Generalization of the Consecutive-ones Property

A THESIS

submitted by

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TABLE OF CONTENTS

[c1] give problem definition etc

[c2] *minor*: (i)make names in bib file uniform style w.r.t. firstname/initials (ii) have back refs - i.e. pages that cite the item.

LIST OF TABLES

LIST OF FIGURES

CHAPTER 2

Consecutive-ones Property – A Survey of Important Results

This chapter surveys several results that are significant to this thesis or to COP in general. These predominantly pertain to characterizations of COP, algorithmic tests to check for COP (COT), optimization problems on binary matrices that do not have COP and some applications of COP.

c1

c1 *important*: Al have a few lines about organization of chapter

2.1 COP in Graph Theory

COP is closely connected to several types of graphs by way of describing certain combinatorial graph properties. There are also certain graphs, like convex bipartite graphs, that are defined solely by some of its associated matrix having COP. In this section we will see the relevance of consecutive-ones property to graphs. To see this we introduce certain binary matrices that are used to define graphs in different ways. While adjacency matrix is perhaps the most commonly used such matrix, Definition 2.1.1 defines this and a few more.

Definition 2.1.1:
Matrices that define graphs. [Dom08, Def. 2.4]. Let G and H be defined as follows. $G = (V, E_G)$ is a graph with vertex set $V = \{v_i \mid i \in [n]\}$ and edge set $E_G \subseteq \{(v_i, v_j) \mid i, j \in [n]\}$ such that $|E_G| = m$. $H = (A, B, E_H)$ is a bipartite graph with partitions $A = \{a_i \mid i \in [n_a]\}$ and $B = \{b_i \mid i \in [n_b]\}$.

2.1.1–i. *Adjacency matrix* of G is the symmetric $n \times n$ binary matrix M with $m_{i,j} = \mathbf{1}$ if and only if $(v_i, v_j) \in E_G$ for all $i, j \in [n]$.

2.1.1–ii. *Augmented adjacency matrix* of G is obtained from its adjacency matrix by setting all main diagonal elements to $\mathbf{1}$, i. e. $m_{i,i} = \mathbf{1}$ for all $i \in [n]$.

2.1.1–iii. *Maximal clique matrix* or *vertex-clique incidence matrix* of G is the $n \times k$ binary matrix M with $m_{i,j} = \mathbf{1}$ if and only if $v_i \in C_j$ for all $i \in [n], j \in [k]$ where $\{C_j \mid j \in [k]\}$ is the set of maximal cliques of G.

2.1.1–iv. *Half adjacency matrix* of H is the $n_a \times n_b$ binary matrix M with $m_{i,j} = \mathbf{1}$ if and only if $(a_i, b_j) \in E_H$.

Now we will see in Definition 2.1.2 certain graph classes that is related to COP or CROP.

Definition 2.1.2:
Graphs that relate to COP.[Dom08, Def. 2.5] Let G be a graph and H be a bipartite graph.

2.1.2–i. G is *convex-round* if its adjacency matrix has the CROP.

2.1.2–ii. G is *concave-round* if its augmented adjacency matrix has CROP. c1 c2

c1 cite BHY00

c2 *minor*: add CROP to glossary

2.1.2–iii. G is an *interval graph* if its vertices can be mapped to intervals on the real line such that two vertices are adjacent if and only if their corresponding intervals overlap c3. G is an interval graph if and only if its maximal clique matrix has COP [FG65][1]

c3 cite Ben59, Haj57

a. G is a *unit interval graph* if it is an interval graph such that all intervals have the same length.[2]

b. G is a *proper interval graph* if it is an interval graph such that no interval properly contains another.[2] c4

c4 cite rob69,gar07 in endnote. pg 33 dom

2.1.2–iv. G is a *circular-arc graph* if its vertices can be mapped to a set of arcs on a circle such that two vertices are adjacent if and only if their corresponding arcs overlap.c5

c5 *pressing*: how is CO/ROP related?

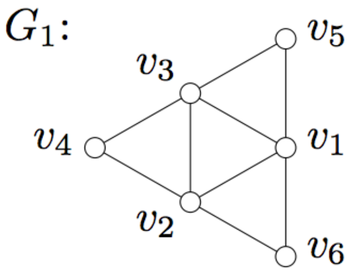
2.1.2–v. H is *convex bipartite on columns (rows)* if its half adjacency matrix has COP on rows (columns).

