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CS 555: Homework # 1
Term: Spring 2023

Problem 1
Proof: The consistency of the implicit C-N scheme with central spatial discretization is consistent with the linear advection equation $\partial_t u + a\partial_x u = 0$.

The crank-Nicolson scheme is written as *Solution:*

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{a}{2}\left(\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} + \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x}\right) = 0 \tag{1}$$

Then, we assume these functions have n derivatives. Each term can be expanded using the Taylor series. And we utilize the modified equation technique to analyze the error type and order.

$$u(x_0, t_0 + \Delta t) = u(x_0, t_0) + u_t\Delta t + u_{tt}\frac{\Delta t^2}{2} + u_{ttt}\frac{\Delta t^3}{6} + \Theta(\Delta t^4) \tag{2}$$

$$u(x_0 + \Delta x, t_0) = u(x_0, t_0) + u_x\Delta x + u_{xx}\frac{\Delta x^2}{2} + u_{xxx}\frac{\Delta x^3}{6} + \Theta(\Delta x^4) \tag{3}$$

$$u(x_0 - \Delta x, t_0) = u(x_0, t_0) - u_x\Delta x + u_{xx}\frac{\Delta x^2}{2} - u_{xxx}\frac{\Delta x^3}{6} + \Theta(\Delta x^4) \tag{4}$$

$$\begin{aligned} u(x_0 + \Delta x, t_0 + \Delta t) = u(x_0, t_0) + u_x\Delta x + u_t\Delta t + u_{xx}\frac{\Delta x^2}{2} + u_{tt}\frac{\Delta t^2}{2} + \\ \frac{1}{6}(u_{xxx}\Delta x^3 + 3u_{xxt}\Delta x^2\Delta t + 3u_{xtt}\Delta x\Delta t^2 + u_{ttt}\Delta t^3) + \Theta(\Delta x^4, \Delta t^4) \end{aligned} \tag{5}$$

$$\begin{aligned} u(x_0 - \Delta x, t_0 + \Delta t) = u(x_0, t_0) - u_x\Delta x + u_t\Delta t + u_{xx}\frac{\Delta x^2}{2} + u_{tt}\frac{\Delta t^2}{2} - \\ \frac{1}{6}(u_{xxx}\Delta x^3 + 3u_{xxt}\Delta x^2\Delta t - 3u_{xtt}\Delta x\Delta t^2 + u_{ttt}\Delta t^3) + \Theta(\Delta x^4, \Delta t^4) \end{aligned} \tag{6}$$

According to the governing equation, we are able to derive other wave equations as follows.

$$u_{tt} = au_{xx}, \quad u_{tx} = -au_{xx}, \quad u_{xxt} = -au_{xxx}, \quad u_{ttx} = a^2u_{xxx}, \quad u_{ttt} = -a^3u_{xxx} \tag{7}$$

Consider equations (1-7), we have

$$u_t + au_x = \left(\frac{5}{12}a^3\Delta t^2 - \frac{a}{6}\Delta x^2\right)u_{xxx} + \Theta(\Delta t^2, \Delta x^2) \tag{8}$$

Thus, we can see that the C-N scheme with central spatial discretization is second order in time and space with dispersion error.

Problem 2
Proof: Use von-Neumann stability theory to prove the C-N scheme is unconditionally stable wrt. different time step.

Solution: We assume the error can be expressed by a Fourier expansion. WLOG, we select an arbitrary term to analyze.

$$\epsilon = e^{at+ik_mx} \tag{9}$$

Where k_m is the wave number, and we also know that ϵ satisfies the governing equation. Taking into the discretized equation, we have

$$\frac{e^{\Delta t} - 1}{\Delta t} + \frac{a}{2}(\frac{e^{ik_m \Delta x} - e^{-ik_m \Delta x}}{2\Delta x} + e^{\Delta t}\frac{e^{ik_m \Delta x} - e^{-ik_m \Delta x}}{2\Delta x}) = 0 \tag{10}$$

Then we can simplify the expression

$$|e^{\Delta t}| = |\frac{1 - i\frac{m}{sin(k_m \Delta x)}}{1 + i\frac{m}{sin(k_m \Delta x)}}| = 1 \tag{11}$$

Here, $m = \frac{a\Delta t}{2\Delta x}$. Thus, the scheme is unconditionally stable at whatever choice of Δt.

Problem 3
Compute: Use the C-N scheme with periodic BC.

Solution: Here, we take the Gaussian curve as the initial condition and set the boundary condition to be periodic. The advection speed is a = 1, and the advection time is 1 second. The only control parameter is the CFL number. While fixing the grid space and changing the time step to vary the CFL number from 20 to 0.2, we find the numerical solution approximates the exact solution better and better. This should be expected because the CN scheme is consistent, the smaller time step should get him a smaller numerical error. In addition, the order of the modified equation on the right-hand side is three, meaning a dispersive error. Thus, we can observe this error as very obvious when the CFL number is large.

