Solution to Chapter 8 Problems

Problem 8.1

1) To show that the waveforms $\psi_n(t)$, $n = 1, 2, 3$ are orthogonal we have to prove that

$$\int_{-\infty}^{\infty} \psi_m(t)\psi_n(t)dt = 0, \qquad m \neq n$$

Clearly,

$$\begin{aligned}
c_{12} &= \int_{-\infty}^{\infty} \psi_1(t)\psi_2(t)dt = \int_0^4 \psi_1(t)\psi_2(t)dt \\
&= \int_0^2 \psi_1(t)\psi_2(t)dt + \int_2^4 \psi_1(t)\psi_2(t)dt \\
&= \frac{1}{4}\int_0^2 dt - \frac{1}{4}\int_2^4 dt = \frac{1}{4}\times 2 - \frac{1}{4}\times(4-2) \\
&= 0
\end{aligned}$$

Similarly,

$$\begin{aligned}
c_{13} &= \int_{-\infty}^{\infty} \psi_1(t)\psi_3(t)dt = \int_0^4 \psi_1(t)\psi_3(t)dt \\
&= \frac{1}{4}\int_0^1 dt - \frac{1}{4}\int_1^2 dt - \frac{1}{4}\int_2^3 dt + \frac{1}{4}\int_3^4 dt \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
c_{23} &= \int_{-\infty}^{\infty} \psi_2(t)\psi_3(t)dt = \int_0^4 \psi_2(t)\psi_3(t)dt \\
&= \frac{1}{4}\int_0^1 dt - \frac{1}{4}\int_1^2 dt + \frac{1}{4}\int_2^3 dt - \frac{1}{4}\int_3^4 dt \\
&= 0
\end{aligned}$$

Thus, the signals $\psi_n(t)$ are orthogonal.

2) We first determine the weighting coefficients

$$x_n = \int_{-\infty}^{\infty} x(t)\psi_n(t)dt, \qquad n = 1, 2, 3$$

$$\begin{aligned}
x_1 &= \int_0^4 x(t)\psi_1(t)dt = -\frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_1^2 dt - \frac{1}{2}\int_2^3 dt + \frac{1}{2}\int_3^4 dt = 0 \\
x_2 &= \int_0^4 x(t)\psi_2(t)dt = \frac{1}{2}\int_0^4 x(t)dt = 0 \\
x_3 &= \int_0^4 x(t)\psi_3(t)dt = -\frac{1}{2}\int_0^1 dt - \frac{1}{2}\int_1^2 dt + \frac{1}{2}\int_2^3 dt + \frac{1}{2}\int_3^4 dt = 0
\end{aligned}$$

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As it is observed, $x(t)$ is orthogonal to the signal waveforms $\psi_n(t)$, $n = 1, 2, 3$ and thus it can not represented as a linear combination of these functions.

Problem 8.2

1) The expansion coefficients $\{c_n\}$, that minimize the mean square error, satisfy

$$c_n = \int_{-\infty}^{\infty} x(t)\psi_n(t)dt = \int_0^4 \sin\frac{\pi t}{4}\psi_n(t)dt$$

Hence,

$$\begin{aligned} c_1 &= \int_0^4 \sin\frac{\pi t}{4}\psi_1(t)dt = \frac{1}{2}\int_0^2 \sin\frac{\pi t}{4}dt - \frac{1}{2}\int_2^4 \sin\frac{\pi t}{4}dt \\ &= -\frac{2}{\pi}\cos\frac{\pi t}{4}\bigg|_0^2 + \frac{2}{\pi}\cos\frac{\pi t}{4}\bigg|_2^4 \\ &= -\frac{2}{\pi}(0-1) + \frac{2}{\pi}(-1-0) = 0 \end{aligned}$$

Similarly,

$$\begin{aligned} c_2 &= \int_0^4 \sin\frac{\pi t}{4}\psi_2(t)dt = \frac{1}{2}\int_0^4 \sin\frac{\pi t}{4}dt \\ &= -\frac{2}{\pi}\cos\frac{\pi t}{4}\bigg|_0^4 = -\frac{2}{\pi}(-1-1) = \frac{4}{\pi} \end{aligned}$$

and

$$\begin{aligned} c_3 &= \int_0^4 \sin\frac{\pi t}{4}\psi_3(t)dt \\ &= \frac{1}{2}\int_0^1 \sin\frac{\pi t}{4}dt - \frac{1}{2}\int_1^2 \sin\frac{\pi t}{4}dt + \frac{1}{2}\int_2^3 \sin\frac{\pi t}{4}dt - \frac{1}{2}\int_3^4 \sin\frac{\pi t}{4}dt \\ &= 0 \end{aligned}$$

Note that c_1, c_2 can be found by inspection since $\sin\frac{\pi t}{4}$ is even with respect to the $x = 2$ axis and $\psi_1(t)$, $\psi_3(t)$ are odd with respect to the same axis.

2) The residual mean square error $E_{\min}$ can be found from

$$E_{\min} = \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{3} |c_i|^2$$

Thus,

$$\begin{aligned} E_{\min} &= \int_0^4 \left(\sin\frac{\pi t}{4}\right)^2 dt - \left(\frac{4}{\pi}\right)^2 = \frac{1}{2}\int_0^4 \left(1 - \cos\frac{\pi t}{2}\right)dt - \frac{16}{\pi^2} \\ &= 2 - \frac{1}{\pi}\sin\frac{\pi t}{2}\bigg|_0^4 - \frac{16}{\pi^2} = 2 - \frac{16}{\pi^2} \end{aligned}$$

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Problem 8.3

1) As an orthonormal set of basis functions we consider the set

$$\psi_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{o.w} \end{cases} \qquad \psi_2(t) = \begin{cases} 1 & 1 \le t < 2 \\ 0 & \text{o.w} \end{cases}$$

$$\psi_3(t) = \begin{cases} 1 & 2 \le t < 3 \\ 0 & \text{o.w} \end{cases} \qquad \psi_4(t) = \begin{cases} 1 & 3 \le t < 4 \\ 0 & \text{o.w} \end{cases}$$

In matrix notation, the four waveforms can be represented as

$$\begin{pmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -2 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \\ \psi_3(t) \\ \psi_4(t) \end{pmatrix}$$

Note that the rank of the transformation matrix is 4 and therefore, the dimensionality of the waveforms is 4

2) The representation vectors are

$$\begin{aligned} \mathbf{s}_1 &= \begin{bmatrix} 2 & -1 & -1 & -1 \end{bmatrix} \\ \mathbf{s}_2 &= \begin{bmatrix} -2 & 1 & 1 & 0 \end{bmatrix} \\ \mathbf{s}_3 &= \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \\ \mathbf{s}_4 &= \begin{bmatrix} 1 & -2 & -2 & 2 \end{bmatrix} \end{aligned}$$

3) The distance between the first and the second vector is

$$d_{1,2} = \sqrt{|\mathbf{s}_1 - \mathbf{s}_2|^2} = \sqrt{\left|\begin{bmatrix} 4 & -2 & -2 & -1 \end{bmatrix}\right|^2} = \sqrt{25}$$

Similarly we find that

$$\begin{aligned} d_{1,3} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_3|^2} = \sqrt{\left|\begin{bmatrix} 1 & 0 & -2 & 0 \end{bmatrix}\right|^2} = \sqrt{5} \\ d_{1,4} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} 1 & 1 & 1 & -3 \end{bmatrix}\right|^2} = \sqrt{12} \\ d_{2,3} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_3|^2} = \sqrt{\left|\begin{bmatrix} -3 & 2 & 0 & 1 \end{bmatrix}\right|^2} = \sqrt{14} \\ d_{2,4} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} -3 & 3 & 3 & -2 \end{bmatrix}\right|^2} = \sqrt{31} \\ d_{3,4} &= \sqrt{|\mathbf{s}_3 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} 0 & 1 & 3 & -3 \end{bmatrix}\right|^2} = \sqrt{19} \end{aligned}$$

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Thus, the minimum distance between any pair of vectors is $d_{\min} = \sqrt{5}$.

Problem 8.4

As a set of orthonormal functions we consider the waveforms

$$\psi_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{o.w} \end{cases} \qquad \psi_2(t) = \begin{cases} 1 & 1 \le t < 2 \\ 0 & \text{o.w} \end{cases} \qquad \psi_3(t) = \begin{cases} 1 & 2 \le t < 3 \\ 0 & \text{o.w} \end{cases}$$

The vector representation of the signals is

$$\begin{aligned} \mathbf{s}_1 &= \begin{bmatrix} 2 & 2 & 2 \end{bmatrix} \\ \mathbf{s}_2 &= \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \\ \mathbf{s}_3 &= \begin{bmatrix} 0 & -2 & -2 \end{bmatrix} \\ \mathbf{s}_4 &= \begin{bmatrix} 2 & 2 & 0 \end{bmatrix} \end{aligned}$$

Note that $s_3(t) = s_2(t) - s_1(t)$ and that the dimensionality of the waveforms is 3.

Problem 8.5

1) The impulse response of the filter matched to $s(t)$ is

$$h(t) = s(T - t) = s(3 - t) = s(t)$$

where we have used the fact that $s(t)$ is even with respect to the $t = \frac{T}{2} = \frac{3}{2}$ axis.

2) The output of the matched filter is

$$\begin{aligned} y(t) &= s(t) \star s(t) = \int_0^t s(\tau)s(t-\tau)d\tau \\ &= \begin{cases} 0 & t < 0 \\ A^2 t & 0 \le t < 1 \\ A^2(2-t) & 1 \le t < 2 \\ 2A^2(t-2) & 2 \le t < 3 \\ 2A^2(4-t) & 3 \le t < 4 \\ A^2(t-4) & 4 \le t < 5 \\ A^2(6-t) & 5 \le t < 6 \\ 0 & 6 \le t \end{cases} \end{aligned}$$

A sketch of $y(t)$ is depicted in the next figure

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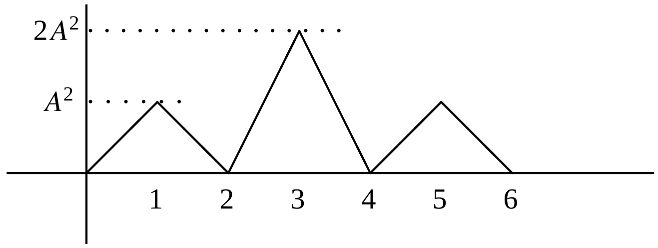

3) At the output of the matched filter and for $t = T = 3$ the noise is

$$\begin{aligned} n_T &= \int_0^T n(\tau)h(T-\tau)d\tau \\ &= \int_0^T n(\tau)s(T-(T-\tau))d\tau = \int_0^T n(\tau)s(\tau)d\tau \end{aligned}$$

The variance of the noise is

$$\begin{aligned} \sigma^2_{n_T} &= E\left[\int_0^T\int_0^T n(\tau)n(v)s(\tau)s(v)d\tau dv\right] \\ &= \int_0^T\int_0^T s(\tau)s(v)E[n(\tau)n(v)]d\tau dv \\ &= \frac{N_0}{2}\int_0^T\int_0^T s(\tau)s(v)\delta(\tau-v)d\tau dv \\ &= \frac{N_0}{2}\int_0^T s^2(\tau)d\tau = N_0A^2 \end{aligned}$$

4) For antipodal equiprobable signals the probability of error is

$$P(e) = Q\left[\sqrt{\left(\frac{S}{N}\right)_o}\right]$$

where $\left(\frac{S}{N}\right)_o$ is the output SNR from the matched filter. Since

$$\left(\frac{S}{N}\right)_o = \frac{y^2(T)}{E[n_T^2]} = \frac{4A^4}{N_0A^2}$$

we obtain

$$P(e) = Q\left[\sqrt{\frac{4A^2}{N_0}}\right]$$

Problem 8.6

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1) Taking the inverse Fourier transform of $H(f)$, we obtain

$$\begin{aligned} h(t) &= \mathcal{F}^{-1}[H(f)] = \mathcal{F}^{-1}\left[\frac{1}{j2\pi f}\right] - \mathcal{F}^{-1}\left[\frac{e^{-j2\pi fT}}{j2\pi f}\right] \\ &= \text{sgn}(t) - \text{sgn}(t-T) = 2\Pi\left(\frac{t-\frac{T}{2}}{T}\right) \end{aligned}$$

2) The signal waveform, to which $h(t)$ is matched, is

$$s(t) = h(T-t) = 2\Pi\left(\frac{T-t-\frac{T}{2}}{T}\right) = 2\Pi\left(\frac{\frac{T}{2}-t}{T}\right) = h(t)$$

where we have used the symmetry of $\Pi\left(\frac{t-\frac{T}{2}}{T}\right)$ with respect to the $t = \frac{T}{2}$ axis.

