical kinetic and potential energy functionals end up being analytic anyways, and if they don't, it is good practice to restrict $x(t)$ to a subspace where the kinetic and potential energies are analytic. For the following derivation I assume that they are as such, using Ref. [6] as a guideline.

Let's begin with the less difficult derivative of the potential energy.

$$\frac{\delta\overline{V}}{\delta x(t)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\frac{1}{\tau}\int_0^\tau V[x(t')+\epsilon\delta(t-t')]dt - \frac{1}{\tau}\int_0^\tau V[x(t')]dt\right] \tag{3}$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\frac{1}{\tau}\int_0^\tau V[x(t')] + \epsilon\delta(t-t')\frac{dV}{dx}[x(t')]dt - \frac{1}{\tau}\int_0^\tau V[x(t')]dt\right] \tag{4}$$

$$= \lim_{\epsilon\to 0}\frac{1}{\tau\epsilon}\int_0^\tau \epsilon\delta(t-t')\frac{dV}{dx}[x(t')]dt \tag{5}$$

$$= \frac{1}{\tau}\int_0^\tau \delta(t-t')\frac{dV}{dx}[x(t')]dt \tag{6}$$

$$= \frac{1}{\tau}\frac{dV}{dx}[x(t)]. \tag{7}$$

Now as a final step, using the definition of force, I rewrite the result explicitly in terms of the trajectory $x(t)$:

$$\frac{\delta\overline{V}}{\delta x(t)} = -\frac{1}{\tau}m\frac{d^2x(t)}{dt^2}.$$

In physics, dot notation is typically used to denote the first and second time derivatives of a function, with one or two dots above the letter, respectively. Hence we can write the functional derivative of $\overline{V}[x(t)]$ as

$$\frac{\delta\overline{V}}{\delta x(t)} = -\frac{1}{\tau}m\ddot{x}(t).$$

Let's analyze the derivation. In step (1), the definition of the functional derivative is applied verbatim. In place of the arbitrary functionals F acting on the arguments in the

brackets, the time-averaged potential functional is used with arguments of $x(t')$ and the variation $x(t')+\epsilon\delta(t-t')$. Here t' is the variable of integration, picked a priori such that in the integration step the Dirac delta function leaves everything as a function of t. In step (2) we take advantage of the Taylor expansion of $V[x(t')+\epsilon\delta(t-t')]$ to cancel out $V[x(t')]$, in the same fashion as in the proof of Theorem 3. The terms $\mathcal{O}(\epsilon^2)$ are omitted because they automatically vanish from the limit.

Next, the two integration terms of $V[x(t')]$ cancel in step (3). In steps (4) and (5) we can evaluate the limit easily as the variational parameter ϵ is divided out to unity. Using the definition of the Dirac delta function, the integration is carried out easily and the integrand is evaluated at t.

Now let's look at the functional derivative of the time-averaged kinetic energy. For this, we will need a lemma.

Lemma 7. *For* $J[f]=\int_{x_1}^{x_2} g[\frac{df}{dx}]dx=\int_{x_1}^{x_2} g[f']dx$, $\frac{\delta J[f(x)]}{\delta f(x)}=-\frac{d}{dx}\left(\frac{dg[f']}{df'}\right)$.

Proof. Begin by changing the integration variable to x' and using the delta function definition of the variational derivative. Let $\frac{\partial f}{\partial x'}=f'$.

$$\begin{aligned}\frac{\delta J[f]}{\delta f}&=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int_{x_1}^{x_2} g\left[\frac{\partial}{\partial x'}(f(x')+\epsilon\delta(x'-x))\right]dx'-\int_{x_1}^{x_2} g\left[\frac{\partial f}{\partial x'}\right]dx'\right)\\&=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int_{x_1}^{x_2} g\Big[f'(x')+\epsilon\delta'(x'-x))\Big]dx'-\int_{x_1}^{x_2} g\Big[f'\Big]dx'\right)\end{aligned}$$

