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(Revised 4/29/15, with minor edits by Craig William French)

MARKET VALUE, TIME, AND RISK

Abstract, by Craig William French

This paper reprints a slightly edited version of Jack L. Treynor's 1961 CAPM manuscript, which has previously been unavailable to the public. The author's facsimile of the original was obtained thanks to the kind generosity of Mr. and Mrs. Treynor. Edits in the present version, which differ from the original, include minor typographical corrections and minor notation differences for some variables in the formulae. Pagination is as in the original.

In 1958, Jack Treynor was employed by Arthur D. Little. That summer he took a three-week vacation to Evergreen, Colorado, during which he produced forty-four pages of mathematical notes on capital asset pricing and capital budgeting. Over the next two years, Treynor refined his notes into what is in all likelihood the first CAPM. Treynor gave a copy of this early model to John Lintner at Harvard in 1960. While in business school at Harvard from 1953 through 1955, Mr. Treynor had taken nearly every finance course offered, and though he signed up for Lintner's economics course he was forced to cancel due to a schedule conflict. In 1960, John Lintner was the only economist he knew even slightly. Treynor refined his 1960 model into the 45-page “Market Value, Time, and Risk” [the present paper]. This paper, Treynor (1961), develops the CAPM using the concept of experiment space to quantify risk and risk relations. Without his knowledge or encouragement, one of Mr. Treynor's colleagues sent the draft to Merton Miller in 1961, after Miller had moved to the University of Chicago from Carnegie Institute of Technology. Miller sent the paper to Franco Modigliani at MIT in the spring of 1962, and Modigliani invited Treynor to embark on a program of graduate work at MIT under his supervision. Treynor did so during the 1962-1963 academic year; in addition to Modigliani's course, he took Bob Bishop's price theory and Ed Kuh's econometrics courses, among others. By the fall of 1962, Treynor had consolidated the first part of Treynor (1961), on the single-period model, into “Toward a Theory of Market Value of Risky Assets,” and presented it to the MIT finance faculty.

A more complete description of the development of the Treynor CAPM may be found in French, Craig W., The Treynor Capital Asset Pricing Model. Journal of Investment Management, Vol. 1, No. 2, pp. 60-72, 2003. Available at SSRN: http://ssrn.com/abstract=447580.

A reprint of Treynor's 1962 paper, “Toward a Theory of Market Value of Risky Assets,” may be found in French, Craig W., Jack Treynor's 'Toward a Theory of Market Value of Risky Assets' (December 28, 2002). Available at SSRN: http://ssrn.com/abstract=628187.

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95-209 MARKET VALUE, TIME, AND RISK

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There are three basic problems in making investment decisions within the widely, actively traded corporation:

a. Defining the decision. This is tantamount to projecting the differential cash flows between alternatives. If the cash flows associated with a particular project depend on how other investment decisions are made, we say that the first decision is financially dependent on the others, following the usage of Bierman and Smidt.[1] Succeeding machines in a replacement chain are usually financially dependent in precisely the same sense. It seems safe to say that the burden of identifying and sorting out alternatives and defining the related cash flows will fall on the decision-maker for some time to come (although certain problems of this type are susceptible of generalization, e.g., inventory and certain replacement problems).

b. Determining the effect of a choice on the market value of the corporation.

c. Appraising the financial risk entailed in the choice.

We choose to confine our discussion to widely held corporations because in such corporations it is relatively easy to keep the portfolio decisions of the owners distinct from investment decisions of corporate management.

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[1] The Capital Budgeting Decision, Bierman, Harold, Jr., and Smidt, Seymour, MacMillan, 1960.

Although it is generally agreed that both the time displacement and riskiness of future cash flows may diminish their present market value, the way in which time and risk together affect market value is not understood. Despite the fact that most of the capital budgeting literature concentrates on problems b. and c., there is remarkably little useful that can be said about these problems in most practical cases in the absence of an understanding of the relations among market value, time, and risk.

Although in recent years there has been a wave of enthusiasm for recognizing the importance of the timing of receipts and disbursements, which quite naturally emphasized the importance of looking at cash flows rather than accruals, the formal techniques of capital budgeting have never dealt with risk explicitly. It is fairly well known that the application of these methods to complex financial decisions--for example, certain lease arrangements--can lead to very unfortunate decisions unless these methods are used with a high degree of financial sophistication. The discounted-cash-flow methods all require the choice of a discount rate. A currently popular notion is that each company has a cost of capital which can be estimated by observing the market prices of the company's securities and the return to each class and computing a weighted average rate of return.[2] It is only fair to defenders of these methods to point out that many of them recognize the necessity of modifying the company's cost of capital to fit the riskiness of a particular

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[2] Bierman and Smidt, *op. cit.*, p. 135.

project. This is a tricky business, however, considering that the range of possible choices is at least broad enough to include the cost of debt money and the cost of equity money, and is probably broader.

The difficulty with computing the weighted cost of capital (apart from the conceptual difficulty, peculiar to the traditional theory, that the current cost of capital may not be the cost of capital at the optimum capital structure) and then modifying it over a very broad range is not merely that the procedure requires judgement, but that the guide-lines for judgement are almost nonexistent. Since the cost of capital applied in a particular problem can be critical in determining whether a decision represents a net gain or loss in present value, considerable controversy has centered on its estimation. A number of writers[3] have recognized that replacing present earnings, or some extrapolation thereof, with the expected future earnings, will generally not account for the market's reaction to the presence of risk.

The limited-risk concept of the corporation actually has no financial meaning in a riskless world; it is therefore not surprising that if, in our quantitative thinking about corporate goals and corporate decisions, we use apparatus which ignores risk, the results will fall short of our intuitive expectations about the nature of the problem.

Like Modigliani and Miller[4] and others, we consider the fundamental financial objective of the corporation to be maximizing present market value. To the extent that market value can be argued to depend on

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[3] The Theory of the Investment of the Firm, Lutz, Freidrich and Vera, Princeton University Press, 1951.

[4] "The Cost of Capital, Corporation Finance, and the Theory of Investment," American Economic Review, June 1958, Modigliani, Franco and Miller, Merton.

certain expected values, expected value has a place in financial appraisal. But the fundamental objective is maximizing the price for which the present owners can sell their interest. We recognize that this objective needs a carefully reasoned defense, especially against writers who argue that it implies seeking "to maximize the share of income going to just one of the several parties that have a stake in the business" and is therefore immoral, and that the proper financial objective is merely "a satisfactory return on capital employed."[5] In this paper, however, we make no attempt at a defense, and confine our inquiry to an investigation of the relations between market value, risk, and time.

The corporate decision-maker is in a curious position, however: he must try to estimate the effect on the market value of his company's stock of his decision, even though the market mechanism remains at least a partial mystery to him. He knows that the prices of various stocks are related, although perhaps loosely. He feels that there is some relationship between stock prices and earnings. Although these relationships are probably far too complex to cope with, if he over-simplifies too much his decisions can cause his company serious financial harm.

This paper is an attempt to develop a body of ideas which can help provide answers to some of the problems. It considers that an understanding of the nature of financial risk lies very near the core of these problems. It is motivated by a conviction that corporate invest-

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[5] "The Trouble with Profit Maximization" Harvard Business Review, November-December 1960, Anthony, Robert N.

ment decisions should be related to consistent, quantifiable, and measurable objectives by conclusions reasoned from a set of axiomatic assumptions about the nature of the problem, rather than related to vague objectives by a collection of loosely rationalized techniques.

We shall try to demonstrate that, given certain parameters which characterize the capital market, our assumptions uniquely determine the present market value of a stream of future cash flows. The assumptions were chosen to be as weak and as economical as possible, and to do the least possible violence to our intuition about the market.

The critical assumptions are:

1. The sole concern of the individual investor is with short-range portfolio performance. This implies that the cost (i.e., brokerage costs and tax effects) of rebalancing his portfolio frequently in order to achieve this objective is small enough to be disregarded.

2. The investor abhors risk to the extent that he will accept a portfolio with somewhat lower expected performance in return for diminished error variance. The implications of assuming that expected performance and error variance are the only portfolio parameters of interest to the investor are discussed at length in Markowitz's[6] book on portfolio balance. As Markowitz shows, this assumption implies that the investor's utility function is quadratic.