Problem 8.7

If $g_T(t) = \text{sinc}(t)$, then its matched waveform is $h(t) = \text{sinc}(-t) = \text{sinc}(t)$. Since, (see Problem 2.17)

$$\text{sinc}(t) \star \text{sinc}(t) = \text{sinc}(t)$$

the output of the matched filter is the same sinc pulse. If

$$g_T(t) = \text{sinc}\left(\frac{2}{T}\left(t - \frac{T}{2}\right)\right)$$

then the matched waveform is

$$h(t) = g_T(T-t) = \text{sinc}\left(\frac{2}{T}\left(\frac{T}{2} - t\right)\right) = g_T(t)$$

where the last equality follows from the fact that $g_T(t)$ is even with respect to the $t = \frac{T}{2}$ axis. The output of the matched filter is

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}[g_T(t) \star g_T(t)] \\ &= \mathcal{F}^{-1}\left[\frac{T^2}{4}\Pi\left(\frac{T}{2}f\right)e^{-j2\pi fT}\right] \\ &= \frac{T}{2}\text{sinc}\left(\frac{2}{T}(t-T)\right) = \frac{T}{2}g_T\left(t - \frac{T}{2}\right) \end{aligned}$$

Thus the output of the matched filter is the same sinc function, scaled by $\frac{T}{2}$ and centered at $t = T$.

Problem 8.8

1) The output of the integrator is

$$\begin{aligned} y(t) &= \int_0^t r(\tau)d\tau = \int_0^t [s_i(\tau) + n(\tau)]d\tau \\ &= \int_0^t s_i(\tau)d\tau + \int_0^t n(\tau)d\tau \end{aligned}$$

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At time $t = T$ we have

$$y(T) = \int_0^T s_i(\tau)d\tau + \int_0^T n(\tau)d\tau = \pm\sqrt{\frac{\mathcal{E}_b}{T}}T + \int_0^T n(\tau)d\tau$$

The signal energy at the output of the integrator at $t = T$ is

$$\mathcal{E}_s = \left(\pm\sqrt{\frac{\mathcal{E}_b}{T}}T\right)^2 = \mathcal{E}_b T$$

whereas the noise power

$$\begin{aligned}
P_n &= E\left[\int_0^T \int_0^T n(\tau)n(v)d\tau dv\right] \\
&= \int_0^T \int_0^T E[n(\tau)n(v)]d\tau dv \\
&= \frac{N_0}{2}\int_0^T \int_0^T \delta(\tau - v)d\tau dv = \frac{N_0}{2}T
\end{aligned}$$

Hence, the output SNR is

$$\text{SNR} = \frac{\mathcal{E}_s}{P_n} = \frac{2\mathcal{E}_b}{N_0}$$

2) The transfer function of the RC filter is

$$H(f) = \frac{1}{1 + j2\pi RCf}$$

Thus, the impulse response of the filter is

$$h(t) = \frac{1}{RC}e^{-\frac{t}{RC}}u_{-1}(t)$$

and the output signal is given by

$$\begin{aligned}
y(t) &= \frac{1}{RC}\int_{-\infty}^t r(\tau)e^{-\frac{t-\tau}{RC}}d\tau \\
&= \frac{1}{RC}\int_{-\infty}^t (s_i(\tau) + n(\tau))e^{-\frac{t-\tau}{RC}}d\tau \\
&= \frac{1}{RC}e^{-\frac{t}{RC}}\int_0^t s_i(\tau)e^{\frac{\tau}{RC}}d\tau + \frac{1}{RC}e^{-\frac{t}{RC}}\int_{-\infty}^t n(\tau)e^{\frac{\tau}{RC}}d\tau
\end{aligned}$$

At time $t = T$ we obtain

$$y(T) = \frac{1}{RC}e^{-\frac{T}{RC}}\int_0^T s_i(\tau)e^{\frac{\tau}{RC}}d\tau + \frac{1}{RC}e^{-\frac{T}{RC}}\int_{-\infty}^T n(\tau)e^{\frac{\tau}{RC}}d\tau$$

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The signal energy at the output of the filter is

$$\begin{aligned}
\mathcal{E}_s &= \frac{1}{(RC)^2} e^{-\frac{2T}{RC}} \int_0^T \int_0^T s_i(\tau) s_i(v) e^{\frac{\tau}{RC}} e^{\frac{v}{RC}} d\tau dv \\
&= \frac{1}{(RC)^2} e^{-\frac{2T}{RC}} \frac{\mathcal{E}_b}{T} \left(\int_0^T e^{\frac{\tau}{RC}} d\tau \right)^2 \\
&= e^{-\frac{2T}{RC}} \frac{\mathcal{E}_b}{T} \left(e^{\frac{T}{RC}} - 1 \right)^2 \\
&= \frac{\mathcal{E}_b}{T} \left(1 - e^{-\frac{T}{RC}} \right)^2
\end{aligned}$$

The noise power at the output of the filter is

$$\begin{aligned}
P_n &= \frac{1}{(RC)^2} e^{-\frac{2T}{RC}} \int_{-\infty}^T \int_{-\infty}^T E[n(\tau)n(v)] d\tau dv \\
&= \frac{1}{(RC)^2} e^{-\frac{2T}{RC}} \int_{-\infty}^T \int_{-\infty}^T \frac{N_0}{2} \delta(\tau - v) e^{\frac{\tau+v}{RC}} d\tau dv \\
&= \frac{1}{(RC)^2} e^{-\frac{2T}{RC}} \int_{-\infty}^T \frac{N_0}{2} e^{\frac{2\tau}{RC}} d\tau \\
&= \frac{1}{2RC} e^{-\frac{2T}{RC}} \frac{N_0}{2} e^{\frac{2T}{RC}} = \frac{1}{2RC} \frac{N_0}{2}
\end{aligned}$$

Hence,

$$\text{SNR} = \frac{\mathcal{E}_s}{P_n} = \frac{4\mathcal{E}_b RC}{T N_0} \left(1 - e^{-\frac{T}{RC}} \right)^2$$

3) The value of RC that maximizes SNR, can be found by setting the partial derivative of SNR with respect to RC equal to zero. Thus, if $a = RC$, then

$$\frac{\vartheta \text{SNR}}{\vartheta a} = 0 = (1 - e^{-\frac{T}{a}}) - \frac{T}{a} e^{-\frac{T}{a}} = -e^{-\frac{T}{a}} \left(1 + \frac{T}{a} \right) + 1$$

Solving this transcendental equation numerically for a, we obtain

$$\frac{T}{a} = 1.26 \Longrightarrow RC = a = \frac{T}{1.26}$$

Problem 8.9

1) The matched filter is

$$h_1(t) = s_1(T - t) = \begin{cases} -\frac{1}{T} t + 1, & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

The output of the matched filter is

$$y_1(t) = \int_{-\infty}^{\infty} s_1(\tau) h_1(t - \tau) d\tau$$

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If $t \leq 0$, then $y_1(t) = 0$, If $0 < t \leq T$, then

$$\begin{aligned}
y_1(t) &= \int_0^{\infty} \frac{\tau}{T}\left(-\frac{1}{T}(t-\tau)+1\right)d\tau \\
&= \int_0^t \tau\left(\frac{1}{T}-\frac{t}{T^2}\right)d\tau + \frac{1}{T^2}\int_0^t \tau^2 d\tau \\
&= -\frac{t^3}{6T^2}+\frac{t^2}{2T}
\end{aligned}$$

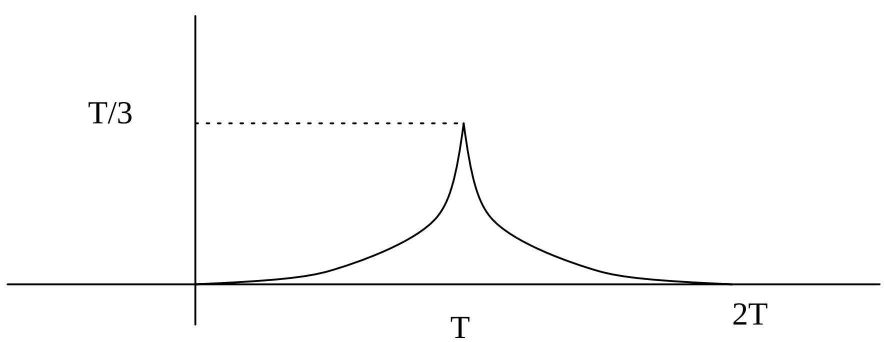
If $T \leq t \leq 2T$, then

$$\begin{aligned}
y_1(t) &= \int_{t-\tau}^{T} \frac{\tau}{T}\left(-\frac{1}{T}(t-\tau)+1\right)d\tau \\
&= \int_{t-\tau}^{T} \tau\left(\frac{1}{T}-\frac{t}{T^2}\right)d\tau + \frac{1}{T^2}\int_{t-\tau}^{T} \tau^2 d\tau \\
&= \frac{(t-T)^3}{6T^2}-\frac{t-T}{2}+\frac{T}{3}
\end{aligned}$$

For $2T < 0$, we obtain $y_1(t) = 0$. In summary

$$y_1(t) = \begin{cases} 0 & t \leq 0 \\ -\frac{t^3}{6T^2}+\frac{t^2}{2T} & 0 < t \leq T \\ \frac{(t-T)^3}{6T^2}-\frac{t-T}{2}+\frac{T}{3} & T < t \leq 2T \\ 0 & 2T < t \end{cases}$$

A sketch of $y_1(t)$ is given in the next figure. As it is observed the maximum of $y_1(t)$, which is $\frac{T}{3}$, is achieved for $t = T$.



