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The weighted consecutive ones problem for a fixed number of rows or columns

Marcus Oswald*, Gerhard Reinelt

Institute of Computer Science, University of Heidelberg, Im Neuenheimer Feld 368, D-69120 Heidelberg, Germany

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Abstract

The NP-hard weighted consecutive ones problem consists of converting a given 0/1-matrix to a consecutive ones matrix by switching entries with a minimum total cost. Here we consider this problem when the number of rows or columns is fixed. We show that in this case the problem can be solved in polynomial time.

Keywords: Consecutive ones problem; Polytopes; Facets

1. Introduction

The input of the *consecutive ones problem* is a 0/1-matrix $A \in \{0,1\}^{m\times n}$ and the task is to find a matrix A' obtained from A by changing as few entries from 0 to 1 or from 1 to 0 as possible such that A' has the *consecutive ones property* (*for rows*). This means that there is an ordering of the columns of A' such that the ones in each row appear consecutively. (For convenience we just say that *A' is C1P* in the following.)

In the *weighted consecutive ones problem* (*WC1P*) each entry has a specified cost for switching from a 0 to a 1 or vice versa. The task is then to find a C1P matrix A' such that the total cost for transforming A into A' is minimal. The WC1P occurs as a model in the physical mapping problem, a central problem in computational biology.

In this paper we study the complexity of WC1P if the number of rows or columns is fixed. The NP-hardness of the general problem is proven in [1].

2. The consecutive ones polytope

For a 0/1-matrix A with m rows and n columns let $\chi^A = (a_{11},\ldots,a_{1n},\ldots,a_{m1},\ldots,a_{mn})$ be its characteristic vector. We define the *consecutive ones polytope* as

$$P_{C1}^{m,n} = \mathrm{conv}\{\chi^A \mid A \text{ is an } (m,n)\text{-matrix with C1P}\}.$$

The WC1P can be solved by optimizing over a C1 polytope in the following way. Let c_{ij} be the cost for switching the entry a_{ij}. Then we construct the objective function by defining

$$c'_{ij} = \begin{cases} c_{ij} & \text{if } a_{ij}=0,\\ -c_{ij} & \text{if } a_{ij}=1. \end{cases}$$

Now the two sums $\sum_{i=1}^m \sum_{j=1}^n c_{ij}|a_{ij} - a'_{ij}|$ and $\sum_{i=1}^m \sum_{j=1}^n c'_{ij}a'_{ij}$ only differ by a constant.

* Corresponding author.
E-mail addresses: marcus.oswald@informatik.uni-heidelberg.de (M. Oswald), gerhard.reinelt@informatik.uni-heidelberg.de (G. Reinelt).

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Therefore, minimizing one sum also minimizes the other and vice versa. And since A' is C1P, minimizing the second sum leads to an optimization problem over $P_{\mathrm{C1}}^{m,n}$.

It is easy to see that $P_{\mathrm{C1}}^{m,n}$ has full dimension $m \cdot n$. Namely, the zero matrix is C1P and, for every $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, the matrix consisting of zeroes only except for a one in position ij is C1P. This gives a set of $n \cdot m + 1$ affinely independent C1P matrices.

Let $a^{\mathrm{T}}x \leqslant a_0$ be a valid inequality for $P_{\mathrm{C1}}^{m,n}$ and let $m' \geqslant m$ and $n' \geqslant n$. We say that the inequality $\bar{a}^{\mathrm{T}}x \leqslant a_0$ for $P_{\mathrm{C1}}^{m',n'}$ is obtained from $a^{\mathrm{T}}x \leqslant a_0$ by *trivial lifting* if

$$\bar{a}_{ij} = \begin{cases} a_{ij} & \text{if } i \leqslant m \text{ and } j \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1. *Let $a^{\mathrm{T}}x \leqslant a_0$ be a facet-defining inequality for $P_{\mathrm{C1}}^{m,n}$ and let $m' \geqslant m$ and $n' \geqslant n$. If $a^{\mathrm{T}}x \leqslant a_0$ is trivially lifted then the resulting inequality defines a facet of $P_{\mathrm{C1}}^{m',n'}$.*

Inequalities that are obviously valid for $P_{\mathrm{C1}}^{m,n}$ are the *trivial inequalities* $x_{ij} \geqslant 0$ and $x_{ij} \leqslant 1$, for all $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$. It is easily seen that they also define facets.

