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A unifying approach to temporal constraint reasoning

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Abstract

We present a formalism, Disjunctive Linear Relations (DLRs), for reasoning about temporal constraints. DLRs subsume most of the formalisms for temporal constraint reasoning proposed in the literature and is therefore computationally expensive. We also present a restricted type of DLRs, Horn DLRs, which have a polynomial-time satisfiability problem. We prove that most approaches to tractable temporal constraint reasoning can be encoded as Horn DLRs, including the ORD-Horn algebra by Nebel and Bürckert and the simple temporal constraints by Dechter et al. Thus, DLRs is a suitable unifying formalism for reasoning about temporal constraints.

Keywords: Temporal constraint reasoning; Disjunctive linear relations; Complexity; Algorithms

1. Introduction

Reasoning about temporal knowledge abounds in artificial intelligence applications and other areas, such as planning [2], natural language understanding [23] and molecular biology [4,11]. In most applications, knowledge of temporal constraints is expressed in terms of collections of relations between time intervals or time points. Typical reasoning tasks include determining the satisfiability of such collections and deducing new relations from those that are known. The research has largely concentrated on two kinds of formalisms; systems of inequalities on time points [6,18,21] to encode quantitative information, and systems of constraints in Allen's algebra [1] to encode qualitative relations between time intervals. Some attempts have been made to integrate quantitative

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and qualitative reasoning into unified frameworks [16,21]. Since the satisfiability problem is NP-complete for Allen's algebra the qualitative and unified approaches have suffered from computational difficulties.

In response to the computational hardness of the full Allen algebra, several polynomial subalgebras have been proposed in the literature [8,9,11,22,24]. Some of these algebras have later been extended with mechanisms for handling quantitative information. For example, the TIMEGRAPH II system [10] extends the *pointisable algebra* [24] with a limited type of quantitative information. Of special interest is the ORD-Horn algebra [22] which is the *unique* maximal tractable subclass of Allen's algebra containing all basic relations. Hence, it would be especially interesting to extend this algebra with quantitative information since the maximality result would carry over to the new algebra, at least with respect to its qualitative expressiveness.

To give a concrete form to the topic of temporal constraint reasoning, consider the following fictious crime scenario. Professor Jones has been found shot on the beach near her house. Rumours tell that she was almost sure of having a proof that $P \neq NP$, but had not yet shown it to any of her colleagues. The graduate student Hill is soon to defend his thesis on his newly invented complexity class, $NRQP_{\Sigma}(\eth)^{\maltese}$, which would unfortunately be of no value were it to be known for certain that $P \neq NP$. Needless to say, Hill is thus one of the prime suspects and inspector Smith is faced with the following facts and observations:

- Professor Jones died between 6 pm and 11 pm, according to the post-mortem.
- Mr Green, who lives close to the beach, is certain that he heard a gunshot sometime in the evening, but certainly after the TV news.
- The TV news is from 7.30 pm to 8.00 pm.
- A reliable neighbour of Hill claims Hill arrived at home sometime between 9.15 pm and 9.30 pm.
- It takes between 10 and 20 min to walk or run from the place of the crime to the closest parking lot.
- It takes between 45 and 60 min to drive from this parking lot to Hill's home.

The first thing to do is verifying that these facts and observations are consistent, which is obviously the case here. We can also draw some further conclusions, like narrowing the time of death to the interval between 8.00 pm and 11 pm, assuming the gunshot heard by Mr Green actually was the killing shot.

Now, suppose inspector Smith adds the hypothesis that Hill was at the place of the murder at the time of the gunshot, which is only known to occur somewhere in the interval from 8.00 pm to 11.00 pm. If the set of facts and observations together with this hypothesis becomes inconsistent, then inspector Smith can rule out Hill as the murderer. [2]

This problem can easily be cast in terms of a temporal-constraint-reasoning problem, involving both quantitative and qualitative relations over time points, intervals and durations. Unfortunately, it seems like this simple example cannot be solved by any of the computationally tractable methods reported in the literature. It can, however, be solved in polynomial time by the method proposed in this paper.

[2] Unfortunately, it seems like Hill will be in need of juridicial assistance.

