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An Extension of a Theorem of Fulkerson and Gross

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ABSTRACT

Fulkerson and Gross proved a theorem regarding 0-1 matrices that have the consecutive ones property. They then used it to test for this property for a given matrix. This was extended to matrices with the circular ones property by Tucker. It was further extended to the one drop property, which occurs in integer programming problems that arise in scheduling. When this result was presented, the question was raised whether there is a hierarchy of such properties and theorems. This paper answers this question in the affirmative. These results may help in testing matrices for these properties and also in solving these integer programs.

This paper is concerned with a generalization of the *consecutive ones property* for 0-1 matrices, introduced and studied by Fulkerson and Gross [4]. This property assures that the matrix is totally unimodular and hence is of

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importance in integer programming. The original application was the study of gene structure, interval graphs, and their relationship. The first generalization of the consecutive ones property, of the kind that is connected to this work, occurred in the study of matrices with the *circular ones property* as in the work of Tucker [7]. These arise naturally in the setting of cyclic scheduling in [1] and [2]. The property was further generalized to the *one drop property* in [5].

In particular, the central theorem in [4] was generalized in [5], and it raised the question of whether there is a hierarchy of such properties and a corresponding hierarchy of such theorems generalizing the results in [4]. This question was raised by A. J. Goldman at the Workshop on Lattice Programming held at Johns Hopkins University in 1989. This paper answers this question at least in part. We give a simple proof of a more general theorem that includes all these results in extending the scope of the theorem of Fulkerson and Gross [4]. The potential for applications in integer programming is explored in several papers, including [1], [2], [3], [5], [7], and [4], and therefore will not be repeated here.

We hope that these results are useful in two ways. The first is towards developing an algorithm to test for these properties in a given 0-1 matrix. The second—more important—direction is to use these properties in solving integer programs with such matrices as constraint matrices. We believe that there is hierarchy of algorithms of increasing order of time complexity that would solve these problems. Thus, matrices with the one drop property result in problems that have the rounding property, and hence the corresponding integer programs are nicely solvable. Those with what we call k-consecutive property would involve algorithms whose complexity grows exponentially with k but polynomially in other parameters. Integer programs mentioned above may be one of the following four types:

$$[\min e^t x : Ax \geqslant b,\ x \geqslant 0,\ x \text{ int}], \tag{1}$$

$$[\max b^t x : Ax \leqslant e,\ x \geqslant 0,\ x \text{ int}], \tag{2}$$

$$[\min b^t x : Ax \geqslant e,\ x \geqslant 0,\ x \text{ int}], \tag{3}$$

$$[\max e^t x : Ax \leqslant b,\ x \geqslant 0,\ x \text{ int}], \tag{4}$$

Our hope is that is for a fixed A with the k-consecutive property all these problems are solvable in time that is polynomial in all parameters except k.

DEFINITION 1. A 0-1 matrix A is said to have the *consecutive ones* property if the columns can be permuted so that the ones in each row occur in consecutive columns (where the first and the last columns are not considered consecutive).

DEFINITION 2. A 0-1 matrix A is said to have the *circular ones* property if the columns can be permuted so that the ones in each row occur in consecutive columns, where the last and first column are considered consecutive.

DEFINITION 3. A 0-1 matrix A is said to have the *one drop* property if the columns can be permuted so that in each row there is at most one instance of a 1 followed by a 0.

For the sake of completeness, let us recall earlier results first. The result below is from [4].

THEOREM 1. *Let A and B be two $m \times n$ matrices whose entries are 0 and 1. If $AA^t = BB^t$, and A has the consecutive ones property, then so does B, and there exists a permutation matrix P such that $B = AP$.*

It was extended in [5] to the one drop property in the following theorem:

THEOREM 2. *Let A and B be two $m \times n$ matrices whose entries are 0 and 1. If $A^{[3]} = B^{[3]}$, and A has the one drop property, then so does B, and there exists a permutation matrix P such that $B = AP$.*

Here the statement $A^{[j]} = B^{[j]}$ refers to the following property: Let the rows of A and B be in one-to-one correspondence. Then for each subset S of corresponding rows from the two matrices, the number of columns with no 0's is the same for both matrices, provided that $|S| \leqslant j$. The case when $j = 2$ is the same as the conditions in [4]. These results were further extended in [6] to matrices with the k-strings property.

We are now ready to state the results of this paper.