3. There is a capital market in which individual investors borrow and lend money and buy and sell corporate securities. Security prices in this market depend entirely on the market's forecasts of

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[6] Portfolio Selection, or Efficient Diversification of Investments, Markowitz, Harry, Wiley, 1959, pp. 116-125. See also "Risk, the Discount Rate, and Investment Decisions, "American Economic Review, May 1961, Hirshleifer, Jack.

short-range performance and the uncertainty in these forecasts. In this market investors and securities compete freely with other investors and other securities. The market is rational in the sense that investors will tend to shift their portfolios into the securities which give them the minimum risk for any given level of performance.

4. Both individuals and corporations can borrow and lend money at a rate which is independent of the amount borrowed, so long as the lender incurs no risk. This rate, which, for simplicity, is assumed to be certain and constant, is called the risk-free rate of interest. We shall assume that creditors will lend only so much as is fully secured by the market value of the corporation's assets.

5. Investors agree in their forecasts of future cash flows. This assumption is interpreted to mean that they agree both in their forecasts of the variables on which cash flows depend and on the functional relationships between cash flows and the relevant variables. We term the forecasts on which investors are assumed to agree <u>market forecasts</u>.

6. Corporate (cash) earnings are differentiable functions of variables such as unit selling price, sales volume, etc. Errors in market forecasts of earnings are entirely the result of errors in forecasting the relevant economic variables. We shall assume that the mean error in market projections is zero; market projections of uncertain economic variables are thus expected values. A given cash flow may be functionally independent of any or all the variables in the set. A "risky"

cash flow is one which is relatively sensitive to changes in one or more economic variables. The set of economic variables breaks down into three overlapping subsets:

Set A: Insurable. Very roughly, an insurable uncertainty is one which has an economic effect which can be absorbed in small shares by a great many investors without appreciably increasing the portfolio uncertainty of any.

Set B: Economically insignificant. If the uncertainty in the total market value of all outstanding investments due to uncertainty in a particular variable is negligible, then that variable is said to be economically insignificant. (Unfortunately, a more precise explanation of the differences between Set A and Set B has to wait until certain subsidiary ideas are developed.)

Set C: Any variable which is not both insurable and economically insignificant is assumed to be in this set. At each point in time the market forecasts the values of these variables for each point in the relevant future. The forecast of a given variable at a fixed point in the future changes as new forecasts replace the old. The forecasts of variables in Set C are continuous functions of time. This means that the difference between successive forecasts of the same event becomes as small as we like, as the time between the forecasts is made small.

A cash flow can be a function of variables in each set. For example, the annual earnings from an undrilled oil well conceivably might be expressed

$$F = (u_1 u_2) - c$$

where u_1 is a forecast of the price of oil and u_2 is a forecast of the rate of flow and c is the cost of operating the well, each in appropriate units. Variable u_1 is typical of Set C; the price of oil affects a significant portion of the economy and is probably intimately related to other economic variables. Variable u_2 is typical of Sets A and B; uncertainty in the forecast of the well's production is probably unrelated to anything except possibly neighboring wells. Furthermore, although uncertainty in u_2 may be very important to an investor who owns the well, its effect would be small on individual portfolios if shares in the well were fairly evenly distributed among all investors. In order for variable u_1 to qualify for Set C, the market forecast of u_1 must be continuous as a function of time. This criterion is satisfied only approximately in reality. Stock prices, which are presumed to be well-behaved functions of market forecasts, as well as commodity prices, approach something like continuity (most, if not all, of the more prominent seeming discontinuities in these time series occur when real time is elapsing but "market" time is not--e.g., week ends). The universal observance of near-continuity by stock prices is consistent with the assumption that forecasts of the more significant economic variables are continuous functions of time. Unlike most of the assumptions in the list, the continuity property for variables in Set C is introduced as a technical convenience to the author; fortunately, the assumption fits reality moderately well. After all, the entire concept of a <u>market</u> forecast is an artifice. Individual investors make forecasts; the market seeks prices at which supply and demand generated by the appraisals of individual investors, based on a range of individual forecasts, resolve. Individual forecasts are perhaps best viewed as discontinuous; resulting market prices may still, indeed do, tend to be continuous.

7. The paper makes no distinction between investments which have already been made and investment opportunities which are being considered. To use the evaluation techniques developed in this paper on an investment not yet made implies that the investment opportunity considered is small enough to have a negligible effect on the demand for funds, hence the market parameters. Although the paper may shed some incidental light on why the marginal parameters of the capital market have certain values, the intended emphasis is on developing a concept of market value which implies certain consistencies among the prices of various securities, taking the market parameters, which we define, as given. These parameters undoubtedly shift as the supply of investments in the market shifts; we assume that the supply of investments, hence the value of the market parameters, shifts sufficiently slowly that any uncertainty in the values of the parameters for the relevant future can be disregarded.

8. Corporate and individual income taxes do not alter the fundamental structure of the problem. The author regrets having to make this simplification; unfortunately, the financial effects of taxation are sufficiently complex to require a separate study.

The development has two parts:

a. A study of market-value relationships over an infinitesimal time interval, and

b. An extension of the results of (a) to evaluation of income streams over time.

Market Value and Risk over the Infinitesimal Time Interval

We begin by considering the effect on an individual investor's portfolio of substituting one security for another, such that the price uncertainties in the respective securities are perfectly correlated. We have assumed that he is concerned primarily with the first and second moments of portfolio performance, where performance is defined, ignoring tax effects, as the sum of the increment in market value and any intervening dividends (assuming the interval is short enough to disregard any present-value effect of the time between receipt of the dividend and the end of the period). The relevant attributes of the two given securities are therefore expected performance, the standard error over the infinitesimal time period, and present market price. The two securities in question may be characterized as follows:

	Security 1	Security 2
Price	p	p'
Performance	μ	μ'
Projected market value	$p + \mu$	$p' + \mu'$
Uncertainty in projected market value	σ	σ'

He might ask what the incremental effect on portfolio performance and uncertainty of a substitution would be. Now the rational investor may have part, or all, of his funds invested in riskless investments, earning the riskless interest rate. But, except for a working inventory investment in cash, none of his funds should be uninvested. (The same is true, incidentally, of corporations.) When he undertakes to bring a security into his portfolio, he must either borrow at the riskless rate, liquidate riskless investment, or liquidate another (risky) security. There is, therefore, an opportunity cost associated with each security investment in the portfolio which is measured by interest at the riskless rate, r, on the funds invested. Thus, for example, if he adds n shares of Security 1 to his portfolio, the incremental effect net of interest will be

$$\Delta \text{ performance} = n(\mu - rp),$$
$$\Delta \text{ uncertainty} = n\sigma. \qquad (1$$

Suppose he currently owns n shares of Security 1 and that for some number (not necessarily integral or even rational) of shares n' of Security 2

$$\sigma' n' = \sigma n. \qquad (2$$

Since the errors in the performance of Security 1 and Security 2 are 100 per cent correlated, then the investor will gain by replacing the latter with the former, if the former has the higher ratio of performance net of interest, to uncertainty. Thus, since

$$n' = \sigma n/\sigma', \qquad (3$$

the contribution to the performance of the portfolio from the latter security is

$$n'(\mu' - rp') = \sigma n(\mu' - rp')/\sigma'. \qquad (4$$

But it is given that

$$(\mu' - rp')/\sigma' < (\mu - rp)/\sigma, \qquad (5$$

so the performance of a portfolio containing the latter security is less than

$$\sigma n(\mu - rp)/\sigma = n(\mu - rp) \; , \qquad (6$$

the performance of a portfolio containing the former security, but with the same degree of risk. The investor can always improve the expected performance of his portfolio without increasing the standard error when this situation exists. We shall assume that, whether guided by trial and error, instinct, hunch, or careful calculation, he will.[7] Investors will continue to sell one security and buy the other until the discrepancy becomes small; that is, until the prices of both securities have adjusted to the point where respective ratios of standard error to performance are equal. In other words, there will be a tendency toward an equilibrium ratio ρ of standard error to performance. Let $\bar{x} = p + \mu$:

$$(\mu - rp)/\sigma = ((\bar{x} - p) - rp)/\sigma \rightarrow \rho , \qquad (7$$

or

$$\bar{x} - (1 + r)p = \rho\sigma$$

$$(1 + r)p = \bar{x} - \rho\sigma \qquad (8$$

$$p = 1/(1+r)\bar{x} + (-\rho/(1+r)\sigma .$$

At this point it is convenient to introduce a more general notation which is suited to a discussion of a capital market in which investment uncertainties are far from perfectly correlated. The new notation requires the definition of a market forecast error. Errors in forecasts of performance are measured by comparing the forecast at the beginning of a short interval with the actual performance over the interval. If we repeated our measurement of the error over a number of such intervals,

