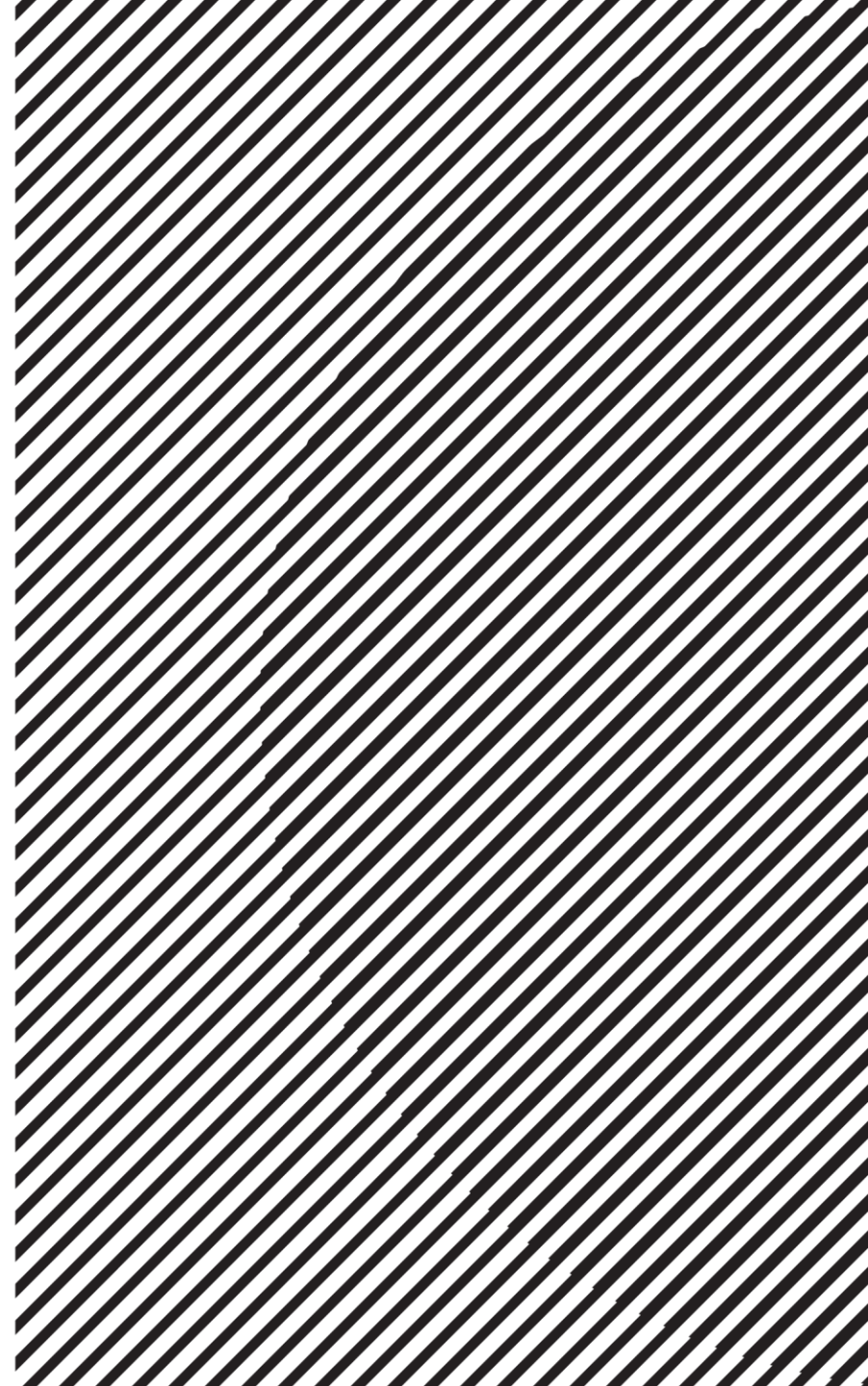

Linear Algebra

주재걸
고려대학교 컴퓨터학과

manifold



ONTO and ONE-TO-ONE

공역 = 치역

전사

공역에서 어떤 원소라도 하나 이상의 정의역을 갖고 있어야 한다.

