Generalization of the Consecutive-ones Property

A THESIS

submitted by

ANJU SRINIVASAN

for the award of the degree of

MASTER OF SCIENCE *by Research*

from the department of

COMPUTER SCIENCE AND ENGINEERING

at

INDIAN INSTITUTE OF TECHNOLOGY MADRAS

Guindy, Chennai - 600036

FEBRUARY 2012

CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 3

TREE PATH LABELING OF PATH HYPERGRAPHS – NEW RESULTS

This chapter documents all the new results obtained by us in the area of tree path labeling of path hypergraphs which is the problem addressed in this thesis. The organization of the chapter is as follows.

Section 3.1 recalls the idea of tree path labeling. The necessary preliminaries with definitions etc. are presented in Section 3.2. Section 3.3 documents a characterization for feasible path labelings. In Section 3.4 we demonstrate the algorithm described in Section 3.3 by working out the solution to an example. Section 3.5 describes two special cases where the target tree is of a particular family of trees. The first one is shown to be equivalent to COP testing in Section 3.5.1. Section 3.5.2 discusses the second special case and presents a polynomial time algorithm to find the tree path labeling of a given set system from a given k-subdivided star. Section 3.6 discusses the plain vanilla version where the target tree has no restrictions and the algorithm to find a feasible TPL, if any, in this case. Finally, in Section 3.7 we discuss the complexity of TPL and we conclude by showing that consecutive-ones property testing, which is a special case of TPL, is in LOGSPACE.

3.1 Summary of Proposed Problems

In Section 1.5 we see that consecutive-ones property and its equivalent problem of interval labeling of a hypergraph is a special case of the general problem of tree path labeling of

path hypergraphs. Moreover, Section 2.2.5 describes how the problem of consecutive-ones property testing can be easily seen as a simple constraint satisfaction problem involving a hypergraph or a system of sets from a universe. Every column (row) of the binary matrix can be converted into a set of non-negative integers which are the indices of the rows (columns) with **1**s in that column (row). When observed in this context, if the matrix has the COP on columns (rows), a reordering of its rows (columns) will result in sets that have only consecutive integers. In other words, the sets after applying the COP row (column) permutation are intervals. In this form, one can see that this is indeed the problem of finding interval assignments to the given set system with a single permutation of the universe (set of row or column indices for COP of columns or rows, respectively) which permutes each set to an interval. The result in [NS09] (see Theorems 2.2.11 and 2.2.12) characterizes interval assignments to the sets which can be obtained from a single permutation of the universe. The cardinality of the interval assigned to it must be same as the cardinality of the set, and the intersection cardinality of any two sets must be same as the intersection cardinality of the corresponding intervals. This is a necessary and sufficient condition.

Intervals are paths from a tree with maximum degree two. Thus the interval assignment problem can be naturally generalized into path assignment problem from any tree. We refer to this as the *tree path labeling problem of path hypergraphs*. This is analogous to the interval labeling problem in literature [KKLV10] to interval hypergraphs. In brief, the problem is defined as follows – given a hypergraph $\mathcal{F}$ from universe U (i. e. hypergraph vertex set) and a target tree T, does there exist a bijection ϕ from U to the vertices of T such that for each hyperedge when ϕ is applied to its elements it gives a path on T. More formally, the problem definition is given by Compute Feasible Tree Path Labeling.

Compute Feasible Tree Path Labeling

`Input`	A hypergraph $\mathcal{F}$ with vertex set U and a tree T.
`Question`	Does there exist a set of paths $\mathcal{P}$ from T and a bijection $\ell : \mathcal{F} \to \mathcal{P}$, such that Feasible Tree Path Labeling returns **true** on $(\mathcal{F}, T, \ell)$.

To characterize a feasible TPL, we consider the case where a tree path labeling is also given as input and we are required to test if the given labeling is feasible. This is defined by the Feasible Tree Path Labeling problem.

FEASIBLE TREE PATH LABELING

Input	A hypergraph $\mathcal{F}$ with vertex set U, a tree T, a set of paths $\mathcal{P}$ from T and a bijection $\ell : \mathcal{F} \to \mathcal{P}$.
Question	Does there exist a bijection $\phi : U \to V(T)$ such that ϕ when applied on any hyperedge in $\mathcal{F}$ will give the path mapped to it by the given tree path labeling ℓ. i.e., $\ell(S) = \{\phi(x) \mid x \in S\}$, for every hyperedge $S \in \mathcal{F}$.

Section 3.3 discusses FEASIBLE TREE PATH LABELING and presents an algorithmic characterization for a feasible TPL.

With respect to computing a feasible TPL, as suggested by COMPUTE FEASIBLE TREE PATH LABELING problem, we were unable to discover an efficient algorithm for it. However, we consider two special cases of the same on special target trees – namely, COMPUTE INTERVAL LABELING and COMPUTE k-SUBDIVIDED STAR PATH LABELING on intervals and k-subdivided stars (see Definition 3.2.6), respectively. Section 3.5 discusses these problems and their polynomial time solutions. COMPUTE INTERVAL LABELING is the known problem of COP testing. The polynomial time solution to COMPUTE k-SUBDIVIDED STAR PATH LABELING is a new result by us.

COMPUTE INTERVAL LABELING

Input	A hypergraph $\mathcal{F}$ with vertex set U and a tree T with maximum degree 2.
Question	Does there exist a set of paths $\mathcal{P}$ from T and a bijection $\ell : \mathcal{F} \to \mathcal{P}$, such that FEASIBLE TREE PATH LABELING returns **true** on $(\mathcal{F}, T, \ell)$.

COMPUTE k-SUBDIVIDED STAR PATH LABELING

Input	A hypergraph $\mathcal{F}$ with vertex set U such that every hyperedge $S \in \mathcal{F}$ is of cardinality at most $k+2$ and a k-subdivided star T.
Question	Does there exist a set of paths $\mathcal{P}$ from T and a bijection $\ell : \mathcal{F} \to \mathcal{P}$, such that FEASIBLE TREE PATH LABELING returns **true** on $(\mathcal{F}, T, \ell)$.

An algorithm for COMPUTE FEASIBLE TREE PATH LABELING on arbitrary trees is presented in Section 3.6 which would run in polynomial time if two subproblems we identify are polynomial time solvable.

3.2 Preliminaries to new results

This section describes ideas and terminologies necessary in this chapter. Some of these may have been seen earlier in this document, but is recalled for the sake of completeness of the chapter.

A note on attribution – the machinery for labeling of hypergraphs has been adopted from [KKLV10] and adapted to the needs of our work. Some classic graph definitions are from [Gol04].

3.2.1 Set systems and Hypergraphs

Consider a universe U with $|U| = n$. The set $\mathcal{F} \subseteq (2^U \setminus \emptyset)$ is a *set system* with universe U. The *support of set system* $\mathcal{F}$ denoted by $supp(\mathcal{F})$, is the union of all the sets in $\mathcal{F}$.

$$supp(\mathcal{F}) = \bigcup_{S \in \mathcal{F}} S$$

By convention, the set system $\mathcal{F}$ "covers" its universe: $supp(\mathcal{F}) = U$. If the sets in $\mathcal{F}$ are intervals or paths of a tree, then it is called an interval system or a path system, respectively.

A set system $\mathcal{F}$ can also be visualized as a *hypergraph* $\mathcal{F}_H$ whose vertex set is U and hyperedges are the sets in $\mathcal{F}$. This is a known representation for interval systems in literature [BLS99, KKLV10]. We extend this definition here to path systems. Due to the equivalence of set system and hypergraph as far as this thesis is concerned, we drop the subscript $_H$ in the notation and refer to both the structures by $\mathcal{F}$. Thus hypergraphs and set systems are synonyms. So are hyperedges and sets.

The notion of *hypergraph isomorphism* is central to the work of this thesis. This is defined by Definition 3.2.1.

Definition 3.2.1 (Hypergraph isomorphism). Let $\mathcal{F}'$, $\mathcal{F}''$ be two hypergraphs. They are said to be *isomorphic* to each other, denoted by $\mathcal{F}' \cong \mathcal{F}''$, if and only if there exists a bijection $\phi : supp(\mathcal{F}') \to supp(\mathcal{F}'')$ such that for all sets $A \subseteq supp(\mathcal{F}')$, A is a hyperedge in $\mathcal{F}'$ if and only if B is a hyperedge in $\mathcal{F}''$ where $B = \{\phi(x) \mid x \in A\}$ (this is also written as $B = \phi(A)$).

Figure 3.1 shows an example of hypergraphs and hypergraph isomorphism.

Recall intersection graph and path graph from Definition 1.3.9 and 1.3.10. The *intersection graph* $\mathbb{I}(\mathcal{F})$ of a hypergraph or set system $\mathcal{F}$ is a graph such that its vertex set has a bijection to $\mathcal{F}$ and there exists an edge between two vertices if and only if their corresponding hyperedges have a non-empty intersection. A graph G is a path graph if it is isomorphic to the intersection graph of a path system.

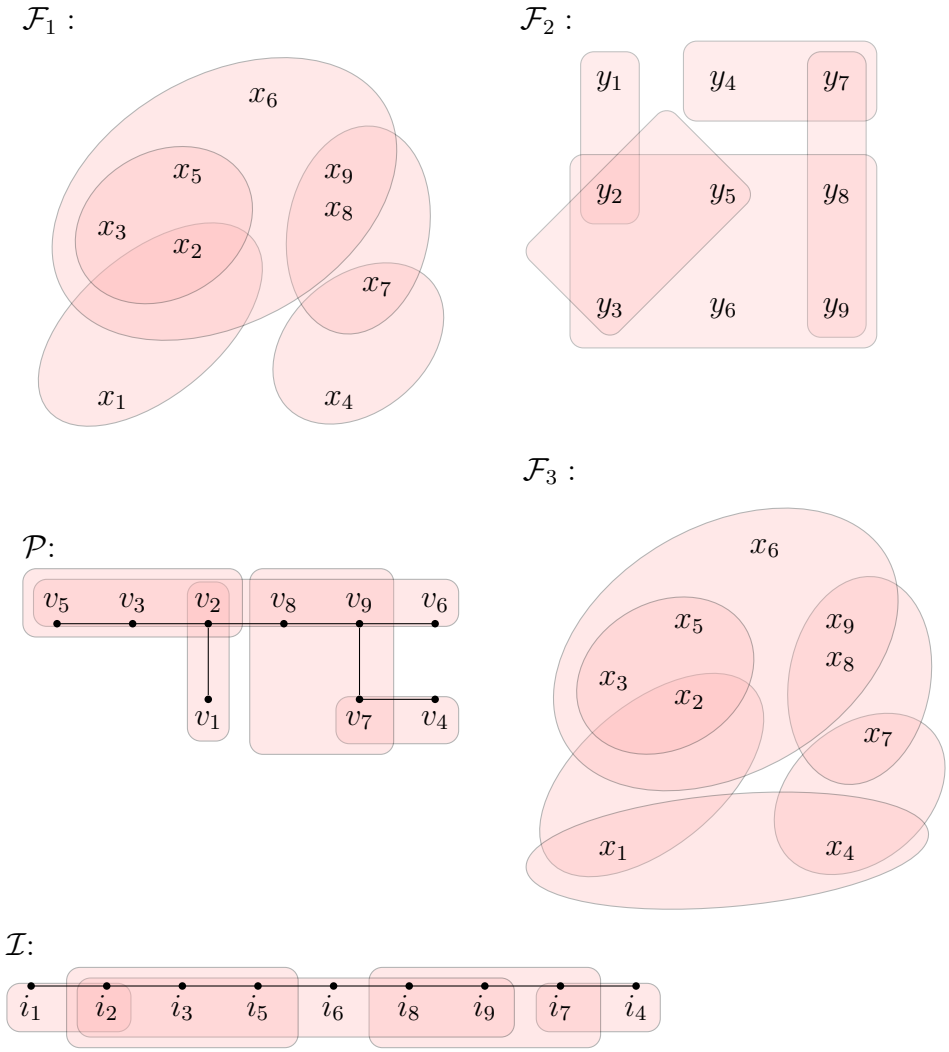


Figure 3.1: Hypergraphs and set systems. $\mathcal{F}_1$ and $\mathcal{F}_2$ are isomorphic hypergraphs. The subscript of each element is such that if ϕ is the hypergraph isomorphism, $\phi(x_i) = y_i$. $\mathcal{I}$ and $\mathcal{P}$ are an interval system and a path system respectively which are both isomorphic to $\mathcal{F}_1$ (and by transitivity, isomorphic to $\mathcal{F}_2$). Here too the subscript of elements indicate the isomorphism bijection. $\mathcal{F}_3$ is a hypergraph that is not isomorphic to any of the other hypergraphs – moreover, it is one that cannot have a feasible path labeling to any tree since it does not have the Helly's property [Gol04, Pr. 4.7] anymore due to the addition of hyperedge $\{x_1, x_4\}$.

