CLIQUE GRAPHS OF CHORDAL AND PATH GRAPHS

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Abstract. Clique graphs of chordal and path graphs are characterized. A special class of graphs named **expanded trees** is discussed. It consists of a subclass of disk-Helly graphs. It is shown that the clique graph of every chordal (hence path) graph is an expanded tree. In addition, every expanded tree is the clique graph of some path (hence chordal) graph. Different characterizations of expanded trees are described, leading to a polynomial time algorithm for recognizing them.

Key words. Algorithms, Chordal Graphs, Clique Graphs, Path Graphs

AMS subject classifications. 05C12, 05C75, 05C85

1. Introduction. We examine clique graphs of chordal graphs. Bandelt and Prisner [2] proved that they are disk-Helly. Chen and Lih [4] and independently Bandelt and Prisner [2] showed that the second iterated clique graph of a chordal graph is again chordal. Here it is shown that clique graphs of chordal graphs correspond to a class named expanded trees, in the sense that the clique graph of a chordal graph is always an expanded tree and every expanded tree is the clique graph of some chordal graph. In addition, the class of clique graphs of (undirected) path graphs is no more restricted than that of chordal graphs. Every expanded tree is also the clique graph of some path graph. Expanded trees are characterized and a polynomial time recognition algorithm is described.

Expanded trees are closely related to dismantlable graphs. The latter were examined by Bandelt and Prisner [2], Prisner [9] and Nowakovski and Winkler [8]. Disk-Helly graphs are a subclass of dismantlable graphs, and can be recognized in polynomial time, according to an algorithm by Bandelt and Pesch [1]. See also Nowakovski and Rival [7] and Quilliot [10].

G denotes a simple undirected graph, $V(G)$ and $E(G)$ are its vertex and edge sets, respectively, $n = |V(G)|$ and $m = |E(G)|$. $N(v)$ is the set of vertices adjacents to $v \in V(G)$, while $N[v] = N(v) \cup \{v\}$. The vertex $v \in V(G)$ is **dominated** by $w \in V(G)$ **in** G when v,w are distinct and $N[v] \subset N[w]$. A **clique** is a subset of vertices inducing a complete subgraph in G. Let F be a family of subsets of some set. The **intersection graph** of F is a graph whose vertices are associated to the subsets of F, two vertices being adjacent if the corresponding pair of subsets intersect. The **clique graph** $K(G)$ of G is the intersection graph of the maximal cliques of G. A **chordal graph** G is the intersection graph of subtrees of a tree T. The subtree of T corresponding to a vertex $v \in V(G)$ is called **representative subtree** of v and denoted by $T(v)$. The tree T together with the representative subtrees form a **tree representation** of G. A **minimal representation** is a tree representation such that $|V(T)|$ is the least possible. Gavril [5] and Buneman [3] showed that a minimal representation is precisely one in which each vertex of T corresponds to a maximal

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clique of G. In addition, for each $v \in V(G)$, the subtree T(v) is formed exactly by the vertices of T corresponding to those maximal cliques of G which contain v. A **path graph** is the intersection graph of paths of a tree. Monma and Wei [6] characterized path graphs and variations of this class in terms of their minimal representations.

Let F be a family of subsets S_i of some set. F satisfies the **Helly property** when every subfamily F' of F in which $S_i \bigcap S_j \neq \emptyset$, for all pairs $S_i, S_j \in F'$, is such that $\bigcap_{S_i \in F'} S_i \neq \emptyset$. Finally, G is a **dismantlable graph** if there exists a sequence $v_1, ..., v_n$ of its vertices such that, for $i < n$, v_i is dominated in $G - \{v_1, ..., v_{i-1}\}$. If additionally the maximal cliques of G satisfy the Helly property then G is **disk-Helly**.

2. Expanded Trees. G is an **expanded tree** when it admits a spanning tree $T(G)$, such that for each edge $(v, w) \in E(G)$ the vertices of the $v - w$ path in T form a clique in G. In this case, $T(G)$ is a **canonical tree** of G.

