Chapter3 The Z-transform

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Objectives of this chapter

- Understand how to represent a sequence of numbers with a function of a complex variable called z-transform
- Change a sequence by manipulating its z-transform and vice versa
- Possess a basic understanding of the concept of system function and use it to investigate
- Determine numerically the response of discrete-time systems described by linear constant coefficient difference equations

Sequence retains shape

- A sequence that retains its shape when it passes through an LTI system
- If complex exponential sequence passes through LTI system, output is also complex exponential sequence
- Complex exponential function is eigen-function of LTI system
- The quantity $H(z)$ is known as the system function or transfer function
- If the input to a LTI system can be expressed as a linear combination of complex exponential functions, then the output is also a linear combination of complex exponential functions whose magnitudes are changed according to the system function

$$
\begin{aligned}
x[n] &= z^n \\
y[n] &= \sum_{k=-\infty}^{\infty} h[k] z^{n-k} = \left(\sum_{k=-\infty}^{\infty} h[k] z^{-k} \right) z^n
\end{aligned}
$$

$$
\begin{aligned}
H(z) &= \sum_{k=-\infty}^{\infty} h[k] z^{-k} \\
y[n] &= H(z) z^n
\end{aligned}
$$

$$
x[n] = \sum_{k} c_k z_k^n
$$

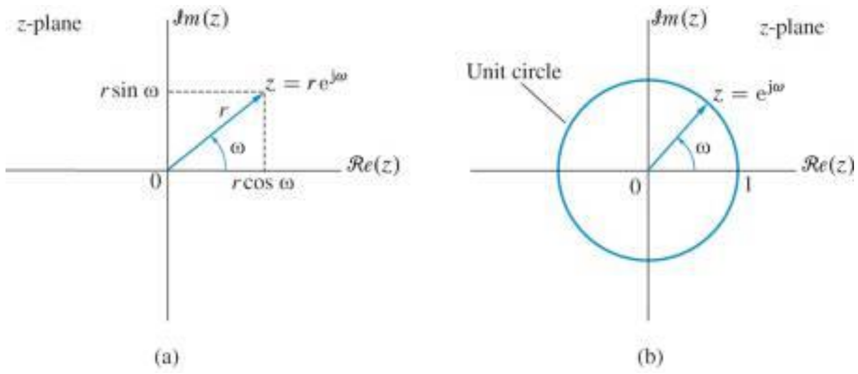
$$
y[n] = \sum_{k=-\infty}^{\infty} c_k H(z_k) z_k^n
$$

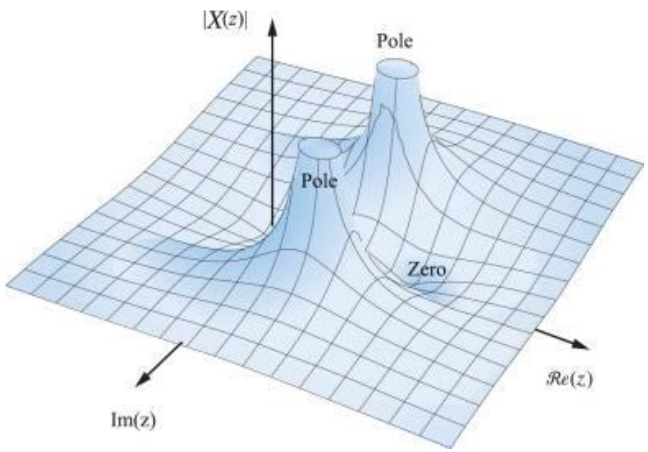
The z-transform

- The z-transform is defined as follows:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- It is convenient to represent a complex number using a polar form
- The z-transform may or may not be finite for all sequences or all values of z
- The set of values of z for which the z-transform exist, ie., the infinite sum converges, is called the region of convergence (ROC)
- The value of z, where its z-transform is 0 called zero
- The value of z, where z-transform does not exist, ie, the series diverges, is called pole
- The ROC does not include any pole