3.2.2 Path Labeling and Path Hypergraphs

The graph T represents a *target tree* with same number of vertices as elements in U; $|V(T)| = |U| = n$. A *path system* $\mathcal{P}$ is a set system of paths from T.

$$\mathcal{P} \subseteq \{P \mid P \subseteq V,\ T[P] \text{ is a path}\}$$

A special mapping of paths from target tree T to hyperedges in $\mathcal{F}$ is called a *path labeling*. Definition 3.2.2 gives details. Note that the term "tree path labeling" is synonymous with "path labeling" in this document since we only consider trees and not general graphs as source of paths.

Definition 3.2.2 (Path labeling of a hypergraph). Consider a hypergraph $\mathcal{F}$ and a path

system $\mathcal{P}$ from tree T such that $\mathbb{I}(\mathcal{F}) \cong \mathbb{I}(\mathcal{P})$. Then the associated bijection $\ell : \mathcal{F} \to \mathcal{P}$ due to this isomorphism is called a **path labeling** of $\mathcal{F}$ from T. The path system $\mathcal{P}$ is alternatively denoted as $\mathcal{F}^{\ell}$. The path labeling may also be denoted by ordered pairs $(\mathcal{F}, \ell)$ or $(\mathcal{F}, \ell, T)$.

To elaborate on the phrase "associated bijection" in Definition 3.2.2, let $g : V(\mathbb{I}(\mathcal{F})) \to V(\mathbb{I}(\mathcal{P}))$ be the given intersection graph isomorphism, where $V(\mathbb{I}(\mathcal{F})) = \{v_S \mid S \in \mathcal{F}\}$ and $V(\mathbb{I}(\mathcal{P})) = \{v_P \mid P \in \mathcal{P}\}$. Then the associated bijection $\ell : \mathcal{F} \to \mathcal{P}$ is defined as follows. For all $S' \in \mathcal{F}$ and $P' \in \mathcal{P}$, $\ell(S') = P'$ if and only if $g(v_{S'}) = v_{P'}$. Thus, it can be seen that a path labeling preserves the non-empty intersection property between every pair of hyperedges in $\mathcal{F}$.

At this point, it is worth comparing Definition 3.2.2 with Definition 3.2.1 seen in the previous section. It may be noted that there are two kinds of isomorphisms here. One is the isomorphism of intersection graphs of $\mathcal{F}$ and $\mathcal{P}$, i.e. $\mathbb{I}(\mathcal{F})$ and $\mathbb{I}(\mathcal{P})$ and second is the isomorphism between the hypergraphs $\mathcal{F}$ and $\mathcal{P}$ themselves. It is easy to verify that given a path labeling of hypergraph $\mathcal{F}$, there need not be a hypergraph isomorphism involved. In fact, checking if an induced hypergraph isomorphism results from a given path labeling is the problem of Compute Feasible Tree Path Labeling. However, a hypergraph isomorphism always induces a path labeling as shown by the following Definition 3.2.3.

Definition 3.2.3 (Hypergraph isomorphism induced path labeling). Let $\mathcal{F}$ be a hypergraph and $\mathcal{P}$ be a path system such that $\mathcal{F} \cong \mathcal{P}$. Let $\phi : supp(\mathcal{F}) \to supp(\mathcal{P})$ be this hypergraph isomorphism function. Then there is an **induced path labeling** denoted by $p_\phi : \mathcal{F} \to \mathcal{P}$ of the hypergraph such that $p_\phi(S) = \phi(S)$ (see Definition 3.2.1) for all $S \in \mathcal{F}$.

Combining the ideas of path labeling, hypergraph isomorphism and its induced path labeling we now define *feasibility* of a path labeling in Definition 3.2.4.

Definition 3.2.4 (Feasible path labeling). Let $\mathcal{F}$ be a hypergraph and $(\mathcal{F}, \ell)$ be a path labeling. $(\mathcal{F}, \ell)$ is called a **feasible path labeling** if the following conditions hold.

- $\mathcal{F} \cong \mathcal{F}^{\ell}$
- if the above hypergraph isomorphism is $\phi : supp(\mathcal{F}) \to supp(\mathcal{F}^{\ell})$, then $p_\phi = \ell$

This isomorphism ϕ is called the **feasibility hypergraph isomorphism** or feasibility isomorphism.

One of the problems solved in this thesis Feasible Tree Path Labeling characterizes feasible tree path labeling.

Now we will see the definition of a *path hypergraph* which is a simple idea extended from interval hypergraphs in [KKLV10].

Definition 3.2.5 (Path hypergraph). Let $\mathcal{F}$ be a set system. $\mathcal{F}$ is called a **path hypergraph** if there exists a target tree T and a path system $\mathcal{P}$ from T such that $\mathcal{F} \cong \mathcal{P}$. If $\mathcal{F}$ is thus a path hypergraph, $\mathcal{P}$ is called **path representation** of $\mathcal{F}$.

In other words, if there exists a feasible tree path labeling for a given hypergraph, it is a path hypergraph.

3.2.3 Overlap Graphs and Marginal Hyperedges

Recall the notion of *overlap* from Definition 1.3.5. Two hyperedges S and S' are said to overlap, denoted by $S \between S'$, if they have a non-empty intersection and neither is contained in the other.

An *overlap graph* $\mathbb{O}(\mathcal{F})$ of a hypergraph $\mathcal{F}$ is a graph such that its vertex set has a bijection to $\mathcal{F}$ and there exists an edge between two of its vertices if and only if their corresponding hyperedges overlap. Thus $\mathbb{O}(\mathcal{F})$ is a spanning subgraph of $\mathbb{I}(\mathcal{F})$ and not necessarily connected. Each connected component of $\mathbb{O}(\mathcal{F})$ is called an *overlap component*.

A hyperedge $S \in \mathcal{F}$ is called *marginal* if for all $S' \between S$, the overlaps $S \cap S'$ form a single inclusion chain [KKLV10] (see Definition1.3.7). Additionally, if S is such that it is contained in no other marginal hyperedge in $\mathcal{F}$, then it is called *super-marginal*.

3.2.4 Miscellaneous

A *star* graph is a complete bipartite graph $K_{1,p}$ (which is clearly a tree with p leaves). The vertex with maximum degree in a star is called its *center* and the edges are called *rays*. This thesis is interested in a star-like graph which is defined in Definition 3.2.6

Definition 3.2.6 (k-subdivided star). A **k-subdivided star** is a star with all its rays subdivided exactly k times. A **ray** of a k-subdivided star is the path from the center vertex to a leaf. It is clear that all rays of a k-subdivided star are of length $k + 2$.

Definition 3.2.7 (In-tree). An **in-tree** is a directed rooted tree in which all edges are directed toward to the root.

In-trees are used in Section 3.6 which describes our solution to Compute Feasible Tree Path Labeling problem.

Finally, note that as a convention in this thesis, the set I represents the index set $[m]$ or $[n]$. If index i is used without further qualification, it is intended that $i \in I$.

3.3 Characterization of Feasible Tree Path Labeling

In this section we discuss the Feasible Tree Path Labeling problem introduced in Section 3.1 and give a set cardinality based characterization for it.

For this characterization, we generalize the notion of the ICPIA [NS09]. We show that for a given hypergraph $\mathcal{F}$ with universe U, a target tree T, and path labeling from T to $\mathcal{F}$, there is a feasible bijection between U and $V(T)$ if and only if all intersection cardinalities among any three sets (not necessarily distinct) is same as the intersection cardinality of the paths assigned to them. This characterization is proved constructively and the algorithm outputs a feasible bijection between U and $V(T)$.

We now introduce a special kind of path labeling called *Intersection Cardinality Preserving Path Labeling (ICPPL)* in Definition 3.3.1.

Definition 3.3.1. Consider a path labeling $(\mathcal{F}, \ell)$ on the given tree T. We call $(\mathcal{F}, \ell)$ an **Intersection Cardinality Preserving Path Labeling (ICPPL)** if it has the following properties.

i. $|S| = |\ell(S)|$ for all $S \in \mathcal{F}$

ii. $|S_1 \cap S_2| = |\ell(S_1) \cap \ell(S_2)|$ for all distinct $S_1, S_2 \in \mathcal{F}$

iii. $|S_1 \cap S_2 \cap S_3| = |\ell(S_1) \cap \ell(S_2) \cap \ell(S_3)|$ for all distinct $S_1, S_2, S_3 \in \mathcal{F}$

In the remaining part of this section we show that $(\mathcal{F}, \ell)$ is feasible if and only if it is an ICPPL. Algorithm 3 takes $(\mathcal{F}, \ell)$ as input and computes the feasibility bijection if it is feasible or else reports failure. This algorithm recursively does two levels of filtering of $(\mathcal{F}, \ell)$ to make it simpler in terms of set intersections while retaining a non-empty subset of isomorphisms, if any, between $\mathcal{F}$ and $\mathcal{F}^\ell$. These two filters are `filter common leaf` in Algorithm 1 and `filter fix leaf` in Algorithm 2.

First, we present Algorithm 1 which describes `filter common leaf`, and prove its correctness. This algorithm refines the path labeling by processing pairs of paths in $\mathcal{F}^\ell$ that share a leaf until no two paths in the new path labeling share any leaf.

In order to prove the correctness of Algorithm 1, we present a few lemma which are useful in subsequent arguments.

The first two lemma below are from results in [NS09]. They have been reworded to use the terminology adopted in this thesis. Lemma 3.3.1 shows that the preservation of pairwise intersection cardinality in an interval assignment is sufficient to preserve three way intersection cardinality. Lemma 3.3.2 shows that if a path labeling (i) has target tree as a path i. e. it is an interval labeling, and (ii) it preserves pairwise intersection cardinalities i. e. it is an ICPIA, then it is a feasible path labeling.

Algorithm 1 Refine ICPPL `filter common leaf` $(\mathcal{F}, \ell, T)$

$\mathcal{F}_0 \leftarrow \mathcal{F}$, $\ell_0(S) \leftarrow \ell(S)$ for all $S \in \mathcal{F}_0$
$j \leftarrow 1$
while there is $S_1, S_2 \in \mathcal{F}_{j-1}$ such that $\ell_{j-1}(S_1)$ and $\ell_{j-1}(S_2)$ have a common leaf in T **do**
  $\mathcal{F}_j \leftarrow (\mathcal{F}_{j-1} \setminus \{S_1, S_2\}) \cup \{S_1 \cap S_2, S_1 \setminus S_2, S_2 \setminus S_1\}$
    `Remove` S_1, S_2 `and add the ``filtered'' sets`
  for every $S \in \mathcal{F}_{j-1}$ s.t. $S \neq S_1$ and $S \neq S_2$ **do** $\ell_j(S) \leftarrow \ell_{j-1}(S)$ **end for**
  $\ell_j(S_1 \cap S_2) \leftarrow \ell_{j-1}(S_1) \cap \ell_{j-1}(S_2)$
    `Carry forward the path labeling for all existing sets other than` S_1, S_2
  $\ell_j(S_1 \setminus S_2) \leftarrow \ell_{j-1}(S_1) \setminus \ell_{j-1}(S_2)$
    `Define path labeling for new sets`
  $\ell_j(S_2 \setminus S_1) \leftarrow \ell_{j-1}(S_2) \setminus \ell_{j-1}(S_1)$
  $j \leftarrow j + 1$
end while
$\mathcal{F}' \leftarrow \mathcal{F}_j$, $\ell' \leftarrow \ell_j$
return $(\mathcal{F}', \ell')$

Lemma 3.3.1 ([NS09, Lem. 1]). *Let P be a path, S_1, S_2, S_3 be 3 sets, and T_1, T_2, T_3 be paths from P, such that $|S_i \cap S_j| = |T_i \cap T_j|, 1 \leq i, j \leq 3$. Then, $|S_1 \cap S_2 \cap S_3| = |T_1 \cap T_2 \cap T_3|$.*

Lemma 3.3.2 ([NS09, Th. 2]). *A path labeling $(\mathcal{F}, \ell)$ with target tree T as a path is feasible if and only if it is an ICPIA.*

Next, we see Lemma 3.3.3 which is a direct consequence of Definition 3.3.1 and is analogous to [NS09, Cor. 1].

Lemma 3.3.3. *If ℓ is an ICPPL, and $S_1, S_2, S_3 \in \mathcal{F}$, then $|S_1 \cap (S_2 \setminus S_3)| = |\ell(S_1) \cap (\ell(S_2) \setminus \ell(S_3))|$.*

Proof. Let $P_i = \ell(S_i)$, for all $1 \leq i \leq 3$. $|S_1 \cap (S_2 \setminus S_3)| = |(S_1 \cap S_2) \setminus S_3| = |S_1 \cap S_2| - |S_1 \cap S_2 \cap S_3|$. Due to properties (ii) and (iii) of ICPPL, $|S_1 \cap S_2| - |S_1 \cap S_2 \cap S_3| = |P_1 \cap P_2| - |P_1 \cap P_2 \cap P_3| = |(P_1 \cap P_2) \setminus P_3| = |P_1 \cap (P_2 \setminus P_3)|$. □

For interval labeling, [NS09] shows that if it is an ICPIA i. e. pairwise intersection cardinalities are preserved, then d-way intersection cardinalities are preserved for $2 < d < n$. Thus, in a way, pairwise intersections "capture" feasibility of an interval labeling. In the same vein, the following lemma indicates that three-way intersection cardinality "captures" d-way intersection cardinalities for $3 < d < n$ in a tree path labeling.