THEOREM 2.1. *The following are equivalent.*

(i) *G is the clique graph of some connected chordal graph H.*

(ii) *G admits a spanning tree T, such that for each $v \in V(G)$, $N_G[v] \cap T$ is a (connected) subtree of T.*

(iii) *G is an expanded tree.*

Proof. (i)$\Rightarrow$(ii): Let H be a chordal graph, T a minimal representation of it and $G = K(H)$. Let G' be the graph obtained from T by adding exactly the edges that transforms each representative subtree $T(w)$ into a $|T(w)|$-clique. $V(G) \simeq V(G')$. Let M_1, M_2 be two maximal cliques of H, v_1, v_2 and v'_1, v'_2 their corresponding vertices in G and G', respectively. Suppose $(v_1, v_2) \in E(G)$. Then there exists $w \in V(H)$ such that $w \in M_1 \cap M_2$. In consequence, the subtree $T(w)$ of the minimal representation of H contains $v'_1, v'_2 \in V(G')$. Hence $(v'_1, v'_2) \in E(G')$. Conversely, if $(v'_1, v'_2) \in E(G')$ then there is a representative subtree $T(w)$ containing v'_1, v'_2. That is, $w \in M_1 \cap M_2$ and consequently $(v_1, v_2) \in E(G)$. Hence $G \simeq G'$ and T is a spanning tree of G. For any $v \in V(G)$, each $u \in N_G[v]$ corresponds to a subtree of T containing v. Hence $N_G[v] \cap T$ is connected.

(ii)$\Rightarrow$(iii): It suffices to show that T is a canonical tree of G. Let $(v, w) \in E(G)$ and $v = v_0, ..., v_r = w$ be the $v - w$ path in T. Suppose by induction that $\{v_0, ..., v_{r-1}\}$ is a clique of G, $r > 1$. Since v and w are adjacent and $N_G[v] \cap T$ is connected it follows that $v_0, .., v_{r-1}$ are all adjacent to w. Consequently $\{v_0, ..., v_r\}$ is a clique of G, that is G is an expanded tree.

(iii)$\Rightarrow$(i): Given an expanded tree G, we construct a connected chordal graph H such that $G = K(H)$. Let M be the set of maximal cliques of G. Define H as the intersection graph of the elements of $M \cup V(G)$. Denote by T a canonical tree of G. We show that each maximal clique $C \in M$ of G induces a subtree in T, which implies that H is the intersection graph of subtrees of a tree. Let v, z be two vertices of C, and P the $v - z$ path in T. For each vertex w of C, the paths $w - v$ and $w - z$ cover P. Since (w, v) and (w, z) are edges of the expanded tree G, it follows that w is adjacent to every vertex of P. By maximality of C, each vertex of P belongs to C. Hence $C \cap T$ is a subtree of T and consequently H is a chordal graph. It remains to show that $G = K(H)$. Clearly, each vertex v of T corresponds to a maximal clique of H, namely that formed by the maximal cliques of G which contain v and by v itself. That is, $V(G) \simeq V(K(H))$. Let C_1, C_2 be two maximal cliques of H and v_1, v_2 their corresponding vertices in G. If $C_1 \cap C_2 \neq \emptyset$ then there is a maximal clique C of G such that $C \in V(H)$ is contained in $C_1 \cap C_2$. In other words, v_1, v_2 are vertices of G belonging to the same clique C. That is, $(v_1, v_2) \in E(G)$. Conversely, if $C_1 \cap C_2 = \emptyset$ then there is no maximal clique $C \in V(H)$ of G containing both $v_1, v_2 \in V(G)$. That is, $(v_1, v_2) \notin E(G)$ and consequently $G = K(H)$. $\square$

COROLLARY 2.2. *G is an expanded tree iff it is the clique graph of some connected path graph.*

