Linear Algebra

주재걸
고려대학교 컴퓨터학과

Orthogonal Projection Perspective

수직인 그림자

- Back to the case of invertible $C = A^T A$, consider the orthogonal projection of $\mathbf{b}$ onto Col A as

수선의 발

$$\hat{\mathbf{b}} = f(\mathbf{b}) = A\hat{\mathbf{x}} = A(A^T A)^{-1} A^T \mathbf{b}$$

역행렬이 존재할 때 $\hat{x} = (A^T A)^{-1} A^T b$

Orthogonal and Orthonormal Sets

→ 모든 벡터 수직일 때

- **Definition**: A set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is an **orthogonal set** if each pair of distinct vectors from the set is orthogonal That is, if $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$.

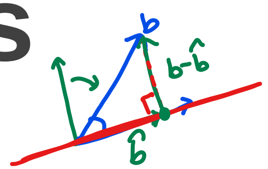
→ orthogonal set의 벡터 길이만 1

- **Definition**: A set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is an **orthonormal set** if it is an orthogonal set of unit vectors.

- Is an orthogonal (or orthonormal) set also a linearly independent set? What about its converse?

자동적으로 선형 독립

Orthogonal and Orthonormal Basis



- Consider basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of a p-dimensional subspace W in $\mathbb{R}^n$.
- Can we make it as an orthogonal (or orthonormal) basis?
  - Yes, it can be done by Gram–Schmidt process. → QR factorization.
- Given the orthogonal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ of W, let's compute the orthogonal projection of $\mathbf{y} \in \mathbb{R}^n$ onto W.

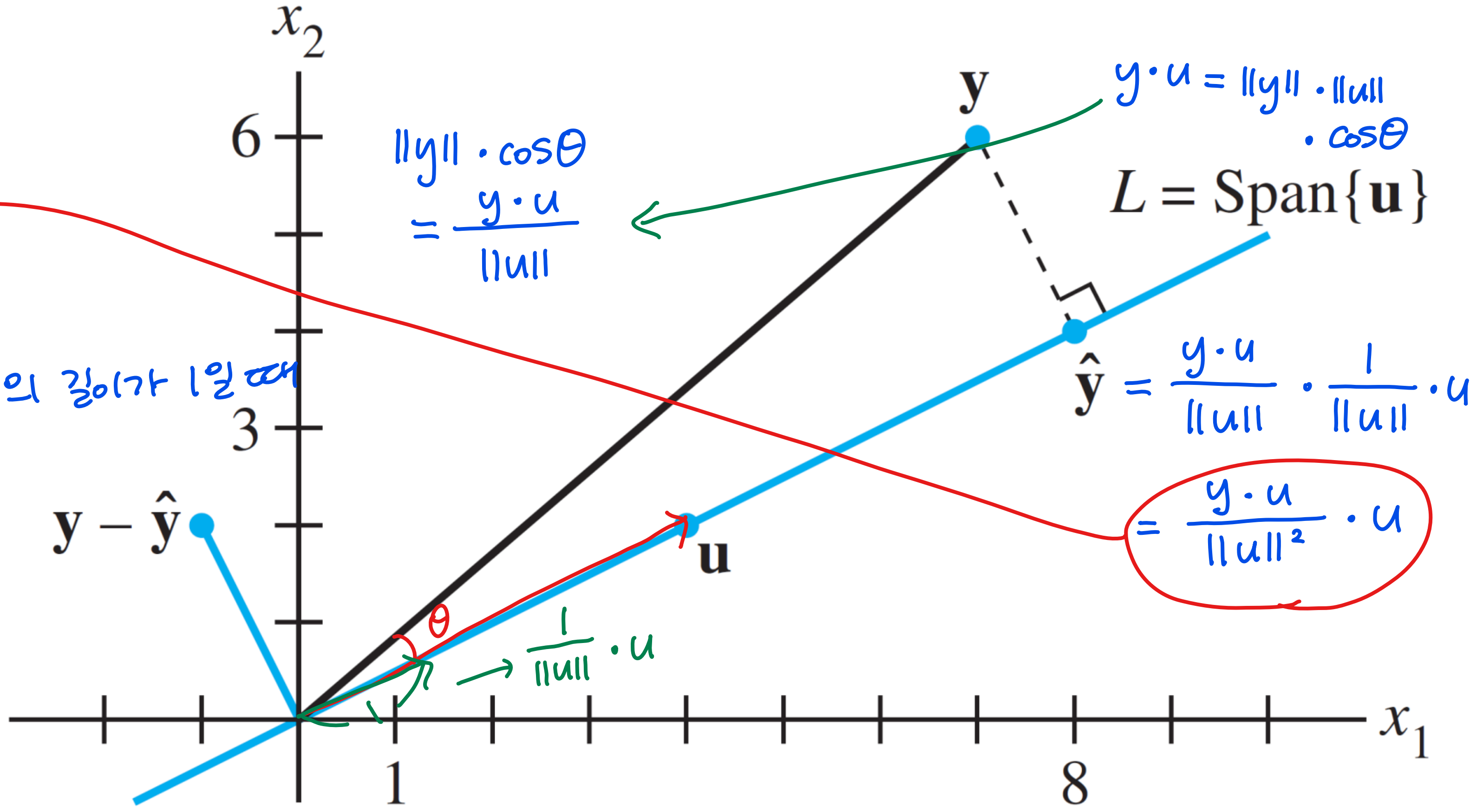
Orthogonal Projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto Line

