# Proper Interval Vertex Deletion

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**Abstract.** Deleting a minimum number of vertices from a graph to obtain a proper interval graph is an *NP*-complete problem. At *WG* 2010 *van Bevern et al.* gave an $O((14k+14)^{k+1}kn^6)$ time algorithm by combining iterative compression, branching, and a greedy algorithm. We show that there exists a simple greedy $O(n+m)$ time algorithm that solves the Proper Interval Vertex Deletion problem on $\{claw, net, tent, C_4, C_5, C_6\}$-free graphs. Combining this with branching on the forbidden structures $claw, net, tent, C_4, C_5$, and $C_6$ enables us to get an $O(kn^6 6^k)$ time algorithm for Proper Interval Vertex Deletion, where $k$ is the number of deleted vertices.

## 1 Introduction

Many problems that are *NP*-hard on general graphs may be polynomial-time solvable on restricted graph classes. Examples of this are maximum induced clique and independent set in chordal graphs [8], and maximum induced forest on *AT*-free graphs [9]. Most famous of these examples is Courcelle's theorem for problems that can be expressed in monadic second order logic on graphs of bounded treewidth [6]. These algorithms can often be generalized to also work for graphs that are somehow close to one of these restricted graph classes.

The closeness of a graph to a graph class can be defined in several ways. One way is to measure the difference in edges, that is, the minimum number of edges to add or remove in order to get a specific graph class. Examples of such algorithms can be found in [17] and [13]. Alternatively the closeness can be defined as a few number of vertices that can be removed, so that a specific graph structure is obtained. A fundamental problem here is to ask if $k$ vertices can be deleted such that a clique or independent set remains. By [11] it is *NP*-complete to check if $k$ vertices can be deleted, such that a proper interval graph remains. Some of these problems are referred to as Fixed Parameter Tractable (FPT), which means that there exists an algorithm that solves the problem in $O(f(k) \cdot n^c)$ time for some function $f$ only depending on $k$ and a constant $c$. A long range of vertex deletion problems have this kind of algorithms. Some examples are deleting $k$ vertices to get an Independent Set [5], Forest [4], Chordal Graph [13], or Proper Interval Graph [16]. Not all $k$-vertex deletion problems are expected to have such an algorithm. For instance the problem of deleting $k$ vertices such that a *wheel*-free graph is obtained is $W[2]$-hard [12]. Thus, it is not expected that an FPT algorithm exists for this problem.

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All the graph classes mentioned above can be characterized by a set of forbidden induced subgraphs. Independent set is exactly the class of graphs where the vertices are pairwise nonadjacent, and forests is the class of graphs without cycles. Whereas an edge is an induced subgraph of constant size, the forbidden cycles may be of any length from 3 to $n$. Cai [3] showed that deleting $k$ vertices such that any hereditary graph class remains is FPT, as long as the number of forbidden induced subgraphs is bounded by some function of $k$. The result of Cai leaves the complexity open for all hereditary graph classes that have an infinite family of forbidden induced subgraphs. A typical example is the forest, where all induced cycles of length three or more are forbidden.

A *hole* is an induced cycle of length at least four, and chordal graphs are exactly the graphs that are *hole*-free. Despite the simple characterization it was not until 2006 that Marx settled the complexity of deleting $k$ vertices to get a chordal graph to be FPT [13]. Wegner [18] (see also Brandstädt et al. [2]) showed that proper interval graphs are exactly the class of graphs that are $\{claw, net, tent, hole\}$-free. *Claw*, *net*, and *tent* are graphs containing at most 6 vertices. Proper interval graphs are hereditary[1], so by combining the results of Wegner, Cai, and Marx, it can be shown that the problem of deleting $k$ vertices to get a proper interval graph is FPT. At WG 2010 van Bevern et al. [16] presented a new algorithm for proper interval vertex deletion using the structure of a problem instance that is already $\{claw, net, tent, C_4, C_5, C_6\}$-free. By this approach an algorithm with running time $O((14k + 14)^{k+1}kn^6)$ was obtained. Furthermore they showed that the problem of deleting a minimum number of vertices from a $\{claw, net, tent\}$-free graph to get a proper interval graph is *NP*-hard. Proper interval graphs are also known as *unit interval* graphs and *indifference* graphs.

