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# Linear Algebra

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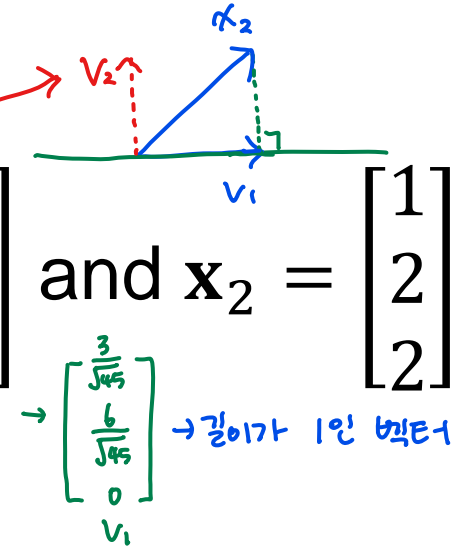


# Gram-Schmidt Orthogonalization

\* 수직이 아닌 벡터들은 어느정도 서로 영향을 줄 수 밖에 없음.

⇒ 서로 영향을 주지 않는 수직인 벡터로 바꿈!  
↳ orthogonal

- **Example 1:** Let  $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$ , where  $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$  and  $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ .  
Construct an orthogonal basis  $\{\mathbf{v}_1, \mathbf{v}_2\}$  for  $W$ .



- **Solution:** Let  $\mathbf{v}_1 = \mathbf{x}_1$ . Next, Let  $\mathbf{v}_2$  the component of  $\mathbf{x}_2$  orthogonal to  $\mathbf{x}_1$ , i.e.,

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{x}_1}{\mathbf{x}_1 \cdot \mathbf{x}_1} \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{15}{45} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

- The set  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is an orthogonal basis for  $W$ .



# Gram-Schmidt Orthogonalization

- **Example 2:** Let  $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ ,  $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ , and  $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ . Then  $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$  is clearly linearly independent and thus is a basis for a subspace  $W$  of  $\mathbb{R}^4$ . Construct an orthogonal basis for  $W$ .



# Gram-Schmidt Orthogonalization

- **Solution:**
- **Step 1.** Let  $\mathbf{v}_1 = \mathbf{x}_1$  and  $W_1 = \text{Span}\{\mathbf{x}_1\} = \text{Span}\{\mathbf{v}_1\}$ .
- **Step 2.** Let  $\mathbf{v}_2$  be the vector produced by subtracting from  $\mathbf{x}_2$  its projection onto the subspace  $W_1$ . That is, let

$$\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{W_1} \mathbf{x}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

- $\mathbf{v}_2$  is the component of  $\mathbf{x}_2$  orthogonal to  $\mathbf{x}_1$ , and  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is an orthogonal basis for the subspace  $W_2$  spanned by  $\mathbf{x}_1$  and  $\mathbf{x}_2$ .



# Gram-Schmidt Orthogonalization

- **Step 2' (optional).** If appropriate, scale  $\mathbf{v}_2$  to simplify later computations, e.g.,

$$\mathbf{v}_2 = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \rightarrow \mathbf{v}'_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

