Programming meets Mathematics: Calculus

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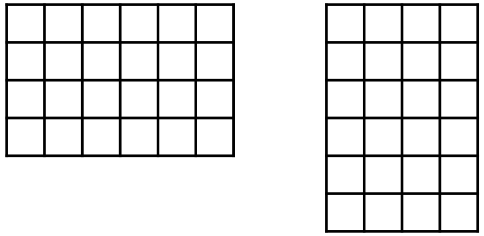
Building Mathematical Intuition

1. **Imagine** (or visualize)
   - e.g. *Numberblocks* (by BBC) [YouTube]
     - 4 x 6 == 6 x 4? (without knowledge of a multiplication table and commutativity)



2. **Ask why** (or think about its applications)
   - e.g. Why did I learn about a matrix?

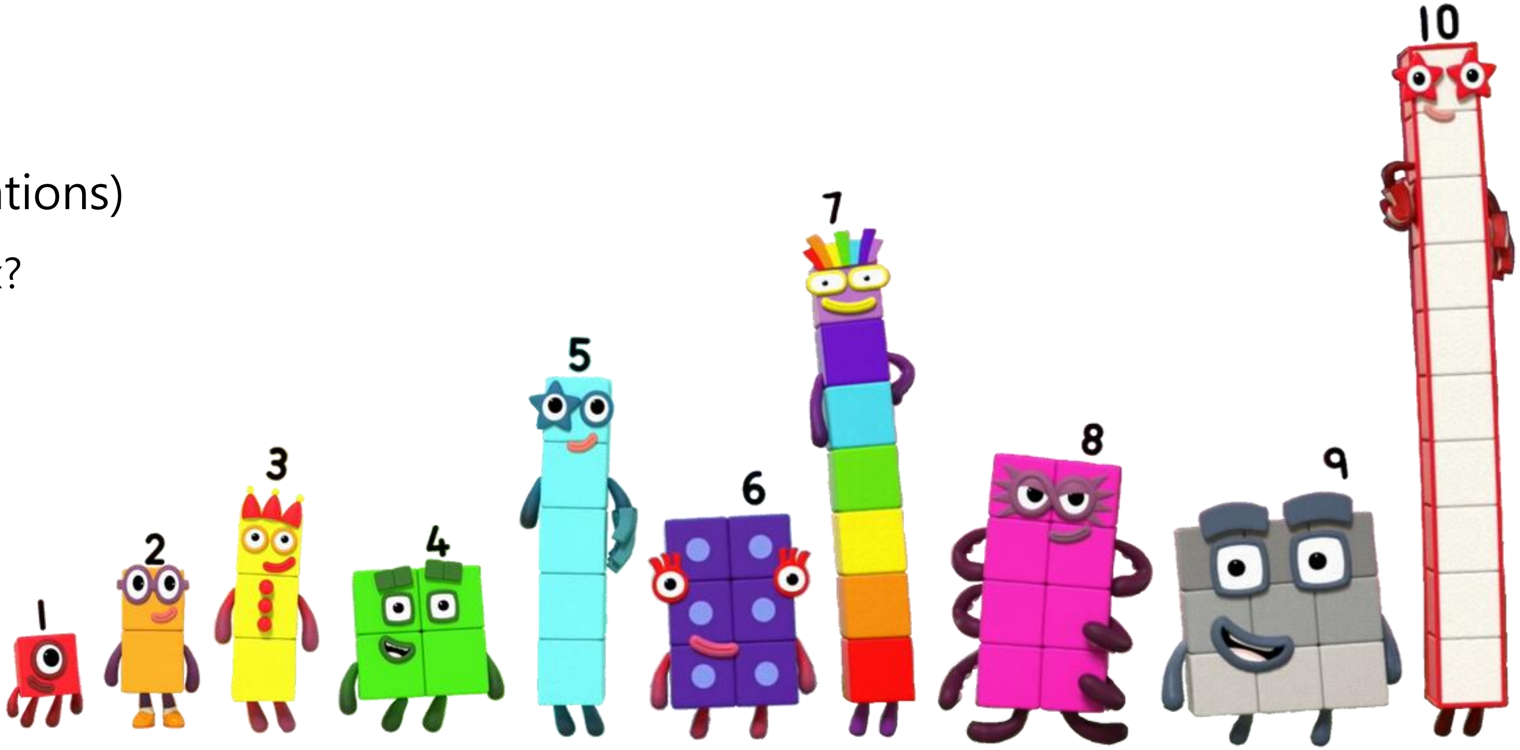


Image: Pinterest

Tools

- **SciPy**
  - A python-based open-source ecosystem for mathematics, science, and engineering.
    - An open-source computing tool against the commercial software, MATLAB
    - Included in Anaconda by default
  - Major components