Proof. Let G be an expanded tree. Construct a path graph H such that $G = K(H)$. By Theorem 1, there exists a chordal graph F such that $G = K(F)$. Let T be a minimal representation of F. Substitute each representative subtree (of a vertex of F) by the collection of all paths between its leaves. The intersection graph of all those paths is a path graph H with minimal tree representation T and such that $K(H) = K(F)$. The converse is clear. $\square$

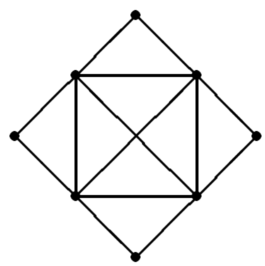

Figure 1

Note the following relation between expanded trees and disk-Helly graphs. The clique graph of every chordal graph is disk-Helly [2]. Hence expanded trees are also disk-Helly, by theorem 1. The inclusion is proper, as the graph of Figure 1 is disk-Helly and not an expanded tree.

3. Recognition of Expanded Trees. A sequence S_k of vertices $v_1, ..., v_k$, $k \leq n$, of a graph G is **canonical** when for each $1 \leq i \leq k$, either $i = n$ or v_i is dominated by some vertex v_j, $i < j$, in the graph $G(S_i)$, defined as

$V(G(S_i)) = V(G)$

$E(G(S_i)) = E(G) - \{(x, y) \in E(G) \text{ s.t. } x \in \{v_1, ..., v_{i-1}\},\ y \in \{v_i, ..., v_n\}$ and $|N_G(x) \cap \{v_i, ..., v_n\}| = 1\}$

Call the vertex v_i **canonical** in $G(S_i)$. The value k is the **length** of S_k. If $k = n$ then S_k is **complete**. S_k is **maximal** when it is complete or otherwise $G(S_{k+1})$ has no canonical vertex. Clearly, if S_k is canonical so is $S_i, i < k$. Define $G(S_0) = G$.

Expanded trees can also be characterized as follows.

THEOREM 3.1. *G is an expanded tree iff it admits a complete canonical sequence.*

Proof. $(\Rightarrow)$: Let T be a canonical tree of G. Let S_n be a sequence $v_1, ..., v_n$ of the vertices of G, such that v_i is a leaf of T_i, $1 \leq i \leq n$, where $T_1 = T$ and for $i > 1$, $T_i = T_{i-1} - v_{i-1}$. We show by induction that S_i is canonical. Assume it true for all subsequences of length $< i$. If $i = n$ there is nothing to prove. Otherwise, let v_j, $j > i$, be the vertex adjacent to v_i in T_i. We claim that v_j dominates v_i in $G(S_i)$. This is clear for $i = 1$. When $i > 1$ suppose the claim false. Then there is a vertex v_p, such that $(v_p, v_i) \in E(G(S_i))$ and $(v_p, v_j) \notin E(G(S_i))$. T is canonical. Hence the following two conditions must hold:

$p < i$ and $(v_p, v_q) \notin E(G)$ for all $q > i$, otherwise (v_p, v_j) would belong to $E(G(S_i))$, a contradiction.

In this case, $|N_G(v_p) \cap \{v_i, ..., v_n\}| = 1$, and $(v_p, v_i) \notin E(G(S_i))$, again a contradiction. Therefore v_j dominates v_i in $G(S_i)$, that is, S_i is canonical.

$(\Leftarrow)$: Let $v_1, ..., v_n$ be a canonical sequence of G. Each v_i, $i < n$, has a dominator v_j in $G(S_i)$, $i < j$, and write $v_j = dom(v_i)$. Let T be a graph defined as

$V(T) = V(G)$

$E(T) = \{(v_i, dom(v_i)) \text{ s.t. } 1 \leq i \leq n\} \subset E(G)$.

