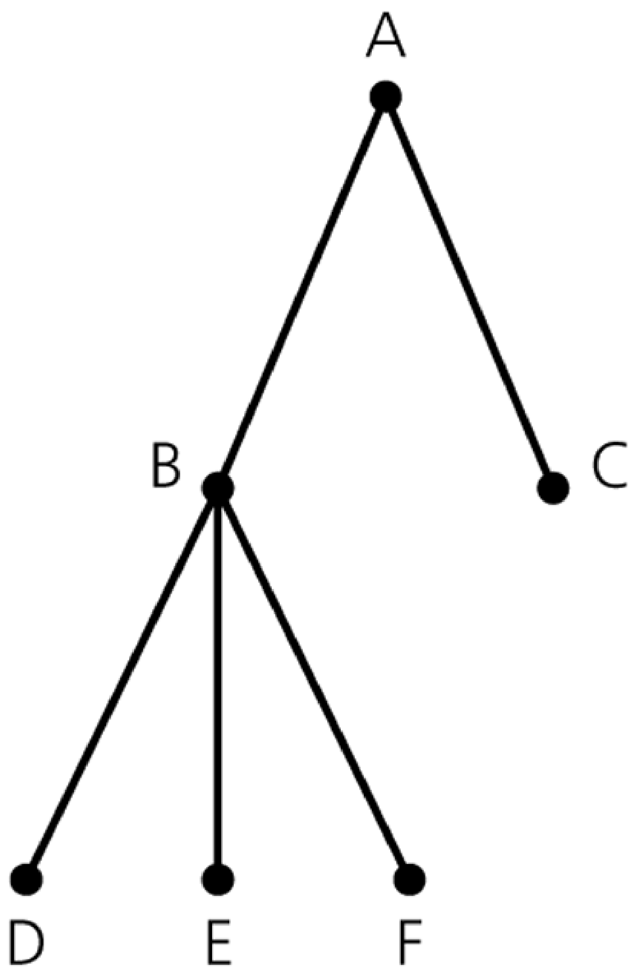
Ch. 9 Trees

사실을 많이 아는 것 보다는
이론적 틀이 중요하고,
기억력보다는
생각하는 법이 더 중요하다.

– 제임스 왓슨

A (rooted) tree

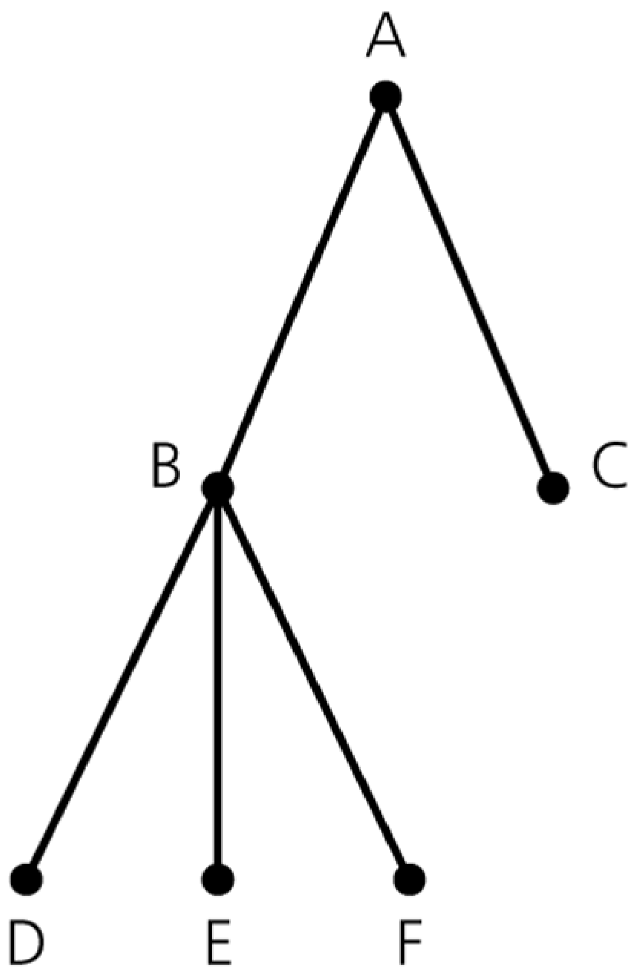


Terminology

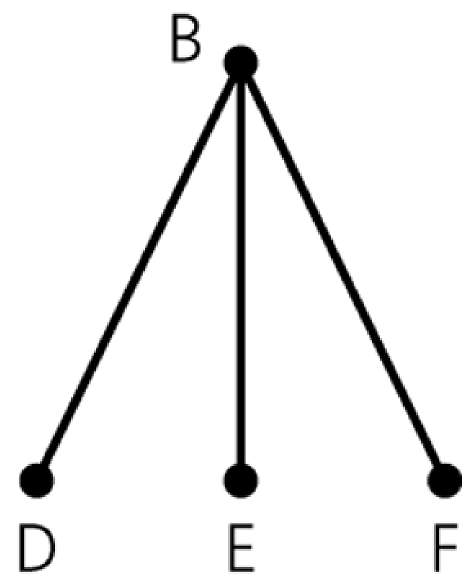
node or vertex
edge
parent
child
siblings
root
leaf
ancestor
descendant
subtree

Definition of Tree

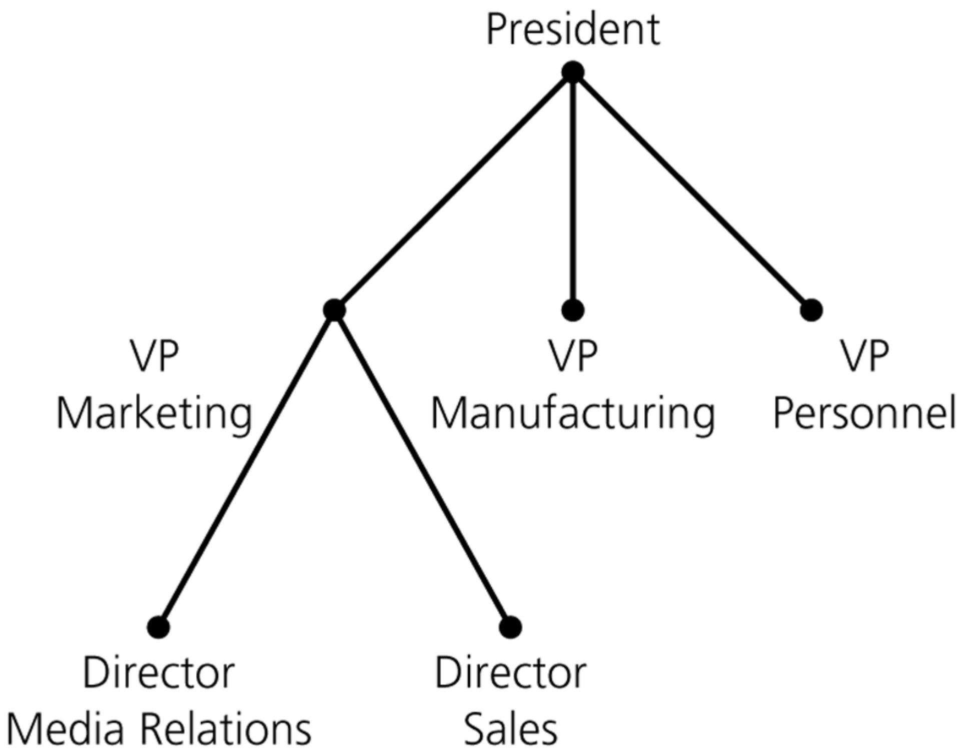
- (Rooted) Tree T is partitioned into disjoint subsets:
  - Empty or
  - Root node + (sub)trees



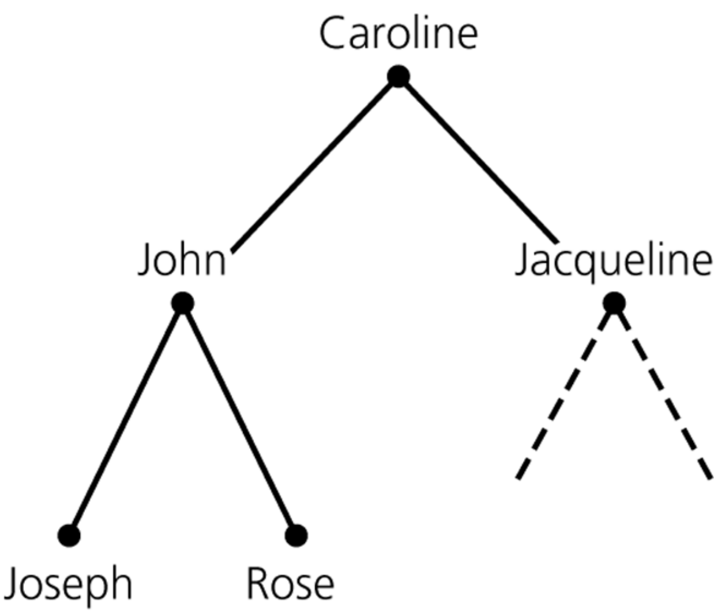
A subtree



An organization chart



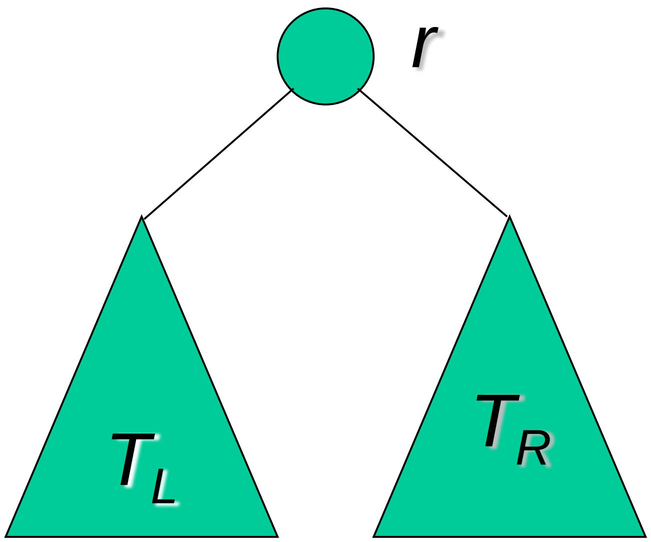
A family tree



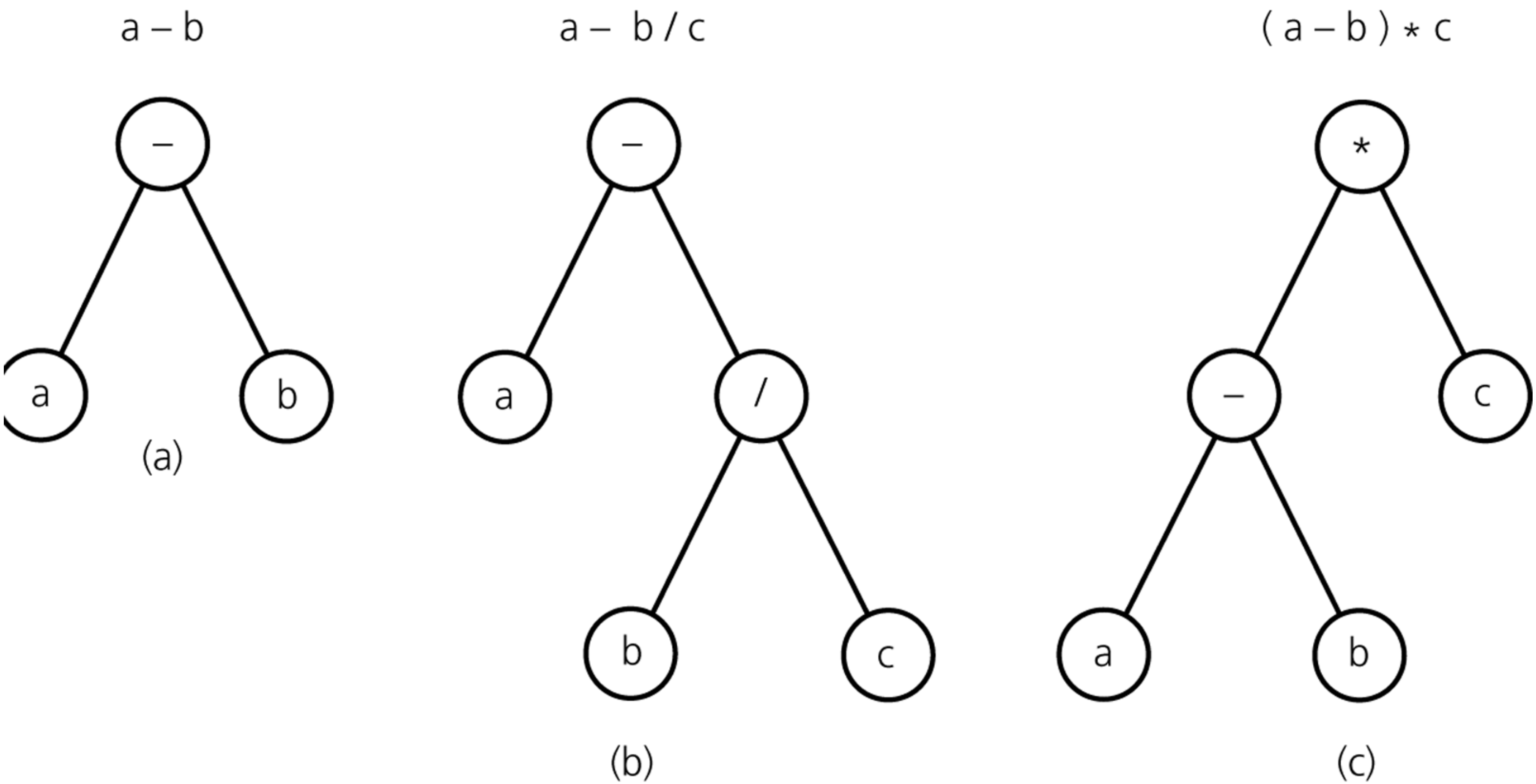
Binary Tree

- T is empty, or
- T is partitioned into three disjoint subsets:
  - Root node *r*
  - Two binary trees, called **left** and **right binary (sub)trees** of *r*

