Reasoning About Temporal Relations: A Maximal Tractable Subclass of Allen's Interval Algebra

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Abstract. We introduce a new subclass of Allen's interval algebra we call "ORD-Horn subclass," which is a strict superset of the "pointisable subclass." We prove that reasoning in the ORD-Horn subclass is a polynomial-time problem and show that the path-consistency method is sufficient for deciding satisfiability. Further, using an extensive machine-generated case analysis, we show that the ORD-Horn subclass is a maximal tractable subclass of the full algebra (assuming P $\neq$ NP). In fact, it is the unique greatest tractable subclass amongst the subclasses that contain all basic relations.

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1. *Introduction*

Temporal information is often conveyed qualitatively by specifying the relative positions of time intervals such as "...point to the figure while explaining the performance of the system..." Further, for natural language understanding

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[Allen 1984; Song and Cohen 1988], general planning [Allen 1991; Allen and Koomen 1983], presentation planning in a multi-media context [André et al. 1993; Feiner et al. 1993], diagnosis of technical systems [Nökel 1991], and knowledge representation [Koubarakis et al. 1987; Weida and Litman 1992], the representation of qualitative temporal relations and reasoning about them is essential. Allen [1983] introduces an algebra of binary relations on intervals (hereafter referred to as *Allen's interval algebra*) for representing qualitative temporal information and addresses the problem of reasoning about such information. In particular, he gives an algorithm for computing an approximation to the *strongest implied relation* for each pair of intervals, which is a simplified version of the *path-consistency algorithm* [Mackworth 1977].

As already noted by Allen [1983], the path-consistency method is in general not sufficient for computing the strongest implied relation for each pair of intervals. Since this problem is NP-hard in the full algebra [Vilain et al. 1989], it is very unlikely that other polynomial-time algorithms will be found that solve this problem in general. Subsequent research has concentrated on designing more efficient reasoning algorithms, on identifying tractable special cases, and on isolating sources of computational complexity.[1] However, it is by no means clear whether the tractable cases that have been identified are maximal and whether the sources of computational complexity found are the only ones.

We extend these previous results in three ways. First, we present a new tractable subclass of Allen's interval algebra, which we call *ORD-Horn subclass* for reasons that will become obvious below. This subclass is considerably larger than all other known tractable subclasses (it contains 10% of the full algebra) and strictly contains the *pointisable subclass* [Ladkin and Maddux 1988; van Beek 1990]. Second, we show that path consistency is sufficient for deciding satisfiability in this subclass. Third, using an extensive machine-generated case analysis, we show that this subclass is a maximal subclass such that satisfiability is tractable (under the assumption that P $\neq$ NP). We finally strengthen this result by showing that the ORD-Horn subclass is in fact the *unique greatest* tractable subclass that contains all the basic relations.

From a practical point of view, these results imply that the path-consistency method has a much larger range of applicability than previously believed, provided we are mainly interested in satisfiability. Further, our results can be used to design backtracking algorithms for the full algebra that are more efficient than those based on other tractable subclasses.

Some words on methodology may be in order at this point. While proving tractability and the applicability of the path-consistency method is a (more or less) straightforward task, showing *maximality* of a subclass with respect to the stated properties requires an extensive case analysis involving a couple of thousand cases, which can only be done by a computer. This case analysis leads to two interesting cases, for which NP-completeness proofs are provided. However, the case analysis itself cannot be reproduced in a research paper or verified manually, either. In order to allow for the verification of our results,

[1] For example, see Freksa [1992], Gerevini and Schubert [1993a, 1993b], Ghallab and Mounir Alaoui [1989], Golumbic and Shamir [1992, 1993], Ladkin and Maddux [1988, 1994], Nökel [1989, 1991], Valdéz-Pérez [1987], van Beek [1989, 1990], van Beek and Cohen [1990], Vilain and Kautz [1986], and Vilain et al. [1989].

we therefore include the abstract form of the programs we used to perform the machine-assisted case analysis.[2]

The paper is structured as follows. Section 2 contains terminology and definitions used in the remainder of the paper. Section 3 introduces the ORD-Horn subclass, which is shown to be tractable. Based on this result, we show in Section 4 that the path-consistency method is sufficient for deciding satisfiability in this subclass. In Section 5, we derive some results on the computational properties of subalgebras. Using these results and an extensive machine-generated case analysis, we show in Section 6 that the ORD-Horn subclass is a maximal tractable subclass of the full algebra and the unique greatest tractable subclass that contains all basic relations.

2. *Reasoning about Interval Relations using Allen's Interval Algebra*

Allen's [1983] approach to reasoning about time is based on the notion of *time intervals* and *binary relations* on them. A *time interval* X is an ordered pair (X^-, X^+) such that $X^- < X^+$, where X^- and X^+ are interpreted as points on the real line.[3] So, if we talk about *interval interpretations* or *I-interpretations* in the following, we mean mappings of time intervals to pairs of distinct real numbers such that the beginning of an interval is strictly before the end of the interval.

Given two interpreted time intervals, their relative positions can be described by exactly one of the elements of the set $\mathbf{B}$ of thirteen *basic interval relations* (denoted by B in the following), where each basic relation can be defined in terms of its *endpoint relations* (see Table I). An atomic formula of the form $X B Y$, where X and Y are intervals and $B \in \mathbf{B}$, is said to be *satisfied* by an interpretation iff the interpretation of the intervals satisfies the endpoint relations specified in Table I.

In order to express indefinite information, unions of the basic interval relations are used, which are written as sets of basic relations leading to 2^{13} binary *interval relations* (denoted by R, S, T)—including the *null relation* $\emptyset$ (also denoted by $\perp$) and the *universal relation* $\mathbf{B}$ (also denoted by $\top$). The set of all binary interval relations $2^{\mathbf{B}}$ is denoted by $\mathscr{A}$.

An atomic formula of the form $X\{B_1, \ldots, B_n\}Y$ (denoted by ϕ) is called *interval formula*. Such a formula is satisfied by an I-interpretation $\mathfrak{I}$ iff $X B_i Y$ is satisfied by $\mathfrak{I}$ for some i, $1 \le i \le n$. Finite sets of interval formulas are denoted by Θ. Such a set Θ is called *I-satisfiable* iff there exists an I-interpretation $\mathfrak{I}$ that satisfies every formula of Θ. Further, such a satisfying I-interpretation $\mathfrak{I}$ is called *I-model* of Θ. If an interval formula ϕ is satisfied by every I-model of a set of interval formulas Θ, we say that ϕ is *logically implied* by Θ, written $\Theta \models_I \phi$.

Fundamental *reasoning problems* in this framework include Golumbic and Shamir [1992; 1993], Ladkin and Maddux [1988], van Beek [1992], and Vilain

[2]The programs we used and an enumeration of the ORD-Horn subclass can be obtained from the authors or by anonymous ftp from `duck.dfki.uni-sb.de` as `/pub/papers/RR-93-11.programs.tar.Z`.

[3]Other underlying models of the time line are also possible, for example, the rationals [Allen and Hayes 1985; Ladkin 1987]. For our purposes these distinctions are not significant, however.

TABLE I. THE SET **B** OF THE THIRTEEN BASIC RELATIONS. THE ENDPOINT RELATIONS $X^- < X^+$ AND $Y^- < Y^+$ THAT ARE VALID FOR ALL RELATIONS HAVE BEEN OMITTED.

Basic Interval Relation	Symbol	Pictorial Example	Endpoint Relations
X before Y	$\prec$	xxx	$X^- < Y^-$, $X^- < Y^+$,
Y after X	$\succ$	yyy	$X^+ < Y^-$, $X^+ < Y^+$
X meets Y	m	xxxx	$X^- < Y^-$, $X^- < Y^+$,
Y met-by X	$\mathsf{m}^\smile$	yyyy	$X^+ = Y^-$, $X^+ < Y^+$
X overlaps Y	o	xxxx	$X^- < Y^-$, $X^- < Y^+$,
Y overlapped-by X	$\mathsf{o}^\smile$	yyyy	$X^+ > Y^-$, $X^+ < Y^+$
X during Y	d	xxx	$X^- > Y^-$, $X^- < Y^+$,
Y includes X	$\mathsf{d}^\smile$	yyyyyyy	$X^+ > Y^-$, $X^+ < Y^+$
X starts Y	s	xxx	$X^- = Y^-$, $X^- < Y^+$,
Y started-by X	$\mathsf{s}^\smile$	yyyyyyy	$X^+ > Y^-$, $X^+ < Y^+$
X finishes Y	f	xxx	$X^- > Y^-$, $X^- < Y^+$,
Y finished-by X	$\mathsf{f}^\smile$	yyyyyyy	$X^+ > Y^-$, $X^+ = Y^+$
X equals Y	$\equiv$	xxxx yyyy	$X^- = Y^-$, $X^- < Y^+$, $X^+ > Y^-$, $X^+ = Y^+$

and Kautz [1986]: Given a set of interval formulas Θ,

(1) decide whether there exists an I-model of Θ (ISAT),
(2) determine for each pair of intervals X, Y the *strongest implied relation* between them (ISI), that is, the smallest set R such that $\Theta \models_I X\,R\,Y$.[4]

In the following, we often consider *restricted reasoning problems* where the relations used in interval formulas in Θ are only from a subclass $\mathscr{S}$ of all interval relations. In this case we say that Θ *is a set of formulas over* $\mathscr{S}$, and we use a parameter in the problem description to denote the subclass considered, for example, ISAT($\mathscr{S}$). As is well-known, ISAT and ISI are equivalent with respect to polynomial Turing-reductions [Vilain and Kautz 1986] and the same holds for other reasoning tasks of interest [Golumbic and Shamir 1992, 1993]. Further, the equivalence also extends to the restricted problems ISAT($\mathscr{S}$) and ISI($\mathscr{S}$) provided $\mathscr{S}$ contains all basic relations.

