Tackling Math Problems in the Classroom

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Teachers often strive to find optimal methods to help their students succeed. However, even the best mathematical minds are challenged in this area. Perhaps, an example could be Albert Einstein, who was one of the greatest physicists of all time but also notorious for being an uninspiring instructor for his students (Grant, 2018). In *How to Solve It* (Polya, 1985), the author discusses meticulous strategies for solving math problems in order to benefit both the teacher and student. In particular, the first section - *Part I. In the Classroom* - consists of twenty sections, which explain strategies that can be used in a math class, followed along with four math problems. Moreover, *Part II. Dialogue* exemplifies how a teacher can adequately answer the student's questions when solving a math problem. Overall, the author proposes that one should constantly address questions or suggestions in the process of solving a math problem. That way, they would proceed with a better understanding of the problem by finding the unknown, data, and conditions. Moreover, they can develop a plan by linking it to useful prior experience or knowledge. Last but not least, one should look back at their solution and reflect on it for further improvements. In applying Polya's three takeaways, specifically to the problem of Grade 10 plane geometry, it is possible to see how they make math more memorable and facilitate growth in a student's ability to solve math problems.

The very initial phase of tackling a math problem is understanding the basics of a problem. It is needless to say that well begun is half done. Not a single student can immediately solve a problem with a lack of comprehension (even if they do, it would be mere luck). Indeed, it is equivalent to participating in war without knowing how to pull the trigger of a gun. In order to avoid such misfortune, the teacher should draw the student's attention to the problem by asking them the following questions: "What is the unknown? What are the data? What is the condition?" (Polya, p.7) These questions are generally applicable to "problems to find." If the student is dealing with a "problem to prove," they should be concerned with the hypothesis and conclusion instead. The teacher can also ask other similar questions tailored to the problem to induce a better understanding. This way, the teacher can guide the student to find principal parts of the problem and connect them together. Furthermore, where applicable, the teacher can request the student to draw diagrams or introduce suitable notation for the principal parts of the math problem.

Working to understand the problem by asking questions helps not only the student but also the teacher. First, it allows the teacher to observe the problem from the student's point of view. While the student answers each question, they would verbally paraphrase the statement, and this would help the teacher verify whether the student has understood the question or not. Moreover, the fact that the questions are effective yet generic allows the teacher to help the student "unobtrusively and naturally" (p.1), which should be the main goal of a teacher. These general questions give room for the student to think independently and increase their ability to solve problems. Indeed, this method also benefits the student by stimulating their mental operation. Questioning the unknown, data, and conditions in a step-wise manner assists the student to organize the principal parts and their connections. After having a fundamental understanding and clarifying details of the problem, the student is ready to make sophisticated plans of their solution.

After understanding the problem, the student must attempt to establish a plan by gathering ideas from prior experience or knowledge. Specifically, Polya suggests to "[l]ook at the unknown! And try to think of a familiar problem having the same or a similar unknown" (p.3). Of course, there could be numerously many problems related to the current problem.

Hence, the author places emphasis on finding problems with the same or similar unknowns in order to discern other problems that are the most relevant. If a student successfully remembers such a problem, they could see whether they could use similar tactics for the current problem. If the student believes they have not seen any similar problems before, they can try solving some simpler related problems first to build experience and knowledge.

Overall, using formerly acquired knowledge or past experience from other problems helps one gain insight on how to make plans for the current problem. The idea one obtains from prior knowledge or experience can be partial or whole. By applying former knowledge or experience, one would come up with some compelling ideas which lead to the solution. Even if the idea ends up being incorrect, it still helps one understand the problem better. In general, the attempt of making use of the past increases one's ability to think flexibly about a problem and the connections between problems. In the final analysis, accumulating one experience after another would develop the student's ability to solve problems more independently in the future. Most importantly, building a plan from various points helps the teacher and student to keep track of their thought process. Since the goal of a teacher is to help the student solve problems, encouraging the student to devise a plan through formerly obtained knowledge or experience is a solid method to guide them towards the solution.