2) The signal waveform matched to $s_2(t)$ is

$$h_2(t) = \begin{cases} -1, & 0 \leq t \leq \frac{T}{2} \\ 2, & \frac{T}{2} < t \leq T \end{cases}$$

The output of the matched filter is

$$y_2(t) = \int_{-\infty}^{\infty} s_2(\tau)h_2(t-\tau)d\tau$$

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If $t \leq 0$ or $t \geq 2T$, then $y_2(t) = 0$. If $0 < t \leq \frac{T}{2}$, then $y_2(t) = \int_0^t (-2)d\tau = -2t$. If $\frac{T}{2} < t \leq T$, then

$$y_2(t) = \int_0^{t-\frac{T}{2}} 4d\tau + \int_{t-\frac{T}{2}}^{\frac{T}{2}} (-2)d\tau + \int_{-\frac{T}{2}}^{t} d\tau = 7t - \frac{9}{2}T$$

If $T < t \leq \frac{3T}{2}$, then

$$y_2(t) = \int_{t-T}^{\frac{T}{2}} 4d\tau + \int_{\frac{T}{2}}^{t-\frac{T}{2}} (-2)d\tau + \int_{t-\frac{T}{2}}^{T} d\tau = \frac{19T}{2} - 7t$$

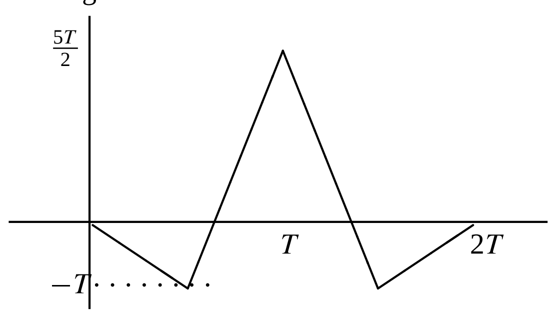
For, $\frac{3T}{2} < t \leq 2T$, we obtain

$$y_2(t) = \int_{t-T}^{T} (-2)d\tau = 2t - 4T$$

In summary

$$y_2(t) = \begin{cases} 0 & t \leq 0 \\ -2t & 0 < t \leq \frac{T}{2} \\ 7t - \frac{9}{2}T & \frac{T}{2} < t \leq T \\ \frac{19T}{2} - 7t & T < t \leq \frac{3T}{2} \\ 2t - 4T & \frac{3T}{2} < t \leq 2T \\ 0 & 2T < t \end{cases}$$

A plot of $y_2(t)$ is shown in the next figure



3) The signal waveform matched to $s_3(t)$ is

$$h_3(t) = \begin{cases} 2 & 0 \leq t \leq \frac{T}{2} \\ 0 & \frac{T}{2} < t \leq T \end{cases}$$

The output of the matched filter is

$$y_3(t) = h_3(t) \star s_3(t) = \begin{cases} 4t - 2T & \frac{T}{2} \leq t < T \\ -4t + 6T & T \leq t \leq \frac{3T}{2} \end{cases}$$

In the next figure we have plotted $y_3(t)$.

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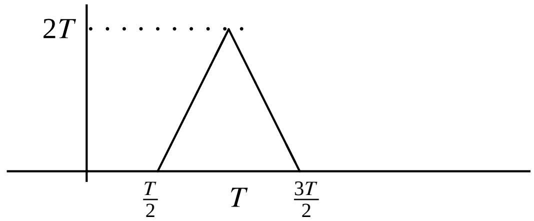

Problem 8.10

Since the rate of transmission is $R = 10^5$ bits/sec, the bit interval T_b is 10^{-5} sec. The probability of error in a binary PAM system is

$$P(e) = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right]$$

where the bit energy is $\mathcal{E}_b = A^2 T_b$. With $P(e) = P_2 = 10^{-6}$, we obtain

$$\sqrt{\frac{2\mathcal{E}_b}{N_0}} = 4.75 \Longrightarrow \mathcal{E}_b = \frac{4.75^2 N_0}{2} = 0.112813$$

Thus

$$A^2 T_b = 0.112813 \Longrightarrow A = \sqrt{0.112813 \times 10^5} = 106.21$$

Problem 8.11

1) For a binary PAM system for which the two signals have unequal probability, the optimum detector is

$$r \underset{s_2}{\overset{s_1}{\gtrless}} \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln \frac{1-p}{p} = \alpha^*$$

Here $\sqrt{\mathcal{E}_b}/N_0 = 10$ and $p = 0.3$. Substituting in the above gives $\alpha^* = 0.025 \times \ln \frac{7}{3} \approx 0.02118$.

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2) The average probability of error is

$$\begin{aligned}
P(e) &= P(e|s_1)P(s_1) + P(e|s_2)P(s_2) \\
&= pP(e|s_1) + (1-p)P(e|s_2) \\
&= p\int_{-\infty}^{\alpha^*} f(r|s_1)dr + (1-p)\int_{\alpha^*}^{\infty} f(r|s_1)dr \\
&= p\int_{-\infty}^{\alpha^*} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-\sqrt{\mathcal{E}_b})^2}{N_0}} dr + (1-p)\int_{\alpha^*}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r+\sqrt{\mathcal{E}_b})^2}{N_0}} dr \\
&= p\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha_1^*} e^{-\frac{x^2}{2}} dx + (1-p)\frac{1}{\sqrt{2\pi}}\int_{\alpha_2^*}^{\infty} e^{-\frac{x^2}{2}} dx
\end{aligned}$$

where

$$\alpha_1^* = -\sqrt{\frac{2\mathcal{E}_b}{N_0}} + \alpha^*\sqrt{\frac{2}{N_0}} \qquad \alpha_2^* = \sqrt{\frac{2\mathcal{E}_b}{N_0}} + \alpha^*\sqrt{\frac{2}{N_0}}$$

Thus,

$$P(e) = pQ\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} - \alpha^*\sqrt{\frac{2}{N_0}}\right] + (1-p)Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} + \alpha^*\sqrt{\frac{2}{N_0}}\right]$$

If $p = 0.3$ and $\frac{\mathcal{E}_b}{N_0} = 10$, then

$$\begin{aligned}
P(e) &= 0.3Q[4.3774] + 0.7Q[4.5668] \\
&= 3.5348 \times 10^{-6}
\end{aligned}$$

If the symbols are equiprobable, then we have

$$P(e) = Q[\sqrt{\frac{2\mathcal{E}_b}{N_0}}] = Q[\sqrt{20}] = 3.8721 \times 10^{-6}$$

Problem 8.12

Assuming that $E[n^2(t)] = \sigma_n^2$, we obtain

$$\begin{aligned}
E[n_1 n_2] &= E\left[\left(\int_0^T s_1(t)n(t)dt\right)\left(\int_0^T s_2(v)n(v)dv\right)\right] \\
&= \int_0^T \int_0^T s_1(t)s_2(v)E[n(t)n(v)]dtdv \\
&= \sigma_n^2 \int_0^T s_1(t)s_2(t)dt \\
&= 0
\end{aligned}$$

where the last equality follows from the orthogonality of the signal waveforms $s_1(t)$ and $s_2(t)$.

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Problem 8.13

1) The optimum threshold is given by

$$\alpha^* = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln \frac{1-p}{p} = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln 2$$

2) The average probability of error is ($\alpha^* = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln 2$)

$$\begin{aligned} P(e) &= p(a_m = -1) \int_{\alpha^*}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-(r+\sqrt{\mathcal{E}_b})^2/N_0} dr \\ &\quad + p(a_m = 1) \int_{-\infty}^{\alpha^*} \frac{1}{\sqrt{\pi N_0}} e^{-(r-\sqrt{\mathcal{E}_b})^2/N_0} dr \\ &= \frac{2}{3} Q\left[\frac{\alpha^* + \sqrt{\mathcal{E}_b}}{\sqrt{N_0/2}}\right] + \frac{1}{3} Q\left[\frac{\sqrt{\mathcal{E}_b} - \alpha^*}{\sqrt{N_0/2}}\right] \\ &= \frac{2}{3} Q\left[\frac{\sqrt{2N_0/\mathcal{E}_b} \ln 2}{4} + \sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] + \frac{1}{3} Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} - \frac{\sqrt{2N_0/\mathcal{E}_b} \ln 2}{4}\right] \end{aligned}$$

3) Here we have $P_e = \frac{2}{3} Q\left[\frac{\sqrt{2N_0/\mathcal{E}_b} \ln 2}{4} + \sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] + \frac{1}{3} Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} - \frac{\sqrt{2N_0/\mathcal{E}_b} \ln 2}{4}\right]$, substituting $\mathcal{E}_b = 1$ and $N_0 = 0.1$ we obtain

$$P_e = \frac{2}{3} Q\left[\frac{\sqrt{0.2} \times \ln 2}{4} + \sqrt{20}\right] + \frac{1}{3}\left[\sqrt{20} + \frac{\sqrt{0.2} \times \ln 2}{4}\right] = \frac{2}{3} Q(4.5496) - \frac{1}{3} Q(4.3946)$$

The result is $P_e = 3.64 \times 10^{-6}$.

Problem 8.14

1) The optimal receiver (see Problem 8.11) computes the metrics

$$C(\mathbf{r}, \mathbf{s}_m) = \int_{-\infty}^{\infty} r(t) s_m(t) dt - \frac{1}{2} \int_{-\infty}^{\infty} |s_m(t)|^2 dt + \frac{N_0}{2} \ln P(\mathbf{s}_m)$$

and decides in favor of the signal with the largest $C(\mathbf{r}, \mathbf{s}_m)$. Since $s_1(t) = -s_2(t)$, the energy of the two message signals is the same, and therefore the detection rule is written as

$$\int_{-\infty}^{\infty} r(t) s_1(t) dt \ \underset{s_2}{\overset{s_1}{\gtrless}} \ \frac{N_0}{4} \ln \frac{P(\mathbf{s}_2)}{P(\mathbf{s}_1)} = \frac{N_0}{4} \ln \frac{p_2}{p_1}$$

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2) If $s_1(t)$ is transmitted, then the output of the correlator is

$$\begin{aligned}\int_{-\infty}^{\infty} r(t)s_1(t)dt &= \int_0^T (s_1(t))^2 dt + \int_0^T n(t)s_1(t)dt \\ &= \mathcal{E}_s + n\end{aligned}$$

where $\mathcal{E}_s$ is the energy of the signal and n is a zero-mean Gaussian random variable with variance

$$\begin{aligned}\sigma_n^2 &= E\left[\int_0^T \int_0^T n(\tau)n(v)s_1(\tau)s_1(v)d\tau dv\right] \\ &= \int_0^T \int_0^T s_1(\tau)s_1(v)E[n(\tau)n(v)]d\tau dv \\ &= \frac{N_0}{2}\int_0^T \int_0^T s_1(\tau)s_1(v)\delta(\tau - v)d\tau dv \\ &= \frac{N_0}{2}\int_0^T |s_1(\tau)|^2 d\tau = \frac{N_0}{2}\mathcal{E}_s\end{aligned}$$

Hence, the probability of error $P(e|\mathbf{s}_1)$ is

$$\begin{aligned}P(e|\mathbf{s}_1) &= \int_{-\infty}^{\frac{N_0}{4}\ln\frac{p_2}{p_1} - \mathcal{E}_s} \frac{1}{\sqrt{\pi N_0 \mathcal{E}_s}} e^{-\frac{x^2}{N_0\mathcal{E}_s}} dx \\ &= Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}} - \frac{1}{4}\sqrt{\frac{2N_0}{\mathcal{E}_s}}\ln\frac{p_2}{p_1}\right]\end{aligned}$$

Similarly we find that

$$P(e|\mathbf{s}_2) = Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}} + \frac{1}{4}\sqrt{\frac{2N_0}{\mathcal{E}_s}}\ln\frac{p_2}{p_1}\right]$$

The average probability of error is

$$\begin{aligned}P(e) &= p_1 P(e|\mathbf{s}_1) + p_2 P(e|\mathbf{s}_2) \\ &= p_1 Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}} - \frac{1}{4}\sqrt{\frac{2N_0}{\mathcal{E}_s}}\ln\frac{1-p_1}{p_1}\right] + (1-p_1)Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}} + \frac{1}{4}\sqrt{\frac{2N_0}{\mathcal{E}_s}}\ln\frac{1-p_1}{p_1}\right]\end{aligned}$$

3) In the next figure we plot the probability of error as a function of p_1, for two values of the SNR $= \frac{2\mathcal{E}_s}{N_0}$. As it is observed the probability of error attains its maximum for equiprobable signals.