PROPOSITION 2.1. *ISAT($\mathscr{S}$) and ISI($\mathscr{S}$) are equivalent under polynomial Turing-reductions, provided $\mathscr{S}$ contains all basic relations.*

PROOF. A solution to ISI($\mathscr{S}$) clearly gives an answer to the ISAT($\mathscr{S}$) decision problem. For the converse direction, one can use an oracle for ISAT($\mathscr{S}$) to check for each pair of intervals X, Y whether $\Theta \cup (X\{B_i\}Y)$ is satisfiable for each $B_i \in \mathbf{B}$. The set of basic relations for which the test succeeds constitutes the strongest implied relation between X and Y. Hence, ISI($\mathscr{S}$) can be solved using a number of calls to the ISAT($\mathscr{S}$) oracle that is polynomial in $|\Theta|$. □

[4]This problems has also been called *deductive closure* problem by Vilain and Kautz [1986] and *minimal labeling* problem (MLP) by van Beek [1989] since it corresponds to finding the minimal network in a general constraint satisfaction problem.

The most prominent method to solve these problems (approximately for all interval relations or exactly for subclasses) is *constraint propagation*[5] using a slightly simplified form of the *path-consistency algorithm* [Mackworth 1977; Montanari 1974]. In the following, we briefly characterize this method without going into details, though. In order to do so, we first have to introduce Allen's interval algebra.

Allen's interval algebra [Allen 1983] consists of the set $\mathscr{A} = 2^{\mathbf{B}}$ of all binary interval relations and the operations unary *converse* (denoted by $\cdot^{\smile}$), binary *intersection* (denoted by $\cap$), and binary *composition* (denoted by $\circ$), which are defined as follows:[6]

$$\forall X, Y\colon XR^{\smile}Y \leftrightarrow YRX$$
$$\forall X, Y\colon X(R \cap S)Y \leftrightarrow XRY \wedge XSY$$
$$\forall X, Y\colon X(R \circ S)Y \leftrightarrow \exists Z\colon (XRZ \wedge ZSY).$$

It follows that the converse of $R = \{B_1, \ldots, B_n\}$ can be expressed by the set of basic relations $R^{\smile} = \{B_1^{\smile}, \ldots, B_n^{\smile}\}$. Further, the intersection of two relations $(R \cap S)$ can be expressed as the set-theoretic intersection of the sets of basic relations that are used to describe the interval relations, that is, $(R \cap S) = \{B \in \mathbf{B} \mid B \in R, B \in S\}$. The composition of two relations cannot be specified straightforwardly, however. Using the definition of composition, it can be derived that

$$R \circ S = \bigcup\{B \circ B' \mid B \in R, B' \in S\},$$

that is, composition is the union of the component-wise composition of basic relations. The results of composing basic relations must in turn be derived from the definition of the basic relations in terms of their endpoint relations.[7] Using Allen's interval algebra, we specify an abstract form of the constraint propagation algorithm that has been proposed for reasoning in this framework.

Assume an operator Γ that maps finite sets of interval formulas to finite sets of interval formulas in the following way:

$$\begin{aligned}\Gamma(\Theta) = \Theta & \\ & \cup\{X \top Y \mid X, Y \text{ appear in } \Theta\} \\ & \cup\{XRY \mid (YR^{\smile}X) \in \Theta\} \\ & \cup\{X(R \cap S)Y \mid (XRY), (XSY) \in \Theta\} \\ & \cup\{X(R \circ S)Y \mid (XRZ), (ZSY) \in \Theta\}.\end{aligned}$$

Since there are only finitely many different interval formulas for a finite set of intervals and Γ is monotone, it follows that for each Θ there exists a natural number n such that $\Gamma^n(\Theta) = \Gamma^{n+1}(\Theta)$. $\Gamma^n(\Theta)$ is called the *closure* of Θ, written $\overline{\Theta}$.

Considering the formulas of the form $(XR_iY) \in \overline{\Theta}$ for given X, Y, it is evident that the R_i's are closed under intersection, and hence there exists $(XSY) \in \overline{\Theta}$ such that S is the *strongest relation* amongst the R_i's, that is,

[5]For example, see Allen [1983], Ladkin and Maddux [1988], Nökel [1991], van Beek [1989], van Beek and Cohen [1990], and Vilain and Kautz [1986].
[6]Note that we obtain a relation algebra if we add *complement* and *union* as operations [Ladkin and Maddux 1988, 1994]. For our purposes, this is irrelevant, however.
[7]Allen [1983] gives a composition table for the basic relations.

$S \subseteq R_i$, for every i. The subset of a closure $\overline{\Theta}$ containing for each pair of intervals only the strongest relations is called the *reduced closure* of Θ and is denoted by $\hat{\Theta}$.

As can easily be shown, every reduced closure of a set Θ is *path consistent* [Mackworth 1977] (or 3-*consistent* [Freuder 1978]), which means that for every three intervals X, Y, Z and for every interpretation $\Im$ that satisfies $(X R Y) \in \hat{\Theta}$, there exists an interpretation $\Im'$ that agrees with $\Im$ on X and Y and in addition satisfies $(X S Z)$, $(Z S' Y) \in \hat{\Theta}$. In other words, for a given triangle of intervals, regardless of how we choose an interpretation for two intervals that satisfies the relation between them, it is still possible to choose an interpretation for the third interval such that the remaining relations are also satisfied.

Under the assumption that $(X R Y) \in \Theta$ implies $(Y R^{\smile} X) \in \Theta$, it is also easy to show that path consistency of Θ implies that $\Theta = \hat{\Theta}$. For this reason, we use the term *path-consistent set* as a synonym for a set that is the reduced closure of itself.

The reduced closure is a path-consistent set that is logically equivalent to the original one, that is, $\Theta \models_I \hat{\Theta}$ and $\hat{\Theta} \models_I \Theta$. Computing $\hat{\Theta}$ is polynomial in the size of Θ. More precisely, let us assume that Θ is a set of interval formulas over n distinct intervals such that $|\Theta| \leq 13 \times n \times (n - 1)$. This assumption is quite reasonable since supposing that for a given pair X, Y there are $c > 13$ different formulas $X R_i Y$ leads to the conclusion that at least $c - 13$ of these are redundant, which can be determined in linear time. For this reason, we assume here and in the following that $|\Theta| \in O(n^2)$, and we specify the *asymptotic runtime behavior* of an algorithm in the number of distinct intervals n. Under these assumptions, an algorithm can be specified that computes the reduced closure of a set of interval formulas in $O(n^3)$ time [Mackworth and Freuder 1985; Montanari 1994].

It should be noted that the path-consistency method provides only an *approximation* to ISI. This means that the relations in a path-consistent set contain the strongest implied relations, but the converse does not hold in general. Similarly for ISAT, the presence of an assertion $X \perp Y$ in a path-consistent set implies that the set is not satisfiable, but the converse does not hold in general. An example of a path-consistent set of interval formulas that is unsatisfiable but does not contain $X \perp Y$ is given by Allen [1983].

3. *The ORD-Horn Subclass*

Previous results on the tractability of ISAT($\mathscr{S}$) (and hence ISI($\mathscr{S}$)) for some subclass $\mathscr{S} \subseteq \mathscr{A}$ made use of the *expressibility* of interval formulas over $\mathscr{S}$ as certain logical formulas involving endpoint relations.

As usual, by a *clause* we mean a disjunction of literals, where a *literal* in turn is an atomic formula or a negated atomic formula. As *atomic formulas*, we allow $a \leq b$ and $a = b$, where a and b denote endpoints of intervals. The negation of $a = b$ is written as $a \neq b$ and the negation of $a \leq b$ as $a \nleq b$. Finite sets of such clauses will be denoted by Ω.

Similarly to the notions of I-interpretation, I-model, and I-satisfiability, we define an *R-interpretation* to be an interpretation that interprets all endpoints in a set of clauses Ω as real numbers, an *R-model* of Ω to be an R-interpretation that satisfies Ω, and *R-satisfiability* of Ω to be the satisfiability of Ω over R-interpretations. If the clause C is logically implied by Ω interpreted over R-interpretations, we write $\Omega \models_R C$.

The *clause form* of an interval formula ϕ is the set of clauses over endpoint relations that is equivalent to ϕ, that is, every I-model of ϕ can be transformed into a R-model of the clause form and vice versa using the obvious transformation. Clearly, it is possible to translate any interval formula into its equivalent clause form.[8]

In the following, we consider a slightly restricted form of clauses, which we call *ORD clauses*. These clauses do not contain negations of atoms of the form $a \leq b$, that is, they only contain literals of the form:

$$a = b, \qquad a \leq b, \qquad a \neq b.$$

The *ORD-clause form* of an interval formula ϕ, written $\pi(\phi)$, is the clause form of ϕ containing only ORD clauses. This restriction does not affect the existence of the clause form because any clause of the form $(a \not\leq b) \vee C$ can be equivalently expressed by the two clauses $a \neq b \vee C$ and $b \leq a \vee C$.

The function $\pi(\cdot)$ is extended to finite sets of interval formulas in the obvious way, that is, for identical intervals in Θ, identical endpoints are used in $\pi(\Theta)$. This implies that any I-model of Θ can be transformed into an R-model of $\pi(\Theta)$ and vice versa.

PROPOSITION 3.1. *Θ is I-satisfiable iff $\pi(\Theta)$ is R-satisfiable.*

While it is obvious that all interval formulas can be translated into their equivalent ORD-clause form, it is not clear that such a translation is worthwhile. However, interestingly, some relations have a very concise ORD-clause form. Consider, for instance, $\pi(X\{\mathsf{d}, \mathsf{o}, \mathsf{s}\}Y)$:[9]

$$\begin{aligned}\{&(X^- \leq X^+), (X^- \neq X^+),\\ &(Y^- \leq Y^+), (Y^- \neq Y^+),\\ &(X^- \leq Y^+), (X^- \neq Y^+),\\ &(Y^- \leq X^+), (X^+ \neq Y^-),\\ &(X^+ \leq Y^+), (X^+ \neq Y^+)\}.\end{aligned}$$

Not all relations permit a translation that leads to a clause form that is as dense as the one shown above, which contains only *unit clauses*, that is, clauses consisting of only one literal. However, in particular those relations that allow for such a clause form have interesting computational properties. For instance, the *continuous endpoint subclass* (which is denoted by $\mathscr{C}$) can be defined as the subclass of interval relations that

(1) permit a clause form that contains only unit clauses, and
(2) for each unit clause $a \neq b$, the clause form contains also a unit clause of the form $a \leq b$ or $b \leq a$.

