# Linear Algebra

주재걸
고려대학교 컴퓨터학과

# Lecture Overview

- Elements in linear algebra
- Linear system
- Linear combination, vector equation,
  Four views of matrix multiplication
- Linear independence, span and subspace
- Linear transformation
- Least squares
- Eigendecomposition
- Singular value decomposition

# Singular Value Decomposition (SVD)

직사각 행렬을 대상으로

- Given a rectangular matrix $A \in \mathbb{R}^{m \times n}$, its singular value decomposition is written as

diagonal matrix

$$A = U\Sigma V^T$$

orthonomal 컬럼들로 이루어짐

where

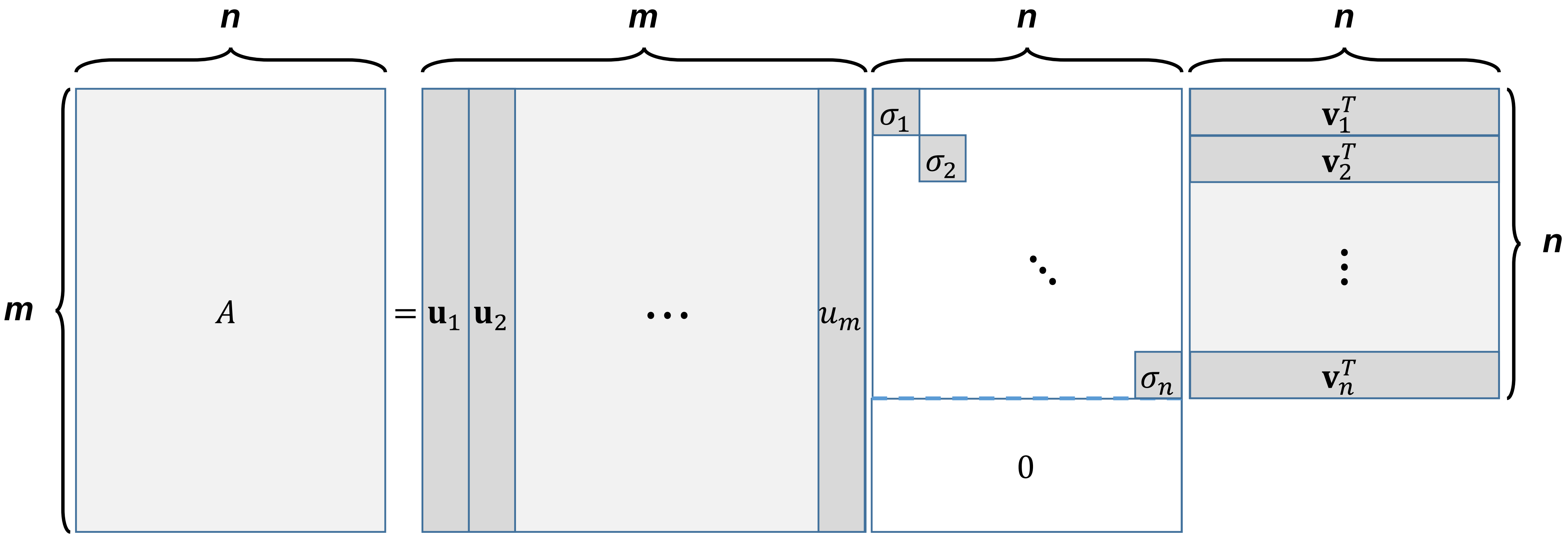
- $U \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{n \times n}$: matrices with orthonormal columns, providing an orthonormal basis of Col $A$ and Row $A$, respectively

- $\Sigma \in \mathbb{R}^{m \times n}$: a diagonal matrix whose entries are in a decreasing order, i.e., $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min(m,n)}$

# Basic Form of SVD

- Given a matrix $A \in \mathbb{R}^{m\times n}$ where $m > n$, SVD gives

$$A = U\Sigma V^T$$




$A$ (m×n) $=$ $U$ (m×m) $\times$ $\Sigma$ (m×n) $\times$ $V^T$ (n×n)