Our result. We present a simpler and faster algorithm for the proper interval vertex deletion problem, having running time $O(6^k kn^6)$. The main result is a proof showing that a connected component of a $\{claw, net, tent, C_4, C_5, C_6\}$-free graph is a proper circular arc graph. Given the structure of a proper circular arc graph it is not hard to show that the problem can be solved by a greedy approach. The FPT algorithm is obtained by combining this algorithm with a simple branching algorithm that produces a family of $\{claw, net, tent, C_4, C_5, C_6\}$-free problem instances. Like the algorithm in [17] this algorithm is a straightforward branching algorithm, where the leaves contain polynomial-time solvable problem instances. Due to space limitations, some of the proofs are not included in this extended abstract.

## 2 Preliminaries

All graphs considered in this text are simple and undirected. For a graph $G = (V, E)$, we use $n$ as the number of vertices and $m$ as the number of edges. Two vertices $u, v \in V$ are *adjacent* if $\{u, v\} \in E$. The neighborhood of a vertex $u$

[1] Since all induced subgraphs of a proper interval graph is also $\{claw, net, tent, hole\}$-free.

is denoted $N(u)$ and vertex $v \in N(u)$ if $\{u, v\} \in E$. The *closed* neighborhood for vertex $u$ is denoted $N[u] = N(u) \cup \{u\}$. A *path* is a sequence of vertices $v_1, v_2, \ldots, v_r$ such that $\{v_i, v_{i+1}\} \in E$ for $1 \leq i < r$. The path is called *induced* if there is no edge $\{v_i, v_j\} \in E$ such that $i + 1 < j$. A *cycle* is a path $v_1, v_2, \ldots, v_r$ where $\{v_1, v_r\} \in E$, and the cycle is induced if 1 and $r$ are the only numbers for $i$ and $j$ where $i + 1 < j$ and $\{v_i, v_j\} \in E$. An induced cycle containing $4 \leq r$ vertices will be denoted $C_r$, and referred to as a *hole*.

A graph is an *interval* graph if each vertex $v \in V$ can be assigned an interval $I_v$ on the real line, such that two vertices are adjacent if and only if their corresponding intervals intersect. The collection of intervals is referred to as the interval model $\mathcal{I}$. The graph $G$ is called a *proper interval* graph if it can be represented by an interval model $\mathcal{I}$ where no interval is a sub-interval of another interval. Alternatively, a graph is a proper interval graph if and only if it is *claw*, *net*, *tent*, and *hole*-free [18]. The forbidden graphs are given in Figure 1.

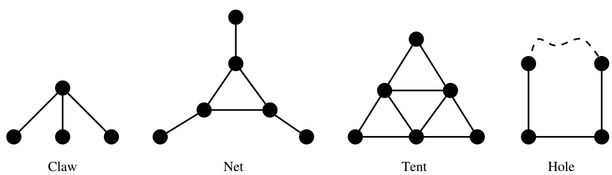


**Fig. 1.** The hole is any induced cycle of length 4 or more

A *circular arc* graph is a graph $G$, where each vertex $v \in V$ is assigned an interval (arc) $I_v$ on a circle, such that two vertices are adjacent if and only if their corresponding intervals intersect. Like for interval graphs the model $\mathcal{I}$ consist of the collection of intervals. A circular arc graph is a *proper circular arc* graph if it can be represented by a model $\mathcal{I}$ where no interval of the model is a sub-interval of another interval. Figure 2 gives a proper circular arc model of the *tent* graph.

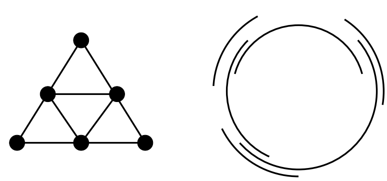

**Fig. 2.** The figure presents a proper circular arc model of the *tent* graph. The *tent* graph is a proper circular arc graph but not an interval graph or a unit circular arc graph. The three degree-two vertices of the *tent* graph from a so-called asteroidal triple, and thus by [10] it is not an interval graph. Tucker [15] has shown that the *tent* graph is not a unit circular arc graph.

For a proper circular arc model $\mathcal{I}$, *left* direction, is defined as the *counter-clockwise* direction along the circle, and *right* direction as the *clockwise* direction. Each interval $I_v$ in model $\mathcal{I}$, has a left *start* point $v^s$ and a right *end* point $v^e$.