Lemma 3.3.4. *Let $(\mathcal{F}, \ell)$ be an ICPPL and $\ell(S_i) = P_i$, $S_i \in \mathcal{F}$, $1 \leq i \leq 4$. Then, $|\cap_{i=1}^4 S_i| = |\cap_{i=1}^4 P_i|$.*

Proof. Consider the set of set intersections with S_1 $\mathcal{F}' = \{S_2 \cap S_1, S_3 \cap S_1, S_4 \cap S_1\}$ and let

ℓ' be a tree path labeling of $\mathcal{F}'$ such that $\ell'(S_i \cap S_1) = P_i \cap P_1, 2 \leq i \leq 4$. Clearly, $P_2 \cap P_1$, $P_3 \cap P_1$, and $P_4 \cap P_1$ are subpaths of path P_1, thus equivalent to intervals. Due to the three way intersection cardinality preservation property of the ICPPL $(\mathcal{F}, \ell)$, this new tree path labeling $(\mathcal{F}', \ell')$ preserves pairwise intersection cardinalities. Now by applying Lemma 3.3.1 to the sets in $\mathcal{F}'$ and their corresponding path images due to ℓ', it follows that $|\cap_{i=1}^{4} S_i| = |\cap_{i=1}^{4} P_i|$. □

Corollary 3.3.5. *Let $(\mathcal{F}, \ell)$ be an ICPPL and $\ell(S_i) = P_i$, $S_i \in \mathcal{F}$, $1 \leq i \leq 4$. If P_i, $1 \leq i \leq 4$ are such that $P_1 \cap P_2$, $P_1 \setminus P_2$, $P_2 \setminus P_1$ are also paths, then:*

$$|(S_1 \setminus S_2) \cap S_3 \cap S_4| = |(P_1 \setminus P_2) \cap P_3 \cap P_4|$$
$$|(S_2 \setminus S_1) \cap S_3 \cap S_4| = |(P_2 \setminus P_1) \cap P_3 \cap P_4|$$

Proof. It is clear that $|(P_1 \setminus P_2) \cap P_3 \cap P_4| = |\cap_{i=1}^{4} P_i| - |P_2 \cap P_3 \cap P_4|$. Since we have an ICPPL, from Lemma 3.3.4, it follows that the intersection cardinalities are preserved. □

Lemma 3.3.6. *Let $(\mathcal{F}, \ell)$ be an ICPPL and $\ell(S) = P$ for some $S \in \mathcal{F}$. If sets S_{priv} and P_{priv} contain only those elements of S and P that occur in no set other than S and P, respectively, then $|S_{priv}| = |P_{priv}|$.*

Proof. Consider the set system $\mathcal{F}' = \{S_i \cap S \mid S_i \in \mathcal{F}, i \in [n]\}$ and path labeling $\ell'(S_i \cap S) = P_i \cap P$. Since $(\mathcal{F}, \ell)$ is an ICPPL, it can be easily verified that $(\mathcal{F}', \ell')$ is an ICPIA on path (interval) P. Let S_{two} and P_{two} be the elements of S and P which are in one *other* set and path, respectively i. e. $S_{two} = S \cap (\cup_{S_i \neq S} S_i)$ and P_{two} is defined analogously. It may be noted that $S_{two} = supp(\mathcal{F}')$ and $P_{two} = supp(\mathcal{F}'^{\ell'})$. From Lemma 3.3.2 it follows that there is a bijection ϕ from S_{two} to P_{two} such that for each i, the image of $S_i \cap S$ under ϕ is $P_i \cap P$ (definition of feasibility). Moreover, this means $|S_{two}| = |P_{two}|$. By definition, $S_{priv} = S \setminus S_{two}$ and $P_{priv} = P \setminus P_{two}$. Thus $|S_{priv}| = |S \setminus S_{two}| = |P \setminus P_{two}| = |P_{priv}|$. □

Now we will prove the effectiveness of Algorithm 1. Lemma 3.3.7 proves that `filter common leaf` does not destroy the feasibility of the input, if it is indeed feasible. Following this, Lemma 3.3.8 and Lemma 3.3.9 show that if the input is an ICPPL the result of `filter common leaf` is also an ICPPL and they both have the same set of feasibility hypergraph isomorphisms.

Lemma 3.3.7. *In Algorithm 1, if input $(\mathcal{F}, \ell)$ is a feasible path labeling then at the end of jth iteration of the* **while** *loop, $j \geq 0$, $(\mathcal{F}_j, \ell_j)$ is a feasible path labeling.*

Proof. We will prove this by mathematical induction on the number of iterations. The base case $(\mathcal{F}_0, \ell_0)$ is feasible since it is the input itself which is given to be feasible. Assume the lemma is true till $j-1$th iteration i. e. every hypergraph isomorphism

$\phi : supp(\mathcal{F}_{j-1}) \to V(T)$ that defines $(\mathcal{F}, \ell)$'s feasibility, is such that the induced path labeling on $\mathcal{F}_{j-1}$, say denoted by $p_{\phi[\mathcal{F}_{j-1}]}$, is equal to ℓ_{j-1}. We will prove that ϕ is also the bijection that makes $(\mathcal{F}_j, \ell_j)$ feasible. Note that $supp(\mathcal{F}_{j-1}) = supp(\mathcal{F}_j)$ since the new sets in $\mathcal{F}_j$ are created from basic set operations to the sets in $\mathcal{F}_{j-1}$ adding or removing no elements. For the same reason and ϕ being a bijection, it is clear that when applying the ϕ-induced path labeling on $\mathcal{F}_j$, $p_{\phi[\mathcal{F}_j]}(S_1 \setminus S_2) = p_{\phi[\mathcal{F}_{j-1}]}(S_1) \setminus p_{\phi[\mathcal{F}_{j-1}]}(S_2)$. Now observe that $\ell_j(S_1 \setminus S_2) = \ell_{j-1}(S_1) \setminus \ell_{j-1}(S_2) = p_{\phi[\mathcal{F}_{j-1}]}(S_1) \setminus p_{\phi[\mathcal{F}_{j-1}]}(S_2)$. Thus the induced path labeling $p_{\phi[\mathcal{F}_j]} = \ell_j$. □

Lemma 3.3.8. *Given that the input to Algorithm 1 is an ICPPL, at the end of jth iteration, $j \geq 0$, of the* **while** *loop, the following invariants are maintained.*

I	$\ell_j(R)$ *is a path in* T,	*for all* $R \in \mathcal{F}_j$
II	$\lvert R\rvert = \lvert\ell_j(R)\rvert$,	*for all* $R \in \mathcal{F}_j$
III	$\lvert R \cap R'\rvert = \lvert\ell_j(R) \cap \ell_j(R')\rvert$,	*for all* $R, R' \in \mathcal{F}_j$
IV	$\lvert R \cap R' \cap R''\rvert = \lvert\ell_j(R) \cap \ell_j(R') \cap \ell_j(R'')\rvert$,	*for all* $R, R', R'' \in \mathcal{F}_j$

Proof. Proof is by induction on the number of iterations, j. In this proof, the term "new sets" will refer to the sets added to $\mathcal{F}_j$ in jth iteration in line 4 of Algorithm 1, $S_1 \cap S_2, S_1 \setminus S_2, S_2 \setminus S_1$ and its images in ℓ_j where $\ell_{j-1}(S_1)$ and $\ell_{j-1}(S_2)$ intersect and share a leaf. The invariants are true in the base case $(\mathcal{F}_0, \ell_0)$, since it is the input ICPPL. Assume the lemma is true till the $j-1$th iteration. Let us consider the possible cases for each of the above invariants for the jth iteration.

✠ *Invariant* I/II

I/IIa | *R is not a new set.* It is in $\mathcal{F}_{j-1}$. Thus trivially true by induction hypothesis.

I/IIb | *R is a new set.* If R is in $\mathcal{F}_j$ and not in $\mathcal{F}_{j-1}$, then it must be one of the new sets added in $\mathcal{F}_j$. In this case, it is clear that for each new set, the image under ℓ_j is a path since by definition the chosen sets S_1, S_2 are from $\mathcal{F}_{j-1}$ and due to the while loop condition, $\ell_{j-1}(S_1)$, $\ell_{j-1}(S_2)$ have a common leaf. Thus invariant I is proven.

Moreover, due to induction hypothesis of invariant III and the definition of l_j in terms of l_{j-1}, invariant II is indeed true in the jth iteration for any of the new sets. If $R = S_1 \cap S_2$, $|R| = |S_1 \cap S_2| = |\ell_{j-1}(S_1) \cap \ell_{j-1}(S_2)| = |\ell_j(S_1 \cap S_2)| = |\ell_j(R)|$. If $R = S_1 \setminus S_2$, $|R| = |S_1 \setminus S_2| = |S_1| - |S_1 \cap S_2| = |\ell_{j-1}(S_1)| - |\ell_{j-1}(S_1) \cap \ell_{j-1}(S_2)| = |\ell_{j-1}(S_1) \setminus \ell_{j-1}(S_2)| = |\ell_j(S_1 \setminus S_2)| = |\ell_j(R)|$. Similarly if $R = S_2 \setminus S_1$.

✠ *Invariant* III

IIIa | *R and R' are not new sets.* It is in $\mathcal{F}_{j-1}$. Thus trivially true by induction hypothesis.

IIIb | *Only one, say R, is a new set.* Due to invariant IV induction hypothesis, Lemma 3.3.3 and definition of ℓ_j, it follows that invariant III is true no matter which of the new sets R is equal to. If $R = S_1 \cap S_2$, $|R \cap R'| = |S_1 \cap S_2 \cap R'| = |\ell_{j-1}(S_1) \cap \ell_{j-1}(S_2) \cap \ell_{j-1}(R')| = |\ell_j(S_1 \cap S_2) \cap \ell_j(R')| = |\ell_j(R) \cap \ell_j(R')|$. If $R = S_1 \setminus S_2$, $|R \cap R'| = |(S_1 \setminus S_2) \cap R'| = |(\ell_{j-1}(S_1) \setminus \ell_{j-1}(S_2)) \cap \ell_{j-1}(R')| = |\ell_j(S_1 \cap S_2) \cap \ell_j(R')| = |\ell_j(R) \cap \ell_j(R')|$. Similarly, if $R = S_2 \setminus S_1$. Note R' is not a new set.

IIIc | *R and R' are new sets.* By definition, the new sets and their path images in path label ℓ_j are disjoint so $|R \cap R'| = |\ell_j(R) \cap \ell_j(R)| = 0$. Thus case proven.

✠ *Invariant* IV

This invariant is ensured at the end of every iteration due to Lemma 3.3.4 and Corollary 3.3.5.

□

Lemma 3.3.9. *If the input ICPPL $(\mathcal{F}, \ell)$ to Algorithm 1 is feasible, then the set of hypergraph isomorphism functions that defines $(\mathcal{F}, \ell)$'s feasibility is the same as the set that defines $(\mathcal{F}_j, \ell_j)$'s feasibility, if any.*

Proof. Since $(\mathcal{F}, \ell)$ is feasible, by Lemma 3.3.7 $(\mathcal{F}_j, \ell_j)$ for every iteration $j > 0$ is feasible. Also, every hypergraph isomorphism $\phi : \mathit{supp}(\mathcal{F}) \to V(T)$ that induces ℓ on $\mathcal{F}$ also induces ℓ_j on $\mathcal{F}_j$, i.e., $\ell_{\phi[\mathcal{F}_j]} = \ell_j$. Thus it can be seen that for all $x \in \mathit{supp}(\mathcal{F})$, for all $v \in V(T)$, if $(x, v) \in \phi$ then $v \in \ell_j(S)$ for all $S \in \mathcal{F}_j$ such that $x \in S$. In other words, `filter common leaf` outputs a filtered path labeling that "preserves" hypergraph isomorphisms of the original path labeling.

□

As a result of `filter common leaf` each leaf v in T is such that there is exactly one set in $\mathcal{F}$ with v as a vertex in the path assigned to it. `filter fix leaf`, in Algorithm 2, identifies elements in $\mathit{supp}(\mathcal{F})$ whose images are leaves in a hypergraph isomorphism if one exists. Let $S \in \mathcal{F}$ be such that $\ell(S)$ is a path with leaf and $v \in V(T)$ is the unique leaf incident on it. We define a new path labeling ℓ_{new} such that $\ell_{new}(\{x\}) = \{v\}$ where x an arbitrary element from $S \setminus \bigcup_{\hat{S} \neq S} \hat{S}$. In other words, x is an element present in no other set in $\mathcal{F}$ except S. This is intuitive since v is present in no other path image under ℓ other than $\ell(S)$. The element x and leaf v are then removed from the set S and path $\ell(S)$ respectively. After doing this for all leaves in T, all path images in the

new path labeling ℓ_{new} except leaf labels (a path that has only a leaf is called the *leaf label* for the corresponding single element hyperedge or set) are paths from a new pruned tree $T_0 = T \setminus \{v \mid v \text{ is a leaf in } T\}$. Algorithm 2 is now presented with details.