2.1.2–vi. H is *biconvex bipartite* or *doubly convex*[YC95] if its half adjacency matrix has COP on both rows and columns.

2.1.2–vii. H is *circular convex* if its half adjacency matrix has CROP.

Interval graphs and circular-arc graphs have a long history in research. The interest around them is due to their very desirable property that several problems that are NP-hardc6 on general graphs are not so in these graph classes, i. e. they

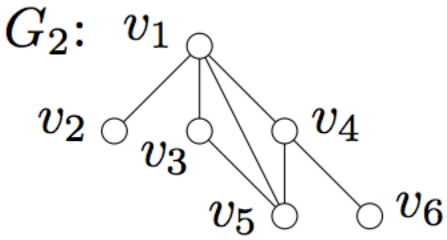
c6 *minor*: all NPC problems are NPH yes?



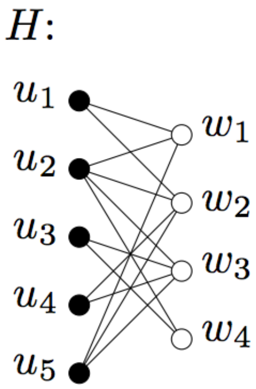
$\mathbf{A_2}$	v_1	v_2	v_3	v_4	v_5	v_6
v_1	**1**	**1**	**1**	0	**1**	**1**
v_2	**1**	**1**	**1**	**1**	0	**1**
v_3	**1**	**1**	**1**	**1**	**1**	0
v_4	0	**1**	**1**	**1**	0	0
v_5	**1**	0	**1**	0	**1**	0
v_6	**1**	**1**	0	0	0	**1**

$\mathbf{A_1}$	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	**1**	**1**	0	**1**	**1**
v_2	**1**	0	**1**	**1**	0	**1**
v_3	**1**	**1**	0	**1**	**1**	0
v_4	0	**1**	**1**	0	0	0
v_5	**1**	0	**1**	0	0	0
v_6	**1**	**1**	0	0	0	0

$\mathbf{A'_2}$	v_1	v_6	v_2	v_4	v_3	v_5
v_1	**1**	**1**	**1**	0	**1**	**1**
v_6	**1**	**1**	**1**	0	0	0
v_2	**1**	**1**	**1**	**1**	**1**	0
v_4	0	0	**1**	**1**	**1**	0
v_3	**1**	0	**1**	**1**	**1**	**1**
v_5	**1**	0	0	0	**1**	**1**



$\mathbf{B}$	c_1	c_2	c_3	c_4
v_1	**1**	**1**	**1**	0
v_2	**1**	0	0	0
v_3	0	**1**	0	0
v_4	0	0	**1**	**1**
v_5	0	**1**	**1**	0
v_6	0	0	0	**1**



$\mathbf{C}$	w_1	w_2	w_3	w_4
u_1	**1**	**1**	0	0
u_2	**1**	**1**	**1**	**1**
u_3	0	0	**1**	**1**
u_4	0	**1**	**1**	0
u_5	**1**	**1**	**1**	0

Figure 2.1: A_1 is the *adjacency matrix* and A_2 is the *augmented adjacency matrix* of G_1. A'_2 is obtained from A_2 by permuting its rows and columns to achieve *CROP order*, i. e. A_2 has CROP – thus G_1 is *a concave-round graph* (Def. 2.1.2 ii) and *a circular-arc graph* (Tab. 2.1) B is the maximal clique matrix of G_2 and has COP – thus G_2 is an *interval graph*(Def. 2.1.2 ii). C is the half adjacency matrix of bipartite graph H and has COP on rows – thus H is a convex bipartite graph. – PLACEHOLDER IMAGES –

are polynomial time solvable on them. In a similar fashion, a lot of problems that are hard on general matrices have efficient solutions on matrics with COP or CROP [Dom08, more citations pg. 33].

Table 2.1 summarises the way these graphs are characterized by their matrices having COP or CROP. Our focus in this chapter (and thesis) is mainly COP and having seen how useful COP is in identifying or characterizing many types of graphs, we will now see results that study recognition of COP in matrices in the following section.

tab 2.1 - have an abridged version of table 2.1 in dom. see notes for the possibility of a "circle diagram".