Notice that this is well defined in a proper circular arc model, since no interval is a subset of another, and thus no interval completes the circle of the model. We will also use the notation $v^s < v^e$ to indicate that $v^s$ is to the left of $v^e$. A interval $I_v$ is left of interval $I_w$ if $v^s < w^s < v^e < w^e$, and it is possible to construct an interval $I_x$, such that $w^e < x^s < x^e < v^s$. For short we will say that vertex $v$ is left of vertex $w$ in the model if interval $I_v$ is left of interval $I_w$.

For two intersecting intervals $I_v$ and $I_w$ where $I_v$ is left of $I_w$ we define the *union* to be an interval $I_{vw}$ having start point $v^s$ and endpoint $w^e$. Let $X \subset V$ be a vertex set such that $G[X]$ is connected, then there exists a vertex ordering $v_1, v_2, \ldots, v_{|X|}$ of the vertices in $X$ such that $G[v_1, v_2, \ldots, v_i]$ is connected for $1 \leq i \leq |X|$. Let $\mathcal{X}$ be the set of intervals such that $I_v \in \mathcal{X}$ if $v \in X$. The *union* of intervals in $\mathcal{X}$ is defined as the interval $I_{1,|X|}$, where $I_{1,2}$ is the union of intervals $I_{v_1}$ and $I_{v_2}$, and by induction $I_{1,i}$ is the union of the intervals $I_{1,i-1}$ and $I_{v_i}$. We say that an interval $I_v$ *covers* the circle of the model if there exists no interval $I_w$ where $w^s < w^e$ such that $I_v$ and $I_w$ do not intersect.

In the proper circular arc model two intervals can not have the same start point since one interval would be a subset of the other. The same goes for the end point. Notice that no point on the circle is both a start and end point since in this case it is not clear if the intervals intersect or not. In the model we can assume that the $2n$ start and end points are evenly distributed over the circle.

For a connected proper circular arc graph $G$ we will define $min(\mathcal{I})$ as a minimum vertex set $X \subset V$ such that $G[V \setminus X]$ is *hole*-free.

## 3 Almost Proper Interval Graphs

An *almost proper interval* graph will be defined as a $\{claw, net, tent, C_4, C_5, C_6\}$-free graph, containing an *induced cycle* of length at least 7. The main result of this section is a proof showing that every connected component of an almost proper interval graph is a proper circular arc graph. Before giving the proof we need some properties of almost proper interval graphs.

**Proposition 1.** *Let graph $G$ be a net, where vertices $x, y, z$ are the vertices of the central clique, and $x', y', z'$ are their corresponding degree one neighbors. Then any $\{net, C_4\}$-free graph $H$ obtained by only adding edges incident to $z'$ in $G$, contains edge $\{z', x\}$ or edge $\{z', y\}$.*

**Lemma 1.** *Let $\mathcal{I}$ be a proper circular arc model of a graph $G$, and let $r$ be the length of the longest induced cycle in $G$, where $4 \leq r$. Then $\lfloor r/2+1/2 \rfloor \leq |\mathcal{X}|$ for any set of intervals $\mathcal{X}$ contained in the model $\mathcal{I}$ where the union of the intervals in $\mathcal{X}$ covers entire the circle of the model.*

**Corollary 1.** *Let $\mathcal{I}$ be a proper circular arc model of a graph $G$, and let $r$ be the length of the longest induced cycle in $G$, where $4 \leq r$. Then $\lfloor r/2 + 1/2 \rfloor \leq |C|$ for any induced cycle $C$ in $G$ where $4 \leq |C|$.*

In the claims 1 to 6 following below, we consider the case where $G$ is a connected almost proper interval graph, and $u$ is a vertex of $G$, such that graph $G' = G[V \setminus \{u\}]$ is a proper circular arc graph and not a proper interval graph. Let $\mathcal{I}$ be a circular arc model of $G'$. Since $G'$ is not a proper interval graph, it contains an induced cycle $C = w_1, w_2, \ldots, w_r$ where $7 \leq r$. Claims 1,2, and 3 have previously been proved in the special case where the graph has diameter at least 4 [1].