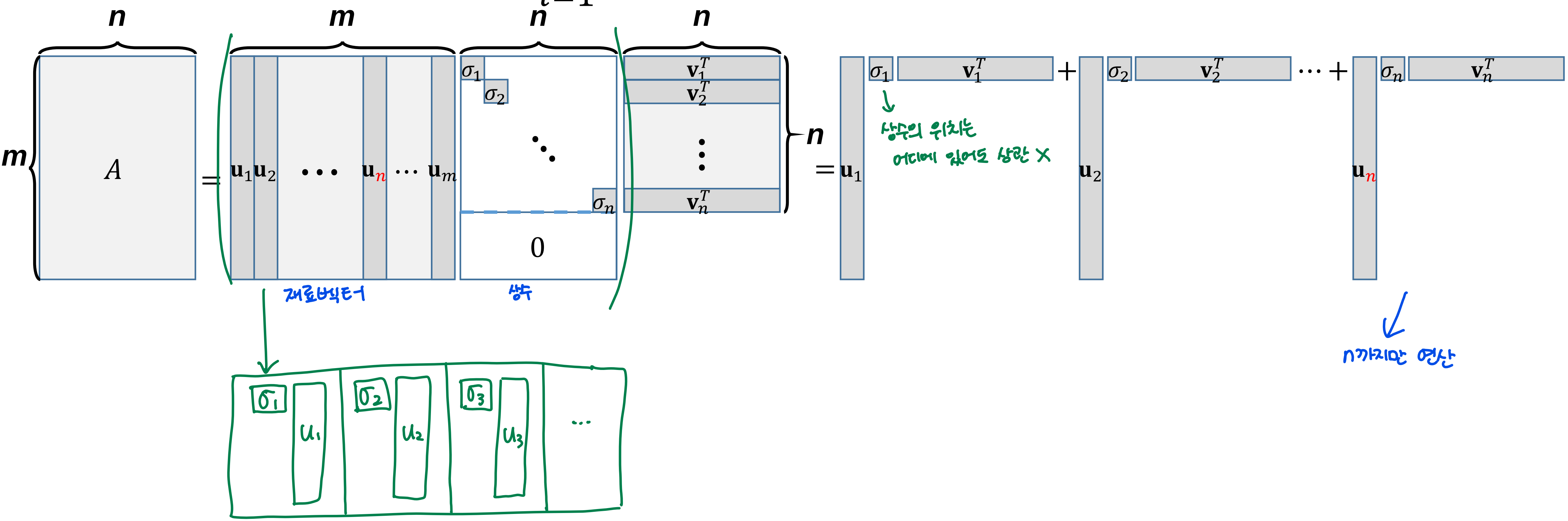
↳ (i,i)만 값을 갖고 나머진 모두 0

# SVD as Sum of Outer Products

(Rank one)

- $A$ can also be represented as the sum of outer products

$$A = U\Sigma V^T = \sum_{i=1}^{n} \sigma_i \mathbf{u}_i \mathbf{v}_i^T, \qquad \text{where } \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$$