Table 2.1: Graph matrices PLACEHOLDER

2.2 Matrices with COP

The most important questions with respect to a particular property desired in a structure/object are perhaps the following.

- Does the desired property exist in the given input?
- If the test is affirmative, what is a certificate of the affirmative?
- If the test is negative, what are the optimization possibilities for the property in the input? In other words, how close to having the property can the input be?
- If the test is negative, what is a certificate of the negative?

In this section and the rest of the chapter we see results that shaped the correponding areas respectively for consecutive-ones property in binary matrices.

a. Does a given binary matrix have COP?

b. What is the COP permutation for the given matrix with COP?

c. What are the optimizations possible and practically useful on the given matrix without COP?

d. If algorithm for (a) returns **false**, can a certificate for this be computed?

Without doubt, besides computing answers to these questions, we are interested in the efficiency of these computations in terms of computational complexity theory. Results towards questions (a) and (b) are surveyed in this section. Those for question (c) are discussed in Section 2.3 and question (d) is discussed in Section 2.4.

It may be noted that one way to design an algorithm to test for COP is by deriving one from any interval graph recognition algorithm using the result HMPV00[c1] [c2] [Dom08] which demonstrates how such a derivation can be done. However, this does not necessarily yield an efficient algorithm. We will see results that directly solve the problem on matrices since it is known that questions (a) and (b) stated above for COP are efficiently solvable. Table 2.2 gives a snapshot of these results.

c1 cite hmpv00

c2 in endnote put the theorem 2.7 - dom pg 43

1899	First mention of COP	[?]
1951	Heuristics for COT	[Rob51]
1965	First polynomial time algorithm for COP testing	[FG65]
1972	Characterization for COP– forbidden submatrices	[Tuc72]
1976	First linear time algorithm for COT – PQ-tree	[BL76]
1992	Linear time algorithm COT without PQ-tree	[Hsu02][3]
2001	PC-tree – a simplification of PQ-tree	[Hsu01, ?]
1996	PQR-tree – generalization of PQ-tree for any binary matrix regardless of its COP status	[MM96]
1998	Almost linear time to construct PQR-tree	[MPT98]
2004	A certifying algorithm for no COP. Generalized PQ-tree.	[McC04]
2009	Set theoretic, cardinality based characterization of COP – ICPIA	[NS09]
2010	Logspace COP testing	[KKLV10]

Table 2.2: A brief history of COP research

2.2.1 Tucker's forbidden submatrices for COP

A polynomial time algorithm for COP testing was seen for the first time in [FG65] which uses overlapping properties of columns with **1**s. A few years later, a deeply significant result based on very different ideas in understanding COP came from Tucker which gave a combinatorial (negative) characterization of matrices with COP [Tuc72]. This result influenced most of the COP results that followed in the literature [c3]including linear time algorithms for COP recognition.

c3 *pressing*: did it? which ones?

[Tuc72] discovered certain forbidden structures for convex bipartite graphs[4] and by definition of this graph class, this translates to a set of forbidden submatrices for matrices with consecutive-ones property. The following are the theorems from [Tuc72] that acheived this characterization.

Theorem 2.2.1 proves that convex bipartite graphs cannot have *asteroidal* *triples*[5] contained in the corresponding vertex partition[6]. Theorem 2.2.2 shows what are the structures in a bipartite graph that force one of its vertex partitions to have asteriodal triples – in other words, it identifies the subgraphs that prevent the graph from being convex bipartite.

Theorem 2.2.1. *([Tuc72, Th. 6], [Dom08, Th. 2.3])*
A bipartite graph $G = (V_1, V_2, E)$ is convex bipartite on columns[7] *if and only if V_1 contains no asteroidal triple of G.*

Theorem 2.2.2. *([Tuc72, Th. 7], [Dom08, Th. 2.4])*
In a bipartite graph $G = (V_1, V_2, E)$ the vertex set V_1 contains no asteroidal triple if and only if G contains none of the graphs G_{I_k}, G_{II_k}, G_{III_k} (with $k \geq 1$), G_{IV}, G_V as shown in Figure 2.2 as subgraphs.