Theorem 2. *For all $m \geqslant 1$, $n \geqslant 1$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, the inequalities $x_{ij} \geqslant 0$ and $x_{ij} \leqslant 1$ define facets of $P_{\mathrm{C1}}^{m,n}$.*

Proofs of these theorems and an integer programming formulation of WC1P that consists only of inequalities which are facet-defining for $P_{\mathrm{C1}}^{m,n}$ are given in [3]. The consecutive ones polytope is not only interesting from a theoretical point of view, it also serves as a basis for branch-and-cut algorithms for solving the WC1P to proven optimality. Successful approaches of this type and their application to the physical mapping problem are presented in [2,3].

3. Feasible sets

In this paper we assume that either the number of rows or the number of columns is fixed.

We will first analyze how many structurally different C1P matrices exist if the number of rows is fixed. To this end, we introduce the concept of feasible sets.

Definition 1. A set $C = \{c_1, \ldots, c_k\}$ where $c_i \in \{0,1\}^{m\times 1}$, for $i = 1, \ldots, k$, of columns is called C1-feasible for columns (C1FC) if the matrix consisting of all columns of C is C1P.

Definition 2. A C1FC set C is called C1-maximal for columns (C1MC) if C is maximal with respect to set inclusion.

Example 1. Let $m = 3$. The set

$$C = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

is C1FC but not C1MC since one can add the column $(1\ 1\ 1)^{\mathrm{T}}$ and the resulting set

$$\bar{C} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

is still C1FC. But any further column would lead to a set whose corresponding matrix is not C1P. Therefore, $\bar{C}$ is C1MC.

Note that C1FC and C1MC sets do not contain duplicate columns. This contrasts with matrices that are C1P.

Lemma 1. *The cardinality of all C1MC sets for m rows is $2m$.*

Proof. Let $C = \{c_1, \ldots, c_k\}$ be C1MC and M a corresponding matrix such that the ones in every row of M occur consecutively and the rows are ordered lexicographically.

Then the first column of M is 0. Assume w.l.o.g. that the columns occur in order $c_1, \ldots, c_k$ in C. Let $|c_i - c_j|$ be the number of entries in which c_i and c_j differ. Since C is C1MC there is at least one 1-entry in every row of M. Otherwise one could add a further column as the first column which has a 1-entry in that row and 0-entries otherwise.

Because the 1-entries occur consecutively in all rows we have the equation

$$\sum_{i=1}^{k} |c_i - c_{i+1}| = 2m$$

(where c_{k+1} is identified with c_1). Since the c_i are pairwise different it follows that $\sum_{i=1}^{k} |c_i - c_{i+1}| \geqslant k$. On the other hand, if there would be a consecutive pair of columns satisfying $|c_i - c_{i+1}| > 1$, one could add an additional different column between these two contradicting the maximality of C. Therefore, we obtain that $\sum_{i=1}^{k} |c_i - c_{i+1}| = k$ which implies that $k = 2m$. □

As immediate consequence of this lemma we can give an upper bound on the number of C1-maximal sets.

Corollary 1. *The number of C1MC sets for m rows is at most* $\binom{2^m}{2m}$.

Proof. The result is clear since the number of all possible columns is 2^m and since the cardinality of all C1MC sets for m rows is $2m$. □

Analogously to the definition of feasible sets for columns we can introduce feasible sets for rows.

Definition 3. A set $R = \{r_1, \ldots, r_k\}$ where $r_i \in \{0,1\}^{1\times n}$, for $i = 1, \ldots, k$, of rows is called C1-feasible for rows (C1FR) if the matrix consisting of all rows of R is C1P.

Definition 4. A C1FR set R is called C1-maximal for rows (C1MR) if R is maximal with respect to set inclusion.

Example 2. Let $n = 3$. The set

$$R = \left\{ \begin{array}{c} (0 \quad 0 \quad 0) \\ (0 \quad 0 \quad 1) \\ (0 \quad 1 \quad 0) \\ (1 \quad 0 \quad 0) \\ (0 \quad 1 \quad 1) \\ (1 \quad 1 \quad 1) \end{array} \right\}$$

is C1FR but not C1MR since one can add the row (1 1 0) and the resulting set

$$\bar{R} = \left\{ \begin{array}{c} (0 \quad 0 \quad 0) \\ (0 \quad 0 \quad 1) \\ (0 \quad 1 \quad 0) \\ (1 \quad 0 \quad 0) \\ (0 \quad 1 \quad 1) \\ (1 \quad 1 \quad 0) \\ (1 \quad 1 \quad 1) \end{array} \right\}$$

is still C1FR. But any further row ((1 0 1) is the only one remaining) would lead to a set which cannot be brought into C1-structure. Therefore, $\bar{R}$ is C1MR. Analogously to the previous case C1FR and C1MR sets do not contain duplicate rows.