As demonstrated above, the relation $\{\mathsf{d}, \mathsf{o}, \mathsf{s}\}$ is a member of the continuous endpoint subclass. This subclass has the favorable property that the path-consistency method solves ISI($\mathscr{C}$) [van Beek 1989; van Beek and Cohen 1990; Vilain et al. 1989].

A slight generalization of the continuous endpoint subclass is the *pointisable subclass* (denoted by $\mathscr{P}$) that is defined in the same way as $\mathscr{C}$, but without

[8] It should be noted that such a translation is not unique.
[9] Note that the fifth and sixth clause are redundant.

condition (2). The relation {d, o} is, for instance, an element of $\mathscr{P} - \mathscr{C}$ because the clause form of $(X\{\mathsf{d}, \mathsf{o}\}Y)$ contains $(X^- \neq Y^-)$ in addition to the clauses of $\pi(X\{\mathsf{d}, \mathsf{o}, \mathsf{s}\}Y)$.

It was claimed that the path-consistency method is also complete for ISI($\mathscr{P}$) [Vilain and Kautz 1986]. However, van Beek [1990] gives a counter-example showing that this claim is wrong. Nevertheless, the path-consistency method is still sufficient for deciding satisfiability [Ladkin and Maddux 1988; Vilain and Kautz 1986]. Using the fact that the path-consistency method needs $O(n^3)$ time and employing the reduction used in the proof of Proposition 2.1, it follows that ISI($\mathscr{P}$) can be solved in $O(n^5)$ time, where n is the number of distinct intervals. It is possible to do better than that, however. Van Beek [1989, 1990] and van Beek and Cohen [1990] give algorithms for solving ISI($\mathscr{P}$) in $O(n^4)$ time and specify an algorithm for deciding ISAT($\mathscr{P}$) in $O(n^2)$ time [van Beek 1990].

We generalize this approach by being more liberal concerning the clause form. We consider the subclass of Allen's interval algebra such that the relations permit an ORD-clause form containing only clauses with *at most one positive literal*, that is, a literal of the form $a = b$ or $a \leq b$, and an arbitrary number of negative literals, that is, literals of the form $a \neq b$. We call such clauses *ORD-Horn clauses* since clauses containing at most one positive literal are called *Horn clauses*. The subclass defined in this way is called *ORD-Horn subclass*, and we use the symbol $\mathscr{H}$ to refer to it. The relation $\{\mathsf{o}, \mathsf{s}, \mathsf{f}^\smile\}$ is, for instance, an element of $\mathscr{H}$, because $\pi(X\{\mathsf{o}, \mathsf{s}, \mathsf{f}^\smile\}Y)$ can be expressed as follows:

$$
\begin{aligned}
\{&(X^- \leq X^+), (X^- \neq X^+),\\
&(Y^- \leq Y^+), (Y^- \neq Y^+),\\
&(X^- \leq Y^-),\\
&(X^- \leq Y^+), (X^- \neq Y^+),\\
&(Y^- \leq X^+), (X^+ \neq Y^-),\\
&(X^+ \leq Y^+), (X^- \neq Y^- \vee X^+ \neq Y^+)\}.
\end{aligned}
$$

By definition, the ORD-Horn subclass contains the pointisable subclass. Further, by the above example, this inclusion is strict.

Consider now the theory *ORD* that axiomatizes "$=$" as an equivalence relation and "$\leq$" as a partial ordering over the equivalence classes:

$$
\begin{array}{lll}
\forall x, y, z: x \leq y \wedge y \leq z & \rightarrow x \leq z & \text{(Transitivity)}\\
\forall x: x \leq x & & \text{(Reflexivity)}\\
\forall x, y: x \leq y \wedge y \leq x & \rightarrow x = y & \text{(Antisymmetry)}\\
\forall x, y: x = y & \rightarrow x \leq y & \\
\forall x, y: x = y & \rightarrow y \leq x. &
\end{array}
$$

Although this theory is much weaker, and hence allows for more models than the intended models of sets of ORD clauses, R-satisfiability of a finite set Ω of ORD clauses is nevertheless equivalent to the satisfiability of $\Omega \cup ORD$ over arbitrary interpretations.

PROPOSITION 3.2. *A finite set of ORD clauses Ω is R-satisfiable iff $\Omega \cup ORD$ is satisfiable.*

PROOF. If Ω has an R-model, then clearly the axioms of *ORD* are also satisfied by this model. Conversely, let $\mathfrak{I}$ be an arbitrary model of $ORD \cup \Omega$. Since transitivity, reflexivity, symmetry, and substitutivity of $=$ follow from the axioms, $=$ is a congruence relation and $\mathfrak{I}/_=$ (i.e., the quotient of $\mathfrak{I}$ modulo $=$) is also a model of Ω. Further, since $\mathfrak{I}/_=$ satisfies *ORD*, it is a set partially ordered by $\leq$. Finally, every partially ordered set can be extended to a linearly ordered set, which in turn can be embedded in the reals. Since in every such linear extension of a partial ordering all formulas of the form $(a = b)$, $(a \neq b)$, and $(a \leq b)$ from Ω are still satisfied, $\mathfrak{I}$ can be transformed into an R-model of Ω. $\square$

It should be noted that the proposition only holds if all clauses in Ω are ORD clauses. Consider, for instance, $\Omega = \{(a \nleq b), (b \nleq a)\}$. This clause set is R-unsatisfiable, but there exists a model of $ORD \cup \Omega$ with a and b interpreted as incomparable elements.

Note that *ORD* is a *Horn theory*, that is, a theory containing only Horn clauses. Since the ORD-clause form of interval formulas over $\mathscr{H}$ is also Horn, tractability of ISAT($\mathscr{H}$) would follow, provided we could replace *ORD* by a propositional Horn theory. In order to decide satisfiability of a set of ORD clauses Ω in *ORD*, however, we can restrict ourselves to Herbrand interpretations, that is, interpretations that have only the endpoints of all intervals mentioned in Ω as objects. In the following, ORD_Ω shall denote the axioms of *ORD* instantiated to all endpoints mentioned in Ω. As a specialization of the Herbrand theorem, we obtain the next proposition.

PROPOSITION 3.3. *$\Omega \cup ORD$ is satisfiable iff $\Omega \cup ORD_\Omega$ is satisfiable.*

From that, polynomiality of ISAT($\mathscr{H}$) is immediate.

THEOREM 3.4. *ISAT($\mathscr{H}$) is polynomial.*

PROOF. For any set Θ over $\mathscr{H}$, a set of propositional Horn clauses $\pi(\Theta)$ can be generated in time linear in Θ. Further, $ORD_{\pi(\Theta)}$, which is a set of propositional Horn clauses, can be computed in time polynomial in Θ. Since satisfiability of a set of propositional Horn clauses can be decided in polynomial time, and since, by Propositions 3.1–3.3, it suffices to decide the satisfiability of $\pi(\Theta) \cup ORD_{\pi(\Theta)}$ in order to decide I-satisfiability of Θ, the claim follows. $\square$

Based on this result and the fact that the best known satisfiability algorithm for propositional Horn theories is linear [Dowling and Gallier 1984], it is possible to give an upper bound for deciding ISAT($\mathscr{H}$). Given a set of interval formula Θ with n distinct intervals, we assume as usual that $|\Theta| \in O(n^2)$.

THEOREM 3.5. *ISAT($\mathscr{H}$) can be decided in $O(n^3)$ time.*

PROOF. Based on the assumption that $|\Theta| \in O(n^2)$, $\pi(\Theta)$ is of size $O(n^2)$ and can be computed in time $O(n^2)$. Similarly, $ORD_{\pi(\Theta)}$ is of size $O(n^3)$ and can be generated in $O(n^3)$ time. Finally, since satisfiability of propositional Horn theories can be decided in linear time, the claim follows. $\square$

Using the reduction employed in the proof of Proposition 2.1, an upper bound for ISI($\mathscr{H}$) follows straightforwardly.

COROLLARY 3.6. *ISI($\mathscr{H}$) can be solved in $O(n^5)$ time.*

4. *The Applicability of Path-Consistency*

Enumerating the ORD-Horn subclass reveals that there are 868 relations (including the null relation $\perp$) in Allen's interval algebra that can be expressed using ORD-Horn clauses. As a side remark, it is interesting to note that the clause form of the interval formulas over $\mathscr{H}$ is less arbitrary than one might expect. Non-unit clauses are only binary and they only contain literals of the form $(X^- \ op_1 \ Y^-)$ and $(X^+ \ op_2 \ Y^+)$, where $op_i \in \{\leq, =, \neq\}$.

Since the full algebra contains $2^{13} = 8192$ relations, $\mathscr{H}$ covers more than 10% of the full algebra. Comparing this with the continuous endpoint subclass $\mathscr{C}$, which contains 83 relations, and the pointisable subclass $\mathscr{P}$, which contains 188 relations,[10] having shown tractability for $\mathscr{H}$ is a clear improvement over previous results. However, there remains the question of whether the "traditional" method of reasoning in Allen's interval algebra, that is, constraint propagation, gives reasonable results.

As we show below, this is indeed the case, ISAT($\mathscr{H}$) is decided by the path-consistency method. Intuitively, the path-consistency method performs *positive unit resolution*, that is, unit resolution involving only *positive* unit clauses, a resolution strategy that is refutation complete for Horn theories [Henschen and Wos 1974]. If a clause C is derivable by positive unit resolution from Ω, we write $\Omega \vdash_{U^+} C$.

In the following, we assume that the clauses $C \in \pi(\phi)$ are *minimal*, that is, there exists no clause C' with fewer literals than C (with respect to set-inclusion) such that $\pi(\phi) \vDash_R C'$. Clearly, if there exists some clause form, there exists also a minimal clause form. Additionally, we assume that

$$\begin{aligned}
(a \leq c) &\in \pi(\phi) \text{ if } (a \leq b), (b \leq c) \in \pi(\phi)\\
(a \leq b), (b \leq a) \in \pi(\phi) &\text{ iff } (a = b) \in \pi(\phi)\\
(a = b) \in \pi(\phi) &\text{ iff } (b = a) \in \pi(\phi)\\
(a = a) &\in \pi(\phi),
\end{aligned}$$

where a, b, and c denote endpoints of the two intervals appearing in ϕ. In other words, we assume that transitivity with respect to $\leq$, antisymmetry for positive unit clauses involving $\leq$ and the "weakening" of $=$, and symmetry and reflexivity of positive unit clauses involving $=$ are explicitly represented in the clause form. We call this the *explicitness* assumption. Note that this assumption is compatible with the assumption that all clauses in $\pi(\phi)$ are minimal.