We introduce a formalism, Disjunctive Linear Relations (DLRs), for reasoning about temporal constraints. DLRs subsumes most of the formalisms for temporal constraint reasoning proposed in the literature including, e.g., Allen's algebra. Consequently, the satisfiability problem for DLRs is NP-complete. To reason efficiently about DLRs, one must impose some type of restriction on the formalism. We present *Horn Disjunctive Linear Relations* (Horn DLRs for short) which allows for polynomial-time satisfiability checking. Horn DLRs subsumes the ORD-Horn algebra *and* most of the formalisms for encoding quantitative information proposed in the literature. The approach is rather different from the commonly used constraint network or graph-theoretic approaches. We base our method upon linear programming which proves to be a convenient tool for managing temporal information. Since most of the low-level handling of time points is thus abstracted away, the resulting algorithm is surprisingly simple.

We strongly believe that Horn DLRs are useful in other areas of computer science than temporal reasoning. One example is in reasoning about action and change where Drakengren and Bjäreland [7] has shown how Horn DLRs can be used to obtain computationally tractable formalisms. It is worth noticing that replacing Horn DLRs with standard linear programming in their approach seems nontrivial; the method needs the ability to express disjunctions. Another example where Horn DLRs may be useful are query languages in deductive databases. For instance, the proposal by Kanellakis et al. [14] has some resemblance with Horn DLRs.

Parts of this article have previously appeared in a conference paper [12]. It should be acknowledged that some of the results were independently discovered by Manolis Koubarakis [19], who published them at another conference only a few weeks after we first presented our results. The paper is structured as follows. We begin by giving the basic terminology and definitions used in the rest of the paper together with a brief introduction to complexity issues in linear programming. We continue by presenting the polynomial-time algorithm for deciding satisfiability of Horn DLRs. As a direct consequence of this algorithm, we show NP-completeness of deciding satisfiability of DLRs. After having stated the complexity results, we compare DLRs and Horn DLRs with a number of temporal constraint formalisms proposed in the literature. The paper concludes with a short discussion of the results.

2. Preliminaries

We begin by defining some different types of relations.

Definition 1. Let $X = \{x_1, \ldots, x_n\}$ be a set of real-valued variables. Let α be a linear polynomial (i.e., a polynomial of degree one) over X and c an integer. A *linear disequation* over X is an expression of the form $\alpha \neq c$. A *linear equality* over X is an expression of the form $\alpha = c$. A *linear relation* over X is an expression of the form $\alpha r c$ where $r \in \{<, \leqslant, =, \neq, \geqslant, >\}$. A *convex linear relation* over X is an expression of the form $\alpha r_c c$ where $r_c \in \{<, \leqslant, =, \geqslant, >\}$. A *disjunctive linear relation* (DLR) is a set of one or more linear relations.

The restriction to integral coeffecients is not important in practice; relations with non-integral (but rational) coeffecients can be transformed to equivalent relations with integral coeffecients by multiplication with suitable factors.

Example 2. The set $\{2x_1 + x_2 - x_3 \leqslant 5, 12x_3 - 7x_2 \neq 0, x_2 = 5\}$ is a DLR over $\{x_1, x_2, x_3\}$.

It is no limitation to assume the right-hand sides of the relations to be constants since a relation of the form $\alpha\, r\, \beta$ where α and β are linear polynomials can be rewritten as an equivalent relation of the form $\alpha'\, r\, c$ where c is a constant.

We assume all sets of DLRs to be finite. The definition of satisfiability for DLRs is then straightforward.

Definition 3. Let $X = \{x_1, \ldots, x_n\}$ be a set of real-valued variables and let $R = \{R_1, \ldots, R_k\}$ be a set of DLRs over X. We say that R is *satisfiable* iff there exists an assignment of real values to the variables in X that makes at least one member of each R_i, $1 \leqslant i \leqslant k$, true.

For DLRs we have the following decision problem.

Definition 4. The decision problem DLRSAT is defined as follows:

INSTANCE: A finite set Θ of DLRs.
QUESTION: Is Θ satisfiable?

We continue by classifying different types of DLRs.

Definition 5. Let γ be a DLR. $\mathcal{C}(\gamma)$ denotes the convex relations in γ and $\mathcal{NC}(\gamma)$ the disequations in γ. We say that γ is *convex* iff $|\mathcal{NC}(\gamma)| = 0$ and that γ is *disequational* iff $|\mathcal{C}(\gamma)| = 0$. If γ is convex or disequational we say that γ is *homogenous* and otherwise *heterogenous*. Furthermore, if $|\mathcal{C}(\gamma)| \leqslant 1$, then γ is *Horn* and if $|\gamma| = 1$, then γ is a *unit DLR*. We extend these definitions to sets of relations in the obvious way. For example, if Γ is a set of DLRs and all $\gamma \in \Gamma$ are Horn, then Γ is Horn.

