Generalization of the Consecutive-ones Property

A THESIS

submitted by

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CHAPTER 2

Consecutive-ones Property – A Survey of Important Results

This chapter surveys several results that are significant to this thesis or to COP in general. These predominantly pertain to characterizations of COP, algorithmic tests to check for COP (COT), optimization problems on binary matrices that do not have COP and some applications of COP.

2.1 COP in Graph Theory

COP is closely connected to several types of graphs by way of describing certain combinatorial graph properties. There are also certain graphs, like convex bipartite graphs, that are defined solely by some of its associated matrix having COP. In this section we will see the relevance of consecutive-ones property to graphs. To see this we introduce certain binary matrices that are used to define graphs in different ways. While adjacency matrix is perhaps the most commonly used such matrix, Definition 2.1.1 defines this and a few more.

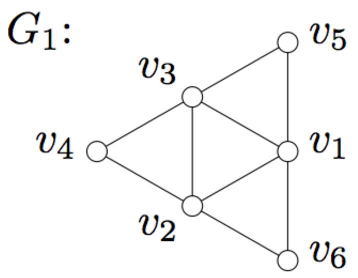
Definition 2.1.1. *Matrices that define graphs. [Dom08, Def. 2.4].* Let G and H be defined as follows. $G = (V, E_G)$ is a graph with vertex set $V = \{v_i \mid i \in [n]\}$ and edge set $E_G \subseteq \{(v_i, v_j) \mid i, j \in [n]\}$ such that $|E_G| = m$. $H = (A, B, E_H)$ is a bipartite graph with partitions $A = \{a_i \mid i \in [n_a]\}$ and $B = \{b_i \mid i \in [n_b]\}$.

2.1.1–i. *Adjacency matrix* of G is the symmetric $n \times n$ binary matrix M with $m_{i,j} = \mathbf{1}$ if and only if $(v_i, v_j) \in E_G$ for all $i, j \in [n]$.

2.1.1–ii. *Augmented adjacency matrix* of G is obtained from its adjacency matrix by setting all main diagonal elements to $\mathbf{1}$, i. e. $m_{i,i} = \mathbf{1}$ for all $i \in [n]$.

2.1.1–iii. *Maximal clique matrix* or *vertex-clique incidence matrix* of G is the $n \times k$ binary matrix M with $m_{i,j} = \mathbf{1}$ if and only if $v_i \in C_j$ for all $i \in [n], j \in [k]$ where $\{C_j \mid j \in [k]\}$ is the set of maximal cliques of G.

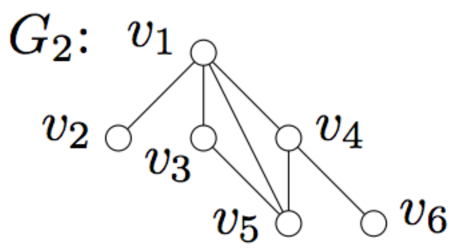
2.1.1–iv. *Half adjacency matrix* of H is the $n_a \times n_b$ binary matrix M with $m_{i,j} = \mathbf{1}$ if and only if $(a_i, b_j) \in E_H$.



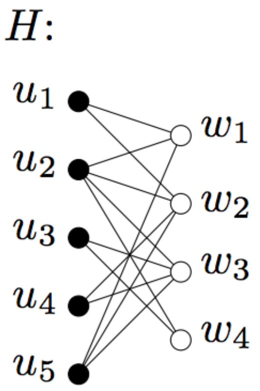
$\mathbf{A_2}$	v_1	v_2	v_3	v_4	v_5	v_6
v_1	**1**	**1**	**1**	0	**1**	**1**
v_2	**1**	**1**	**1**	**1**	0	**1**
v_3	**1**	**1**	**1**	**1**	**1**	0
v_4	0	**1**	**1**	**1**	0	0
v_5	**1**	0	**1**	0	**1**	0
v_6	**1**	**1**	0	0	0	**1**

$\mathbf{A_1}$	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	**1**	**1**	0	**1**	**1**
v_2	**1**	0	**1**	**1**	0	**1**
v_3	**1**	**1**	0	**1**	**1**	0
v_4	0	**1**	**1**	0	0	0
v_5	**1**	0	**1**	0	0	0
v_6	**1**	**1**	0	0	0	0

$\mathbf{A_2'}$	v_1	v_6	v_2	v_4	v_3	v_5
v_1	**1**	**1**	**1**	0	**1**	**1**
v_6	**1**	**1**	**1**	0	0	0
v_2	**1**	**1**	**1**	**1**	**1**	0
v_4	0	0	**1**	**1**	**1**	0
v_3	**1**	0	**1**	**1**	**1**	**1**
v_5	**1**	0	0	0	**1**	**1**



$\mathbf{B}$	c_1	c_2	c_3	c_4
v_1	**1**	**1**	**1**	0
v_2	**1**	0	0	0
v_3	0	**1**	0	0
v_4	0	0	**1**	**1**
v_5	0	**1**	**1**	0
v_6	0	0	0	**1**



$\mathbf{C}$	w_1	w_2	w_3	w_4
u_1	**1**	**1**	0	0
u_2	**1**	**1**	**1**	**1**
u_3	0	0	**1**	**1**
u_4	0	**1**	**1**	0
u_5	**1**	**1**	**1**	0

Figure 2.1: A_1 is the *adjacency matrix* and A_2 is the *augmented adjacency matrix* of G_1. A_2' is obtained from A_2 by permuting its rows and columns to achieve *CROP order*, i. e. A_2 has CROP – thus G_1 is *a concave-round graph* (Def. 2.1.2 ii) and *a circular-arc graph* (Tab. 2.1) B is the maximal clique matrix of G_2 and has COP – thus G_2 is an *interval graph*(Def. 2.1.2 ii). C is the half adjacency matrix of bipartite graph H and has COP on rows – thus H is a convex bipartite graph. – PLACEHOLDER IMAGES –

Now we will see in Definition 2.1.2 certain graph classes that is related to COP or CROP.