**Claim 1.** *Each vertex $x$ of the proper circular arc graph $G'$ is a vertex of cycle $C$ or adjacent to at least two consecutive vertices on $C$.*

**Claim 2.** *Vertex $u$ has two consecutive neighbors on the cycle $C$.*

**Claim 3.** *All neighbors of $u$ on $C$ are consecutive, and there are at most 4 of them.*

From now on let $i, j$ be numbers such that for each number $q$ where $i \leq q \leq j$ vertex $w_q$ is a neighbor of $u$.

**Claim 4.** *There exist two vertices $w_t, w_{t+1}$ on the cycle $C$ such that $w_t, w_{t+1} \in N(u)$ and $N(u) \subseteq (N(w_t) \cup N(w_{t+1}))$.*

**Claim 5.** *Vertex set $G[N(u)]$ is connected, and the union of the intervals representing vertices in $N(u)$ does not cover the entire circle of the model.*

**Claim 6.** *Every induced cycle $C$ in $G'$ where the union of the intervals representing vertices of $C$ covers the entire circle of the model contains at least 7 vertices.*

**Lemma 2.** *A connected almost proper interval graph $G$ is a proper circular arc graph.*

*Proof.* By definition, an almost proper interval graph is not a proper interval graph, and thus there exists an induced cycle of length at least 7. Let $C$ be this cycle. The proof is by induction, and the *base case* is the cycle $C$, which clearly is a proper circular arc graph and thus has a proper circular arc model. Let $v_{r+1}, v_{r+2}, \ldots, v_n$ be an ordering of the vertices $V \setminus V(C)$ such that $G[V(C) \cup \{v_{r+1}, v_{r+2} \ldots, v_k\}]$ is connected for $r+1 \leq k \leq n$. Let $G_k$ be the graph $G[V(C) \cup \{v_{r+1}, v_{r+2} \ldots, v_k\}]$. Assume now that $G_{k-1}$ is a proper circular arc graph, and let us argue that $G_k$ is also a proper circular arc graph. This will be done by modifying the proper circular arc model $\mathcal{I}_{k-1}$ of $G_{k-1}$ such that an interval $I_{v_k}$ for vertex $v_k$ can be added without violating the properties of a proper circular arc model.

By Claim 4, $N(v_k)$ contains two consecutive vertices $w_i, w_{i+1}$ of cycle $C$, where all vertices of $N(v_k)$ are contained in $N(w_i) \cup N(w_{i+1})$. Let $z_1$ be the vertex in $N(v_k)$ with the leftmost end point $z_1^e$ ($z_1 \in N[w_i]$), and let $z_2$ be the vertex in $N(v_k)$ with the rightmost start point $z_2^s$ ($z_2 \in N[w_{i+1}]$). Let $I_{v_k}$ be an interval with start point $v_k^s$ and end point $v_k^e$. Interval $I_{v_k}$ will first be placed on the model $\mathcal{I}_{k-1}$, and then the model will be adapted such that a proper circular arc model $\mathcal{I}_k$ for $G_k$ is obtained. If $\{z_1, z_2\} \notin E$ then $v_k^s$ and $v_k^e$ are placed on model $\mathcal{I}_{k-1}$ such that there exists no start or end point $p_1$ in model $\mathcal{I}_{k-1}$ where $v_k^s < p_1 < z_1^e$ and there exists no start or end point $p_2$ where $z_2^s < p_2 < v_k^e$. If $\{z_1, z_2\} \in E$ then start point $v_k^s$ and end point $v_k^e$ are placed such that there exists no start or end point $p_1$ in model $\mathcal{I}_{k-1}$ where $v_k^s < p_1 < z_2^s$ and there exists no start or end point $p_2$ where $z_1^e < p_2 < v_k^e$. If $v_k$ is adjacent to a vertex $x$ if and only if intervals $I_{v_k}$ and $I_x$ intersect, and there is no interval $I_x$ such that $v_k^s < x^s < x^e < v_k^e$ or $x^s < v_k^s < v_k^e < x^e$, then adding interval $I_{v_k}$ to model $\mathcal{I}_{k-1}$ makes a proper circular arc model for $G_k$, and thus $G_k$ is a proper circular arc graph.