Let S^n be the 0-1 matrix with n rows and 2^n distinct columns. Let $S^n = [O^n, E^n]$ where the columns of O^n are the columns of S^n with an odd number of ones, and those of E^n are the even columns. Note that E^n has the zero vector, and $E^n = J^n - O^n$ whenever n is odd; here J^n is a matrix all of whose entries are equal to 1 and is of the same size as E^n and O^n.

DEFINITION 4. We say that $R_k(A, B)$ holds for a pair of 0-1 matrices A and B of the same size if the number $n(S)$, of columns that have no zero in any subset S of j rows is the same in both matrices $\forall S \ni |S| \leqslant k$; i.e., $A^{[j]} = B^{[j]}$ for $1 \leqslant j \leqslant k$. [In particular, $R_k(O^n, E^n)$ holds $\forall k < n$.] Here we suppose that there is one-to-one correspondence between the rows of A and B.

DEFINITION 5. A property Q of a matrix is said to be an *inherited property* if [A has Q, and D is a submatrix of A] $\Rightarrow$ [D has Q]. We are only interested in properties that are preserved under permutations of rows and columns in this paper. Let $\mathscr{Q}$ be the class of all such inherited properties defined on matrices.

Examples: Total unimodularity; balancedness; k-strings property; k-circular property; k-drop property. While the first two are well known, the last three are defined in [6], and their definitions are repeated below for convenience. For $k = 1$, these three properties are called *consecutive ones*, *circular ones*, and *one drop*, respectively.

DEFINITION 6. A 0-1 matrix is said to have the *k-strings* property if its columns can be permuted so that there are at most k strings of consecutive ones. Here we do not consider the first and the last columns as consecutive.

DEFINITION 7. A 0-1 matrix A is said to have the *k-circular* property if its columns can be permuted so that there are at most k strings of consecutive ones, with the first and the last columns being considered as consecutive.

DEFINITION 8. A 0-1 matrix A is said to have the *k-drop* property if its columns can be permuted so that there are at most k instances of a 1 followed by a 0 in each row.

THEOREM 3. *Let q be the largest number such that at least one of O^q and E^q has a specified property $Q \in \mathscr{Q}$. Let A and B be* 0-1 *matrices of the same size such that $R_q(A, B)$ holds. Then if A has property Q, so does B and there exists a permutation matrix P such that $B = AP$.*

Proof. Proof is by induction on n, the number of rows of the two matrices. Suppose the theorem is false; let A and B be minimal violators of the theorem. Thus, the theorem is valid if any row or column is removed (inheritance is used here). Thus the matrices are of the form

$$A = \begin{bmatrix} r_a \\ C \end{bmatrix} \text{ and } B = \begin{bmatrix} r_b \\ C \end{bmatrix} \qquad \text{with} \quad r_a + r_b = e.$$

Thus, for every column in A there is a column in B with every entry complemented, and for every column in B there is one in A with every entry complemented. Hence, one of the two matrices is a matrix with n rows with all columns having an even number of 1's (there are columns with all possible even numbers up to n), and the other is the matrix with the remaining columns from the set of all 0-1 columns of size n. Hence, one of these is O^n and the other is E^n for some n. Since $R_q(A, B)$ holds, $n > q$. But for $n > q$, neither has property Q— a contradiction to the hypothesis that A has the property. Hence there are no minimal violators to the theorem, and hence, no violators. ■

The value of q depends on the property $Q \in \mathscr{Q}$. For example:

(1) If Q is consecutive ones, when $n = 3$ neither matrix of the above type has Q; when $n = 2$, both have the property. Hence, $q = 2$. Thus, if $R_2(A, B)$ holds, and A has consecutive ones, then so does B, and $B = AP$ for some P. This is the result in [4]. The same result also holds for the circular ones property.

(2) If Q is one drop, for $n = 4$ neither matrix has Q; for $n = 3$, both have the property. Hence, $q = 3$. Hence, the theorem is valid if $R_3(A, B)$ holds.

(3) If Q has the k-strings property, for $k = 2$, both O^4 and E^4 have the property; neither O^5 nor E^5 has the property, and hence $q = 4$ and we need $R_4(A, B)$. For $k > 3$, we show below that $q \leqslant k + 1$ and hence we do not need more than $R_{k+1}(A, B)$ to hold.

