Linear Algebra

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Lecture Overview

- Elements in linear algebra
- Linear system
- Linear combination, vector equation,
  Four views of matrix multiplication
- Linear independence, span, and subspace
- Linear transformation
- Least squares
- Eigendecomposition
- Singular value decomposition

Scalar, Vector, and Matrix

스칼라 : 하나의 숫자

- Scalar: a single number $s \in \mathbb{R}$ (lower case), e.g., 3.8

↔ set (순서가 정해져 있지 않음)

- Vector: an ordered list of numbers, e.g. $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ (boldface, lower-case), e.g., $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \in \mathbb{R}^3$

≒ array ↳ 순서가 정해져 있는

컬럼벡터 / 로우벡터

행렬 : 두 개의 축을 가지는 숫자들의 집합

- Matrix: a two-dimensional array of numbers, e.g. $A = \begin{bmatrix} 1 & 6 \\ 3 & 4 \\ 5 & 2 \end{bmatrix} \in \mathbb{R}^{3\times 2}$ (capital letter)

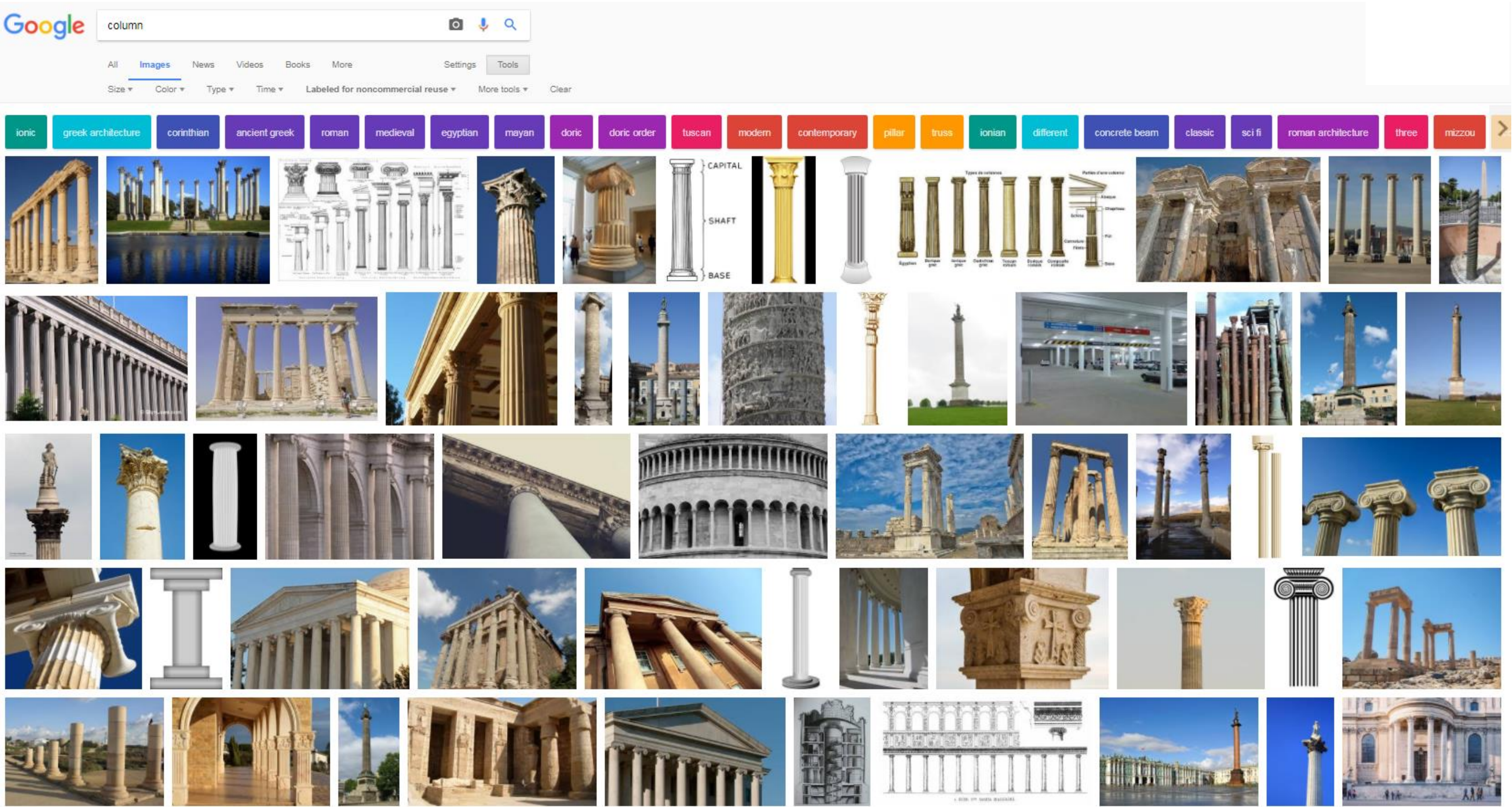
  row

  column

  - Matrix size: 3×2 means 3 rows and 2 columns
  - Row vector: a horizontal vector
  - 디폴트 → Column vector: a vertical vector