NumPy Base N-dimensional array package	SciPy library Fundamental library for scientific computing	Matplotlib Comprehensive 2-D plotting
IPython Enhanced interactive console	SymPy Symbolic mathematics	pandas Data structures & analysis

- Online references
  - Please visit mint-lab/know-where > Programming > Python.

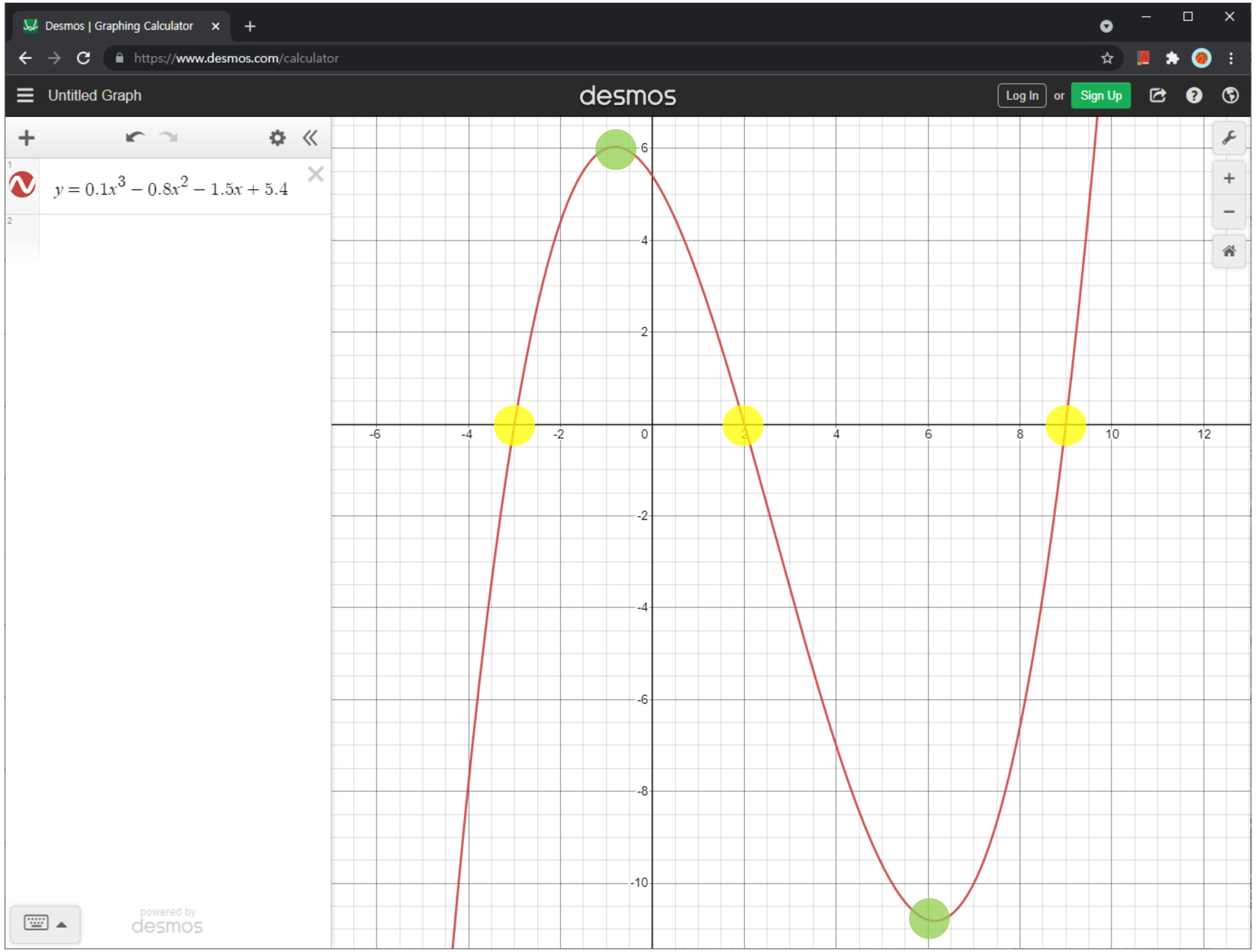
Image: SciPy

~~Programming~~ Python/SciPy meets Mathematics