- **Definition:** A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **onto** \mathbb{R}^m if each $\mathbf{b} \in \mathbb{R}^m$ is the image of **at least** one $\mathbf{x} \in \mathbb{R}^n$. $n \geq m$ ★
- That is, the range is equal to the co-domain.

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

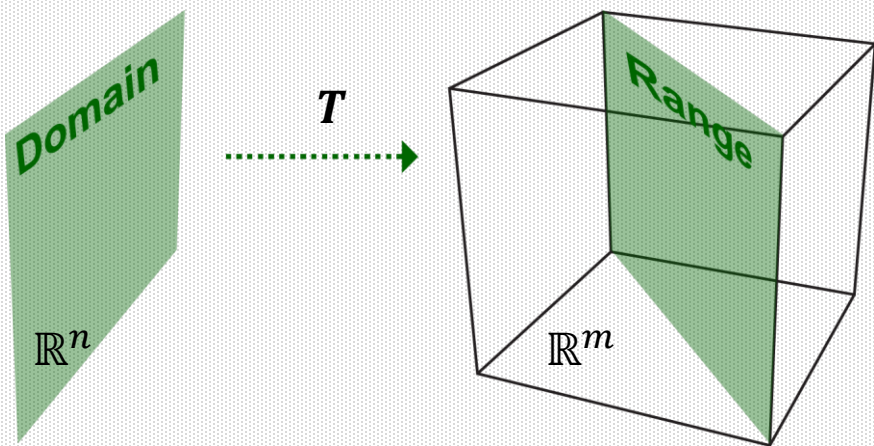


치역

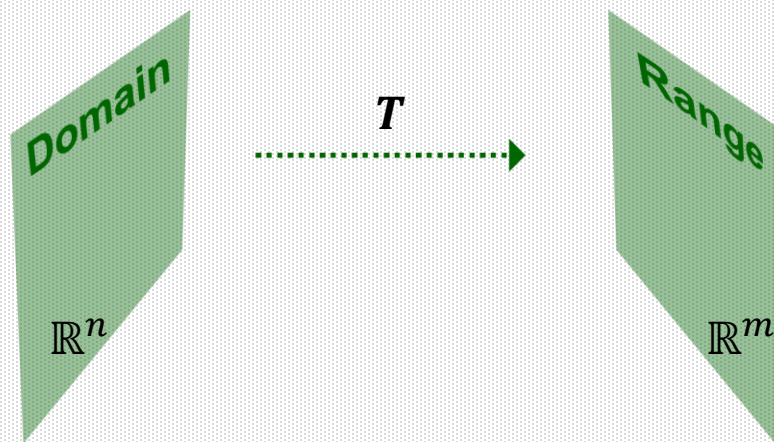
$$\rightarrow \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \text{의 span이 치역이므로}$$

onto가 안됨!!

\rightarrow ex) GAN, decoding, ...



T is NOT onto \mathbb{R}^m



T is onto \mathbb{R}^m

ONTO and ONE-TO-ONE

↓
선형 독립

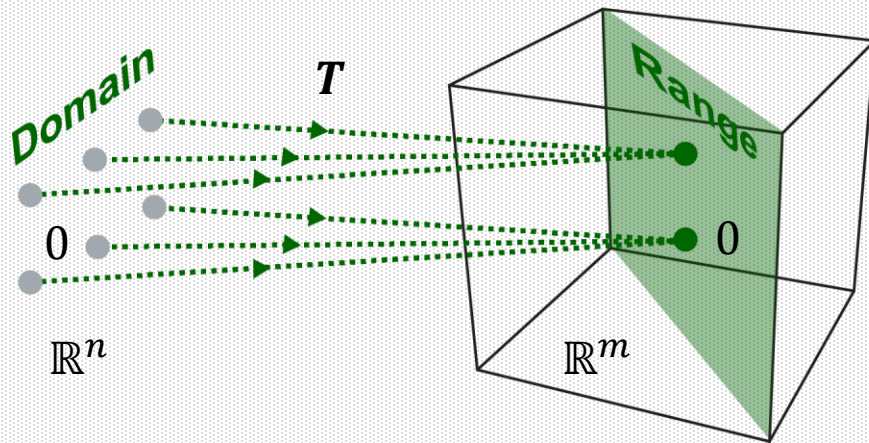
$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

