Linear Algebra

주재걸
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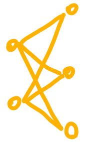
manifold

ONTO and ONE-TO-ONE

공역 = 치역

전사

$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\2\\3\end{bmatrix}$, $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}4\\5\\6\end{bmatrix}$

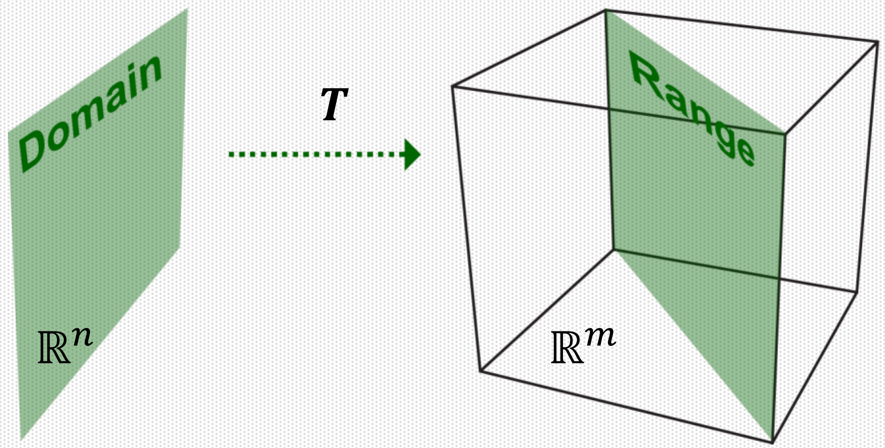


형태 → $\begin{bmatrix}1&4\\2&5\\3&6\end{bmatrix}$의 span이 치역이므로 onto가 안됨!!

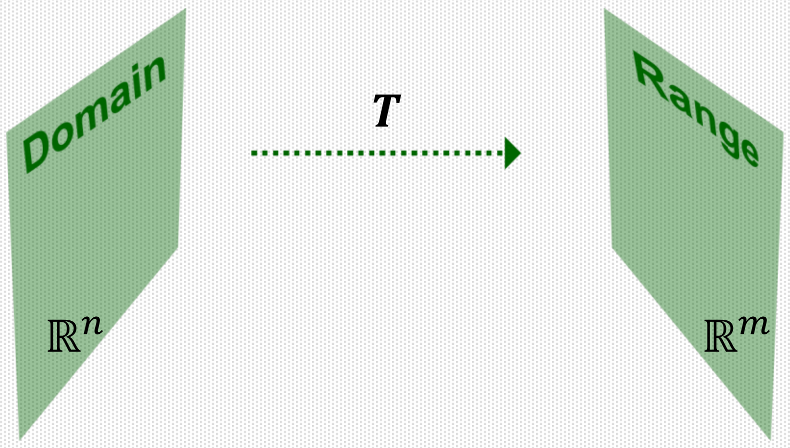
→ ex) GAN, decoding, ...

공역에서 어떤 원소라도 하나 이상의 정의역을 갖고 있어야 한다.

- **Definition:** A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto $\mathbb{R}^m$ if each $\mathbf{b} \in \mathbb{R}^m$ is the image of at least one $\mathbf{x} \in \mathbb{R}^n$. $n \geq m$ ☆
  That is, the range is equal to the co-domain.



T is NOT onto $\mathbb{R}^m$



T is onto $\mathbb{R}^m$

$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\2\end{bmatrix}$, $T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}3\\4\end{bmatrix}$, $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}5\\8\end{bmatrix}$

$T: y=\begin{bmatrix}1&3&5\\2&4&8\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