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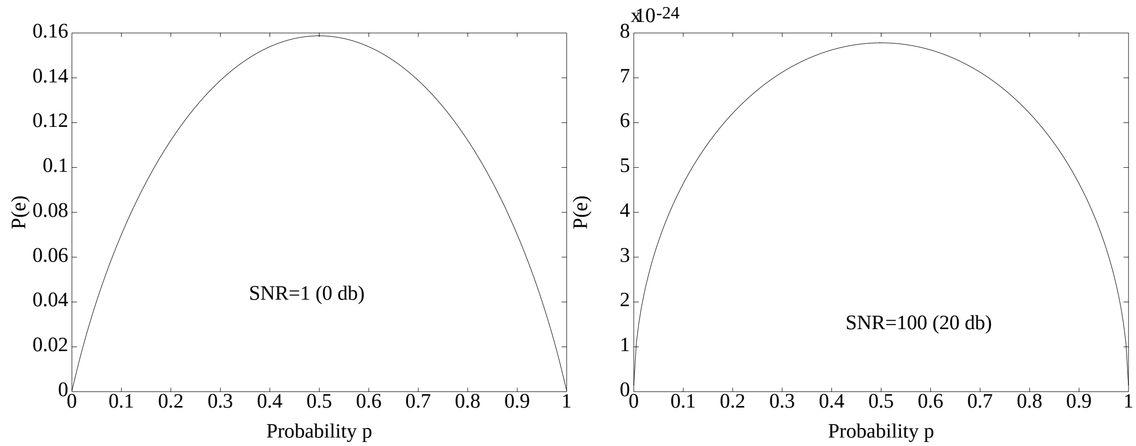

Problem 8.15

1) The two equiprobable signals have the same energy and therefore the optimal receiver bases its decisions on the rule

$$\int_{-\infty}^{\infty} r(t)s_1(t)dt \overset{s_1}{\underset{s_2}{\gtrless}} \int_{-\infty}^{\infty} r(t)s_2(t)dt$$

2) If the message signal $s_1(t)$ is transmitted, then $r(t) = s_1(t) + n(t)$ and the decision rule becomes

$$\begin{aligned}
&\int_{-\infty}^{\infty} (s_1(t) + n(t))(s_1(t) - s_2(t))dt \\
&= \int_{-\infty}^{\infty} s_1(t)(s_1(t) - s_2(t))dt + \int_{-\infty}^{\infty} n(t)(s_1(t) - s_2(t))dt \\
&= \int_{-\infty}^{\infty} s_1(t)(s_1(t) - s_2(t))dt + n \overset{s_1}{\underset{s_2}{\gtrless}} 0
\end{aligned}$$

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where n is a zero mean Gaussian random variable with variance

$$\begin{aligned}
\sigma_n^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(s_1(\tau)-s_2(\tau))(s_1(v)-s_2(v))E[n(\tau)n(v)]d\tau dv \\
&= \int_0^T\int_0^T (s_1(\tau)-s_2(\tau))(s_1(v)-s_2(v))\frac{N_0}{2}\delta(\tau-v)d\tau dv \\
&= \frac{N_0}{2}\int_0^T (s_1(\tau)-s_2(\tau))^2 d\tau \\
&= \frac{N_0}{2}\int_0^T\int_0^T\left(\frac{2A\tau}{T}-A\right)^2 d\tau \\
&= \frac{N_0}{2}\frac{A^2T}{3}
\end{aligned}$$

Since

$$\begin{aligned}
\int_{-\infty}^{\infty} s_1(t)(s_1(t)-s_2(t))dt &= \int_0^T \frac{At}{T}\left(\frac{2At}{T}-A\right)dt \\
&= \frac{A^2T}{6}
\end{aligned}$$

the probability of error $P(e|\mathbf{s}_1)$ is given by

$$\begin{aligned}
P(e|\mathbf{s}_1) &= P(\frac{A^2T}{6}+n<0) \\
&= \frac{1}{\sqrt{2\pi\frac{A^2TN_0}{6}}}\int_{-\infty}^{-\frac{A^2T}{6}}\exp\left(-\frac{x^2}{2\frac{A^2TN_0}{6}}\right)dx \\
&= Q\left[\sqrt{\frac{A^2T}{6N_0}}\right]
\end{aligned}$$

Similarly we find that

$$P(e|\mathbf{s}_2) = Q\left[\sqrt{\frac{A^2T}{6N_0}}\right]$$

and since the two signals are equiprobable, the average probability of error is given by

$$\begin{aligned}
P(e) &= \frac{1}{2}P(e|\mathbf{s}_1)+\frac{1}{2}P(e|\mathbf{s}_2) \\
&= Q\left[\sqrt{\frac{A^2T}{6N_0}}\right] = Q\left[\sqrt{\frac{\mathcal{E}_s}{2N_0}}\right]
\end{aligned}$$

where $\mathcal{E}_s$ is the energy of the transmitted signals.

Problem 8.16

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For binary phase modulation, the error probability is

$$P_2 = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = Q\left[\sqrt{\frac{A^2T}{N_0}}\right]$$

With $P_2 = 10^{-6}$ we find from tables that

$$\sqrt{\frac{A^2T}{N_0}} = 4.74 \Longrightarrow A^2T = 44.9352 \times 10^{-10}$$

If the data rate is 10 Kbps, then the bit interval is $T = 10^{-4}$ and therefore, the signal amplitude is

$$A = \sqrt{44.9352 \times 10^{-10} \times 10^4} = 6.7034 \times 10^{-3}$$

Similarly we find that when the rate is 10^5 bps and 10^6 bps, the required amplitude of the signal is $A = 2.12 \times 10^{-2}$ and $A = 6.703 \times 10^{-2}$ respectively.

Problem 8.17

The energy of the two signals $s_1(t)$ and $s_2(t)$ is

$$\mathcal{E}_b = A^2T$$

The dimensionality of the signal space is one, and by choosing the basis function as

$$\psi(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \leq t < \frac{T}{2} \\ -\frac{1}{\sqrt{T}} & \frac{T}{2} \leq t \leq T \end{cases}$$

we find the vector representation of the signals as

$$s_{1,2} = \pm A\sqrt{T} + n$$

with n a zero-mean Gaussian random variable of variance $\frac{N_0}{2}$. The probability of error for antipodal signals is given by, where $\mathcal{E}_b = A^2T$. Hence,

$$P(e) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left[\sqrt{\frac{2A^2T}{N_0}}\right]$$

Problem 8.18

Plots of $s(t)$ and $h(t)$ are shown on left and right, respectively.

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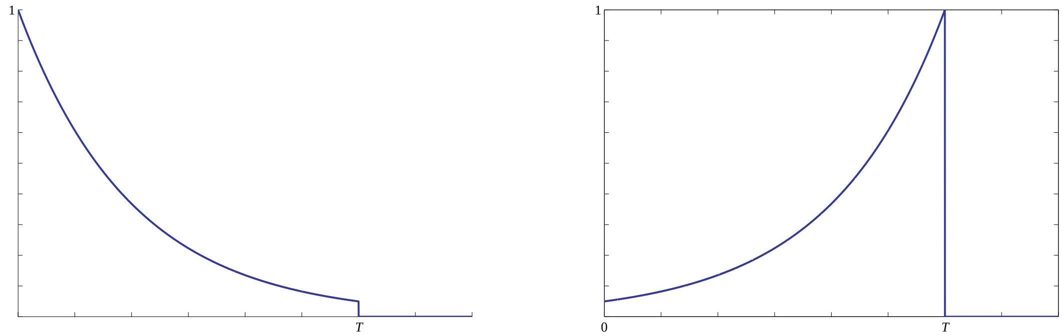
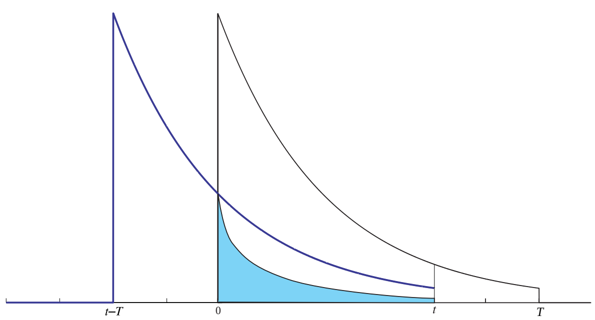


The output of the matched filter is

$$y(t) = \int_{-\infty}^{\infty} s(\tau)h(t-\tau)\,d\tau$$

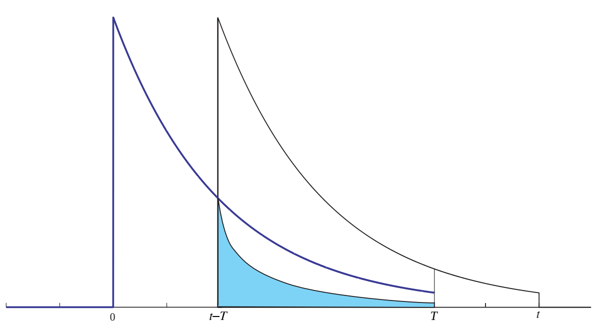
For $t < 0$, there is no overlap and the integral is zero. For $0 < t \le T$ we have the following figure, where the the product of the two signals in the overlapping region is $s(\tau)h(t-\tau) = e^{-\tau} \times e^{-(T-t+\tau)} = e^{t-T-2\tau}$ and the integral is the area of the shaded region.



For this case we have

$$\begin{aligned} y(t) &= e^{t-T} \int_0^t e^{-2\tau}\,d\tau \\ &= e^{t-T}\left[-\frac{1}{2}e^{-2\tau}\right]_0^t \\ &= \frac{1}{2}e^{-T}\left(e^t - e^{-t}\right) \end{aligned}$$

For $T < t \le 2T$ we have the following figure



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and

$$\begin{aligned} y(t) &= e^{t-T}\int_{t-T}^{T} e^{-2\tau}\,d\tau \\ &= e^{t-T}\left[-\frac{1}{2}e^{-2\tau}\right]_{t-T}^{T} \\ &= \frac{1}{2}e^{-t+T}-\frac{1}{2}e^{t-3T} \end{aligned}$$

Therefore

$$y(t)=\begin{cases} \frac{1}{2}e^{-T}\left(e^{t}-e^{-t}\right) & 0<t\le T \\ \frac{1}{2}e^{-t+T}-\frac{1}{2}e^{t-3T} & T<t\le 2 \\ 0 & \text{otherwise} \end{cases}$$

Problem 8.19

We have $P_{av}=R\mathcal{E}_b=2\times 10^6\mathcal{E}_b$, hence

$$P_b=Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)=Q\left(\sqrt{\frac{2P_{av}}{RN_0}}\right)=10^{-6}$$

Using the Q-function table (page 220) we have $Q(4.77)\approx 10^{-6}$, therefore

$$\sqrt{\frac{2P_{av}}{RN_0}}=\sqrt{\frac{2P_{av}}{2\times 10^6N_0}}=Q(4.77)\Rightarrow\frac{P_{av}}{10^6N_0}=4.77^2$$

From this we have $\frac{P_{av}}{N_0}=22.753\times 10^6$.