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[7] It is interesting to compare this argument with the arbitrage argument in Modigliani and Miller, op. cit.

we would obtain a set of numbers which can be termed an error vector. That is, the component e_{ij} of the error vector

$$E_1 = \begin{matrix} e_{11} \\ e_{12} \\ \cdot \\ \cdot \\ \cdot \end{matrix} \tag{9}$$

is the error in the performance forecast corresponding to the j^{th} infinitesimal time period in the set of periods for which measurements were taken (the ordering implied by the values of j is not necessarily the time order of these intervals; it is significant only in the sense that, for example, the components e_{1j} and e_{2j} of error vectors E_1 and E_2 are error measurements for the respective securities for the same interval). We shall consider the statement "The market considers the error vectors for the performance of two securities perfectly correlated" to mean that it expects that, if actual measurements of errors were taken over a number of time intervals, the following expression for the correlation coefficient τ_{12}, involving the error vectors, say E_1 and E_2, for the respective securities, would hold.

$$\tau_{12} = (E_1 \bullet E_2)/(\sqrt{[E_1 \bullet E_1]}\sqrt{[E_2 \bullet E_2]}) = 1, \tag{10}$$

or, equivalently,

$$E_1 = dE_2 \tag{11}$$

where d is any positive scalar. The use of an error vector to characterize the uncertainty in a security (investment) is not meant to imply that the market's "subjective" assessment of the uncertainty is to be discarded in favor of a purely empirical measurement of the uncertainty. Rather, the error vector is regarded as the form which the market's assessment of the uncertainty will take, whether based purely on hunch, purely on empiricism, or a mixture of both.

In order to develop an expression which provides price consistency among the large number of error vectors corresponding to the investments in an actual capital market, we introduce the concept of a basis--that is, a set of vectors which is comprehensive in the sense that every vector in the market can be expressed as a (weighted) linear combination of the basis vectors, yet economical in the sense (i.e., the basis "spans" the market) that no basis vector can be similarly expressed in terms of the other basis vectors. For our present purpose it is only necessary to show that, whatever the number of vectors required to form a basis,* the market value relationships among the investments in the capital market satisfy an expression akin to (8). Our argument is essentially inductive, though we argue only the first step in the induction explicitly.

Expression (8) suggests that for any class of investments, errors in the performance of which are perfectly correlated, the present market value (price) will tend to be equal to a linear function of expected value and uncertainty.

We define two classes of investments, within which performance errors are perfectly correlated. Errors may be imperfectly correlated between the classes. Perfect correlation means that the corresponding error

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* If we restricted ourselves to error vectors empirically obtained, the number of intervals for which errors were measured would limit the number of basis vectors required to span the market (i.e., the dimensions of the market).

vector E for any member of class 1 can be expressed

$$E = \alpha_1 E_1 \ , \qquad (12$$

and, similarly, for any member of class 2

$$E = \alpha_2 E_2 \ , \qquad (13$$

where α_1 and α_2 are appropriate scalars.

Let

$$\sigma^*_1 = \sqrt{[(E_1 \bullet E_1)/n]} \ , \qquad (14$$

so that we have for any member of class 1

$$\sigma_1 = \sqrt{[(\alpha_1 E_1 \bullet \alpha_1 E_1)/n]} = \alpha_1 \sigma^*_1 \ . \qquad (15$$

Then if we let b = 1/1 + r, the expression

$$P_1 = b\bar{x}_1 + (-\rho 1/(1+r))\sigma_1 \qquad (16$$

can be rewritten,

$$P_1 = b\bar{x}_1 + (-\rho_1/(1+r))\alpha_1\sigma^*_1 \qquad (17$$

$$= b\bar{x}_1 + ((-\rho_1\sigma^*_1)/(1+r))\alpha_1 = b\bar{x}_1 + a_1\alpha_1 \ ,$$

and for class 2

$$P_2 = b\bar{x}_2 + ((-\rho_2\sigma^*_2)/(1+r))\alpha_2 = b\bar{x}_2 + a_2\alpha_2 \ . \qquad (18$$

Expression (18) summarizes the relationship between the previous notation and the vector notation for investment classes within which performance errors are perfectly correlated.

Given that the expressions

$$P_1 = b\bar{x}_1 + a_1\alpha_1$$
$$P_1 = b\bar{x}_2 + a_2\alpha_2 \qquad (19$$

apply to members of the respective classes, it can be argued that the market price for any investment with projected value $\bar{x}$ and error vector E such that

$$E = \alpha_1 E_1 + \alpha_2 E_2 \qquad (20$$

is determined.

Suppose, for example, that an investor owns such an investment. He should be unwilling to sell for less than

$$p = b\bar{x} + a_1\alpha_1 + a_2\alpha_2, \qquad (21$$

since he can realize at least the amount p by selling short "pure" investments in class 1, class 2, respectively, and by borrowing or lending, in amounts which hedge his current investment perfectly.

We consider investments from the respective classes such that the respective expected future values are $\bar{x}_1$ and $\bar{x}_2$ and the respective error vectors are B_1E_1 and B_2E_2. In order to hedge the given investment (let us refer to it as $(\bar{x}, \alpha_1, \alpha_2)$) perfectly, the amounts q_1 and q_2 * in which $(\bar{x}_1, B_1)$ and $(\bar{x}_2, B_2)$ are sold must satisfy the following system of equations, in which y is an amount borrowed at the risk-free rate of interest:

$$q_1B_1E_1 = \alpha_1E_1 \quad , \quad q_1B_1 = \alpha_1$$
$$q_2B_2E_2 = \alpha_2E_2 \quad , \quad q_2B_2 = \alpha_2 \qquad (22$$
$$q_1\bar{x}_1 + q_2\bar{x}_2 + y = \bar{x}$$

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* Assuming infinitely divisible shares.

The price p the investor can realize by selling short $(\bar{x}_1, B_1)$ and $(\bar{x}_2, B_2)$ and borrowing an amount y (a negative value for y would correspond to lending) is

$$p = q_1 p_1 + q_2 p_2 + by \;, \tag{23}$$

where p_1 and p_2 are determined according to expression (19) and b is the risk-free discount rate, as before. Hence we have

$$\begin{aligned} p &= q_1(b\bar{x}_1 + a_1 B_1) + q_2(b\bar{x}_2 + a_2 B_2) + by \\ &= b(q_1\bar{x}_1 + q_2\bar{x}_2 + y) + a_1 q_1 B_1 + a_2 q_2 B_2 \\ &= b\bar{x} + a_1\alpha_1 + a_2\alpha_2 \;, \end{aligned} \tag{24}$$

at which price he has effectively liquidated his investment. Thus he need not sell the investment he actually holds for less.

On the other hand, it is equally clear that no one should be willing to pay more than

$$p = b\bar{x} + a_1\alpha_1 + a_2\alpha_2 \;, \tag{25}$$

since an equivalent investment can be formed of "pure" investments

$$(\bar{x}_1,\ B_1) \text{ and } (\bar{x}_2,\ B_2), \tag{26}$$

by purchasing $(\bar{x}_1, B_1)$ and $(\bar{x}_2, B_2)$ in amounts q_1 and q_2, respectively, where q_1, q_2 and y must satisfy the same system of equations as before. The total cost p to the investor of the equivalent investment is

$$p = q_1 p_1 + q_2 p_2 + by \;, \tag{27}$$

in which p_1 and p_2 also satisfy expression (19), as before. Substituting, we obtain

$$p = b\bar{x} + a_1\alpha_1 + a_2\alpha_2 \;. \tag{28}$$

This represents the maximum price that the market will pay for $(\bar{x}, \alpha_1, \alpha_2)$.