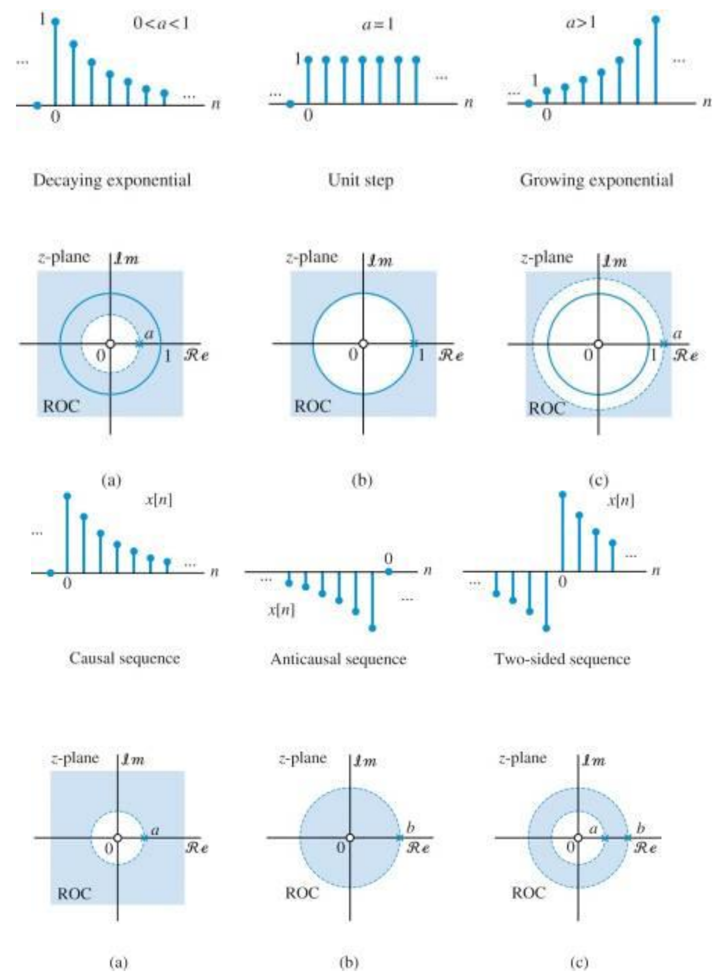
z-transform of exponential sequences

- The z-transform of causal exponential sequences:

$$
\begin{aligned}
x[n] &= a^n u[n] \\
X(z) &= \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \\
&\quad \text{ROC:} |z| > |a|
\end{aligned}
$$

- The z-transform of anti-causal exponential sequences:

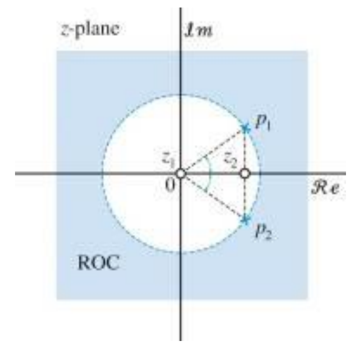
$$
\begin{aligned}
y[n] &= -b^n u[-n-1] \\
Y(z) &= -\sum_{n=-\infty}^{-1} b^n z^{-n} = -b^{-1} z (1 + b^{-1} z + b^{-2} z^{-2} + \ldots), \\
&= \frac{-b^{-1} z}{1-b^{-1} z} = \frac{1}{1-bz^{-1}} = \frac{z}{z-b} \\
&\quad \text{ROC:} |z| < |b|
\end{aligned}
$$



z-transform of exponentially oscillating sequences

- The z-transform of exponentially oscillating sequences:

$$
\begin{aligned}
x[n] &= r^n(\cos\omega_0 n)u[n] \\
X(z) &= \sum_{n=0}^{\infty} r^n \cos\omega_0 n z^{-n} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}(re^{j\omega_0}z^{-1})^n + \frac{1}{2}\sum_{n=0}^{\infty}(re^{-j\omega_0}z^{-1})^n, \\
&= \frac{1}{2}\frac{1}{1-re^{j\omega_0}z^{-1}} + \frac{1}{2}\frac{1}{1-re^{-j\omega_0}z^{-1}} \\
& \quad \text{ROC:}|z| > |r|
\end{aligned}
$$