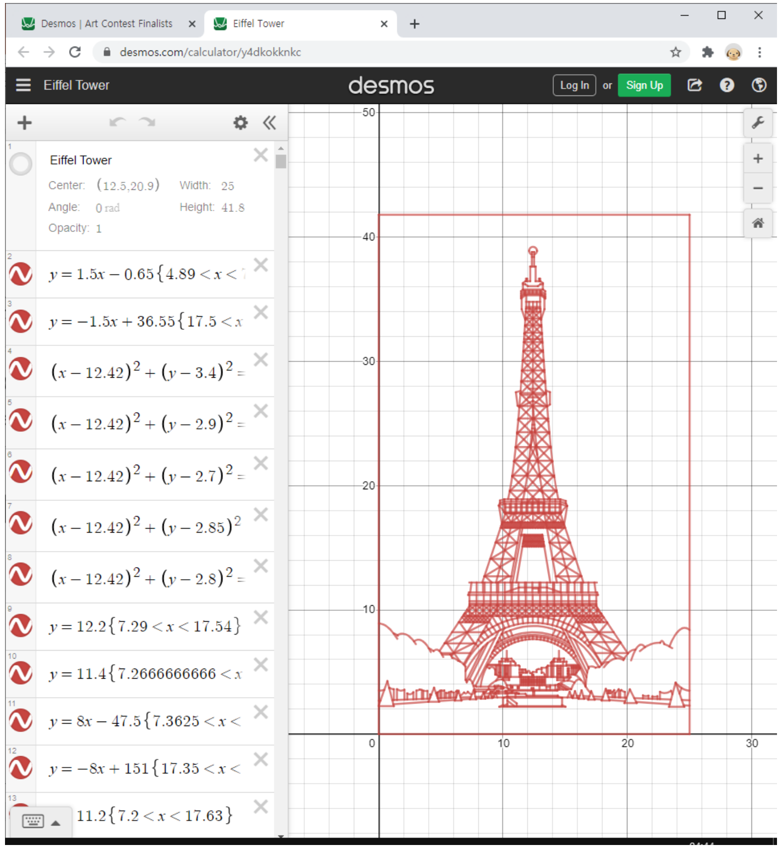
- **Calculus**
- Linear Algebra
- Optimization
- Probability
- Information Theory

Getting Started from Drawing an Equation

- Example) Drawing $y = 0.1x^3 - 0.8x^2 - 1.5x + 5.4$
- Visualization with Desmos Graphing Calculator



Desmos Global Math Art Contest (Age 13-14)



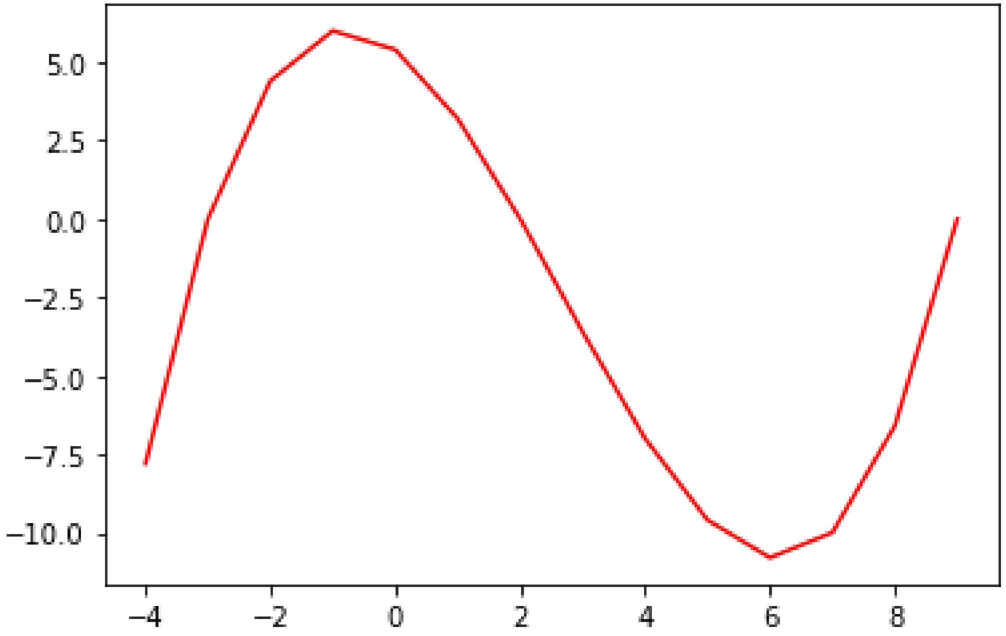
Getting Started from Drawing an Equation