직관적 의미:
root가 있고 각 node가 최대 2개의
children을 가질 수 있는 tree



Binary Trees for Algebraic Expressions

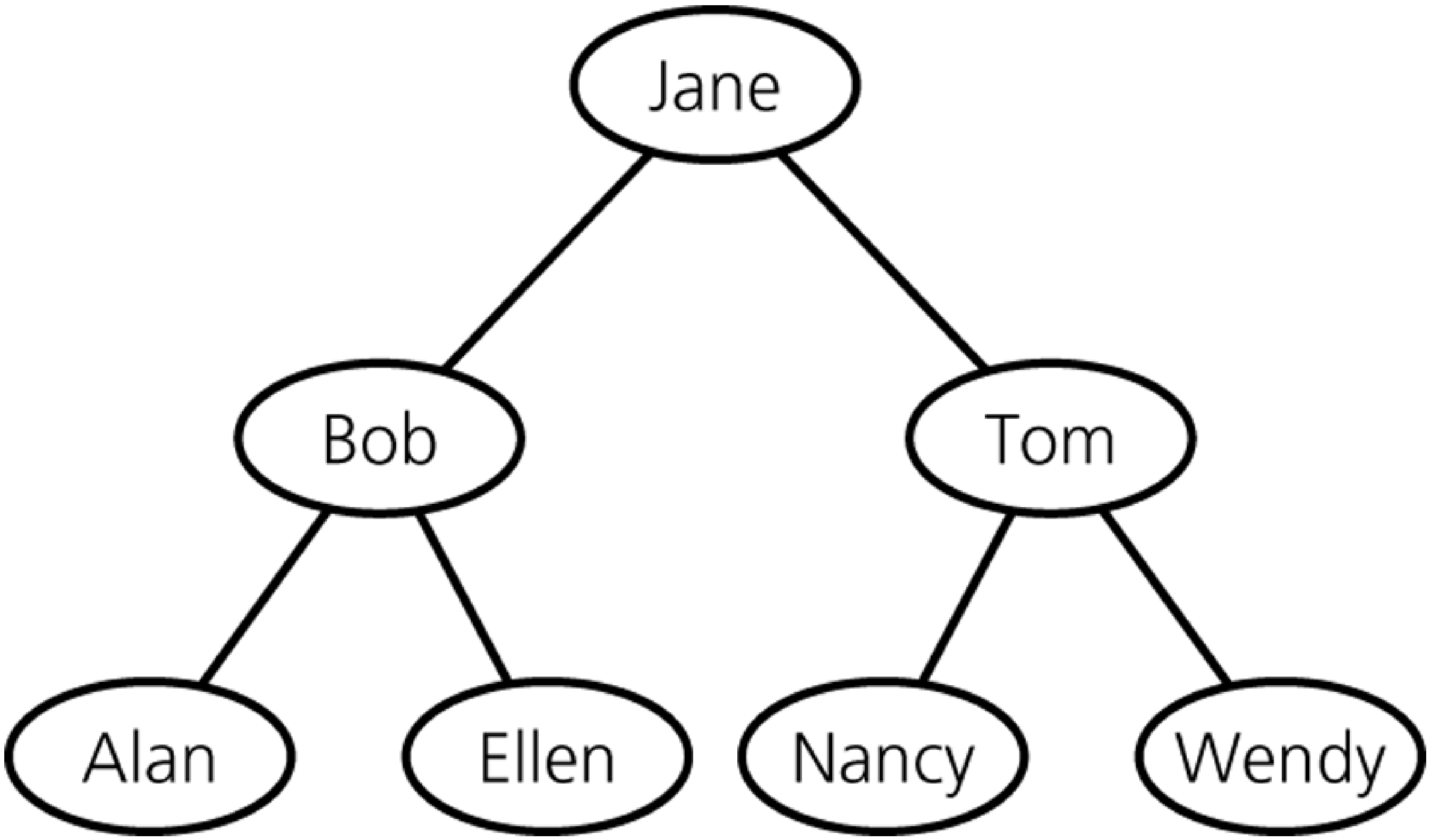


Binary Search Tree

- A binary tree for indexing by values
- For each node ***d***, it satisfies:
  - All values in the nodes are different one another
  - ***d***'s value is greater than all values in its left subtree T_L
  - ***d***'s value is less than all values in its right subtree T_R
  - Both T_L and T_R are binary search trees

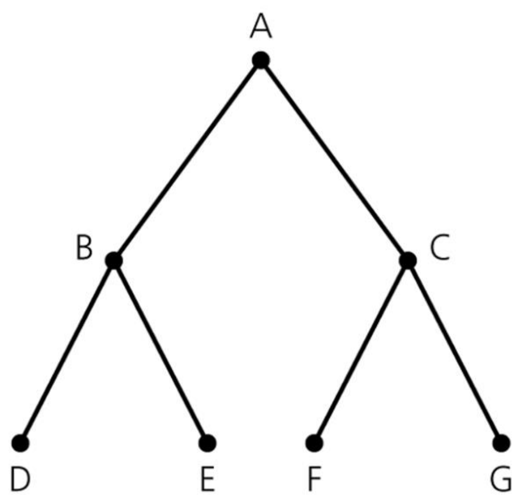
✓ In this class,
it is the 1st non-trivial data structure for **index**

A Binary Search Tree of Names

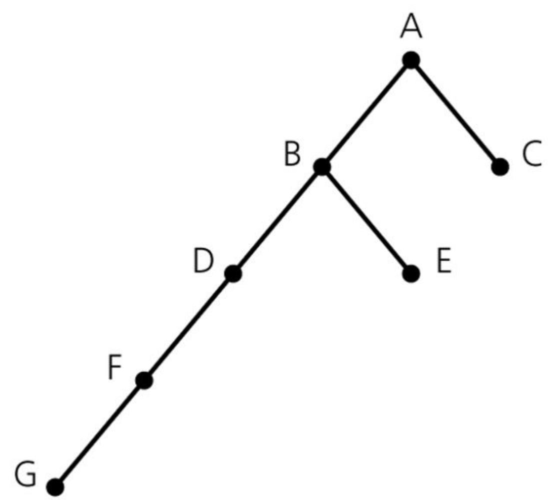


Height of a Tree

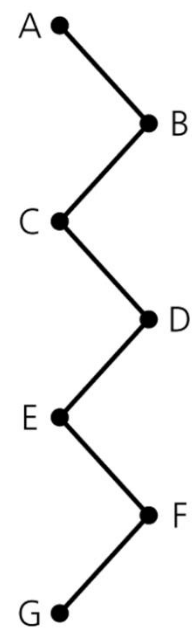
- The number of nodes on the longest path from the root to a leaf



Height 3



Height 5



Height 7

Full Binary Tree

- If T is empty,

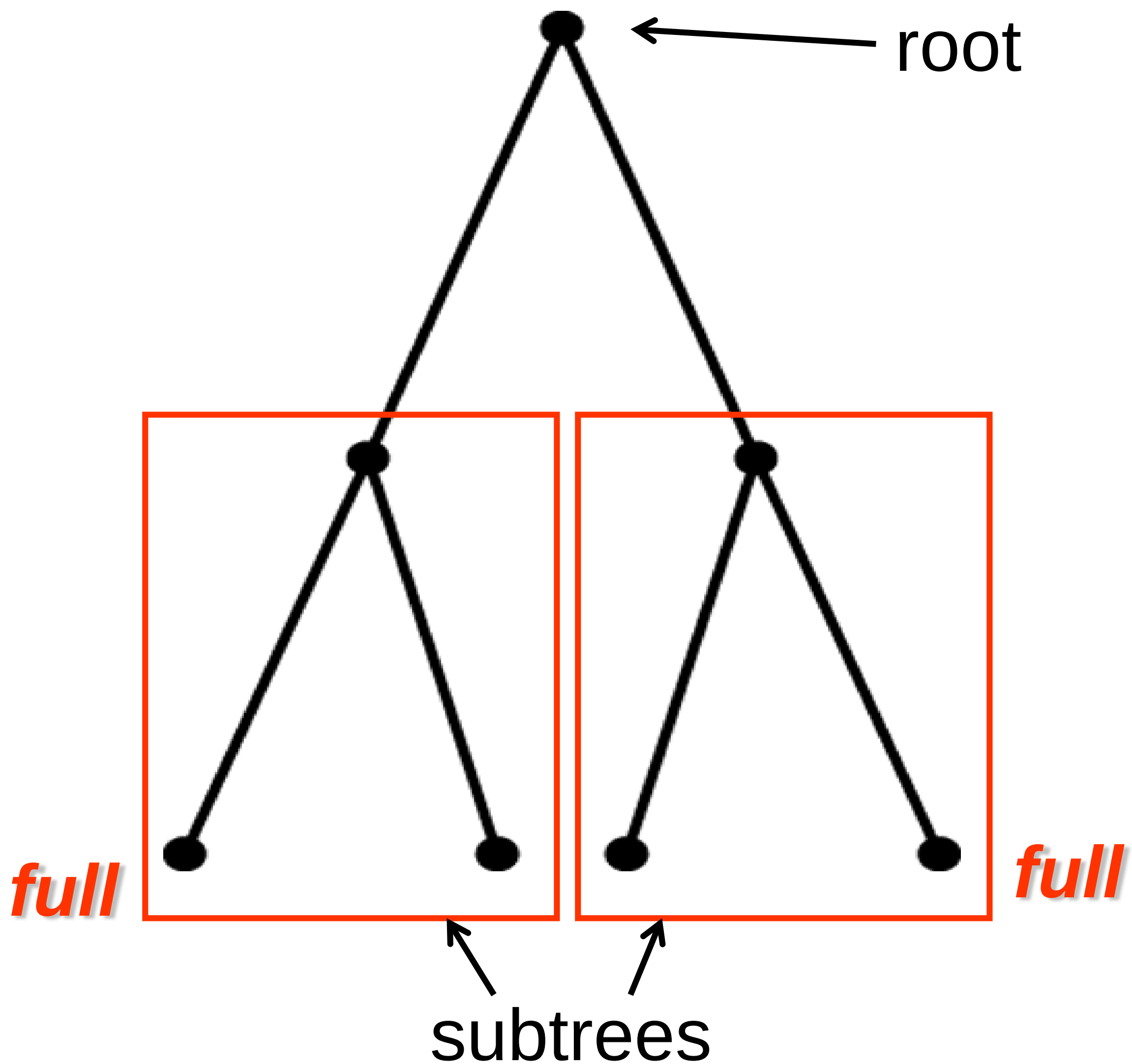
  T is a full binary tree of height 0
- If T is not empty and has height h,

  T is a full binary tree

  if the root's subtrees are both full binary trees of height h–1

직관적 의미:
모든 leaf node가 같은 레벨에 위치하고
leaf를 제외한 모든 node가 정확히 2개씩의 children을 갖는 tree

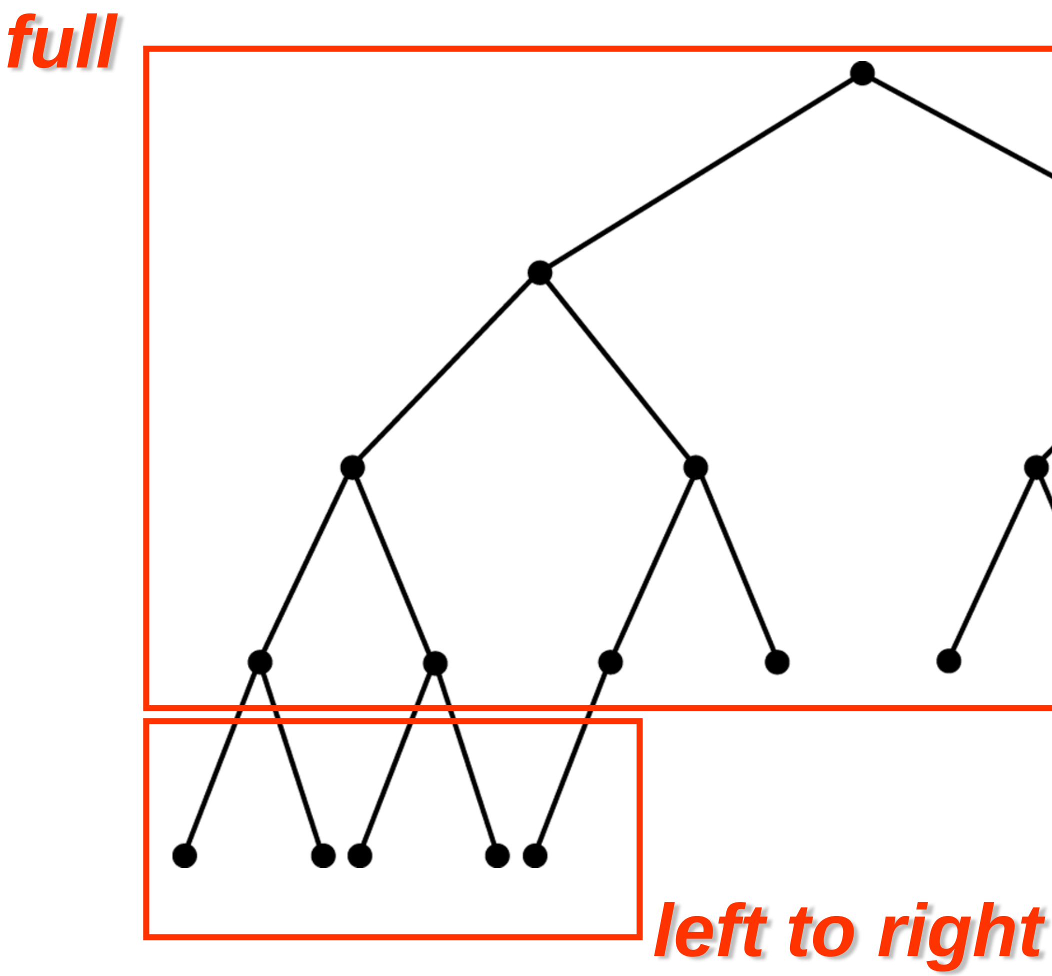
A Full Binary Tree of Height 3



Complete Binary Tree

- A complete binary tree of height h is
  a binary tree
  that is full down to level h–1
  with level h filled in from left to right

A Complete Binary Tree

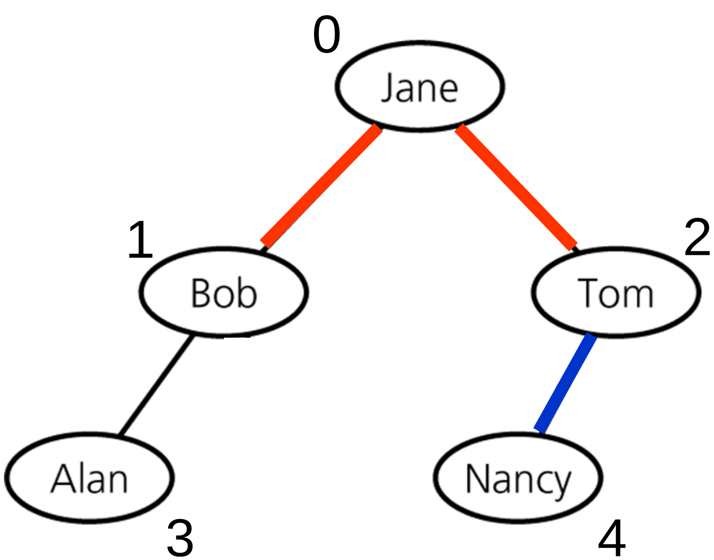


ADT *Binary Tree* Operations

- Create an empty binary tree
- Create a one-node binary tree
- Remove all nodes from a binary tree
- Determine whether a binary tree is empty
- Determine what data is the binary tree's root

Incomplete!