Here $\delta'(x'-x)$ is the derivative of the delta function with respect to x'. Applying a Taylor expansion on $g\Big[f'(x')+\epsilon\delta'(x'-x))\Big]$ up to order ϵ and evaluating the limit yields

$$\begin{aligned}\frac{\delta J[f]}{\delta f}&=\left(\int_{x_1}^{x_2}\Big(g[f']+\epsilon\delta'(x'-x)\frac{d}{df'}g[f']\Big)dx'-\int_{x_1}^{x_2} g[f']dx'\right)\\&=\int_{x_1}^{x_2}\Big(\epsilon\delta'(x'-x)\frac{d}{df'}g[f']\Big)dx'\\&=\left[\delta(x'-x)\frac{d}{df'}g[f']\right]_{x_1}^{x_2}-\int_{x_1}^{x_2}\delta(x'-x)\frac{d}{dx'}\frac{dg[f']}{df'}dx'\end{aligned}$$

The step above is an integration by parts. Notice that the boundary term, after evaluation, contains $\delta(x_1-x)$ and $\delta(x_2-x)$. These vanish since $x\neq x_1,x_2$ as long as $x\in[x_1,x_2]$, which is assumed. Evidently,

$$\begin{aligned}\frac{\delta J[f]}{\delta f}&=-\int_{x_1}^{x_2}\delta(x'-x)\frac{d}{dx'}\frac{dg[f']}{df'}dx'\\&=-\frac{d}{dx}\frac{dg[f']}{df'}\end{aligned}$$

where the rule for integrating over the delta function makes the replacement $x' \to x$. $\square$

Using the lemma, the functional derivative of the time-averaged kinetic energy can be evaluated, with the replacements $g(f') \to \frac{1}{2\tau} m[\dot{x}(t)]^2$, $f(x) \to x(t)$, and $f' \to \dot{x}(t)$ where $\dot{x}(t) = \frac{dx(t)}{dt}$.

$$\frac{\delta \overline{T}[x]}{\delta x(t)} = -\frac{d}{dt}\left(\frac{d}{d\dot{x}} \frac{1}{2\tau} m[\dot{x}(t)]^2\right) \tag{8}$$

$$= -\frac{1}{\tau} m \frac{d}{dt} \dot{x}(t) \tag{9}$$

$$= -\frac{1}{\tau} m \frac{d^2 x(t)}{dt^2} \tag{10}$$

Using dot notation, $\overline{T}[x(t)]$ is written as

$$\frac{\delta \overline{T}[x]}{\delta x(t)} = -\frac{1}{\tau} m \ddot{x}(t).$$

Now we can compare the potential and kinetic energy variations. Amazingly, it is found that

$$\frac{\delta \overline{T}[x]}{\delta x(t)} = \frac{\delta \overline{V}[x]}{\delta x(t)} \tag{11}$$

Although in a few more lines and with two definitions we will arrive at the equation $\delta S = 0$, which is the usual way of stating Hamilton's principle, I should pause here. Equation (9) is much more readily suited for physical interpretation than $\delta S = 0$, because it states clairvoyantly that the kinetic and potential energies vary in function space *by the exact same amount*. This fact gets watered down in the following statements. However, what the following formulation lacks in intuitive qualities it makes up for in mathematical power.

$$\frac{\delta \overline{T}[x]}{\delta x(t)} - \frac{\delta \overline{V}[x]}{\delta x(t)} = 0 \tag{12}$$

$$\frac{\delta}{\delta x(t)}\left(\overline{T}[x] - \overline{V}[x]\right) = 0$$

$$\frac{\delta}{\delta x(t)} \frac{1}{\tau}\left(\int_0^\tau T[\dot{x}(t)] - V[x(t)] dt\right) = 0$$

To concatenate the result above, consider the following useful objects.