When $n > 1$, every vertex of G is incident to some edge of T. Since $|E(T)| = n-1$, T is a spanning tree of G. We show that T is canonical. Suppose the contrary. Then G contains an edge (v_a, v_b) such that the $v_a - v_b$ path P in T is not a clique of G. We can choose (v_a, v_b) such that $(v_i, v_b) \in E(G)$ for all vertices v_i of P, except the neighbor v_c of v_a in P, which satisfies $(v_c, v_b) \notin E(G)$. Let v_d, $d \neq a, b$, be the second adjacent vertex to v_c in P. Let $\vec{T}$ be the in-tree obtained by directing each edge $(v_i, dom(v_i))$ of T from v_i to $dom(v_i)$.

Case 1: $a < b$

The edge (v_a, v_c) must be oriented in $\vec{T}$ from v_a to v_c. Otherwise, $\vec{T}$ contains the directed path $v_b - v_a$, implying $b < a$, a contradiction. Then $a < c$ and $(v_a, v_c) \in E(\vec{T})$ implies that v_c dominates v_a in $G(S_a)$. Since $a < b$, $(v_a, v_b) \in E(G(S_a))$. Hence $(v_b, v_c) \in E(G)$, a contradiction.

Case 2: $a > b$

Suppose (v_a, v_c) is directed from v_c to v_a in $\vec{T}$. Then P is directed from v_b to v_a. That is $a > c > d > b$ and consequently $(v_a, v_b), (v_d, v_b) \in E(G(S_d))$. But v_c dominates v_d in $G(S_d)$. Hence $(v_c, v_b) \in E(G)$, a contradiction.

Finally, let (v_a, v_c) be directed from v_a to v_c in $\vec{T}$. Since $a > b$, P contains some vertex v_i, $i \neq a, b$, having in-degree > 1. Such v_i satisfies $i > a$ and $(v_a, v_b), (v_i, v_b) \in E(G(S_a))$. But v_c dominates v_a in $G(S_a)$. Hence $(v_c, v_b) \in E(G)$, again a contradiction. □

LEMMA 3.2. *All maximal sequences have the same length.*

Proof. Suppose the contrary. Then there is a graph G with two maximal sequences $S_k = \{v_1, ..., v_k\}$ and $S'_\ell = \{v'_1, ..., v'_\ell\}$ satisfying $0 < k < \ell$. Clearly, $0 = k < \ell$ can not occur, as v'_1 is canonical en $G(S_0) = G$. If $S_k = S'_k$ then S_k is not maximal and the lemma holds. Otherwise, let q be the smallest integer satisfying $v_q \neq v'_q$. We construct a canonical sequence S''_k in which $S''_{q+1} = S_{q+1}$ and such that S''_k is maximal *iff* S_k is so, leading to a contradiction.

In general, for a vertex v_i, let $dom(i, S_j)$ denote its set of dominators in $G(S_j)$, not belonging to S_{j-1}. First, we show that if w, x are vertices $\notin S_{j-1}$ such that $w \in dom(x, S_i)$ then $w \in dom(x, S_j)$, for $i < j$. Suppose this domination condition is not true. Then either $(x, w) \notin E(G(S_j))$ or there exists some vertex y adjacent to x and not to w in $G(S_j)$. But $w \in dom(x, S_i)$. Then $(x, w) \in E(G(S_i))$. Since $i < j$ and $x, w \notin S_{j-1}$ it follows $(x, w) \in E(G(S_j))$. For the second alternative, suppose (y, x) is an edge of $G(S_j)$. Then (y, x) is an edge of $G(S_\ell)$, $\ell < j$. That is, y is incident in $G(S_\ell)$ to at least two vertices $w, x \notin S_{j-1}$. This implies (y, w) to be an edge of $G(S_j)$, a contradiction. Hence w dominates x in S_j and the assertion is proved.

Examine the unmatched vertex v'_q. The following can occur.

Case 1: $v'_q \in S_k$

Then $v'_q = v_j$, for some $j > q$.