Definition 2.1.2. *Graphs that relate to COP.[Dom08, Def. 2.5]* Let G be a graph and H be a bipartite graph.

2.1.2–i. G is *convex-round* if its adjacency matrix has the CROP.

2.1.2–ii. G is *concave-round* if its augmented adjacency matrix has CROP.

2.1.2–iii. G is an *interval graph* if its vertices can be mapped to intervals on the real line such that two vertices are adjacent if and only if their corresponding intervals overlap . G is an interval graph if and only if its maximal clique matrix has COP [FG65][1]

   a. G is a *unit interval graph* if it is an interval graph such that all intervals have the same length.[2]

   b. G is a *proper interval graph* if it is an interval graph such that no interval properly contains another.[2]

2.1.2–iv. G is a *circular-arc graph* if its vertices can be mapped to a set of arcs on a circle such that two vertices are adjacent if and only if their corresponding arcs overlap.

2.1.2–v. H is *convex bipartite on columns (rows)* if its half adjacency matrix has COP on rows (columns).

2.1.2–vi. H is *biconvex bipartite* or *doubly convex*[YC95] if its half adjacency matrix has COP on both rows and columns.

2.1.2–vii. H is *circular convex* if its half adjacency matrix has CROP.

Interval graphs[3]and circular-arc graphs have a long history in research. The interest around them is due to their very desirable property that several problems

[1]This follows [GH64] which states that the maximal cliques of interval graph G can be linearly ordered such that for all $v \in V(G)$, cliques containing v are consecutive in the ordering [Gol04, Th. 8.1].

[2]The set of unit interval graphs and the set of proper interval graphs are the same

[3][McC04] cites that the problem of recognizing interval graphs has significance in molecular biology. Interestingly, in the late 1950s, before the structure of DNA was well-understood, Seymour Benzer was able to show that the intersection graph of a large number of fragments of genetic material was an interval graph [Ben59]. This was regarded as compelling evidence that genetic information was somehow arranged inside a structure that had a linear topology which we now know to be true from the discovery of linear structure of DNA.

that are NP-hard on general graphs, like finding a maximum clique or minimum coloring or independent set, are polynomial time solvable in these graph classes [CLRS01]. In a similar fashion, a lot of problems that are hard on general matrices have efficient solutions on matrices with COP or CROP [Dom08, more citations pg. 33].

Table 2.1 summarises the way these graphs are characterized by their matrices having COP or CROP. Our focus in this chapter (and thesis) is mainly COP and having seen how useful COP is in identifying or characterizing many types of graphs[4], we will now see results that study recognition of COP in matrices in the following section.

tab 2.1 - have an abridged version of table 2.1 in dom. see notes for the possibility of a "circle diagram".

Table 2.1: Graph matrices PLACEHOLDER

2.2 Matrices with COP

The most important questions with respect to a particular property desired in a structure/object are perhaps the following.

- Does the desired property exist in the given input?
- If the test is affirmative, what is a certificate of the affirmative?
- If the test is negative, what are the optimization possibilities for the property in the input? In other words, how close to having the property can the input be?
- If the test is negative, what is a certificate of the negative?

In this section and the rest of the chapter we see results that shaped the correponding areas respectively for consecutive-ones property in binary matrices.

a. Does a given binary matrix have COP?

b. What is the COP permutation for the given matrix with COP?

c. What are the optimizations possible and practically useful on the given matrix without COP?

[4]COP has several other applications fill up.

d. If algorithm for (a) returns **false**, can a certificate for this be computed?

Without doubt, besides computing answers to these questions, we are interested in the efficiency of these computations in terms of computational complexity theory. Results towards questions (a) and (b) are surveyed in this section. Those for question (c) are discussed in Section 2.3 and question (d) is discussed in Section 2.2.3.

It may be noted that one way to design an algorithm to test for COP is by deriving one from any interval graph recognition algorithm using the result HMPV00 [Dom08] which demonstrates how such a derivation can be done. However, this does not necessarily yield an efficient algorithm. We will see results that directly solve the problem on matrices since it is known that questions (a) and (b) stated above for COP are efficiently solvable. Table 2.2 gives a snapshot of these results.

1899	First mention of COP (archaeology)	
1951	Heuristics for COT	[Rob51]
1965	First polynomial time algorithm for COP testing	[FG65]
1972	Characterization for COP– forbidden submatrices	[Tuc72]
1976	First linear time algorithm for COT – PQ-tree	[BL76]
1992	Linear time algorithm COT without PQ-tree	[Hsu02]
2001	PC-tree – a simplification of PQ-tree	[Hsu01, HM03]
1996	PQR-tree – generalization of PQ-tree for any binary matrix regardless of its COP status	[MM96]
1998	Almost linear time to construct PQR-tree	[MPT98]
2004	A certifying algorithm for no COP. Generalized PQ-tree.	[McC04]
2009	Set theoretic, cardinality based characterization of COP – ICPIA	[NS09]
2010	Logspace COP testing	[KKLV10]

Table 2.2: A brief history of COP research

The first polynomial time algorithm for COP testing was by [FG65] which uses overlapping properties of columns with **1**s. Their result has close relations to the characterization of interval graphs by [GH64]. A graph G is an interval graph if and only if all its maximal cliques can be linearly ordered such that for any vertex v in G, all the cliques that v is incident on are consecutive in this order. Clearly, this means that the maximal clique incidence matrix[5] must have COP on rows.