This classification provides the basis for the forthcoming proofs. One detail to note is that if a Horn DLR is convex, then it is a unit DLR. For Horn DLRs we have the following decision problem.

Definition 6. The decision problem HORNDLRSAT is defined as follows:

INSTANCE: A finite set Θ of Horn DLRs.
QUESTION: Is Θ satisfiable?

For Horn DLRs, we restrict ourselves to use only $\leqslant$ and $\neq$ in the relations. This is no loss of generality since we can express all the other relations in terms of these two. For example, a DLR of the form $\{\alpha < c\} \cup D$ can be replaced by the two DLRs $\{\alpha \leqslant c\} \cup D$

and $\{x \neq y\} \cup D$. Observe that the resulting set can contain at most twice as many members as the original one so this is a polynomial-time transformation.

Our method for reasoning about DLRs is based on linear programming techniques so we begin by providing the basic facts needed. The linear programming problem is defined as follows.

Definition 7. Let A be an arbitrary $m \times n$ matrix of integers and let $x = (x_1, \ldots, x_n)$ be an n-vector of variables over the real numbers. Then an instance of the *linear programming* (LP) problem is defined by: $\{\min c^{\mathrm{T}}x$ subject to $Ax \leqslant b\}$ where b is an m-vector of integers and c an n-vector of integers. The computational problem is as follows:
(1) Find an assignment to the variables $x_1, \ldots, x_n$ such that the condition $Ax \leqslant b$ holds and $c^{\mathrm{T}}x$ is minimial subject to these conditions, or
(2) Report that there is no such assignment, or
(3) Report that there is no lower bound for $c^{\mathrm{T}}x$ under the conditions.

Analogously, we can define an LP problem where the objective is to maximize $c^{\mathrm{T}}x$ under the condition $Ax \leqslant b$. We have the following theorem.

Theorem 8. *The linear programming problem is solvable in polynomial time* (*cf. Khachiyan* [17] *or Karmarkar* [15]).

Note that convex unit DLRs can be expressed as LP problems in a straightforward way. Next, we recapitulate some standard mathematical concepts.

Definition 9. Given two points $x, y \in \mathbb{R}^n$, a *convex combination* of x and y is any point of the form $z = \lambda x + (1 - \lambda)y$ where $0 \leqslant \lambda \leqslant 1$. A set $S \subseteq \mathbb{R}^n$ is *convex* iff it contains all convex combinations of all pairs of points $x, y \in S$.

Definition 10. A *hyperplane* H in $\mathbb{R}^n$ is a non-empty set defined as

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid a_1x_1 + \cdots + a_nx_n = b\}$$

for some $a_1, \ldots, a_n, b \in \mathbb{R}$.

Definition 11. Let A be an arbitrary $m \times n$ matrix and b be an m-vector. The *polyhedron* defined by A and b is the set $\{x \in \mathbb{R}^n \mid Ax \leqslant b\}$.

The connection between polyhedrons and convex sets is expressed in the following well-known fact.

Fact 12. *Every non-empty polyhedron is convex.*

3. Complexity of reasoning about DLRs

In this section we prove DLRSAT to be NP-complete and present a polynomial algorithm for HORNDLRSAT. We claim that the algorithm in Fig. 1 correctly solves HORNDLRSAT in polynomial time. The concept of *blocking* is defined as follows.

Definition 13. Let A be a satisfiable set of DLRs and let γ be a DLR. We say that γ *blocks* A iff $A \cup \{d\}$ is not satisfiable for any $d \in \mathcal{NC}(\gamma)$.

Observe that if $A \cup \{\gamma\}$ is satisfiable and γ blocks A, then there must exist a relation $\delta \in \mathcal{C}(\gamma)$ such that $A \cup \{\delta\}$ is satisfiable. This observation will be of great importance later on. Another important property of blocking is that it can be decided in polynomial time, provided that A consists of convex unit DLRs only.