Lemma 2. *The cardinality of all C1MR sets for n columns is* $n(n+1)/2 + 1$. *The total number of different C1MR sets for n columns is* $n!/2$.

Proof. Let R be a C1MR set of rows with n columns. Further let π be a permutation of the columns that establishes C1P for each row. W.l.o.g. we assume that π is the identity. Then all rows of R must be of the form

$$(0 \quad \ldots \quad 0 \quad 1 \quad \ldots \quad 1 \quad 0 \quad \ldots \quad 0).$$

There is one such row with no 1-entry, n rows with one 1-entry, $n-1$ rows with two 1-entries and so on up to a single row with n 1-entries. Because of the maximality of R none of these rows must be missing. Therefore, by adding all these numbers we obtain that the total number of rows in R is equal to $n(n+1)/2+1$.

For two different column permutations π_1 and π_2 the corresponding C1MR sets are equal if and only if π_2 is the reverse ordering of π_1. This can be seen by viewing the subsets of the corresponding C1MR sets consisting of the $n-1$ rows containing exactly 2 ones. These rows correspond to pairs of neighboring elements of π_1 or π_2, respectively. These 2 sets of pairs are identical if and only if π_1 and π_2 define the same permutation or one defines the reverse of the other. As a consequence, the number of different C1MR sets is $n!/2$. □

Note that due to the definition of C1P, a matrix is C1P if and only if it consists of (an arbitrary number of) rows from a set R which is C1MR (columns from a set C which is C1MC, resp.). Further, for any matrix $M \in \{0,1\}^{m\times n}$ which is C1P there is at least one set R which is C1MR and contains all different rows of M as a subset and at least one set C which is C1MC and contains all different columns of M.

4. The number of facets of $P_{\mathrm{C1}}^{m,n}$

Since in the consecutive ones problem we actually deal with matrices we will denote a facet defining inequality for $P_{\mathrm{C1}}^{m,n}$ by $B \circ x \leqslant b_0$, where B is the coefficient matrix, x is the matrix of the variables and $B \circ x$ is the product

$$B \circ x = \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}x_{ij}.$$

Further, we will use $B_{.j}^{\mathrm{T}}x_{.j}$ instead of $\sum_{i=1}^{m} b_{ij}x_{ij}$.

Now using the observations of the previous section we are able to prove the surprising result that for a fixed number of rows (columns) the number of facets of $P_{\mathrm{C1}}^{m,n}$ grows only polynomially in the number of columns (rows). This contrasts with many combinatorial optimization problems which are polynomially solvable but where the corresponding polytope has an exponential number of facets.

Theorem 3. *The number of facets of $P_{\mathrm{C1}}^{m,n}$ for fixed m is* $\mathrm{O}(n^{\binom{2^m}{2m}})$.

Proof. Let m be the fixed number of rows.

Suppose $B \circ x \leqslant b_0$ is a non-trivial facet-defining inequality for $P_{\mathrm{C1}}^{m,n}$. Since the zero matrix and all matrices consisting of zeroes except for one entry are feasible solutions it follows that $b_0 > 0$.

Let l be the number of C1MC sets C consisting of columns $\in \{0,1\}^{m\times 1}$ with the property that there exists a matrix $M \in \{0,1\}^{m\times n}$ with $B\circ M = b_0$ and all columns of M are in C. According to Corollary 1 we have $l \leqslant \binom{2^m}{2m}$. Our goal is to show that the support of B has at most l columns.

For each of these chosen sets $C \in \{C_1,\dots,C_l\}$ and every column j of B we define

$$m_j^C(B) = \max\{B_{.j}^{\mathrm{T}}v \mid v \in C\}$$

as the maximum possible contribution of column j to the left-hand side of the facet.

Furthermore, for all C1P matrices M with $B\circ M = b_0$ and all C1MC sets C containing all columns of M the relation

$$B_{.j}^{\mathrm{T}}M_{.j} = m_j^C(B)$$

holds for every column j. The relation $B_{.j}^{\mathrm{T}}M_{.j} \leqslant m_j^C(B)$ is clear from the definition of $m_j^C(B)$. If we assume that $B_{.j}^{\mathrm{T}}M_{.j} < m_j^C(B)$ then we can construct a new C1P matrix M' by replacing the column j of M by the maximum column v from the above definition. But then $B \circ M' > b_0$ contradicting the validity of $B \circ x \leqslant b_0$.

