factor : spanning subgraph of G

k- factor : 〃 k regular

sub graph / set of edges

1 - factor ≃ perfect - matching

odd component : component of odd order

Tutte's condition : For all $S \subseteq V(G)$, $o(G-S) \leq |S|$.

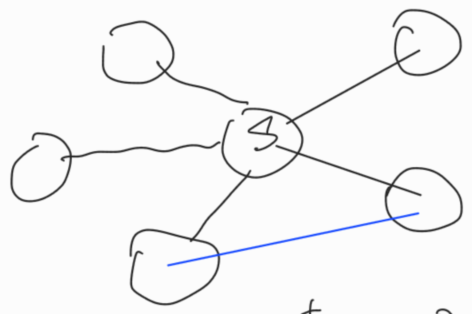
also true for $S = \emptyset$

Thm. $\Updownarrow$

A graph G has a 1-factor

$\Rightarrow$ trivial (1-factor)

$\Leftarrow$



e o e   + + +   o o e   o e e

① if $G' = G + e$

$S \subseteq V(G)$ then

$o(G'-S) \leq o(G-S) \leq |S|$.

$o(G)$ does not increase

② if G' has no 1-factor then G has no 1-factor

$\therefore$ The theorem holds unless $\exists$ simple G s.t

G satisfies tutte's condition

has no 1-factor

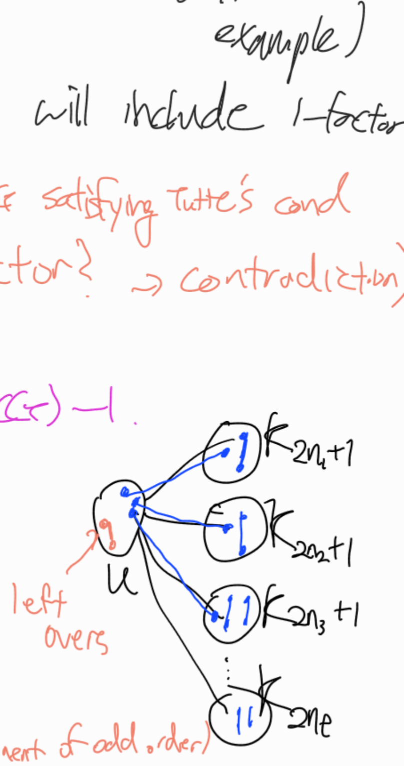
adding any missing edges to G $\rightarrow$ 1-factor (minimal counter example)

Try to find above condition graph but every graph will include 1-factor

Thus proving the theorem (This step is: what if $\exists G$ satisfying Tutte's cond but not having 1-factor? $\rightarrow$ contradiction)

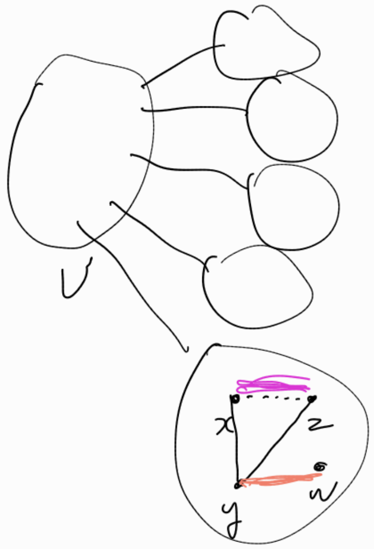
Let U be the set of vertices that have degree $n(G)-1$.

Case1) $G-U$ consists of disjoint complete graphs



$\Rightarrow n(G)$ is even $\Rightarrow S = \emptyset \Rightarrow o(G) \leq 0$ (a graph of odd order would have a component of odd order)

Case 2) $G-U$ has a non-clique component.



By choice of G.
adding an edge to G creates 1-factor

M_1 be 1-factor in $G+xz$

M_2 be 1-factor in $G+yw$

$M_1 \triangle M_2$ contains a 1-factor avoiding xz and yw $\Rightarrow$ 1-factor in G

$F = M_1 \triangle M_2$

xz and $yw \in F$

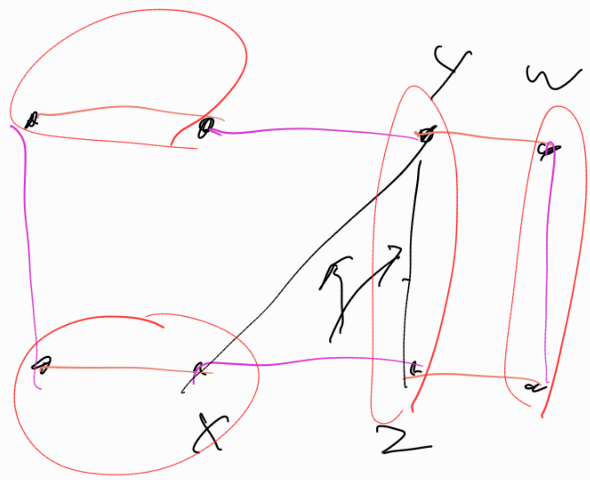
Since every vertex of G has degree 1 in M_1/M_2

$d(v \in V(G) \mid F) = 0$ or $2 \rightarrow$ even cycle / Isolated vertices

C: Cycle of F containing xz

- If C does not contain yw: desired 1-factor consists of the edges of M_2 from C and all of M_1 not in C
- If C does contain yw

then to avoid them we use yw or yz



in portion of C starting from y along yw, use edges of M_1 to avoid using yw.

When we reach $\{x, z\}$, use zy of xy. remainder use edge of M_2. $\rightarrow$ we have 1-factor of G