Lemma 14. *Let A be an arbitrary $m \times n$ matrix, b be an m-vector and $x = (x_1, \ldots, x_n)$ be an n-vector of variables over the real numbers. Let α be a linear polynomial over $x_1, \ldots, x_n$ and c an integer. Deciding whether the system $S = \{Ax \leqslant b, \alpha \neq c\}$ is satisfiable or not is polynomial.*

Proof. Consider the following instances of LP:

$$\mathrm{LP1} = \{\min \alpha \text{ subject to } Ax \leqslant b\},$$

$$\mathrm{LP2} = \{\max \alpha \text{ subject to } Ax \leqslant b\}.$$

If LP1 and LP2 have no solutions, then S is not satisfiable. If both LP1 and LP2 yield the same optimal value c, then S is not satisfiable since every solution y to LP1 and LP2 satisfies $\alpha(y) = c$. Otherwise S is obviously satisfiable. Since we can solve the LP problem in polynomial time by Theorem 8, the lemma follows. □

Corollary 15. *Let A be a satisfiable set of convex unit DLRs and let γ be a DLR. Deciding whether γ blocks A or not can be decided in polynomial time.*

Observe that the convex relations in a set of Horn DLRs define a convex set in $\mathbb{R}^n$. Furthermore, we can identify each disequation with a hyperplane in $\mathbb{R}^n$. These observations motivate the next lemma.

algorithm SAT(Γ)
$A \leftarrow \bigcup\{\mathcal{C}(\gamma) \mid \gamma \in \Gamma \text{ is convex}\}$
if A not satisfiable **then reject**
if $\exists \gamma \in \Gamma$ that blocks A and is disequational **then reject**
if $\exists \gamma \in \Gamma$ that blocks A and is heterogenous **then** SAT($(\Gamma - \{\gamma\}) \cup \mathcal{C}(\gamma)$)
accept

Fig. 1. Algorithm for deciding satisfiability of Horn DLRs.

Lemma 16. *Let $S \subseteq \mathbb{R}^n$ be a convex set and let $H_1, \ldots, H_k \subseteq \mathbb{R}^n$ be distinct hyperplanes. If $S \subseteq \bigcup_{i=1}^{k} H_i$, then there exists a j, $1 \leqslant j \leqslant k$, such that $S \subseteq H_j$.*

Proof. If it is possible to drop one or more hyperplanes from H and still have a union containing S, then do so. Let $H' = \{H_1', \ldots, H_m'\}$, $m \leqslant k$, be the resulting minimal set of hyperplanes. Every $H_i' \in H'$ contains some point x_i of S not in any other $H_j' \in H'$. We want to prove that there is only one hyperplane in H'.

If this is not the case, consider the line segment L adjoining x_1 and x_2. (The choice of x_1 and x_2 is not important. Every choice of x_i and x_j, $1 \leqslant i, j \leqslant m$ and $i \neq j$, would do equally well.) By convexity it holds that $L \subseteq S$, so each $H_i' \in H'$ either contains L or meets it in at most one point. But no $H_i' \in H'$ can contain L, since it would then contain both x_1 and x_2. Thus, each H_i' can have at most one point in common with L, so the rest of L cannot be a subset of $\bigcup_{i=1}^{m} H_i'$ which contradicts that $L \subseteq S \subseteq \bigcup_{i=1}^{m} H_i'$. Hence, the lemma follows. □

A more complicated proof of the previous lemma appears in Lassez and McAloon [20]. We can now tie together the results and end up with a sufficient condition for satisfiability of Horn DLRs.

Lemma 17. *Let Γ be a set of arbitrary Horn DLRs. Let $C \subseteq \Gamma$ be the set of convex DLRs in Γ and let $D = \{D_1, \ldots, D_k\} \subseteq \Gamma$ be the set of DLRs that are not convex. Under the condition that C is satisfiable, Γ is satisfiable if D_i does not block C for any $1 \leqslant i \leqslant k$.*

Proof. Pick one disequation d_i out of every D_i such that $\{C, d_i\}$ is satisfiable. This is possible since no D_i blocks C. We show that $\Gamma' = \{C, d_1, \ldots, d_k\}$ is satisfiable and, hence, Γ is satisfiable. Assume that $d_i = (\alpha_i \neq c_i)$ for each i, $1 \leqslant i \leqslant k$. Define the hyperplanes $H_1, \ldots, H_k$ such that

$$H_i = \{x \in \mathbb{R}^n \mid \alpha_i(x) = c\}.$$

Since every $\{C, d_i\}$ is satisfiable, the polyhedron P defined by C (which is non-empty and hence convex by Fact 12) is not a subset of any H_i. Suppose Γ' is not satisfiable. Then

$$P - \bigcup_{i=1}^{k} H_i = \emptyset$$

which is equivalent to

$$P \subseteq \bigcup_{i=1}^{k} H_i.$$

By Lemma 16, there exists a H_j, $1 \leqslant j \leqslant k$, such that $P \subseteq H_j$. Clearly, this contradicts our initial assumptions, so the lemma holds. □

It is important to note that the previous lemma does not give a necessary condition for satisfiability of Horn DLRs. Now, we need an auxiliary lemma which is a formal version of an observation previously made.