Let us list the possible obstructions for getting a proper circular arc model when adding interval $I_{v_k}$. Consider now a vertex $x$ and the interval $I_x$ starting at $x^s$ and ending at $x^e$. Among the four points $z_1^e, z_2^s, x^s, x^e$ there are 4! permutations. All permutations that list $x^e$ before $x^s$ are not plausible in the model $\mathcal{I}_{k-1}$. Furthermore the cases where both $x^s$ and $x^e$ are left or right of $z_1^e$ and $z_2^s$ can be ignored, since $x \notin N(v_k)$ by the definition of $z_1$ and $z_2$, and $I_x$ does not intersect $I_{v_k}$. This leaves the eight cases listed in Figure 3.

1. $x^s < z_1^e < z_2^s < x^e$,
2. $x^s < z_1^e < x^e < z_2^s$,
3. $x^s < z_2^s < z_1^e < x^e$,
4. $x^s < z_2^s < x^e < z_1^e$,
5. $z_1^e < x^s < z_2^s < x^e$,
6. $z_1^e < x^s < x^e < z_2^s$,
7. $z_2^s < x^s < z_1^e < x^e$, and
8. $z_2^s < x^s < x^e < z_1^e$.

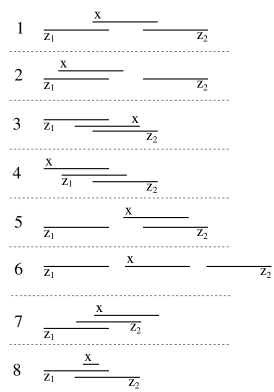

**Fig. 3.** The figure enumerates the 8 remaining cases

For each of these cases we have two sub-cases depending on the existence of the edge $\{v_k, x\}$. We will argue for the proof case by case. Before starting we can notice that due to Claim 5 the intervals $I_{z_1}$ and $I_{z_2}$ can only intersect if $z_2^s < z_1^e$. Furthermore, due to Claim 6 we need the union of at least 7 intervals in order to cover the entire circle of the model. This means that the model will appear as an proper interval model as long as at most 6 intervals are considered.

**Case 1:**$(x^s < z_1^e < z_2^s < x^e)$ $\mathbf{x} \notin \mathbf{N}(\mathbf{v_k})$. Vertices $z_1$ and $z_2$ are not adjacent, but both of them are adjacent to $x$. In this case $G[\{z_1, x, z_2, v_k\}]$ induces a $C_4$, a contradiction. $\mathbf{x} \in \mathbf{N}(\mathbf{v_k})$. If $x$ has neighbors $x_1$ and $x_2$, such that $x_1^s < x^s < x_1^e < z_1^e < z_2^s < x_2^s < x^e < x_2^e$, then $x_1, x_2 \notin N(v_k)$ by definition of $z_1$ and $z_2$. By definition vertices $x_1$ and $x_2$ are not adjacent, since $x_1^e < x_2^s$. Contradiction is now obtained since $G[\{x_1, x, x_2, v_k\}]$ is a *claw*.

Otherwise vertex $x_1$ or vertex $x_2$ do not exists, which implies that that $z_1$ is the leftmost neighbor of $x$ or $z_2$ is the rightmost. Without loss of generality assume that $z_1$ is the leftmost neighbor of $x$. This implies that only start points appear between $x^s$ and $v_k^s$, since otherwise a vertex $x_1$ would exists. Let $y_1^s$ be the point directly to the left of $v_k^s$ on the model $\mathcal{I}_{k-1}$ where points $v_k^s$ and $v_k^e$ are added. Interval $I_{y_1}$ is not a sub-interval of $I_x$, so $x^s < y_1^s < z_1^e < z_2^s < x^e < y_1^e$. Move $v_k^s$ to the left of $y_1^s$, and obtain a model with one obstruction less. Repeat until no such obstruction exists.