We have argued that the owner of this security should not sell for less than

$$p = b\bar{x} + a_1\alpha_1 + a_2\alpha_2 \,, \tag{28}$$

and no one should be willing to pay more. Thus the value p of an investment with projected future value $\bar{x}$ and error vector $E = \alpha_1 E_1 + \alpha_2 E_2$ is given by expression (28), provided expressions (19) hold for any "pure" investments in classes 1 and 2. This result extends to any number of linearly independent classes. For an investment with projected future value $\bar{x}$ and uncertainty E related to a set of basis vectors E_i

$$E = \sum_i \alpha_i E_i \,, \tag{29}$$

a consistent market price p is

$$p = b\bar{x} + \sum_i a_i \alpha_i \,, \tag{30}$$

where b is the risk-free discount factor and the a_i are market parameters relating to risk.

Given a set of basis vectors E_i in terms of which the uncertainty E in any investment can be expressed

$$E = \sum_i \alpha_i E_i \,, \tag{31}$$

then the expected performance υ, over and above the cost of interest, of an investment with components α_i is obtained by transposing (30):

$$p = b\bar{x} + \sum_i a_i \alpha_i$$

$$b\bar{x} - p = -\sum_i a_i \alpha_i \qquad (32$$

$$\upsilon = \bar{x} - p/b = -\sum_i (a_i/b)\alpha_i \ .$$

Since this expression is linear in the components α_i , it applies equally well to a portfolio containing a number of investments, each of which satisfies (32) (which guarantees price consistency among investments of varying expected performance and risk). The α_i then represent the components of the portfolio with respect to the basis vectors E_i. The variance of the portfolio can be expressed in terms of the basis vectors and the components α_i.

$$S^2 = 1/n \ \sum_i \alpha_i E_i \bullet \sum_j \alpha_j E_j = \sum_{ij} \alpha_i \alpha_j ((E_i \bullet E_j)/n), \qquad (33$$

assuming that the error components of particular E_i have zero mean (since we have assumed that the market's forecasts are unbiased). By definition, we have

$$A_{ij} = \sum_{ij} ((E_i \bullet E_j)/n), \qquad (34$$

where A_{ij} is the matrix of variances and covariances among the basis vectors E_i. Then the portfolio variance S^2 can be expressed

$$S^2 = \sum_{ij} \alpha_i A_{ij} \alpha_j \ . \qquad (35$$

Presumably, the individual investor's object is to diversify in such a way as to achieve the minimum variance for any given level of expected performance. If a portfolio is altered by a pure change of scale such that the uncertainty E' in the new portfolio equals

$$E' = kE = k \sum \alpha_i E_i = \sum (k\alpha_i) E_i \ , \tag{36}$$

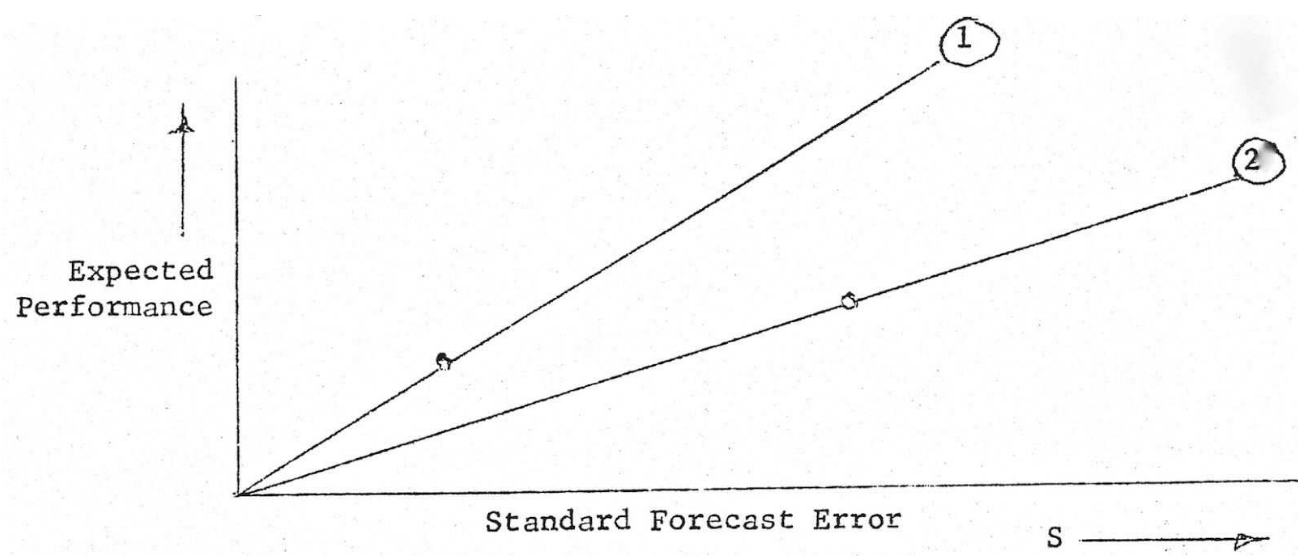
then expected performance υ' after interest equals

$$\upsilon' = \sum_i (a_i/b) \ (k\alpha_i) = -k \sum_i (a_i/b) \alpha_i = k\upsilon, \tag{37}$$

and the variance $(S')^2$ equals

$$\begin{aligned}(S')^2 &= \sum_{ij} (k\alpha_i) A_{ij} (k\alpha_j) = k^2 \sum_{ij} \alpha_i A_{ij} \alpha_j \\ &= k^2 s^2 \ ,\end{aligned} \tag{38}$$

whence both the standard error $S' = kS$ and expected performance are altered proportionately. The locus of combinations of standard error and expected performance that can be achieved by a change of scale is therefore a straight line on the following graph.



(A pure change of scale is accomplished by risk-free borrowing or lending at the risk-free rate.) Line 1 corresponds to one set of portfolios which are identical to within a change of scale, and Line 2 corresponds to another, distinct, set. It is obvious from the graph that if the relative proportions of portfolio 1 make it superior to portfolio 2 for any specified level of risk or profit, then it is superior for any other level of risk or profit, in the sense of providing more expected profit for a given level of risk, or a lower level of risk for a given expected profit. Thus no matter what the investor's attitude toward risk, within the single restriction that, other things equal, he prefers less risk to more for the same expected performance, the relative proportions of portfolio 1 are then superior to the relative proportions of portfolio 2.

It is natural to inquire what relative proportions will give the least risk for any given level of expected performance. We suppose that the investor's object is to minimize

$$S^2 = \sum_{ij} \alpha_i A_{ij} \alpha_j \ , \tag{39}$$

subject to

$$\upsilon = \bar{x} - p/b = -\sum_i (a_i/b)\alpha_i = k \ , \tag{40}$$

where k is some arbitrary constant. If we let

$$\mu_i = -(a_i/b) \ , \tag{41}$$

then (40) becomes

$$\upsilon = \sum_i \mu_i \alpha_i = k \ . \tag{42}$$

Applying the method of Lagrange Multipliers, we obtain the expressions

$$2 \sum_j A_{ij}\alpha_j - \lambda\mu_i = 0$$

$$\sum_i \mu_i\alpha_i - k = 0 \qquad (43$$

Instead of treating the α_i as unknowns and solving for the optimal portfolio balance, expressed in terms of the α_j, let us turn the problem around. If every investor were rational in the sense that he owned securities in the optimal proportions, each individual portfolio would have to satisfy these expressions. The individual is still free to blend the optimal portfolio with riskless investment, or to borrow, to satisfy his own attitudes toward risk; by optimal we mean that portfolio which gives the minimum degree of risk for any specified expected performance. But if every investor held the available (risky) securities in the proportions dictated by the market forecast, the total holdings of all investors would have to be in the optimal proportions. On the other hand, the total holdings in each security are necessarily equal to the amount of that security available.

We may reasonably ask: what expected performance over and above the cost of interest* (μ_i) would each basis investment have to have in order for supply and demand to resolve? The α_j then become the amounts in which the corresponding securities are available in total. Thus we may take the α_j as given and solve for the μ_i.