Lemma 18. *Let Γ be a set of Horn DLRs and let $C \subseteq \Gamma$ be the set of convex DLRs in Γ. If there exists a heterogenous DLR $\gamma \in \Gamma$ such that γ blocks C, then Γ is satisfiable iff $(\Gamma - \{\gamma\}) \cup \mathcal{C}(\gamma)$ is satisfiable.*

Proof. *If*: Trivial.

Only-if: If Γ is satisfiable, then γ has to be satisfiable. Since γ blocks C, $\mathcal{C}(\gamma)$ must be satisfied in any solution of Γ. □

Proving soundness and completeness of SAT is now straightforward.

Lemma 19. *Let Γ be a set of Horn DLRs. If SAT(Γ) accepts, then Γ is satisfiable.*

Proof. Induction over n, the number of heterogenous DLRs in Γ.

Basis step: If $n = 0$ and SAT(Γ) accepts, then the formulae in A are satisfiable and there does not exist any $\gamma \in \Gamma$ that blocks A. Consequently, Γ is satisfiable by Lemma 17.

Induction hypothesis: Assume the claim holds for $n = k$, $k \geqslant 0$.

Induction step: Γ contains $k + 1$ heterogenous DLRs. If SAT accepts in line 5, then $(\Gamma - \{\gamma\}) \cup \mathcal{C}(\gamma)$, which contains k heterogenous DLRs, is satisfiable by the induction hypothesis. By Lemma 18, this is equivalent to Γ being satisfiable. If SAT accepts in line 6, then there does not exist any disequational or heterogenous $\gamma \in \Gamma$ which blocks A. By Lemma 17, this means that Γ is satisfiable. □

Lemma 20. *Let Γ be a set of Horn DLRs. Let $C \subseteq \Gamma$ be the set of convex DLRs in Γ. If there exists a disequational DLR $\gamma \in \Gamma$ that blocks C, then Γ is not satisfiable.*

Proof. In any solution to Γ, the relations in $C \cup \{\gamma\}$ must be satisfied. Since γ is disequational and blocks C this is not possible and the lemma follows. □

Lemma 21. *Let Γ be a set of Horn DLRs. If SAT(Γ) rejects, then Γ is not satisfiable.*

Proof. Induction over n, the number of heterogenous DLRs in Γ.

Basis step: If $n = 0$ then, SAT can reject in lines 3 and 4. If SAT rejects in line 3, then, trivially, Γ is not satisfiable. If SAT rejects in line 4, then there exists a disequational $\gamma \in \Gamma$ that blocks A. Hence, Γ is not satisfiable by Lemma 20.

Induction hypothesis: Assume the claim holds for $n = k$, $k \geqslant 0$.

Induction step: Γ contains $k + 1$ heterogenous DLRs. If SAT rejects in line 3, then Γ is not satisfiable. If SAT rejects in line 4, then Γ is not satisfiable by Lemma 20. If SAT rejects in line 5, then $(\Gamma - \{\gamma\}) \cup \mathcal{C}(\gamma)$, which contains k heterogenous DLRs, is not satisfiable by the induction hypothesis. By Lemma 18, this is equivalent to Γ not being satisfiable. □

Finally, we can show that SAT is a polynomial-time algorithm and, thus, show that HORNDLRSAT is a polynomial-time problem.

Theorem 22. *Horn DLRSat can be solved in polynomial time.*

Proof. By Lemmata 19 and 21, it is sufficient to show that SAT is polynomial. The number of recursive calls is bounded by the number of heterogenous DLRs in the given input. By Corollary 15, we can decide the blocking property in polynomial time. Since this has to be decided only a polynomial number of times in each recursion, the theorem follows. □

By using the previous theorem, we can now show that DLRSAT is NP-complete.