Column is Vertical Vector
(Don’t be Confused!)

column : 기둥 (세로)



Column Vector and Row Vector

- A vector of n-dimension is usually a column vector, i.e., a matrix of the size $n \times 1$
  - $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n = \mathbb{R}^{\boxed{n \times 1}}$
- Thus, a row vector is usually written as its transpose, i.e.,
  - $\mathbf{x}^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^T = [x_1 \quad x_2 \quad \cdots \quad x_n] \in \mathbb{R}^{\boxed{1 \times n}}$

ex) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

$(m, n) \longrightarrow (n, m)$

Matrix Notations

- $A \in \mathbb{R}^{n\times n}$: Square matrix (#rows = #columns)
  - e.g., $B = \begin{bmatrix} 1 & 6 \\ 3 & 4 \end{bmatrix}$
- $A \in \mathbb{R}^{m\times n}$: Rectangular matrix (possible: #rows ≠ #columns)
  - e.g., $A = \begin{bmatrix} 1 & 6 \\ 3 & 4 \\ 5 & 2 \end{bmatrix}$

전치

- A^T: Transpose of matrix (mirroring across the main diagonal)
  - e.g., $A^T = \begin{bmatrix} 1 & 3 & 5 \\ 6 & 4 & 2 \end{bmatrix}$
- A_{ij} : (i, j)-th component of A, e.g., $A_{2,1} = 3$ → i행 j열
- $A_{i,:}$: i-th row vector of A, e.g., $A_{2,:} = [3 \quad 4]$ → i행 모든 열
- $A_{:,i}$: i-th column vector of A, e.g., $A_{:,2} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$ → 모든 행 j열

Vector/Matrix Additions and Multiplications

☆ 모두 같은 사이즈 ☆

- $C = A + B$: Element-wise addition, i.e., $C_{ij} = A_{ij} + B_{ij}$
  - A, B, C should have the same size, i.e., $A, B, C \in \mathbb{R}^{m\times n}$
- $c\mathbf{a}$, cA : Scalar multiple of vector/matrix

  모든 원소에 c를 곱해줌
  - e.g., $2\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}6\\4\\2\end{bmatrix}$, $2\begin{bmatrix}1 & 6\\3 & 4\\5 & 2\end{bmatrix} = \begin{bmatrix}2 & 12\\6 & 8\\10 & 4\end{bmatrix}$
- $C = AB$: Matrix-matrix multiplication, i.e., $C_{ij} = \sum_k A_{i,k}B_{k,j}$
  - e.g., $\begin{bmatrix}1 & 6\\3 & 4\\5 & 2\end{bmatrix}\begin{bmatrix}1 & -1\\2 & 1\end{bmatrix} = \begin{bmatrix}13 & 5\\11 & 1\\9 & -3\end{bmatrix}$, $\begin{bmatrix}3 & 2 & 1\end{bmatrix}\begin{bmatrix}1\\3\\5\end{bmatrix} = [14]$, 내적 $\begin{bmatrix}1\\3\\5\end{bmatrix}\begin{bmatrix}1 & 2\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 6\\5 & 10\end{bmatrix}$ 외적

Size: $(3\times \cancel{2})(\cancel{2}\times 2) = 3\times 2$, $(1\times \cancel{3})(\cancel{3}\times 1) = 1\times 1$, $(3\times \cancel{1})(\cancel{1}\times 2) = 3\times 2$

Matrix multiplication is NOT commutative

교환법칙

- $AB \neq BA$: Matrix multiplication is NOT commutative.
- e.g., Given $A \in \mathbb{R}^{2\times 3}$ and $B \in \mathbb{R}^{3\times 5}$, AB is defined, but BA is not even defined.

  $R^{2\times 5}$ 곱셈도 불가!

- What if BA is defined, e.g., $A \in \mathbb{R}^{2\times 3}$ and $B \in \mathbb{R}^{3\times 2}$? Still, the sizes of $AB \in \mathbb{R}^{2\times 2}$ and $BA \in \mathbb{R}^{3\times 3}$ does not match, so $AB \neq BA$.
- What if the sizes of AB and BA match, e.g., $A \in \mathbb{R}^{2\times 2}$ and $B \in \mathbb{R}^{2\times 2}$? Still in this case, generally, $AB \neq BA$.
- E.g., $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$

  $\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$

  $R^{2\times 2}$로 형태는 같으나 값이 다름

Other Properties

- $A(B + C) = AB + AC$: Distributive 분배 법칙
- $A(BC) = (AB)C$: Associative 결합 법칙
- $(AB)^T = B^T A^T$: Property of transpose 전치 특성