Tree Path Labeling of Path Hypergraphs: A Generalization of Consecutive Ones Property

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We consider the following constraint satisfaction problem. Given (i) a set system $\mathcal{F} \subseteq (powerset(U) \setminus \emptyset)$ of a finite set U of cardinality n, (ii) a tree T of size n and (iii) a bijection ℓ, defined as *tree path labeling*, mapping the sets in $\mathcal{F}$ to paths in T, does there exist at least one bijection $\phi : U \rightarrow V(T)$ such that for each $S \in \mathcal{F}$, $\{\phi(x) \mid x \in S\} = \ell(S)$? A tree path labeling of a set system is called *feasible* if there exists such a bijection ϕ. In this paper, we characterize feasible tree path labeling of a given set system to a tree. This result is a natural generalization of results on matrices with the Consecutive Ones Property. Moreover, we pose some interesting algorithmic questions which extend from this work.

Consecutive ones property (COP) of binary matrices is its property of rearrangment rows (columns) in such a way that every column (row) has its 1s occuring consecutively. The problem of COP testing is also a constraint satisfaction problem of a set system as follows. In a binary matrix, if every column is represented as a set of indices of the rows with 1s in that column, then if the matrix has the COP, a reordering of its rows will result in sets that are intervals. The COP is equivalent to the problem of finding interval assignments to a given set system [3] with a single permutation of the universe which permutes each set to its interval. Clearly COP is a special instance of tree path labeling problem described above when T is a path. The result in [3] characterize interval assignments to the sets which can be obtained from a single permutation of the rows - the cardinality of the interval assigned to it must be same as the cardinality of the set, and the intersection cardinality of any two sets must be same as the interesction cardinality of the corresponding intervals - Intersection Cardinality Preserving Interval Assignment (ICPIA). This is obviously necessary and was discovered to be sufficient.

We focus on the question of generalizing the notion of an ICPIA [3] to characterize feasible path assignments. We show that for a given set system $\mathcal{F}$, a tree T, and an assignment of paths from T to the sets, there is a bijection ϕ between U and $V(T)$ if and only if all intersection cardinalities among any three sets (not necessarily distinct) is same as the intersection cardinality of the paths assigned to them and the input successfully executes our filtering algorithm (described in this paper) without prematurely exiting. Aside from finding the bijection ϕ mentioned above for a given path labeling, we also present a characterization of set systems which have a feasible tree path labeling on a given tree T and present an algorithm to find a path labeling if it exists when T is a *k-subdivided star*. A k-subdivided star is a star with all its rays subdivided exactly k times. The path from the center to a leaf is called a ray of a k-subdivided star and they are all of length $k+2$. A *star* graph is a complete bipartite graph $K_{1,l}$ which is clearly a tree and l is the number of leaves. The vertex with maximum degree is called the *center* of the star and the edges are called *rays* of the star.

The following questions are extensions to this work.

1. The intersection graph of a set system with a feasible tree path labeling from a tree T must be a *path graph* which is a subclass of chordal graphs. This can be checked efficiently because path graph recognition is polynomial time solvable[1, 4]. However, this is only a necessary condition. It is possible to have a pair of set system and tree $(\mathcal{F}, T)$, such that the intersection graph of $\mathcal{F}$ is a path graph, but there is no feasible tree path labeling to T. Therefore, the following questions.

   (a) What is the maximal set system $\mathcal{F}' \subseteq \mathcal{F}$ such that $(\mathcal{F}', T)$ has a feasible tree path labeling?

   (b) What is the maximal subtree $T' \subseteq T$ such that $(\mathcal{F}, T)$ has a feasible tree path labeling?

   (c) Path graph isomorphism is known be isomorphism-complete[2]. An interesting area of research would be to see what this result tells us about the complexity of the tree path labeling problem.

2. A set system $\mathcal{F}$ can be alternatively represented by a *hypergraph* $\mathcal{H}_{\mathcal{F}}$ whose vertex set is *supp*$(\mathcal{F})$ and hyperedges are the sets in $\mathcal{F}$. This is a known representation for interval systems in literature [2]. We extend this definition here to path systems. Two hypergraphs $\mathcal{H}$, $\mathcal{K}$ are said to be isomorphic to each other, denoted by $\mathcal{H} \cong \mathcal{K}$, iff there exists a bijection $\phi : supp(\mathcal{H}) \rightarrow supp(\mathcal{K})$ such that for all sets $H \subseteq supp(\mathcal{H})$, H is a hyperedge in $\mathcal{H}$ iff K is a hyperedge in $\mathcal{K}$ where $K = \{\phi(x) \mid x \in H\}$. If $\mathcal{H}_{\mathcal{F}} \cong \mathcal{H}_{\mathcal{P}}$ where $\mathcal{P}$ is a path system, then $\mathcal{H}_{\mathcal{F}}$ is called a *path hypergraph* and $\mathcal{P}$ is called *path representation* of $\mathcal{H}_{\mathcal{F}}$. If isomorphism is $\phi : supp(\mathcal{H}_{\mathcal{F}}) \rightarrow supp(\mathcal{H}_{\mathcal{P}})$, then it is clear that there is an induced path

labeling $l_\phi : \mathcal{F} \rightarrow \mathcal{P}$ to the set system. So our problem of finding if a given path labeling is a feasible path labeling is a path hypergraph isomorphism problem.

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