Theorem 23. DLRSAT *is NP-complete.*

Proof. Let $\Theta = \{\theta_1, \ldots, \theta_n\}$ be an arbitrary finite set of DLRs. If Θ is satisfiable, then for each i, $1 \leqslant i \leqslant n$, there exists a linear relation $\gamma_i \in \theta_i$ such that $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ is satisfiable. Checking the satisfiability of Γ is polynomial by Theorem 22. Thus, given Θ we can let Γ be the non-deterministic guess which can be checked in polynomial time. Consequently, DLRSAT is in NP. NP-hardness follows trivially by reduction from 3-COLOURABILITY. For each vertex v we introduce a DLR of the form $\{v = 0, v = 1, v = 2\}$ to ensure that v is coloured by one of three colours. To guarantee that adjacent vertices are not coloured with the same colour, add a DLR $\{v \neq w\}$ for each edge (v, w). □

4. Temporal constraint reasoning

We begin this section by showing that DLRs subsumes several proposed methods for general temporal constraint reasoning. Furthermore, we show that Horn DLRs subsumes most of the proposed methods for tractable reasoning about temporal constraints.

In the sequel, let x, y be real-valued variables, c, d constants and $\mathcal{A}$ Allen's algebra [1]. It is trivial to see that the DLR language subsumes Allen's algebra. Furthermore, it subsumes the universal temporal language by Kautz and Ladkin which is defined as follows.

Definition 24 (Kautz and Ladkin [16]). The *universal temporal language* consists of $\mathcal{A}$ augmented with formulae of the form $-c r_1 (x - y) r_2 d$ where $r_1, r_2 \in \{<, \leqslant\}$ and x, y are endpoints of intervals.

DLRs also subsumes the *qualititative algebra* (QA) by Meiri [21]. In QA, a qualitative constraint between two objects O_i and O_j (each may be a point or an interval), is a disjunction of the form

$$(O_i r_1 O_j) \vee \cdots \vee (O_i r_k O_j)$$

where each one of the r_i's is a *basic relation* that may exist between two objects. There are three types of basic relations.

(1) Interval-interval relations that can hold between a pair of intervals. These relations correspond to Allen's algebra.
(2) Point-point relations that can hold between a pair of points. These relations correspond to the point algebra [25].
(3) Point-interval and interval-point relations that can hold between a point and an interval and vice versa. These relations were introduced by Vilain [25].

Obviously, DLRs subsumes QA. Meiri also considers QA extended with quantitative constraints of the following two forms: let $x_1, \ldots, x_n$ be time points or endpoints of intervals.

- $(c_1 \leqslant x_1 \leqslant d_1) \vee \cdots \vee (c_1 \leqslant x_n \leqslant d_1)$;
- $(c_1 \leqslant x_n - x_1 \leqslant d_1) \vee \cdots \vee (c_1 \leqslant x_n - x_{n-1} \leqslant d_1)$.

Also this extension to QA can easily be expressed as DLRs. It has been shown that the satisfiability problem for all of these formalisms is NP-complete [16,21,26]. In retrospect, the different restrictions imposed on these formalisms seem quite artificial when compared to DLRs, especially since they do not reduce the complexity of the problem.

We continue by showing that Horn DLRs subsume several tractable methods for temporal constraint reasoning.

Definition 25 (Nebel and Bürckert [22]). An *ORD clause* is a disjunction of relations of the form xry where $r \in \{\leqslant, =, \neq\}$. The *ORD-Horn* subclass $\mathcal{H}$ contains those relations in $\mathcal{A}$ that can be written as ORD clauses containing only disjunctions with at most one relation of the form $x = y$ or $x \leqslant y$ and an arbitrary number of relations of the form $x \neq y$.

Note that the ORD-Horn class subsumes both the continuous endpoint algebra [26] and the pointisable endpoint algebra [24].

Definition 26 (Koubarakis [18]). Let $r \in \{\leqslant, \geqslant, \neq\}$. A *Koubarakis formula* is a formula on either of the forms (1) $(x - y)rc$, (2) xrc or (3) a disjunction of formulae of the form $(x - y) \neq c$ or $x \neq c$.

Definition 27 (Dechter et al. [6]). A *simple temporal constraint* is a formula on the form $c \leqslant (x - y) \leqslant d$.

Definition 28 (Kautz and Ladkin [16]). A *simple metric constraint* is a formula on the form $-cr_1(x - y)r_2d$ where $r_1, r_2 \in \{<, \leqslant\}$.