Region of convergence

- The ROC cannot include any poles
- The ROC is a connected (that is, a single contiguous) region
- For finite duration sequence the ROC is the entire z-plane, with the possible exceptions of 0 or infinity
- The z-transform of a sequence consists of an algebraic formula and its associated ROC. Thus to uniquely specify a sequence, we need both the z-transform and its ROC
- The z-transformation is legitimate only for z within the ROC.
- For causal infinite duration sequence, ROC has the following form $|z| > r$
- For anti-causal infinite duration sequence, ROC has the following form $|z| < r$
- For two-sided infinite duration sequence, ROC has the following form, $a < |z| < b$

Inverse Z-transform

- The inverse z-transform is defined as follows:

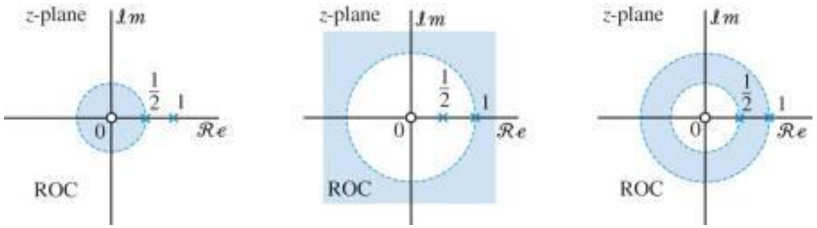
$$x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

- However, for many z-transforms, partial fraction expansion can be effectively used to determine associated inverse z-transform
- Note the following relationship

$$x[n] = \sum_{k=1}^{N} A_k(p_k)^n \stackrel{Z}{\longleftrightarrow} X(z) = \sum_{k=1}^{N}\frac{A_k}{1-p_k z^{-1}}$$

- Ex:

$$\begin{aligned} X(z) &= \frac{1+z^{-1}}{(1-z^{-1})(1-0.5z^{-1})} \\ &= \frac{0.4}{1-z^{-1}} - \frac{3}{1-0.5z^{-1}} \end{aligned}$$



$$\begin{aligned} \text{Causal:} \quad x[n] &= 4u[n] - 3\left(\frac{1}{2}\right)^n u[n], \text{ROC:}|z| > 1 \\ \text{Anti-causal:} \quad x[n] &= -4u[-n-1] + 3\left(\frac{1}{2}\right)^n u[-n-1], \text{ROC:}|z| < 0.5 \\ \text{Double sided:} \quad x[n] &= -4u[-n-1] - 3\left(\frac{1}{2}\right)^n u[n], \text{ROC:}0.5 < |z| < 1 \end{aligned}$$

Linearity and time shifting

- Linearity: The z-transform is a linear operator, that is

$$a_1x_1[n] + a_2x_2[n] \stackrel{Z}{\longleftrightarrow} a_1X_1(z) + z_2X_2(z)$$

- The ROC of the linear combination is at least the intersection of the two ROCs

- Time shifting: The following relationship holds

$$x[n-k] \stackrel{Z}{\longleftrightarrow} z^{-k}X(z)$$

- The ROC for the time shifted sequence is the same as the one of original sequence except at zero or infinity

Convolution and multiplication by an exponential sequence

- Convolving two sequences is equivalent to multiplying their z-transform

$$x_1[n] * x_2[n] \stackrel{Z}{\longleftrightarrow} X_1(z)X_2(z)$$

- The ROC of the convolved sequence is the intersection of the two ROCs

$$\begin{aligned} Y(z) &= \sum_n \left(\sum_k x_1[k]x_2[n-k] \right) z^{-n} \\ &= \sum_n \sum_k x_1[k]x_2[n-k]z^{-(n-k)}z^{-k} \end{aligned}$$

- Multiplication by an exponential sequence

$$a^n x[n] \stackrel{Z}{\longleftrightarrow} X(z/a),\ \mathrm{ROC} = |a|R_x$$

$$Y(z) = \sum_{n=-\infty}^{\infty} a^n x[n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n](z/a)^{-n} = X(z/a)$$

- Since the resulting z-transform is the scaled version of the original Z-transform, the ROC is also scaled