Definition 8. The Classical *Lagrangian*

$$\mathcal{L}[t, x(t), \dot{x}(t)] = T[\dot{x}(t)] - V[x(t)]$$

Definition 9. The Classical *Action*

$$\mathcal{S}[x(t)] = \int_{t_1}^{t_2} \mathcal{L}[t, x(t), \dot{x}(t)] dt$$

With these definitions, it is evident that

$$\frac{\delta \mathcal{S}[x(t)]}{\delta x(t)} = 0 \quad \text{(Hamilton's Principle)} \tag{13}$$

Equivalently, many will write $\delta \mathcal{S} = 0$ as a shorthand. Hamilton's principle is simply the statement that the Action is a functional which is always extremized for a physical trajectory $x(t)$. By physical, I mean that the mathematical curve $x(t)$ models to some degree of accuracy the trajectory of a particle or system of particles in the real world. This is because real trajectories must satisfy their kinetic and potential energy functionals, whose definitions are contrived such that the total energy $E = T + V$ is constant with time. This is known as the law of conservation of energy; Nöether's theorem gives a statement about the connection between mathematical symmetries and conservation laws, and one of those connections is directly evident here.

Compare equations 11 and 12. On the one hand, equation 11 is a symmetry equation; it says that two things change in the same way. On the other hand, equation 12 says that a quantity does not change with the variation of $x(t)$; this quantity, which turns out to be the Action, is conserved. It is no accident that the action and the total energy are physically conserved quantities, and here I conjecture that there is a bijection between the two statements.

Now that we know that the functional $\mathcal{S}$ is extremized, Euler-Lagrange suddenly becomes very applicable. For trajectories which satisfy the assumptions of the Euler-Lagrange theorem,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}(t)}\right) - \frac{\partial \mathcal{L}}{\partial x(t)} = 0.$$

This is an amazing tool. The Euler-Lagrange theorem for the classical Action provides us with a differential equation whose solution $x(t)$ is the trajectory for a system described with the Lagrangian $\mathcal{L}$. For most problems in classical mechanics, writing down the Lagrangian is easy once the potential energies and kinetic energies of the system are known.

It is also possible that the Lagrangian is a functional which depends on several variables which describe the configuration of a physical system. These variables are also known as the degrees of freedom of a system. For instance, a system of n particles would be described with coordinates $\{x_1, ..., x_n\}$, where each x_i is a degree of freedom. The degrees of freedom may be more general, and they are able to describe configurations beyond just the locations of particles. So, in general they are often labeled $q_i(t)$ [4]. I'll show you some examples of this soon. Then the Lagrangian becomes

$$\mathcal{L} = \mathcal{L}[t, q_1, ..., q_n, q_1', ..., q_n']$$

and we obtain the plural Euler-Lagrange *equations*:

$$\left\{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i(t)}\right) - \frac{\partial \mathcal{L}}{\partial q_i(t)} = 0\right\},$$

a system of n ordinary differential equations.

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5. Applications and Examples

5.1. Classical Mechanics.

Example 10. *Brachistochrone*

The Euler-Lagrange equations provide a formalism with which to tackle integral equations that depend on arbitrary functions. Returning now to the problem mentioned in the first section, the Brachistochrone problem is posed as such: Suppose a point-particle of mass m is constrained to move without friction along a path $y(x) \in \mathbb{R}^2$ of total arc length L. Its movement along the path is parameterized by the time t, such that $x = x(t)$ and $y = y(t)$. Find a path $y(x)$ which passes through the points $(x_1, y_1), (x_2, y_2)$ such that the particle slides in a gravitational potential $V(y(x)) = mgy(x)$, with the force of gravity pointing along the y-direction, in the shortest possible time. The integral equation in question is

$$\begin{aligned} T &= \int_{t_1}^{t_2} dt \\ &= \int_0^L \frac{ds}{v(s)} \end{aligned} \tag{14}$$

where the infinitesimal time element is related to the arc length of the curve by $dt = \frac{ds}{v(s)}$. It is preferable to abandon the time-parameterization immediately in favor of a space integral. We can cast the time integral T in Cartesian coordinates via the following relationship. Express the total energy E as the sum of the kinetic and potential energies;

$$E = V + T$$

$$E = \underbrace{mgy(x)}_{potential} + \underbrace{\frac{1}{2}mv^2(x)}_{kinetic}$$

where E, being conserved, is a constant. This yields

$$v(x) = \sqrt{\frac{2E}{m} - 2gy(x)}.$$

Now fully cast in Cartesian coordinates, the time integral in Equation 14 is

$$T = \int_{x_0}^{x_1} \frac{\sqrt{1 + y'^2}}{\sqrt{\frac{2E}{m} - 2gy(x)}} dx = \int_{x_0}^{x_1} F[x, y, y'] dx$$

