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※ Your answer must be provided with descriptions how to get the answer.

1.(5 point) Let $\{\overrightarrow{v_n}\}_{n=1}^{\infty}$ be a sequence of vectors generated by the recurrence relation : $\overrightarrow{v_n}$ be the vector projection of $\overrightarrow{v_{n-1}}$ onto $\overrightarrow{v_{n-2}}$ for $n \geq 3$, where two initial $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ are vectors with $|\overrightarrow{v_1}|=2$, $|\overrightarrow{v_2}|=3$, and $\overrightarrow{v_1} \cdot \overrightarrow{v_2}=5$.

Compute $\sum_{n=1}^{\infty} |\overrightarrow{v_n}|$.

Sol.) $|V_3| = \frac{V_1 \cdot V_2}{|V_1|} = \frac{5}{2}$, $V_3 = \frac{5}{2^2} V_1$

$|V_4| = \frac{V_2 \cdot V_3}{|V_2|} = \frac{5^2}{2^2 \cdot 3}$, $V_4 = \frac{5^2}{2^2 \cdot 3^2} V_2$

$|V_5| = \frac{V_3 \cdot V_4}{|V_3|} = \frac{5^3}{2^3 \cdot 3^2}$, $V_5 = \frac{5^3}{2^4 \cdot 3^2} V_1$

$|V_6| = \frac{5^4}{2^4 \cdot 3^3}$ . . .

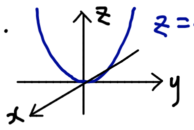
$$\therefore \sum_{n=1}^{\infty} |V_n| = 2+3+\frac{5}{2}\sum_{k=1}^{\infty}\left(\frac{5}{6}\right)^{k-1}$$

$$= 5 + \frac{5}{2} \cdot \frac{1}{1-\frac{5}{6}}$$

$$= 20.$$

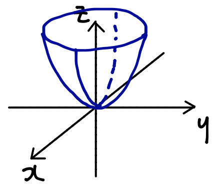
2.(5 point) Suppose a surface is generated by rotating the parabola $z=4y^2, x=0$ about the z-axis.
Write an equation of the surface in spherical coordinates.

시험 범위 아님!

Sol. 

$z=4y^2$: 포물선 (in plane $x=0$)

$\Rightarrow$ z축 회전

원포물곡면 (paraboloid) 

$z=4(x^2+y^2)$

$\Rightarrow$ 구면 방정식으로 변환하면

$\rho\cos\phi = 4\rho^2\sin^2\phi$

3.(6 point) Let L be the line of intersection of the planes $x+y+z=c$ and $cx+cy+z=c$, where c is a real number, not equal to 1.

(1)(3 point) Find parametric equations for L.

Sol.) Let $\alpha_1 : x+y+z=c$
$\alpha_2 : cx+cy+z=c$, $c\neq 1$

$\Rightarrow$ α_1의 법선벡터 $\vec{n_1}=\langle 1,1,1\rangle$
α_2의 〃 $\vec{n_2}=\langle c,c,1\rangle$

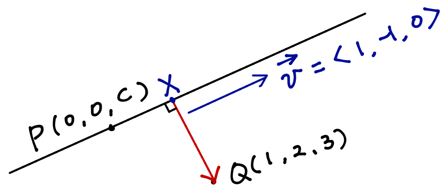
① 교선 L의 방향 $\vec{v} \,/\!/\, \vec{n_1}\times\vec{n_2}=\langle 1-c, -(1-c), 0\rangle$
$\therefore \vec{v}=\langle 1,-1,0\rangle$

② 교선 L위의 한점을 결정하면 된다.
$(0,0,c)\in\alpha_1$ & α_2
∴ 교선 L의 매개방정식 :

$$L\begin{cases} x=t \\ y=-t \\ z=c \end{cases},\ t\in\mathbb{R}.$$

(2)(3 point) Find the point on the line L that is closest to the point $(1,2,3)$.

Sol.)



$X(t,-t,c)\in L$

$\Rightarrow \vec{XQ}=\langle 1-t, 2+t, 3-c\rangle$

$\Rightarrow \vec{XQ}\cdot\vec{v}=0$

$\therefore (1-t)-(2+t)=0$

$\therefore t=-\frac{1}{2}$

∴점 $X(-\frac{1}{2}, \frac{1}{2}, c)$ 에서 최단거리.

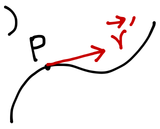
4.(6 point) Let C be the curve represented by

$$r(t)=\left\langle 2t^3-t^2+2, \frac{3}{2}t^2, 3t^3-1\right\rangle,\ t<0.$$

Find the curvature at a point P that has normal plane of C parallel to the plane $4x-3y+3z+5=0$.

(Note : The normal plane is parallel to $4x-3y+3z+5=0$.)

Sol.)



* 점 P에서의 법평면 (normal plane)
; $\vec{r}'$ (또는 $\vec{T}$)을 법선벡터로 갖는 평면.

$\Rightarrow \vec{r}'(t)=\langle 6t^2-2t, 3t, 9t^2\rangle \,/\!/\, \langle 4,-3,3\rangle$

$\Rightarrow 6t^2-2t=4k$
$3t=-3k$
$9t^2=3k$ $\Big) \Rightarrow 3t=-9t^2$
$\therefore t=\cancel{0}, -\frac{1}{3}$
$(k=\frac{1}{3}$

$P \leftrightarrow \vec{r}(-\frac{1}{3})$

이제 P에서의 곡률 $K(-\frac{1}{3})$을 구하자.