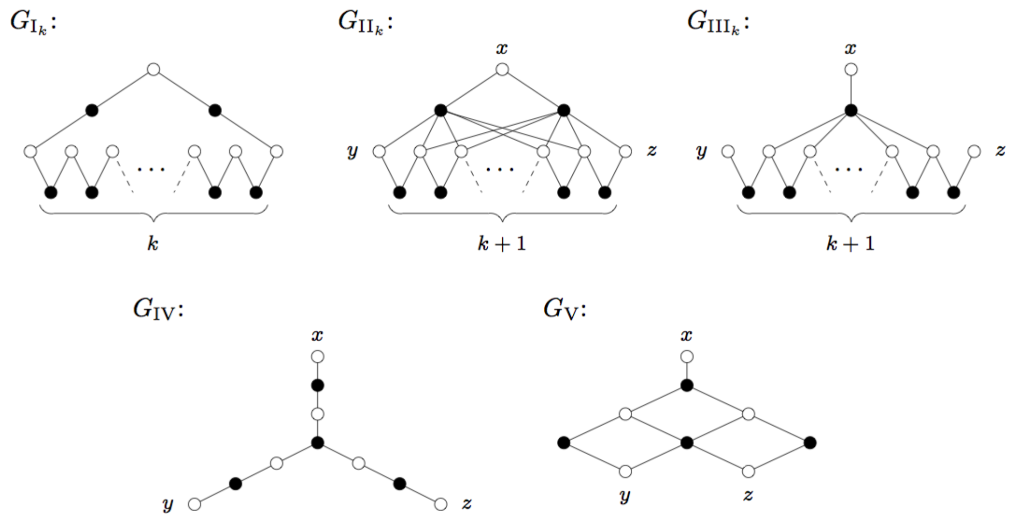


Figure 2.2: Tucker's forbidden subgraphs for convex bipartite graphs. PLACEHOLDER IMG

Theorem 2.2.1 and Theorem 2.2.2 result in the following Theorem 2.2.3 which characterizes matrices with COP.

Theorem 2.2.3. *([Tuc72, Th. 9], [Dom08, Th. 2.5])*
A matrix M has COP if and only if it contains none of the matrices M_{I_k}, M_{II_k}, M_{III_k} (with $k \geq 1$), M_{IV}, M_V as shown in Figure 2.3 as submatrices.

It can be verified that the matrices in Figure 2.3 are the half adjacency matrices of the graphs in Figure 2.2 respectively which is not surprising due to Definition 2.1.2 v.

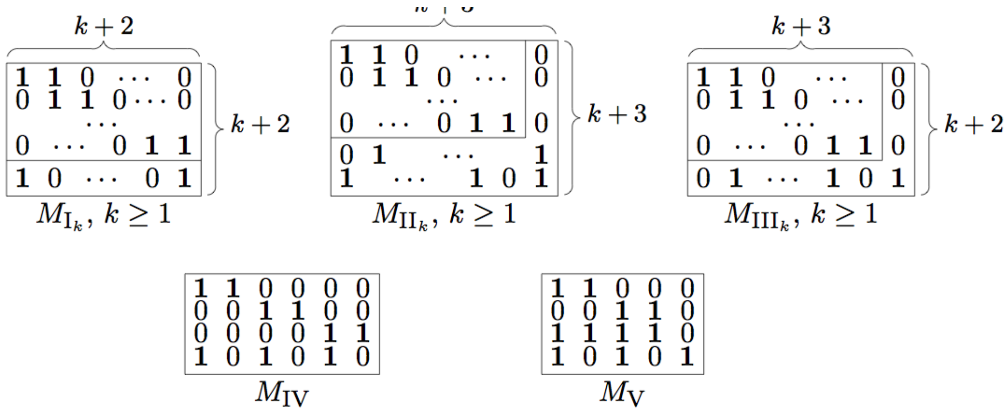


Figure 2.3: Tucker's forbidden submatrices for convex bipartite graphs. PLACEHOLDER IMG

2.2.2 PQ tree – a linear COT algorithm

Booth and Lueker in their paper [BL76] gave the first linear algorithm for consecutive-ones property testing by way of an interval graph recognition algorithm. Their result has close relations to the characterization of interval graphs by [GH64]. A graph G is an interval graph if and only if all its maximal cliques can be linearly ordered such that for any vertex v in G, all the cliques that v is incident on are consecutive in this order. Clearly, this means that the maximal clique incidence matrix[c0] must have COP on rows. This algorithm constructs a data structure called PQ-tree. A PQ-tree represents all the COP orderings of the matrix it is associated with. [BL76]'s algorithm uses the fact that if a matrix has COP, a PQ-tree for it can be constructed. It is interesting to note that aside from interval graph recognition and COP testing, PQ-tree is also useful in other applications like finding planar embeddings of planar graphs [?, McC04] and recognizing CROP in a matrix.