Algorithm 2 Leaf labeling from an ICPPL `filter fix leaf` $(\mathcal{F}, \ell, T)$

1: $\mathcal{F}_0 \leftarrow \mathcal{F}$, $\ell_0(S) \leftarrow \ell(S)$ for all $S \in \mathcal{F}_0$
   `| Path images are such that no two path images share a leaf.`
2: $j \leftarrow 1$
3: **while** there is a leaf v in T and a unique $S_1 \in \mathcal{F}_{j-1}$ such that $v \in \ell_{j-1}(S_1)$ **do**
4: $\quad \mathcal{F}_j \leftarrow \mathcal{F}_{j-1} \setminus \{S_1\}$
5: $\quad$ for all $S \in \mathcal{F}_{j-1}$ such that $S \neq S_1$ set $\ell_j(S) \leftarrow \ell_{j-1}(S)$
6: $\quad X \leftarrow S_1 \setminus \bigcup_{S \in \mathcal{F}_{j-1}, S \neq S_1} S$
7: $\quad x \leftarrow$ arbitrary element from X
8: $\quad \mathcal{F}_j \leftarrow \mathcal{F}_j \cup \{\{x\}, S_1 \setminus \{x\}\}$
9: $\quad \ell_j(\{x\}) \leftarrow \{v\}$
10: $\quad \ell_j(S_1 \setminus \{x\}) \leftarrow \ell_{j-1}(S_1) \setminus \{v\}$
11: $\quad j \leftarrow j + 1$
12: **end while**
13: $\mathcal{F}' \leftarrow \mathcal{F}_j$, $\ell' \leftarrow \ell_j$
14: **return** $(\mathcal{F}', \ell')$

Suppose the input $(\mathcal{F}, \ell)$ is feasible, yet set X in Algorithm 2 is empty in some iteration of the **while** loop. Then clearly, the algorithm will have a problem in line 7. The following Lemma 3.3.10 shows that this cannot happen. Following that Lemma 3.3.11 shows that given an ICPPL input, the filtered output path labeling after `filter fix leaf` is also an ICPPL.

Lemma 3.3.10. *If the input ICPPL $(\mathcal{F}, \ell)$ to Algorithm 2 is feasible, then for all iterations $j > 0$ of the* **while** *loop, the set X will be non-empty.*

Proof. This directly follows from Lemma 3.3.6 since X is nothing but $S_{1_{priv}}$. □

Lemma 3.3.11. *In Algorithm 2, for all $j > 0$, at the end of the jth iteration of the* **while** *loop the four invariants given in Lemma 3.3.8 hold.*

Proof. Following Lemma 3.3.10, we know that set X will not be empty in any iteration of the **while** loop if input ICPPL $(\mathcal{F}, \ell)$ is feasible and thus ℓ_j is always computed for all $j > 0$. Also note that removing a leaf from any path keeps the new path connected. Thus invariant I is obviously true. In every iteration $j > 0$, we remove exactly one element x from one set S in $\mathcal{F}$ and exactly one vertex v which is a leaf from one path $\ell_{j-1}(S)$ in T. This is because x is exclusive to S and v is exclusive to $\ell_{j-1}(S)$. Due to this fact, it is clear that the intersection cardinality equations do not change, i.e., invariants II, III, IV remain true. □

We have seen two filtering algorithms above, namely, Algorithm 1 for `filter common`

`leaf` and Algorithm 2 for `filter fix leaf` which when executed in that order result in a new ICPPL on the same universe U and target tree T. We also proved that if the input is indeed feasible, these algorithms executed in sequence give an ICPPL that preserves a non-empty subset of set of feasibility hypergraph isomorphisms of the original input. Now we present Algorithm 3 which uses these two filters to compute one such isomorphism.

Algorithm 3 computes a hypergraph isomorphism ϕ recursively using Algorithm 1 and Algorithm 2 and pruning the leaves of the input tree. In brief, it is done as follows. If the input is feasible, the output of `filter common leaf`, say $(\mathcal{F}_1, \ell_1)$, gives a new ICPPL which preserves all the original hypergraph isomorphisms. This ICPPL $(\mathcal{F}_1, \ell_1)$ is then given to `filter fix leaf` which gives the second filtered ICPPL, say $(\mathcal{F}_2, \ell_2)$ such that a feasible leaf labeling is in singleton elements of $\mathcal{F}_2$. These leaf labels are the elements in $supp(\mathcal{F})$ that map to leaves in T in at least one hypergraph isomorphism of the input ICPPL ℓ. It must be noted that at this point, the algorithm "chooses" one (or more - there could be more than one bijections that have the same leaf labels) hypergraph isomorphism ϕ', from the set of all isomorphisms resulting from the feasibility of the original input $(\mathcal{F}, \ell)$. However, if there exists one (i. e. input is feasible), it will be found – thus performing the test required in Feasible Tree Path Labeling.

Since their preimages have been found, all leaves in T are then pruned away. The leaf labels are removed from the path labeling ℓ_2 and the corresponding elements are removed from the corresponding sets in $\mathcal{F}_2$. It is now clear that we have a subproblem with a new hypergraph $\mathcal{F}'$, new tree path labeling ℓ' and target tree T'. The tree pruning algorithm is recursively called on $\mathcal{F}', \ell', T'$. The recursive call returns the bijection ϕ'' for the rest of the elements in $supp(\mathcal{F})$ which along with the leaf labels ϕ' computed earlier gives us a hypergraph isomorphism ϕ for the input $\mathcal{F}, \ell, T$. Lemma 3.3.12 proves the correctness of this computation.

Lemma 3.3.12. *If $(\mathcal{F}, \ell)$ is an ICPPL from a tree T and Algorithm 3, on input $(\mathcal{F}, \ell, T)$ returns a non-empty function ϕ, then ϕ is a hypergraph isomorphism $\phi : supp(\mathcal{F}) \to V(T)$ such that the ϕ-induced tree path labeling is equal to ℓ; $p_\phi = \ell$.*

Proof. It is clear that in the end of every recursive call to Algorithm 3, the function ϕ' is one-to-one involving all the leaves in the input target tree T (of the current recursive call). Moreover, by Lemma 3.3.9 and Lemma 3.3.10 it is consistent with the input tree path labeling ℓ (of the current recursive call). The tree pruning is done by only removing leaves in each call to the function and is done till the tree becomes empty. Thus the returned function $\phi : supp(\mathcal{F}) \to V(T)$ is a union of mutually exclusive one-to-one functions exhausting all vertices of the tree. In other words, it is a bijection from $supp(\mathcal{F})$ to $V(T)$ inducing the given path labeling ℓ and thus a hypergraph isomorphism. □

Algorithm 3 `get-hypergraph-isomorphism` $(\mathcal{F}, \ell, T)$

1: **if** T is empty **then**
2: **return** $\emptyset$
3: **end if**
4: $L \leftarrow \{v \mid v \text{ is a leaf in } T\}$
5: $(\mathcal{F}_1, \ell_1) \leftarrow$ `filter common leaf` $(\mathcal{F}, \ell, T)$
6: $(\mathcal{F}_2, \ell_2) \leftarrow$ `filter fix leaf` $(\mathcal{F}_1, \ell_1, T)$
7: $(\mathcal{F}', \ell') \leftarrow (\mathcal{F}_2, \ell_2)$
8: $\phi' \leftarrow \emptyset$
9: **for** every $v \in L$ **do**
10: $\phi'(x) \leftarrow v$ where $x \in \ell_2^{-1}(\{v\})$
`Copy the leaf labels to a one to one function` $\phi' : supp(\mathcal{F}) \to L$
11: Remove $\{x\}$ and $\{v\}$ from $\mathcal{F}'$, ℓ' appropriately
12: **end for**
13: $T' \leftarrow T \setminus L$
14: $\phi'' \leftarrow$ `get-hypergraph-isomorphism` $(\mathcal{F}', \ell', T')$
15: $\phi \leftarrow \phi'' \cup \phi'$
16: **return** ϕ

Finally we have Theorem 3.3.13 which puts together all that we saw so far in this section to prove that ICPPL is indeed a characterization of a feasible path labeling.

Theorem 3.3.13. *A path labeling $(\mathcal{F}, \ell)$ on tree T is feasible if and only if it is an ICPPL and Algorithm 3 with $(\mathcal{F}, \ell, T)$ as input returns a non-empty function.*

Proof. From Lemma 3.3.12, we know that if $(\mathcal{F}, \ell)$ is an ICPPL and if Algorithm 3 with $(\mathcal{F}, \ell, T)$ as input returns a non-empty function, then $(\mathcal{F}, \ell)$ is feasible. Now consider the case where $(\mathcal{F}, \ell)$ is feasible, i. e. there exists a hypergraph isomorphism ϕ such that $p_\phi = \ell$. Algorithm 3 then clearly returns a non-empty function because the target tree is non-empty. □

3.4 Solution to study group accommodation problem

In this section we run through the filtering and tree pruning algorithm seen in Section 3.3 on an example. The example problem is the study group accommodation problem posed in Section 1.2. The solution is depicted in Figure 3.2, Figure 3.3, Figure 3.4, Figure 3.5, Figure 3.6, Figure 3.7, Figure 3.8 and the figures along with their captions are self-explanatory.

In Figure 3.2, the original problem from Section 1.2 is restated more formally. The target tree is T with vertex set $\{1, 2, \ldots, 11\}$ on the left hand side along with the path system (hypergraph) – $\{7, 2, 6, 5\}, \{8, 2, 4\}, \{10, 6, 5, 3\}, \{9, 1, 5, 3, 11\}$. The hypergraph is $\mathcal{F}$ with hyperedges $\{\mathbf{Pa}, \mathbf{Pi}, \mathbf{Vi}, \mathbf{Ch}\}, \{\mathbf{Sn}, \mathbf{Wo}, \mathbf{Pi}\}, \{\mathbf{Ch}, \mathbf{Fr}, \mathbf{Vi}, \mathbf{Li}\}, \{\mathbf{Lu}, \mathbf{Sa}, \mathbf{Sc}, \mathbf{Ch}, \mathbf{Fr}\}$ shown in the right hand side. In subsequent illustrations, the gray boxes which contain

the elements of universe U, will be assigned to nodes and removed from the right hand side figure and placed next to the vertex of T in the left hand side figure. The path labeling is implicitly given by the color-coded Venn diagrams (the order of listing of the hyperedges in given above is also done respective to the input path labeling).

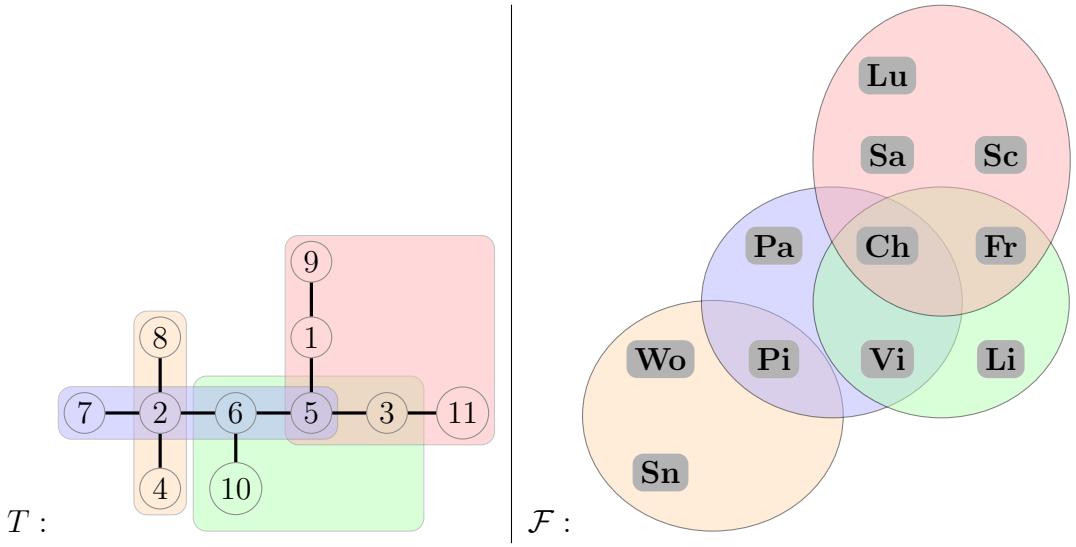


Figure 3.2: The problem is shown again here using the input described in Section 1.2.

Now see Figure 3.3 to see how the algorithm proceeds. Before that it can be observed in Figure 3.2, that none of the leaves are shared by more than one path. Hence, `filter common leaf` does not alter the hypergraphs. `filter fix leaf` finds the leaf assignments as shown – elements **Pa**, **Sn**, **Lu**, **Sc**, **Li**, **Wo** are assigned to vertices 7, 8, 9, 11, 10, 4 respectively. Computation of these leaf labels are deterministic except for **Lu**, **Sc** where the algorithm arbitrarily chooses them from $X = \{\mathbf{Lu}, \mathbf{Sa}, \mathbf{Sc}\}$ and for **Sn**, **Wo** arbitrarily choosen from $X = \{\mathbf{Sn}, \mathbf{Wo}\}$. Here we show only one such choice and clearly one could have chosen another leaf label (e. g. **Sa** on 9 or **Wo** on 8) to get a different feasible hypergraph isomorphism. After this, the leaves of T are pruned and the hypergraph updated accordingly by `get-hypergraph-isomorphism` subroutine – the pruned leaves are shown with dotted edges and the corresponding Venn diagrams show the updated hypergraphs. ϕ_0 is this leaf assignment and the algorithm proceeds to solve the subproblem with the new hypergraphs and path labeling which is: $\{2, 6, 5\}, \{2\}, \{6, 5, 3\}, \{1, 5, 3\}$ labeled with $\{\mathbf{Pi}, \mathbf{Vi}, \mathbf{Ch}\}, \{\mathbf{Pi}\}, \{\mathbf{Ch}, \mathbf{Fr}, \mathbf{Vi}\}, \{\mathbf{Sa}, \mathbf{Ch}, \mathbf{Fr}\}$ respectively.