Case 1.1: $dom(v_j, S_q) - S_{j-1} \neq \emptyset$

Let $w \in dom(v_j, S_q) - S_{j-1}$. Let S''_k be the sequence obtained from S_k by moving v_j to the $q-$th position, while maintaining the relative ordering of the remaining vertices. We show that S''_k is canonical. Let $z \in dom(v_i, S_i)$, $q \leq i < j$. If $z \neq v_j$ it follows that z also dominates $v_i = v''_{i+1}$ in $G(S''_{i+1})$. Consider $z = v_j$. By the above domination preserving condition, $w \in dom(v_j, S_j)$. If there is v_ℓ, $q + 1 \leq \ell \leq j - 1$, such that $v_j \in dom(v_\ell, S_\ell)$ then because $w \in dom(v_j, S_\ell)$ it follows $w \in dom(v_\ell, S_\ell)$. Hence S''_k is canonical. In addition, the vertices of S''_k and S_k are the same, i.e., S''_k is maximal. However, $S''_{q+1} = S'_{q+1}$, while $S_{q+1} \neq S'_{q+1}$.

Case 1.2: $dom(v_j, S_q) - S_{j-1} = \emptyset$

By the domination condition, there exists some $v_i \in \{v_q, ..., v_{j-1}\} \cap dom(v_j, S_i)$. Moreover, choose v_i so that no v_ℓ, $\ell = i+1, ..., j-1$, dominates v_j in $G(S_\ell)$. Let $S^\star_k$ be the sequence obtained by interchanging the positions of v_i and v_j in S_k. Let $z \in dom(v_i, S_i)$. If $z = v_j$ then $N_{G(S_i)}[v_i] = N_{G(S_i)}[v_j]$ and $S^\star_k$ is canonical. If $z \neq v_j$ then $z \neq v_\ell$, $\ell = i+1, ..., j-1$, otherwise, v_ℓ dominates v_j in $G(S_\ell)$, a contradiction. Hence $v_j \in dom(v_p, S_p)$ implies $z \in dom(v_p, S_p)$, $i \leq p \leq j$. Therefore $S^\star_k$ is canonical and maximal. Then Case 1.1 applies.

Case 2: $v'_q \notin S_k$

Let $w \in dom(v'_q, S_q)$. Then $w = v_i$, for some $i \leq k$, otherwise S_k is not maximal, by the domination property. Moreover, choose v_i such that no v_ℓ, $\ell > i$ dominates v'_q in $G(S_\ell)$. Let $z \in dom(v_i, S_i)$. Suppose $z \neq v'_q$. It follows $z \in dom(v'_q, S_i)$. If $z \notin S_k$ then $z \in dom(v'_q, S_k)$ and S_k is not maximal. Then $z \in S_k$. But, now $z = v_\ell$ for some $\ell > i$ and $v_\ell \in dom(v'_q, S_\ell)$, a contradiction. Hence $z = v'_q$. In this case $N_{G(S_i)}[{v_q}'] = N_{G(S_i)}[v_i]$. Replace v_i by v'_q in S_k. The new sequence so obtained is also canonical and maximal. Then Case 1 applies. $\square$

Theorem 2 and Lemma 1 lead to a greedy algorithm for recognizing expanded trees. Construct a maximal canonical sequence S_k of vertices $v_1, ..., v_k$ of the graph G. Clearly, G is an expanded tree *iff* $k = n$. For $i < n$ each v_i can be arbitrarily chosen among the dominated vertices in $G(S_i)$, if existing. The algorithm terminates within $O(n^2 m)$ steps. A canonical tree T can be obtained as a by-product: for $i < n$, include in $E(T)$ the edge (v_i, w) where w is the dominator of v_i in $G(S_i)$.

NOTE: A referee has pointed out that expanded trees were before investigated by F.F.Dragan in his Ph.D. thesis (Centers of Graphs and the Helly Property, Minsk, 1989, in russian) under the name *HT-graphs*, where the equivalence (ii)⇔(iii) of Theorem 1 was already established. We are grateful to the referees for the suggestions that improved this article. In special, H. Bandelt provided new proofs of Corollary 1 and part of Theorem 1, which have been adopted.

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