LEMMA 4.1. *Let $\hat{\Theta}$ be a path-consistent set over $\mathscr{H}$ such that $(X \perp Y) \notin \hat{\Theta}$. Then $\pi(\hat{\Theta}) \cup ORD_{\pi(\hat{\Theta})}$ does not allow the derivation of new unit clauses by positive unit resolution.*

PROOF. A new unit clause U can only be derived if there exists a non-unit clause $C \in \pi(\hat{\Theta}) \cup ORD_{\pi(\hat{\Theta})}$ and a set of *positive unit clauses* $D \subseteq \pi(\hat{\Theta}) \cup$

[10] An enumeration of $\mathscr{C}$ and $\mathscr{P}$ is given by van Beek and Cohen [1990].

$ORD_{\pi(\hat{\Theta})}$ such that for all literals in C except U there is a complementary positive unit clause in D. We proceed by case analysis:

(1) Suppose C is an instance of the *transitivity* axiom.

(a) Positive units resulting from the reflexivity axiom cannot lead to new units if resolved with the transitivity axiom.

(b) Assume $D \subseteq \pi(\{\phi_i\})$, for some interval formulas $\phi_i \in \hat{\Theta}$ over the intervals X, Y. Since $\hat{\Theta}$ is path consistent, for any given pair X, Y there exist only two interval formulas of the form XRY and $YR^{\smile}X \in \hat{\Theta}$. Since $\pi(XRY)$ is logically equivalent to $\pi(YR^{\smile}X)$, we can assume that $D \subseteq \pi(XRY)$, for some pair of intervals X, Y. By minimality and explicitness of the clause form, it follows that $U \in \pi(XRY)$.

(c) Consider two different interval formulas, say $XRY, YSZ \in \hat{\Theta}$. By the above arguments, there do not exist other interval formulas over the same intervals that are not logically equivalent. Assume that each of the ORD-clause forms of these interval formulas contains one positive unit $U_{xy} \in \pi(XRY)$, $U_{yz} \in \pi(YSZ)$ and $D = \{U_{xy}, U_{yz}\}$. Consider now $(XTZ) \in \hat{\Theta}$. Since $\hat{\Theta}$ is a path-consistent set, it holds that $T \subseteq (R \circ S)$. Further, because $\pi(\{XRY, YSZ\}) \models_R U$, and because U mentions only endpoints of X and Z, it follows that $\pi(\{X(R \circ S)Z\}) \models_R U$, and, since $T \subseteq (R \circ S)$, $\pi(XTZ) \models_R U$. Since by assumption $\hat{\Theta}$ is over $\mathscr{H}$, it must be the case that $T \in \mathscr{H}$. Finally, since all ORD clause forms are minimal and explicit, it follows that $U \in \pi(XTZ)$.

(2) C cannot be an instance of the *reflexivity* axiom because we assumed that C is a non-unit clause.

(3) Suppose C is an instance of the *antisymmetry* axiom.

(a) Assume $D = \{(a \leq a), (a \leq a)\} \subseteq \pi(\hat{\Theta})$. However, by the explicitness assumption $(a = a) \in \pi(\hat{\Theta})$.

(b) So assume, $D = \{(a \leq b), (b \leq a)\} \subseteq \pi(\hat{\Theta})$. However, again by the explicitness and minimality assumptions, $(a = b) \in \pi(\hat{\Theta})$.[11]

(4) Suppose that C is an instance of one of the two axioms

$$\forall x, y\colon x = y \rightarrow x \leq y,$$
$$\forall x, y\colon x = y \rightarrow y \leq x.$$

Again, by the explicitness assumption, no new unit can be derived.

(5) Finally, suppose that $C \in \pi(\hat{\Theta})$. Since the only units in $ORD_{\pi(\hat{\Theta})}$ are $a \leq a$ and no clause in $\pi(\hat{\Theta})$ contains a literal of the form $(a \not\leq a)$, we must have $D \subseteq \pi(\hat{\Theta})$. Assume that $C \in \pi(XRY)$. Since D contains unit clauses over the same endpoints, and since path-consistency of $\hat{\Theta}$ implies that there is no other non-equivalent formula over the same intervals, it must be the case that $D \subseteq \pi(XRY)$. Now, by minimality and explicitness, it follows that $U \in \pi(XRY)$. Hence, also in this case, no new unit clause is derivable.

Hence, it is impossible to derive a new unit clause from any clause $C \in \pi(\hat{\Theta}) \cup ORD_{\pi(\hat{\Theta})}$ by positive unit resolution. □

[11] Note that it might be possible to derive the new unit clause $(b \not\leq a)$ if $D = \{(a \leq b), (a \neq b)\}$. However, this would not be a *positive* unit resolution step.

Since the only interval formulas that have the empty clause as their ORD-clause form are those involving $\perp$, it follows by refutation completeness of positive unit resolution that any path-consistent set over $\mathscr{H}$ without any formula involving $\perp$ is satisfiable.

THEOREM 4.2. *Let $\hat{\Theta}$ be a path-consistent set of interval formulas over $\mathscr{H}$. Then $\hat{\Theta}$ is I-satisfiable iff $(X \perp Y) \notin \hat{\Theta}$.*

PROOF

"$\Rightarrow$:" Obvious.

"$\Leftarrow$:" Assume that $(X \perp Y) \notin \hat{\Theta}$. Since the only interval formulas that have the empty clause in the clause form are formulas of the form $(X \perp Y)$, it follows that $\pi(\hat{\Theta})$ does not contain the empty clause. By Lemma 4.1 and refutation completeness of positive unit resolution, it follows that $\pi(\hat{\Theta}) \cup ORD_{\pi(\hat{\Theta})}$ is satisfiable. By Propositions 3.1–3.3, it follows that $\hat{\Theta}$ has an interval model. □

The only remaining part we have to show is that transforming Θ over $\mathscr{H}$ into its equivalent path-consistent form $\hat{\Theta}$ does not result in a set that contains relations not in $\mathscr{H}$. In order to show this we prove that $\mathscr{H}$ is closed under converse, intersection, and composition, that is, $\mathscr{H}$ (together with these operations) defines a *subalgebra* of Allen's interval algebra.

At first sight, this looks like a straightforward consequence of the fact that minimal clauses implied by a Horn theory are Horn clauses. Unfortunately, this fact cannot be exploited in our case. As long as we interpret $\pi(\Theta)$ over the reals, this fact is not applicable and Proposition 3.2 only guarantees the equivalence of satisfiability of ORD-Horn clauses, not the equivalence of logical implication. As a matter of fact, in our case, the mentioned fact does not hold, as the following example demonstrates:

$$\{(a \leq b)\} \models_R (a \leq c \vee c \leq b).$$

In order to show that $\mathscr{H}$ is nevertheless a subalgebra, we first need two technical lemmas.

LEMMA 4.3. *Let Ω be a set of ORD-Horn clauses such that $\Omega \cup \{(c \neq d)\}$ is R-satisfiable and $\Omega \cup \{(c \neq d), (a \leq b), (a \neq b)\}$ is R-unsatisfiable. Then $\Omega \cup \{(a \leq b), (a \neq b)\}$ is already R-unsatisfiable.*

PROOF. By Propositions 3.2 and 3.3, $ORD_{\Omega} \cup \Omega \cup \{(c \neq d), (a \leq b), (a \neq b)\}$ must be unsatisfiable. Since a set of Horn clauses is unsatisfiable iff it contains an unsatisfiable subset with exactly one negative clause [12], it follows that $ORD_{\Omega} \cup \Omega \cup \{(a \leq b)\}$, $ORD_{\Omega} \cup \Omega \cup \{(a \leq b), (a \neq b)\}$, or $ORD_{\Omega} \cup \Omega \cup \{(c \neq d), (a \leq b)\}$ is already unsatisfiable. If one of the former two cases holds, then the claim follows by Propositions 3.2 and 3.3. Hence, let us assume that the latter case holds.

By refutation completeness of positive unit resolution $ORD_{\Omega} \cup \Omega \cup \{(a \leq b)\} \vdash_{U^+} (c = d)$. By that it follows that $ORD_{\Omega} \cup \Omega \cup \{(a \leq b)\} \vdash_{U^+} (c \leq d), (d \leq c)$. Further, at most one of these atoms can be derived from $ORD_{\Omega} \cup \Omega$ since otherwise the empty clause could be derived from $ORD_{\Omega} \cup \Omega \cup \{(c \neq d)\}$. Hence, $(a \leq b)$ must be involved in deriving $c \leq d$ or $d \leq c$. Without loss of generality, we assume the first of these alternatives. If the *transitivity* axiom is used in deriving $c \leq d$, there must be a sequence of unit clauses derivable from $ORD_\Omega \cup \Omega \cup \{(a \leq b)\}$ by positive unit resolution such that $c \leq \cdots \leq d$. If $c \leq d$ is derived from $c = d$ or from a clause in Ω, then this chain is simply $c \leq d$.

Suppose that $a \leq b$ is one of the unit clauses in the above chain, that is, $c \leq \cdots \leq a \leq b \leq \cdots \leq d$. Since $ORD_\Omega \cup \Omega \cup \{(a \leq b)\} \vdash_{U^+} (c = d)$, it follows that $ORD_\Omega \cup \Omega \cup \{(a \leq b)\} \vdash_{U^+} (a = b)$. This means that the empty clause is derivable from $ORD_\Omega \cup \Omega \cup \{(a \leq b), (a \neq b)\}$. Applying Propositions 3.2 and 3.3, the claim follows in this case.