In the subproblem shown in Figure 3.3, it can be noted that there are two leaves which are shared by more than one paths – 2 and 3. Hence `filter common leaf` filters this out one by one. In Figure 3.4, we show the handling of leaf 2 which was shared by paths $\{2, 6, 5\}$ $(= P_1$, say) and $\{2\} (= P_2$, say). `filter common leaf` removes P_1 and adds $\{6, 5\}$. Note that only one set is added and removed since $P_2 \subset P_1$. Accordingly, `filter common leaf` alters the hypergraph on the right hand side by removing and adding $\{\mathbf{Pi}, \mathbf{Vi}, \mathbf{Ch}\}$ and $\{\mathbf{Vi}, \mathbf{Ch}\}$ respectively.

After handling leaf 2, `filter common leaf` proceeds to handle leaf 3 which is

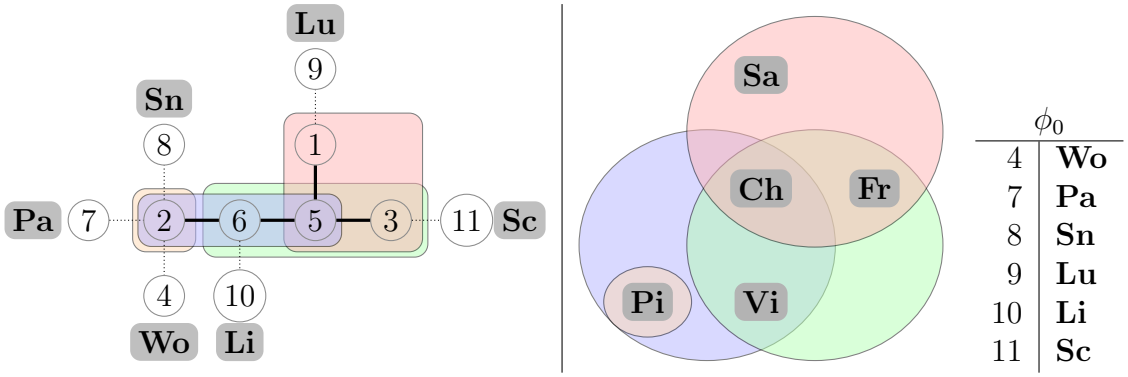


Figure 3.3: `filter fix leaf` finds leaf assignments as shown.

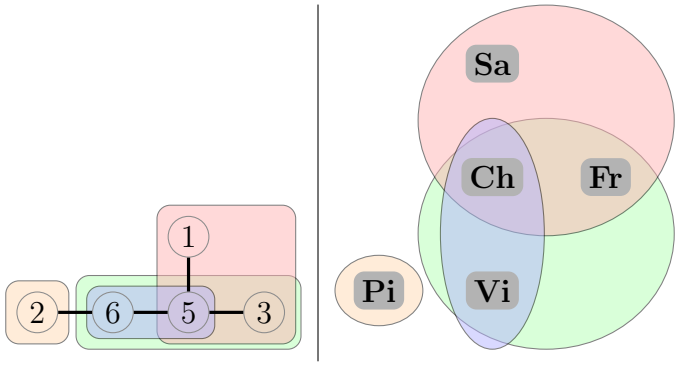


Figure 3.4: `filter common leaf` handling shared leaf 2.

shared by paths $\{1, 5, 3\}$ and $\{6, 5, 3\}$. This is illustrated in Figure 3.5. `filter common leaf` removes these two paths and adds their intersection and set differences – $\{5, 3\}$, $\{1\}$ and $\{6\}$. Correspondingly, hyperedges $\{\mathbf{Sa}, \mathbf{Ch}, \mathbf{Fr}\}$ and $\{\mathbf{Vi}, \mathbf{Ch}, \mathbf{Fr}\}$ are removed and new hyperedges $\{\mathbf{Ch}, \mathbf{Fr}\}$, $\{\mathbf{Sa}\}$ and $\{\mathbf{Vi}\}$ respectively are added. There are no more leaves shared and the algorithm proceeds to `filter fix leaf`.

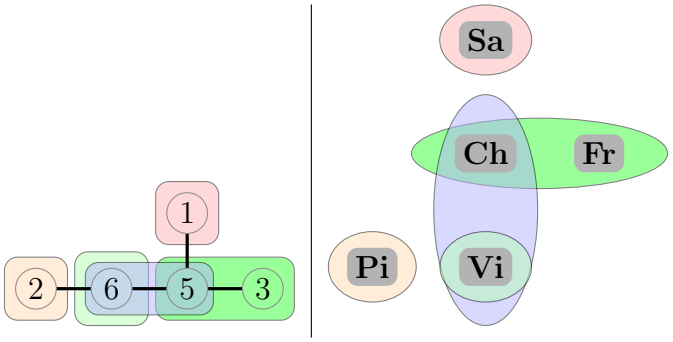
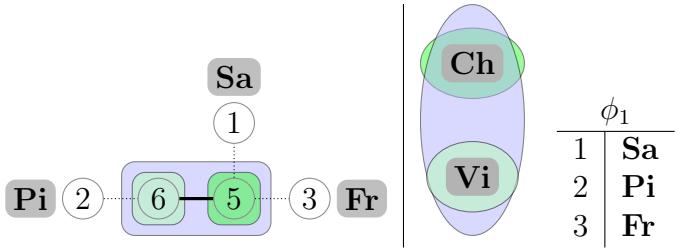


Figure 3.5: `filter common leaf` handles shared leaf 3.

Next, Figure 3.6 shows how `filter fix leaf` computes the leaf assignment ϕ_1 – **Pi**, **Sa**, **Fr** on 2,1,3 respectively. These assignments are deterministic since X consisted of only single elements for each of these leaves. After this the rest of the labeling is fairly obvious but we present it in Figure 3.7 for the sake of completeness.

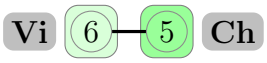
In Figure 3.7, the last part of the isomorphism is computed as the leaf label ϕ_2 which maps **Ch**, **Vi** to 5, 6 respectively. Figure 3.8 shows the final feasibility isomorphism which is the union of all the leaf labels. The feasibility hypergraph isomorphism ϕ is computed



ϕ_1	
1	Sa
2	Pi
3	Fr

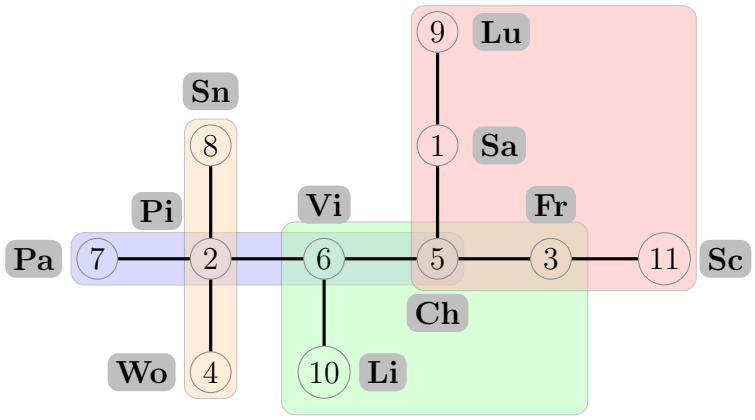
Figure 3.6: `filter fix leaf` computes the leaf assignment ϕ_1.

as the union of the leaf labels ϕ_0, ϕ_1, ϕ_2. One can easily verify that this isomorphism does indeed induce the path labeling given in the original input.



ϕ_2	
5	Ch
6	Vi

Figure 3.7: Last part of solution computed.



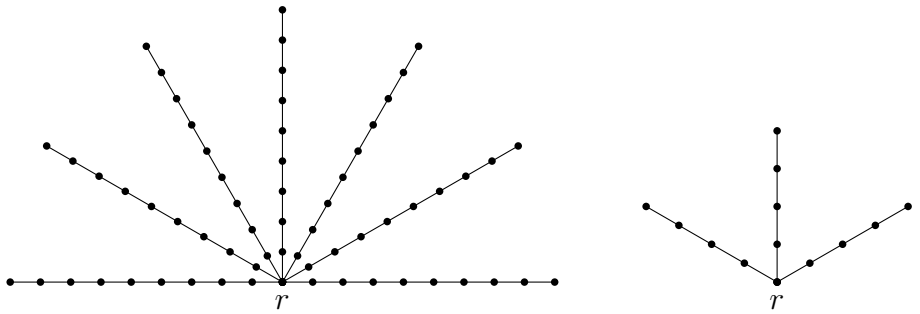
$\phi = \phi_0 \cup \phi_1 \cup \phi_2$	
1	Sa
2	Pi
3	Fr
4	Wo
5	Ch
6	Vi
7	Pa
8	Sn
9	Lu
10	Li
11	Sc

Figure 3.8: The solution to the problem presented in Figure 3.2.

3.5 Computing feasible TPL with special target trees

Section 3.3 described properties that a TPL must have for it to be feasible. The next problem of interest is to test if a given hypergraph is a path hypergraph with respect to a given target tree [1]. In other words, the problem is to find out if a feasible tree path labeling exists from a given target tree for a given hypergraph. In this section we will see two special cases of this problem where the target tree is from a particular family of trees. The first one, where the tree is a path as described by the Compute Interval Labeling problem (see Section 3.1 for definition) in Section 3.5.1 is known to be equivalent to the well studied problem of consecutive-ones. The second one, where the tree is a k-subdivided tree as described by Compute k-subdivided Star Path Labeling (see Section 3.1 for definition), has been solved using a polynomial time algorithm. The

1. A larger problem would be to test if a given hypergraph is a path hypergraph where target tree is not given as input. This problem is not addressed in this thesis and is discussed as further research in Chapter 4.



(a) $k = 8$ (b) $k = 3$

Figure 3.9: (a) 8-subdivided star with 7 rays (b) 3-subdivided star with 3 rays

latter problem enforces some conditions on the hypergraph too which will be seen in Section 3.5.2.

3.5.1 Target tree is a Path

In Section 1.5 it is described how COP is a special case of TPL as described by the Compute Interval Labeling problem. This special case of TPL is where the target tree T is a path. All path labels can now be viewed as intervals assigned to the hyperedges in input hypergraph $\mathcal{F}$. It is shown, in [NS09], that the filtering algorithms outlined above need only preserve pairwise intersection cardinalities, and higher level intersection cardinalities are preserved by the Helly Property of intervals. Thus ICPIA as described in Section 2.2.5 is a sufficient and necessary condition for COP and this is solvable in polynomial time.

Moreover, this special case structure and its algorithm is used in the next section for finding tree path labeling from a k-subdivided star due to this tree's close relationship with intervals.

3.5.2 Target tree is a k-subdivided Star

In this section we consider the problem of assigning paths from a k-subdivided star (see Definition 3.2.6) T to a given set system $\mathcal{F}$ as described by the Compute k-subdivided Star Path Labeling problem. We consider $\mathcal{F}$ for which the overlap graph $\mathbb{O}(\mathcal{F})$ is connected. The overlap graph is well-known from the work of [KKLV10, NS09, Hsu02]. We use the notation in [KKLV10]. Recall from Section 3.2 that hyperedges S and S' are said to overlap, denoted by $S \between S'$, if S and S' have a non-empty intersection but neither of them is contained in the other. The overlap graph $\mathbb{O}(\mathcal{F})$ is a graph in which the vertices correspond to the sets in $\mathcal{F}$, and the vertices corresponding to the hyperedges S and S' are adjacent if and only if they overlap. Note that the intersection graph of $\mathcal{F}$, $\mathbb{I}(\mathcal{F})$ is different from $\mathbb{O}(\mathcal{F})$ and $\mathbb{O}(\mathcal{F}) \subseteq \mathbb{I}(\mathcal{F})$. A connected component of $\mathbb{O}(\mathcal{F})$ is

called an overlap component of $\mathcal{F}$. An interesting property of the overlap components is that any two distinct overlap components, say $\mathcal{O}_1$ and $\mathcal{O}_2$, are such that any two sets $S_1 \in \mathcal{O}_1$ and $S_2 \in \mathcal{O}_2$ are disjoint, or, w.l.o.g, all the sets in $\mathcal{O}_1$ are contained within one set in $\mathcal{O}_2$. This containment relation naturally determines a decomposition of the overlap components into rooted containment trees. We consider the case when there is only one rooted containment tree, and we first present our algorithm when $\mathbb{O}(\mathcal{F})$ is connected. It is easy to see that once the path labeling to the overlap component in the root of the containment tree is achieved, the path labeling to the other overlap components in the rooted containment tree is essentially finding a path labeling when the target tree is a path: each target path is a path that is allocated to sets in the root overlap component. Therefore, for the rest of this section, $\mathbb{O}(\mathcal{F})$ is a connected graph. We also assume that all hyperedges are of cardinality at most $k+2$.