Array-Based Representation



tree

	item	leftChild	rightChild
0	Jane	1	2
1	Bob	3	-1
2	Tom	4	–1
3	Alan	–1	–1
4	Nancy	–1	–1
5	?	–1	–1
6	?	–1	7
7	?	–1	8
8	?	–1	9
⋮	⋮	⋮	⋮

root

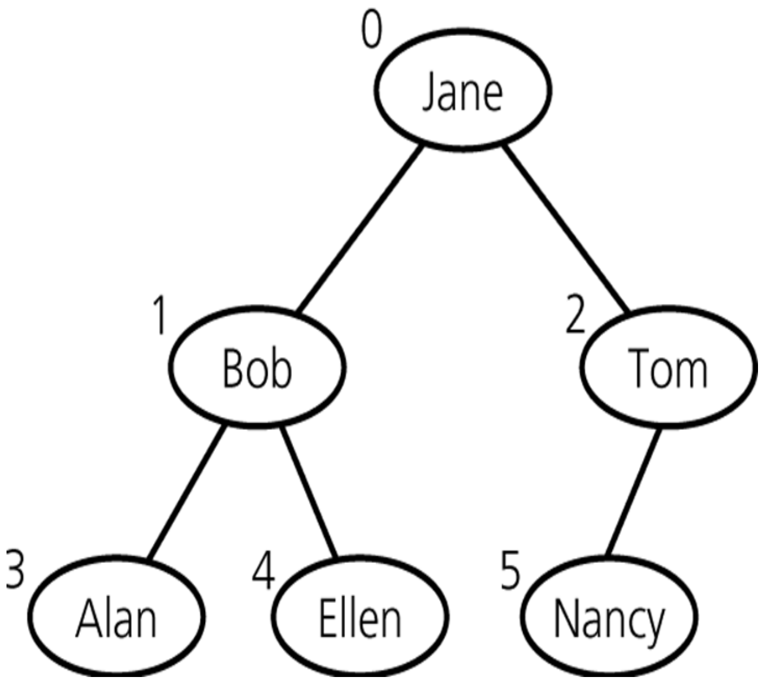
0

free

6

Free list

In Case of Complete Binary Trees



0	Jane
1	Bob
2	Tom
3	Alan
4	Ellen
5	Nancy
6	
7	

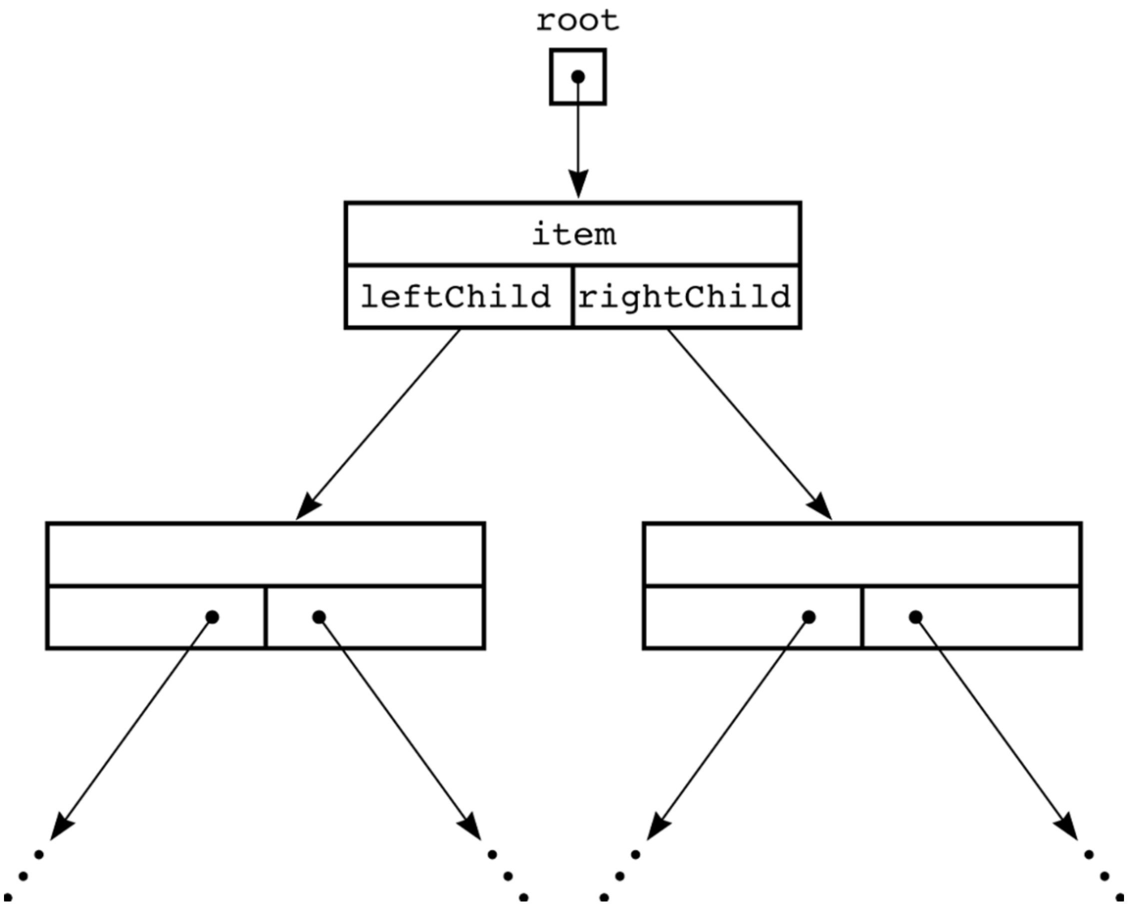
Node i's children:

$$2i + 1, 2i + 2$$

Node i's parent: $\left\lfloor \frac{i-1}{2} \right\rfloor$

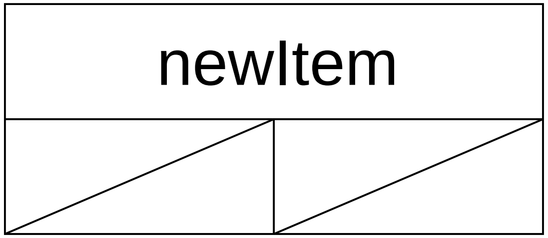
No link needed!

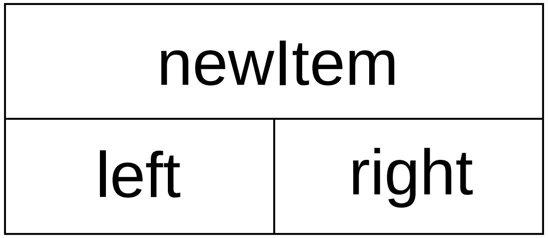
Reference-Based Representation



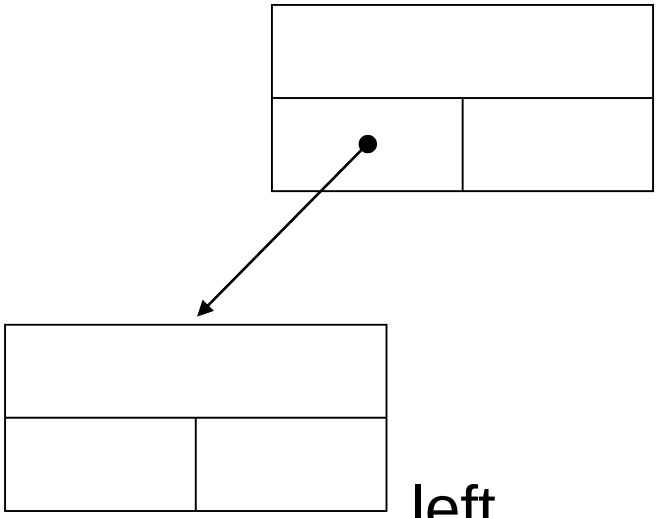
Reference-Based Implementation of Binary Tree

```
public class TreeNode {
    private Object item;
    private TreeNode leftChild;
    private TreeNode rightChild;
    public TreeNode(Object newItem) {
            item = newItem;
            leftChild = rightChild = null;
    }
    public TreeNode(Object newItem, TreeNode left, TreeNode right) {
            item = newItem;
            leftChild = left;
            rightChild = right;
    }
```





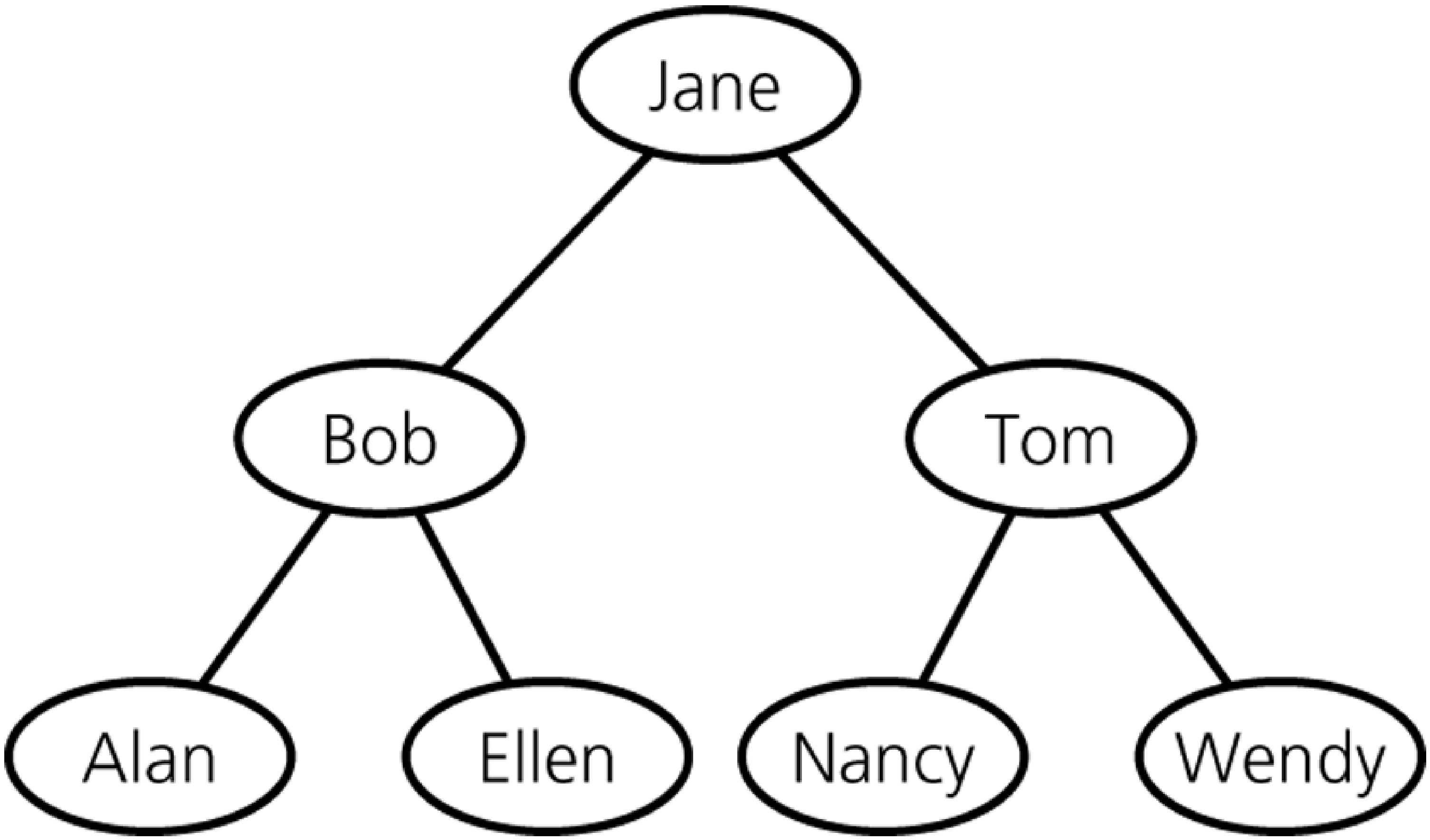
```
public Object getItem( ) {
            return item;
}
public void setItem(Object newItem) {
            item = newItem;
}
public TreeNode getLeft( ) {
            return leftChild;
}
public TreeNode getRight( ) {
            return rightChild;
}
public setLeft(TreeNode left) {
            leftChild = left;
}
public setRight(TreeNode right) {
            rightChild = right;
}
} // end TreeNode
```



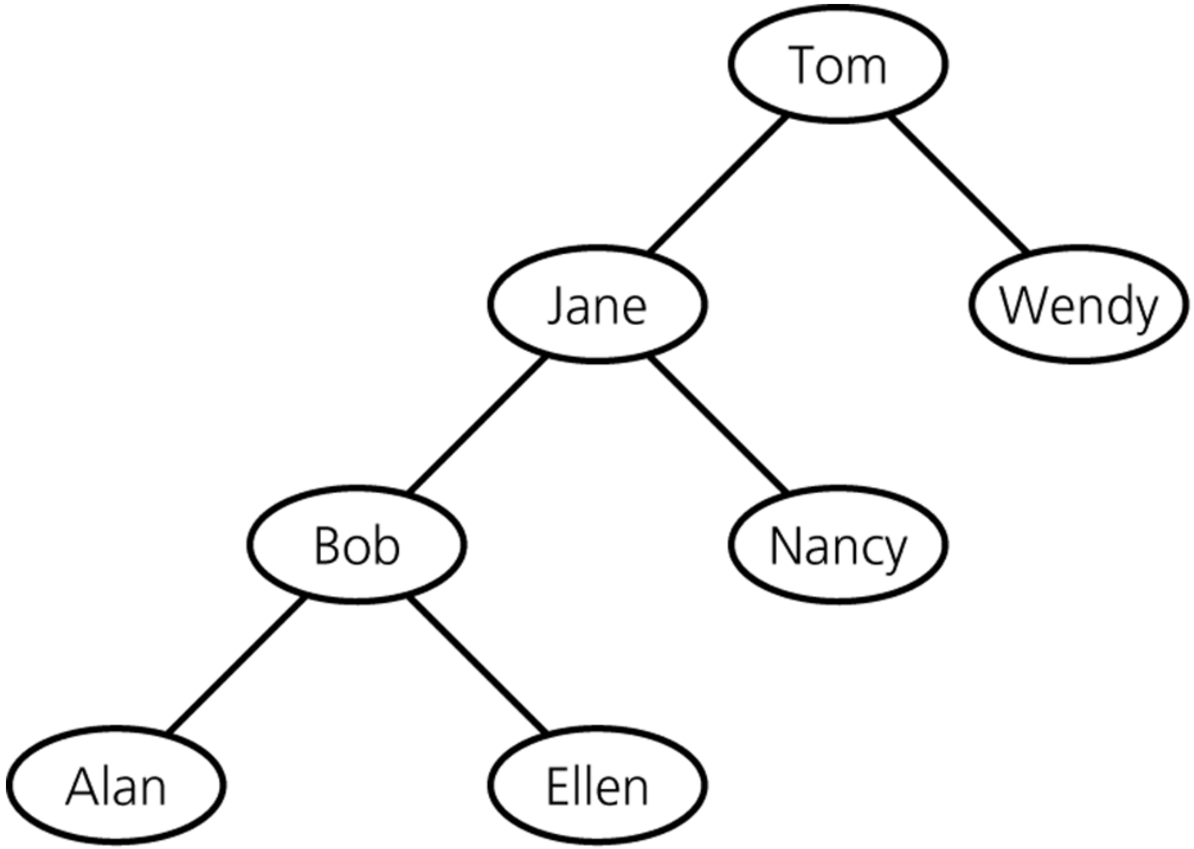
Binary Search Tree

- Each node has a search key
  - There are no duplications among the search keys in a binary search tree
- For each node ***n***, it satisfies:
  - ***n***'s key is greater than all keys in its left subtree T_L
  - ***n***'s key is less than all keys in its right subtree T_R
  - Both T_L and T_R are binary search trees