Suppose that $(a \leq b)$ does not appear as a unit participating in a chain as specified above. Since $(a \leq b)$ is nevertheless necessary for deriving $(c \leq d)$, some positive unit resolution steps involving clauses from Ω are necessary. Consider the first such step where $(a \leq b)$ is involved as an ancestor. Since all negative literals have the form $e \neq f$, a sequence of units as follows must be derivable from $ORD_\Omega \cup \Omega \cup \{(a \leq b)\}$:

$$e \leq \cdots \leq a \leq b \leq \cdots \leq f.$$

Since $e = f$ is also derivable by positive unit resolution, by the same arguments as above, it follows that $\Omega \cup \{(a \leq b), (a \neq b)\}$ must be R-unsatisfiable. □

LEMMA 4.4. *Let Ω be a set of ORD-Horn clauses such that $\Omega \cup \{(a_1 \leq b_1), (a_1 \neq b_1), (a_2 \leq b_2), (a_2 \neq b_2)\}$ is R-unsatisfiable, but $\Omega \cup \{(a_i \leq b_i), (a_i \neq b_i)\}$, for $i = 1, 2$, is R-satisfiable. Then $\Omega \models_R (b_1 \leq a_2), (b_2 \leq a_1)$.*

PROOF. Let Ω' be the subset of Ω that contains all clauses of Ω except the negative ones. By Lemma 4.3, it follows that $\Omega' \cup \{(a_1 \leq b_1), (a_2 \leq b_2), (a_2 \neq b_2)\}$ is already R-unsatisfiable. Using the same arguments as in the proof of Lemma 10, it follows that $ORD_\Omega \cup \Omega' \cup \{(a_1 \leq b_1)\} \vdash_{U^+} (b_2 \leq a_2)$. Further, $ORD_\Omega \cup \Omega' \nvdash_{U^+} (b_2 \leq a_2)$ since otherwise $\Omega \cup \{(a_2 \leq b_2), (a_2 \neq b_2)\}$ would be already R-unsatisfiable. Hence, $(a_1 \leq b_1)$ is used in the positive unit derivation of $(b_2 \leq a_2)$. As in the proof of Lemma 4.3, there are two cases.

(1) There exists a sequence of unit clauses derivable from $ORD_\Omega \cup \Omega' \cup \{(a_1 \leq b_1\}$ such that

$$b_2 \leq \cdots \leq a_1 \leq b_1 \leq \cdots \leq a_2.$$

Hence, $b_2 \leq a_1$ and $b_1 \leq a_2$ are derivable by unit resolution. By soundness of positive unit resolution, the claim follows in this case.

(2) There is no unit $(a_1 \leq b_1)$ in the sequence of unit clauses above. Since $(a_1 \leq b_1)$ is involved in the derivation of $(b_2 \leq a_2)$, a positive unit resolution step involving an ancestor of $(a_1 \leq b_1)$ with a clause from Ω' must be involved. Since the only negative literals in such clauses have the form $c \neq d$, $a_1 = b_1$ must be derivable from $ORD_\Omega \cup \Omega' \cup \{(a_1 \leq b_1)\}$ by positive unit resolution. However, this contradicts our assumption that $\Omega \cup \{(a_1 \leq b_1), (a_1 \neq b_1)\}$ is R-satisfiable.

Hence, the first case must apply, and the claim holds. □

THEOREM 4.5. *$\mathscr{H}$ is closed under converse, intersection, and composition.*

PROOF. Suppose $R \in \mathscr{H}$, that is, $\pi(XRY)$ is a set of ORD-Horn clauses. Clearly, $\pi(YRX)$ is a set of ORD-Horn clauses; hence, $\pi(XR^\smile Y)$ is as well. Therefore, $R^\smile \in \mathscr{H}$.

Suppose $R, S \in \mathscr{H}$; hence, $\pi(\{XRY, XSY\})$ is a set of ORD-Horn clauses. Since $\pi(\{XRY, XSY\})$ is logically equivalent to $\pi(X(R \cap S)Y)$, the latter can be expressed as a set of ORD-Horn clauses, so $(R \cap S) \in \mathscr{H}$.

Suppose $R, S \in \mathscr{H}$. Given XRZ, ZSY, $R \circ S$ is the *strongest implied relation* between X and Y, that is, $\{XRZ, ZSY\} \models_I X(R \circ S)Y$, for any X, Y, Z, such that $(R \circ S)$ is the strongest relation satisfying this relation. Assume that it is impossible to find a clause form for $\pi(X(R \circ S)Y)$ that is ORD-Horn. This means that $\pi(X(R \circ S)Y)$ must contain at least one clause C with more than one positive literal. Let $C = C_{\leq} \vee C_{=} \vee C_{\neq}$, where $C_{\leq}$, $C_{=}$, and $C_{\neq}$ are clauses containing only literals over $\leq$, $=$, and $\neq$, respectively. Without loss of generality, we assume that C is minimal. Since C follows logically from $\pi(\{XRZ, ZSY\})$, the negation of C together with this clause form is R-unsatisfiable. Let us consider the set of unit ORD-clauses D that is logically equivalent to the negation of C under interpretation of the endpoints as reals, where $D = D_{\leq} \cup D_{=} \cup D_{\neq}$ such that the respective clause sets correspond to the clause parts in C.

As the first step, we show that $C_{=}$ must be empty. Assume that $D_{=} = \{(a_1 \neq b_1), \ldots, (a_k \neq b_k)\}$, where $k \geq 2$. By Propositions 3.2 and 3.3, it follows that $ORD_{\Omega} \cup \pi(\{XRZ, ZSY\}) \cup D$ is unsatisfiable. Since a set of Horn clauses is unsatisfiable iff it contains an unsatisfiable subset with exactly one negative clause [Gallier and Raatz 1985], it follows that $ORD_{\Omega} \cup \pi(\{XRZ, ZSY\}) \cup D_{\leq} \cup D_{\neq} \cup \{(a_i \neq b_i)\}$, for some i, $1 \leq i \leq k$, must be already unsatisfiable, hence, by Propositions 3.2 and 3.3, $\pi(\{XRZ, ZSY\}) \cup D_{\leq} \cup D_{\neq} \cup \{(a_i \neq b_i)\}$ is already R-unsatisfiable; hence, the clause C is not minimal, contradicting the assumption.

Assume that $C_{=} = (c = d)$, that is, $D_{=} = \{(c \neq d)\}$. In this case, $C_{\leq}$ cannot be empty since otherwise C would be an ORD-Horn clause, contradicting our assumption. Thus, $D_{\leq}$ contains the two unit clauses $(a \leq b)$, $(a \neq b)$ resulting from the literal $(b \leq a)$ in $C_{\leq}$. Applying Lemma 4.3 leads to the consequence that $\Omega \cup D_{\leq} \cup D_{\neq}$ is already R-unsatisfiable, contradicting the assumption that C is minimal. Hence, it must be the case that $C_{=}$ is the empty clause.

As the second step, we show that for any clause C containing more than one literal in $C_{\leq}$, we can construct two clauses C_1 and C_2 with fewer positive literals than C such that $\pi(\{XRZ, ZSY\}) \models_R C_1, C_2$ and $\{C_1, C_2\} \models_R C$.

Let $(b_1 \leq a_1)$, $(b_2 \leq a_2)$ be two literals from $C_{\leq}$, let $C'_{\leq}$ be $C_{\leq}$ without those two literals, and let $C' = C'_{\leq} \vee C_{=} \vee C_{\neq}$. Similarly, let $D'_{\leq}$ be $D_{\leq}$ without the units $(a_1 \leq b_1)$, $(a_1 \neq b_1)$, $(a_2 \leq b_2)$, $(a_2 \neq b_2)$, and let $D' = D'_{\leq} \cup D_{=} \cup D_{\neq}$.

By the assumption that C is a minimal clause logically implied by $\pi(\{XRZ, ZSY\})$, it follows that $\pi(\{XRZ, ZSY\}) \cup D' \cup \{(a_1 \leq b_1), (a_1 \neq b_1), (a_2 \leq b_2), (a_2 \neq b_2)\}$ is R-unsatisfiable, but if $\{(a_i \leq b_i), (a_i \neq b_i)\}$, for some $i \in \{1, 2\}$, is omitted from the set of clauses, it becomes R-satisfiable. Applying Lemma 4.4 yields $\pi(\{XRZ, ZSY\}) \cup D' \models_R (b_1 \leq a_2), (b_2 \leq a_1)$.

Set $C_1 = C' \vee (b_1 \leq a_2)$ and $C_2 = C' \vee (b_2 \leq a_1)$. First, the clauses C_1 and C_2 have fewer positive literals than C. Second, we obviously have $\pi(\{XRZ, ZSY\}) \models_R C_1, C_2$. Third, we also have $\{C_1, C_2\} \models_R C$, because

$$\{(C' \vee (b_1 \leq a_2)), (C' \vee (b_2 \leq a_1))\}$$
$$\cup \{(a_1 \leq b_1), (a_1 \neq b_1), (a_2 \leq b_2), (a_2 \neq b_2)\} \cup D'$$

is R-unsatisfiable.

By induction over the number of positive literals in C, it follows that if there exists a clause C such that $\pi(\{XRZ, ZSY\}) \models_R C$, then there exists a set of ORD-Horn clauses $\{C_i\}$ that is logically implied by $\pi(\{XRZ, ZSY\})$ and implies C. Hence, $\pi(X(R \circ S)Y)$ can be expressed as a set of ORD-Horn clauses, hence $(R \circ S) \in \mathscr{H}$. □

From that it follows immediately that ISAT($\mathscr{H}$) is decided by the path-consistency method.

THEOREM 4.6. *If Θ is a set over $\mathscr{H}$, then Θ is satisfiable iff $(X \perp Y) \notin \hat{\Theta}$ for all intervals X, Y.*

PROOF. Since $\hat{\Theta}$ is logically equivalent to Θ, satisfiability of Θ implies $(X \perp Y) \notin \hat{\Theta}$, for all X, Y.