Recall from Section 3.2 that a k-subdivided star is a star with each edge subdivided k times. Therefore, a k-subdivided star has a central vertex which we call the *root*, and each root to leaf path is called a *ray*. First, we observe that by removing the root r from T, we get a collection of p vertex disjoint paths of length $k+1$, p being the number of leaves in T. We denote the rays by $R_1, \ldots, R_p$, and the number of vertices in R_i, $i \in [p]$ is $k+2$. Let $\langle v_{i1}, \ldots, v_{i(k+2)} = r \rangle$ denote the sequence of vertices in R_i, where v_{i1} is the leaf. Note that r is a common vertex to all R_i.

Description of the Algorithm. In this section the given hypergraph $\mathcal{F}$, the k-subdivided star and the root of the star are denoted by $\mathcal{O}$, T and vertex r, respectively. In particular, note that the vertices of $\mathcal{O}$ correspond to the sets in $\mathcal{F}$, and the edges correspond to the overlap relation.

For each hyperedge $X \in \mathcal{O}$, we will maintain a 2-tuple of non-negative numbers $\langle p_1(X), p_2(X) \rangle$. The numbers satisfy the property that $p_1(X)+p_2(X) \leq |X|$, and at the end of path labeling, for each X, $p_1(X) + p_2(X) = |X|$. This signifies the algorithm tracking the lengths of subpaths of the path assigned to X from at most two rays. We also maintain another parameter called the *residue* of X denoted by $s(X) = |X| - p_1(X)$. This signifies the residue path length that must be assigned to X which must be from another ray. For instance, if X is labeled a path from only one ray, then $p_1(X) = |X|$, $p_2(X) = 0$ and $s(X) = 0$.

The algorithm proceeds in iterations, and in the i-th iteration, $i > 1$, a single hyperedge X that overlaps with a hyperedge that has been assigned a path is considered. At the beginning of each iteration hyperedges of $\mathcal{O}$ are classified into the following disjoint sets.

$\mathcal{L}_1^i$ *Labeled without* r. Those that have been labeled with a path which does not contain r in one of the previous iterations.
$\mathcal{L}_1^i = \{X \mid p_1(X) = |X| \text{ and } p_2(X) = 0 \text{ and } s(X) = 0, X \in \mathcal{O}\}$

$\mathcal{L}_2^i$ *Labeled with r.* Those that have been labeled with two subpaths of $\ell(X)$ containing r from two different rays in two previous iterations.
$\mathcal{L}_2^i = \{X \mid 0 < p_1(X), p_2(X) < |X| = p_1(X) + p_2(X) \text{ and } s(X) = 0, X \in \mathcal{O}\}$

$\mathcal{T}_1^i$ *Type 1 / partially labeled.* Those that have been labeled with one path containing r from a single ray in one of the previous iterations. Here, $p_1(X)$ denotes the length of the subpath of $\ell(X)$ that X has been so far labeled with.
$\mathcal{T}_1^i = \{X \mid 0 < p_1(X) < |X| \text{ and } p_2(X) = 0 \text{ and } s(X) = |X| - p_1(X), X \in \mathcal{O}\}$

$\mathcal{T}_2^i$ *Type 2 / not labeled.* Those that have not been labeled with a path in any previous iteration.
$\mathcal{T}_2^i = \{X \mid p_1(X) = p_2(X) = 0 \text{ and } s(X) = |X|, X \in \mathcal{O}\}$

The set $\mathcal{O}_i$ refers to the set of hyperedges $\mathcal{T}_1^i \cup \mathcal{T}_2^i$ in the ith iteration. Note that $\mathcal{O}_1 = \mathcal{O}$. In the ith iteration, hyperedges from $\mathcal{O}_i$ are assigned paths from T using the following rules. Also the end of the iteration, $\mathcal{L}_1^{i+1}, \mathcal{L}_2^{i+1}, \mathcal{T}_1^{i+1}, \mathcal{T}_2^{i+1}$ are set to $\mathcal{L}_1^i, \mathcal{L}_2^i, \mathcal{T}_1^i, \mathcal{T}_2^i$ respectively, along with some case-specific changes mentioned in the rules below.

I. **Iteration 1:** Let $S = \{X_1, \ldots, X_s\}$ denote the super-marginal hyperedges from $\mathcal{O}_1$. If $|S| = s \neq p$, then exit reporting failure. Else, assign to each $X_j \in S$, the path from R_j such that the path contains the leaf in R_j. This path is referred to as $\ell(X_j)$. Set $p_1(X_j) = |X|, p_2(X_j) = s(X_j) = 0$. Hyperedges in S are not added to $\mathcal{O}_2$ but are added to $\mathcal{L}_1^2$ and all other hyperedges are added to $\mathcal{O}_2$.

II. **Iteration i:** Let X be a hyperedge from $\mathcal{O}_i$ such that there exists $Y \in \mathcal{L}_1^i \cup \mathcal{L}_2^i$ and $X \between Y$. Further let $Z \in \mathcal{L}_1^i \cup L_2^i$ such that $Z \between Y$. If $X \in \mathcal{T}_2^i$, and if there are multiple Y candidates then any Y is selected. On the other hand, if $X \in \mathcal{T}_1^i$, then X has a partial path assignment, $\ell'(X)$ from a previous iteration, say from ray R_j. Then, Y is chosen such that $X \cap Y$ has a non-empty intersection with a ray different from R_j. The key things that are done in assigning a path to X are as follows. The *end* of path $\ell(Y)$ where $\ell(X)$ would overlap is found, and then based on this the existence of a feasible assignment is decided. It is important to note that since $X \between Y$, $\ell(X) \between \ell(Y)$ in any feasible assignment. Therefore, the notion of the *end* at which $\ell(X)$ and $\ell(Y)$ overlap is unambiguous, since for any path, there are two end points.

   (a) *End point of $\ell(Y)$ where $\ell(X)$ overlaps depends on $X \cap Z$:* If $X \cap Z \neq \emptyset$, then $\ell(X)$ has an overlap of $|X \cap Y|$ at that end of $\ell(Y)$ at which $\ell(Y)$ and $\ell(Z)$ overlap. If $X \cap Z = \emptyset$, then $\ell(X)$ has an overlap of $|X \cap Y|$ at that end of $\ell(Y)$ where $\ell(Y)$ and $\ell(Z)$ do not intersect.

   (b) *Any path of length $s(X)$ at the appropriate end contains r:* If $X \in \mathcal{T}_1^i$ then after finding the appropriate end as in step IIa this the unique path of length

$s(X)$ should end at r. If not, we exit reporting failure. Else, $\ell(X)$ is computed as union of $\ell'(X)$ and this path. If any three-way intersection cardinality is violated with this new assignment, then exit, reporting failure. Otherwise, X is added to $\mathcal{L}_2^{i+1}$. On the other hand, if $X \in \mathcal{T}_2^i$, then after step IIa, $\ell(X)$ or $\ell'(X)$ is unique up to the root and including it. Clearly, the vertices $\ell(X)$ or $\ell'(X)$ contains depends on $|X|$ and $|X \cap Y|$. If any three way intersection cardinality is violated due to this assignment, exit, reporting failure. Otherwise, $p_1(X)$ is updated as the length of the assigned path, and $s(X) = |X| - p_1(X)$. If $s(X) > 0$, then X is added to $\mathcal{T}_1^{i+1}$. If $s(X) = 0$, then X is added to $\mathcal{L}_1^{i+1}$.

(c) *The unique path of length $s(X)$ overlapping at the appropriate end of Y does not contain r:* In this case, $\ell(X)$ is updated to include this path. If any three way intersection cardinality is violated, exit, reporting failure. Otherwise, update $p_1(X)$ and $p_2(X)$ are appropriate, X is added to $\mathcal{L}_1^{i+1}$ or $\mathcal{L}_2^{i+1}$, as appropriate.

Proof of Correctness and Analysis of Running Time: It is clear that the algorithm runs in polynomial time, as at each step, at most three-way intersection cardinalities need to be checked. Further, finding super-marginal hyperedges can also be done in polynomial time, as it involves considering the overlap regions and checking if the inclusion partial order contains a single minimal element. In particular, once the super-marginal edges are identified, each iteration involves finding the next hyperedge to consider, and testing for a path to that hyperedge. To identify the next hyperedge to consider, we consider the breadth first layering of the hyperedges with the zeroth layer consisting of the super-marginal hyperedges. Since $\mathcal{O}$ is connected, it follows that all hyperedges of $\mathcal{O}$ will be considered by the algorithm. Once a hyperedge is considered, the path to be assigned to it can also be computed in constant time. In particular, in the algorithm the path to be assigned to X depends on $\ell(Y), \ell(Z)$, $s(X)$ and the presence or absence of r in the candidate partial path $\ell'(X)$. Therefore, once the super-marginal edges are identified, the running time of the algorithm is linear in the size of the input. By the technique used for constructing prime matrices [Hsu02], the super-marginal edges can be found in linear time in the input size. Therefore, the algorithm can be implemented to run in linear time in the input size.

The proof of correctness uses the following main properties:

1. The k-subdivided star has a very symmetric structure. This symmetry is quantified based on the following observation – either there are no feasible path labelings of $\mathcal{O}$ using paths from T, or there are exactly $p!$ feasible path labelings. In other words, there is either no feasible assignment, or effectively a unique assignment modulo symmetry.

2. The p super-marginal hyperedges, if they exist, will each be assigned a path from distinct rays, and each such path contains the leaf.

3. For a candidate hyperedge X, the partial path assignment $\ell'(X)$ is decided by its overlap with $\ell(Y)$ and cardinality of intersection with $\ell(Z)$.

These properties are formalized as follows:

Lemma 3.5.1. *If $X \in \mathcal{F}$ is super-marginal and ℓ is a feasible tree path labeling to tree T, then $\ell(X)$ will contain a leaf in T.*

Proof. Suppose $X \in \mathcal{F}$ is super-marginal and $(\mathcal{F}, \ell)$ is a feasible path labeling from T. Assume $\ell(X)$ does not have a leaf. Let R_i be one of the rays (or the only ray) $\ell(X)$ is part of. Since X is in a connected overlap component, there exists $Y_1 \in \mathcal{F}$ and $X \nsubseteq Y_1$ such that $Y_1 \between X$ and Y_1 has at least one vertex closer to the leaf in R_i than any vertex in X. Similarly with the same argument there exists $Y_2 \in \mathcal{F}$ with same subset and overlap relation with X except it has has at least one vertex farther away from the leaf in R_i than any vertex in X. Clearly $Y_1 \cap X$ and $Y_2 \cap X$ cannot be part of same inclusion chain which contradicts that assumption X is super-marginal. Thus the claim is proved. □

Lemma 3.5.2. *If $\mathcal{O}$ does not have any super-marginal edges, then in any feasible path labeling ℓ of $\mathcal{O}$ with paths from T is such that, for any hyperedge X for which $\ell(X)$ contains a leaf, $|X| \geq k + 3$.*

Proof. The proof of this lemma is by contradiction. Let X be a hyperedges such that $|X| \leq k+2$ and that $\ell(X)$ has a leaf. This implies that the overlap regions with X, which are captured by the overlap regions with $\ell(X)$, will form a single inclusion chain. This shows that X is a marginal hyperedge which contradicts the assumption that $\mathcal{O}$ does not have super-marginal hyperedges. □

This lemma is used to prove the next lemma for the case when for all $X \in \mathcal{O}$, $|X| \leq k + 2$. The proof is left out as it just uses the previous lemma and the fact that the hyperedges in X have at most $k + 2$ elements.

Lemma 3.5.3. *If there is a feasible path labeling for $\mathcal{O}$ in T, then there are exactly p super-marginal hyperedges.*

These lemmas now are used to prove the following theorem.

Theorem 3.5.4. *Given $\mathcal{O}$ and a k-subdivided star T, the above algorithm decides correctly if there is a feasible path labeling ℓ.*

Proof. Outline. If the algorithm outputs a path labeling ℓ, then it is clear that it is an ICPPL. The reason is that the algorithm checks that three-way intersection cardinalities are preserved in each iteration which ensures ICPPL Property iii. Moreover, it is clear that $\ell(X)$ for any $X \in \mathcal{O}$ is computed by maintaining ICPPL Property i and ICPPL

Property ii. For such a labeling ℓ, the proof that it is feasible is by induction on k. What needs to be shown is that Algorithm 3 successfully runs on input $(\mathcal{O}, \ell)$. In base case $k = 0$, T is a star. Also every set is at most size 2 $(k + 2)$ size and thus overlaps are at most 1. If two paths share a leaf in `filter common leaf` one must be of length 2 and the other of length 1. Thus the exit condition is not met. Further, it is also clear that the exit condition in `filter fix leaf` is also not met. Thus claim proven for base case. Now assume the claim to be true when target tree is a $(k-1)$-subdivided star. Consider the case of a k-subdivided star. We can show that after `filter common leaf` and one iteration of a modified `filter fix leaf` *all* leaves are assigned pre-images. Removing the leaves from T and the pre-images from support of $\mathcal{O}$, results in an ICPPL to a $(k-1)$-subdivided star. Now we apply the induction hypothesis, and we get an isomorphism between the hypergraphs $\mathcal{O}$ and $\mathcal{O}^{\ell}$.