**Case 2:**$(x^s < z_1^e < x^e < z_2^s)$ $\mathbf{x} \notin \mathbf{N}(\mathbf{v_k})$. If vertices $N_{G_{k-1}}[z_1] = N_{G_{k-1}}[x]$ then we can simply swap the intervals assigned to $z_1$ and $x$ such that $z_1$ is assigned interval $I_x$ and $x$ is assigned interval $I_{z_1}$. After the swap there will be one obstruction less, since $x^s < z_1^s < x^e < v_k^s < z_1^e$ and thus $I_x$ do not intersect $I_{v_k}$. Note that the swap may result in a vertex $z_1'$ being the left most neighbor of $v_k$, but any such vertex have a starting and end points before the swap where $z_1^s < z_1'^s < x^s < z_1^e < z_1'^e < x^e$ and thus $N_{G_{k-1}}[z_1] = N_{G_{k-1}}[x] = N_{G_{k-1}}[z_1']$. So $N_{G_{k-1}}[z_1] \neq N_{G_{k-1}}[x]$. There are two cases, either there exists a vertex $z_{1'} \in N(z_1) \setminus N(x)$ or there exists a vertex $x' \in N(x) \setminus N(z_1)$. Vertex $z_{1'}$ has its interval assigned to the left of the interval for $z_1$, so by the definition of $z_1$, we get $z_{1'} \notin N(v_k)$, and thus we have a contradiction since $G[\{z_{1'}, z_1, x, v_k\}]$ is a *claw*. In the remaining case $x$ has a neighbor $x'$ to the right not adjacent to $z_1$. Vertex $x'$ is not adjacent to $v_k$, since this would make $G[\{v_k, z_1, x, x'\}]$ an induced 4-*cycle*. By the existence of the cycle $C$, there exists a vertex to the left of $z_1$ in the model. Let $x_\ell$ be the rightmost vertex to the left of $z_1^s$ and let $w_\ell$ be the vertex of $C$ such that $x_\ell^s < w_\ell^s < x_\ell^e < z_1^s < x^s < w_\ell^e$. Interval $I_{w_\ell}$ can not intersect interval $I_{x'}$ to the right since this makes $I_{z_1}$ a sub-interval of $I_{w_\ell}$, and by Claim 6, it can not intersect to the left. Interval $I_{x_\ell}$ can only intersect $I_{w_\ell}$ to the right, and by Claim 6, $I_{x_\ell}$ does not intersect $I_x$ or $I_{x'}$ to the left. Now we have a contradiction since $G[\{x_\ell, w_\ell, z_1, x, x', v_k\}]$ is a *net*. $\mathbf{x} \in \mathbf{N}(\mathbf{v_k})$. We only have to argue that $I_x$ is not a sub-interval of $I_{v_k}$, since $I_{v_k}$ contains point $z_2^s$ and $I_x$ does not contain this point. Initially $v_k^s$ is just left of $z_1^e$ and $v_k^e$ is just to the right of $z_2^s$, so in this case $I_x$ is not a sub-interval of $I_{v_k}$. Since we only move $v_k^s$ to the left or $v_k^e$ to the right of a point $y^s$ or $y^e$ in the case where $I_{v_k}$ is a sub-interval of $I_y$, this will never occur for $I_x$ and vertex $x$.

**Case 3:**$(x^s < z_2^s < z_1^e < x^e)$ $\mathbf{x} \notin \mathbf{N}(\mathbf{v_k})$. Let $v_a$ be the rightmost vertex such that $v_a^e < z_1^s$, let $v_b$ be the leftmost vertex such that $z_2^e < v_b^s$, let $w_a$ and $w_b$ be vertices of $C$ such that $w_a^s < v_a^e < z_1^s < w_a^e$ and $w_b^s < z_2^e < v_b^s < w_b^e$. Vertices $w_a, w_b, v_a, v_b$ exists due to the fact that the intervals of $C$ cover the entire circle of the model, and $v_a, v_b$ are the closest vertices on either side. Vertex $w_b$ is adjacent to $z_2$ but also to $x$, since otherwise $G[\{x, z_2, v_k, w_b\}]$ is a *claw*. Vertex $w_a$ is adjacent to $z_1$ but also to $x$, since otherwise $G[\{w_a, z_1, x, v_k\}]$ is a *claw*. One of the edges $\{w_a, z_2\}, \{w_a, w_b\}, \{z_1, w_b\}$ are present since otherwise $G[\{w_a, x, w_b, z_1, z_2, v_k\}]$ is a *tent*. If edge $\{w_a, z_2\}$ is present, then $G[\{w_a, z_2, w_b, v_k\}]$ is a *claw* unless edge $\{w_a, w_b\}$ is present. If edge $\{w_b, z_1\}$ is present, then $G[\{w_a, z_1, w_b, v_k\}]$ is a *claw* unless edge $\{w_a, w_b\}$ is present. Thus, we can conclude that edge $\{w_a, w_b\}$