$$T: y = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

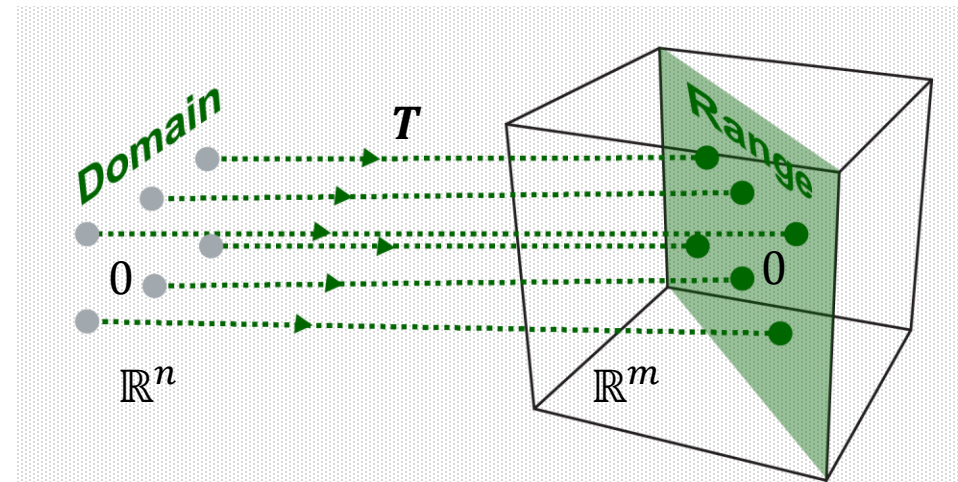
- Definition:** A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **one-to-one** if each $b \in \mathbb{R}^m$ is the image of **at most** one $x \in \mathbb{R}^n$.
 That is, each output vector in the range is mapped by only one input vector, no more than that.