Differentiation of the transform time reversal, initial value theorem

- Multiplying the value of each sample by its index is equivalent to differentiating its transform

$$nx[n] \xleftrightarrow{Z} -z\frac{dX(z)}{dz},\ \mathrm{ROC} = R_x$$

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x[n](-n)z^{-n-1} = -z^{-1}\sum_{n=-\infty}^{\infty} nx[n]z^{-n}$$

- Time reversal

$$x[-n] \xleftrightarrow{Z} X(1/z),\ \mathrm{ROC} = \frac{1}{R_x}$$

$$\sum_{n=-\infty}^{\infty} x[-n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n](z)^{n} = X(1/z)$$

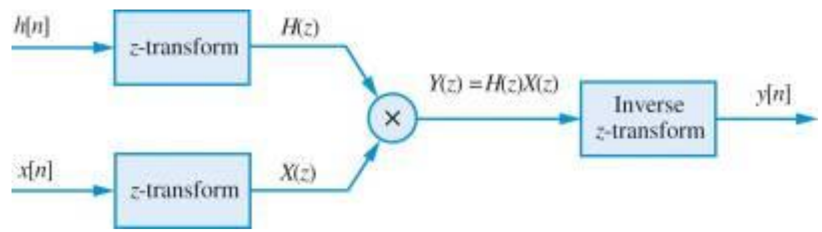
- Initial value theorem: For a causal sequence, the following holds

$$x[0] = \lim_{z\to\infty} X(z)$$

$$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + ...$$

System function of LTI systems

- Z-transforms of input, output and impulse response
  $$Y(z) = H(z)X(z)$$
- ROC of input and output should overlap
- Causality: A causal system has the ROC that is the exterior of a circle extending to infinity (this is not a sufficient condition)
- Stability: An LTI system is stable if and only if the ROC of the system function includes unit circle
- Causal and stable system: An LTI system whith rational system function is both causal and stable if and only if all the poles are inside the unit circle and its ROC is on the exterior of a circle, extending to infinity



$$H(z) = \sum_{n=0}^{\infty} h[n]z^{-n}$$

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} |h[n]z^{-n}| < \infty$$

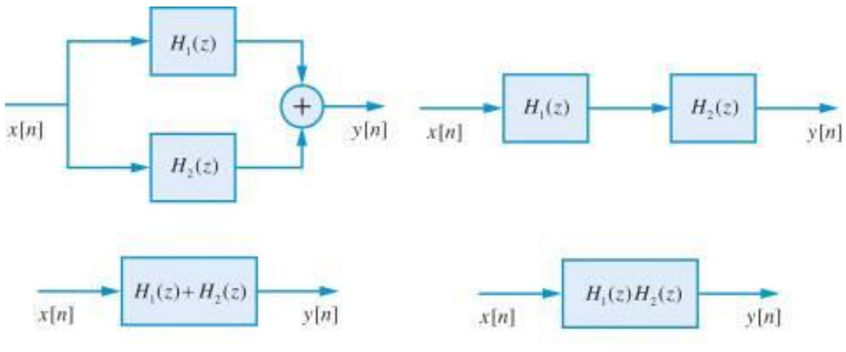
System function algebra

- Parallel interconnection:

$$
\begin{aligned}
h[n] &= h_1[n] + h_2[n] \\
H(z) &= H_1(z) + H_2(z)
\end{aligned}
$$

- Serial interconnection

$$
\begin{aligned}
h[n] &= h_1[n] * h_2[n] \\
H(z) &= H_1(z)H_2(z)
\end{aligned}
$$



(a) (b)

LTI System characterized by linear constant coefficient difference equation

- Causal system:

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

- If system is assumed to be initially at rest:

$$x[-1] = ...x[-M] = 0, y[-1] = ... = y[-N] = 0$$

$$\left(1 + \sum_{k=1}^{N} a_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right) X(z)$$