Now let k be the number of non-zero columns of B. We create the $l \times k$-matrix

$$\mathscr{M}(B) = (m_{ij}(B)) = (m_j^{C_i}(B)).$$

Of course, the rank of $\mathscr{M}(B)$ is at most l independently of the number k of columns.

Now assume that a facet-defining inequality $B \circ x \leqslant b_0$ is given with the number of non-zero columns k of B greater than l. Because of rank $\mathscr{M}(B) \leqslant l$ at least one column j of $\mathscr{M}(B)$ can be written as a linear combination $\mathscr{M}(B)_{.j} = \sum_{j'\neq j} d_{j'}\mathscr{M}(B)_{.j'}$. And since $B_{.j}^{\mathrm{T}}M_{.j} = m_j^{C_i}(B)$ holds for any C1P matrix M with $B\circ M = b_0$ and a suitable C1MC set C_i containing the different columns of M we have

$$\begin{aligned} B_{.j}^{\mathrm{T}}M_{.j} = m_j^{C_i}(B) &= \sum_{j'\neq j} \mathrm{d}_{j'} m_{j'}^{C_i}(B) \\ &= \sum_{j'\neq j} \mathrm{d}_{j'} B_{.j'}^{\mathrm{T}} M_{.j'}. \end{aligned}$$

This equation holds for every vertex M of $\{x \in P_{\mathrm{C1}}^{m,n} \mid B\circ x = b_0\}$. And since it contains no constant coefficient it cannot be obtained by scaling the equation $B \circ x = b_0$ with $b_0 > 0$. Therefore, the inequality $B \circ x \leqslant b_0$ cannot be facet-defining.

Thus the support of every facet-defining inequality for $P_{\mathrm{C1}}^{m,n}$ has at most $l \leqslant \binom{2^m}{2m}$ columns and therefore each of these facets can be obtained by trivial lifting from a facet of $P_{\mathrm{C1}}^{m,\binom{2^m}{2m}}$. Since the number of facets of $P_{\mathrm{C1}}^{m,\binom{2^m}{2m}}$ is constant in n and the number of lifting possibilities for one facet is at most $\binom{n}{\binom{2^m}{2m}}$ the total number of facet-defining inequalities for $P_{\mathrm{C1}}^{m,n}$ is $\mathrm{O}(n^{\binom{2^m}{2m}})$. □

Theorem 4. *The number of facets of $P_{\mathrm{C1}}^{m,n}$ for fixed n is* $\mathrm{O}(m^{n!/2})$.

Proof. The proof follows along the same lines as the proof of the previous theorem. Since for fixed n the number of C1MR sets R is equal to $n!/2$, the total number of facet-defining inequalities for $P_{\mathrm{C1}}^{m,n}$ for fixed n is of order $\mathrm{O}(m^{n!/2})$. □

5. Polynomial solvability

As a consequence of the results on the number of facets we obtain that WC1P is solvable in polynomial time for fixed n or fixed m.

Corollary 2. *WC1P is solvable in polynomial time for fixed n or fixed m.*

Proof. Consider the case that the number n of columns is fixed.

According to the discussion above all facets of $P_{\mathrm{C1}}^{m,n}$ can be obtained by trivial lifting from facets of $P_{\mathrm{C1}}^{n!/2,n}$. Computing all of these facets takes time constant in m. And for each of these facets there are at most $\binom{m}{n!/2}$ possibilities for trivial lifting. Thus, we need time $\mathrm{O}(m^{n!/2})$ to create a complete listing of all facets of $P_{\mathrm{C1}}^{m,n}$.

Since the number of facets as well as their encoding length is polynomial in m, we can list them all explicitly in polynomial time and use a polynomial algorithm for linear programming to minimize the objective function over the exact linear description of $P_{\mathrm{C1}}^{m,n}$.

An analogous argumentation applies to the case where m is fixed. □

Linear programming provides one means of solving the W1CP for fixed m or n in polynomial time. However, looking more closely into the combinatorial structure of the problem, we can even derive a linear time algorithm.

Theorem 5. *WC1P is solvable in linear time for fixed n or fixed m.*

Proof. We only consider the case that m is fixed as the discussion for fixed n is similar.