1대1 함수 (공역 ≠ 치역일 수 있음)
반대!! 1개의 치역엔 1개의 정의역

$$n \leq m \quad \star$$



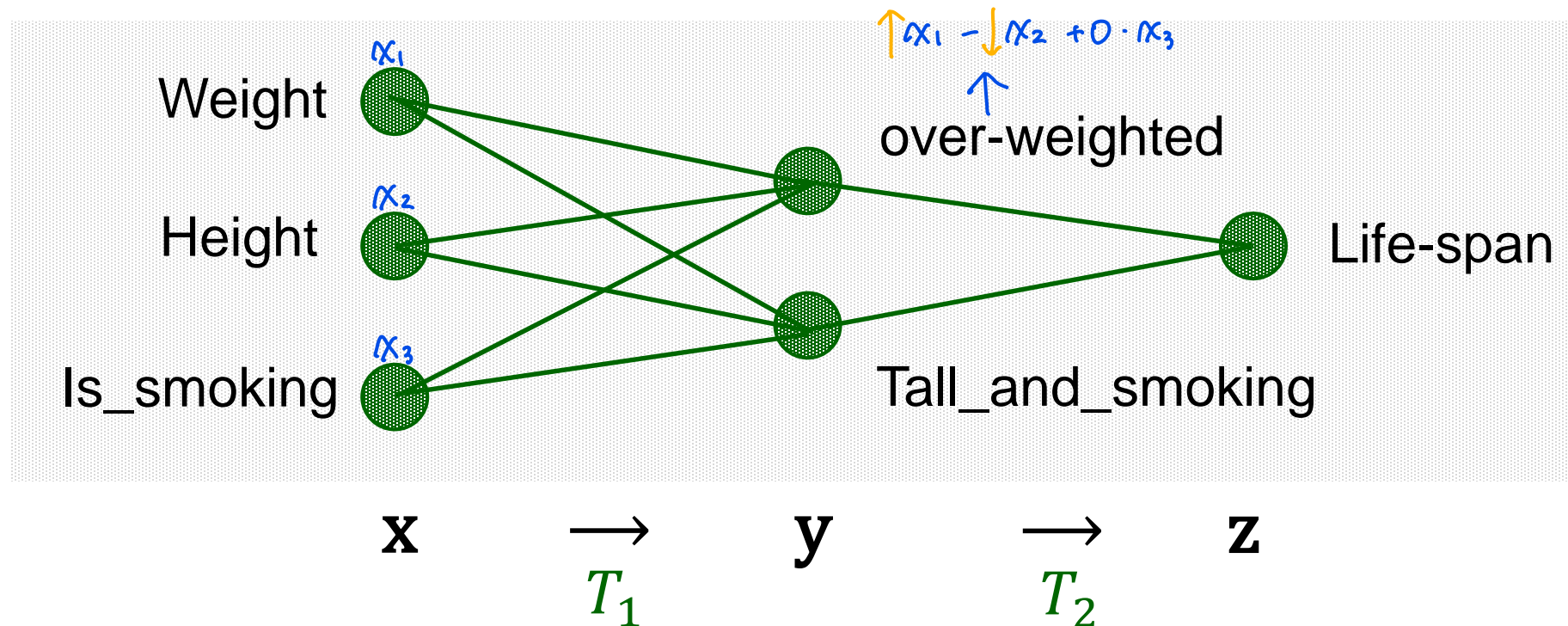
T is NOT one-to-one



T is one-to-one

Neural Network Example

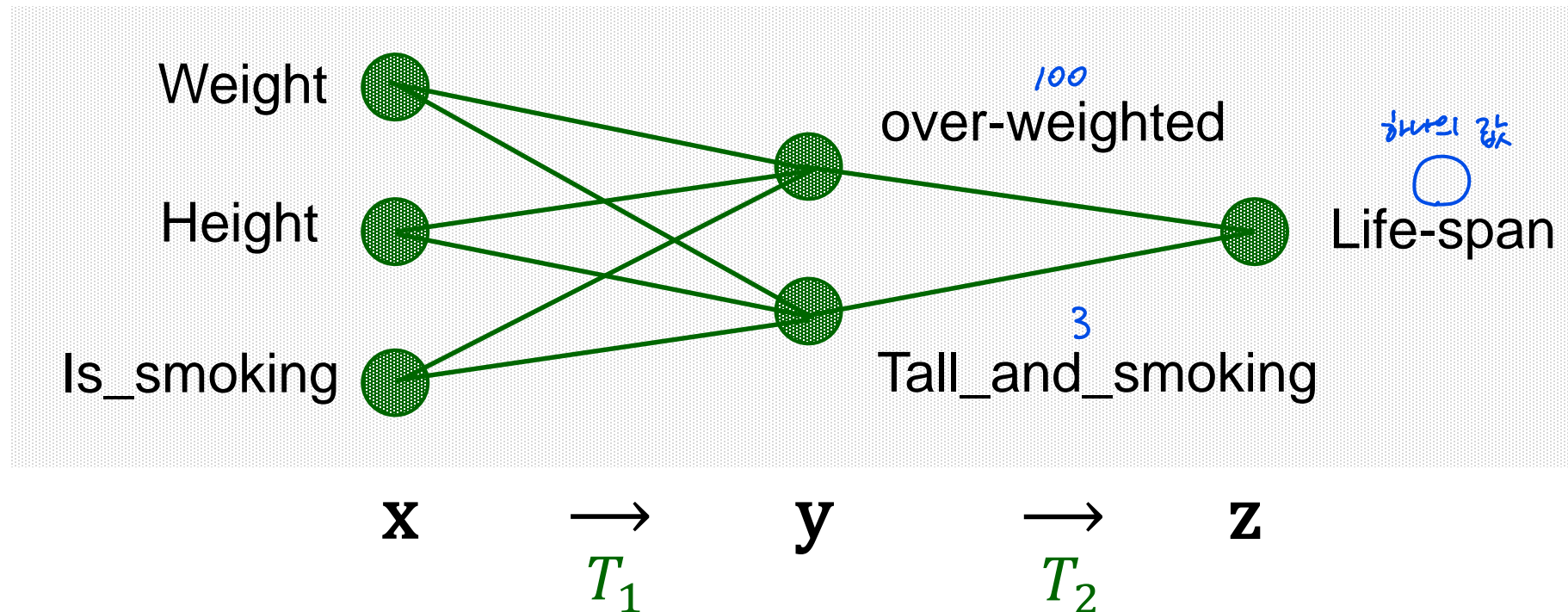
- Fully-connected layers



Neural Network Example: ONE-TO-ONE

$\mathbb{R}^3 \rightarrow \mathbb{R}^2$ 이므로 1대1함수 X

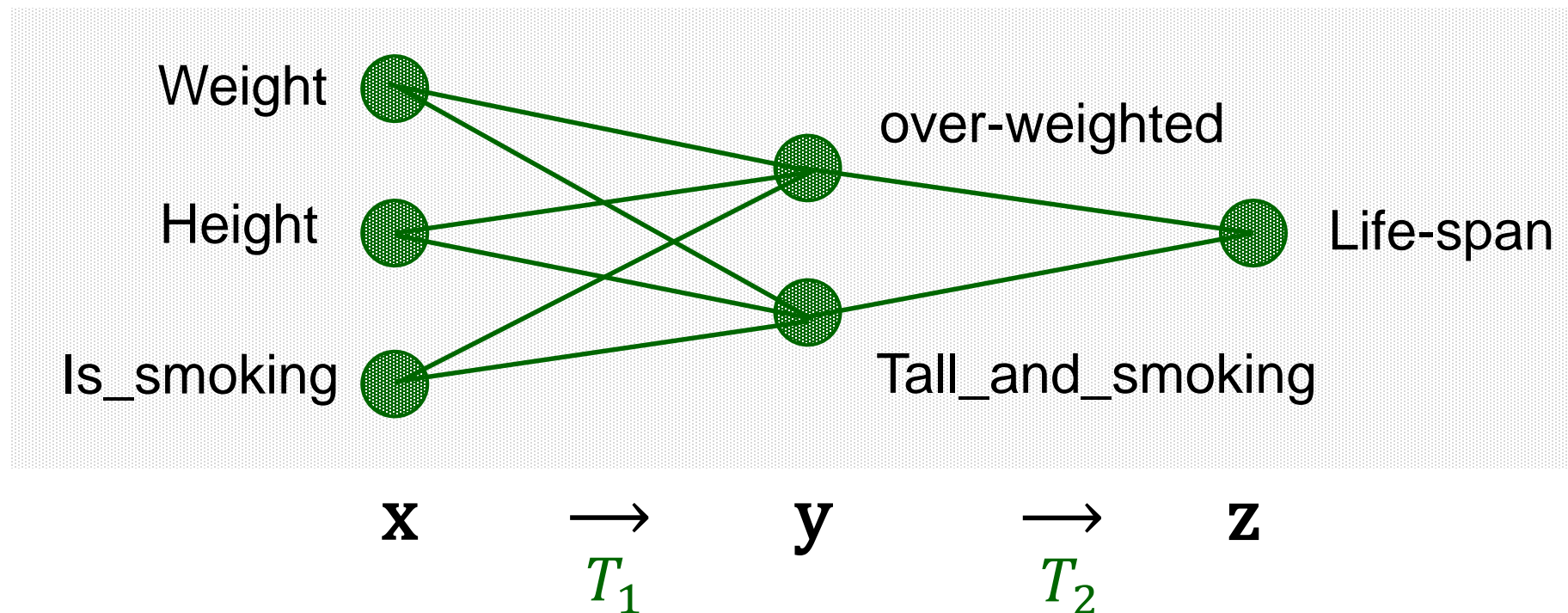
- Will there be many (or unique) people mapped to the same (over_weighted, tall_and_smoking)?



Neural Network Example: ONTO

전사함수가 될 가능성은 有 \rightarrow 항상 전사함수는 \times

- Is there any (over_weighted, tall_and_smoking) that does not exist at all?





ONTO and ONE-TO-ONE

- Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation, i.e.,

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n.$$

- T is **one-to-one** if and only if the columns of A are **linearly independent**.
- T maps \mathbb{R}^n **onto** \mathbb{R}^m if and only if the columns of A **span** \mathbb{R}^m .



ONTO and ONE-TO-ONE

- **Example:**

$$\text{Let } T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Is T one-to-one?
- Does T map \mathbb{R}^2 onto \mathbb{R}^3 ?



ONTO and ONE-TO-ONE

- **Example:**

$$\text{Let } T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- Is T one-to-one?
- Does T map \mathbb{R}^3 onto \mathbb{R}^2 ?



Further Study

- Gaussian elimination, row reduction, echelon form
 - Lay Ch1.2,
- LU factorization: efficiently solving linear systems
 - Lay Ch2.5
- Computing invertible matrices
 - Lay Ch2.2
- Invertible matrix theorem for square matrices
 - Lay Ch2.3, Ch2.9