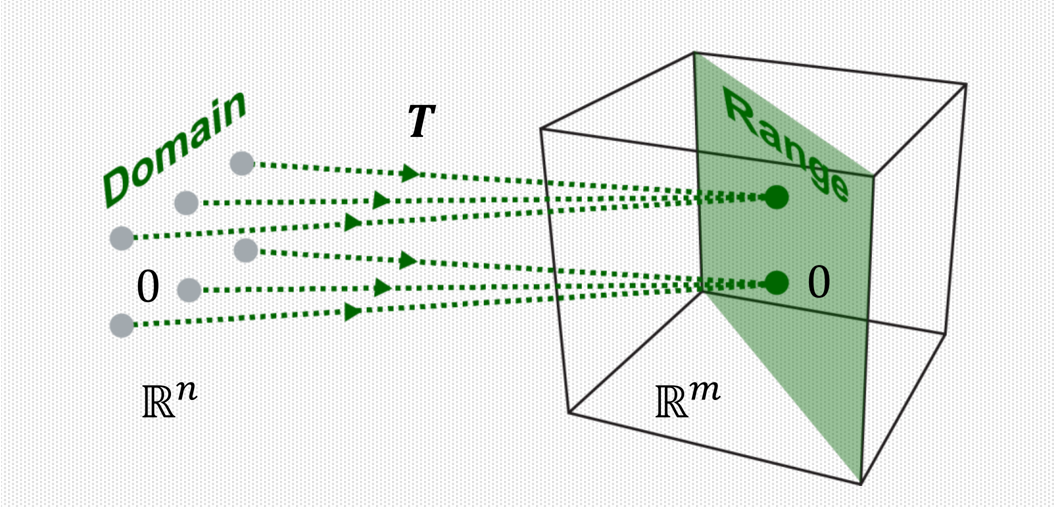
ONTO and ONE-TO-ONE

↓ 선형 독립

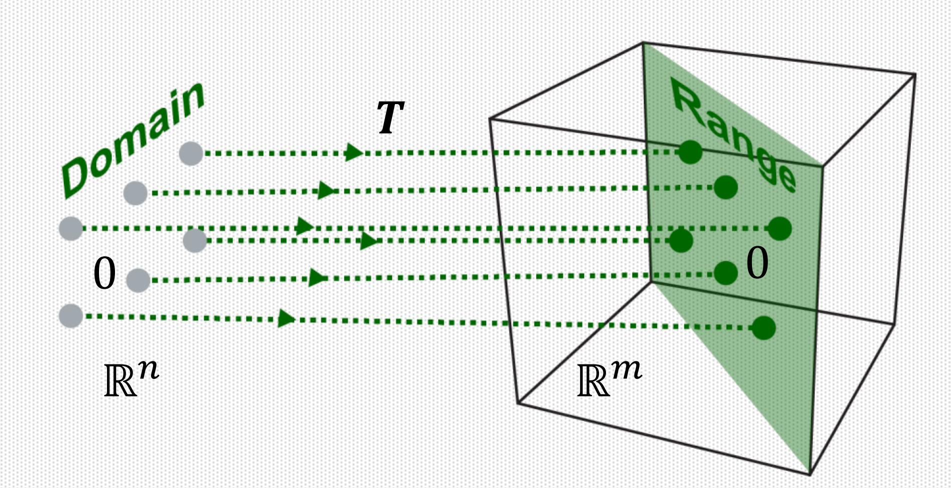
1대1 함수 (공역 ≠ 치역일 수 有
반대!! 1개의 치역엔 1개의 정의역

- **Definition:** A mapping $T:\mathbb{R}^n \to \mathbb{R}^m$ is said to be one-to-one if each $\mathbf{b} \in \mathbb{R}^m$ is the image of at most one $\mathbf{x} \in \mathbb{R}^n$.
  That is, each output vector in the range is mapped by only one input vector, no more than that.