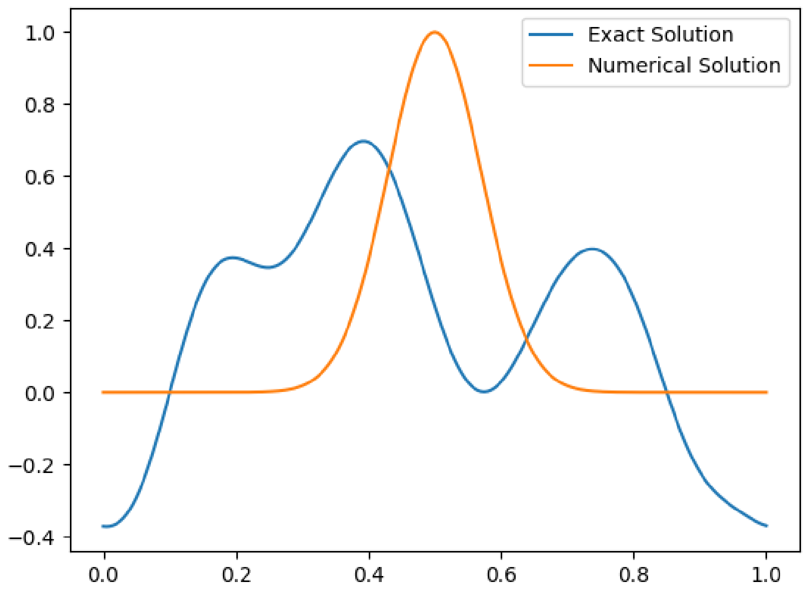

Figure 1: Linear advection equation CFL = 20

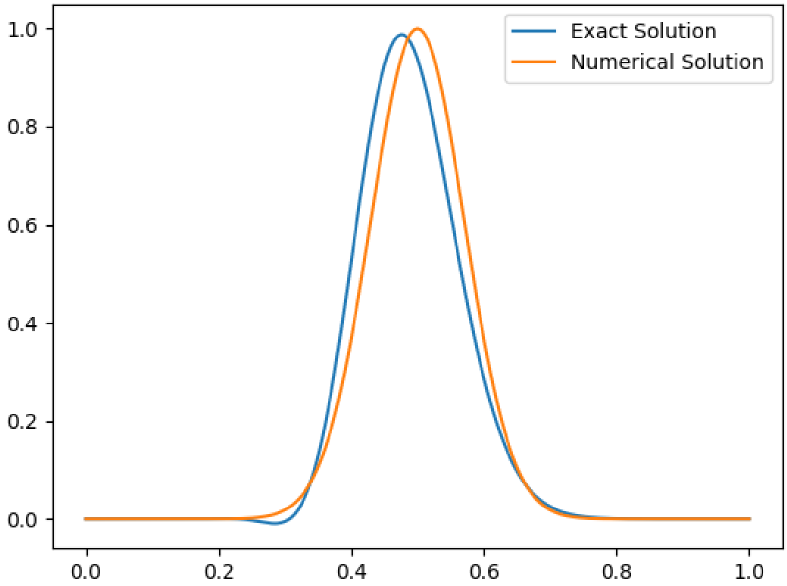

Figure 2: Linear advection equation CFL = 2

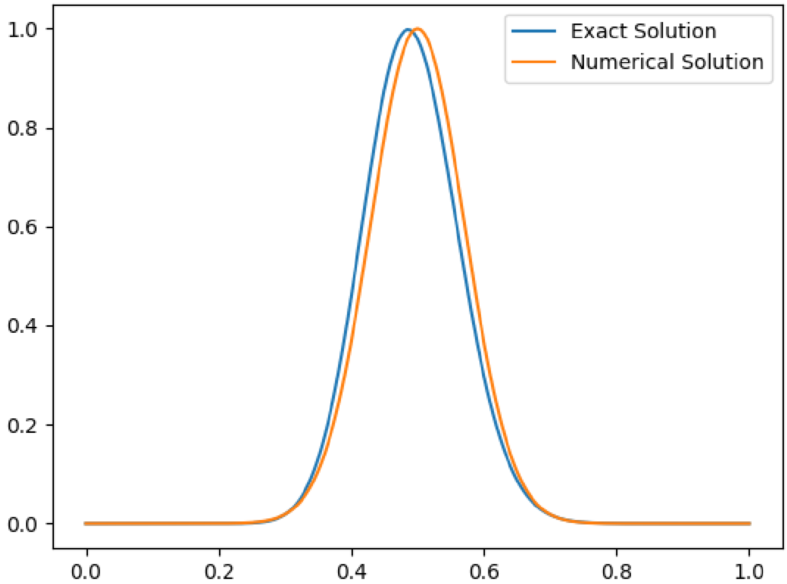

Figure 3: Linear advection equation CFL = .2