- Consider the orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto one-dimensional subspace L.

- $\hat{\mathbf{y}} = \text{proj}_L \mathbf{y} = \dfrac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$

- If $\mathbf{u}$ is a unit vector, → u의 길이가 1일 때

  $\hat{\mathbf{y}} = \text{proj}_L \mathbf{y} = (\mathbf{y}\cdot\mathbf{u})\mathbf{u}$



Orthogonal Projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto Plane

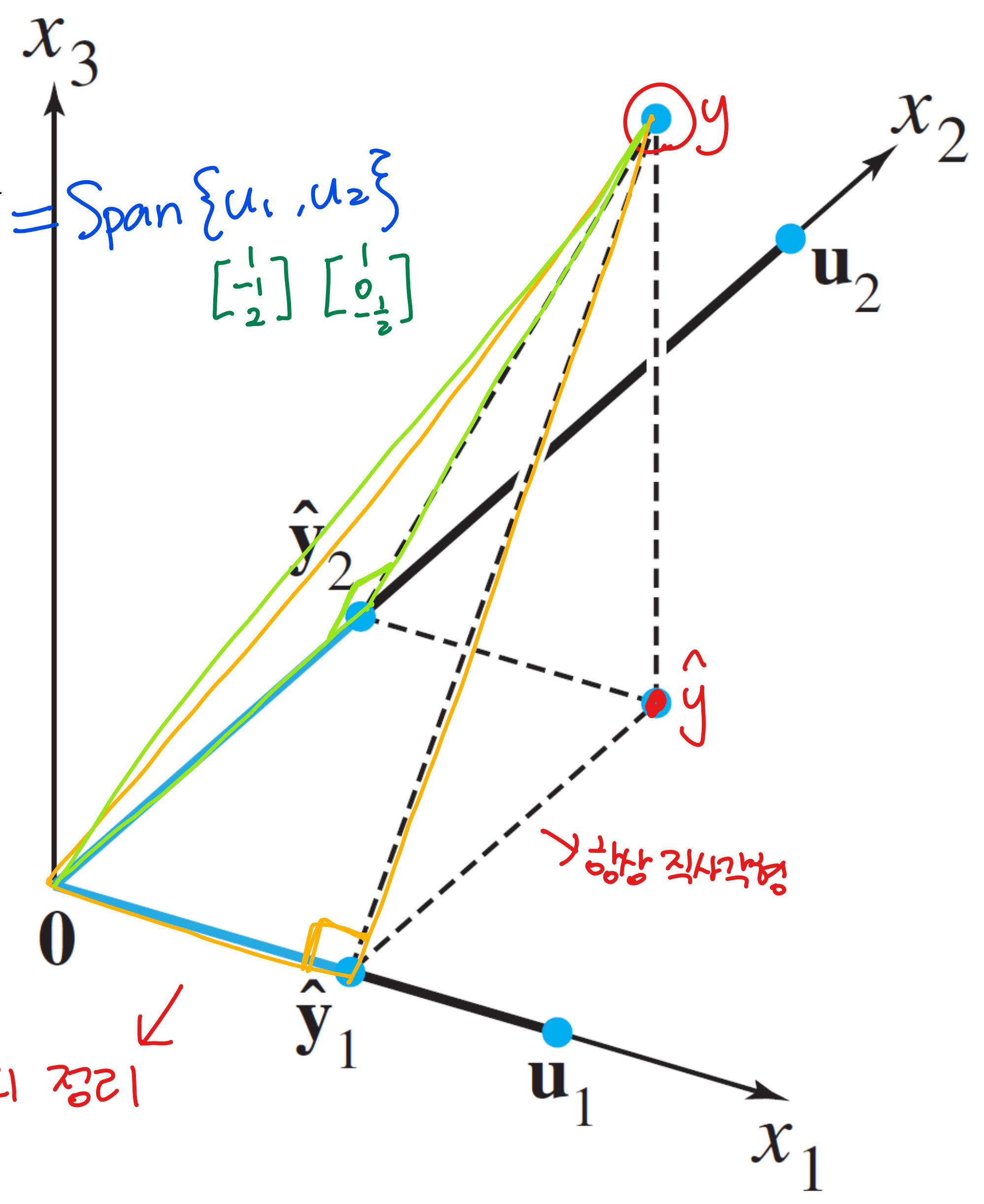
- Consider the orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto two-dimensional subspace W = Span{u₁ ,u₂}