A few years later, a deeply significant result based on very different ideas in understanding COP came from Tucker which gave a combinatorial (negative) characterization of matrices with COP [Tuc72]. This result influenced most of the COP results that followed in the literature including linear time algorithms for COP recognition.

[5] Definition 2.1.1 iii

2.2.1 Tucker's forbidden submatrices for COP

[Tuc72] discovered certain forbidden structures for convex bipartite graphs[6] and by definition of this graph class, this translates to a set of forbidden submatrices for matrices with consecutive-ones property. The following are the theorems from [Tuc72] that acheived this characterization.

Theorem 2.2.1 states that convex bipartite graphs cannot have *asteroidal triples*[7] contained in the corresponding vertex partition[8]. Theorem 2.2.2 lists the structures in a bipartite graph that force one of its vertex partitions to have asteriodal triples – in other words, it identifies the subgraphs that prevent the graph from being convex bipartite.

Theorem 2.2.1. *([Tuc72, Th. 6], [Dom08, Th. 2.3])*
A bipartite graph $G = (V_1, V_2, E)$ is convex bipartite on columns[9] *if and only if V_1 contains no asteroidal triple of G.*

Theorem 2.2.2. *([Tuc72, Th. 7], [Dom08, Th. 2.4])*
In a bipartite graph $G = (V_1, V_2, E)$ the vertex set V_1 contains no asteroidal triple if and only if G contains none of the graphs G_{I_k}, G_{II_k}, G_{III_k} (with $k \geq 1$), G_{IV}, G_V as shown in Figure 2.2 as subgraphs.

Theorem 2.2.1 and Theorem 2.2.2 result in the following Theorem 2.2.3 which characterizes matrices with COP.

Theorem 2.2.3. *([Tuc72, Th. 9], [Dom08, Th. 2.5])*
A matrix M has COP if and only if it contains none of the matrices M_{I_k}, M_{II_k}, M_{III_k} (with $k \geq 1$), M_{IV}, M_V as shown in Figure 2.3 as submatrices.

It can be verified that the matrices in Figure 2.3 are the half adjacency matrices of the graphs in Figure 2.2 respectively which is not surprising due to Definition 2.1.2 v.

[6]The terminology in [Tuc72] differs. It uses the term *graphs with V_1-consecutive arragement* instead of *convex bipartite graphs.*

[7]If $G = (V, E)$ is a graph, a set of three vertices from V form an *asteroidal triple* if between any two of them there exists a path in G that does not contain any vertex from the closed neighborhood of the third vertex.

[8]The partition corresponds to columns (rows) if its half adjacency matrix has COP columns (rows).

[9]Abridged to match terminology adopted in this document. See previous note.

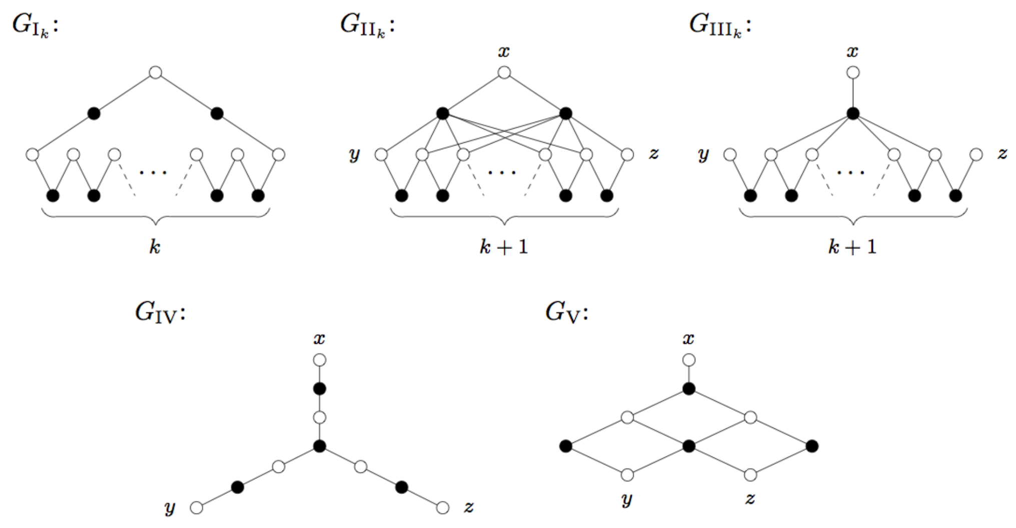


Figure 2.2: Tucker's forbidden subgraphs for convex bipartite graphs. PLACEHOLDER IMG