The form of the numerator and the change of the integration variable to x from s is facilitated by the identity $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2} dx$, where $y' = \frac{dy}{dx}$. Now T is ready for the application of Euler-Lagrange. It is required that T be extremal at $y(x)$ in order for the time to be minimized. Hence,

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0.$$

Solving this differential equation is rather easy and leads to the well known parameterized curve known as the cycloid, which is illustrated in Figure 2. The method used in this

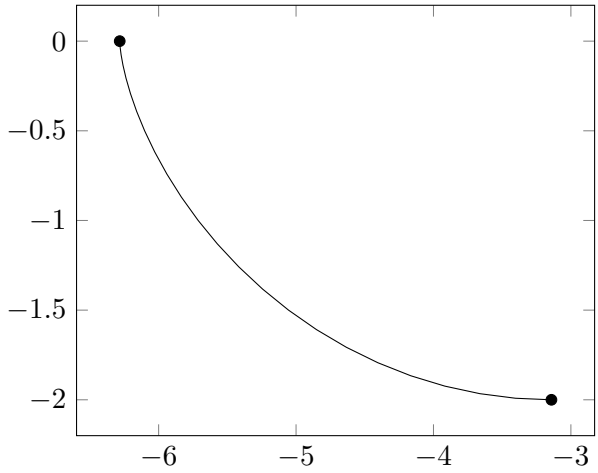

FIGURE 2. The path of shortest time for the particle to slide down under the force of gravity. It is actually an upside-down cycloid, with this particular curve generated by $(x(t), y(t)) = (t - \sin t, -1 + \cos t)$ over $t \in [-2\pi, -\pi]$.

example is very algorithmic; the extremized functional T was identified, an appropriate choice of coordinates were made, and the integrand functional was cast into Equation 2. The Brachistochrone curve can be compared to other curves to see how quickly a mass m particle completes its trajectory in each curve. See Ref. [7] for a wonderful animation of this comparison.

Example 11. *Coupled Oscillators*

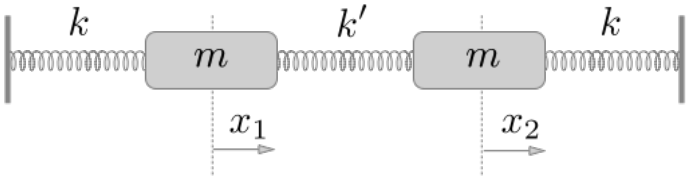

FIGURE 3. A coupled two-spring system of oscillators.

Here's an example where Hamilton's principle can be applied. In fact, in most physical systems in which the trajectories of particles (or equations of motion, as they are called) need to be found, Hamilton's principle in conjunction with Euler-Lagrange is often used. Consider two particles, labeled 1 and 2, each of mass m coupled together by springs between two barriers, as in Figure 3. Our goal is to write down the Lagrangian, which must be extremized, and then solve the Euler-Lagrange equations. Each mass is given a degree

of freedom, $x_1(t)$ and $x_2(t)$, which is the coordinate basis for their motions. The springs provide the sources of potential energy which determine the motion; the potential energy in general for springs is $V(x) = \frac{1}{2}k'(x - x_0)^2$, where x_0 is the equilibrium position of the spring. So, if the equilibrium positions relative to x_1 and x_2 are x_1^0 and x_2^0, the three spring potentials are

$$V(x_1) = \frac{1}{2}k(x_1 - x_1^0)^2$$
$$V(x_2) = \frac{1}{2}k(x_2 - x_2^0)^2$$
$$V(x_2 - x_1) = \frac{1}{2}k'(x_2 - x_1 - x_2^0 + x_1^0)^2.$$

k and k' are the spring constants which correspond to the "stiffness" of the spring as k or k' increases. $V(x_1)$ and $V(x_2)$ correspond to the potentials of the left and right springs, respectively, and $V(x_2 - x_1)$ is the potential due to the coupling between the two masses by the middle spring. The kinetic energies are trivial to write down, so we have the Lagrangian

$$\mathcal{L} = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_1^2 - [V(x_1) + V(x_2) + V(x_2 - x_1)].$$

The Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1(t)}\right) - \frac{\partial \mathcal{L}}{\partial x_1(t)} = 0$$
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2(t)}\right) - \frac{\partial \mathcal{L}}{\partial x_2(t)} = 0$$

The approach now is to set this system of equations up as a matrix equation and solve for the eigenvalues. The eigenvectors correspond to the *normal modes* of oscillation, which are the stable configurations of the system of springs, and the eigenvalues will be the frequencies of each mode. The Euler-Lagrange equations take the form

$$\ddot{x}_1 = -\kappa_{11}x_1 - \kappa_{12}x_2$$
$$\ddot{x}_2 = -\kappa_{21}x_1 - \kappa_{22}x_2$$

or,

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} \kappa_{11}x_1 & \kappa_{12}x_2 \\ \kappa_{21}x_1 & \kappa_{22}x_2 s \end{pmatrix}$$

where each κ_{ij} are functions of the spring constants k and k'. Since a general spring constant can be expressed in terms of the mass and the frequency of oscillation ω, $k_{1,2} = m\omega_{1,2}^2$, the above matrix equation can be solved for the eigenvalues ω_1 and ω_2.

$$\omega_1 = \pm\sqrt{\frac{k + 2k'}{m}}$$
$$\omega_2 = \pm\sqrt{\frac{k}{m}}$$

These correspond to the two frequencies of the normal modes. The corresponding eigenvectors are

$$\vec{X}^{\pm} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}^{\pm} = \frac{1}{2}\begin{pmatrix} \eta_1 \\ \pm\eta_2 \end{pmatrix} \tag{15}$$

where η_1 and η_2 are linear combinations of the solutions;

$$\eta_1(t) = A^+ e^{i\omega_1 t} + A^- e^{i\omega_1 t}$$
$$\eta_2(t) = B^+ e^{i\omega_2 t} + B^- e^{i\omega_2 t}$$

$A^{\pm}$ and $B^{\pm}$ depend on the initial amplitudes of the oscillators. So, Equation 15 tells us the what the normal modes are in terms of the relative motions of particles 1 and 2 by x_1 and x_2. Pictographically, the modes can be represented as

$\longleftarrow$	$\longleftarrow$	$+$	Symmetric motion
$\longrightarrow$	$\longrightarrow$	$+$	
$\longleftarrow$	$\longrightarrow$	$-$	Antisymmetric motion
$\longrightarrow$	$\longleftarrow$	$-$	

[4].

† † †

6. Conclusion

There are a myriad of examples in classical mechanics that depend on the joining of Hamilton's principle with the Euler-Lagrange equations. These include many other examples of coupled systems, such as double pendulums, pendulums whose masses are coupled by springs, vibrating strings, and so on. The applications also extend far beyond classical mechanics. For instance, one can use an extremal arc length integral to find geodesics in Euclidean space using the Euler-Lagrange equation; but, there is no need for restrictions to flat space. In fact, one can express the Lagrangian in terms of the metric of any general curved space. Using the Schwarzschild metric of Albert Einstein's theory of general relativity, one can solve the Euler-Lagrange equations for geodesics in curved four-dimensional spacetime [10].

Lagrangian functionals are perpetuated in modern physics, especially in the theory of quantum fields. With the starting point of an extremal Action functional, the groundwork of particle physics and condensed matter physics has been laid out [5], [9]. This formalism is the building block of functional analysis, and while it is deeply appreciated for its applications, it has led to a sophistication of analysis unlike any other formalism.

† † †

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