# Gram-Schmidt Orthogonalization

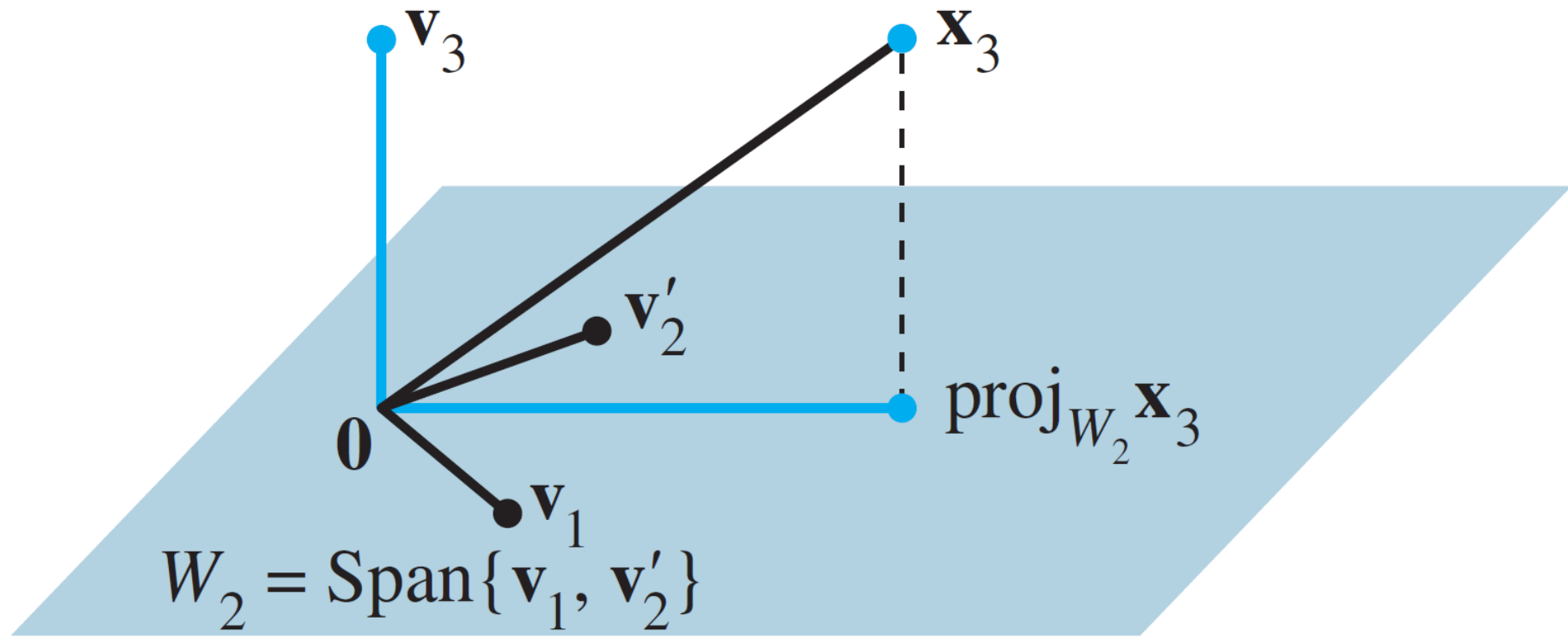
- **Step 3.** Let  $\mathbf{v}_3$  be the vector produced by subtracting from  $\mathbf{x}_3$  its projection onto the subspace  $W_2$ . Use the orthogonal basis  $\{\mathbf{v}_1, \mathbf{v}_2'\}$  to compute this projection onto  $W_2$ :

$$\text{proj}_{W_2} \mathbf{x}_3 = \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_3 \cdot \mathbf{v}_1} \mathbf{v}_1 + \frac{\mathbf{x}_3 \cdot \mathbf{v}_2'}{\mathbf{v}_3 \cdot \mathbf{v}_2'} \mathbf{v}_2' = \begin{bmatrix} 0 \\ 2/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

- Then  $\mathbf{v}_3$  is the component of  $\mathbf{x}_3$  orthogonal to  $W_2$ , namely,

$$\mathbf{v}_3 = \mathbf{x}_3 - \text{proj}_{W_2} \mathbf{x}_3 = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

# Gram-Schmidt Orthogonalization



**FIGURE 2** The construction of  $\mathbf{v}_3$  from  $\mathbf{x}_3$  and  $W_2$ .

Figure from Lay Ch6.4



# QR Factorization

- If  $A$  is an  $m \times n$  matrix with linearly independent columns, then  $A$  can be factored as  $A = QR$ , where  $Q$  is an  $m \times n$  matrix whose columns form an orthonormal basis for  $\text{Col } A$  and  $R$  is an  $n \times n$  upper triangular invertible matrix with positive entries on its diagonal.