- System transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

- Poles and zeros

$$\begin{aligned} B(z) &= b_0 z^{-M} \left(z^M + \frac{b_1}{b_0} z^{M-1} + ... + \frac{b_M}{b_0} \right) \\ &= b_0 z^{-M} (z - z_1)...(z - z_M) \end{aligned}$$

$$\begin{aligned} A(z) &= z^{-N} \left(z^N + a_1 z^{N-1} + ... + a_N \right) \\ &= z^{-N} (z - p_1)...(z - p_N) \end{aligned}$$

$$H(z) = \frac{A(z)}{B(z)} = b_0 \frac{z^{-M} \prod_{k=1}^{M} (z - z_k)}{z^{-N} \prod_{k=1}^{N} (z - p_k)} = b_0 \frac{\prod_{k=1}^{M} (1 - z_k z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})}$$

Impulse response, causality, stability of IIR system

- Impulse response: If the system is assumed to be a causal system, then the ROC is the exterior of a circle starting at the outmost pole

$$
\begin{aligned}
H(z) &= \sum_{k=0}^{M-N} C_k z^{-k} + \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} \\
h[n] &= \sum_{k=0}^{M-N} C_k \delta[n-k] + \sum_{k=1}^{N} A_k (p_k)^n u[n]
\end{aligned}
$$

- Causality and stability: If we assume the system is causal, the stability condition is as follows:

$$
\sum_{n=0}^{\infty} |h[n]| \leq \sum_{k=0}^{M-N} |C_k| + \sum_{k=1}^{N} |A_k| \sum_{n=0}^{\infty} |p_k|^n < \infty
$$

- A causal LTI system with a rational system function is stable if and only if all poles are inside the unit circle in the z-plane

System classification

- Length of impulse response: If at least one nonzero pole exists, the system is called infinite impulse response (IIR) system. Otherwise, the system is called finite impulse response (FIR) system

$$A_k(p_k)^n u[n]$$

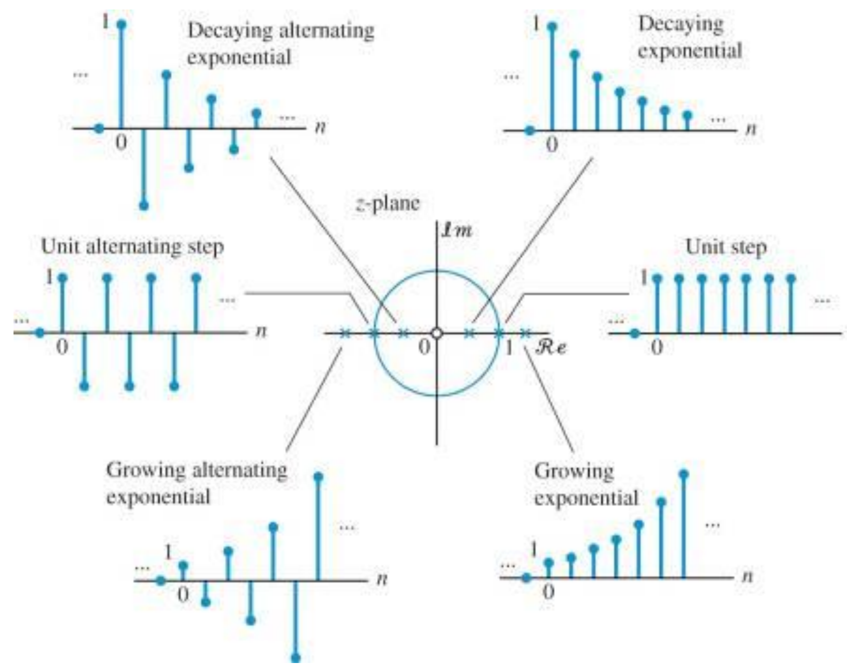
- Feedback in implementation: If N is not zero, the output of a system is fed back into the input and the system is known a recursive system. Otherwise, the system is a non-recursive system.
- Poles and zeros: All-zero system (FIR). All-pole system

Connection between pole-zero locations and time-domain behavior

- Rational system function can be decomposed

$$
\begin{aligned}
H(z) &= \sum_{k=0}^{M-N} C_k z^{-k} + \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} \\
A(k) &= (1 - p_k z^{-1}) H(z)|_{z=p_k}
\end{aligned}
$$