Let the WC1P be formulated as

$$\max\{B \circ x \mid x \in \{0,1\}^{m\times n} \text{ is C1P}\}.$$

Now for each C1MC set C and for each column c of B we compute $m_c^C(B)$. One computation takes time $\mathrm{O}(2m^2)$. Therefore, all of these calculations take time $\mathrm{O}(2m^2\binom{2^m}{2m}n)$. Since,

$$\max\{B \circ x \mid x \in \{0,1\}^{m\times n} \text{ is C1P}\}$$
$$= \max\left\{\sum_j m_j^C(B) \mid C \text{ is a C1MC set}\right\}$$

holds, we are done. The total running time of this algorithm is $\mathrm{O}(2m^2\binom{2^m}{2m}n)$ and thus linear in n. □

When fixing the number n of columns, we obtain a running time $\mathrm{O}(n^3 n! m)$ which can be improved to $\mathrm{O}(n! nm)$ by making use of a scan line algorithm.

Example 3. We illustrate our algorithm by an example.

Consider the problem

$$\max\{B \circ x \mid x \in \{0,1\}^{3\times 8} \text{ is C1P}\},$$

where

$$B = \begin{pmatrix} 1 & -2 & 2 & 0 & 1 & -1 & 3 & -2 \\ -1 & 1 & 0 & 1 & 2 & 1 & -1 & -3 \\ 2 & 1 & 1 & -2 & -1 & 3 & 2 & 1 \end{pmatrix}.$$

There are 12 different C1MC sets for three rows, namely

$$C_1 = \left\{\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right\},$$

$$C_2 = \left\{\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\},$$

$$C_3 = \left\{\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\},$$

$$C_4 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\},$$

$$C_5 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\},$$

$$C_6 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\},$$

$$C_7 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\},$$

$$C_8 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\},$$

$$C_9 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\},$$

$$C_{10} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\},$$

$$C_{11} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\},$$

$$C_{12} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Having computed $m_c^C(B)$ for each of these sets and each column of B we obtain the following matrix $\mathscr{M}(B)$:

$$\mathscr{M}(B) = \begin{pmatrix} 2 & 2 & 2 & 1 & 3 & 4 & 3 & 1 \\ 3 & 1 & 3 & 1 & 3 & 3 & 5 & 1 \\ 3 & 2 & 3 & 1 & 2 & 4 & 5 & 1 \\ 2 & 2 & 3 & 1 & 3 & 4 & 4 & 0 \\ 2 & 2 & 3 & 1 & 3 & 4 & 4 & 1 \\ 2 & 2 & 3 & 1 & 3 & 4 & 4 & 1 \\ 3 & 1 & 3 & 1 & 3 & 3 & 5 & 0 \\ 3 & 1 & 3 & 1 & 3 & 3 & 5 & 1 \\ 3 & 1 & 3 & 1 & 3 & 3 & 5 & 1 \\ 3 & 2 & 3 & 1 & 2 & 4 & 5 & 1 \\ 3 & 2 & 3 & 1 & 2 & 4 & 5 & 0 \\ 3 & 2 & 3 & 0 & 2 & 4 & 5 & 1 \end{pmatrix}.$$

The highest value of the sum of the entries in one row is 21 choosing C_3 or C_{10} as C1MC set. Therefore, 21 is also the optimum value of the WC1P. Furthermore, from set C_3 or C_{10} one can also construct a corresponding optimum solution

$$x^* = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

6. Concluding remarks

While the number of C1MR sets with n columns is exactly $n!/2$ the upper bound $\binom{2^m}{2m}$ on the number of C1MC sets with m rows is fairly weak. We have computed the exact numbers up to $m = 5$ and got one set for $m = 2$, 12 sets for $m = 3$, 324 sets for $m = 4$ and 14 880 different sets for $m = 5$. The huge gap between 14 880 and $\binom{32}{10} = 64\,512\,240$ suggests that there is much potential for improving this upper bound and with it for improving the upper bound for the number of facets of $P_{\mathrm{C1}}^{m,n}$ for fixed m.

The concept of maximal feasible sets of columns and rows can also be applied to the *weighted simultaneous consecutive ones problem* (*WSC1P*) where the feasible solutions are 0/1-matrices which satisfy the consecutive ones property for rows as well as for columns. We, therefore, obtain that also WSC1P can be solved in linear time for a fixed number of columns or rows.

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References

[1] K.S. Booth, PQ-tree algorithms, Ph.D. Thesis, University of California, Berkeley, 1975.

[2] T. Christof, M. Oswald, G. Reinelt, Consecutive ones and a betweenness problem in computational biology, in: Proceedings of the Sixth IPCO Conference, Houston, 1998, pp. 213–228.

[3] M. Oswald, G. Reinelt, Polyhedral aspects of the consecutive ones problem, in: Proceedings of the Fifth Conference on Computing and Combinatorics, Sydney, 2000, pp. 373–382.