① $\vec{r}'(-\frac{1}{3})=\langle \frac{4}{3}, -1, 1\rangle = \frac{1}{3}\langle 4,-3,3\rangle$

② $\vec{r}''(-\frac{1}{3})=\langle 12t-2, 3, 18t\rangle_{t=-\frac{1}{3}}$
$=\langle -6, 3, -6\rangle$

③ $\vec{r}'\times\vec{r}''=\langle 3, +2, -2\rangle$

$$\therefore K(-\tfrac{1}{3})=\frac{\sqrt{17}}{\frac{1}{27}\cdot 34\sqrt{34}}=\frac{27\sqrt{17}}{34\sqrt{34}}=\frac{27}{34\sqrt{2}}$$

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5.(6 point) Let a function f be defined on $\mathbb{R}^2$ by

$$f(x,y)=\begin{cases}\dfrac{xy^2}{\sqrt{x^4+y^2}} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

(1)(3 point) Investigate the continuity of f at (0, 0).

Sol.)

$\begin{pmatrix} x^4 \geq 0 \\ \text{이므로} \end{pmatrix}$ $0 \leq \dfrac{y^2}{x^4+y^2} \leq 1$

$\Rightarrow 0 \leq \dfrac{|y|}{\sqrt{x^4+y^2}} \leq 1$

$\Rightarrow 0 \leq \dfrac{|xy^2|}{\sqrt{x^4+y^2}} \leq |xy|$

따라서

$$-|xy| \leq \frac{xy^2}{\sqrt{x^4+y^2}} \leq |xy|$$

Squeeze thm 에 의해

$$0=\lim_{(x,y)\to(0,0)} -|xy| \leq \lim_{(x,y)\to(0,0)} \frac{xy^2}{\sqrt{x^4+y^2}} \leq \lim_{(x,y)\to(0,0)} |xy| = 0$$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{xy^2}{\sqrt{x^4+y^2}} = 0 = f(0,0)$$

$\therefore$ f는 (0,0)에서 연속.

(2)(3 point) Do the partial derivative $D_y f(0,0)=\dfrac{\partial}{\partial y}f(0,0)$ exist? If it exists, find it by <u>**the definition of partial derivative**</u>.

Sol.)

$$\frac{\partial}{\partial y}f(0,0)=\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{0}{\sqrt{h^2}}-0}{h}$$

$$=\lim_{h\to 0}\frac{0}{h}=0 \quad : \text{존재함.}$$

6.(6 point) Let $z=\sin^{-1}(x+y)+\tanh(xy)$

If x changes from 0.6 to 0.61 and y changes from 0 to 0.01, find the value of dz.

Sol.) $\begin{cases} x: 0.6 \to 0.61 & \therefore \Delta x = 0.01 \\ y: 0 \to 0.01 & \therefore \Delta y = 0.01 \end{cases}$

$$dz = z_x(0.6,0)\Delta x + z_y(0.6,0)\Delta y$$

① $z_x(0.6,0) = \dfrac{1}{\sqrt{1-(x+y)^2}} + \operatorname{sech}^2(xy)\cdot y \Big|_{(0.6,0)}$

$= \dfrac{1}{\sqrt{1-0.36}} = \dfrac{1}{\sqrt{0.64}} = \dfrac{1}{0.8} = 1.25$

② $z_y(0.6,0) = \dfrac{1}{\sqrt{1-(x+y)^2}} + \operatorname{sech}^2(xy)\cdot x \Big|_{(0.6,0)}$

$= \dfrac{1}{0.8} + 0.6 = 1.85$

$\therefore dz = 1.25\times 0.01 + 1.85 \times 0.01$

$= 0.031$

문제 5번 > 별해 ($\varepsilon-\delta$기법) : 참고만 하세요

먼저, $|xy|=|x||y| \leq \sqrt{x^2+y^2}\sqrt{x^2+y^2}$ 이므로

$0 \leq \dfrac{|xy^2|}{\sqrt{x^4+y^2}} \leq |xy| \leq (x^2+y^2)$ 가 성립.

따라서, 임의의 양수 ε에 대하여 $\delta=\sqrt{\varepsilon}$ 라 하면 $\delta>0$ 이고

$0<\sqrt{x^2+y^2}<\delta$ 인 모든 (x,y)에 대하여

$|f(x,y)| = \dfrac{|xy^2|}{\sqrt{x^4+y^2}} \leq |xy| \leq (x^2+y^2) < \varepsilon$

을 만족한다. 따라서

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$$

$\therefore$ f는 (0,0)에서 연속.

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7.(6 point) If $z = f(x+\sqrt{2}t) + g(x-\sqrt{2}t)$, find the value of k satisfying the equation

$$\frac{\partial^2 z}{\partial t^2} = k\frac{\partial^2 z}{\partial x^2}$$

→ f와 g는 1변수함수이고 z는 2변수함수임.

Sol.)

① $\underbrace{\frac{\partial z}{\partial t}}_{x\text{를 고정}} = f'(x+\sqrt{2}t)\cdot\sqrt{2} + g'(x-\sqrt{2}t)\cdot(-\sqrt{2})$

$\therefore \frac{\partial^2 z}{\partial t^2} = 2\cdot f''(x+\sqrt{2}t) + 2\cdot g''(x-\sqrt{2}t)$

② $\underbrace{\frac{\partial z}{\partial x}}_{t\text{를 고정}} = f'(x+\sqrt{2}t) + g'(x-\sqrt{2}t)$

$\therefore \frac{\partial^2 z}{\partial x^2} = f''(x+\sqrt{2}t) + g''(x-\sqrt{2}t)$

따라서, $k=2$.

(만약 $u = f(x+\sqrt{2}t)$라 하면

$u = f(X)$, $X = x+\sqrt{2}t$

여기서, X는 중개변수

(x와 t는 독립변수.