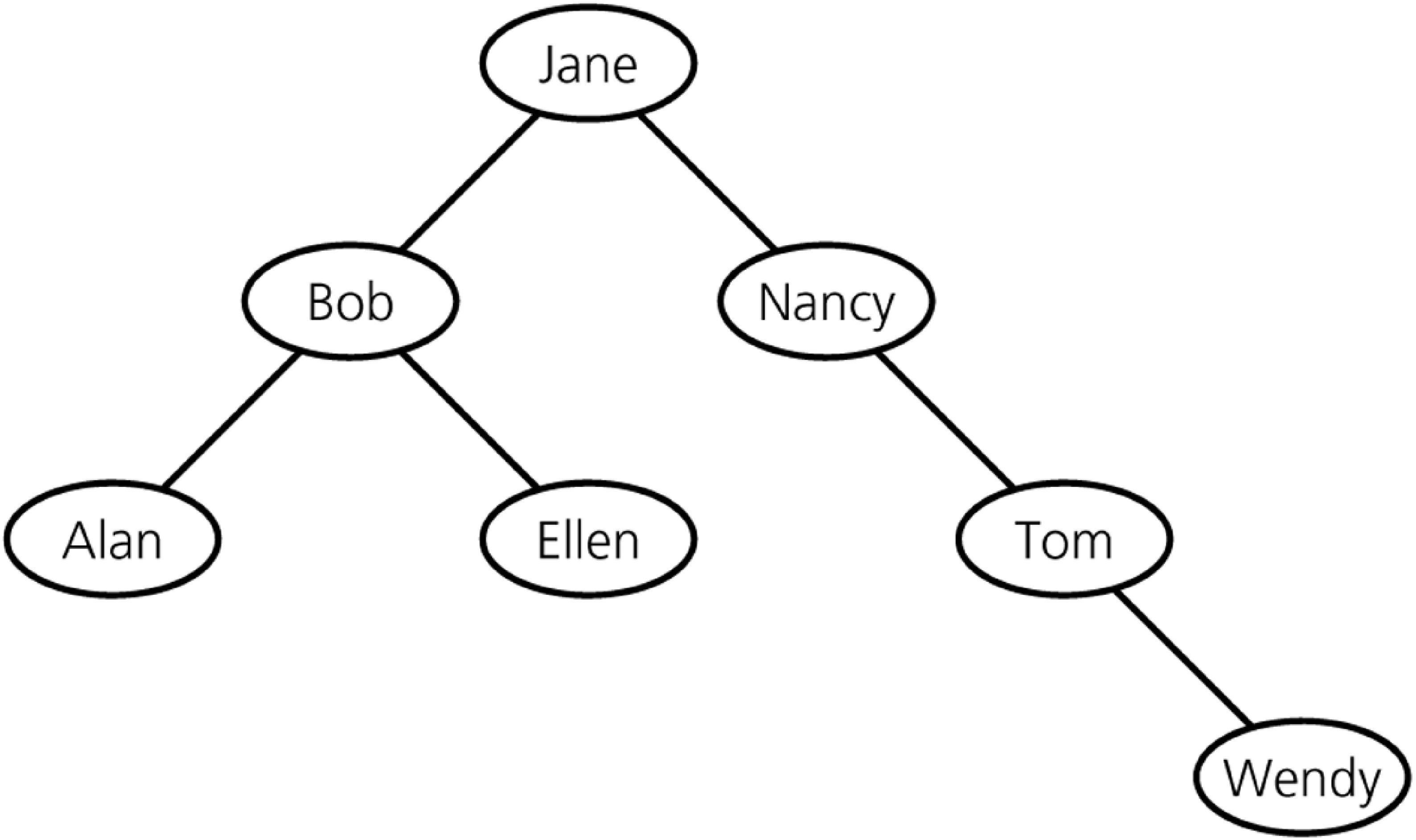
A Binary Search Tree of Names



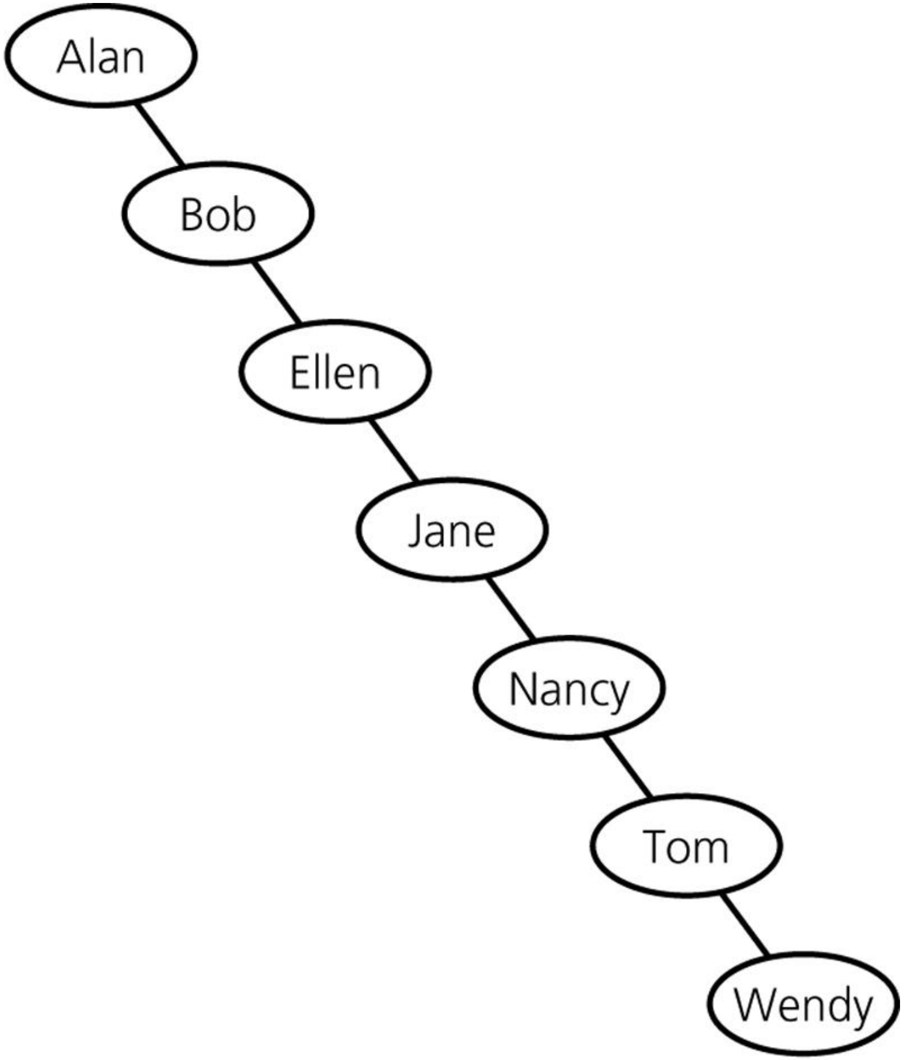
Another Binary Search Tree w/ the Same Data



Yet Another



Yet Another



ADT *Binary Search Tree* Operations

- …
- Insert a new item into a binary search tree
- Delete the item w/ a given search key from a binary search tree
- Retrieve the item w/ a given search key from a binary search tree
- …

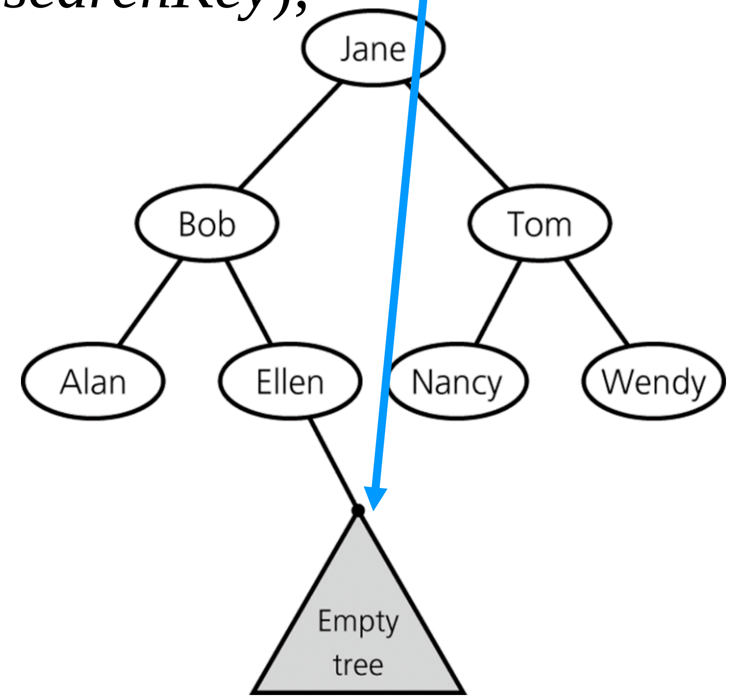
✓ Binary search tree는 index(색인, 찾아보기)용으로 유용하다

```
public class BinarySearchTree {
        private TreeNode root;
        // private int numItems; (이런 게 필요할 수도)
        public BinarySearchTree( … ) {
                        …
        }
        public TreeNode search(TreeNode root, Object searchKey) {
                …
        }
        public void insert(TreeNode root, Object newKey) {
                …
        }
        public void delete(TreeNode rootNode, Object searchKey) {
                …
        }
        private TreeNode deleteItem(TreeNode rootNode, Object searchKey) {
                …
        }
        private TreeNode deleteNode(TreeNode tNode) {
                …
        }
        …
} // end BinarySearchTree
```

Search in a Binary Search Tree

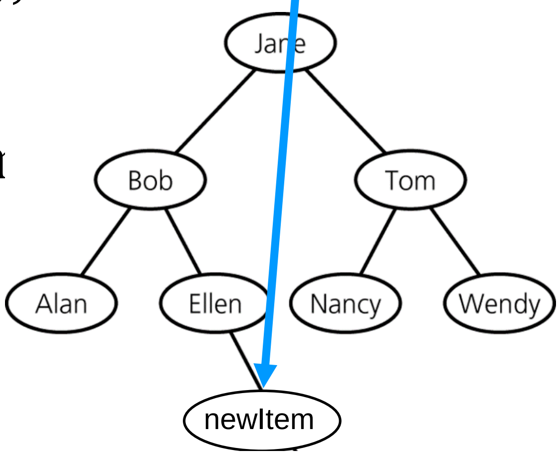
```
search(root, searchKey) {
        if (root is empty) return "Not found!";
        else if (searchKey == root's key) return root;
        else if (searchKey < root's key)
                return search(root's left child, searchKey);
        else
                return search(root's right child, searchKey);
}
```

```
search(root, searchKey) {
    if (root is empty) return "Not found!";
    else if (searchKey == root's key) return root;
    else if (searchKey < root's key)
        return search(root's left child, searchKey);
    else
        return search(root's right child, searchKey);
}
```



Insertion in a Binary Search Tree

```
insert (root, newItem) {
        if (root is null) {
                newItem을 key로 가진 새 node를 매단다;
        }
        else if (newItem < root's key)
                insert(root's left child, newItem);
        else
                insert(root's right child, newIten
}
```



✓ Search()와 구조가 거의 같다

Deletion in a Binary Search Tree

deleteItem (*root*, searchKey) {

dNode = search(*root*, searchKey);

deleteNode(*dNode*);

}

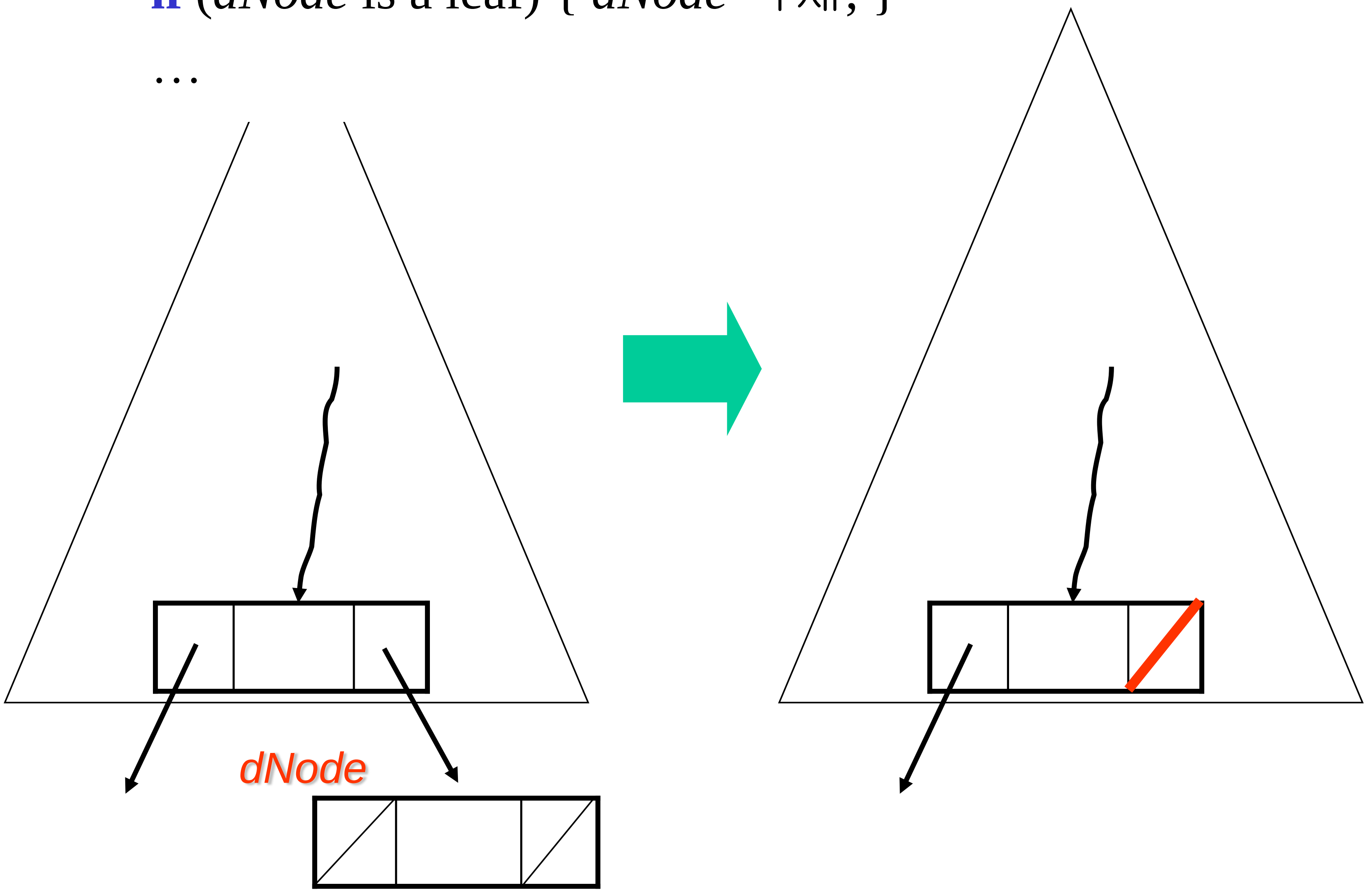
✓ Binary search tree의 operation들 중 상대적으로 복잡

```
deleteNode (dNode) {
        if (dNode is a leaf) { dNode 삭제; } // case 1
        else if (dNode has only one child c) {   // case 2
                c replaces dNode;
        } else { // dNode has two children          // case 3
                minNode =  dNode' right subtree의 leftmost node;
                // minNode has at most one right child
                minNode replaces dNode;
                deleteNode(minNode);
        }
}
```

deleteNode (*dNode*) { // case 1

if (*dNode* is a leaf) { *dNode* 삭제; }

. . .