Finally, Polya suggests that one should review their solution after finishing the problem. After establishing and carrying out the plan, the student could be confident enough to skip this step. Nonetheless, one should be aware that solutions are written by humans who are not completely error-free. Thus, it is essential for one to verify their solution. Any related test, formula, question, or procedure can be implemented to check one's result or procedure of the solution. For instance, one could be asking whether they "use[d] all the data" (p.16) when deriving the result of the unknown. Since the data is one of the principal components - basically a hint in the problem - required to figure out the unknown, it is necessary to use them in the solution[1]. Also, the teacher could ask the student (or the student can ask them-

[1] This is analogous to the situation in Python where the program warns one if there is an unused argument within the function.

selves) the following: "Can you derive the result differently? Can you see it in a glance?" (p.15) This could be somewhat difficult for inexperienced students. However, if they successfully arrive at the same result by using a different method (whether it be intuitive or formal), the student could find more appealing solutions by observing new, interesting facts. Ultimately, looking back at the solution would help develop robust knowledge and ability to solve problems, which would make the student even more confident than before.

Looking back at the solution yields various advantages. Unfortunately, this is a step that many people often neglect. The author states that "they miss an important and instructive phase of the work" (p.14) by doing so. One can gain more than just detecting errors by developing a habit of proofreading. It is not only a review but also a good method to solidify one's knowledge and ability to solve math problems. When a student successfully checks over their solution through some formal or intuitive test, there's a higher chance of the solution staying in their long-term memory. Sometimes, the student can think of more ideal solutions than their first copy and decide to make improvements. In fact, it is important for the teacher to make the students realize that a single math problem can be viewed in multiple perspectives. By questioning whether the student can yield the result differently, the student gets to recognise that there are more than one possible solution. As a result, the student would be able to think flexibly instead of getting stuck in a fixed mindset.

Now, let us demonstrate how the three takeaways are applied to a math problem. The following example is a plane geometry problem intended for Grade 10 students. In our scenario, assume that the student solving this problem knows Pythagorean theorem (which is typically learnt in Grade 8) but does not recall the formula of the area of a trapezoid. In fact, it's been a while since the student has solved a math problem after a long summer break. Also, note that the dimension of the trapezoid is not accurately illustrated (e.g., Although it seems like it, the height of the trapezoid is not 10).

The area of the trapezoid shown below is equal to 270 square units. Find its perimeter and round your answer to the nearest unit (Analyze Math, n.d).

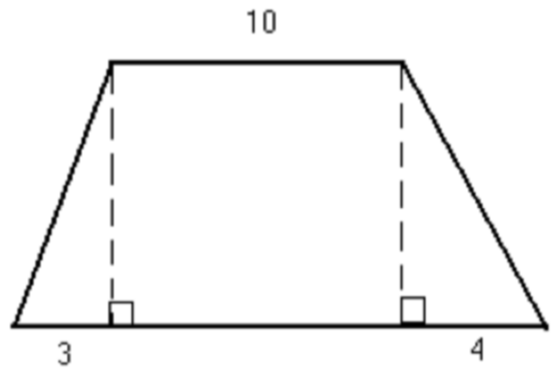

Figure 1: Problem

This problem can be solved by the guidance of a teacher who will be walking through the problem with the three takeaways. First, the teacher would question about the problem for a better understanding: *What is the unknown?* In response, the student can easily pinpoint to the perimeter of this trapezoid. *What are the given data?* The student would mention the area of the trapezoid which equals to 270, top width of the trapezoid equalling 10, and the base of the triangles on the left and right hand side equalling 3 and 4, respectively. Next, the teacher would ask to introduce suitable notation and might encourage the student to label them on the diagram. Then, the student could label the trapezoid as Figure 2[2]. Though it is not written on the figure, it is also essential to denote P as the perimeter of the trapezoid. *Look at the unknown. What does it mean to find a perimeter?* Here, it means to calculate $P = a + b + c + d + 2t$, but the values of c and d remain mysterious for now. Hence, this is a great timing to ask: *What is the condition linking the data to the unknown?* Thanks to the generously depicted diagram, the student can identify that the unknown two sides are equivalent to the hypotenuse of the two triangles. By Pythagorean theorem, the hypotenuse can be found by computing $c = \sqrt{a^2 + h^2}$ and $d = \sqrt{b^2 + h^2}$. However, the student could be wondering the following: *How do we find the hypotenuse when we don't know the height of the triangle?*

[2]To be specific, $A = 270$ denotes the area of the trapezoid, $t = 10$ is the top-base of the trapezoid, h is the height of the trapezoid, c and d are the sides of the trapezoid, and $a = 3$ and $b = 4$ are the bases of the triangles within the trapezoid.