Definition 2.2.1:
[*PQ-tree [BL76, McC04]*]
A PQ-tree of matrix M with COP, is a tree with the following properties.

i. Each leaf uniquely represents a row (column) of M. The leaf order of the tree gives a COP order for column (row)[8] for M.

ii. Every non-leaf node in the tree are labeled P or Q.

iii. The children of P nodes are unordered. They can be permuted in any fashion to obtain a new COP order for M.

iv. The children of Q nodes are linearly ordered. Their order can be reversed to get obtain a new COP order for M.

[c0] Definition 2.1.1 iii

Thus, effectively, there exists a bijection between set of matrices with COP and the set of PQ-trees (accurately speaking, each matrix with COP bijectively maps to an equivalence class of PQ-trees due to properties (iii) and (iv)). See Figure 2.4 for an example of PQ-tree.

r_1	**1**	0	**1**	0	0	0	0
r_2	**1**	0	**1**	0	0	0	0
r_3	**1**	0	0	**1**	0	0	0
r_4	**1**	0	0	**1**	0	0	0
r_5	**1**	0	0	0	0	0	0
r_6	**1**	**1**	0	0	0	0	0
r_7	0	**1**	0	0	0	**1**	0
r_8	0	**1**	0	0	**1**	**1**	0
r_9	0	**1**	0	0	**1**	**1**	0
r_{10}	0	**1**	0	0	**1**	**1**	**1**
r_{11}	0	**1**	0	0	**1**	0	**1**

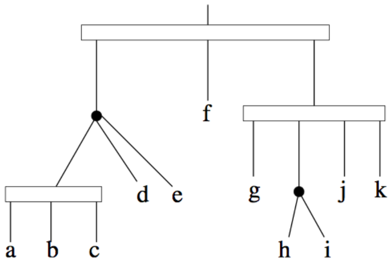


Figure 2.4: An example for PQ-tree. Permuting the order of the left child of the root, we see that (d,a,b,c,e,f,g,h,i,j,k) is a COP order. Reversing the order of the right child of the root, we see that (a,b,c,d,e,f,k,j,h,i,g) is yet another COP order. PLACEHOLDER IMG. [McC04, Fig. 1[9]]

The [BL76] algorithm with input $n \times m$ matrix M starts with a PQ-tree for a vacuous $n \times 0$ matrix M' (submatrix induced by 0 columns). This is known as a *universal* PQ-tree which is one with its root as a P node and only leaves as its children – each leaf representative of a row of input (by definition of COP for columns). This induced submatrix M' vacuously has COP. Each column is then added iteratively to M' to check if the new M' has COP. By a complicated, but linear, procedure the algorithm does one of the following actions in each iteration: (a) declare that M has no COP, or (b) modify the current PQ-tree to represent the new M' (which clearly, must have COP, since if not, option (a) would have been executed).

After the invention of PQ-trees, there has been several variants of the same with either simpler procedure of construction, like PC-tree [Hsu01] or a generalization of it, like generalized PQ-tree [McC04]. PQR-tree is another generalization of PQ-tree by [MM96, MPT98]. There algorithm was improved by [TM05] [c1]. c1 improved in terms of what?

Expand a little more on PC trees and PQR trees.?

We will see more of generalized PQ-tree in Section 2.4.

-I- Parallel algorithms [AS95, BS03, CY91]

c1

c2

c3

2.3 Optimization problems in COP

c4

So far we have been concerned about matrices that have the consecutive ones property. However in real life applications, it is rare that data sets represented by binary matrices have COP, primarily due to the noisy nature of data available. At the same time, COP is not arbitrary and is a desirable property in practical data representation [COR98, JKC⁺04, Kou77]. In this context, there are several interesting problems when a matrix does not have COP but is "close" to having COP or is allowed to be altered to have COP. These are the optimization problems related to a matrix which does not have COP. Some of the significant problems are surveyed in this section.