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* See expression (32).

$$\mu_i = (2/\lambda)\sum_j A_{ij}\alpha_j \ , \tag{44}$$

where λ has the same value for all investments,

$$\lambda = (2/k) \sum_{ij} \alpha_i A_{ij} \alpha_j \ . \tag{45}$$

Total market uncertainty E* equals

$$E^* = \sum_i \alpha_i E_i \ . \tag{46}$$

By writing expression (44) in the following form

$$\begin{aligned}\mu_i &= (2/\lambda)\sum_j A_{ij}\alpha_j = (2/\lambda n)\sum_j E_i \cdot E_j \alpha_j \\ &= (2/\lambda n) E_i \cdot \sum_j \alpha_j E_j = (2/\lambda n) E_i \cdot E^*,\end{aligned} \tag{47}$$

we see that the risk premium μ_i extracted by the market on basis investment (i) with uncertainty E_i equals $2/\lambda$ times the covariance of E_i with total market uncertainty E^*. The same statement naturally holds true for any investment with general uncertainty characterized by

$$E = \sum_i \alpha_i E_i \ , \tag{48}$$

since

$$\begin{aligned}\upsilon &= \sum_i \alpha_i \mu_i = \sum_i (2/\lambda n)\alpha_i E_i \cdot E^* \\ &= (2/\lambda n)\ E \cdot E^*.\end{aligned} \tag{49}$$

We shall treat λ, which expresses the over-all attitude of the market toward risk, as a constant.

We have considered that the uncertainty E in any market value $V(u_1, u_2, \ldots)$ is the result of uncertainty in the economic variables $u_1, u_2, \ldots$, of which V is a function. A small error in the projections of the u_j will cause a corresponding error in V that can be approximated by

$$\Delta V = \sum_j (\delta V/\delta u_j)\,\Delta u_j \tag{50}$$

hence to error vectors E'_j for the u_j corresponds the error vector E for V:

$$E = \sum_j (\delta V/\delta u_j)\ E'_j\ . \tag{51}$$

A like expression holds for every investment V_k in the market.

$$E_k = \sum_j (\delta V_k/\delta u_j)\ E'_j\ .\ ^* \tag{52}$$

We can substitute these expressions in (49) to obtain a relation between the risk premium υ on the investment V and the partial derivatives $\delta V/\delta u_j$.

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* With the introduction of the error vectors $E'j$ characterizing the uncertainty in the variables on which cash flows and market values depend, a subtle broadening in the definition of an error vector is necessary: for market values, cash flows, and variables on which cash flows depend, the relevant error is the difference between the forecast at the beginning of the interval and the actual outcome at the end, as defined previously. For variables on which market values at the end of the interval depend, however, the relevant error is the difference between the forecast at the beginning, and the forecast at the end.

$$\nu = (2/\lambda n)E \bullet \sum_k E_k = (2/\lambda n)\sum_j ((\delta V)/(\delta u_j))E'_j \bullet \sum_k \sum_m ((\delta V_k)/(\delta u_m))E'_m$$

$$= (2/\lambda n) \sum_{jkm} E'_j \bullet E'_m((\delta V_k)/(\delta u_m))((\delta V)/(\delta u_j)) \qquad (53$$

$$= \sum_j [(2/\lambda n) \sum_{km} E'_j \bullet E'_m((\delta V_k)/(\delta u_m))]((\delta V)/(\delta u_j)) = \sum_j a'_j((\delta V)/(\delta u_j)),$$

letting

$$a'_j = (2/\lambda n) \sum_{km} E'_j \bullet E'_m((\delta V_k)/(\delta u_m)). \qquad (54$$

The expression in brackets is independent of the particular investment V, except as $\delta V/\delta u$ enters into summations with all the other investments in the market. For purposes of determining the risk premium, therefore, the riskiness of an investment can be characterized by the partial derivatives $\delta V/\delta u_j$. No explicit consideration of probability distributions is necessary, if the values of the expressions in brackets are known. As long as investment V is a differentiable function of economic variables u_j, expression (53) can be applied to estimate the risk premium. The cost of capital problem thus ceases to be a problem which has to be solved anew for each investment decision. Estimating the values of the a'_j may be a formidable problem; on the other hand, the general usefulness of a knowledge of the a'_j and the simplicity of their application ought to justify whatever effort is required.

Insurability and Economic Significance

The coefficients in the risk premium expression can be written

$$a'_j = (2/\lambda n)E'_j \cdot \left[\sum_{km} E'_m((\delta V_k)/(\delta u_m)) \right] \tag{55}$$

where the E'_j and the E'_m characterize the uncertainties in the corresponding economic variables, and $(\delta V_k)/(\delta u_m)$ is the partial derivative of investment V_k with respect to economic variable u_m. In these expressions, the index k ranges over all the investments in the capital market, and the indices m and j range over all the economic variables on which the future values of these investments depend. The vector in brackets is the same for all economic variables u_j. Since presumably not all the a'_j are zero, particular a'_j are zero only when the projection of E'_j on the vector in brackets is zero. When, for a given u_j, a'_j is zero or nearly zero, we term u_j insurable.

If a'_j is zero, then no risk premium is attached to the sensitivity of any future value V to the economic variable u_j. (The future value V may, of course, be sensitive to other economic variables for which there is a substantial risk premium.) If a future value is certain, or dependent only on insurable economic variables, then the market should tend to discount it at the risk-free rate of interest. Consequently, it is necessary to retain only those terms in the expression for the risk premium ν for which the a'_j are nonnegligible.

In the sum in brackets

$$\sum_{km} E'_m((\delta V_k)/(\delta u_m)) = \sum_m \sum_k E'_m((\delta V_k)/(\delta u_m)) \tag{56}$$

the term corresponding to u_m measures the uncertainty of the total value in the capital market one period hence due to uncertainty in u_m.

If the contribution of the term

$$\sum_k E'_m((\delta V_k)/(\delta u_m)) \tag{57}$$

to the sum is too large to disregard, we say u_m is <u>economically significant</u>.

Thus, for a given economic variable there are four possible cases of interest.

		Insurable: No	Insurable: Yes
Economically significant	No	1	2
	Yes	3	4

Case 1 is vacuous; any uncertainty which is uninsurable is economically significant. An example of case 2 is the (as yet unknown) rate of flow from an oil well not yet drilled, which presumably is correlated with very few other variables in the economy (possibly excepting the rates of flow of neighboring wells). An example of case 3 is the retail price of gasoline, which affects the earnings of a substantial piece of the economy, and is probably uninsurable, if only because it correlates perfectly with that piece. Examples of case 4 probably come least easily to mind. Possible examples are a commodity price for which the effect of fluctuations on the profits of buyers offsets the effect on sellers and the share-of-market fraction for one among several competing manufacturers. Both variables may be economically significant, but neither is reflected in total market uncertainty. (Such variables may or may not be correlated with other variables which have an effect.) If a significant fraction of the uncertainty in the economy depends on u_m, then, by definition, the expression

$$\sum_k E'_m((\delta V_k)/(\delta u_m)) \tag{58}$$

is large. Since, however, there is probably a high degree of linear dependence among the E_m, it is always possible that the component of

$$\sum_m \sum_k E'_m((\delta V_k)/(\delta u_m)) \tag{59}$$

parallel to E'_m will be offset by other market-value uncertainties. Hence

$$E'_j \bullet \left[\sum_{km} E'_m((\delta V_k)/(\delta u_m))\right] \tag{60}$$

may be zero, or insignificantly small, even though u_j is economically significant.

We have assumed that variables which are economically significant have projections which are continuous functions of time. As we observed earlier, we are not making any logical defense for this assumption, although counterexamples are difficult (but not necessarily impossible) to find. For economically significant variables, therefore,

$$\lim_{\Delta t \to 0} \Delta u = 0, \tag{61}$$

where Δu is the difference between forecasts separated by a time interval Δt. Provided the functions $V_k(u_1, u_2, \ldots)$ are continuous, the approximate expressions

$$\Delta V_k = ((\delta V_k)/(\delta u_j))\, \Delta u_j$$

$$E_k = ((\delta V_k)/(\delta u_j))\, E'_j \tag{62}$$

become exact in the limit, for economically significant u_j. By definition, the magnitude of these expressions for variables which are not economically significant is small enough to disregard. Hence the expression for total market uncertainty in the risk premium υ

$$\upsilon = (2/\lambda n)E \bullet \sum_k E_k = (2/\lambda n) \sum_j [E'_j \bullet \sum_{km} E'_m((\delta V_k)/(\delta u_m))]((\delta V)/(\delta u_j)) \tag{63}$$

also tends to become exact in the limit, since the term

$$\sum_k E'_m((\delta V_k)/(\delta u_m))$$

tends to become exact if u_m is economically significant, and, by definition, is small enough to disregard if u_m is not economically significant.

A Simplification

In general, future value is some function of economic variables of several types. Market value one interval earlier then depends on:

1. "expected" future market value, computed, in principle, by integrating over the joint probability distribution of the relevant variables (u_1, u_2, . . .), and
2. the uncertainty in future market value due to uncertainty in the projections of the economic variables.