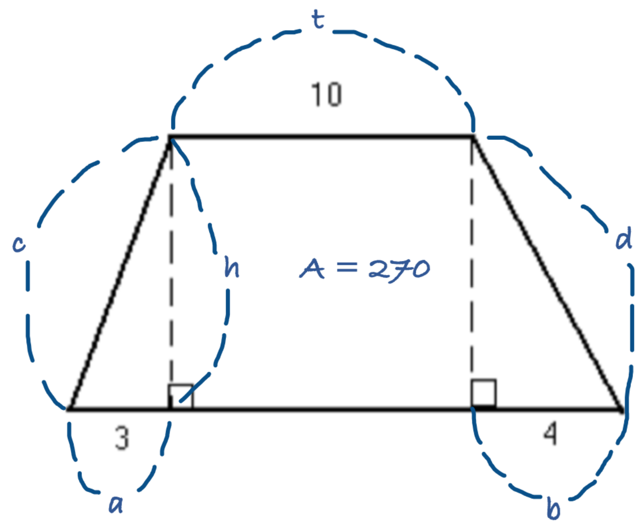

Figure 2: Labeling notations

When the student has understood the basics but is stuck on developing further ideas, it is ideal to ask whether they can think of any related simpler problems. Here, the student could be thinking of a simpler 2-dimensional plane than a trapezoid. Again, the nicely drawn diagram gives us a hint that rectangles and triangles are some examples of simpler planes. *Can you think of a problem with a rectangle that involves the perimeter as the unknown?* The student can mention a case where a square has a side equalling 10, which would then have 40 as the perimeter. However, if the sides were unknown and the area was given instead - like the current problem in concern - one would need to utilize the simple formula of the square's area[3] first, in order to find the length of the side before calculating the perimeter[4]. After processing this idea, a light bulb would flash in the student's head. It seems like A needs to be used first to find the height of the trapezoid. Although the student does not recall the formula of the area of a trapezoid, they will know that the areas of the two triangles and

[3]i.e., $A_S = s^2$ where s and A_S are the side and the area of a square, respectively.
[4]$P_S = 4s$, where P_S is the perimeter of a square.

the rectangle added altogether will form A.

$$
\begin{aligned}
A &= \overbrace{\frac{1}{2} \times a \times h}^{\text{Triangle on the LHS}} + \underbrace{10h}_{\text{Rectangle}} + \overbrace{\frac{1}{2} \times b \times h}^{\text{Triangle on the RHS}} \\
270 &= \frac{3}{2} \times h + 10h + 2h \\
270 &= \frac{27}{2}h \\
h &= 20
\end{aligned}
$$

Now that the height is known, the student can easily use Pythagorean theorem to find $c = \sqrt{20^2 + 3^2} = \sqrt{409}$ and $d = \sqrt{20^2 + 4^2} = \sqrt{416}$. Therefore, we compute the following to find P:

$$
\begin{aligned}
P &= a + b + c + d + 2t \\
&= 3 + 4 + \sqrt{409} + \sqrt{416} + 20 \\
&= 27 + \sqrt{409} + 4\sqrt{26} \\
&\approx 68
\end{aligned}
$$

Finally, the teacher needs to encourage the student to look back at their solution. *Did you use all the data?* In this scenario, yes, the student did. *Can you check the result?* The student can plug in the values and see if there are not any calculation mistakes. Perhaps, if the student ever remembers the formula of finding a trapezoid's area, $\frac{a+b+2t}{2} \times h$, they can use that as a tool to check for higher accuracy. In general, going over the solution boosts the student's confidence.

As observed in the case study, making use of all three strategies helped the teacher guide the student with ease. This way, the teacher was also able to track the student's thought process and address step-wise questions based on the student's stance. The teacher unobtrusively helped by addressing general questions instead of giving concrete solutions to the problem. Every teacher's goal should be helping the student naturally so that the student can acquire better problem-solving skills. With the use of these techniques that Polya mentioned, any teacher can provoke a student's mental operation of solving math problems.

Despite having affluent knowledge, teaching others is not simple as it may seem. The teacher should strategically stimulate one's curiosity by asking questions about math problems, which would eventually enhance their ability to think independently. All of the three main takeaways discussed in Polya's book involve asking questions in regard to the problem. To begin with, the student must have a decent understanding of the problem by identifying principal parts of the problem. Second, the student should build their plan by using formerly acquired knowledge or experience, which includes linking the current problem to problems with the similar or same unknown. Lastly, the student should remember to reexamine their solution in order to fully solidify their ability to solve problems. The important part is that these three points are distinct yet inseparable. As a consequence of using such strategies in the classroom, the teacher would have a higher chance of effectively helping the student.

References

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