$M_{I_k}, k \geq 1$

$$\left.\overbrace{\begin{array}{|cccccc|}
\hline
\mathbf{1} & \mathbf{1} & 0 & & \dots & 0 \\
0 & \mathbf{1} & \mathbf{1} & 0 & \dots & 0 \\
 & & \dots & & & \\
0 & \dots & 0 & & \mathbf{1} & \mathbf{1} \\
\hline
\mathbf{1} & 0 & \dots & & 0 & \mathbf{1} \\
\hline
\end{array}}^{k+2}\right\} k+2$$

$M_{II_k}, k \geq 1$

$$\left.\overbrace{\begin{array}{|cccccc|c|}
\hline
\mathbf{1} & \mathbf{1} & 0 & & \dots & & 0 \\
0 & \mathbf{1} & \mathbf{1} & 0 & \dots & & 0 \\
 & & \dots & & & & \\
0 & \dots & & 0 & \mathbf{1} & \mathbf{1} & 0 \\
\hline
0 & \mathbf{1} & & \dots & & & \mathbf{1} \\
\mathbf{1} & & \dots & & \mathbf{1} & 0 & \mathbf{1} \\
\hline
\end{array}}^{k+3}\right\} k+3$$

$M_{III_k}, k \geq 1$

$$\left.\overbrace{\begin{array}{|cccccc|c|}
\hline
\mathbf{1} & \mathbf{1} & 0 & & \dots & & 0 \\
0 & \mathbf{1} & \mathbf{1} & 0 & \dots & & 0 \\
 & & \dots & & & & \\
0 & \dots & & 0 & \mathbf{1} & \mathbf{1} & 0 \\
\hline
0 & \mathbf{1} & \dots & & \mathbf{1} & 0 & \mathbf{1} \\
\hline
\end{array}}^{k+3}\right\} k+2$$

M_{IV}

$$\begin{array}{|cccccc|}
\hline
\mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\
\mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\
\hline
\end{array}$$

M_V

$$\begin{array}{|ccccc|}
\hline
\mathbf{1} & \mathbf{1} & 0 & 0 & 0 \\
0 & 0 & \mathbf{1} & \mathbf{1} & 0 \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\
\mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} \\
\hline
\end{array}$$

Figure 2.3: Tucker's forbidden submatrices for convex bipartite graphs. [Tuc72]

2.2.2 Booth and Lueker's PQ tree – a linear COT algorithm

Booth and Lueker in their paper [BL76] gave the first linear algorithm[10] for consecutive-ones property testing while given a linear time interval graph recognition algorithm by a simplification of 's planarity test algorithm. [BL76] introduces a data structure called PQ-tree and their COP testing algorithm is a constructive one that outputs a PQ-tree if the input has COP. A PQ-tree represents all the COP orderings of the matrix it is associated with. [BL76]'s algorithm uses the fact that if a matrix has COP, a PQ-tree for it can be constructed. It is interesting to note that aside from interval graph recognition and COP testing, PQ-tree is also useful in other applications like finding planar embeddings of planar graphs [**?**, McC04] and recognizing CROP in a matrix.

Definition 2.2.1 (PQ-tree [BL76, McC04]). A PQ-tree of matrix M with COP on columns (rows), is a tree with the following properties.

i. Each leaf uniquely represents a row (column) of M. The leaf order of the tree gives a COP order for column (row)[11] for M.

ii. Every non-leaf node in the tree is labeled P or Q.

iii. The children of P nodes are unordered. They can be permuted in any fashion to obtain a new COP order for M.

iv. The children of Q nodes are linearly ordered. Their order can be reversed to obtain a new COP order for M.

See Figure 2.4 for an example of PQ-tree. It may be noted that there is no way an empty set of COP orderings can be represented in this data structure. For this reason, PQ-tree is undefined for matrices that do not have COP. Thus effectively, there exists a bijection between set of matrices with COP and the set of PQ-trees (accurately speaking, each matrix with COP bijectively maps to an equivalence class of PQ-trees resulting from properties (iii) and (iv)).

The [BL76] algorithm with input $n \times m$ matrix M starts with a PQ-tree for a vacuous $n \times 0$ matrix M' (submatrix induced by 0 columns). This is known as a *universal* PQ-tree which is one with its root as a P node and only leaves as its children – each leaf representative of a row of input (by definition of COP for

[10]Time complexity is $O(m + n + f)$ where $m \times n$ is the order of the input matrix and f is the number of **1**s in it.

[11]Note that COP order for column requires permutation of rows and vice versa.

r_1	**1**	0	**1**	0	0	0	0
r_2	**1**	0	**1**	0	0	0	0
r_3	**1**	0	0	**1**	0	0	0
r_4	**1**	0	0	**1**	0	0	0
r_5	**1**	0	0	0	0	0	0
r_6	**1**	**1**	0	0	0	0	0
r_7	0	**1**	0	0	0	**1**	0
r_8	0	**1**	0	0	**1**	**1**	0
r_9	0	**1**	0	0	**1**	**1**	0
r_{10}	0	**1**	0	0	**1**	**1**	**1**
r_{11}	0	**1**	0	0	**1**	0	**1**

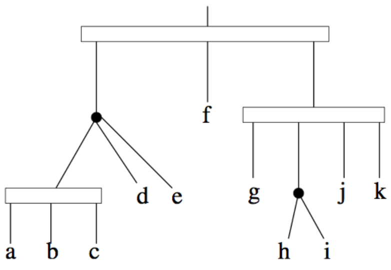


Figure 2.4: An example for PQ-tree. Permuting the order of the left child of the root, we see that (d,a,b,c,e,f,g,h,i,j,k) is a COP order. Reversing the order of the right child of the root, we see that (a,b,c,d,e,f,k,j,h,i,g) is yet another COP order. PLACEHOLDER IMG. [McC04, Fig. 1^{10}]

columns). This induced submatrix M' vacuously has COP. Each column is then added iteratively to M' to check if the new M' has COP. By a complicated, but linear, procedure the algorithm does one of the following actions in each iteration: (a) declare that M has no COP, or (b) modify the current PQ-tree to represent the new M' (which clearly, must have COP, since if not, option (a) would have been executed).