Conversely, for any set Θ over $\mathscr{H}$, $\hat{\Theta}$ is a set over $\mathscr{H}$ by Theorem 4.5. Since the absence of $\perp$ from $\hat{\Theta}$ over $\mathscr{H}$ implies its satisfiability by Theorem 4.2, and since Θ is logically equivalent to $\hat{\Theta}$, the absence of $\perp$ from $\hat{\Theta}$ implies satisfiability of Θ. □

5. *Subalgebras and Their Computational Properties*

Although the introduction of the algebraic structure on the set of expressible interval relations may have seemed to be motivated only by the particular approximation algorithm employed, this structure is also useful when we explore the computational properties of restricted problems. As it turns out, it is not necessary to explore the entire space of subclasses of the interval algebra (consisting of $2^{2^{13}}$ or approximately 10^{2400} subsets), but we can restrict ourselves to subalgebras of Allen's interval algebra. For any arbitrary subset $\mathscr{S} \subseteq \mathscr{A}$, $\overline{\mathscr{S}}$ shall denote the *closure* of $\mathscr{S}$ under converse, intersection, and composition. In other words, $\overline{\mathscr{S}}$ is the carrier of the *least subalgebra generated by* $\mathscr{S}$.

THEOREM 5.1. *ISAT($\overline{\mathscr{S}}$) can be polynomially transformed to ISAT($\mathscr{S}$).*

PROOF. Let $\mathscr{T} = \overline{\mathscr{S}} - \mathscr{S}$. Every element of $R \in \mathscr{T}$ is equivalent to some expression ϵ_R over $\mathscr{S}$ involving converse, intersection, and composition. Let m be the maximum number of operators appearing in these expressions.

We will show by induction that for any set of intervals Θ over $\overline{\mathscr{S}}$, we can construct a set Θ' over $\mathscr{S}$ such that $|\Theta'| \leq (2^m \times |\Theta|)$ and Θ is I-satisfiable iff Θ' is. Since m is fixed for given $\mathscr{S}$, this is a polynomial transformation.

Base step: $m = 1$. For any interval formula $(XRY) \in \Theta$ such that $R \in \mathscr{T}$ one of the following cases applies:

(1) $R = S^{\smile}$ and $S \in \mathscr{S}$. In this case, the interval formula (XRY) in Θ is replaced by (YSX).
(2) $R = S \cap T$ and $S, T \in \mathscr{S}$. In this case, the interval formula (XRY) in Θ is replaced by the two formulas (XSY), (XTY).
(3) $R = S \circ T$ and $S, T \in \mathscr{S}$. In this case, the interval formula (XRY) in Θ is replaced by (XSZ), (ZTY), where Z is a fresh interval.

Clearly, if Θ is I-satisfiable, then Θ' is and vice versa. Further $|\Theta'| \leq 2^1 \times |\Theta|$.

Inductive step: We assume that the hypothesis holds for $m = k$ and assume that the maximum number of operators appearing in expressions ϵ_R for $R \in \mathscr{T}$ is $k + 1$. Let $\mathscr{T}' \subseteq \mathscr{T}$ be the relations R such that the expressions ϵ_R involve $k + 1$ operators. For all these relations, we can find expressions ϵ'_R over $\bar{\mathscr{S}} - \mathscr{T}'$ that contain only one operator.

Applying now the above transformation for all $R \in \mathscr{T}'$ using ϵ'_R yields a set Θ'' over $\bar{\mathscr{S}} - \mathscr{T}'$ of size $2 \times |\Theta|$ that is equivalent to Θ with respect to I-satisfiability. Applying the induction hypothesis yields that it is possible to construct a set Θ' of size $2^{k+1} \times |\Theta|$ that is equivalent to Θ with respect to I-satisfiability, which proves the induction claim. □

In other words, once we have proven that satisfiability is polynomial for some set $\mathscr{S} \subseteq \mathscr{A}$, this result extends to the least subalgebra generated by $\mathscr{S}$.

COROLLARY 5.2. *ISAT($\mathscr{S}$) is polynomial iff ISAT($\bar{\mathscr{S}}$) is polynomial.*

Conversely, NP-hardness for a subalgebra is "inherited" by all subsets that generate this subalgebra. Since ISAT($\mathscr{A}$) $\in$ NP, NP-completeness follows.

COROLLARY 5.3. *ISAT($\mathscr{S}$) is NP-complete iff ISAT($\bar{\mathscr{S}}$) is NP-complete.*

It should be noted that these results do not hold in their full generality if the interval satisfiability problem is defined somewhat differently. Often, this problem is defined over "binary constraint networks."[12] Such networks correspond to what we will call *normalized sets* of interval formulas, where for each pair of intervals X, Y we have *exactly one* interval formula. The corresponding decision problem for the satisfiability of normalized sets of interval formulas is denoted by $\mathrm{ISAT}_N(\mathscr{S})$. Provided the subclass $\mathscr{S}$ of Allen's interval algebra contains $\top$ and $\{\equiv\}$, which is usually true, then a slight modification of the reduction used in the proof of Theorem 5.1 leads to identical results.

THEOREM 5.4. *$\mathrm{ISAT}_N(\bar{\mathscr{S}})$ can be polynomially transformed to $\mathrm{ISAT}_N(\mathscr{S})$, provided $\{\top, \{\equiv\}\} \subseteq \mathscr{S}$.*

PROOF. The reduction for converses and composition can be done as in the proof of Theorem 5.1. Interval formulas XRY that involve a relation R that can only be expressed as an intersection $(S \cap T)$ are transformed into sets of formulas of the following form $\{(XSY), (X\{\equiv\}Z), (ZTY)\}$, where Z is a fresh interval, which leads to a set of interval formulas that is equivalent to the original set with respect to I-satisfiability. □

However, if $\top \notin \mathscr{S}$ or $\{\equiv\} \notin \mathscr{S}$, the reduction does not apply any longer. In such a case, polynomiality of a set does not automatically extend to the least subalgebra generated by this set. In fact, Golumbic and Shamir [1992, 1993] show that for $\mathscr{S}_0 = \{\{\prec\}, \{\succ\}, \{\prec, \succ\}, \mathbf{B} - \{\prec, \succ\}\}$ the problem $\mathrm{ISAT}_N(\mathscr{S}_0)$ is polynomial, while $\mathrm{ISAT}_N(\mathscr{S}_0 \cup \{\top\})$ is NP-complete, despite the fact that $\mathscr{S}_0 \cup \{\top\} \subseteq \bar{\mathscr{S}}_0$.

We believe that for the applications mentioned in the Introduction the definition of the interval satisfiability problem over arbitrary sets of interval formulas is more appropriate than over normalized sets because it allows leaving some relations between intervals unspecified and permits incremental

[12] For example, see Golumbic and Shamir [1992, 1993], Nökel [1989], van Beek and Cohen [1990], and Vilain et al. [1989].

refinements of constraints between intervals (by adding interval formulas to an already existing set). However, the problem definition of ISAT_N is certainly worthwhile in cases where the problem solving process is nonincremental and constraints between all intervals are known.

6. *The Borderline between Tractable and NP-Complete Subclasses*

Having identified the tractable fragment $\mathcal{H}$ that contains the previously identified tractable fragment $\mathcal{P}$ and that is considerably larger than $\mathcal{P}$ is satisfying in itself. However, such a result also raises the questions of whether there may exist other tractable fragments that contain $\mathcal{H}$ or whether there are other incomparable tractable fragments. In other words, we want to know the boundary between polynomiality and NP-completeness in Allen's interval algebra.

Although we have narrowed down the space of possible candidates in the previous section from arbitrary subsets of $\mathcal{A}$ to subalgebras, it still takes some effort to prove that a given fragment $\mathcal{S}$ is a *maximal* tractable subclass of Allen's interval algebra. First, using Corollary 5.2, one has to show that $\mathcal{S} = \overline{\mathcal{S}}$. For the ORD-Horn subclass, this has been done in Theorem 4.5. Second, employing Corollary 5.3, it suffices to prove that $\mathrm{ISAT}(\mathcal{T})$ is NP-complete for all *minimal* subalgebras $\mathcal{T}$ that strictly contain $\mathcal{S}$. This, however, means that the minimal subalgebras containing $\mathcal{S}$ have to be identified. The only way to solve this problem seems to be to enumerate all subalgebras generated by $\mathcal{S} \cup \{R\}$, for $R \in \mathcal{A} - \mathcal{S}$, and to filter out the minimal ones—a process that involves a case analysis with a couple of thousand cases. Certainly, such a case analysis cannot be done manually. In fact, we used a program to identify the minimal subalgebras strictly containing $\mathcal{H}$. An analysis of the clause form of the relations appearing in these subalgebras leads us to the formulation of the following machine-verifiable lemma.

LEMMA 6.1. *Let $\mathcal{S} \subseteq \mathcal{A}$ be any set of interval relations that strictly contains $\mathcal{H}$. Then* $\{\mathsf{d}, \mathsf{d}^{\smile}, \mathsf{o}^{\smile}, \mathsf{s}^{\smile}, \mathsf{f}\}$ *or* $\{\mathsf{d}^{\smile}, \mathsf{o}, \mathsf{o}^{\smile}, \mathsf{s}^{\smile}, \mathsf{f}^{\smile}\}$ *is an element of* $\overline{\mathcal{S}}$.

PROOF. In order to verify the claim, a machine-assisted case analysis of the following form is necessary:

(1) Generate all subalgebras $\mathcal{T}_R = \overline{\mathcal{H} \cup \{R\}}$, for all $R \in \mathcal{A} - \mathcal{H}$.
(2) Test: $\{\mathsf{d}, \mathsf{d}^{\smile}, \mathsf{o}^{\smile}, \mathsf{s}^{\smile}, \mathsf{f}\} \in \mathcal{T}_R$ or $\{\mathsf{d}^{\smile}, \mathsf{o}, \mathsf{o}^{\smile}, \mathsf{s}^{\smile}, \mathsf{f}^{\smile}\} \in \mathcal{T}_R$.