Definition 29 (Meiri [21]). A *CPA/single interval* formula is a formula on one of the following two forms: (1) $c\, r_1\, (x - y)\, r_2\, d$; or (2) $x\, r\, y$ where $r \in \{<, \leqslant, =, \neq, \geqslant, >\}$ and $r_1, r_2 \in \{<, \leqslant\}$.

Definition 30 (Gerevini et al. [10]). A *TG-II* formula is a formula on one of the following forms: (1) $c \leqslant x \leqslant d$, (2) $c \leqslant x - y \leqslant d$ or (3) $x\, r\, y$ where $r \in \{<, \leqslant, =, \neq, \geqslant, >\}$.

The tractable formalisms defined in Definitions 25–30 can trivially be expressed as Horn DLRs. Beside these six classes, other temporal classes that can be expressed as Horn DLRs have been identified by different authors. Examples include the approach by Barber [3], the algebra $\mathcal{V}^{23}$ for relating points and intervals by Jonsson et al. [13] and the temporal part of TMM by Dean and Boddy [5]. Also note that Golumbic and Shamir [11] and Drakengren and Jonsson [8,9] consider further tractable classes that cannot (in any obvious way) be transformed into Horn DLRs. The finding that the ORD-Horn algebra can be expressed as Horn DLRs is especially important in the light of the following theorem.

Theorem 31 (Nebel and Bürckert [22]). *Let $\mathcal{S}$ be any subclass of $\mathcal{A}$ that contains all basic relations. Then either*

(1) *$\mathcal{S} \subseteq \mathcal{H}$ and the satisfiability problem for $\mathcal{S}$ is polynomial, or*

(2) *satisfiability for $\mathcal{S}$ is NP-complete.*

By the previous theorem, we cannot expect to find tractable classes that are able to handle all basic relations in $\mathcal{A}$ and, at the same time, are able to handle any single relation that cannot be expressed as a Horn DLR.

5. Discussion

Several researchers in the field of temporal constraint reasoning have expressed a feeling that their proposed methods should be extended so they can express relations between more than two time points. As a first example, Dechter et al. [6] write "The natural extension of this work is to explore TCSPs with higher-order expressions (e.g., "John drives to work at least 30 minutes more than Fred does"; $X_2 - X_1 + 30 \leqslant X_4 - X_3$)...". Even though they do not define the exact meaning of "higher-order expressions" we can notice that their example is a simple Horn DLR. Something similar can be found in [18] where Koubarakis wants to express "the duration of interval I exceeds the duration of interval J". Once again, this can easily be expressed as a Horn DLR. These claims seem to indicate that the use of Horn DLRs is a significant contribution to temporal reasoning.

We have shown that the satisfiability problem for Horn DLRs can be solved in polynomial time. However, the method builds on solving linear programs and it is a widespread belief that such calculations are computationally heavy. The commercial packages for solving linear programs which are available today shows that this is not an absolute truth any longer. These packages easily solve linear programs containing thousands of variables and tens of thousands of constraints. Nevertheless, it is fairly obvious that the proposed method cannot outperform highly specialized algorithms for severely restricted classes. It should be likewise obvious that the specialized methods cannot compete with Horn DLRs in terms of expressivity. We are, as always in tractable reasoning, facing the trade-off between expressivity and computational complexity. We believe, though, that the complexity of deciding satisfiability can be drastically improved by devising better algorithms than SAT. The algorithm SAT is constructed in a way that facilitates its correctness proofs and it is not optimized with respect to execution time in any way.

Throughout this paper we have assumed that time is linear, dense and unbounded but this may not be the case in real applications. For example, in a sampled system we cannot assume time to be dense. One question to answer in the future is what the effects of changing the assumptions of time are. Switching to discrete time will probably make reasoning computationally harder. There are some positive results concerning discrete time, however. Meiri [21] presents a class of temporal constraint reasoning problems where integer time satisfiability is polynomial.

6. Conclusion

We have suggested DLRs as a formalism for reasoning about temporal constraints. We have shown that DLRs subsumes most of the formalisms for temporal constraint reasoning proposed in the literature. A restricted type of DLRs, Horn DLRs, has been shown to have a polynomial-time satisfiability problem. We have proved that most approaches to tractable temporal constraint reasoning can be encoded as Horn DLRs, including the ORD-Horn algebra by Nebel and Bürckert and the simple temporal constraints by Dechter et al.

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