Judging from notes in literature, this algorithm is apparently notoriously difficult to program. In the procedure to modify the PQ-tree at each iteration, nodes are considered from leaves to tree. At each node considered, it uses one of nine templates to determine how the tree must be altered in the vicinity of this node. Recognition of this template poses a difficult challenge in terms of implementing it. Each template is actually a representative of a larger class of similar templates, which must be dealt with explicitly by a program[McC04].

After the invention of PQ-trees, presumably due to the implementation challenge it posed, there has been several variants of the same in the literature, like PC-tree [SH99, Hsu01, HM03], generalized PQ-tree [KM89, McC04], PQR-tree [MM96, MPT98] etc. Most of these are generalizations of PQ-tree– for instance, PC-tree is generalized to matrices with CROP, PQR-tree and generalized PQ-tree are generalized to matrices and set systems with or without COP. [KM89] invented a modified form of PQ-tree a simpler incremental update of the tree only for recognizing interval graphs. [KR88] constructed efficient parallel algorithms for manipulating PQ trees. dom chapter 2 pg 40 Variations of PQTrees first para - summarize.

In the next few following sections we will see some of these variations.

2.2.3 PQR-tree – COP for set systems

Section 1.6 mentions how a binary matrix naturally maps to a system of sets. A set can be constructed for each column of matrix with its elements being those row indices at which the column has **1**s. Thus the collection of sets corresponding each column of the matrix forms a set system with universe as the set of all row indices of the matrix. This simple construction is formally described in Definition 2.2.2 along with the idea of consecutive-ones property for set systems[12].

Definition 2.2.2 (*Consecutive-ones property for set systems*). Let M be a binary matrix of order $n \times m$ and $\{c_i \mid i \in [m]\}$ be the columns in M. A set system $\mathcal{F}_M = \{S_i \mid S_i \subseteq [n], i \in [m]\}$ is defined such that for every column c_i of M, set $S_i = \{j \mid m_{ji} = \mathbf{1}\}$. The collection $\mathcal{F}_M$ is the *set system of binary matrix M*. The *binary matrix for set system $\mathcal{F}$* is conversely constructed and denoted by $M^{\mathcal{F}}$. Thus, $M^{\mathcal{F}_M} = M$.

A set system $\mathcal{F}$ from universe U, $|U| = n$ has the *consecutive-ones property* if there exists a linear order or permutation $\sigma = w_1 w_2 \dots w_n$ that can be applied to U such that each set $S \in \mathcal{F}$ becomes a consecutive subsequence[13] $w_i w_{i+1} \dots w_{i+k-1}$ on σ for some positive integer $i \leq n + 1 - k$ where $k = |S|$.

The set $\texttt{valid}(U, \mathcal{F})$ respresents all COP orders of $\mathcal{F}$ in U. $\boxed{\maltese}$

It is easy to see the equivalence of this definition to COP for matrices in Definition 1.3.3.

Before proceeding to describe PQR-tree per se, we will see a few more terminologies that will make the subsequent discussion in this section simpler.

Definition 2.2.3 (*Orthogonal sets [Nov89, MM96, McC04]*[14]). Let $\mathcal{F}$ be a set system with universe U and sets $A, B \in \mathcal{F}$.

1. A and B are said to have a *trivial intersection* if $A \cap B$ is one of the following.

   i. $\emptyset$

   ii. A

[12] As seen in Section 1.2.1

[13] Also termed an *interval*

[14] [McC04] does not use the term "mutually orthogonal" but refers to the same idea as "sets that do not overlap". This terminology is also used in other literature like [NS09, Hsu02].

iii. B[15]

2. A and B are called *mutually orthogonal* or A is *orthogonal to* B and vice versa, if they have a trivial intersection.

3. *Trivial subsets* of a universe U, denoted by $\mathcal{T}(U)$, are sets that have trivial intersections with any set in 2^U. These sets are U, singleton sets in 2^U and $\emptyset$ [Nov89, MM96]. Thus, $\mathcal{T}(U) = \{U\} \bigcup \{\{v\} \mid v \in U\} \bigcup \{\emptyset\}$.

4. *Orthogonal sets of a set system* $\mathcal{F}$ with universe U are subsets of U that are orthogonal to all sets in $\mathcal{F}$. The set of all orthogonal sets to $\mathcal{F}$ is denoted by $\mathcal{F}^{\perp}$[16].

5. $\mathcal{F}$ is called *complete*[17] if the following hold true for every pair of non-orthogonal[18] sets A, B in $\mathcal{F}$.

   i. $\mathcal{T}(U) \subset \mathcal{F}$

   ii. $A \cup B \in \mathcal{F}$

   iii. $A \cap B \in \mathcal{F}$

   iv. $A \setminus B \in \mathcal{F}$

   v. $B \setminus A \in \mathcal{F}$

   In other words, $\mathcal{F}$ contains all the trivial subsets of U, $A \cup B$ and the partitions of $A \cup B$ defined by intersection and set difference.