[c5][c6] [Tuc72] showed that a matrix that does not have COP have certain substructures that prevent it from having COP. Tucker classified these forbidden substructures into five classes of submatrices. This result is presented in the context of convex bipartite graphs which [Tuc72] proved to be AT-free[c7]. By definition, convex bipartite graph have half adjacency matrices that have COP on either rows or columns (graph is biconvex if it has COP on both)[Dom08]. A half adjacency matrix is a binary matrix representing a bipartite graph as follows. The set of rows and the set of columns form the two partitions of the graph. Each row node is adjacent to those nodes that represent the columns that have **1**s in the corresponding row. [Tuc72] proves that this bipartite graph has no asteroidal triple if and only if the matrix has COP and goes on to identify the forbidden substructures for these bipartite graphs. The matrices corresponding to these substructures are the forbidden submatrices.

Once a matrix has been detected to not have COP (using any of the COT algorithms mentioned earlier), it is naturally of interest to find out the smallest forbidden substructure (in terms of number of rows and/or columns and/or number of entries that are **1**s). [Dom08] discusses a couple of algorithms which are efficient if the number of **1**s in a row is small. This is of significance in the case of sparse matrices where this number is much lesser than the number of columns. $(*, \Delta)$-*matrices* are matrices with no restriction on number of **1**s in any column but has at most Δ **1**s in any row. Min COS-R (Min COS-C), Max COS-R (Max COS-C) are similar problems which deals with inducing COP on a matrix. In Min COS-R (Min COS-C) the question is to find the minimum number of rows (columns) that must be deleted to result in a matrix with COP. In the dual problem Max COS-R (Max COS-C) the search is for the maximum number of rows (columns) that induces a submatrix with COP. Given a matrix M with no COP, [Boo75] shows that finding a submatrix M' with all columns[c8] but a maximum cardinality subset of rows such that M' has COP is NP complete. [HG02] corrects an error of the abridged proof of this reduction as given in [GJ79]. [Dom08] discusses all these problems in detail giving an extensive survey of the previously existing results which are almost exhaustively all approximation results and hardness results. Taking this further, [Dom08] presents new results in the area of parameterized algorithms for this problem[c9].

Another problem is to find the minimum number of entries in the matrix that can be toggled to result in a matrix with COP. [Vel85] discusses approximation of COP Augmentation which is the problem of changing of the minimum number of zero entries to **1**s so that the resulting matrix has COP. As mentioned earlier, this problem is known to be NP complete due to [Boo75]. [Vel85] also proves, using a reduction to the longest path problem, [c10] that finding a Tucker's forbidden submatrix of at least k rows is NP complete. [c11] [c12]

[JKC⁺04] discusses the use of matrices with almost-COP (instead of one block of consecutive **1**s, they have x blocks, or *runs*, of consecutive **1**s and x is not too large) in the storage of very large databases. The problem is that of reordering of a binary matrix such that the resulting matrix has at most k runs of **1**s. This is proved to be NP hard using a reduction from the

c1 *pressing*: The results in cite bl76 on COT are based on the result that interval graphs are AT-free chordal graphs. - Tucker's?

c2 *pressing*: cite f uses these results to give the first polynomial time algorithm for COT. WHICH RESULTS? check. how do they use it?

c3 *pressing*: ADD peo exists iff chordal. lexicographic BFS [tag:chordalGraph]

c4 Expand on ref:sec:optcop

c5 – sect 4.1 in cite:d08phd has many results surveyed. hardness results, approx. results. results are usually for a class of matrices (a, b) where number 1s in columns and rows are restricted to a and b. – problem of flipping at most k entries of M to make it attain COP. this is NP complete cite:b75-phd

c6 (1) scite:lb62 showed that interval graphs are AT-free. describe AT (2) show the close relationship b/w COP and graphs sec 2.2, pg 31

c7 check this up. give details. - doms'

c8 check if b75 deals with COP col or COP row. also is it any submatrix with k less than r rows or submatrix must have all columns?

c9 elaborate - what are the results?

c10 or is it a survey of another result? check.

c11 how is this different from booth's 75 result??

c12 where should this go? cite—tz04 (approx submatrix with COP sparse matrices)

Hamiltonian path problem.[c13] [c14][c15] [c16] [c17]

2.4 COP for set systems

Section **??** mentions how a binary matrix naturally maps to a system of sets. A set can be constructed for each column of matrix with its elements being those row indices at which the column has **1**s. Thus the collection of sets corresponding each column of the matrix forms a set system with universe as the set of all row indices of the matrix. This simple construction is formally described in Definition 2.4.1.