$n \le m$ ☆



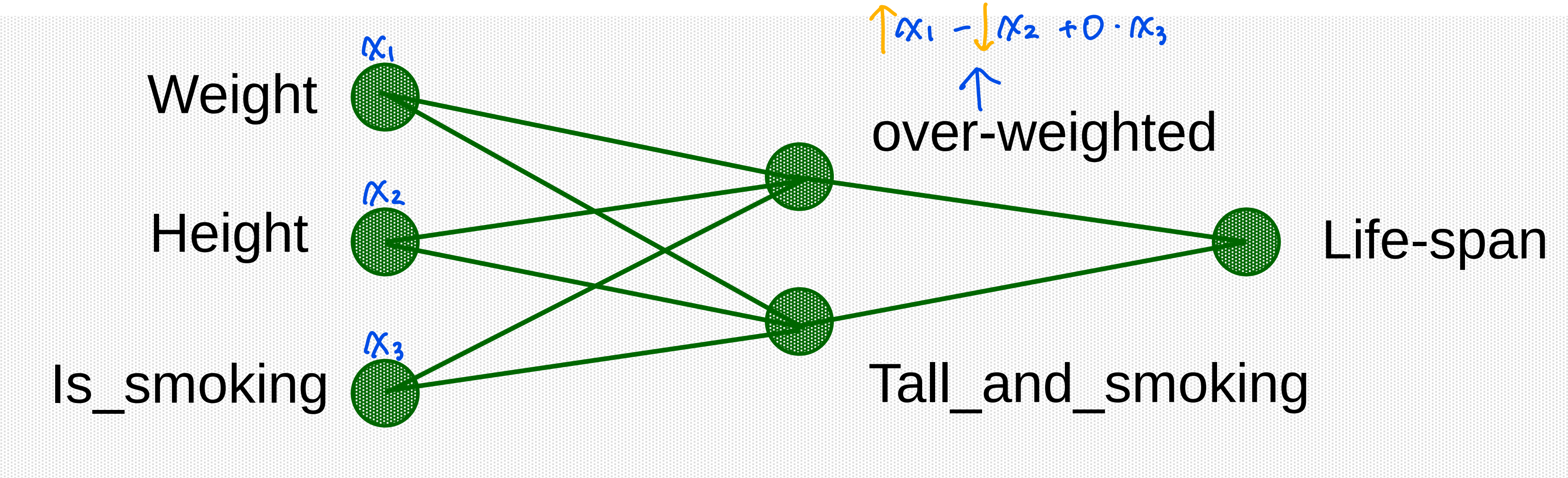
T is NOT one-to-one



T is one-to-one

Neural Network Example

- Fully-connected layers

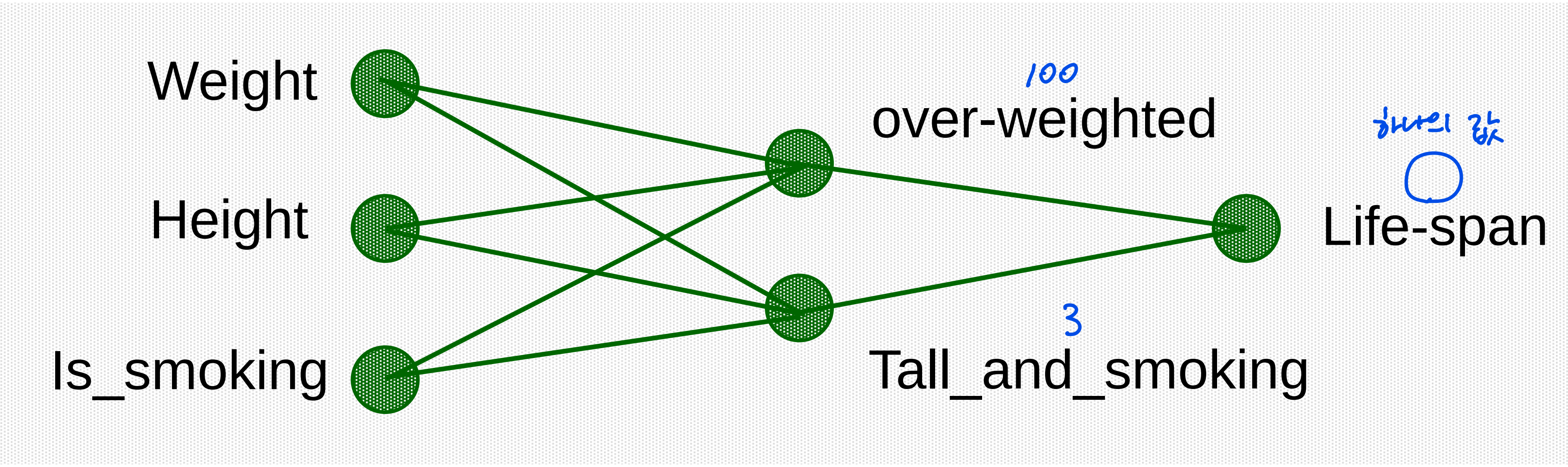


$$\mathbf{x} \underset{T_1}{\longrightarrow} \mathbf{y} \underset{T_2}{\longrightarrow} \mathbf{z}$$

Neural Network Example: ONE-TO-ONE

$\mathbb{R}^3 \rightarrow \mathbb{R}^2$ 이므로 1대1함수 X

- Will there be many (or unique) people mapped to the same (over_weighted, tall_and_smoking)?