is present. Edge $\{w_a, z_2\}$ is present to prevent $G[\{w_a, z_2, w_b, v_b\}]$ from inducing a *claw*, and edge $\{w_b, z_1\}$ is present to prevent $G[\{w_a, z_1, w_b, v_a\}]$ from inducing a *claw*. Now we have a contradiction since $G[\{v_a, w_a, w_b, v_b, z_1, v_k\}]$ induces a *net*. $\mathbf{x} \in \mathbf{N}(\mathbf{v_k})$. We have to argue that $I_{v_k}$ is not a sub-interval of $I_x$. There are no two vertices $x_1$ and $x_2$ such that $x_1^s < x^s < x_1^e < v_k^s$ and $v_k^e < x_2^s < x^e < x_2^e$, since $G[\{x_1, x, x_2, v_k\}]$ is a *claw*. Let us without loss of generality assume that all points between $x^s$ and $v_k^s$ are start points. Let $y^s$ be the point to the left of $v_k^s$ in the model, then $x^s < y^s < v_k^s < v_k^e < x^e < y^e$. Vertex $y \notin N(v_k)$ is covered by first part of this case. Swap points $y^s$ and $v_k^s$ and reduce the number of conflicting vertices by one. Repeat until $x^s$ and $v_k^s$ are swapped.

**Case 4:**$(x^s < z_2^s < x^e < z_1^e)$ $\mathbf{x} \notin \mathbf{N}(\mathbf{v_k})$. Let $v_a$ be the rightmost vertex such that $v_a^e < x^s$, let $v_b$ be the leftmost vertex such that $z_2^e < v_b^s$, let $w_a$ and $w_b$ be vertices of $C$ such that $w_a^s < v_a^e < x^s < w_a^e$ and $w_b^s < z_2^e < v_b^s < w_b^e$. Vertices $w_a$ and $w_b$ exist due to the fact that the intervals of $C$ cover the entire circle of the model, and $v_a, v_b$ are the closest vertices on either side. Vertex $w_b$ is adjacent to $z_2$ and thus also to $x$, since otherwise $G[\{x, z_2, v_k, w_b\}]$ is a *claw*. Vertex $w_a$ is adjacent to $x$. In order to prevent $G[\{w_a, x, w_b, v_b, z_2, v_k\}]$ from inducing a *net*, edge $\{w_a, w_b\}$ or edge $\{w_a, z_2\}$ is present since $v_b$ is only adjacent to $w_b$, and $v_k$ is only adjacent to $z_2$. Edge $\{w_a, w_b\}$ makes $G[\{w_a, w_b, v_b, z_2\}]$ a *claw* and forces edge $\{w_a, z_2\}$, and edge $\{w_a, z_2\}$ makes $G[\{w_a, w_b, z_2, v_k\}]$ a *claw* and forces edge $\{w_a, w_b\}$. In order to prevent $G[\{v_a, w_a, x, w_b, v_b, z_1\}]$ from inducing a *net*, one of the edge $\{w_a, z_1\}$ or edge $\{z_1, w_b\}$ is present since $v_b$ is only adjacent to $w_b$, $v_a$ is only adjacent to $w_a$. Edge $\{w_a, z_1\}$ makes $G[\{w_a, w_b, v_a, z_1\}]$ a *claw* and forces edge $\{z_1, w_b\}$, and edge $\{z_1, w_b\}$ makes $G[\{w_a, w_b, z_1, v_b\}]$ a *claw* and forces edge $\{w_a, z_1\}$. Now we have a contradiction since $G[\{v_a, w_a, w_b, v_b, z_1, v_k\}]$ induces a *net*. $\mathbf{x} \in \mathbf{N}(\mathbf{v_k})$. This is a contradiction to the definition of $z_1$ as the leftmost neighbor of $v_k$.