We have assumed that projections of variables which are not both insurable and economically insignificant are continuous functions of time. As the forecasting interval becomes infinitesimal, some simplification in our representation of the problem becomes possible, granted the continuity assumption:

a. One can substitute for future market value as a function of the relevant economic variables the expected future value, with respect to the insurable variables, as a function of the uninsurable variables, since the contribution of insurable variables to the risk premium is, by definition, negligible. The resulting function is dependent only on the uninsurable variables, projections of which are, by assumption, continuous functions of time.

b. In discussions of future market value, it might seem important to distinguish between the value of the (new) function of the market projections of the remaining (uninsurable) variables u_j, and the expected value of the function $F(u_1, u_2, . . .)$.

It is easy to show that the distinction disappears in the limiting case when the time interval approaches zero. The expected value of the function $F(u_1, u_2, \ldots)$ is

$$\bar{F} = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} F(u_1, u_2, \ldots) p(u_1, u_2, \ldots) du_1 du_2 \ldots, \qquad (64$$

given the joint frequency function implied by current projections of the u_j and the joint frequency function of projection errors in the u_j. In the limit, the frequency function of the future u_j becomes the so-called Dirac delta function $\delta(u_1 - u'_1, u_2 - u'_2, \ldots)$, if the u'_j are defined to be the current projections of the u_j (since the u_j are now restricted to uninsurable economic variables, projections of which, by assumption, are continuous functions of time). Hence the limiting value of $\bar{F}$ is

$$\begin{aligned} \bar{F} &= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} F(u_1, u_2, \ldots) \delta(u_1 - u'_1, u_2 - u'_2, \ldots) du_1 du_2 \ldots \\ &= F(u'_1, u'_2, \ldots) \qquad (65 \end{aligned}$$

Henceforth in our discussion of a projected future value F we shall assume (1) that it is an expected value with respect to any relevant insurable economic variables, and (2) a function of market projections of any relevant uninsurable variables.

Evaluation of Income Streams Over Time

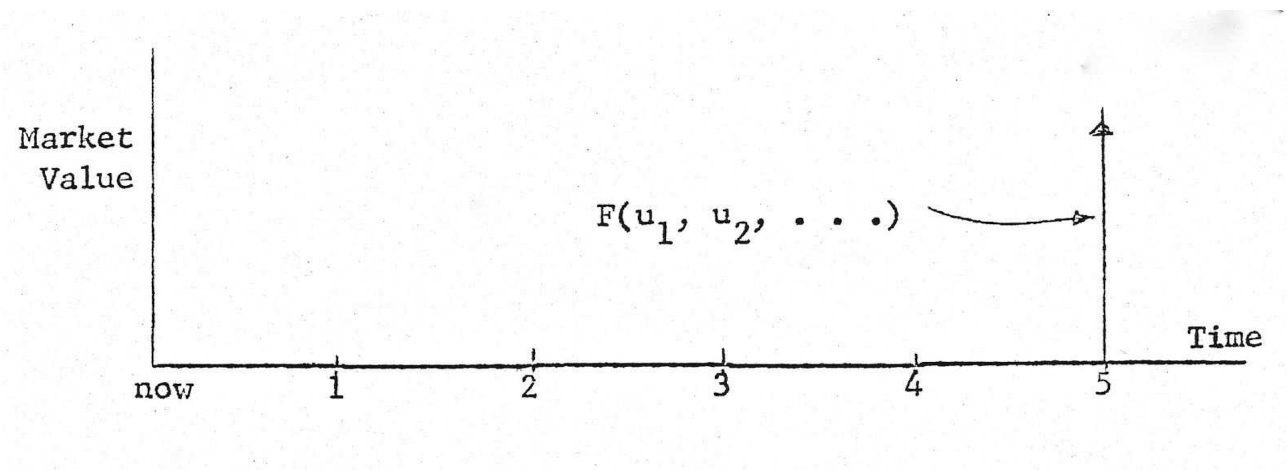
We have proved that as the time interval becomes infinitesimal, the expression for risk premium,

$$\nu = (2/\lambda n) \sum_j [E'_j \cdot \sum_{km} E'_m((\delta V_k)/(\delta u_m))]((\delta V)/(\delta u_j)) = \sum_j a'_j((\delta V)/(\delta u_j)), \qquad (66$$

and the relation between expected future value $\bar{x}$ and present market value v,

$$v = b\bar{x} + \nu = b\bar{x} + b \sum_j a'_j((\delta V)/(\delta u_j)), \qquad (67$$

tend to become exact. In order to extend these results to an interval of finite length, we first consider the example shown in the following diagram. A sequence of unit intervals separates the present from an investment of given future value* $F(u_1, u_2, \ldots)$ at time 5. We are interested in the value of F at the prior times 0, 1, 2, 3 and 4. At any time prior to time 5, the current market projection of F depends on current market projections of the u_j. In general, the projected



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* In the absence of brokerage and tax effects, the individual investor, hence the market, will have no reason to discriminate between future market values and future cash flows.

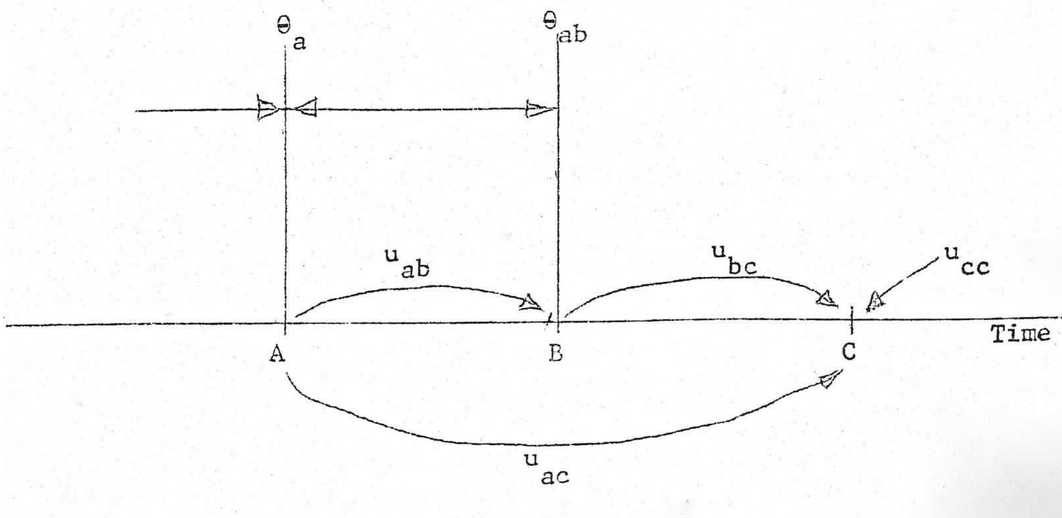
values of the u_j at time 5, hence the projected value of F, will be different at each point in time preceding time 5. In particular, they will probably differ between time 4 and time 3.

What is the value of F at time 3? It depends only on the projected value at time 4 and the uncertainty in that value. The actual value at time 4 depends on the market's projections at time 4 of the u_j. If we knew at time 3 what those projections would be at time 4, there would be no uncertainty in owning rights to F from time 3 to time 4. In fact, however, we don't know. Since we are given the value F at time 5 as a function of the u_j, however, for particular projections of the u_j we can obtain the corresponding projection of F and the first partial derivatives $\delta F/\delta u_j$, and apply expression (67), with market parameters a'_j and r appropriate to the interval between times 4 and 5, to obtain an approximation to the market value at time 4 corresponding to these projections. Thus knowledge of the function $F_5(u_j)$ leads to a new function $F_4(u_j)$--market value at time 4 as a function of projections at time 4 of the values of the u_j at time 5. As we pointed out earlier, at time 3 we don't know with certainty what the projections at time 4 will be. If we have a projection at time 3 of the projections at time 4 (of the u_j at time 5), however, then we can apply expression (67), with market parameters a'_j and r appropriate to the interval between times 3 and 4, to the market value at time 4, as a function of projections at time 4, to determine the market value of F at time 3. Like the market value of F at time 4, the market value at time 3 (a) can be deduced as a function of the u_j without reference to specific values, and (b) is a function only of the projections of the next period projections.