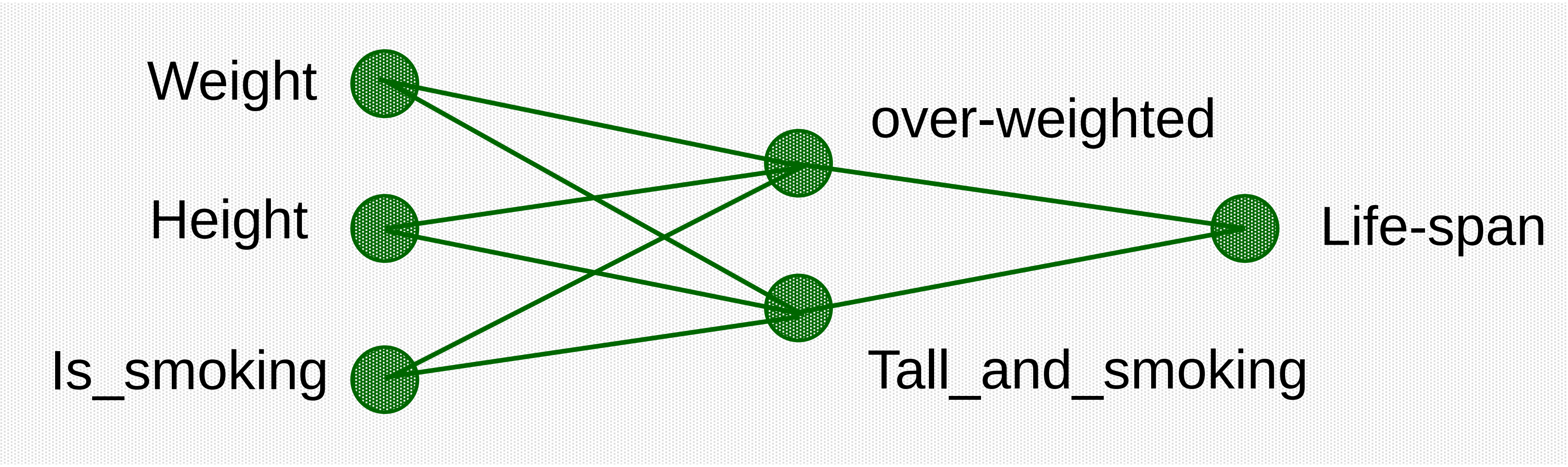


$$\mathbf{x} \xrightarrow[T_1]{} \mathbf{y} \xrightarrow[T_2]{} \mathbf{z}$$

Neural Network Example: ONTO

전사함수가 될 가능성은 有 → 항상 전사함수는 X

- Is there any (over_weighted, tall_and_smoking) that does not exist at all?



$$\mathbf{x} \xrightarrow[T_1]{} \mathbf{y} \xrightarrow[T_2]{} \mathbf{z}$$

ONTO and ONE-TO-ONE

- Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, i.e.,

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n.$$

- T is one-to-one if and only if the columns of A are linearly independent.

- T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.

ONTO and ONE-TO-ONE

- **Example:**

  Let $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

- Is T one-to-one?
- Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$?

ONTO and ONE-TO-ONE

- **Example:**

  Let $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

- Is T one-to-one?
- Does T map $\mathbb{R}^3$ onto $\mathbb{R}^2$?

Further Study

- Gaussian elimination, row reduction, echelon form
  - Lay Ch1.2,
- LU factorization: efficiently solving linear systems
  - Lay Ch2.5
- Computing invertible matrices
  - Lay Ch2.2
- Invertible matrix theorem for square matrices
  - Lay Ch2.3, Ch2.9