Problem 8.20

a) The received signal may be expressed as

$$r(t)=\begin{cases} n(t) & \text{if } s_0(t) \text{ was transmitted} \\ A+n(t) & \text{if } s_1(t) \text{ was transmitted} \end{cases}$$

Assuming that $s(t)$ has unit energy, then the sampled outputs of the crosscorrelators are

$$r=s_m+n,\qquad m=0,1$$

where $s_0=0$, $s_1=A\sqrt{T}$ and the noise term n is a zero-mean Gaussian random variable with variance

$$\begin{aligned} \sigma_n^2 &= E\left[\frac{1}{\sqrt{T}}\int_0^T n(t)dt\frac{1}{\sqrt{T}}\int_0^T n(\tau)d\tau\right] \\ &= \frac{1}{T}\int_0^T\int_0^T E\left[n(t)n(\tau)\right]dt\,d\tau \\ &= \frac{N_0}{2T}\int_0^T\int_0^T \delta(t-\tau)dt\,d\tau=\frac{N_0}{2} \end{aligned}$$

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The probability density function for the sampled output is

$$
\begin{aligned}
f(r|s_0) &= \frac{1}{\sqrt{\pi N_0}} e^{-\frac{r^2}{N_0}} \\
f(r|s_1) &= \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-A\sqrt{T})^2}{N_0}}
\end{aligned}
$$

Since the signals are equally probable, the optimal detector decides in favor of s_0 if

$$\mathrm{PM}(\mathbf{r}, \mathbf{s}_0) = f(r|s_0) > f(r|s_1) = \mathrm{PM}(\mathbf{r}, \mathbf{s}_1)$$

otherwise it decides in favor of s_1. The decision rule may be expressed as

$$\frac{\mathrm{PM}(\mathbf{r}, \mathbf{s}_0)}{\mathrm{PM}(\mathbf{r}, \mathbf{s}_1)} = e^{\frac{(r-A\sqrt{T})^2 - r^2}{N_0}} = e^{-\frac{(2r-A\sqrt{T})A\sqrt{T}}{N_0}} \underset{s_1}{\overset{s_0}{\gtrless}} 1$$

or equivalently

$$r \underset{s_0}{\overset{s_1}{\gtrless}} \frac{1}{2}A\sqrt{T}$$

The optimum threshold is $\frac{1}{2}A\sqrt{T}$.
b) The average probability of error is

$$
\begin{aligned}
P(e) &= \frac{1}{2}P(e|s_0) + \frac{1}{2}P(e|s_1) \\
&= \frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} f(r|s_0)dr + \frac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} f(r|s_1)dr \\
&= \frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{r^2}{N_0}} dr + \frac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-A\sqrt{T})^2}{N_0}} dr \\
&= \frac{1}{2}\int_{\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx + \frac{1}{2}\int_{-\infty}^{-\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= Q\left[\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}\right] = Q\left[\sqrt{\mathrm{SNR}}\right]
\end{aligned}
$$

where

$$\mathrm{SNR} = \frac{\frac{1}{2}A^2T}{N_0}$$

Thus, the on-off signaling requires a factor of two more energy to achieve the same probability of error as the antipodal signaling.

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Problem 8.21

1) Since $m_2(t) = -m_3(t)$ the dimensionality of the signal space is two.

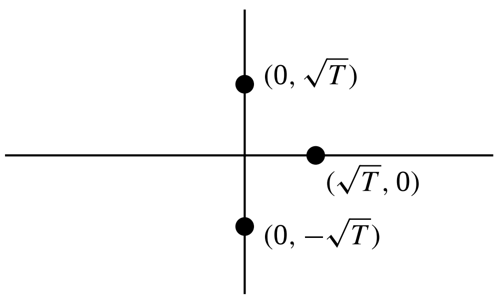
2) As a basis of the signal space we consider the functions

$$\psi_1(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases} \qquad \psi_2(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \le t \le \frac{T}{2} \\ -\frac{1}{\sqrt{T}} & \frac{T}{2} < t \le T \\ 0 & \text{otherwise} \end{cases}$$

The vector representation of the signals is

$$\begin{aligned} \mathbf{m}_1 &= [\sqrt{T}, \;\; 0] \\ \mathbf{m}_2 &= [0, \;\; \sqrt{T}] \\ \mathbf{m}_3 &= [0, \; -\sqrt{T}] \end{aligned}$$

3) The signal constellation is depicted in the next figure



4) The three possible outputs of the matched filters, corresponding to the three possible transmitted signals are $(r_1, r_2) = (\sqrt{T} + n_1, n_2)$, $(n_1, \sqrt{T} + n_2)$ and $(n_1, -\sqrt{T} + n_2)$, where n_1, n_2 are zero-mean Gaussian random variables with variance $\frac{N_0}{2}$. If all the signals are equiprobable the optimum decision rule selects the signal that maximizes the metric

$$C(\mathbf{r} \cdot \mathbf{m}_i) = 2\mathbf{r} \cdot \mathbf{m}_i - |\mathbf{m}_i|^2$$

or since $|\mathbf{m}_i|^2$ is the same for all i,

$$C'(\mathbf{r} \cdot \mathbf{m}_i) = \mathbf{r} \cdot \mathbf{m}_i$$

Thus the optimal decision region R_1 for $\mathbf{m}_1$ is the set of points (r_1, r_2), such that $(r_1, r_2) \cdot \mathbf{m}_1 > (r_1, r_2) \cdot \mathbf{m}_2$ and $(r_1, r_2) \cdot \mathbf{m}_1 > (r_1, r_2) \cdot \mathbf{m}_3$. Since $(r_1, r_2) \cdot \mathbf{m}_1 = \sqrt{T} r_1$, $(r_1, r_2) \cdot \mathbf{m}_2 = \sqrt{T} r_2$ and $(r_1, r_2) \cdot \mathbf{m}_3 = -\sqrt{T} r_2$, the previous conditions are written as

$$r_1 > r_2 \quad \text{and} \quad r_1 > -r_2$$

Similarly we find that R_2 is the set of points (r_1, r_2) that satisfy $r_2 > 0$, $r_2 > r_1$ and R_3 is the region such that $r_2 < 0$ and $r_2 < -r_1$. The regions R_1, R_2 and R_3 are shown in the next figure.

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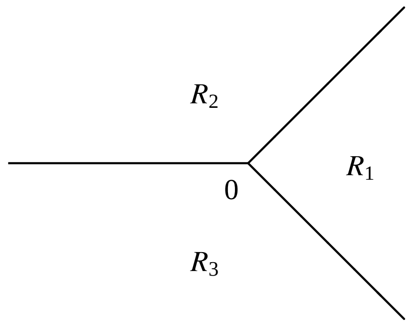

5) If the signals are equiprobable then,

$$P(e|\mathbf{m}_1) = P(|\mathbf{r}-\mathbf{m}_1|^2 > |\mathbf{r}-\mathbf{m}_2|^2 \big| \mathbf{m}_1) + P(|\mathbf{r}-\mathbf{m}_1|^2 > |\mathbf{r}-\mathbf{m}_3|^2 \big| \mathbf{m}_1)$$

When $\mathbf{m}_1$ is transmitted then $\mathbf{r} = [\sqrt{T}+n_1, n_2]$ and therefore, $P(e|\mathbf{m}_1)$ is written as

$$P(e|\mathbf{m}_1) = P(n_2 - n_1 > \sqrt{T}) + P(n_1 + n_2 < -\sqrt{T})$$

Since, n_1, n_2 are zero-mean statistically independent Gaussian random variables, each with variance $\frac{N_0}{2}$, the random variables $x = n_1 - n_2$ and $y = n_1 + n_2$ are zero-mean Gaussian with variance N_0. Hence,

$$\begin{aligned} P(e|\mathbf{m}_1) &= \frac{1}{\sqrt{2\pi N_0}} \int_{\sqrt{T}}^{\infty} e^{-\frac{x^2}{2N_0}} dx + \frac{1}{\sqrt{2\pi N_0}} \int_{-\infty}^{-\sqrt{T}} e^{-\frac{y^2}{2N_0}} dy \\ &= Q\left[\sqrt{\frac{T}{N_0}}\right] + Q\left[\sqrt{\frac{T}{N_0}}\right] = 2Q\left[\sqrt{\frac{T}{N_0}}\right] \end{aligned}$$

When $\mathbf{m}_2$ is transmitted then $\mathbf{r} = [n_1, n_2 + \sqrt{T}]$ and therefore,

$$\begin{aligned} P(e|\mathbf{m}_2) &= P(n_1 - n_2 > \sqrt{T}) + P(n_2 < -\sqrt{T}) \\ &= Q\left[\sqrt{\frac{T}{N_0}}\right] + Q\left[\sqrt{\frac{2T}{N_0}}\right] \end{aligned}$$

Similarly from the symmetry of the problem, we obtain

$$P(e|\mathbf{m}_2) = P(e|\mathbf{m}_3) = Q\left[\sqrt{\frac{T}{N_0}}\right] + Q\left[\sqrt{\frac{2T}{N_0}}\right]$$

Since $Q[\cdot]$ is monotonically decreasing, we obtain

$$Q\left[\sqrt{\frac{2T}{N_0}}\right] < Q\left[\sqrt{\frac{T}{N_0}}\right]$$

and therefore, the probability of error $P(e|\mathbf{m}_1)$ is larger than $P(e|\mathbf{m}_2)$ and $P(e|\mathbf{m}_3)$. Hence, the message $\mathbf{m}_1$ is more vulnerable to errors.