Where does this seemingly endless chain begin and end? Working backward in time, it begins with a cash flow at some point in the future, defined as a function of the u_j. It ends at the present, merely because the objective is to determine current market value. Our ability to make use of this chain of reasoning to determine current value depends, however, on the availability of a current projection of the next-period projection (of the projections of the period after <u>that</u>, etc.) of the u_j at the time of the ultimate cash flow.

Although it may be possible in some cases to deduce estimates of current next-period projections of the u_j from observations of related current market prices, it is important to establish a relationship between one-period projections of the type required and the more familiar projections of ultimate values of the u_j at the time of the cash flow. It is this relationship which guarantees consistency between two points of view, both of which are valid: one, which we have explored at length, regards current market value as a function of market value one period hence; the other regards current value as a function of the ultimate cash flow. We have just demonstrated that market value one period hence, as a function of the current next-period projections, is uniquely determined by the ultimate cash flow, as a function of the projections of the ultimate values of the u_j at the time of the cash flow. If it can be shown that the latter projections must equal the former, then the reconciliation will be complete.

If the projections in question are all expected values, proof is tantamount to showing that the expected value of a later(expected value) projection of the ultimate event is the expected value of the ultimate event. Some people will accept this as obvious.



The general situation is pictured in the above diagram.* At time A, a projection (u_{ac}) of the value (u_{cc}) of the economic variable of interest u at time C is made, using all the information (θ_a) available at time A. At time B, the next projection (u_{bc}) is made, using any additional information (θ_{ab}) received between A and B. At time A, with only information θ_a available, a projection (u_{ab}) is also made of the value of the projection u_{bc}.

Knowledge about the projected variables can be expressed in terms of probability distributions; thus we may define

$$P_1 = P_1(u_{cc} | \theta_a) \qquad (68$$

as the probability that u_{cc} will be the actual value of the variable u at time C, given only the information θ_a available at time A. In a similar fashion we define

$$P_2 = P_2(u_{cc} | \theta_a, \theta_{ab}) \qquad (69$$

as the analogous probability given the information available at time B,

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* The following proof is due to R.A. Howard and J.D. Herniter. A proof employing a different, and less generally familiar, symbolism can be found on p. 107 of Applied Statistical Decision Theory, Raiffa, Howard, and Schlaifer, Robert, Harvard University Press, 1961.

and

$$P_3 = P_3(u_{ab} | \theta_a) \tag{70}$$

as the probability of obtaining the particular additional information θ_{ab} between times A and B, given prior information θa. Assuming that the criterion is minimal mean-square error, then the appropriate projections in each case will be the means of the probability distributions of the variables u_{cc} and u_{bc} being projected. Hence we have

$$u_{ac} = \int_{-\infty}^{\infty} u_{cc} P_1(u_{cc}|\theta_a) du_{cc}, \tag{71}$$

and

$$u_{bc} = \int_{-\infty}^{\infty} u_{cc} P_2(u_{cc}|\theta_a, \theta_{ab}) du_{cc}. \tag{72}$$

Similarly, the appropriate projection u_{ab} of u_{bc} is the mean of the probability distribution of u_{bc} given only the information available at time A. The actual projection u_{bc} at time B is, of course, entirely a function of θ_a and the further information θ_{ab} received prior to time B. Hence we have

$$\begin{aligned} u_{ab}(\theta_a) &= \int_{\theta ab} u_{bc} P_3(\theta_{ab}|\theta_a) d\theta_{ab} \\ &= \int_{\theta ab} \int_{-\infty}^{\infty} u_{cc} P_2(u_{cc}|\theta_a, \theta_{ab}) P_3(\theta_{ab}|\theta_a) d\theta_{ab}\ du_{cc} \end{aligned} \tag{73}$$

where the integration with respect to the information θ_{ab} ranges over all possible "messages" received between A and B. Reversing the order of integration, we obtain

$$u_{ab}(\theta_a) = \int_{-\infty}^{\infty}\int_{\theta ab} u_{cc}P_2(u_{cc}|\theta_a, \theta_{ab})P_3(\theta_{ab}|\theta_a)d\theta_{ab}du_{cc} . \quad (74$$

Now this integral reduces to

$$\int_{\theta ab} P_2(u_{cc}|\theta_a, \theta_{ab})P_3(\theta_{ab}|\theta_a)d\theta_{ab} = P_1(u_{cc}|\theta_a) . \ * \quad (75$$

Hence we obtain the equality

$$u_{ab}(\theta_a) = \int_{-\infty}^{\infty} u_{cc}P_1(u_{cc}|\theta_a)du_{cc} = u_{ac}(\theta_a) , \quad (76$$

which completes the proof.

We are now in a position to develop a relatively simple description of market value of a future cash flow as a function of the length of time until the occurrence of the cash flow.

We have developed the idea that the relation

$$v = bV + \sum_j a'_j((\delta V)/(\delta u_j)) \quad (77$$

applies approximately between market value v at one point in time and market value V at the next point in time, where most generally the u_j

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* Consider the following general identities in the probability distributions of combinations of x, y, and z (probability functions of distinct arguments are distinct functions):

$$p(x,y,z) \equiv p(x,y,z)p(z)$$

and also,

$$p(x,y,z) \equiv p(x|y,z)p(y,z)$$

$$\equiv p(x|y,z)p(y|z)p(z).$$

Equating the right-hand members, and dividing by $p(z)$ we have,

$$p(x,y|z) \equiv p(x|y,z)p(y|z),$$

hence also

$$\int_y p(x|y,z)p(y|z)dy = \int_y p(x,y|z)dy = p(x|z)$$

are market projections at the next point in time of market projections still further removed into the future, and where V and v are based on market projections at the first point in time; and that it tends to apply exactly as the interval becomes infinitesimal.

We can rewrite (77) as

$$(1 - b)v = b(V - v) + \sum a'_j((\delta V)/(\delta u_j)), \tag{78}$$

or

$$v = b/(1 - b)\ (V - v) + 1/(1 - b) \sum a'_j((\delta V)/(\delta u_j)). \tag{79}$$

Multiplying the second term top and bottom by Δt, the length of one period, we obtain

$$v = [(b\Delta t)/(1-b)]\bullet[(V-v)/\Delta t] + [1/(1 - b)] \sum a'_j((\delta V)/(\delta u_j)). \tag{80}$$

Now v is the market value of V, one period earlier. Let Δv equal the projected change in value, over one period.

$$\Delta v = V - v\ . \tag{81}$$

The coefficient b is the risk-free discount factor; that is, in the absence of risk, the partial derivatives $(\delta V)/(\delta u_j)$ are all zero and expression (77) becomes

$$v = bV\ . \tag{82}$$

In the risk-free case the same relationship can be expressed in terms of a nominal rate of return r and the length of the discount period Δt.

$$v + vr\Delta t = v(1+r\Delta t) = V\ . \tag{83}$$

Using (81) and the resulting equivalence

$$b = 1/(1+r\Delta t), \qquad (84$$

we can rewrite (80),

$$v = \Delta t/[(1+r\Delta t)-1] \bullet \Delta v/\Delta t + 1/(1-b) \sum_{j=1}^{n} a'_j((\delta V)/(\delta u_j)),$$

$$= 1/r \bullet \Delta v/\Delta t + 1/(1-b) \sum_{j=1}^{n} a'_j((\delta V)/(\delta u_j)). \qquad (85$$

In the limit, as Δt approaches zero,

$$\Delta v/\Delta t \rightarrow \delta v/\delta t$$

$$(\delta V)/(\delta u_j) \rightarrow \delta v/\delta u_j, \qquad \delta V/\delta t \rightarrow \delta v/\delta t \qquad (86$$

and

$$b(\Delta t) = 1/(1+r\Delta t) \rightarrow 1 . \qquad (87$$

In the limit, the coefficients

$$a'_j = (2/\lambda n)E'_j \bullet \sum_{km} E'_m((\delta V_k)/(\delta u_m)) \qquad (88$$

of the uninsurable variables u_j approach zero, because of the assumed continuity in market projections of these variables: since we have for an individual error

$$\lim_{\Delta t \to 0} E_j = \underline{0} \qquad (89$$

then

$$\lim_{\Delta t \to 0} a'_j = 0, \quad j \neq 0 . \qquad (90$$

If the following limits exist,

$$\lim_{\Delta t \to 0} a'_j/(1-b) = a''_j \tag{91}$$

the substitutions (86), (87), and (91) enable us to write the limiting form of (85), hence (77), as

$$\lim_{\Delta t \to 0} v = 1/r \ \delta v/\delta t + \sum_{j=1}^{n} a''_j((\delta v)/(\delta u_j)) \tag{92}$$

Expression (92) is the partial differential equation which market value $v(t, u_1, u_2, \ldots, u_n)$ must satisfy.