6. $\overline{\mathcal{F}}$ represents the smallest super set system of $\mathcal{F}$ that is complete[19]

✠

Generalized PQ-tree or gPQ-tree is a data structure defined in [Nov89] to represent all orthogonal sets of a set system $\mathcal{F}$. A data structure with the same name was later defined in [McC04] as part of a *substitution decomposition* for a set system $\mathcal{F}$ and subsequently [McC04] gives a new characterization of $\mathcal{F}$ using a so-called incompatibility graph. PQR-tree is defined in [MM96] as a data structure to represent any set system $\mathcal{F}$ with additional information in their R-nodes if $\mathcal{F}$ has no COP. All three of these data structures are proposed as generalizations of [BL76]'s PQ-tree and hence produce the PQ-tree if $\mathcal{F}$ has COP. As a whole, all these three data strutures are largely identical with their differences being

[15]In other words, A and B are either disjoint or one is the subset of the other.
[16][McC04, Def. 3.1] uses the term *non-overlapping family* of $\mathcal{F}$ and denotes it by $\mathcal{N}(\mathcal{F})$.
[17][McC04] calls this a *weakly partative family.*
[18]Or overlapping.
[19][McC04, Def. 3.2] calls this the *weak closure* of $\mathcal{F}$ denoted by $\mathcal{W}(\mathcal{F})$.

notional. In this section, we will discuss the basic theory that they all hold. We will predominantly use the terminology from [MM96] and refer to the data structure as PQR-tree.

An important observation made by [MM96] is presented now along with a few theorems that help in decomposing the COP problem on $\mathcal{F}$ into subproblems.

*Observation 2.2.1 (**[MM96, Sec. 3]**).* If $\mathcal{F}$ is a set system with COP then, after applying the COP order, not only must every set in $\mathcal{F}$ be consecutive but the following sets must also be consecutive for any $A, B \in \mathcal{F}$.

1. The intersection $A \cap B$
2. The union $A \cup B$ if $A \cap B \neq \emptyset$
3. The relative complements $A \setminus B$ and $B \setminus A$ if $B \nsubseteq A$ and $A \nsubseteq B$ respectively.
4. Also note that trivially, sets in $\mathcal{T}(U)$ are consecutive in any permutation of U[20].

Theorem 2.2.4 ([MM96, Th. 3,6]). *For any set system $\mathcal{F}$ we have the following.*

$$\texttt{valid}\,(\mathcal{F}) = \texttt{valid}\left(\overline{\mathcal{F}}\right)$$
$$\mathcal{F}^{\perp} = \overline{\mathcal{F}^{\perp}} = (\overline{\mathcal{F}})^{\perp}$$

Proposition 2.2.5. Any set that is consecutive on all the COP permutations of $\mathcal{F}$ is present in $\overline{\mathcal{F}}$.

Proposition 2.2.5 is owing to the following theorem by which [MM96] describes a way to decompose the problem of finding all COP orders of $\mathcal{F}$ into two subproblems using sets in $\overline{\mathcal{F}} \cap \mathcal{F}^{\perp}$.

Theorem 2.2.6 ([MM96, Th. 7]). *For any set system $\mathcal{F}$, and $\emptyset \neq H \in \overline{\mathcal{F}} \cap \mathcal{F}^{\perp}$ we have the following.*

$$\texttt{valid}\,(U, \mathcal{F}) = \texttt{valid}\,(U/H, \mathcal{F}/H) * \texttt{valid}\left(H, \mathcal{F} \cap 2^{H}\right)$$

The idea behind Theorem 2.2.6 is as follows. A permutation α of U is a composition of two permutations with respect to H - i) a permutation γ of H and (ii) a permutation β of U/H.

For two mutually orthogonal sets A, B such that $A \nsubseteq B$, A/B is defined as the set obtained by removing all elements of B from A and adding a represen-

[20] $\emptyset$ is considered consecutive by convention.

tative element for B in A. Being orthogonal, results in only the following three possibilities.

1. A and B are disjoint: $A/B = A$
2. A is a subset of B: A/B is not defined
3. B is a subset of A: $A/B = A \setminus B \cup \{b\}$, where b is a new element not in U added to represent B.

We observe that this idea of decomposing the COP problem into two subproblems in [MM96] is very similar to the substitution decomposition of a set system given in [McC04].[21]

The following corollary states how PQR-tree elegantly fits into this whole theory and help in computing all COP orders of $\mathcal{F}$.

Corollary 2.2.7 ([MM96, Cor. 8]). *Let $\mathcal{F}$ be a set system with universe U and H is a non-empty orthogonal set $H \in \overline{\mathcal{F}} \cap \mathcal{F}^{\perp}$. If there is a PQR-tree T_1 that encodes all permutations in* `valid`$(U/H, \mathcal{F}/H)$ *and also a PQR-tree T_2' that encodes all permutations* `valid`$\left(H, \mathcal{F} \cap 2^H\right)$*, then a PQR-tree T for $\mathcal{F}$ can be obtained by replacing the leaf h in T_1 by T_2.*

Thus we have a recursive algorithm that can compute the PQR-tree for $\mathcal{F}$ provided we find an element from $\overline{\mathcal{F}} \cap \mathcal{F}^{\perp}$ in each iteration. The non-empty sets in $\overline{\mathcal{F}} \cap \mathcal{F}^{\perp}$ are called *node sets* since they form the nodes in the PQR-tree. They are calculated as follows. One is by computing the overlap components of $\mathcal{F}$. The overlap components is the partition that results from the overlap equivalence relation which is nothing but non-orthogonal equivalence relation[22]. Overlap components are linearly computable[MM95, Hsu92]. Once these elements are factored out, the rest of the node sets are obtained by identifying *twin* elements[23]. Two elements $a, b \in U$ are twins if their membership in every set of $\mathcal{F}$ is in tandem with each other, i. e. $\{a, b\} \perp \mathcal{F}$. This is clearly an equivalence relation and their equivalence classes is known to be computable in linear time[Hsu01, ?] and even in logspace[KKLV10].