6) From the union bound

$$P_e \le (M-1)Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

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Here the minimum distance of the constellation is between $\mathbf{m}_1$ and $\mathbf{m}_2$ (or $\mathbf{m}_1$ and $\mathbf{m}_3$) and is equal to $\sqrt{2T}$. Therefore,

$$P_e \leq 2Q\left(\sqrt{\frac{T}{N_0}}\right)$$

Problem 8.22

The three symbols A, 0 and $-A$ are used with equal probability. Hence, the optimal detector uses two thresholds, which are $\frac{A}{2}$ and $-\frac{A}{2}$, and it bases its decisions on the criterion

$$\begin{array}{rl} A: & r > \frac{A}{2} \\ 0: & -\frac{A}{2} < r < \frac{A}{2} \\ -A: & r < -\frac{A}{2} \end{array}$$

If the variance of the AWG noise is σ_n^2, then the average probability of error is

$$\begin{aligned} P(e) &= \frac{1}{3}\int_{-\infty}^{\frac{A}{2}} \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{(r-A)^2}{2\sigma_n^2}} dr + \frac{1}{3}\left(1 - \int_{-\frac{A}{2}}^{\frac{A}{2}} \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{r^2}{2\sigma_n^2}} dr\right) \\ &\quad + \frac{1}{3}\int_{-\frac{A}{2}}^{\infty} \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{(r+A)^2}{2\sigma_n^2}} dr \\ &= \frac{1}{3}Q\left[\frac{A}{2\sigma_n}\right] + \frac{1}{3}2Q\left[\frac{A}{2\sigma_n}\right] + \frac{1}{3}Q\left[\frac{A}{2\sigma_n}\right] \\ &= \frac{4}{3}Q\left[\frac{A}{2\sigma_n}\right] \end{aligned}$$

Problem 8.23

1) The PDF of the noise n is

$$f(n) = \frac{\lambda}{2} e^{-\lambda|n|}$$

The optimal receiver uses the criterion

$$\frac{f(r|A)}{f(r|-A)} = e^{-\lambda[|r-A|-|r+A|]} \underset{-A}{\overset{A}{\gtrless}} 1 \Longrightarrow r \underset{-A}{\overset{A}{\gtrless}} 0$$

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The average probability of error is

$$\begin{aligned}
P(e) &= \frac{1}{2}P(e|A) + \frac{1}{2}P(e|-A) \\
&= \frac{1}{2}\int_{-\infty}^{0} f(r|A)dr + \frac{1}{2}\int_{0}^{\infty} f(r|-A)dr \\
&= \frac{1}{2}\int_{-\infty}^{0} \lambda_2 e^{-\lambda|r-A|}dr + \frac{1}{2}\int_{0}^{\infty} \lambda_2 e^{-\lambda|r+A|}dr \\
&= \frac{\lambda}{4}\int_{-\infty}^{-A} e^{-\lambda|x|}dx + \frac{\lambda}{4}\int_{A}^{\infty} e^{-\lambda|x|}dx \\
&= \frac{\lambda}{4}\frac{1}{\lambda}e^{\lambda x}\Big|_{-\infty}^{-A} + \frac{\lambda}{4}\left(-\frac{1}{\lambda}\right)e^{-\lambda x}\Big|_{A}^{\infty} \\
&= \frac{1}{2}e^{-\lambda A}
\end{aligned}$$

2) The variance of the noise is

$$\begin{aligned}
\sigma_n^2 &= \frac{\lambda}{2}\int_{-\infty}^{\infty} e^{-\lambda|x|}x^2dx \\
&= \lambda\int_{0}^{\infty} e^{-\lambda x}x^2dx = \lambda\frac{2!}{\lambda^3} = \frac{2}{\lambda^2}
\end{aligned}$$

Hence, the SNR is

$$\text{SNR} = \frac{A^2}{\frac{2}{\lambda^2}} = \frac{A^2\lambda^2}{2}$$

and the probability of error is given by

$$P(e) = \frac{1}{2}e^{-\sqrt{\lambda^2A^2}} = \frac{1}{2}e^{-\sqrt{2\text{SNR}}}$$

For $P(e) = 10^{-5}$ we obtain

$$\ln(2\times 10^{-5}) = -\sqrt{2\text{SNR}} \Longrightarrow \text{SNR} = 58.534 = 17.6741 \text{ dB}$$

If the noise was Gaussian, then

$$P(e) = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = Q\left[\sqrt{\text{SNR}}\right]$$

where SNR is the signal to noise ratio at the output of the matched filter. With $P(e) = 10^{-5}$ we find $\sqrt{\text{SNR}} = 4.26$ and therefore SNR = 18.1476 = 12.594 dB. Thus the required signal to noise ratio is 5 dB less when the additive noise is Gaussian.

Problem 8.24

The points in the constellation are at distance $\pm d, \pm 3d, \pm 5d, \ldots, \pm(M-1)d$ from the origin. Since the square of the distance of a point in the constellation from the origin is equal to the energy of the signal

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corresponding to that point, we have two signals with energy d^2, two signals with energy $9d^2$, two signals with energy $25d^2$,...., and two signals with energy $(M-1)^2d^2$. The average energy is

$$E_{av} = \frac{1}{M}\sum_i E_i = \frac{2d^2}{M}\left(1+9+25+\cdots+(M-1)^2\right)$$

Using the well known relation

$$1^2+2^2+3^2+\cdots+M^2 = \frac{M(M+1)(2M+1)}{6}$$

we have (note that $M=2^k$ is even)

$$2^2+4^2+\cdots+(M)^2 = 4\left(1^2+2^2+\cdots+\left(\frac{M}{2}\right)^2\right) = \frac{M(M+1)(M+2)}{6}$$

subtracting the latter two series gives

$$1^2+3^2+\cdots+(M-1)^2 = \frac{M(M+1)(2M+1)}{6} - \frac{M(M+1)(M+2)}{6} = \frac{M\left(M^2-1\right)}{6}$$

Therefore,

$$E_{av} = \frac{2d^2}{M}\times\frac{M\left(M^2-1\right)}{6} = \frac{d^2(M^2-1)}{3}$$

Problem 8.25

The correlation coefficient between the m^{th} and the n^{th} signal points is

$$\gamma_{mn} = \frac{\mathbf{s}_m\cdot\mathbf{s}_n}{|\mathbf{s}_m||\mathbf{s}_n|}$$

where $\mathbf{s}_m = (s_{m1}, s_{m2}, \ldots, s_{mN})$ and $s_{mj} = \pm\sqrt{\frac{\mathcal{E}_s}{N}}$. Two adjacent signal points differ in only one coordinate, for which s_{mk} and s_{nk} have opposite signs. Hence,

$$\begin{aligned}\mathbf{s}_m\cdot\mathbf{s}_n &= \sum_{j=1}^{N} s_{mj}s_{nj} = \sum_{j\neq k} s_{mj}s_{nj} + s_{mk}s_{nk}\\ &= (N-1)\frac{\mathcal{E}_s}{N} - \frac{\mathcal{E}_s}{N} = \frac{N-2}{N}\mathcal{E}_s\end{aligned}$$

Furthermore, $|\mathbf{s}_m| = |\mathbf{s}_n| = (\mathcal{E}_s)^{\frac{1}{2}}$ so that

$$\gamma_{mn} = \frac{N-2}{N}$$

The Euclidean distance between the two adjacent signal points is

$$d = \sqrt{|\mathbf{s}_m-\mathbf{s}_n|^2} = \sqrt{\left|\pm2\sqrt{\mathcal{E}_s/N}\right|^2} = \sqrt{4\frac{\mathcal{E}_s}{N}} = 2\sqrt{\frac{\mathcal{E}_s}{N}}$$

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Problem 8.26

The energy of the signal waveform $s'_m(t)$ is

$$\begin{aligned}
\mathcal{E}' &= \int_{-\infty}^{\infty} \left|s'_m(t)\right|^2 dt = \int_{-\infty}^{\infty} \left|s_m(t) - \frac{1}{M}\sum_{k=1}^{M} s_k(t)\right|^2 dt \\
&= \int_{-\infty}^{\infty} s_m^2(t)dt + \frac{1}{M^2}\sum_{k=1}^{M}\sum_{l=1}^{M}\int_{-\infty}^{\infty} s_k(t)s_l(t)dt \\
&\quad -\frac{1}{M}\sum_{k=1}^{M}\int_{-\infty}^{\infty} s_m(t)s_k(t)dt - \frac{1}{M}\sum_{l=1}^{M}\int_{-\infty}^{\infty} s_m(t)s_l(t)dt \\
&= \mathcal{E} + \frac{1}{M^2}\sum_{k=1}^{M}\sum_{l=1}^{M}\mathcal{E}\delta_{kl} - \frac{2}{M}\mathcal{E} \\
&= \mathcal{E} + \frac{1}{M}\mathcal{E} - \frac{2}{M}\mathcal{E} = \left(\frac{M-1}{M}\right)\mathcal{E}
\end{aligned}$$

The correlation coefficient is given by

$$\begin{aligned}
\gamma_{mn} &= \frac{\int_{-\infty}^{\infty} s'_m(t)s'_n(t)dt}{\left[\int_{-\infty}^{\infty}|s'_m(t)|^2dt\right]^{\frac{1}{2}}\left[\int_{-\infty}^{\infty}|s'_n(t)|^2dt\right]^{\frac{1}{2}}} \\
&= \frac{1}{\mathcal{E}'}\int_{-\infty}^{\infty}\left(s_m(t) - \frac{1}{M}\sum_{k=1}^{M}s_k(t)\right)\left(s_n(t) - \frac{1}{M}\sum_{l=1}^{M}s_l(t)\right)dt \\
&= \frac{1}{\mathcal{E}'}\left(\int_{-\infty}^{\infty} s_m(t)s_n(t)dt + \frac{1}{M^2}\sum_{k=1}^{M}\sum_{l=1}^{M}\int_{-\infty}^{\infty} s_k(t)s_l(t)dt\right) \\
&\quad -\frac{1}{\mathcal{E}'}\left(\frac{1}{M}\sum_{k=1}^{M}\int_{-\infty}^{\infty} s_n(t)s_k(t)dt + \frac{1}{M}\sum_{l=1}^{M}\int_{-\infty}^{\infty} s_m(t)s_l(t)dt\right) \\
&= \frac{\frac{1}{M^2}M\mathcal{E} - \frac{1}{M}\mathcal{E} - \frac{1}{M}\mathcal{E}}{\frac{M-1}{M}\mathcal{E}} = -\frac{1}{M-1}
\end{aligned}$$

Problem 8.27

The optimal receiver bases its decisions on the metrics

$$\text{PM}(\mathbf{r}, \mathbf{s}_m) = f(\mathbf{r}|\mathbf{s}_m)P(\mathbf{s}_m)$$

For an additive noise channel $\mathbf{r} = \mathbf{s}_m + \mathbf{n}$, so

$$\text{PM}(\mathbf{r}, \mathbf{s}_m) = f(\mathbf{n})P(\mathbf{s}_m)$$

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where $f(\mathbf{n})$ is the N-dimensional PDF for the noise channel vector. If the noise is AWG, then

$$f\mathbf{n}) = \frac{1}{(\pi N_0)^{\frac{N}{2}}} e^{-\frac{|\mathbf{r}-\mathbf{s}_m|^2}{N_0}}$$

Maximizing $f(\mathbf{r}|\mathbf{s}_m)P(\mathbf{s}_m)$ is the same as minimizing the reciprocal $e^{\frac{|\mathbf{r}-\mathbf{s}_m|^2}{N_0}}/P(\mathbf{s}_m)$, or by taking the natural logarithm, minimizing the cost

$$D(\mathbf{r}, \mathbf{s}_m) = |\mathbf{r} - \mathbf{s}_m|^2 - N_0 P(\mathbf{s}_m)$$

This is equivalent to the maximization of the quantity

$$C(\mathbf{r}, \mathbf{s}_m) = \mathbf{r} \cdot \mathbf{s}_m - \frac{1}{2}|\mathbf{s}_m|^2 + \frac{N_0}{2} \ln P(\mathbf{s}_m)$$

If the vectors $\mathbf{r}$, $\mathbf{s}_m$ correspond to the waveforms $r(t)$ and $s_m(t)$, where

$$\begin{aligned} r(t) &= \sum_{i=1}^{N} r_i \psi_i(t) \\ s_m(t) &= \sum_{i=1}^{N} s_{m,i} \psi_i(t) \end{aligned}$$

then,

$$\begin{aligned} \int_{-\infty}^{\infty} r(t) s_m(t) dt &= \int_{-\infty}^{\infty} \sum_{i=1}^{N} r_i \psi_i(t) \sum_{j=1}^{N} s_{m,j} \psi_j(t) dt \\ &= \sum_{i=1}^{N} \sum_{j=1}^{N} r_i s_{m,j} \int_{-\infty}^{\infty} \psi_i(t) \psi_j(t) dt \\ &= \sum_{i=1}^{N} \sum_{j=1}^{N} r_i s_{m,j} \delta_{i,j} = \sum_{i=1}^{N} r_i s_{m,i} \\ &= \mathbf{r} \cdot \mathbf{s}_m \end{aligned}$$

Similarly we obtain

$$\int_{-\infty}^{\infty} |s_m(t)|^2 dt = |\mathbf{s}_m|^2 = \mathcal{E}_{s_m}$$

Therefore, the optimal receiver can use the costs

$$\begin{aligned} C(\mathbf{r}, \mathbf{s}_m) &= \int_{-\infty}^{\infty} r(t) s_m(t) dt - \frac{1}{2} \int_{-\infty}^{\infty} |s_m(t)|^2 dt + \frac{N_0}{2} \ln P(\mathbf{s}_m) \\ &= \int_{-\infty}^{\infty} r(t) s_m(t) dt + c_m \end{aligned}$$

to base its decisions. This receiver can be implemented using M correlators to evaluate $\int_{-\infty}^{\infty} r(t) s_m(t) dt$. The bias constants c_m can be precomputed and added to the output of the correlators. The structure of the receiver is shown in the next figure.