  $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 0 \\ -\frac{1}{2} \end{bmatrix}$

- $\hat{\mathbf{y}} = \text{proj}_L \mathbf{y} = \underline{\dfrac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \dfrac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2}$

- If $\mathbf{u}_1$ and $\mathbf{u}_2$ are unit vectors,
  $\hat{\mathbf{y}} = \text{proj}_L \mathbf{y} = (\mathbf{y}\cdot\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y}\cdot\mathbf{u}_2)\mathbf{u}_2$

- Projection is done independently on each orthogonal basis vector.



삼수선의 정리

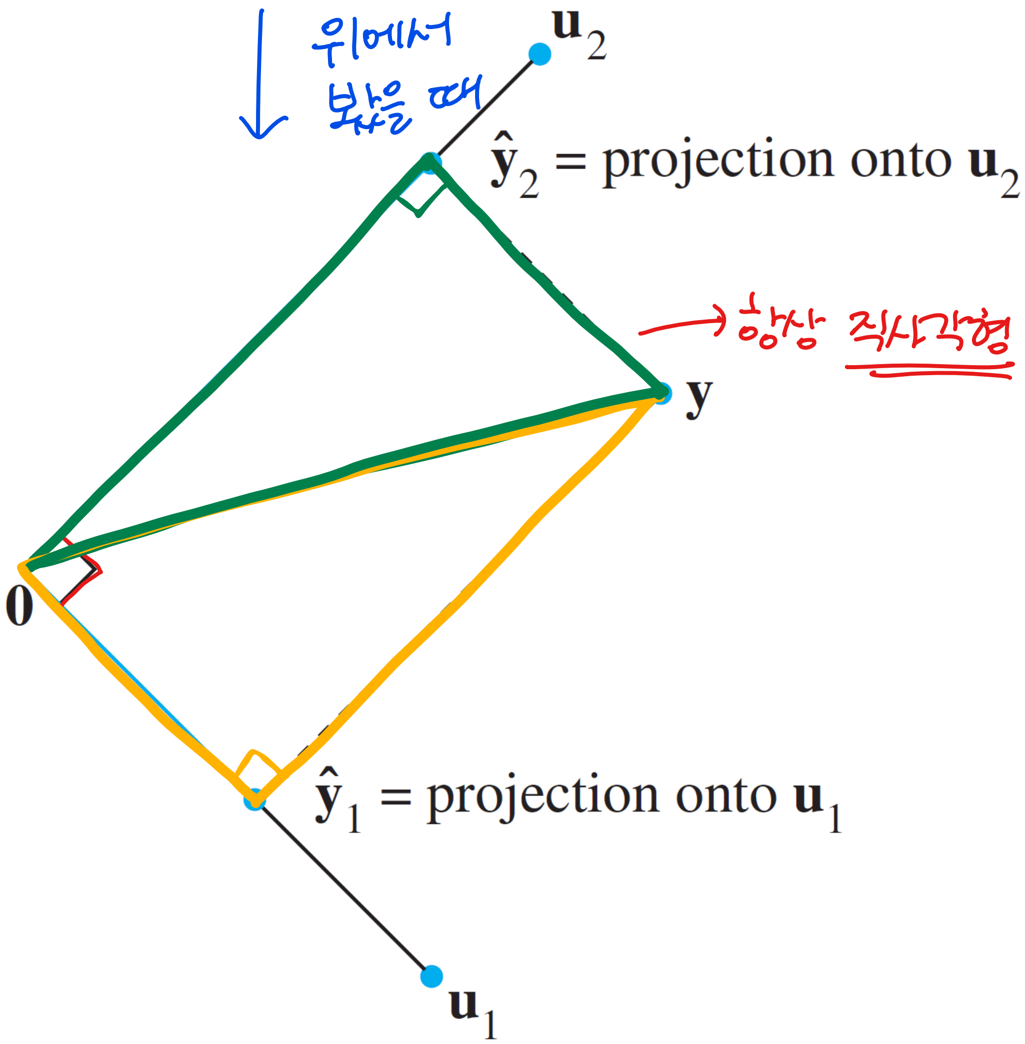
Orthogonal Projection when $\mathbf{y} \in W$