The recursion end condition is when one cannot find any more sets from $\overline{\mathcal{F}} \cap \mathcal{F}^{\perp}$ that are non-trivial. This is when $\overline{\mathcal{F}} \cap \mathcal{F}^{\perp} = \mathcal{T}(U)$. This is the point where the

[21]Using the theory cited in footnotes 16,17,19.

[22]It is easy to verify that this relation is indeed an equivalence relation.

[23][KKLV10, Sec. 3] calls this indistinguishable elements. The equivalence class is called a *slot*.

parents of the leaves of the PQR-tree are created. The following theorem helps the algorithm decide whether a P, Q or R node must be created.

Theorem 2.2.8 ([MM96, Th. 9],[McC04, Th. 2.1]). *If $\mathcal{F}$ is a set system with universe U and $|U| \geq 3$ then one of the following statements hold.*

1. $\overline{\mathcal{F}} = \mathcal{T}(U)$
2. $\overline{\mathcal{F}} = consec(\alpha)$ *for some permutation α on U*
3. $\overline{\mathcal{F}} = 2^U$

In case (1) all elements in U are made children of a P node. In case (2) all elements in U are made children of a Q node in the order given by α. Finally case (3) is the one when no permutation of U gives COP. All elements of U are in this case made children in an R node.

Theorem 2.2.8 and the theory leading to it is very similar to [McC04, Th. 2.1, 3.5. Also Th. 3.2, 3.3, 3.4] which categorizes the nodes in the above the three cases as *prime*, *linear* and *degenerate* respectively. Their generalized PQ-tree is created in similar ways as PQR-tree above. This tree is in essense the Hasse diagram of what they call *strong elements* of $\overline{\mathcal{F}}$. Strong elements of a set family are elements that do not overlap with any other elements in the family, i. e. it is orthogonal to all other sets[McC04, Def. 3.3].

Theorem 2.2.9 ([MM96], [McC04, Th. 3.6]). *The set system $\mathcal{F}$ has COP if and only if its PQR-tree has no R nodes.*

Certifying algorithm (Generalized PQ trees) [McC04] –

Set theoretic characterizations [Hsu02, NS09]
[Hsu02] describes the simpler algorithm for COT.
[NS09] describes a characterization of consecutive-ones property solely based on the cardinality properties of the set representations of the columns (rows); every column (row) is equivalent to a set that has the row (column) indices of the rows (columns) that have one entries in this column (row). This is interesting and relevant, especially to this thesis because it simplifies COT to a great degree by reducing the solution search space owing to the a simple set theoretic characterization.

[McC04] describes a different approach to COT. While all previous COT algorithms gave the COP order if the matrix has the property but exited stating negative if otherwise, this algorithm gives an evidence by way of a certificate of matrix even when it has no COP. This enables a user to verify the algorithm's result even when the answer is negative. This is significant from an implementation perspective because automated program verification is hard and manual verification is more viable. Hence having a certificate reinforces an implementation's credibility. Note that when the matrix *has* COP, the COP order is the certificate. The internal machinery of this algorithm is related to the weighted betweenness problem addressed in [COR98].

2.2.4 PC-tree– a generalization of PQ-tree

PC-tree is a generalization of PQ-tree. It is a data structure that is analogous to PQ-tree but for matrices with circular-ones property. PC-tree was introduced by [SH99] for the purpose of planarity testing where this data structure represents partial embeddings of planar graphs. In [Hsu01], Hsu reintroduces PC-tree as a generalization of PQ-tree and shows how it simplifies [BL76]'s planarity test by making the PQ-tree construction much less complicated. Later [HM03], discovers that PC-tree is a representation of all circular-ones property orders of a matrix when it is unrooted. PC-tree presented in [Hsu01] is rooted; however the construction of PC-tree is the same in both results. The property of being unrooted is necessary in order to use PC-tree as a data structure for encoding circular ordering. Definition 2.2.4 defines PC-tree.

Definition 2.2.4. [*PC-tree* [Hsu01, Dom08].] A PC-tree of matrix M with CROP on columns (rows), is a tree with the following properties.

i. Is unrooted – thus it has (a) no parent child relationship between nodes (b) there is no left to right (or vice versa) ordering.

ii. Each leaf uniquely represents a row (column) of M. The leaf order of the tree gives a CROP order for column (row) for M. Moreover, any sequence obtained by considering the leaves in clockwise or counter-clockwise order describes a CROP order for M.

iii. Every non-leaf node in the tree is labeled P or C.

iv. The neighbors of P nodes can be permuted in any fashion to obtain a new CROP order for M.

v. The tree can be changed by applying the following "mirroring" operation to obtain a new CROP order for M. Root the PC-tree at a neighbor of a C-node, v and mirror the subtree whose root is v and finally unrooting the tree. Mirroring a subtree is done by putting the children of every node of the subtree in reverse order.