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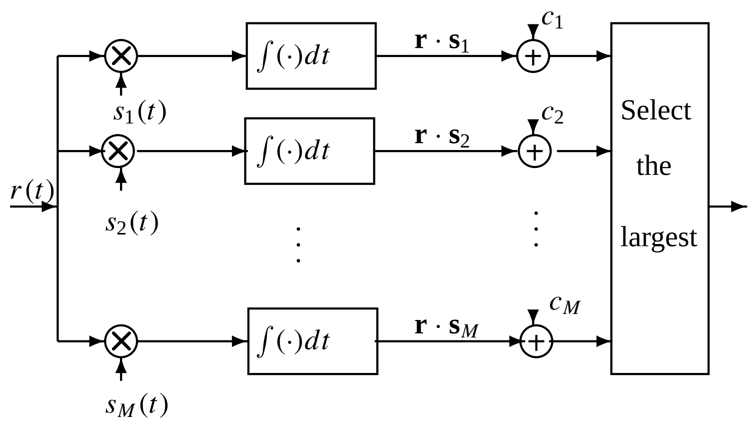
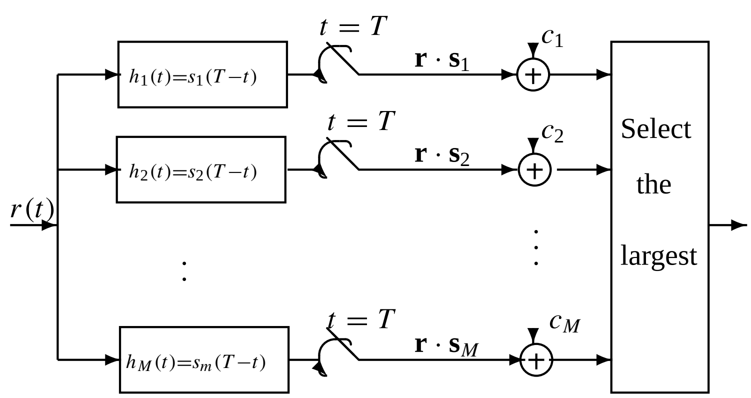


Parallel to the development of the optimal receiver using N filters matched to the orthonormal functions $\psi_i(t)$, $i = 1, \ldots, N$, the M correlators can be replaced by M equivalent filters matched to the signal waveforms $s_m(t)$. The output of the m^{th} matched filter $h_m(t)$, at the time instant T is

$$\begin{aligned}
\int_0^T r(\tau)h_m(T-\tau)d\tau &= \int_0^T r(\tau)s_m(T-(T-\tau))d\tau \\
&= \int_0^T r(\tau)s_m(\tau)d\tau \\
&= \mathbf{r}\cdot\mathbf{s}_m
\end{aligned}$$

The structure of this optimal receiver is shown in the next figure. The optimal receivers, derived in this problem, are more costly than those derived in the text, since N is usually less than M, the number of signal waveforms. For example, in an M-ary PAM system, $N = 1$ always less than M.



Problem 8.28

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The biorthogonal signal set has the form

$$\begin{aligned}
\mathbf{s}_1 &= [\sqrt{\mathcal{E}_s}, 0, 0, 0] & \mathbf{s}_5 &= [-\sqrt{\mathcal{E}_s}, 0, 0, 0] \\
\mathbf{s}_2 &= [0, \sqrt{\mathcal{E}_s}, 0, 0] & \mathbf{s}_6 &= [0, -\sqrt{\mathcal{E}_s}, 0, 0] \\
\mathbf{s}_3 &= [0, 0, \sqrt{\mathcal{E}_s}, 0] & \mathbf{s}_7 &= [0, 0, -\sqrt{\mathcal{E}_s}, 0] \\
\mathbf{s}_4 &= [0, 0, 0, \sqrt{\mathcal{E}_s}] & \mathbf{s}_8 &= [0, 0, 0, -\sqrt{\mathcal{E}_s}]
\end{aligned}$$

For each point $\mathbf{s}_i$, there are $M - 2 = 6$ points at a distance

$$d_{i,k} = |\mathbf{s}_i - \mathbf{s}_k| = \sqrt{2\mathcal{E}_s}$$

and one vector $(-\mathbf{s}_i)$ at a distance $d_{i,m} = 2\sqrt{\mathcal{E}_s}$. Hence, the union bound on the probability of error $P(e|\mathbf{s}_i)$ is given by

$$P_{\mathrm{UB}}(e|\mathbf{s}_i) = \sum_{k=1,k\neq i}^{M} Q\left[\frac{d_{i,k}}{\sqrt{2N_0}}\right] = 6Q\left[\sqrt{\frac{\mathcal{E}_s}{N_0}}\right] + Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right]$$

Since all the signals are equiprobable, we find that

$$P_{\mathrm{UB}}(e) = 6Q\left[\sqrt{\frac{\mathcal{E}_s}{N_0}}\right] + Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right]$$

With $M = 8 = 2^3$, $\mathcal{E}_s = 3\mathcal{E}_b$ and therefore,

$$P_{\mathrm{UB}}(e) = 6Q\left[\sqrt{\frac{3\mathcal{E}_b}{N_0}}\right] + Q\left[\sqrt{\frac{6\mathcal{E}_b}{N_0}}\right]$$

Problem 8.29

It is convenient to find first the probability of a correct decision. Since all signals are equiprobable

$$P(C) = \sum_{i=1}^{M} \frac{1}{M} P(C|\mathbf{s}_i)$$

All the $P(C|\mathbf{s}_i)$, $i = 1, \ldots, M$ are identical because of the symmetry of the constellation. By translating the vector $\mathbf{s}_i$ to the origin we can find the probability of a correct decision, given that $\mathbf{s}_i$ was transmitted, as

$$P(C|\mathbf{s}_i) = \int_{-\frac{d}{2}}^{\infty} f(n_1)dn_1 \int_{-\frac{d}{2}}^{\infty} f(n_2)dn_2 \ldots \int_{-\frac{d}{2}}^{\infty} f(n_N)dn_N$$

where the number of the integrals on the right side of the equation is N, d is the minimum distance between the points and

$$f(n_i) = \frac{1}{\pi N_0} e^{-\frac{n_i^2}{N_0}}$$

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Hence,

$$\begin{aligned} P(C|\mathbf{s}_i) &= \left(\int_{-\frac{d}{2}}^{\infty} f(n)dn\right)^N = \left(1 - \int_{-\infty}^{-\frac{d}{2}} f(n)dn\right)^N \\ &= \left(1 - Q\left[\frac{d}{\sqrt{2N_0}}\right]\right)^N \end{aligned}$$

and therefore, the probability of error is given by

$$\begin{aligned} P(e) &= 1 - P(C) = 1 - \sum_{i=1}^{M} \frac{1}{M}\left(1 - Q\left[\frac{d}{\sqrt{2N_0}}\right]\right)^N \\ &= 1 - \left(1 - Q\left[\frac{d}{\sqrt{2N_0}}\right]\right)^N \end{aligned}$$

Note that since

$$\mathcal{E}_s = \sum_{i=1}^{N} s_{m,i}^2 = \sum_{i=1}^{N} (\frac{d}{2})^2 = N\frac{d^2}{4}$$

the probability of error can be written as

$$P(e) = 1 - \left(1 - Q\left[\sqrt{\frac{2\mathcal{E}_s}{NN_0}}\right]\right)^N$$

Problem 8.30

Consider first the signal

$$y(t) = \sum_{k=1}^{n} c_k \delta(t - kT_c)$$

The signal $y(t)$ has duration $T = nT_c$ and its matched filter is

$$\begin{aligned} g(t) &= y(T-t) = y(nT_c - t) = \sum_{k=1}^{n} c_k \delta(nT_c - kT_c - t) \\ &= \sum_{i=1}^{n} c_{n-i+1}\delta((i-1)T_c - t) = \sum_{i=1}^{n} c_{n-i+1}\delta(t - (i-1)T_c) \end{aligned}$$

that is, a sequence of impulses starting at $t = 0$ and weighted by the mirror image sequence of $\{c_i\}$. Since,

$$s(t) = \sum_{k=1}^{n} c_k p(t - kT_c) = p(t) \star \sum_{k=1}^{n} c_k \delta(t - kT_c)$$

the Fourier transform of the signal $s(t)$ is

$$S(f) = P(f)\sum_{k=1}^{n} c_k e^{-j2\pi f kT_c}$$

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and therefore, the Fourier transform of the signal matched to $s(t)$ is

$$\begin{aligned}
H(f) &= S^*(f)e^{-j2\pi fT} = S^*(f)e^{-j2\pi fnT_c} \\
&= P^*(f)\sum_{k=1}^{n} c_k e^{j2\pi fkT_c} e^{-j2\pi fnT_c} \\
&= P^*(f)\sum_{i=1}^{n} c_{n-i+1} e^{-j2\pi f(i-1)T-c} \\
&= P^*(f)\mathcal{F}[g(t)]
\end{aligned}$$

Thus, the matched filter $H(f)$ can be considered as the cascade of a filter,with impulse response $p(-t)$, matched to the pulse $p(t)$ and a filter, with impulse response $g(t)$, matched to the signal $y(t) = \sum_{k=1}^{n} c_k \delta(t - kT_c)$. The output of the matched filter at $t = nT_c$ is

$$\begin{aligned}
\int_{-\infty}^{\infty} |s(t)|^2 &= \sum_{k=1}^{n} c_k^2 \int_{-\infty}^{\infty} p^2(t - kT_c)dt \\
&= T_c \sum_{k=1}^{n} c_k^2
\end{aligned}$$

where we have used the fact that $p(t)$ is a rectangular pulse of unit amplitude and duration T_c.