# Computing QR Factorization

- **Step 1 (Construction of  $Q$ ):** The columns of  $A$  form a basis for  $\text{Col } A$  since they are linearly independent. Let these columns be  $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ . Then, we can construct the orthonormal basis  $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$  for  $\text{Col } A$  by the Gram-Schmidt process described by Theorem 11. Using this basis, we can construct  $Q$  as

$$Q = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n]$$

# Computing QR Factorization

- **Step 2 (Construction of  $R$ ):** From (1) in Theorem 11, for  $k = 1, \dots, n$ ,  $\mathbf{x}_k$  is in  $\text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\} = \text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ . Therefore, there exist constants  $r_{1k}, \dots, r_{kk}$  such that

$$\mathbf{x}_k = r_{1k}\mathbf{u}_1 + \dots + r_{kk}\mathbf{u}_k + 0 \cdot \mathbf{u}_{k+1} + \dots + 0 \cdot \mathbf{u}_n$$

- We can always make  $r_{kk} \geq 0$  because if  $r_{kk} < 0$ , then we can multiply both  $r_{kk}$  and  $\mathbf{u}_k$  by -1. Using this linear combination representation, we can construct  $\mathbf{r}_k$ , the  $k$ -th column of  $R$ , as

$$\mathbf{r}_k = \begin{bmatrix} r_{1k} \\ \vdots \\ r_{kk} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$



# Computing QR Factorization

- That is,  $\mathbf{x}_k = Q\mathbf{r}_k$  for  $k = 1, \dots, n$ . Let  $R = [\mathbf{r}_1 \ \cdots \ \mathbf{r}_n]$ . Then,  
$$A = [\mathbf{x}_1 \ \cdots \ \mathbf{x}_n] = [Q\mathbf{r}_1 \ \cdots \ Q\mathbf{r}_n] = QR$$
- The fact that  $R$  is invertible follows easily from the fact that the columns of  $A$  are linearly independent (Exercise 19). Since  $R$  is clearly upper triangular (from the previous slide) and invertible, the diagonal entries  $r_{kk}$ 's should be nonzero. By combining this with the fact that  $r_{kk} \geq 0$ ,  $r_{kk}$ 's must be positive.

# Example: QR Factorization

- **Example 4:** Find a QR factorization of  $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ .
- **Solution:** Let  $A = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_3]$ . We first obtain  $\mathbf{v}_1 = \mathbf{x}_1$  and its normalized vector is  $\mathbf{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$ .
- Thus,  $\mathbf{x}_1 = 2\mathbf{u}_1$ , which gives us  $\mathbf{r}_{11} = 2$ , i.e.,  $\mathbf{r}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ .

# Example: QR Factorization

- Next, we obtain  $\mathbf{v}_3$  as  $\mathbf{v}_3 = \mathbf{x}_3 - \text{proj}_{W_2} \mathbf{x}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 -$

$$\frac{\mathbf{x}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} - \frac{2}{\sqrt{12}} \begin{bmatrix} -3/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} \text{ and its}$$

normalized vector  $\mathbf{u}_2$  as  $\mathbf{u}_2 = \begin{bmatrix} 0 \\ -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$ .

- Thus,  $\mathbf{x}_3 = 1\mathbf{u}_1 + \frac{2}{\sqrt{12}}\mathbf{u}_2 + \frac{2}{\sqrt{6}}\mathbf{u}_3$ , i.e.,  $\mathbf{r}_3 = \begin{bmatrix} 1 \\ 2/\sqrt{12} \\ 2/\sqrt{6} \end{bmatrix}$ .

# Example: QR Factorization

matrix A의 orthonormal 화시킨 matrix

• In conclusion,  $Q = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] = \begin{bmatrix} 1/2 & -3/\sqrt{12} & 0 \\ 1/2 & 1/\sqrt{12} & -2/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \end{bmatrix}$

and  $R = [\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{r}_3] = \begin{bmatrix} 2 & -3/2 & 1 \\ 0 & -3/\sqrt{12} & 2/\sqrt{12} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}.$

$\rightarrow A = QR$