deleteNode (*dNode*) { // case 2

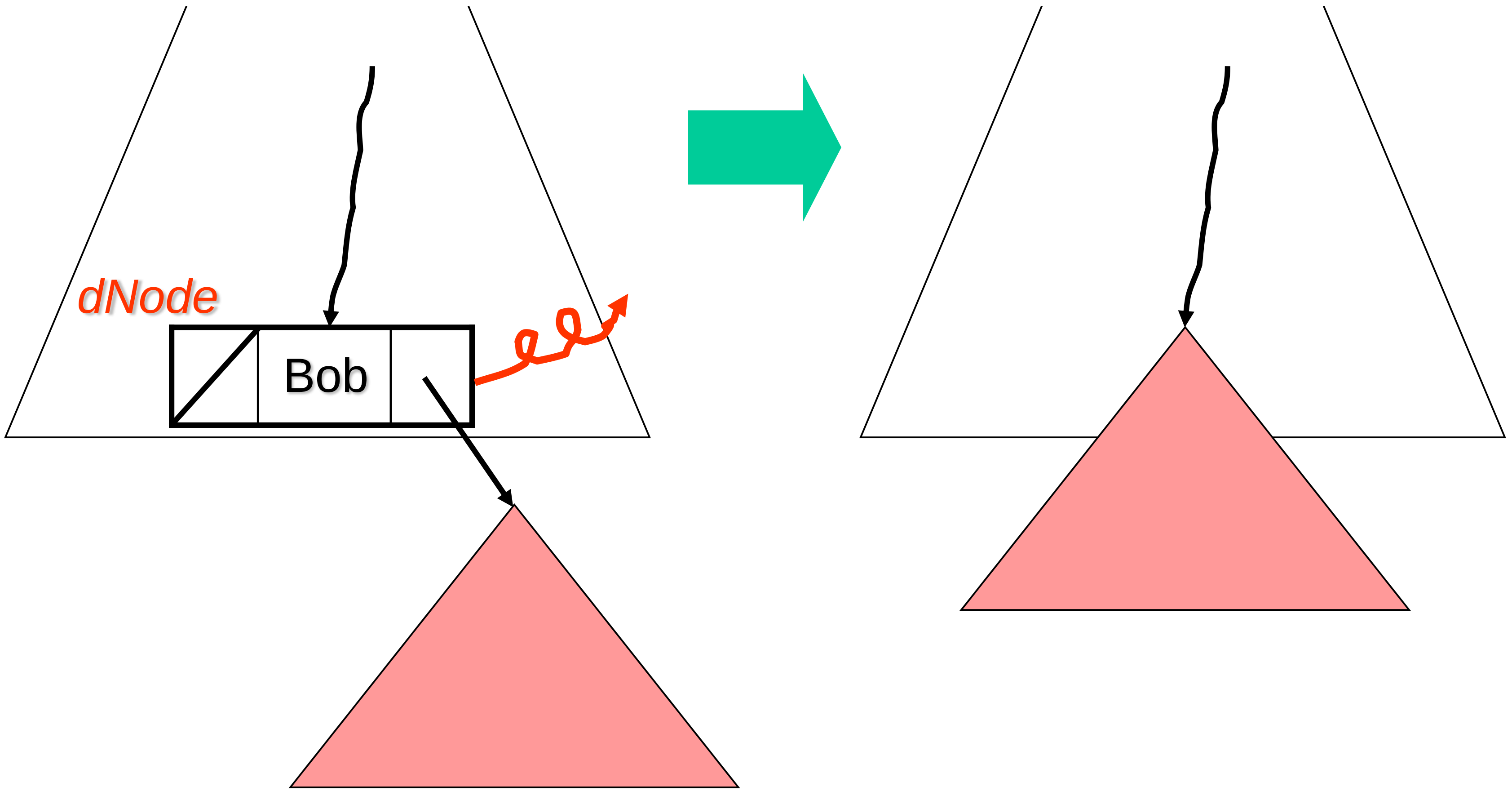
...

else if (*dNode* has only one child ***c***) {

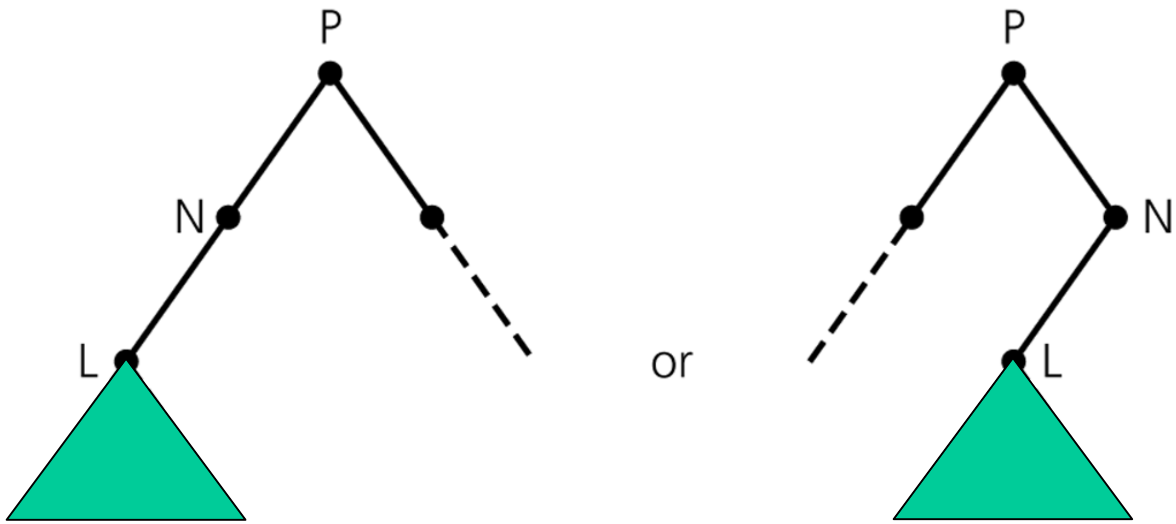
c replaces *dNode*;

}

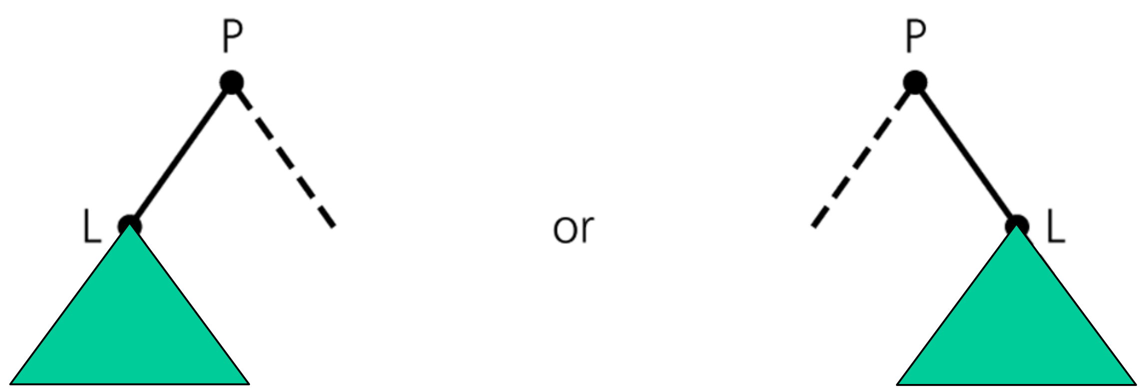
...



A Case 2 Example

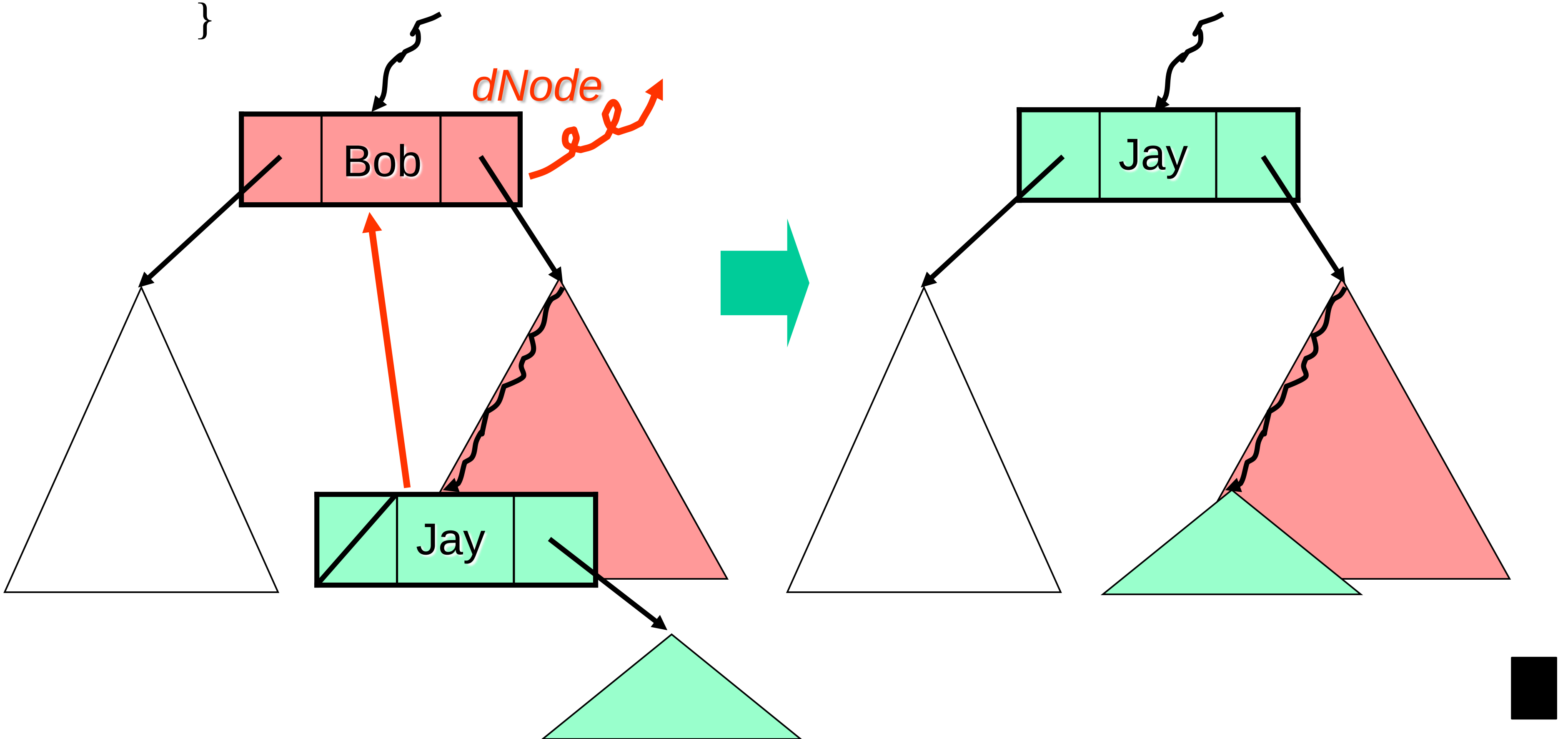


N can be either the left or right child of *P*



After deleting node *N*

deleteNode (*dNode*) { // case 3
    . . .
    } **else** { // *dNode* has two children
        minNode = *dNode*’ right subtree의 leftmost node;
        // minNode has at most one right child
        minNode replaces *dNode*;
        deleteNode(*minNode*); // case 1 or 2
    }