Consider a cash flow

$$v_0 = v_0(u_1, u_2, \ldots, u_n). \tag{93}$$

which is some given function of the economic variables $(u_1, u_2, \ldots, u_n)$, and occurs at time t_0 . Expression (94) is a boundary condition on $v(t, u_1, u_2, \ldots, u_n)$. Since at time t_0

$$v(t_0, u_1, u_2, \ldots, u_n) = v_0(u_1, u_2, \ldots, u_n), \tag{94}$$

expressions (92) and (94) together completely determine market value as a function of time and the relevant economic variables. The solution is

$$v(t, u_1, u_2, \ldots, u_n) = v_0(u'_1, u'_2, \ldots, u'_n)\exp\ [r(t-t_0)] \tag{95}$$

where

$$u'_j = u_j - r\int_{t0}^{t} a''_j(\tau)d\tau \tag{96}$$

In order to determine the market value v at time t prior to the occurrence of the cash flow $v(u_1, u_2, . . ., u_n)$ at time t_0, one

a) forecasts the u_j at time t_0

b) computes the u'_j as a function of $(t-t_0)$

c) substitutes the resulting values of the u'_j in (95)

d) discounts the result in c) by the factor $\exp[r(t-t_0)]$.

At this point it is convenient to introduce a slight simplification. Thus far the effort has been confined to evaluating a future cash flow or market value $F(u_1, u_2, . . .)$ occurring at a single point in time. We have focussed attention on the uncertainties due to errors between successive forecasts of the uninsurable variables u_j, where, for example, $F(u_1, u_2, . . .)$ might be the (cash) earnings from a project in a certain future time interval, and a particular variable u_{j0} might be the sales volume during that interval. We are now interested in determining the present market value of a stream of earnings (where, for the moment, we consider the stream to be rather arbitrarily broken down into a series of discrete cash flows) dependent on a set of variables which, although physically the same set from one point in time to the next, is nevertheless a distinct set of numbers with distinct forecasts at any point in time, and distinct errors between succeeding forecasts. Now the fact that sales volume, u_{j0}, for example, in 1965 is distinct from sales volume in 1966 causes us no formal difficulty, since we can simply assign another economic variable u_{j1} to sales volume in 1966, and still another to sales volume in 1967, and so forth. It is probably clear intuitively that in most practical cases no information is gained by retaining such distinctions, and the device of treating the same physical variable at different times as different variables is obviously clumsy. Consider the terms in the general expression for risk premium ν

$$\nu = (2/\lambda n) \sum_{j} [E'_j \bullet \sum_{km} E'_m((\delta V_k)/(\delta u_m))]((\delta V)/(\delta u_j)) \tag{97}$$

corresponding to the same variable at two different times, u_{j0} and u_{j1}. If errors between succeeding forecasts of u_{j0} and u_{j1} are highly correlated and of approximately the same magnitude (and since u_{j0} and u_{j1} represent the same "physical" variable, this is highly likely), then the respective uncertainty vectors may be taken as equal;

$$E'_{j0} = E'_{j1} \text{ ,} \tag{98}$$

and the coefficients corresponding to $\delta V/\delta u_{j0}$ and $\delta V/\delta u_{j1}$ in expression (97) are then equal.

$$E'_{j0} \bullet \sum_{km} E'_m((\delta V_k)/(\delta u_m)) = E'_{j1} \bullet \sum_{km} E'_m((\delta V_k)/(\delta u_m)). \tag{99}$$

Hence the terms in ν corresponding to u_{j0} and u_{j1} equal

$$(2/\lambda n)[E'_{j0} \bullet \sum_{km} E'_m((\delta V_k)/(\delta u_m))]((\delta V)/(\delta u_{j0})) + [E'_{j1} \bullet \sum_{km} E'_m((\delta V_k)/(\delta u_m))]((\delta V)/(\delta u_{j1})) \tag{100}$$

$$= (2/\lambda n)[E'_j \bullet \sum_{km} E'_m((\delta V_k)/(\delta u_m))][((\delta V)/(\delta u_{j0})) + ((\delta V)/(\delta u_{j1})).$$

But if no distinction is made between u_{j0} and u_{j1} then in place of the partial derivatives $\delta V/\delta u_{j0}$ and $\delta V/\delta u_{j1}$ we will have

$$\delta V/\delta u_j = \delta V/\delta u_{j0} + \delta V/\delta u_{j1} \text{ ,} \tag{101}$$

and the value of the terms corresponding to u_{j0} and u_{j1} becomes

$$(2/\lambda n)[E'_j \bullet \sum_{km} E'_m((\delta V_k)/(\delta u_m))]((\delta V)/(\delta u_j)). \quad (102$$

which shows that disregarding the distinction between physically identical economic variables at different points in time is permissible, as long as (98) is satisfied.

Conclusions

With this simplification, we are ready to consider streams of cash flows $f_0(u_1, u_2, \ldots)$, $f_1(u_1, u_2, \ldots)$, . . ., f_k, . . . occurring at unit intervals, where u_j in $f_k(u_1, u_2, \ldots, u_j, \ldots)$ represents the value that the variable u_j takes on at time k. Such a stream can be regarded as a vector of which the individual cash flow functions $f_k(u_1, u_2, \ldots)$ are components. It is possible to speak in this sense of the sum of two or more cash flow streams

$$f_k + g_k \quad , \quad k = 0, 1, 2, \ldots,$$

where the indicated operation is the addition of cash flow functions corresponding to the same point in time. We shall show that the present market value of cash flow streams f_k and g_k has the linearity property

$$V(f_k + g_k) = V(f_k) + V(g_k), \quad (103$$

where $V(f_k)$ represents the application of a present-market-value operator V to the stream of cash flows f_k, $k = 0, 1, 2, \ldots$.

Define S_k as an auxiliary operator which determines the present (time 0) market value function $v(u_1, u_2, \ldots)$ from a future market value $f_k(u_1, u_2, \ldots)$ at time k in accordance with the following modifications of equation (95),

$$v(u_1, u_2, \ldots) = f_k(u'_1, u'_2, \ldots)\exp(-rk) , \tag{104}$$

where

$$u'_j = u_j + r\int_0^k a''_j(\tau)d\tau . \tag{105}$$

Then V is defined as follows in terms of S,

$$V(f_k) = V(f_0, f_1, f_2, \ldots) = \sum_k S_k f_k . \tag{106}$$

Inspection of equation (104) shows that for any k_0, S_k has the property

$$S_{k0}(f_{k0} + g_{k0}) = S_{k0}\, f_{k0} + S_{k0}\, g_{k0} . \tag{107}$$

Hence we have

$$V(f_k + g_k) = \sum_k S_k(f_k + g_k) = \sum_k S_k f_k + S_k g_k$$

$$= \sum_k S_k f_k + \sum_k S_k g_k \tag{108}$$

$$= V(f_k) + V(g_k) . \tag{109}$$

This result has several consequences for the cost of capital problem, provided company managements avoid projects which lower the companies' market value (and given our assumption about the absence of complications due to taxes).

1. A project with uninsurably risky cash flows is not worth as much as a project with certain cash flows of the same expected value. The risk premiums exacted by the market should tend to observe certain rules of consistency developed in this paper.

2. The incremental effect on the market value of a company of undertaking a given project is the market value of the project (including initial investment). The cost of capital for a project is entirely a function of the cash flows associated with the project, and is independent of the investing company.

3. If a company borrows money at the risk-free rate of interest and offsets the proceeds and repayments against the cash flows associated with a project, the market value of the project will not change.

4. The risk in an investment project can be defined as the increase in the sensitivity of the market value of the investing company to the relevant variables which results if the project is undertaken. This risk can be assessed by reference to the present-market-value function $V(u_1, u_2, \ldots)$. The cumulative contribution of future cash flows to present value is usually far greater than the contribution of current cash flows. Hence the short-term risk due to the uninsurable variables is usually largely the risk due to the possibility of changes in the market forecasts of these variables, rather than uncertainty in the future cash flows themselves.