- The roots of a polynomial with real coefficients either must be real or must occur in complex conjugate
- Therefore, system can be decomposed into an FIR system, first-order system with real poles and second order system



$$H(z) = \sum_{k=0}^{M-N} C_k z^{-k} + \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^{K_2} \frac{b_{k0} + b_{k1} z^{-1}}{1 + a_{k1} z^{-1} + a_{k2} z^{-2}}$$

$$\frac{A}{1 - p z^{-1}} + \frac{A^*}{1 - p^* z^{-1}} = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

Second order systems

- Second order system function

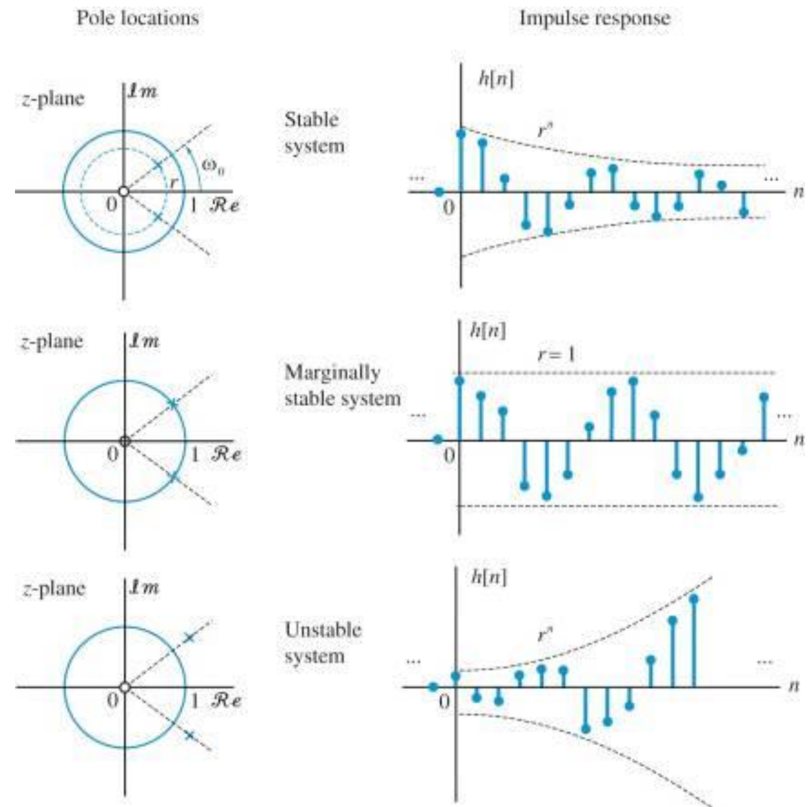
$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{z(b_0 z + b_1)}{z^2 + a_1 z + a_2}$$

- Poles of the system function

$$p_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

- Impulse response of a causal system with a pair of complex conjugate poles

$$\begin{aligned} h[n] &= Ap^n u[n] + A^*(p^*)^n u[n] \\ &= |A|e^{j\theta} r^n e^{j\omega_0} u[n] + |A|e^{-j\theta} r^n e^{-j\omega_0} u[n] \\ &= |A| r^n \left[e^{j(\omega_0 n + \theta)} + e^{-j(\omega_0 n + \theta)} \right] u[n] \\ &= 2|A| r^n \cos(\omega_0 n + \theta) u[n] \end{aligned}$$



One sided Z-transform

- One-sided or unilateral Z-transform

$$X^+(z) \triangleq Z^+[x[n]] \triangleq \sum_{n=0}^{\infty} x[n] z^{-n}$$

- The ROC is always the exterior of a circle
- Time shifting property can be used to solve linear constant coefficient difference equations with nonzero initial conditions

$$Z^+\{x[n-k]\} = z^{-k} X^+(z) + \sum_{m=1}^{k} x[-m] z^{m-k}$$

- Example: $y[n] = ay[n-1] + bx[n]$

$$\begin{aligned} Y^+(z) &= ay[-1] + az^{-1} Y^+(z) + bX^+(z) \\ &= \underbrace{\frac{ay[-1]}{1-az^{-1}}}_{\text{initial condition}} + \underbrace{\frac{b}{1-az^{-1}} X^+(z)}_{\text{zero state}} \end{aligned}$$