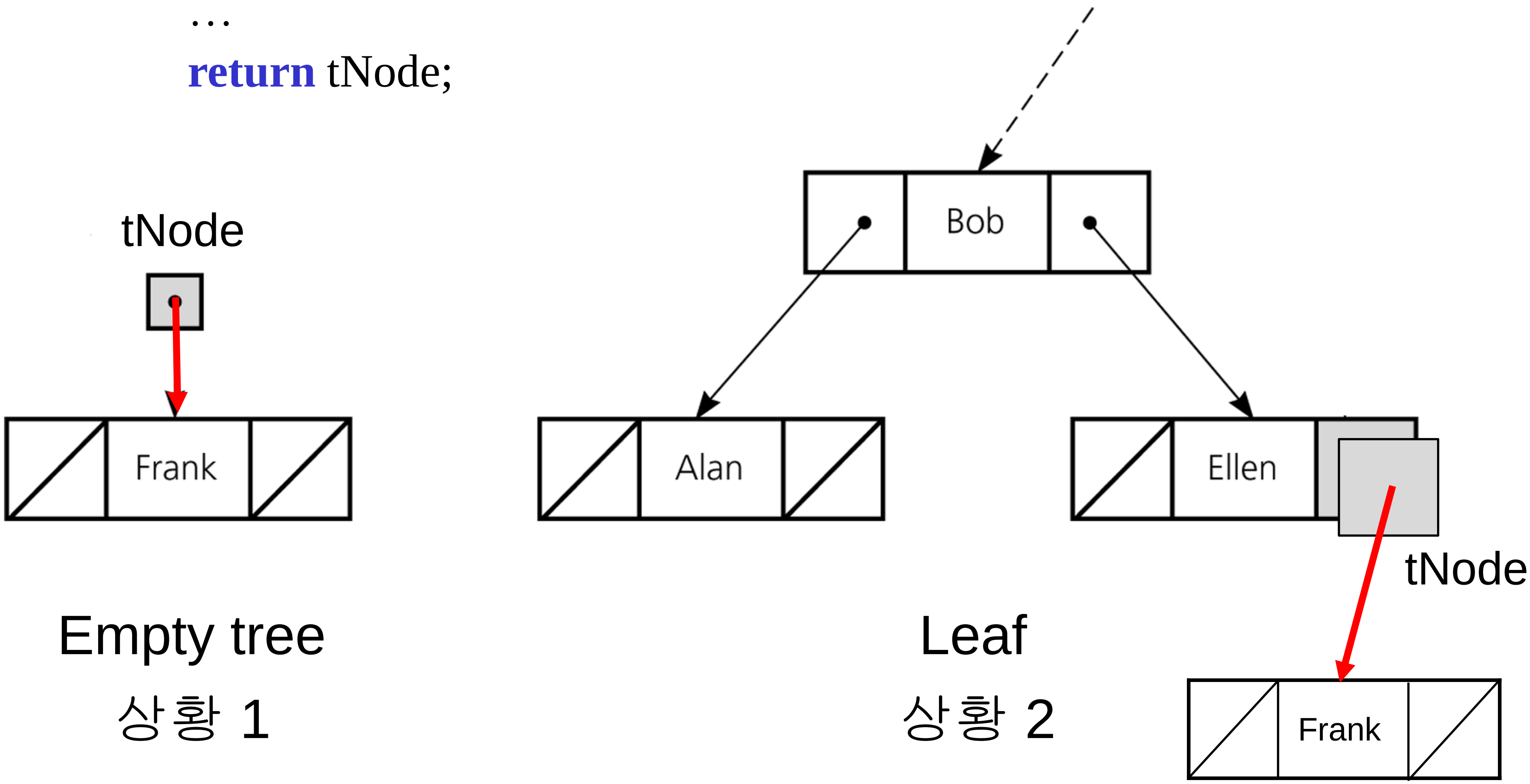


More Detailed Pseudo-Code (Reference-Based)

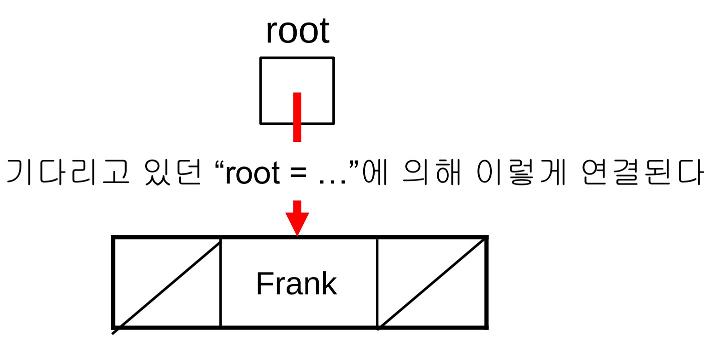
```
insert(... newItem) {
        root = insertItem(root, newItem);
}
TreeNode insertItem(TreeNode tNode, ... newItem) {
        if (tNode == null) { // insert after a leaf  (or into an empty tree)
                tNode = new TreeNode(newItem, null, null);
        } else  if (newItem < tNode's item) { // branch left
                tNode.setLeft( insertItem(tNode.getLeft( ), newItem) );
        } else { // branch right
                tNode.setRight( insertItem(tNode.getRight( ), newItem) );
        }
        return tNode;
} // end insertItem
```

✓ tNode는 null일 때만 값이 바뀐다

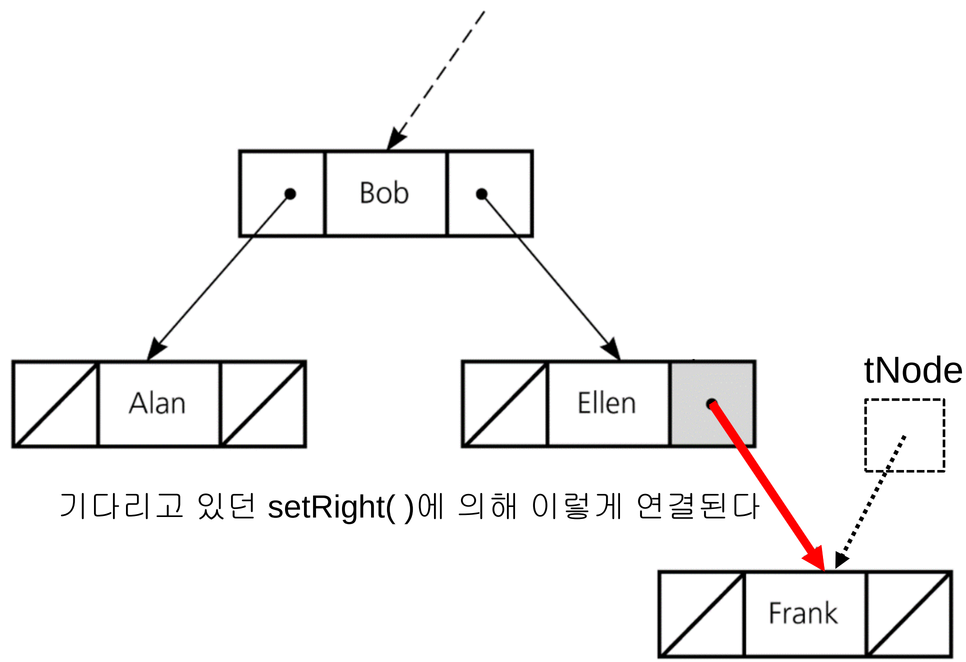
```
TreeNode insertItem(TreeNode tNode, ... newItem) {
        if (tNode == null) { // insert after a leaf (or into an empty tree)
                tNode = new TreeNode(newItem, null, null);
        }
        …
        return tNode;
```



```
TreeNode insertItem(TreeNode tNode, ... newItem) {
        if (tNode == null) { // insert after a leaf (or into an empty tree)
                tNode = new TreeNode(newItem, null, null);
        }
        …
        return tNode;
```

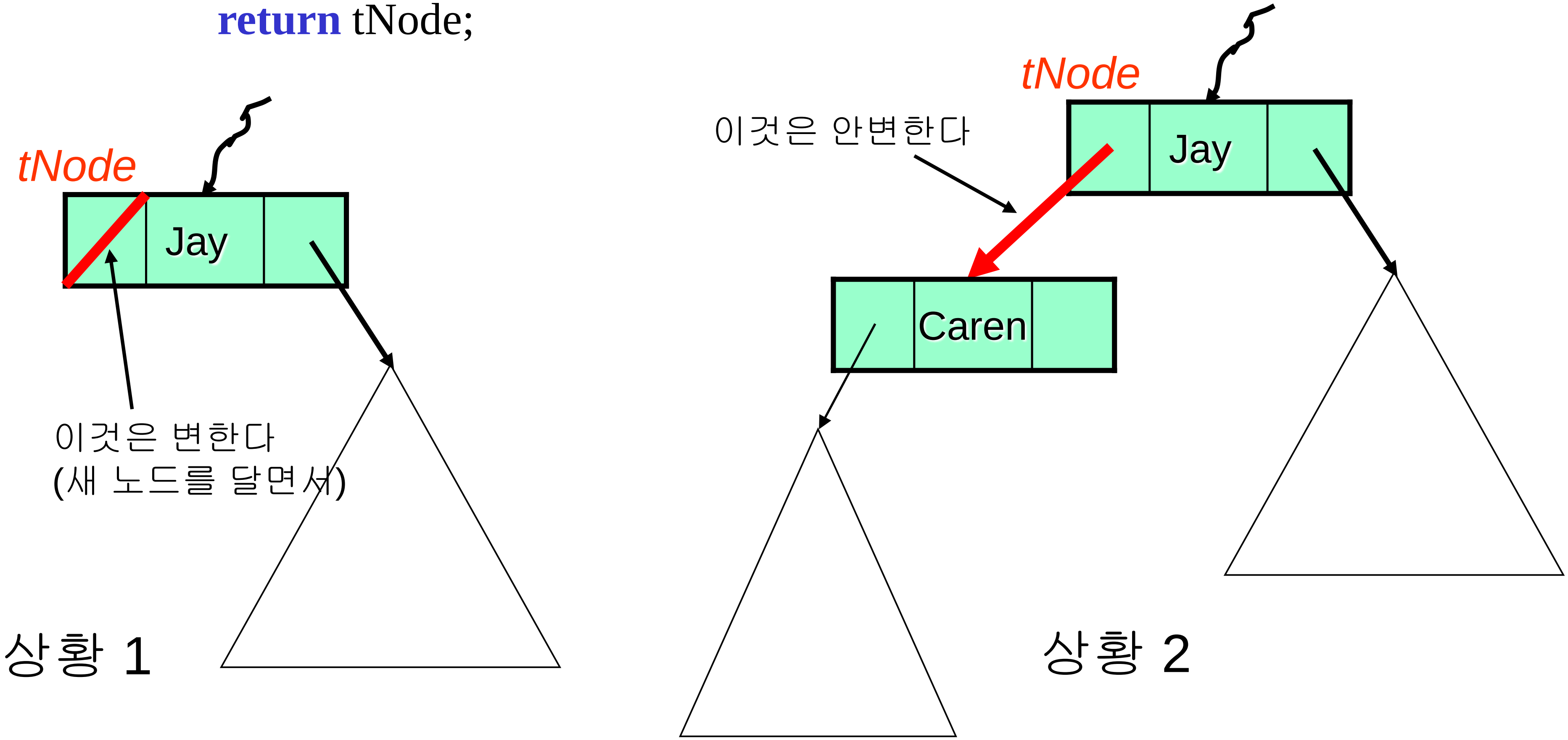


상황 1



상황 2

```
TreeNode insertItem(TreeNode tNode, ... newItem) {
        …
        if (newItem <  tNode's item) { // branch left
                tNode.setLeft( insertItem(tNode.getLeft( ), newItem) );
        }
        …
        return tNode;
```



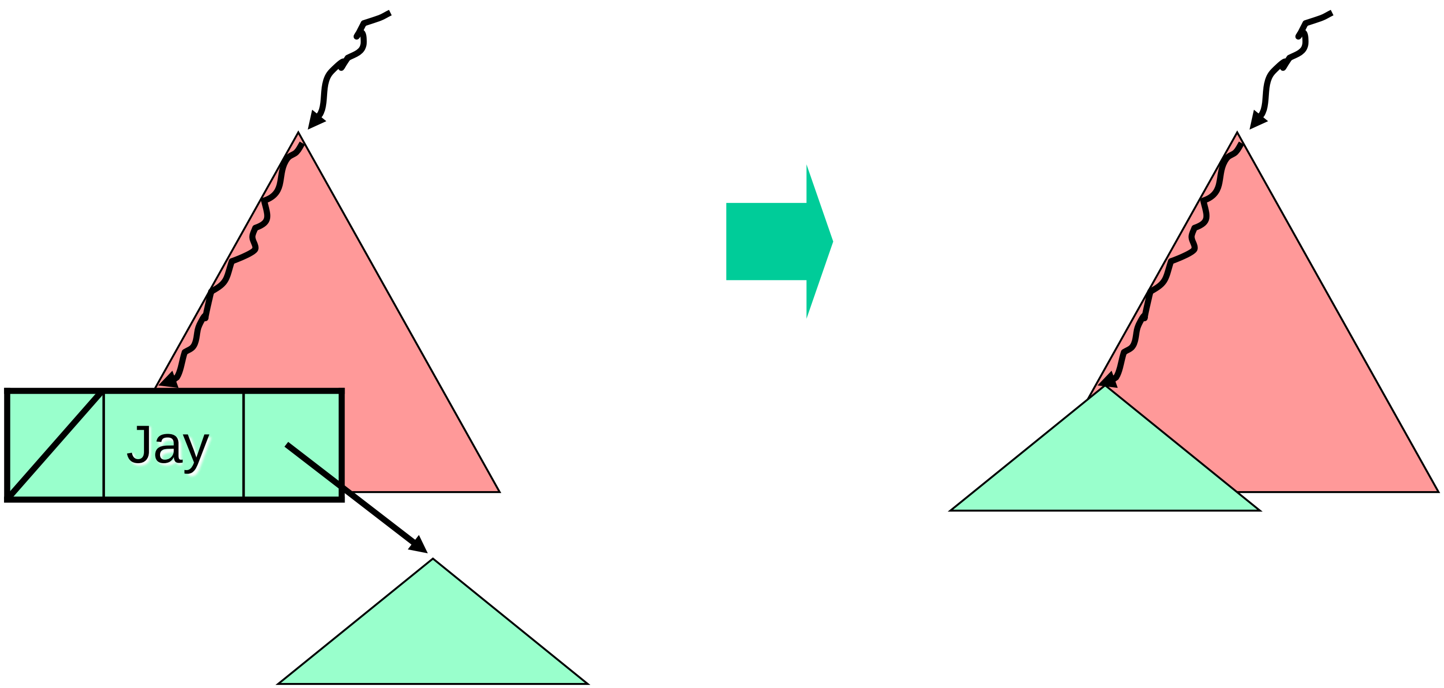
```
delete(... searchKey) {
                root = deleteItem(root, searchKey);
}

TreeNode deleteItem (TreeNode tNode, ... searchKey) {
    if (tNode == null) {exception 처리}; // item not found!
    else {
        if (searchKey == tNode's key) { // item found!
            tNode = deleteNode(tNode);
        } else if (searchKey < tNode's key) {
            tNode.setLeft(deleteItem(tNode.getLeft( ), searchKey));
        } else {
            tNode.setRight(deleteItem(tNode.getRight( ), searchKey) );
        }
    }
    return tNode; // tNode: parent에 매달리는 노드
}
```

```
TreeNode deleteNode(TreeNode tNode) {
    // Three cases
    //    1. tNode is a leaf
    //    2. tNode has only one child
    //    3. tNode has two children

    if ( (tNode.getLeft( ) == null)  && (tNode.getRight( ) == null)) { // case 1
            return null;
    } else if (tNode.getLeft( ) == null ) {  // case 2 (only right child)
            return tNode.getRight( );
    } else if (tNode.getRight( ) == null) {  // case 2 (only left child)
            return tNode.getLeft( );
    } else {  // case 3 – two children
            tNode.setItem(minimum item of tNode's right subtree);
            tNode.setRight(deleteMin(tNode.getRight( ));
            return tNode; // tNode survived
    }
}
```