As a data structure when PC-tree is compared with PQ-tree, the differences are, (i) it is unrooted, (ii) it represents all CROP order of a matrix (iii) it has C nodes instead of Q nodes which can be "mirrored" (operation defined in Definition 2.2.4 v). The algorithms of construction of PQ-tree in [BL76] and that

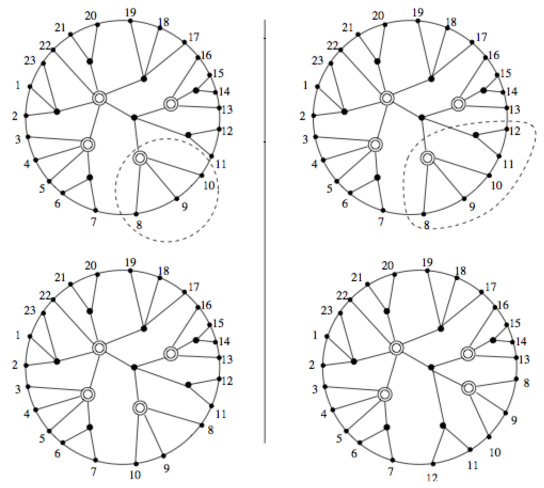


Figure 2: The PC tree can be viewed as a gadget for generating the circular-ones permutations of the columns. The C nodes are represented by double circles and the P nodes are represented by black dots. The subtree lying at one side of an edge can be flipped over to reverse the order of its leaves. The order of leaves of a consecutive set of subtrees that would result from the removal of a P node can also be reversed. All circular-ones arrangements can be obtained by a sequence of such reversals.

$$\begin{array}{cccccccccc}
c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} \\
\mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\
0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1}
\end{array}$$

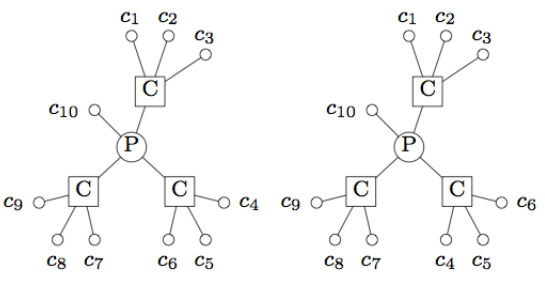


Figure 2.5: *PC*-tree PLACEHOLDER IMGS [HM03, Dom08]

of PC-tree in [Hsu01, HM03] starkly differ since the latter is a much simplified procedure.

2.3 Optimization problems in COP

So far we have been concerned about matrices that have the consecutive ones property. However in real life applications, it is rare that data sets represented by binary matrices have COP, primarily due to the noisy nature of data available. At the same time, COP is not arbitrary and is a desirable property in practical data representation [COR98, JKC$^+$04, Kou77]. In this context, there are several interesting problems when a matrix does not have COP but is "close" to having COP or is allowed to be altered to have COP. These are the optimization problems related to a matrix which does not have COP. Some of the significant problems are surveyed in this section.

couple of lines refering to tucker's submatrices. refer earlier section.

Once a matrix has been detected to not have COP (using any of the COT algorithms mentioned earlier), it is naturally of interest

1. to find out the smallest forbidden substructure (in terms of number of rows and/or columns and/or number of entries that are **1**s). [Dom08] discusses a couple of algorithms which are efficient if the number of **1**s in a row is small. This is of significance in the case of sparse matrices where this number is much lesser than the number of columns.
2. $(*, \Delta)$-*matrices* are matrices with no restriction on number of **1**s in any column but has at most Δ **1**s in any row. Min COS-R (Min COS-C), Max COS-R (Max COS-C) are similar problems which deals with *inducing COP* on a matrix.

   (a) In the dual problem Max COS-R (Max COS-C) the search is for the maximum number of rows (columns) that induces a submatrix with COP.

   (b) In Min COS-R (Min COS-C) the question is to find the minimum number of rows (columns) that must be deleted to result in a matrix with COP.

   Given a matrix M with no COP, [Boo75] shows that finding a submatrix M' with all columns but a maximum cardinality subset of rows such that M' has COP is NP complete. [HG02] corrects an error of the abridged proof of this reduction as given in [GJ79]. [Dom08] discusses all these problems in detail giving an extensive survey of the previously existing results which are almost exhaustively all approximation results and hardness results. Taking this further, [Dom08] presents new results in the area of parameterized algorithms for this problem.
3. Another problem is to find the minimum number of entries in the matrix that can be toggled to result in a matrix with COP. [Vel85] discusses approximation of COP Augmentation which is the problem of changing of the minimum number of zero entries to **1**s so that the resulting matrix has COP. As mentioned earlier, this problem is known to be NP complete due to [Boo75]. [Vel85] also proves, using a reduction to the longest path problem, that finding a Tucker's forbidden submatrix of at least k rows is NP complete.

4. [JKC$^+$04] discusses the use of matrices with almost-COP (instead of one block of consecutive **1**s, they have x blocks, or *runs*, of consecutive **1**s and x is not too large) in the storage of very large databases. The problem is that of reordering of a binary matrix such that the resulting matrix has at most k runs of **1**s. This is proved to be NP hard using a reduction from the Hamiltonian path problem.

2.4 COP in Graph Isomorphism

The survey from kklv10 conclusion.

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