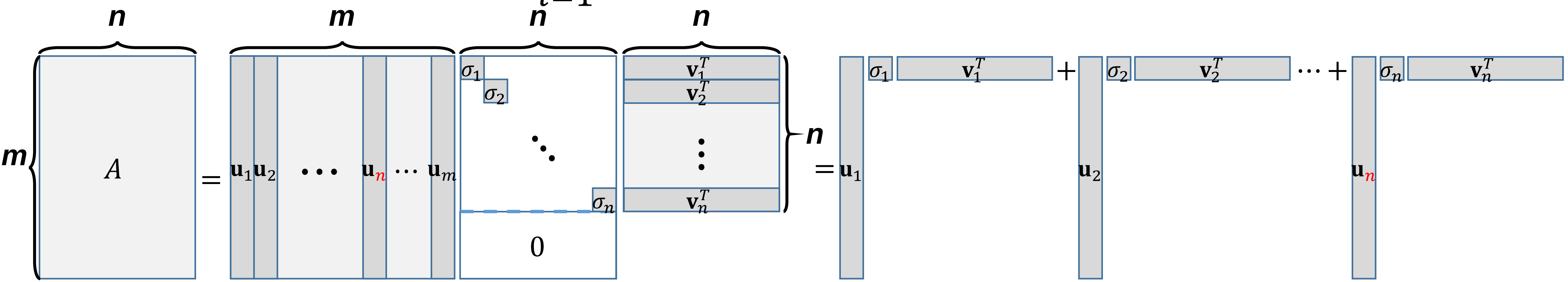


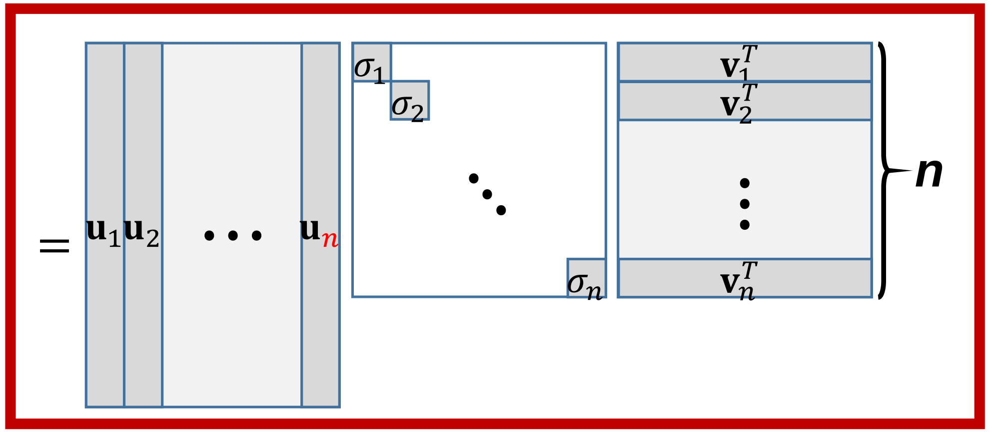
# Reduced Form of SVD

$A^T = V\Sigma^T U^T$

- $A$ can also be represented as the sum of outer products

$$\boxed{A = U\Sigma V^T} = \sum_{i=1}^{n} \sigma_i \mathbf{u}_i \mathbf{v}_i^T, \qquad \text{where } \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$$



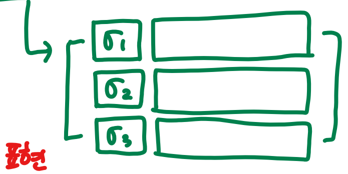

다시 matrix로 표현하면

(QR Factorization)

A = (m×n) (n×n) (n×m) 으로 구성됨

basis vector

(m×n)의 선형 결합으로 표현



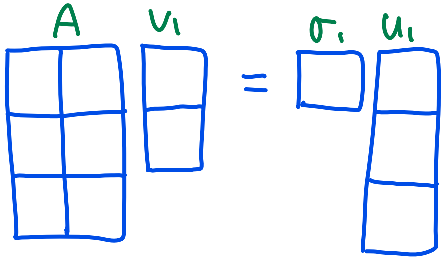
# Another Perspective of SVD

- We can easily find two orthonormal basis sets, $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ for Col $A$ and $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ for Row $A$, by using, say, Gram–Schmidt orthogonalization.
- Are these unique orthonormal basis sets?

- No. Then, can we jointly find them such that

***n***

$$A\mathbf{v}_i = \sigma_i \mathbf{u}_i, \qquad \forall\, i = 1, \dots, n$$



# Another Perspective of SVD

$A = U\Sigma V^T$

- Let us denote $U = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n] \in \mathbb{R}^{m\times n}$, $V = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n] \in \mathbb{R}^{n\times n}$, and $\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_n \end{bmatrix} \in \mathbb{R}^{n\times n}$
- Consider $AV = A[\boxed{\mathbf{v}_1} \quad \boxed{\mathbf{v}_2} \quad \cdots \quad \boxed{\mathbf{v}_n}] = [A\mathbf{v}_1 \quad A\mathbf{v}_2 \quad \cdots \quad A\mathbf{v}_n]$ and

$$U\Sigma = [\boxed{\mathbf{u}_1} \quad \boxed{\mathbf{u}_2} \quad \cdots \quad \boxed{\mathbf{u}_n}] \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_n \end{bmatrix}$$

$$= [\sigma_1\mathbf{u}_1 \quad \sigma_2\mathbf{u}_2 \quad \cdots \quad \sigma_n\mathbf{u}_n]$$

역행렬 존재

- $AV = U\Sigma \Leftrightarrow [A\mathbf{v}_1 \quad A\mathbf{v}_2 \quad \cdots \quad A\mathbf{v}_n] = [\sigma_1\mathbf{u}_1 \quad \sigma_2\mathbf{u}_2 \quad \cdots \quad \sigma_n\mathbf{u}_n]$
- $V^{-1} = V^T$ since $V \in \mathbb{R}^{n\times n}$ has orthonormal columns.
- Thus $A\cancel{V V^T} = U\Sigma V^T \Leftrightarrow A = U\Sigma V^T$