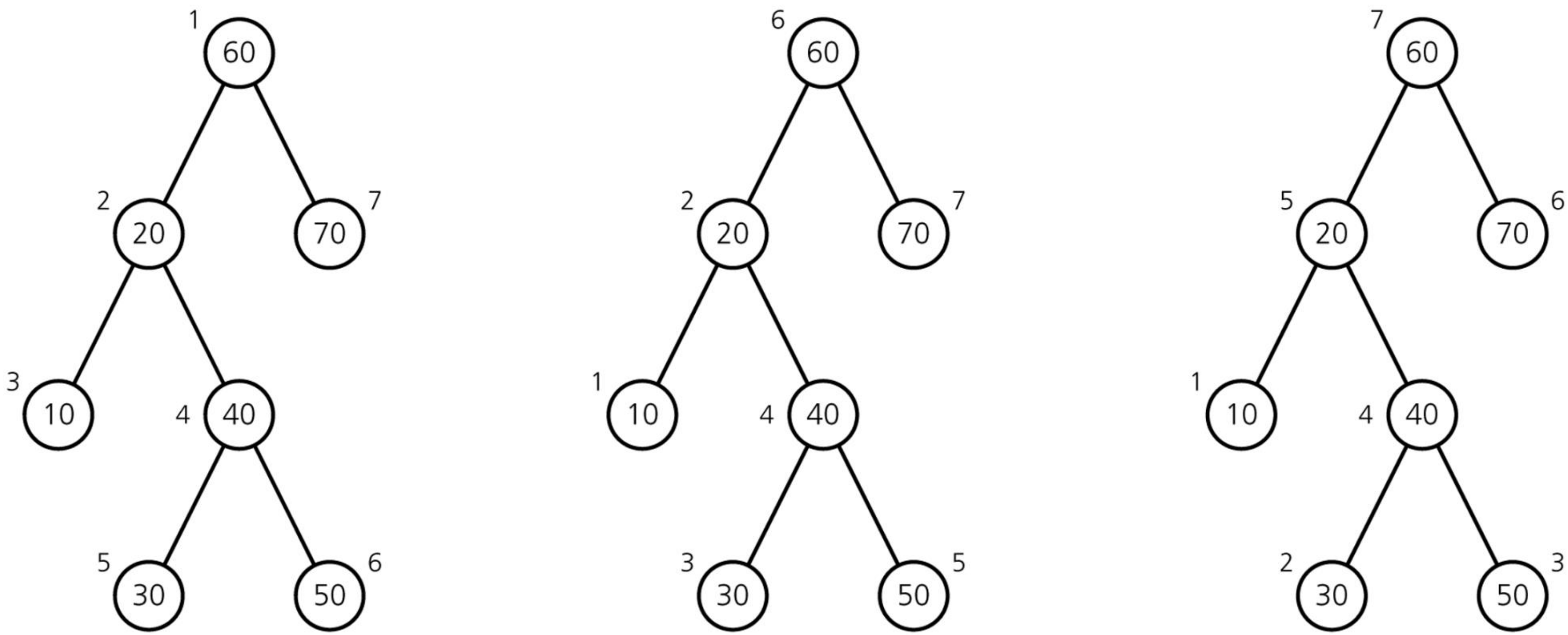
```
TreeNode deleteMin (TreeNode tNode) {
        if (tNode.getLeft( ) == null) { // found min
                return tNode.getRight( ); // right child moves to min's place
        } else { // branch left, then backtrack
                tNode.setLeft(deleteMin(tNode.getLeft( )));
                return tNode;
        }
}
```



Traversal of Binary Tree

- A traversal algorithm visits every node in the tree
- There are three representative traversal algorithms for binary trees
  - Preorder traversal
  - Inorder traversal
  - Postorder traversal

Preorder, Inorder, Postorder



(a) Preorder: 60, 20, 10, 40, 30, 50, 70 (b) Inorder: 10, 20, 30, 40, 50, 60, 70 (c) Postorder: 10, 30, 50, 40, 20, 70, 60

(Numbers beside nodes indicate traversal order.)

Preorder Traversal

preorder(*root*)
{
    if (*root* is not empty) {
        Mark *root*;
        preorder(Left subtree of *root*);
        preorder(Right subtree of *root*);
    }
}

Inorder Traversal

```
inorder(root)
{
    if (root is not empty) {
        inorder(Left subtree of root);
        Mark root;
        inorder(Right subtree of root);
    }
}
```

Postorder Traversal

```
postorder(root)
{
    if (root is not empty) {
        postorder(Left subtree of root);
        postorder(Right subtree of root);
        Mark root;
    }
}
```

Operations' Efficiency on B.S.T.

Operation	Average case	Worst case
Retrieval	$\Theta(\log n)$	$\Theta(n)$
Insertion	$\Theta(\log n)$	$\Theta(n)$
Deletion	$\Theta(\log n)$	$\Theta(n)$
Traversal	$\Theta(n)$	$\Theta(n)$

Properties of Binary Trees

Theorem 1

The *inorder* traversal of a binary search tree T visits its nodes in sorted search-key order.

<Proof>

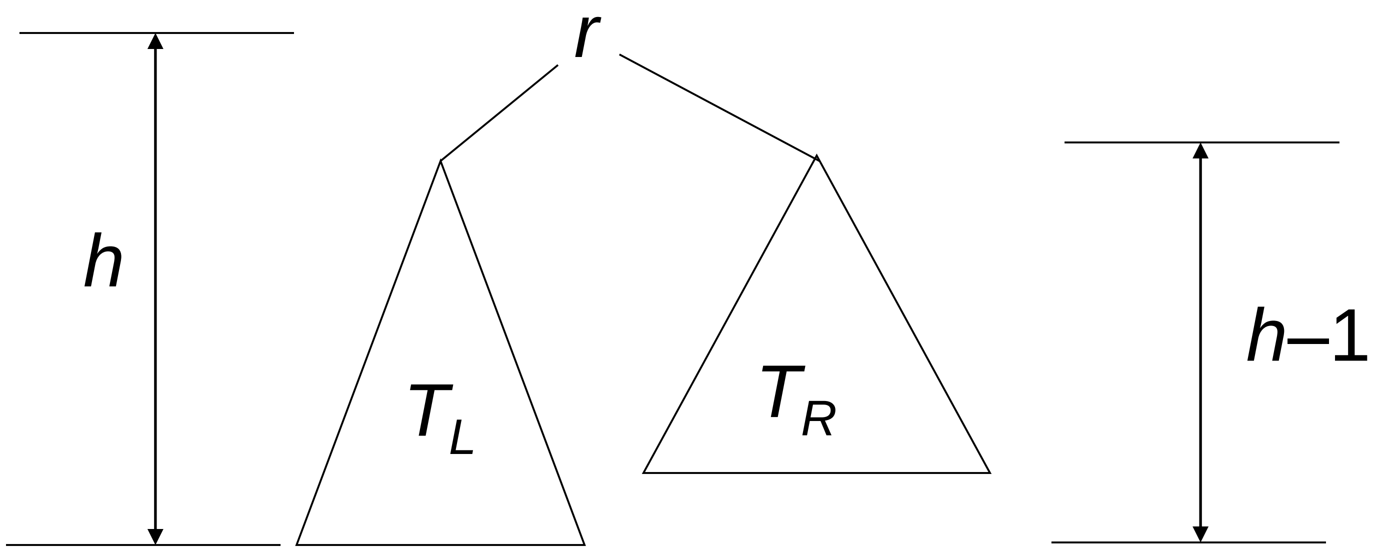
Basis: $h = 1$.

T consists of only one node, the root.

Visiting the only node is obviously in sorted order.

Inductive hypothesis: Assume that the theorem is true for all $k < h$.

Inductive conclusion: Want to show that the theorem is true for $k = h$. T is of the form



Inorder visits T_L and T_R in sorted order, respectively, by the inductive hypothesis. Because keys in T_L < r's key and keys in T_R > r's key, the *inorder* traversal of $T_L \rightarrow r \rightarrow T_R$ is in sorted order. ■

Height

Theorem 2

A full binary tree of height $h \geq 0$ has $2^h - 1$ nodes.

Corollary 1

The number of nodes in a binary tree of height h is at most $2^h - 1$.

Proofs are simple!

Theorem 3

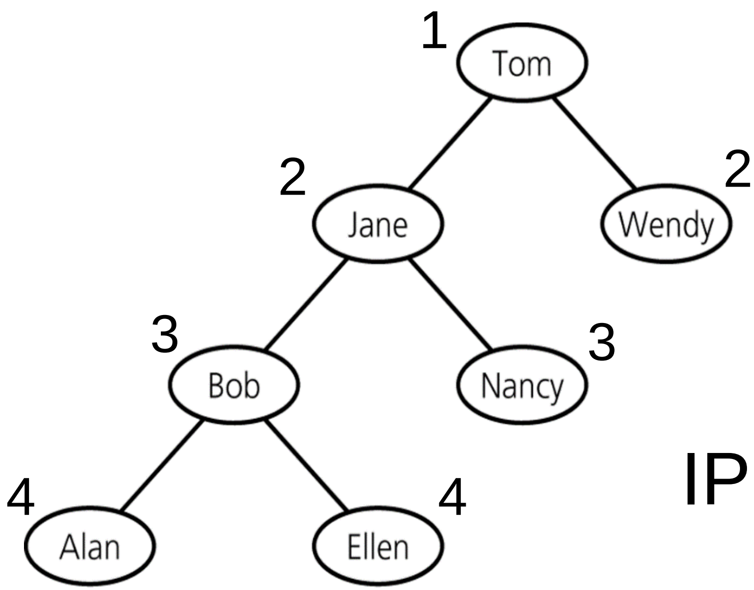
The minimum height of a binary tree with n nodes is $\lceil \log_2(n+1) \rceil$.

<Proof>

Straightforward by Corollary 1

Depth

- Definition: Internal Path Length (IPL)
  - Sum of depths of its nodes



IPL = 19

Theorem 4

The expected IPL of a binary tree with n nodes is $O(n \log n)$ under the assumption that all permutations are equally likely.

<Proof> Chapter11-IPL증명.doc

✓ Meaning: Average search time for an item is $O(\log n)$.

Tree Size 구하기

```
int size(TreeNode t)
{
        if (t == null) return 0;
        else return (1 + size(t.getLeft( )) + size(t.getRight( )));
}
```

An Example Use of Binary Tree: Huffman Code

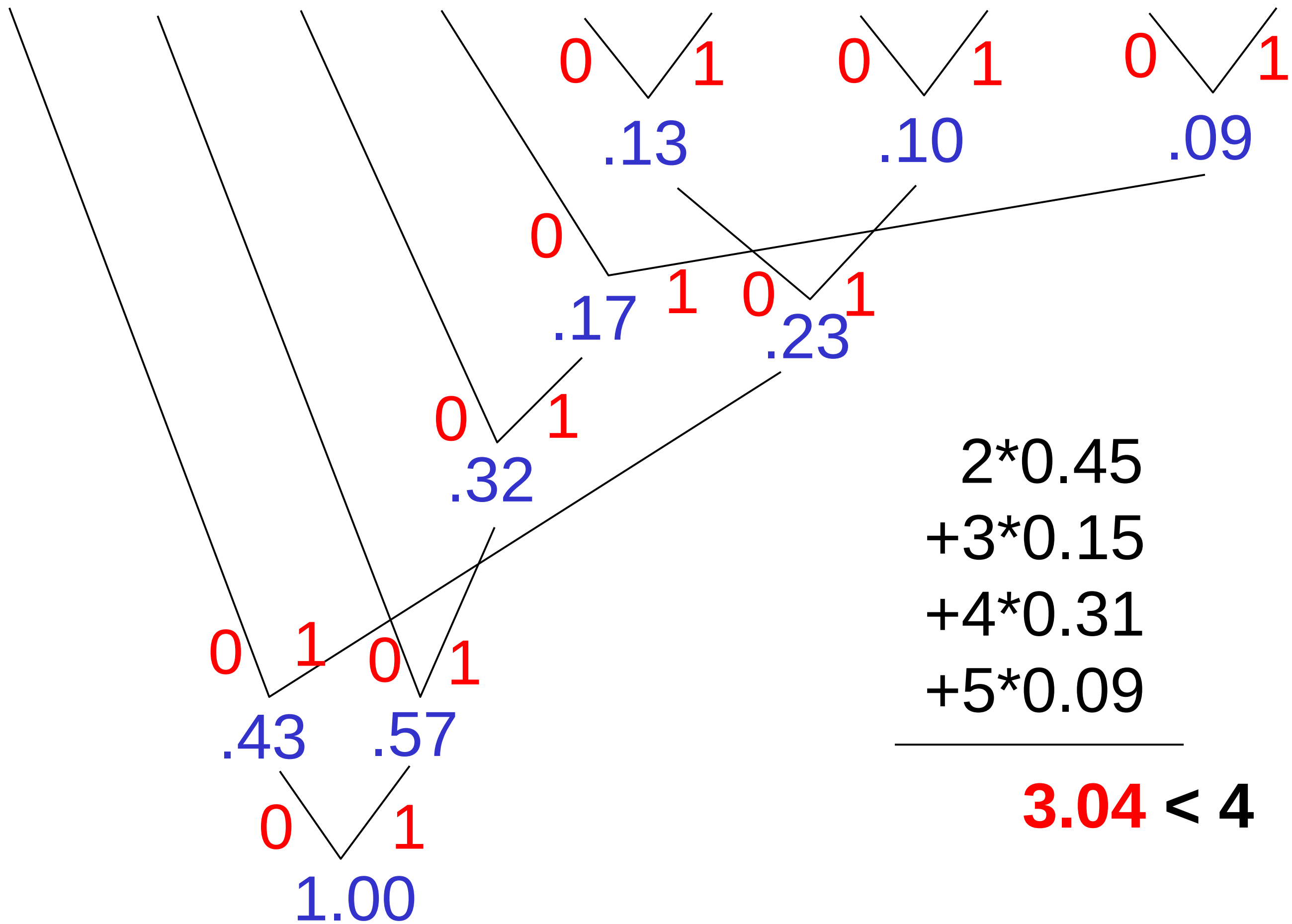
- A Simple data compression
- Examine the frequencies of each digit in the file
- Then, determine the code for each digit w/ a binary tree

✓ Optimal in symbol-by-symbol encoding with given probabilities

✓ cf: an interesting history in relation to Shannon-Fano algorithm (top-down. Frequency sorting 후 frequency 합이 반반에 가깝게 좌우로 분할하는 일을 반복.)