The test succeeds for all $R \in \mathcal{A} - \mathcal{H}$. Since for any set $\mathcal{S}$ that strictly contains $\mathcal{H}$, $\overline{\mathcal{S}}$ contains $\mathcal{T}_R$ for some $R \in \mathcal{A} - \mathcal{H}$, the claim must be true. □

For reasons of simplicity, we will not use the ORD-clause form in the following, but a clause form that also contains literals over the relations $\geq$, $<$, $>$. Then, the clause form for the relations mentioned in the lemma can be given as follows:

$$
\begin{aligned}
\pi(X\{\mathsf{d}, \mathsf{d}^{\smile}, \mathsf{o}^{\smile}, \mathsf{s}^{\smile}, \mathsf{f}\}Y) = \{&(X^- < X^+), (Y^- < Y^+),\\
&(X^- < Y^+), (X^+ > Y^-),\\
&(X^- > Y^- \vee X^+ > Y^+)\},\\
\pi(X\{\mathsf{d}^{\smile}, \mathsf{o}, \mathsf{o}^{\smile}, \mathsf{s}^{\smile}, \mathsf{f}^{\smile}\}Y) = \{&(X^- < X^+), (Y^- < Y^+),\\
&(X^- < Y^+), (X^+ > Y^-),\\
&(X^- < Y^- \vee X^+ > Y^+)\}.
\end{aligned}
$$

In plain words, {d, d$^\smile$, o$^\smile$, s$^\smile$, f} expresses the relation "strictly intersects and (starts after or ends after)" and {d$^\smile$, o, o$^\smile$, s$^\smile$, f$^\smile$} expresses the relation "strictly intersects and (starts before or ends after)." We will show that each of these relations together with the two relations {≺, d$^\smile$, o, m, f$^\smile$} and {≺, d, o, m, s}, which are elements of $\mathscr{C}$, are enough for making the interval satisfiability problem NP-complete. The clause form of these relations looks as follows:

$$\pi(X\{\prec, \mathsf{d}^\smile, \mathsf{o}, \mathsf{m}, \mathsf{f}^\smile\}Y) = \{(X^- < X^+), (Y^- < Y^+), (X^- < Y^-), (X^- < Y^+)\}$$

$$\pi(X\{\prec, \mathsf{d}, \mathsf{o}, \mathsf{m}, \mathsf{s}\}Y) = \{(X^- < X^+), (Y^- < Y^+), (X^+ < Y^+), (X^- < Y^+)\}.$$

LEMMA 6.2. *ISAT($\mathscr{S}$) is NP-complete if*

(1) $\mathscr{N}_1$ = {(≺, d$^\smile$, o, m, f$^\smile$}, {≺, d, o, m, s}, {d, d$^\smile$, o$^\smile$, s$^\smile$, f}} $\subseteq \mathscr{S}$, *or*
(2) $\mathscr{N}_2$ = {{≺, d$^\smile$, o, m, f$^\smile$}, {≺, d, o, m, s}, {d$^\smile$, o, o$^\smile$, s$^\smile$, f$^\smile$}} $\subseteq \mathscr{S}$.

PROOF. Since ISAT($\mathscr{A}$) ∈ NP, membership in NP follows.

For the NP-hardness part we will show that 3SAT can be polynomially transformed to ISAT($\mathscr{N}_k$). This implies that any set containing $\mathscr{N}_k$ has this property. We will first prove the claim for $\mathscr{N}_1$.

Let $D = \{C_i\}$ be a set of clauses, where $C_i = l_{i,1} \vee l_{i,2} \vee l_{i,3}$ and the $l_{i,j}$'s are literal occurrences. We will construct a set of interval formulas Θ over $\mathscr{N}_1$ such that Θ is I-satisfiable iff D is satisfiable.

For each literal occurrence $l_{i,j}$ a pair of intervals $X_{i,j}$ and $Y_{i,j}$ is introduced, and the following first group of interval formulas is put into Θ:

$$\left(X_{i,j}\{\mathsf{d}, \mathsf{d}^\smile, \mathsf{o}^\smile, \mathsf{s}^\smile, \mathsf{f}\}Y_{i,j}\right).$$

This implies that $\pi(\Theta)$ contains among other things the following clauses:

$$\left(X_{i,j}^- > Y_{i,j}^- \vee X_{i,j}^+ > Y_{i,j}^+\right).$$

Additionally, we add a second group of formulas for each clause C_i:

$$(X_{i,2}\{\prec, \mathsf{d}^\smile, \mathsf{o}, \mathsf{m}, \mathsf{f}^\smile\}Y_{i,1}),$$
$$(X_{i,3}\{\prec, \mathsf{d}^\smile, \mathsf{o}, \mathsf{m}, \mathsf{f}^\smile\}Y_{i,2}),$$
$$(X_{i,1}\{\prec, \mathsf{d}^\smile, \mathsf{o}, \mathsf{m}, \mathsf{f}^\smile\}Y_{i,3}),$$

which leads to the inclusion of the following clauses in $\pi(\Theta)$:

$$(Y_{i,1}^- > X_{i,2}^-), (Y_{i,2}^- > X_{i,3}^-), (Y_{i,3}^- > X_{i,1}^-).$$

This construction leads to the situation that there is no model of Θ that satisfies for given i all disjuncts of the form $(X_{i,j}^- > Y_{i,j}^-)$ in the clause form of $\pi(X_{i,j}\{\mathsf{d}, \mathsf{d}^\smile, \mathsf{o}^\smile, \mathsf{s}^\smile, \mathsf{f}\}Y_{i,j})$, since otherwise a cycle $X_{i,1}^- > Y_{i,1}^- > X_{i,2}^- > \cdots > Y_{i,3}^- > X_{i,1}^-$ would be satisfied, which is impossible.

If the jth disjunct $(X_{i,j}^- > Y_{i,j}^-)$ is unsatisfied in an I-model of Θ, we will interpret this as the satisfaction of the literal occurrence $l_{i,j}$ in C_i of D.

In order to guarantee that if a literal occurrence $l_{i,j}$ is interpreted as satisfied, then all complementary literal occurrences in D are interpreted as unsatisfied, the following third group of interval formulas is added. Assume that $l_{i,j}$ and $l_{g,h}$ are complementary literal occurrences, then the following interval formulas are added to Θ:

$$\left(X_{g,h}\{\prec, \mathsf{d}, \mathsf{o}, \mathsf{m}, \mathsf{s}\}Y_{i,j}\right),$$
$$\left(X_{i,j}\{\prec, \mathsf{d}, \mathsf{o}, \mathsf{m}, \mathsf{s}\}Y_{g,h}\right),$$

which leads to the inclusion of the following clauses in $\pi(\Theta)$:

$$\left(Y_{i,j}^{+} > X_{g,h}^{+}\right), \left(Y_{g,h}^{+} > X_{i,j}^{+}\right).$$

Now there exists no model of Θ that makes the disjuncts $(X_{i,j}^{-} > Y_{i,j}^{-})$ and $(X_{g,h}^{-} > Y_{g,h}^{-})$ simultaneously false, which would correspond to the simultaneous satisfaction of $l_{i,j}$ and $l_{g,h}$, since otherwise the disjuncts $(X_{i,j}^{+} > Y_{i,j}^{+})$ and $(X_{g,h}^{+} > Y_{g,h}^{+})$ would be satisfied by this model, which implies that the chain $X_{i,j}^{+} > Y_{i,j}^{+} > X_{g,h}^{+} > Y_{g,h}^{+} > X_{i,j}^{+}$ would be satisfied by the model, which is impossible.

Now we will show that Θ is I-satisfiable iff D is satisfiable.

If Θ has a model $\mathfrak{I}$, then by the above arguments it is possible to satisfy each clause C_i by (at least) one literal occurrence $l_{i,j}$ such that the corresponding disjunct $(X_{i,j}^{-} > Y_{i,j}^{-})$ is unsatisfied in $\mathfrak{I}$. Further, if the literal occurrence $l_{i,j}$ is used for the satisfaction of clause C_i, all complementary literal occurrences in D cannot be satisfied. This, however, means that it is possible to construct a satisfying truth assignment for D.

For the converse direction assume that there exists a satisfying truth assignment of D. Using this assignment, we will construct as set of clauses Ω from $\pi(\Theta)$ by eliminating from each nonunit clause one disjunct. The remaining set will then only contain unit clauses of the form $(a < b)$, which can easily be shown to be satisfiable.

If the literal l is interpreted as true in D by the satisfying truth assignment, then we eliminate for all $l_{i,j} = l$ the disjunct $(X_{i,j}^{-} > Y_{i,j}^{-})$ from the clause $(X_{i,j}^{-} > Y_{i,j}^{-} \vee X_{i,j}^{+} > Y_{i,j}^{+})$, and for all $l_{i,j}$ that are complementary to l eliminate $(X_{i,j}^{+} > Y_{i,j}^{+})$ from the clause $(X_{i,j}^{-} > Y_{i,j}^{-} \vee X_{i,j}^{+} > Y_{i,j}^{+})$. Since either l or its complementary form is true, this leads to a set Ω that contains only unit clauses.

Further, since all clauses $C_i \in D$ are satisfied, there cannot be a "$>$"-cycle over the X^{-}, Y^{-} endpoints. Since no complementary literals can have the same truth value, there cannot be any "$>$"-cycle over the X^{+}, Y^{+} endpoints.

It may be the case, however, that Ω contains a cycle using beginnings and ends of intervals, for instance: $X_1^{-} < Y_2^{+} < \cdots < X_1^{-}$. Note, however, that such a cycle must contain at least one unit of the form $X^{+} < Y^{-}$. Since none of the relations we used in the proof has a clause form that contains such a literal, such a cycle is not possible. Hence, Ω does not contain a cycle of the form $a < \cdots < a$. This, however, means that Ω is satisfiable by a partially ordered set, and by Proposition 3.2, Ω is R-satisfiable. Since any R-model of Ω is by construction an R-model of $\pi(\Theta)$, Θ must be I-satisfiable by Proposition 3.1.

Hence, D is satisfiable iff Θ is, and since Θ is polynomial in D, 3SAT can be polynomially transformed to ISAT($\mathscr{N}_1$).