**Case 5:**$(z_1^e < x^s < z_2^s < x^e)$ This case is symmetric to Case 2.

**Case 6:**$(z_1^e < x^s < x^e < z_2^s)$ $\mathbf{x} \notin \mathbf{N}(\mathbf{v_k})$. Vertices $z_1$ and $z_2$ are not adjacent to each other or to vertex $x$. By Claim 4, there exists one or two vertices of $C \cap N(v_k)$ that make a path from $z_1$ to $z_2$. No vertex in $N(v_k)$ has an interval starting before $z_1^e$ and ending after $z_2^s$ since this vertex would have a super-interval of $I_x$ in the model $\mathcal{I}_{k-1}$. Thus, there is an induced path $z_1, w_t, w_{t+1}, z_2$ in $N(v_k)$ such that $w_t, w_{t+1}$ are vertices of $C$. Interval $I_x$ is not a sub-interval of $I_{w_t}$ or $I_{w_{t+1}}$, so $x$ is adjacent to both $w_t$ and $w_{t+1}$. Now we have a contradiction since $G[\{z_1, z_2, w_t, w_{t+1}, x, v_k\}]$ is a *tent*. $\mathbf{x} \in \mathbf{N}(\mathbf{v_k})$. Vertices $z_1$ and $z_2$ are not adjacent to each other or to vertex $x$. This is a contradiction since $G[\{z_1, x, z_2, v_k\}]$ is a *claw*.

**Case 7:**$(z_2^s < x^s < z_1^e < x^e)$ This case is symmetric to Case 4.

**Case 8:**$(z_2^s < x^s < x^e < z_1^e)$ This case is not plausible since $I_x$ is a sub-interval of both $I_{z_1}$ and $I_{z_2}$, which is a contradiction to $\mathcal{I}_{k-1}$ being a proper circular arc model of $G_{k-1}$. □

## 4 FPT Algorithm for Proper Interval Graphs

First we give a polynomial-time algorithm that finds a vertex set of minimum cardinality in an almost proper interval graph, such that the removal of this set makes the graph *hole*-free. Figure 4 gives a full description of the algorithm for general graphs, including the branching to get the set of $\{claw, net, tent, C_4, C_5, C_6\}$-free problem instances. The problem addressed is defined as follows:

PROPER INTERVAL $k$-VERTEX DELETION PROBLEM
**Problem instance:** A graph $G$, and parameter $k$
**Question:** Does there exist a vertex set $X \subset V$ such that $|X| \leq k$ and $G[V \setminus X]$ is a proper interval graph?

For a connected almost proper interval graph $G = (V, E)$, a vertex set $X$ such that $G[V \setminus X]$ is a proper interval graph is called a *hole cut.* A *minimal* hole cut is a hole cut $X$, such that $G[V \setminus X']$ contains a hole for every proper subset $X'$ of $X$.

**Lemma 3.** *A connected almost proper interval graph $G$ contains at most $n$ minimal hole cuts.*

**Lemma 4.** *For a connected almost proper interval graph $G$, there exists an algorithm that finds a minimum hole cut $X$ in $O(n + m)$ time.*

**Theorem 1.** *There exists an algorithm that solves the* PROPER INTERVAL $k$-VERTEX DELETION PROBLEM *in $O(kn^6 \cdot 6^k)$ time.*

$MaxProperIntervalSubgraph(G, k)$
**Input:** A graph $G$, and parameter $k$
**Output:** A vertex set $X$ of size at most $k$
such that $G[V \setminus X]$ is a proper interval graph or answer "No"

```
if G[U] is a claw, net, tent, C4, C5 or C6 for U ⊂ V then
    if 0 < k then
        for each vertex v of U
            X = MaxProperIntervalSubgraph(G[V \ {v}], k − 1)
            if X ≠ "No" then return X ∪ {v}
    else
        return "No"
else
    X = ∅
    for each connected component C of G
        if G[C] is not a proper interval graph then
            Let I be a proper circular arc model of G[C]
            X = X ∪ min(I)
    if |X| ≤ k then return X
    else return "No"
```

**Fig. 4.** The algorithm checks if there exists a vertex set $X$ of size at most $k$, such that $G[V \setminus X]$ is a proper interval graph

# 5 Conclusion

By recognizing that $\{claw, net, tent, C_4, C_5, C_6\}$-free graphs are indeed the disjoint union of proper circular arc graphs, we have obtained a simple $O(6^k kn^6)$ time algorithm for proper interval vertex deletion.

Finally we can mention some related open problems. Is the problem of deleting $k$ vertices from a graph $G$ to get an interval graph or proper circular arc graph fixed parameter tractable? Deciding the complexity for interval graphs is probably the more interesting of the two.

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