- E.g., Want to handle a file w/ only 10 digits

Digits	0	1	2	3	4	5	6	7	8	9
Freq.	.20	.25	.15	.08	.07	.06	.05	.05	.05	.04



0	00
1	10
2	110
3	1110
4	0100
5	0101
6	0110
7	0111
8	11110
9	11111

2*0.45
+3*0.15
+4*0.31
+5*0.09

3.04 < 4

Treesort

- Inorder traversal을 이용한 sorting 방법
  1. Element들을 모두 binary search tree로 넣는다
  2. Inorder traversal 순서대로 print 한다
- Performance
  - Average case: $\Theta(n \log n)$
  - Worst case: $\Theta(n^2)$

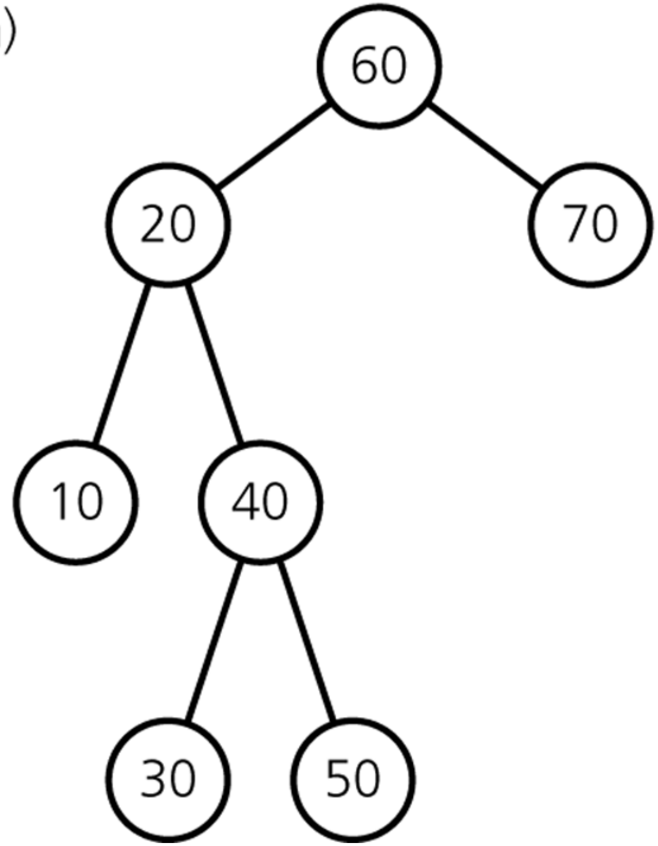
Saving a B.S.T. in a File

- Preserving the original shape
  - Use preorder for saving
- Restoring to a balanced shape
  - Use inorder for saving
  - Restoring

```
recursiveRestore (L) { // L: an array
    if (L is null) { return null; }
    else {
        Set the median item r to be the root;
        r.leftChild = recursiveRestore(the left part of median);
        r.rightChild = recursiveRestore(the right part of median);
        return r ;
    }
}
```

a) A binary search tree ***bst***; b) the sequence of insertions that result in this tree

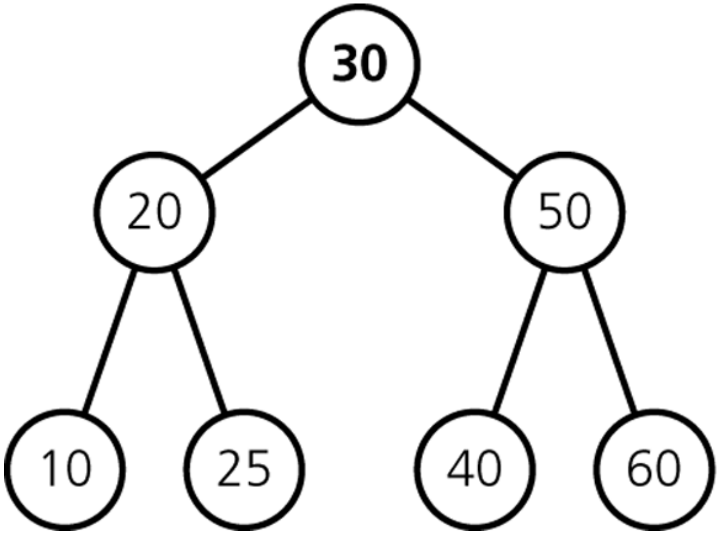
(a)



(b)

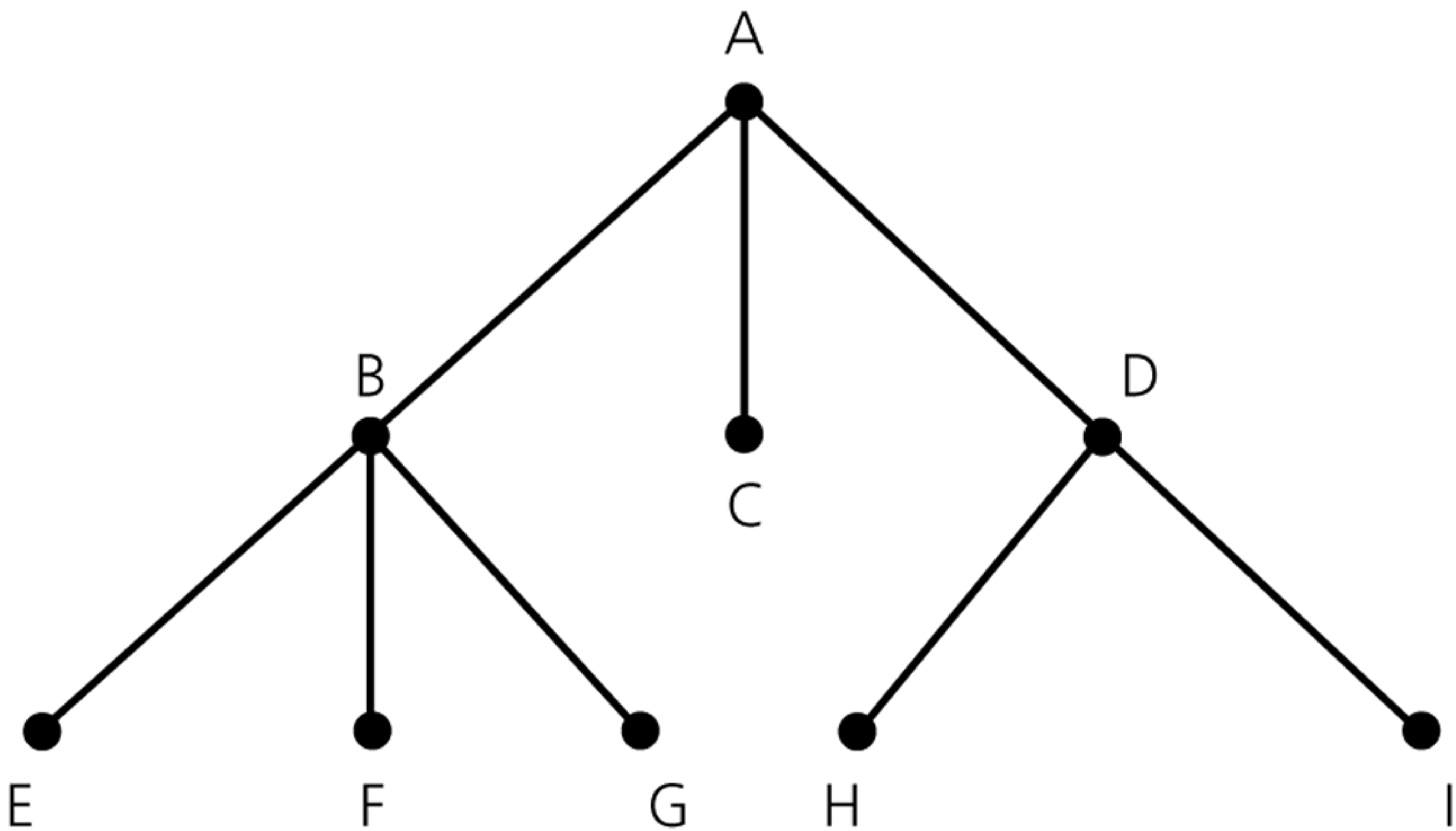
```
bst.insert(60);
bst.insert(20);
bst.insert(10);
bst.insert(40);
bst.insert(30);
bst.insert(50);
bst.insert(70);
```

A full tree saved in a file by using inorder traversal

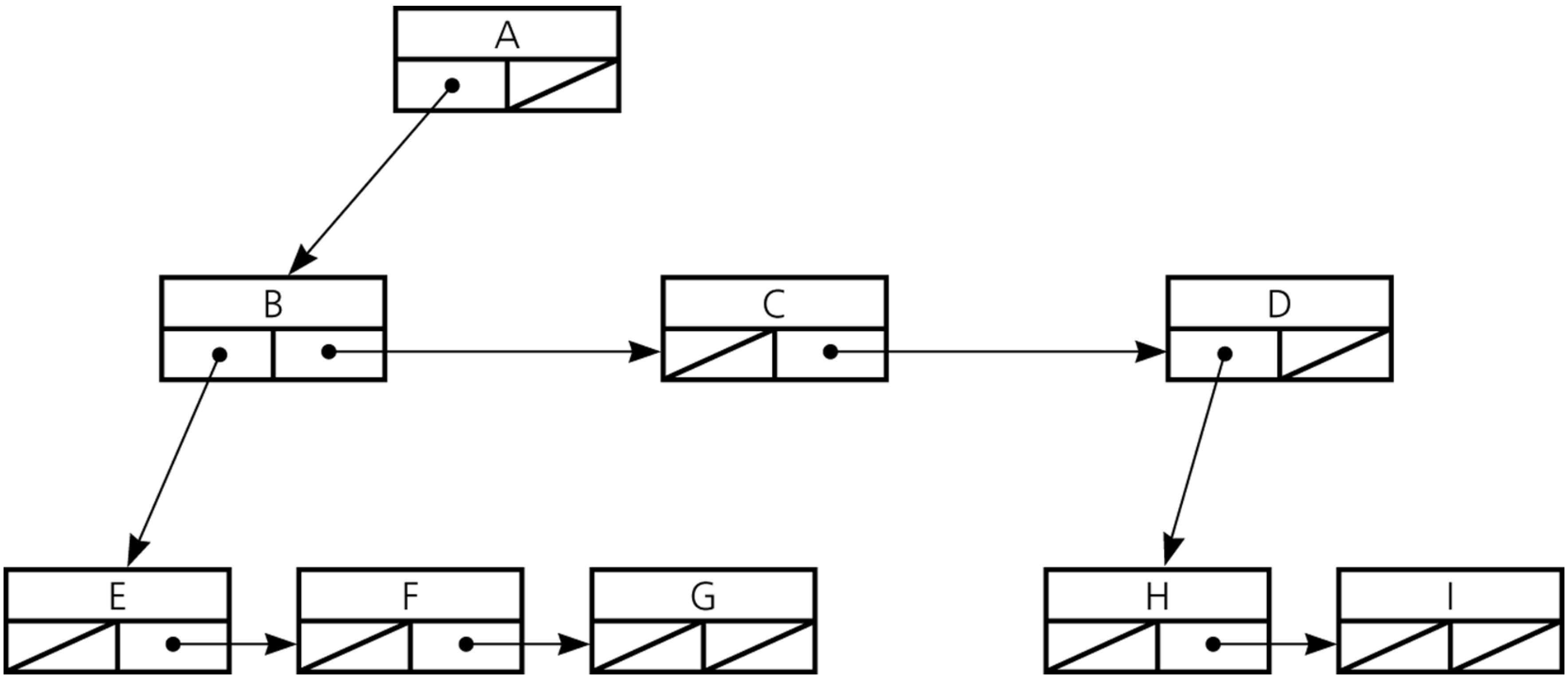


File

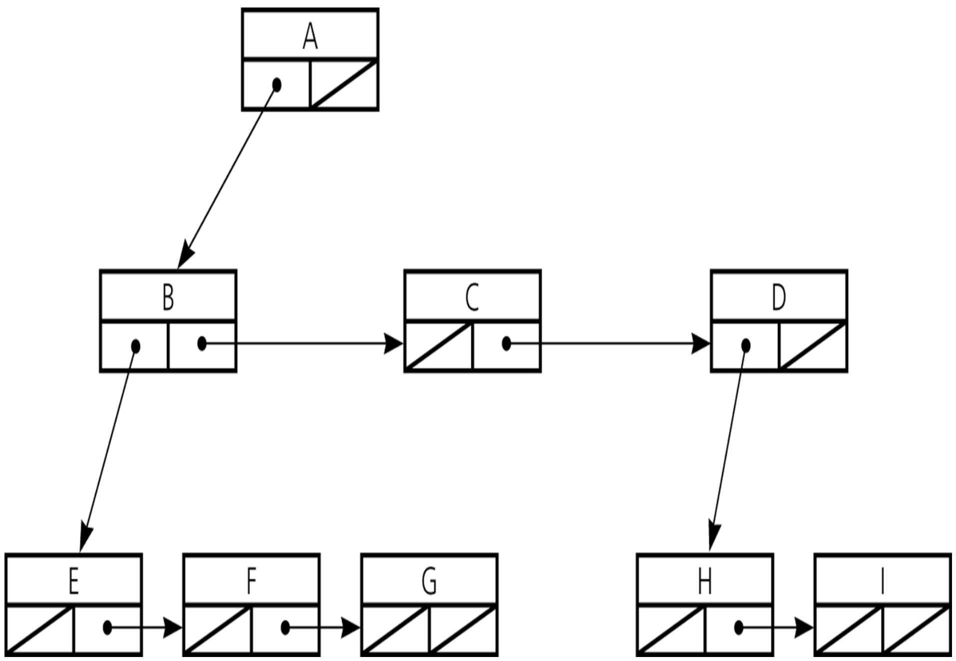
General Trees

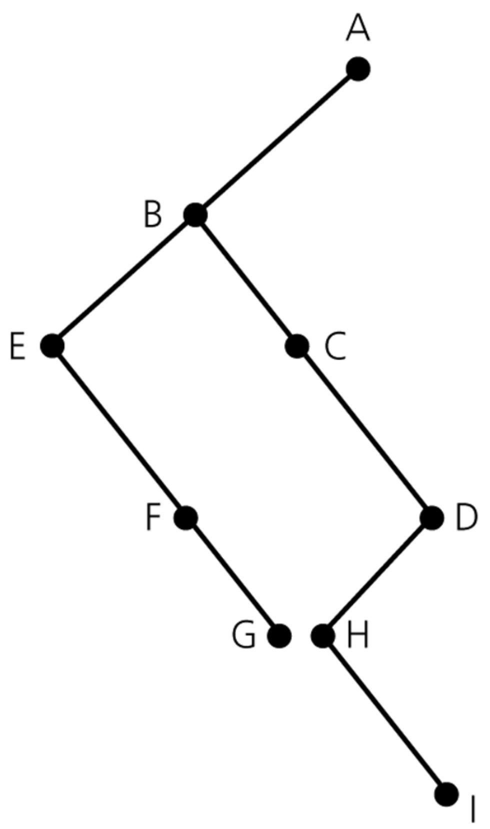


A Reference-Based Implementation of General Trees



A General Tree and Corresponding Binary Tree





n-ary Tree