Problem 8.31

The bandwidth required for transmission of an M-ary PAM signal is

$$W = \frac{R_b}{2\log_2 M} \text{ Hz}$$

Since,

$$R_b = 8 \times 10^3 \frac{\text{samples}}{\text{sec}} \times 8 \frac{\text{bits}}{\text{sample}} = 64 \times 10^3 \frac{\text{bits}}{\text{sec}}$$

we obtain

$$W = \begin{cases} 16 \text{ KHz} & M = 4 \\ 10.667 \text{ KHz} & M = 8 \\ 8 \text{ KHz} & M = 16 \end{cases}$$

Problem 8.32

The vector $\mathbf{r} = [r_1, r_2]$ at the output of the integrators is

$$\mathbf{r} = [r_1, r_2] = [\int_0^{1.5} r(t)dt, \quad \int_1^2 r(t)dt]$$

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If $s_1(t)$ is transmitted, then

$$\begin{aligned}\int_0^{1.5} r(t)dt &= \int_0^{1.5} [s_1(t)+n(t)]dt = 1 + \int_0^{1.5} n(t)dt \\ &= 1+n_1 \\ \int_1^2 r(t)dt &= \int_1^2 [s_1(t)+n(t)]dt = \int_1^2 n(t)dt \\ &= n_2\end{aligned}$$

where n_1 is a zero-mean Gaussian random variable with variance

$$\sigma_{n_1}^2 = E\left[\int_0^{1.5}\int_0^{1.5} n(\tau)n(v)d\tau dv\right] = \frac{N_0}{2}\int_0^{1.5} d\tau = 1.5$$

and n_2 is is a zero-mean Gaussian random variable with variance

$$\sigma_{n_2}^2 = E\left[\int_1^2\int_1^2 n(\tau)n(v)d\tau dv\right] = \frac{N_0}{2}\int_1^2 d\tau = 1$$

Thus, the vector representation of the received signal (at the output of the integrators) is

$$\mathbf{r} = [1+n_1,\ n_2]$$

Similarly we find that if $s_2(t)$ is transmitted, then

$$\mathbf{r} = [0.5+n_1,\ 1+n_2]$$

Suppose now that the detector bases its decisions on the rule

$$r_1 - r_2 \underset{s_2}{\overset{s_1}{\gtrless}} T$$

The probability of error $P(e|s_1)$ is obtained as

$$\begin{aligned}P(e|s_1) &= P(r1 - r_2 < T|s_1) \\ &= P(1+n_1-n_2 < T) = P(n_1-n_2 < T-1) \\ &= P(n < T)\end{aligned}$$

where the random variable $n = n_1 - n_2$ is zero-mean Gaussian with variance

$$\begin{aligned}\sigma_n^2 &= \sigma_{n_1}^2 + \sigma_{n_2}^2 - 2E[n_1 n_2] \\ &= \sigma_{n_1}^2 + \sigma_{n_2}^2 - 2\int_1^{1.5}\frac{N_0}{2}d\tau \\ &= 1.5 + 1 - 2\times 0.5 = 1.5\end{aligned}$$

Hence,

$$P(e|s_1) = \frac{1}{\sqrt{2\pi\sigma_n^2}}\int_{-\infty}^{T-1} e^{-\frac{x^2}{2\sigma_n^2}}dx$$

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Similarly we find that

$$\begin{aligned} P(e|s_2) &= P(0.5 + n_1 - 1 - n_2 > T) \\ &= P(n_1 - n_2 > T + 0.5) \\ &= \frac{1}{\sqrt{2\pi\sigma_n^2}} \int_{T+0.5}^{\infty} e^{-\frac{x^2}{2\sigma_n^2}} dx \end{aligned}$$

The average probability of error is

$$\begin{aligned} P(e) &= \frac{1}{2}P(e|s_1) + \frac{1}{2}P(e|s_2) \\ &= \frac{1}{2\sqrt{2\pi\sigma_n^2}} \int_{-\infty}^{T-1} e^{-\frac{x^2}{2\sigma_n^2}} dx + \frac{1}{2\sqrt{2\pi\sigma_n^2}} \int_{T+0.5}^{\infty} e^{-\frac{x^2}{2\sigma_n^2}} dx \end{aligned}$$

To find the value of T that minimizes the probability of error, we set the derivative of $P(e)$ with respect to T equal to zero. Using the Leibnitz rule for the differentiation of definite integrals, we obtain

$$\frac{\vartheta P(e)}{\vartheta T} = \frac{1}{2\sqrt{2\pi\sigma_n^2}} \left[e^{-\frac{(T-1)^2}{2\sigma_n^2}} - e^{-\frac{(T+0.5)^2}{2\sigma_n^2}} \right] = 0$$

or

$$(T-1)^2 = (T+0.5)^2 \Longrightarrow T = 0.25$$

Thus, the optimal decision rule is

$$r_1 - r_2 \underset{s_2}{\overset{s_1}{\gtrless}} 0.25$$

Problem 8.33

1) The inner product of $s_i(t)$ and $s_j(t)$ is

$$\begin{aligned} \int_{-\infty}^{\infty} s_i(t)s_j(t)dt &= \int_{-\infty}^{\infty} \sum_{k=1}^{n} c_{ik} p(t - kT_c) \sum_{l=1}^{n} c_{jl} p(t - lT_c) dt \\ &= \sum_{k=1}^{n} \sum_{l=1}^{n} c_{ik} c_{jl} \int_{-\infty}^{\infty} p(t - kT_c) p(t - lT_c) dt \\ &= \sum_{k=1}^{n} \sum_{l=1}^{n} c_{ik} c_{jl} \mathcal{E}_p \delta_{kl} \\ &= \mathcal{E}_p \sum_{k=1}^{n} c_{ik} c_{jk} \end{aligned}$$

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The quantity $\sum_{k=1}^{n} c_{ik}c_{jk}$ is the inner product of the row vectors $\underline{C}_i$ and $\underline{C}_j$. Since the rows of the matrix H_n are orthogonal by construction, we obtain

$$\int_{-\infty}^{\infty} s_i(t)s_j(t)dt = \mathcal{E}_p \sum_{k=1}^{n} c_{ik}^2 \delta_{ij} = n\mathcal{E}_p \delta_{ij}$$

Thus, the waveforms $s_i(t)$ and $s_j(t)$ are orthogonal.

2) Using the results of Problem 8.30, we obtain that the filter matched to the waveform

$$s_i(t) = \sum_{k=1}^{n} c_{ik} p(t - kT_c)$$

can be realized as the cascade of a filter matched to $p(t)$ followed by a discrete-time filter matched to the vector $\underline{C}_i = [c_{i1}, \ldots, c_{in}]$. Since the pulse $p(t)$ is common to all the signal waveforms $s_i(t)$, we conclude that the n matched filters can be realized by a filter matched to $p(t)$ followed by n discrete-time filters matched to the vectors $\underline{C}_i$, $i = 1, \ldots, n$.

Problem 8.34

1) The optimal ML detector selects the sequence $\underline{C}_i$ that minimizes the quantity

$$D(\mathbf{r}, \underline{C}_i) = \sum_{k=1}^{n} (r_k - \sqrt{\mathcal{E}_b} C_{ik})^2$$

The metrics of the two possible transmitted sequences are

$$D(\mathbf{r}, \underline{C}_1) = \sum_{k=1}^{w} (r_k - \sqrt{\mathcal{E}_b})^2 + \sum_{k=w+1}^{n} (r_k - \sqrt{\mathcal{E}_b})^2$$

and

$$D(\mathbf{r}, \underline{C}_2) = \sum_{k=1}^{w} (r_k - \sqrt{\mathcal{E}_b})^2 + \sum_{k=w+1}^{n} (r_k + \sqrt{\mathcal{E}_b})^2$$

Since the first term of the right side is common for the two equations, we conclude that the optimal ML detector can base its decisions only on the last $n - w$ received elements of $\mathbf{r}$. That is

$$\sum_{k=w+1}^{n} (r_k - \sqrt{\mathcal{E}_b})^2 - \sum_{k=w+1}^{n} (r_k + \sqrt{\mathcal{E}_b})^2 \underset{\underline{C}_1}{\overset{\underline{C}_2}{\gtrless}} 0$$

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or equivalently

$$\sum_{k=w+1}^{n} r_k \begin{array}{c} \underline{C}_1 \\ > \\ < \\ \underline{C}_2 \end{array} 0$$

2) Since $r_k = \sqrt{\mathcal{E}_b} C_{ik} + n_k$, the probability of error $P(e|\underline{C}_1)$ is

$$\begin{aligned} P(e|\underline{C}_1) &= P\left(\sqrt{\mathcal{E}_b}(n-w) + \sum_{k=w+1}^{n} n_k < 0\right) \\ &= P\left(\sum_{k=w+1}^{n} n_k < -(n-w)\sqrt{\mathcal{E}_b}\right) \end{aligned}$$

The random variable $u = \sum_{k=w+1}^{n} n_k$ is zero-mean Gaussian with variance $\sigma_u^2 = (n-w)\sigma^2$. Hence

$$P(e|\underline{C}_1) = \frac{1}{\sqrt{2\pi(n-w)\sigma^2}} \int_{-\infty}^{-\sqrt{\mathcal{E}_b}(n-w)} \exp(-\frac{x^2}{2\pi(n-w)\sigma^2})dx = Q\left[\sqrt{\frac{\mathcal{E}_b(n-w)}{\sigma^2}}\right]$$

Similarly we find that $P(e|\underline{C}_2) = P(e|\underline{C}_1)$ and since the two sequences are equiprobable

$$P(e) = Q\left[\sqrt{\frac{\mathcal{E}_b(n-w)}{\sigma^2}}\right]$$

3) The probability of error $P(e)$ is minimized when $\frac{\mathcal{E}_b(n-w)}{\sigma^2}$ is maximized, that is for $w = 0$. This implies that $\underline{C}_1 = -\underline{C}_2$ and thus the distance between the two sequences is the maximum possible.

Problem 8.35

In the non decision-directed timing recovery method we maximize the function

$$\Lambda_2(\tau) = \sum_m y_m^2(\tau)$$

with respect to τ. Thus, we obtain the condition

$$\frac{d\Lambda_2(\tau)}{d\tau} = 2\sum_m y_m(\tau)\frac{dy_m(\tau)}{d\tau} = 0$$

Suppose now that we approximate the derivative of the log-likelihood $\Lambda_2(\tau)$ by the finite difference

$$\frac{d\Lambda_2(\tau)}{d\tau} \approx \frac{\Lambda_2(\tau+\delta) - \Lambda_2(\tau-\delta)}{2\delta}$$

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Then, if we substitute the expression of $\Lambda_2(\tau)$ in the previous approximation, we obtain

$$\begin{aligned}
&\frac{d\Lambda_2(\tau)}{d\tau} \\
&= \frac{\sum_m y_m^2(\tau+\delta) - \sum_m y_m^2(\tau-\delta)}{2\delta} \\
&= \frac{1}{2\delta}\sum_m \left[\left(\int r(t)u(t-mT-\tau-\delta)dt\right)^2 - \left(\int r(t)u(t-mT-\tau+\delta)dt\right)^2\right]
\end{aligned}$$

where $u(-t) = g_R(t)$ is the impulse response of the matched filter in the receiver. However, this is the expression of the early-late gate synchronizer, where the lowpass filter has been substituted by the summation operator. Thus, the early-late gate synchronizer is a close approximation to the timing recovery system.

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