In the reverse direction if there is a feasible path labeling ℓ, then we know that ℓ is unique up to isomorphism. Therefore, again by induction on k it follows that the algorithm finds ℓ. □

3.6 TPL on arbitrary trees

In this section we describe an algorithm for the Compute Feasible Tree Path Labeling problem where the target tree can be any arbitrary tree. In essence, we provide a characterization for path hypergraphs (see Section 3.2.2 for definition). We first study some properties of the overlap components of the input hypergraph and introduce some theory in that area. This is in the same vein as the theory used in [Hsu02, NS09] but we extend it to TPL. We use this theory to decompose Compute Feasible Tree Path Labeling into subproblems. Each subproblem is on a sub-hypergraph of the input, in which for each hyperedge there is another hyperedge in the sub-hypergraph that overlaps with it. These sub-hypergraphs are called overlap components (see Section 3.2.3 for definition). As a consequence of this characterization, aside from the overlap component subproblems, we identify two other subproblems that must be solved to obtain an ICPPL. The inefficiency of our algorithm in terms of polynomial time solvability comes from these two subproblems. We leave these subproblems open to be solved in P and only mention the obvious brute force method of solution.

As described by Definition 2.2.3 a set system or hypergraph can be concisely represented by a binary matrix such that the row indices of the matrix denote the vertex set of the hypergraph and each column bijectively correspond to a hyperedge. And thus the ideas of feasible TPL and ICPPL are also defined for a binary matrix. If a TPL $(\mathcal{F}, \ell, T)$ is feasible, i. e. it is an ICPPL (see Section 3.3) and if M is the representative matrix for hypergraph $\mathcal{F}$, then we say that M has an ICPPL and vice versa.

Consider the overlap graph and overlap components of $\mathbb{O}(\mathcal{F})$ defined in Section 3.2.3. We use this to decompose M as described in [Hsu02, NS09]. A *prime submatrix* of M is defined as the matrix formed by a set of columns of M which correspond to an overlap component. Let us denote the prime submatrices of M by $M_1, \ldots, M_p$, each corresponding to one of the p overlap components of $\mathcal{F}$. Clearly, two distinct prime submatrices have mutually exclusive sets of columns of M. Let $col(M_i)$ be the set of columns in the submatrix M_i. The support of a prime submatrix M_i and support of a set of prime submatrices X are defined as follows. Note that for each i, $supp(M_i) \subseteq U$ where U is the vertex set of hypergraph $\mathcal{F}$.

$$supp(M_i) = \bigcup_{j \in col(M_i)} S_j$$
$$supp(X) = \bigcup_{M \in X} supp(M)$$

Consider the binary relation $\preccurlyeq$ defined on the set of prime submatrices $\mathbb{P} = \{M_i \mid i \in [p]\}$ as follows.

$$\begin{aligned} \preccurlyeq = &\{(M_i, M_j) \mid \exists S \in M_i \text{ and } \exists S' \in M_j \text{ such that } S \subseteq S'\} \\ &\cup \{(M_i, M_i) \mid i \in [p]\} \end{aligned}$$

This relation is the same as that defined in [NS09]. The prime submatrices and the above relation can be defined for any hypergraph since an overlap graph is defined for any hypergraph. We will use this structure of prime submatrices to present our results on an ICPPL for a hypergraph $\mathcal{F}$. Now we present a few lemma and a theorem that show that $\preccurlyeq$ is a partial order. These results are from [NS09] but are independent of the COP property of M and is valid for any binary matrix M except Theorem 3.6.5 which we subsequently extend in Theorem 3.6.6 to accommodate ICPPL. Note that an alternate notation for $(M_i, M_j) \in \preccurlyeq$ is $M_i \preccurlyeq M_j$.

Lemma 3.6.1 ([NS09, Lem. 3]). *If $M_i \preccurlyeq M_j$, then there is a set $S' \in M_j$ such that for each $S \in M_i$, $S \subseteq S'$.*

Lemma 3.6.2 ([NS09, Lem. 4]). *For each pair of prime submatrices M_i, M_j, either $M_i \not\preccurlyeq M_j$ or $M_j \not\preccurlyeq M_i$.*

Lemma 3.6.3 ([NS09, Lem. 5]). *If $M_i \preccurlyeq M_j$ and $M_i \preccurlyeq M_k$, then $M_i \preccurlyeq M_k$.*

Lemma 3.6.4 ([NS09, Lem. 6]). *If $M_i \preccurlyeq M_j$ and $M_i \preccurlyeq M_k$, then either $M_j \preccurlyeq M_k$ or $M_k \preccurlyeq M_j$.*

Theorem 3.6.5 ([NS09, Th. 4]). *Let M be a binary matrix and $\mathbb{P}$ be the set of prime submatrices of M. Then the following hold true.*

i. $\preccurlyeq$ *is a partial order on* $\mathbb{P}$.

ii. $\preccurlyeq$ *uniquely partitions* $\mathbb{P}$ *into* $\mathbb{P}_1, \mathbb{P}_2 \ldots, \mathbb{P}_r$ *for some* $r \geq 1$ *as follows.*

iii. If M *has COP, then for each* $i \in [r]$, $\preccurlyeq$ *induces a total order in* $\mathbb{P}_i$.

[NS09] does not explicitly mention how the partitioning of $\mathbb{P}$ is done in Theorem 3.6.5 and merely mention that the total order in (iii) is obvious. We call this partitioning in Theorem 3.6.5 point (ii) as *chain partitioning*. We explain what chain partitioning is in the following text. For that, first we construct the Hasse diagram of $\preccurlyeq$ which we observe is a collection of in-trees in the following Theorem 3.6.6. Each of these in-trees have disjoint vertex sets, which in this case would be disjoint sets of prime submatrices $\mathbb{P}_i$, $i \in [r]$ for some $r > 0$. For each $\mathbb{P}_i$, the in-tree is rooted at a maximal upper bound under $\preccurlyeq$. The in-trees are specified by selecting the appropriate edges from the Hasse diagram associated with $\preccurlyeq$. This is defined in Theorem 3.6.6 before which we see the covering relation for $\preccurlyeq$ defined below.

$$\begin{aligned} \preccurlyeq_c \;&=\; \{(M_i, M_j) \mid M_i \preccurlyeq M_j \text{ such that } \nexists M_k \text{ s.t. } M_i \preccurlyeq M_k,\, M_k \preccurlyeq M_j\} \\ &\cup\; \{(M_i, M_i), i \in [p]\} \end{aligned}$$

It is easy to see that $\preccurlyeq_c$ is the covering relation of the partial order $\preccurlyeq$.

Theorem 3.6.6. *Let* M *be a binary matrix and* $\mathbb{P}$ *be the set of prime matrices of* M. *The partial order* $\preccurlyeq$ *and its covering relation* $\preccurlyeq_c$ *are as defined earlier. Then the directed graph* $X = (\mathbb{P}, \preccurlyeq_c)$ *whose vertex set is the set of prime submatrices and the edges are given by the covering relation* $\preccurlyeq_c$, *is a vertex disjoint collection of in-trees and the root of each in-tree is a maximal upper bound in* $\preccurlyeq$.

Proof. To observe that X is a collection of in-trees, we observe that for vertices corresponding to maximal upper bounds, no outgoing edge is present in $\preccurlyeq_c$. Secondly, for each other element, exactly one outgoing edge is chosen, and for the minimal lower bound, there is no incoming edge. Consequently, X is acyclic, and since each vertex has at most one edge leaving it, it follows that X is a collection of vertex disjoint in-trees, and for each in-tree, the root is a maximal upper bound in $\preccurlyeq$. □

We call the partitioning of set of prime matrices $\mathbb{P}$ given by Theorem 3.6.6 as *containment partitioning*. The term signifies the containment property described in Lemma 3.6.1 and the fact that each containment parts are disjoint from one another.

Let the containment partition of set of prime matrices $\mathbb{P}$ be:

$$\{\mathbb{P}_i \mid i \in [r], \mathbb{P}_i \text{ is the vertex set of an in-tree of } X\}$$

Further, the vertices of each in-tree can be layered based on their distance from the root, the root being the maximal upper bound. The root is considered to be at level zero. For $j \geq 0$, let $\mathbb{P}_{i,j}$ denote the set of prime matrices in level j of in-tree $\mathbb{P}_i$. The set $\mathbb{P}_{i,0}$ has only one prime submatrix which is the maximal upper bound in $\mathbb{P}_i$. Hence $\mathbb{P}_{i,0}$ is denoted by $mub(\mathbb{P}_i)$.

Now we describe how the chain partitioning alluded to in Theorem 3.6.5 is obtained. Each in-tree described above is decomposed by the following rules into chains. Traverse the tree level by level from bottom to top. At each level, if there is a node with degree greater than 2, the chain that contains this node and all its descendants is one chain part. Remove this chain from the tree and recurse till reaching the root. Now it is clear why each of these partitions are totally ordered – it is so because they are chains of the Hasse diagram. Now we continue with the description of $\preccurlyeq$ for TPLs.

The next Lemma 3.6.7 and Lemma 3.6.8 show why it is necessary to assign a subtree of the target tree only to the *mub* of each in-tree and all other matrices reduce to interval assignment problems.

Lemma 3.6.7. *Let M be a matrix and let X be the directed graph whose vertices are in correspondence with the prime submatrices of M. Further let $\{\mathbb{P}_i \mid i \in [r]\}$ be the partition of X into in-trees as defined above. Then, matrix M has an ICPPL on target tree T if and only if T can be partitioned into vertex disjoint subtrees $\{T_i \mid i \in [r]\}$ such that, for each $i \in [r]$, the set of prime submatrices $\mathbb{P}_i$ has an ICPPL on subtree T_i.*

Proof. Let us consider the reverse direction first. Let us assume that T can be partitioned into $T_1, \ldots, T_r$ such that for each $i \in [r]$, the set of prime submatrices $\mathbb{P}_i$ has an ICPPL on T_i. It is clear from the properties of the partial order $\preccurlyeq$ that this naturally yield an ICPPL of M on T. The main property used in this inference is that for each $1 \leq i \neq j \leq r$, $supp(X_i) \cap supp(X_j) = \emptyset$. To prove the forward direction, we show that if M has an ICPPL, say ℓ, on T then there exists a partition of T into vertex disjoint subtree $T_1, \ldots, T_r$ such that for each $i \in [r]$, the set of prime submatrices $\mathbb{P}_i$ has an ICPPL on T_i. For each $i \in [r]$, we define based on ℓ a subtree T_i corresponding to X_i. We then argue that the subtrees thus defined are vertex disjoint and prove the claim. In partition $\mathbb{P}_i$, consider prime submatrix $mub(\mathbb{P}_i)$ and the paths labeled under ℓ to the hyperedges in the prime submatrix $mub(\mathbb{P}_i)$. Since the component in $\mathbb{O}(M)$ corresponding to this matrix (this notation is analogous to overlap graph $\mathbb{O}(\mathcal{F})$ for a hypergraph $\mathcal{F}$) is a connected component, it follows that the paths labeled to hyperedges in this prime submatrix form a (connected) subtree of T. We call this subtree T_i. All other prime submatrices in $\mathbb{P}_i$ are assigned merely paths in T_i. This follows from Lemma 3.6.1 along with the facts that ℓ is an ICPPL and $M_x \preccurlyeq mub(\mathbb{P}_i)$ for any prime submatrix $M_x \in \mathbb{P}_i$. Secondly, for each $1 \leq i \neq j \leq r$, $supp(\mathbb{P}_i) \cap supp(\mathbb{P}_j) = \emptyset$ and ℓ is an ICPPL. Thus it follows that T_i and T_j are vertex disjoint. Finally, since $|U| = |V(T)|$, it follows that $T_1, \ldots, T_r$ is indeed a

partition of T. Thus the claim is proven. □

The essence of the following lemma is that an ICPPL only needs to be assigned to the prime submatrix corresponding to the root of each in-tree, and all the other prime submatrices only need to have an Intersection Cardinality Preserving Interval Assignments (ICPIA). Recall, an ICPIA is an assignment of intervals to sets such that the cardinality of an assigned interval is same as the cardinality of the interval, and the cardinality of intersection of any two sets is same as the cardinality of the intersection of the corresponding intervals. It is shown in [NS09] that the existence of an ICPIA is a necessary and sufficient condition for a matrix to have COP. We present the pseudocode to test if M has an ICPPL in T.