- Consider the orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto two-dimensional subspace W, where $\boxed{\mathbf{y} \in W}$

- $\hat{\mathbf{y}} = \text{proj}_L \mathbf{y} = \mathbf{y} = \dfrac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \dfrac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2$

- If $\mathbf{u}_1$ and $\mathbf{u}_2$ are unit vectors,
  $\hat{\mathbf{y}} = \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2)\mathbf{u}_2$

- The solution is the same as before. Why?

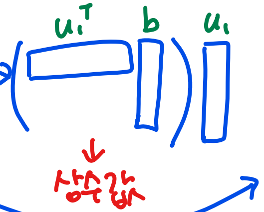


Transformation: Orthogonal Projection

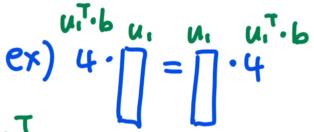
- Consider a transformation of orthogonal projection $\hat{\mathbf{b}}$ of $\mathbf{b}$, given orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ of a subspace W:

$$\hat{\mathbf{b}} = f(\mathbf{b}) = (\mathbf{b}\cdot\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{b}\cdot\mathbf{u}_2)\mathbf{u}_2 \rightarrow \frac{b\cdot u_1}{u_1\cdot u_1}u_1 + \frac{b\cdot u_2}{u_2\cdot u_2}u_2$$

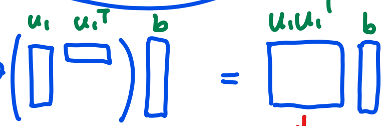
$$= (\mathbf{u}_1^T\mathbf{b})\mathbf{u}_1 + (\mathbf{u}_2^T\mathbf{b})\mathbf{u}_2$$



$$= \mathbf{u}_1(\mathbf{u}_1^T\mathbf{b}) + \mathbf{u}_2(\mathbf{u}_2^T\mathbf{b})$$



$$= (\mathbf{u}_1\mathbf{u}_1^T)\mathbf{b} + (\mathbf{u}_2\mathbf{u}_2^T)\mathbf{b}$$



$$= (\mathbf{u}_1\mathbf{u}_1^T + \mathbf{u}_2\mathbf{u}_2^T)\mathbf{b}$$

$$= \underbrace{[\mathbf{u}_1 \quad \mathbf{u}_2]}_{U}\underbrace{\begin{bmatrix}\mathbf{u}_1^T \\ \mathbf{u}_2^T\end{bmatrix}}_{U^T}\mathbf{b} = UU^T\mathbf{b}$$

$\Rightarrow$ linear transformation!

Orthogonal Projection Perspective

→ u₁과 u₂가 수직 관계

- Let's verify the following, when $A = U = [\mathbf{u}_1 \quad \mathbf{u}_2]$ has orthonormal columns:
  Back to the case of invertible $C = A^T A$, consider the orthogonal projection of $\mathbf{b}$ onto Col A as
$$\hat{\mathbf{b}} = A\hat{\mathbf{x}} = A(A^T A)^{-1} A^T \mathbf{b} = f(\mathbf{b})$$

$$= \begin{bmatrix} u_1^T u_1 & u_1^T u_2 \\ u_2^T u_1 & u_2^T u_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ → 길이가 1인 orthonormal 이므로

- $C = A^T A = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \end{bmatrix} [\mathbf{u}_1 \quad \mathbf{u}_2] = I$. Thus,
$$\hat{\mathbf{b}} = A\hat{\mathbf{x}} = A(A^T A)^{-1} A^T \mathbf{b} = A(I)^{\cancel{-1}} A^T \mathbf{b} = A A^T \mathbf{b} = U U^T \mathbf{b}$$