Definition 2.4.1:
[*Set system of a binary matrix.*] Let M be a binary matrix of order $n \times m$ and $\{c_i \mid i \in [m]\}$ be the columns in M. A set system $\mathcal{F}_M = \{S_i \mid S_i \subseteq [n], i \in [m]\}$ is defined such that for every column c_i of M, set $S_i = \{j \mid m_{ji} = \mathbf{1}\}$. The collection $\mathcal{F}_M$ is the set system of matrix M. ✠

Thus the idea of consecutive-ones property naturally maps to set sytems as well[10] as Definition 2.4.2 states. This is equivalent to COP for matrices seen in Definition **??**.

Definition 2.4.2:
[*Consecutive-ones property for set systems.*] A set system $\mathcal{F}$ with universe U has the consecutive-ones property if it is possible to assign a linear order to U where each set $S \in \mathcal{F}$ is an interval in the linear order.[c1] ✠

Certifying algorithm (Generalized PQ trees) [McC04] – Move this to section on certifying algo.

Set theoretic characterizations [Hsu02, NS09]
[Hsu02] describes the simpler algorithm for COT.[c2]
[NS09] describes a characterization of consecutive-ones property solely based on the cardinality properties of the set representations of the columns (rows); every column (row) is equivalent to a set that has the row (column) indices of the rows (columns) that have one entries in this column (row). This is interesting and relevant, especially to this thesis because it simplifies COT to a great degree by reducing the solution search space owing to the a simple set theoretic characterization.

[McC04] describes a different approach to COT. While all previous COT algorithms gave the COP order if the matrix has the property but exited stating negative if otherwise, this algorithm gives an evidence by way of a certificate of matrix even when it has no COP. This enables a user to verify the algorithm's result even when the answer is negative. This is significant from an implementation perspective because automated program verification is hard and manual verification is more viable. Hence having a certificate reinforces an implementation's credibility. Note that when the matrix *has* COP, the COP order is the certificate. The internal machinery of this algorithm is related to the weighted betweenness problem addressed[c3] in [COR98]. [c4]

[c13] Theorem 2.1 in jkckv

[c14] (1) A connection of COP problem to the travelling salesman problem is also introduced. what does this mean? – COP can be used as a tool to reorder $0.5T \leq runs(M) \leq$ (2) The optimization version of the k-run problem, i.e. minimization of number of blocks of ones is proven to be NP complete by cite:k77

[c15] are these two the same?

[c16] what is the reduction?

[c17] other problems similar to COP – cite:ckl96 (ILP, circ ones, one drop) – cite:th98 (generalization of COP - minimax, biotonic column) Tucker

[c1] *important*: No formal enough.

[c2] *pressing*: in terms of what?

[c3] in what way??

[c4] expand on the COP order graph creation and it having to be bipartite for M to have COP. and thus an odd cycle being an evidence of no COP.

2.5 COP in Relational Database Model

c5 c6 c5 Expand on sec:apprdbm

c6 (set systems theme)

2.6 COP in Graph Isomorphism

c1 c2 c1 Expand on sec:appgraphiso

The survey from kklv10 conclusion. c2 (canonization theme)

Chapter Notes

[1]This follows [GH64] which states that the maximal cliques of interval graph G can be linearly ordered such that for all $v \in V(G)$, cliques containing v are consecutive in the ordering [Gol04, Th. 8.1].

[2]The set of unit interval graphs and the set of proper interval graphs are the same

[3]First published in [**?**]

[4]The terminology in [Tuc72] differs. It uses the term *graphs with V_1-consecutive arragement* instead of *convex bipartite graphs.*

[5]If $G = (V, E)$ is a graph, a set of three vertices from V form an *asteroidal triple* if between any two of them there exists a path in G that does not contain any vertex from the closed neighborhood of the third vertex.

[6]The partition corresponds to columns (rows) if its half adjacency matrix has COP columns (rows).

[7]Abridged to match terminology adopted in this document. See previous note.

[8]Note that COP order for column requires permutation of rows and vice versa.

[9][McC04]illustrates COP on rows. Our convention in this document is COP on columns and thus example matrix has been transposed.

[10]As seen in Section **??**

c3 c3 remove if none.

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