Now we want to find $q = \max[n : \text{at least one of } E^n, O^n \text{ has a specified inherited property } Q]$. The properties of interest are: (1) k-strings, (2) k-circular, and (3) k-drop. Clearly, all of these are inherited and independent of permutations of rows and/or columns. Given a 0-1 matrix A, we can count the instances where $a_{i,j} = 0$, $a_{i,j+1} = 1$ and those where $a_{i,j} = 1$, $a_{i,j+1} = 0$. Let $N^i_{0,1}(A) = |\{j : a_{i,j} = 1,\ a_{i,j+1} = 0\}|$ and $N^i_{1,0}(A) = |\{j : a_{i,j} = 0,\ a_{i,j+1} = 1\}|$. Let $N_{0,1}(A) = \Sigma_i N^i_{0,1}(A)$ and $N_{1,0}(A) = \Sigma_i N^i_{1,0}(A)$, and let $N(A) = N_{0,1}(A) + N_{1,0}(A)$. An $m \times n$ matrix has the k-strings property if for some permutation matrix P and all rows $1 \leqslant i \leqslant m$, we have $N^i_{0,1}(AP) \leqslant k$, $N^i_{1,0}(AP) \leqslant k$; and if both these are equalities, then either the first or the last entry in this row of AP must be a zero. It has the k-drop property if for some permutation matrix P and all rows $1 \leqslant i \leqslant m$, $N^i_{1,0}(AP) \leqslant k$. It is k-circular if each row of AP satisfies the k-string condition or the $(k+1)$-strings and k-drop conditions for some permutation matrix P.

Lemma 1. *Suppose at least one of $\{E^n, O^n\}$ has the k-strings property. Then $2^{n-1} \leqslant kn$.*

Proof. $\Sigma_i |A_{i,j} - A_{i,k}| \geqslant 2\ \forall j \neq k$ with $A = E^n$ or $A = O^n$. Hence, $N(AP) \geqslant 2(2^{n-1} - 1)$ for both these matrices for any permutation matrix P. However, if a 0-1 matrix A has the k-strings property, then for some permutation matrix P, we have $N^i_{0,1}(AP) \leqslant k$, $N^i_{1,0}(AP) \leqslant k$; and if both these are equalities, then either the first or the last entry in this row of AP must be a zero. Thus, $N(AP) \leqslant 2nk$; for equality to hold in this, we must have two columns of A be zero vectors—which is not true for either matrix under consideration. Hence, for the matrices under consideration, $N(AP) \leqslant 2nk - 2$. Combining these lower and upper bounds no $N(AP)$, we get the desired result. ■

For $k = 2$, the above lemma implies that $q \leqslant 4$. It can easily be verified that both O^4 and E^4 both have the 2-strings property and hence $q = 4$ for $k = 2$. Lemma also implies that $q \leqslant 4$ and hence equal to 4 for $k = 3$. Moreover, both $\{O^5, E^5\}$ have the 4-strings property and hence $q = 5$ for $k = 4$. In general, we have:

Lemma 2. *For the k-strings property with $k \geqslant 3$, $q \leqslant k + 1$*

Proof. $2^k \leqslant k(k+1) \Rightarrow k \leqslant 4$. Hence, $2^{n-1} \leqslant kn \Rightarrow n \leqslant k + 1$. ■

The result is also true if, instead of requiring at least one to have the k-strings property, we require both of them to have it. Hence, in this case, both have the property for $n \leqslant n^*(k)$ and neither has it for $n > n^*(k) + 1$. This lemma implies a result in [6].

Definition 9. Let $n^+(k) = \max[n : \text{both } O^n \text{ and } E^n \text{ have the } k\text{-strings property}]$.

Definition 10. Let $n^\#(k) = \max[n : \text{at least one of } O^n, E^n \text{ has the } k\text{-strings property}]$.

DEFINITION 11. If $n^+(k) = n^\#(k)$, then let $n^*(k) = n^+(k) = n^\#(k)$.

LEMMA 3. *Suppose both E^n and O^n have the k-drop property. Then $2^n \leqslant (2k + 1)n$. For $k = 1$, one must have $n \leqslant 3$ for this to happen.*

LEMMA 4. *Suppose one of E^n or O^n has the k-drop property. Then $2(2^{n-1} - 1) \leqslant n(2k + 1)$. For $k = 1$, $n \leqslant 3$.*

Here both matrices have the k-drop property if $n \leqslant n^+(k)$; only one has the property if $n \leqslant n^\#(k) \leqslant n^+(k) + 1$. For some values of k, $n^\#(k) \neq n^+(k)$; for example, $k = 1$. For $k = 2$, $n^\#(k) = n^+(k)$. All of this still is an overestimate of q.

We hope that these results will help in testing a zero-one matrix for these properties just as the theorem in [4] helped in testing for the consecutive ones property.

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