The transformation for $\mathcal{N}_2$ is identical, except that we use $\{d^{\smile}, o, o^{\smile}, s^{\smile}, f^{\smile}\}$ in the first group of interval formulas added to Θ and we exchange the order of $X_{i,j}$'s and $Y_{i,j}$'s in the second group. □

It should be noted that the above NP-completeness result does not refer to the relation $\{\prec, \succ\}$, which has been used in all NP-completeness proofs so far [Golumbic and Shamir 1992, 1993; Vilain and Kautz 1986; Vilain et al. 1989]. Vilain et al. [1989] have pointed out that this relation was crucial for their NP-completeness result and mention this relation as an instance of a *truly* disjunctive relation. However, as we have seen above, even relations that do not require having an interval before or after another interval may still have enough "disjunctive" potential to allow for encoding "real" disjunctions. Based on this result, it follows straightforwardly that $\mathcal{H}$ is indeed a maximal tractable subclass of $\mathcal{A}$.

THEOREM 6.3. *If $\mathcal{S}$ strictly contains $\mathcal{H}$, then ISAT($\mathcal{S}$) is NP-complete.*

PROOF. By Corollary 5.3, it suffices to consider only subalgebras that strictly contain $\mathcal{H}$. By Lemma 6.1, we know that each such subalgebra contains $\{d, d^{\smile}, o^{\smile}, s^{\smile}, f\}$ or $\{d^{\smile}, o, o^{\smile}, s^{\smile}, f^{\smile}\}$. Together with the fact that $\{\prec, d^{\smile}, o, m, f^{\smile}\}, \{\prec, d, o, m, s\} \in \mathcal{C} \subset \mathcal{H}$ and Lemma 6.2, the claim follows. □

The next question is whether there are other maximal tractable subclasses that are incomparable with $\mathcal{H}$. One example of an incomparable tractable subclass is $\mathcal{U} = \{\{\prec, \succ\}, \top\}$. Since $\{\prec, \succ\}$ has no ORD-Horn clause form, this subclass is incomparable with $\mathcal{H}$, and since all sets of interval formulas over $\mathcal{U}$ are trivially satisfiable (by making all intervals disjoint), ISAT($\mathcal{U}$) can be decided in constant time.

The subclass $\mathcal{U}$ is, of course, not a very *interesting* fragment. Thus, we may restate the above question as asking for other *interesting* incomparable tractable subclasses. Although interestingness is a more or less subjective category, it seems nevertheless possible to narrow down the space of possible candidates. Provided we are interested in temporal reasoning in the framework as described by Allen [1983], one necessary requirement is that *all basic relations* are contained in the subclass. Otherwise, we will not be able to specify *complete* information, that is, the exact relationship between two intervals. It is possible to deviate from Allen's framework, for instance, by considering *macro relations* of Allen's relations, as done by Golumbic and Shamir [1992, 1993]. However, in this case we base our representation on different assumptions than those spelled out by Allen [1983]. For this reason, we will only look for other tractable subclasses in the space of subclasses that contain the thirteen basic relations. Since tractability (and NP-completeness) are properties of subalgebras, we can actually restrict ourselves to subclasses that contain the least subalgebra generated by the basic relations:

$$\mathcal{B} = \overline{\{\{B\} \mid B \in \mathbf{B}\}}.$$

LEMMA 6.4. *If $\mathcal{S}$ is a subclass that contains the thirteen basic relations, then one of the following alternatives hold:*

(1) $\overline{\mathcal{S}} \subseteq \mathcal{H}$, *or*
(2) $\{d, d^{\smile}, o^{\smile}, s^{\smile}, f\}$ *or* $\{d^{\smile}, o, o^{\smile}, s^{\smile}, f^{\smile}\}$ *is an element of* $\overline{\mathcal{S}}$.

PROOF. In order to verify the claim, a machine-assisted case analysis of the following form is necessary:

(1) Generate all sets $\mathcal{T}_R = \overline{\mathcal{B} \cup \{R\}}$, for all $R \in \mathcal{A} - \mathcal{H}$.
(2) Test: $\{\mathsf{d}, \mathsf{d}^{\smile}, \mathsf{o}^{\smile}, \mathsf{s}^{\smile}, \mathsf{f}\} \in \mathcal{T}_R$ or $\{\mathsf{d}^{\smile}, \mathsf{o}, \mathsf{o}^{\smile}, \mathsf{s}^{\smile}, \mathsf{f}^{\smile}\} \in \mathcal{T}_R$.

The test succeeds for all $R \in \mathcal{A} - \mathcal{H}$.

Now suppose that the claim does not hold, that is, there exists a subclass $\mathcal{S}$ that contains all basic relations such that (1) $\overline{\mathcal{S}}$ does not contain one of the two relations mentioned in the lemma and (2) $\overline{\mathcal{S}} \not\subseteq \mathcal{H}$. Because of (1) and the machine-assisted case analysis, $\mathcal{S}$ cannot contain any element from $\mathcal{A} - \mathcal{H}$, hence, because all basic relations are elements of $\mathcal{H}$, we have $\mathcal{S} \subseteq \mathcal{H}$. This, however, implies $\overline{\mathcal{S}} \subseteq \overline{\mathcal{H}}$, contradicting (2). Thus, the claim must be true. □

Using the fact that $\{\prec, \mathsf{d}^{\smile}, \mathsf{o}, \mathsf{m}, \mathsf{f}^{\smile}\}, \{\prec, \mathsf{d}, \mathsf{o}, \mathsf{m}, \mathsf{s}\} \in \mathcal{B}$ and employing Lemma 6.2 again, we obtain the quite satisfying result that $\mathcal{H}$ is in fact the unique greatest tractable subclass amongst the subclasses containing all basic relations.

THEOREM 6.5. *Let $\mathcal{S}$ be any subclass of $\mathcal{A}$ that contains all basic relations. Then either*

(1) *$\mathcal{S} \subseteq \mathcal{H}$ and ISAT($\mathcal{S}$) is polynomial, or*
(2) *ISAT($\mathcal{S}$) is NP-complete.*

PROOF. If $\mathcal{S} \subseteq \mathcal{H}$, then ISAT($\mathcal{S}$) is polynomial by Theorem 3.4. So, suppose $\mathcal{S} \not\subseteq \mathcal{H}$. By Lemma 6.4 and the fact that $\mathcal{S}$ contains all basic relations, it follows that $\{\mathsf{d}, \mathsf{d}^{\smile}, \mathsf{o}^{\smile}, \mathsf{s}^{\smile}, \mathsf{f}\}$ or $\{\mathsf{d}^{\smile}, \mathsf{o}, \mathsf{o}^{\smile}, \mathsf{s}^{\smile}, \mathsf{f}^{\smile}\}$ is an element of $\overline{\mathcal{S}}$. Since $\{\prec, \mathsf{d}^{\smile}, \mathsf{o}, \mathsf{m}, \mathsf{f}^{\smile}\}, \{\prec, \mathsf{d}, \mathsf{o}, \mathsf{m}, \mathsf{s}\} \in \mathcal{B}$, and since $\mathcal{S}$ contains the basic relations, $\{\prec, \mathsf{d}^{\smile}, \mathsf{o}, \mathsf{m}, \mathsf{f}^{\smile}\}, \{\prec, \mathsf{d}, \mathsf{o}, \mathsf{m}, \mathsf{s}\} \in \overline{\mathcal{S}}$. Using Lemma 6.2, it follows that ISAT($\overline{\mathcal{S}}$) is NP-complete. By Corollary 5.3, it follows that ISAT($\mathcal{S}$) is NP-complete, which completes the proof. □

In other words, $\mathcal{H}$ presents an optimal trade-off between expressiveness and tractability [Levesque and Brachman 1987] in the framework of reasoning about qualitative temporal relations using Allen's interval algebra.

7. *Conclusion*

We have identified a new tractable subclass of Allen's interval algebra, which we call *ORD-Horn subclass* and which contains the previously identified *continuous endpoint* and *pointisable* subclasses. Enumerating the ORD-Horn subclass reveals that this subclass contains 868 elements out of 8192 elements in the full algebra, that is, more than 10% of the full algebra. Comparing this with the continuous endpoint subclass that covers approximately 1% and with the pointisable subclass that covers 2%, our result is a clear improvement in quantitative terms.

Furthermore, we showed that the "traditional" method of reasoning in Allen's interval algebra, namely, the *path-consistency method*, is sufficient for deciding satisfiability in the ORD-Horn subclass. In other words, our results indicate that the path-consistency method has a much larger range of applicability for reasoning in Allen's interval algebra than previously believed—provided we are mainly interested in satisfiability. An interesting open question is whether the upper bound of $O(n^3)$ for deciding satisfiability (see Theorem 3.5) and the upper bound of $O(n^5)$ for computing the *strongest implied relations* between all intervals (see Corollary 3.6) can be significantly strengthened for the ORD-Horn subclass.

Finally, we showed that it is impossible to improve on our results. By enumerating the minimal subalgebras strictly containing the ORD-Horn subclass, we identified two relations that allow us to prove that satisfiability in these subalgebras is NP-complete. Interestingly, the NP-completeness proofs do not make use of the relation $\{\prec, \succ\}$ that has been used in all other NP-completeness proofs for reasoning in (subclasses of) Allen's interval algebra so far. Using this result and employing the fact that NP-hardness of a subalgebra is inherited by all subclasses that generate the subalgebra, we proved that the ORD-Horn subclass is a *maximal* tractable subclass of Allen's interval algebra and even the *unique greatest* tractable subclass in the set of subclasses that contain all basic relations. In other words, the ORD-Horn subclass presents an optimal tradeoff between expressiveness and tractability.

From a practical point of view, our results may appear to be quite limited at first sight. All applications of Allen's framework we cited in this paper either require relations that are outside of the ORD-Horn subclass or use only relations that fall into the previously known pointisable subclass, so the tractability of the ORD-Horn subclass does not seem to help much. However, there are two important practical consequences of our results. First of all, our results can be used to determine the complexity of reasoning for novel applications of Allen's framework by employing the fact that the ORD-Horn subclass is the unique greatest subclass (among the subclasses containing all basic relations) that is polynomial. In other words, determining the complexity reduces to checking whether all relations fall into the ORD-Horn subclass or not. Second, provided that a restriction to the ORD-Horn subclass is not possible in an application, our results may be employed in designing faster backtracking algorithms for the full algebra [Valdéz-Pérez 1987; van Beek 1990]. Since our subclass contains significantly more relations than other tractable subclasses, the branching factor in a backtrack search can be considerably decreased if the ORD-Horn subclass is used.

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