Lemma 3.6.8. *Let M be a matrix, X be the directed graph whose vertices are in correspondence with the prime submatrices of M and $\{\mathbb{P}_1, \ldots, \mathbb{P}_r\}$ be the partition of X into in-trees by $\preccurlyeq$ defined earlier. Let T be the target tree and let $\{T_1, \ldots, T_r\}$ be a given partition of T into vertex disjoint subtrees. Then, for each $1 \leq i \leq r$, the set of prime submatrices $\mathbb{P}_i$ has an ICPPL in T_i if and only if the prime submatrix $mub(\mathbb{P}_i)$ has an ICPPL on T_i and all other matrices in $\mathbb{P}_i$ have an ICPIA.*

Proof. The proof is based on the fact that $\preccurlyeq$ is a partial order and X is a directed graph which is a disjoint union of in-trees. Each edge in the in-tree is a containment relationship among the supports of the corresponding submatrices. Therefore, any ICPPL to a prime submatrix that is not the *mub* is contained in a path assigned to the sets in the parent matrix. This follows from Lemma 3.6.1 and since $M_x \preccurlyeq mub(\mathbb{P}_i)$ for any prime submatrix $M_x \in \mathbb{P}_i$. Consequently, any ICPPL to the prime submatrix that is not $mub(\mathbb{P}_i)$ is an ICPIA, and any ICPIA can be used to construct an ICPPL to the matrices corresponding to nodes in $\mathbb{P}_i$ provided $mub(\mathbb{P}_i)$ has an ICPPL in T_i. □

Lemma 3.6.7 and Lemma 3.6.8 point out two algorithmic challenges in finding an ICPPL for a given hypergraph $\mathcal{F}$ on a target tree T.

1. Finding X and its partition into in-trees $\{\mathbb{P}_1, \ldots, \mathbb{P}_r\}$, can be done in polynomial time. On the other hand, as per Lemma 3.6.7 we need to partition T into vertex disjoint subtrees $\{T_1, \ldots, T_r\}$ such that for each $i \in [r]$, the prime submatrices in $\mathbb{P}_i$ have an ICPPL in T_i (to be accurate, the prime submatrix $mub(\mathbb{P}_i)$ has an ICPPL in T_i and the rest of the prime submatrices have an ICPIA as claimed in Lemma 3.6.8). This seems to be a challenging step, and it must be remarked that this step is easy when T itself is a path, as each individual T_i would be sub-paths.

2. The second algorithmic challenge is identified by Lemma 3.6.8 which is to compute an ICPPL from the chosen subtree T_i to the matrix associated with $mub(\mathbb{P}_i)$.

Figure 3.10 and Figure 3.11 illustrate the containment partitions of a binary matrix and the claim of Lemma 3.6.8. In Figure 3.10, the feasibility bijection is explicitly shown by labeling the rows (which represents elements in hypergraph's universe) with the vertices of the target tree T, $v_1, \ldots, v_{12}$. M decomposed into prime submatrices $M_1, \ldots, M_6$ which is collectively represented by $\mathbb{P}$. It can be verified that they correspond to the overlap components in $\mathbb{O}(\mathcal{F})$. The containment partitions of $\mathbb{P}$ are $\mathbb{P}_1, \mathbb{P}_2$ as shown. Note that the empty cells of the matrix represent 0s – they are left empty to make the containment partitions more obvious. Figure 3.11 shows the target tree of this TPL. The rectangles depict the tree paths allocated to the columns in the *mub* of M's prime matrices, namely M_1 and M_4. The rest of the columns are allocated intervals illustrating Lemma 3.6.8.

Note that in this example the matrix has been explicitly labeled with the vertices of the tree which indicates the solution but arriving at this solution poses the above mentioned challenges. Challenge 1 in this example can be seen as follows. Figure 3.11 shows that $\mathbb{P}_1$ has been allocated the top subtree and $\mathbb{P}_2$ has been allocated the bottom subtree. Computationally this is not an easy decision. There is no obvious indication (as far as we know) that the tree must be decomposed at edge (v_3, v_{11}). If this was given, then one can make the decision based on the difference between the subtrees, namely sizes of the containment parts and the subtrees and conclude that $\mathbb{P}_1$ must be allocated the subtree $\{v_1, \ldots, v_7\}$ and $\mathbb{P}_2$ to the rest. What if the subtrees were of the same sizes, say vertex v_{14} in T, the fourteenth row in M and column S_9 in M were not present? One has to try to solve challenge 2 given above to conclude that the bottom (top) subtree indeed does not give an ICPPL to $\mathbb{P}_1$ ($\mathbb{P}_2$) and thus $\mathbb{P}_1$ ($\mathbb{P}_2$) but be allocated the top (bottom) subtree. Solving challenge 2 is not easy (as far as we know) either. By brute force, one can try all possible path labeling and check for ICPPL. If all of them fail, one must backtrack to challenge 1 and try another subtree for the containment part $\mathbb{P}_i$.

Algorithm 4 consolidates all that we described in this section and gives the procedure to solve Compute Feasible Tree Path Labeling. It may be noted that all operations other than in line 5 and line 7 are known to be polynomial time solvable.

	M											
$X:$	$\mathbb{P}_1$						$\mathbb{P}_2$					
$\mathbb{P}:$	M_1			M_2		M_3	M_4				M_5	M_6
$\mathcal{F}:$	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}
v_1	0	0	**1**	0	0	0						
v_2	0	**1**	0	0	0	0						
v_3	**1**	**1**	**1**	0	0	0						
v_4	**1**	0	**1**	0	0	0						
v_5	**1**	0	0	**1**	**1**	0						
v_6	**1**	0	0	**1**	**1**	**1**						
v_7	**1**	0	0	**1**	0	**1**						
v_8							**1**	**1**	0	0	0	0
v_9							**1**	**1**	**1**	0	0	0
v_{10}							**1**	0	0	0	0	0
v_{11}							**1**	0	0	**1**	**1**	**1**
v_{12}							**1**	0	0	**1**	**1**	**1**
v_{13}							**1**	0	0	**1**	**1**	0
v_{14}							0	0	**1**	0	0	0
v_{15}							0	0	0	**1**	0	0

Figure 3.10: Partial order on prime submatrices. The table illustrates the structural properties of prime matrices and partial order $\preccurlyeq$. A hypergraph $\mathcal{F}$ with FTPL has hyperedges $S_1, \ldots, S_{12}$ which are represented by the binary matrix M. Empty entries denote zeros.

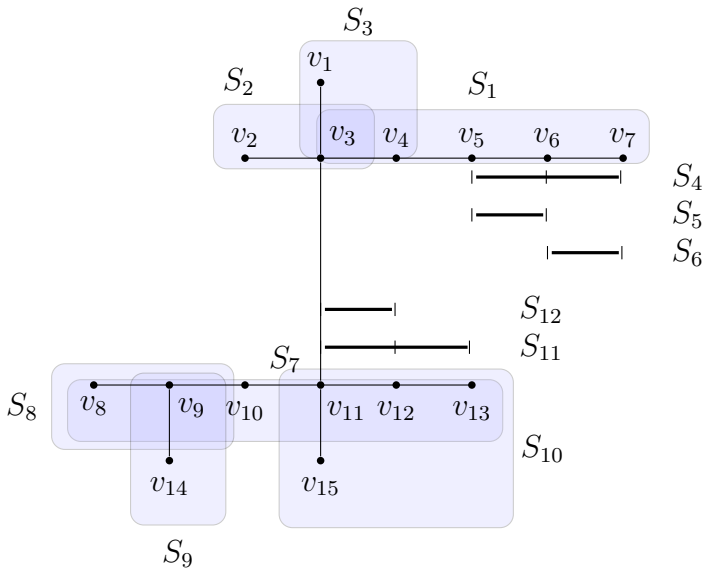


Figure 3.11: This graph is the target tree for the matrix in Fig. 3.10 with each column $S_i, 1 \leq i \leq 12$ shown as tree paths and intervals.

Algorithm 4 Algorithm to find an ICPPL for a matrix M on tree T: $main\_ICPPL(M, T)$

1: Identify the prime submatrices. This is done by constructing the strict overlap graph and identify connected components. Each connected component yields a prime submatrix.
2: Construct the partial order $\preccurlyeq$ on the set of prime submatrices.
3: Construct the partition $\mathbb{P}_1, \ldots, \mathbb{P}_r$ of the prime submatrices induced by $\preccurlyeq$
4: For each $1 \leq i \leq r$, check if all matrices except $mub(\mathbb{P}_i)$ has an ICPIA. If a matrix does not have ICPIA exit with a negative answer. To check for the existence of ICPIA, use the result in [NS09].
5: Find a partition of $T_1, \ldots, T_r$ such that matrices in $mub(\mathbb{P}_i)$ has an ICPPL in T_i. If not such partition exists, exit with negative answer.
6: **for** each $1 \leq i \leq r$ **do**
7: Find the ICPPL for $mub(\mathbb{P}_i)$ on T_i
8: **for** each prime submatrix M_x other than $mub(\mathbb{P}_i)$ in $\mathbb{P}_i$ **do**
9: Find ICPIA for M_x
10: **end for**
11: **end for**

3.7 Complexity of Tree Path Assignment – A Discussion

In this section, we briefly make some observations about complexity of TPL and COP. First we present our conclusions on time complexity of TPL and then on space complexity of COP. The latter is a conclusion based on existing literature. However to the best of our knowledge this particular observation about COP's space complexity has not been made earlier.

3.7.1 Time complexity of TPL

Complexity of Compute Feasible Tree Path Labeling for arbitrary trees is in NP. If a solution to Compute Feasible Tree Path Labeling is given, it can be verified using ICPPL and computing bounded (in this case, up to 3-way) intersection cardinalities is polynomial time solvable. Hence the claim. Beyond this, however, we do not have conclusive complexity results for Compute Feasible Tree Path Labeling.

Nonetheless, the problems Compute Interval Labeling and Compute k-subdivided Star Path Labeling are polynomially solvable as we saw in Section 3.5.

Further, we observe that COP testing can in fact be done in logspace. We use two different results in the literature [KKLV10, McC04] to conclude that COP is in logspace. We see this in the next section.

3.7.2 Consecutive Ones Testing is in Logspace

TPL is polynomial time solvable when the given tree is a path. This is the case that is equivalent to COP testing of binary matrices. The known approaches to testing for COP fall into two categories: those that provide a witness when the input matrix does not have COP, and those that do not provide a witness. We mention this aspect of COT in Section 2.3. Section 2.2.3 described generalized PQ-tree where P and Q nodes are called prime and linear nodes. Aside from that, it has a third type of node called degenerate nodes which is present only if the set system does not have COP. Section 2.2.3 further described that using the idea of generalized PQ-tree, [McC04] proves that checking for bipartiteness in the certain incomparability graph is sufficient to check for COP. [McC04] invented a certificate to confirm when a binary matrix does not have COP. Recall [McC04]'s incompatibility graph of a set system $\mathcal{F}$ from Definition 2.3.1 – it has vertices $(a, b), a \neq b$ for every $a, b \in U$, U being the universe of the set system. There are edges $((a, b), (b, c))$ and $((b, a), (c, b))$ if there is a set $S \in \mathcal{F}$ such that $a, c \in S$ and $b \notin S$. In other words the vertices of an edge in this graph represents two orderings that

cannot occur in a consecutive ones ordering of $\mathcal{F}$. In the same section, we also discuss Theorem 2.3.6 which describes the linear certificate for no COP on the incompatibility graph. This is restated below in Theorem 3.7.1.

Theorem 3.7.1 ([McC04, Th. 6.1], [Dom08, Corrected by Th. 2.6, Proof p. 44–47], Theorem 2.3.6). *Let $\mathcal{F}$ be a set family with universe U. Then $\mathcal{F}$ has COP if and only if its incompatibility graph is bipartite and if it does not have the COP, the incompatibility graph has an odd cycle of length at most $n + 3$.*

Thus checking if a set system/hypergraph/binary matrix's incompatibility graph is bipartite is sufficient to conclude the existence of COP. The following result makes this possible in logspace. [Rei84] showed that checking for bipartiteness can be done in logspace. Thus we conclude that consecutive ones testing can be done in logspace.

We also observed logspace complexity of COP using a more recent and independent result – [KKLV10] showed that interval graph isomorphism can be done in logspace. Their paper proves that a canon for interval graphs can be calculated in logspace using an interval hypergraph representation of the interval graph with each hyperedge being a set to which an interval shall be assigned by the canonization algorithm. An overlap graph (subgraph of intersection graph, edges define only strict intersections and no containment) of the hyperedges of the hypergraph is created and canons are computed for each overlap component. The overlap components define a tree like relation due to the fact that two overlap components are such that either all the hyperedges of one is disjoint from all in the other, or all of them are contained in one hyperedge in the other. This is similar to the containment tree defined in [NS09] and in this paper. Finally the canon for the whole graph is created using logspace tree canonization algorithm from [Lin92]. The interval labeling done in this process of canonization is exactly the same as the problem of assigning feasible intervals to a set system, and thus the problem of finding a COP ordering in a binary matrix [NS09].

Theorem 3.7.2 ([KKLV10, Th. 4.7]). *Given an interval hypergraph $\mathcal{H}$, a canonical interval labeling ℓ for $\mathcal{H}$ can be computed in* FL.

We know the equivalence of hypergraphs and set systems (see Definition 2.2.3). Thus interval labeling of hypergraphs obviously translates to finding COP order of a binary matrix which leads to the following corollary.

Corollary 3.7.3. *If a binary matrix M has COP then the interval assignments to each of its columns can be calculated in* FL.

Thus we again conclude that COP testing is indeed in logspace.