- Example) Drawing $y = 0.1x^3 - 0.8x^2 - 1.5x + 5.4$
- Visualization with Matplotlib

```
import matplotlib.pyplot as plt

xs = [x for x in range(-4, 10)]
ys = [0.1*x**3 - 0.8*x**2 - 1.5*x + 5.4 for x in xs]

plt.plot(xs, ys, 'r-')
plt.show()
```



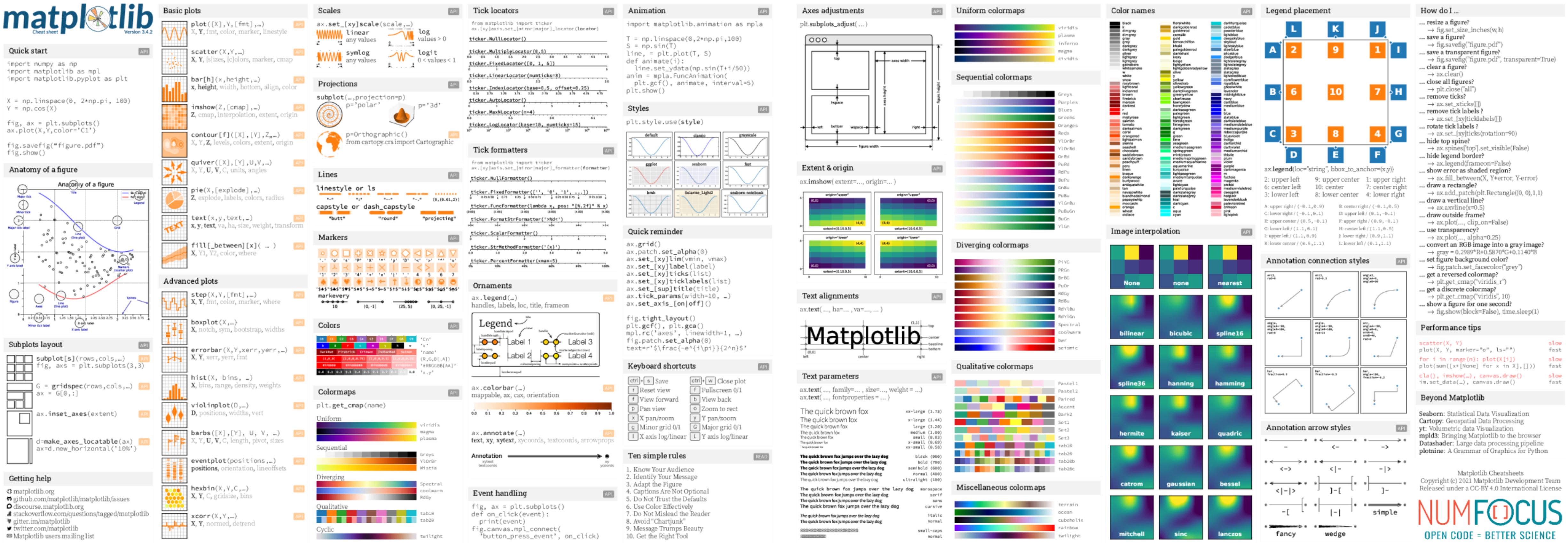
Further improvement)

- How to make the graph smooth?
- How to draw the grid?
- How to display labels on X and Y axes?
- How to make its aspect ratio equal?

Image: Matplotlib cheatsheets

Matplotlib: A Plotting Library in Python

- It is a plotting library for creating static, animated, and interactive visualization in Python.
  - References: Documentation, Examples (Gallery), Tutorials, and **Cheatsheets** (made by Matplotlib)



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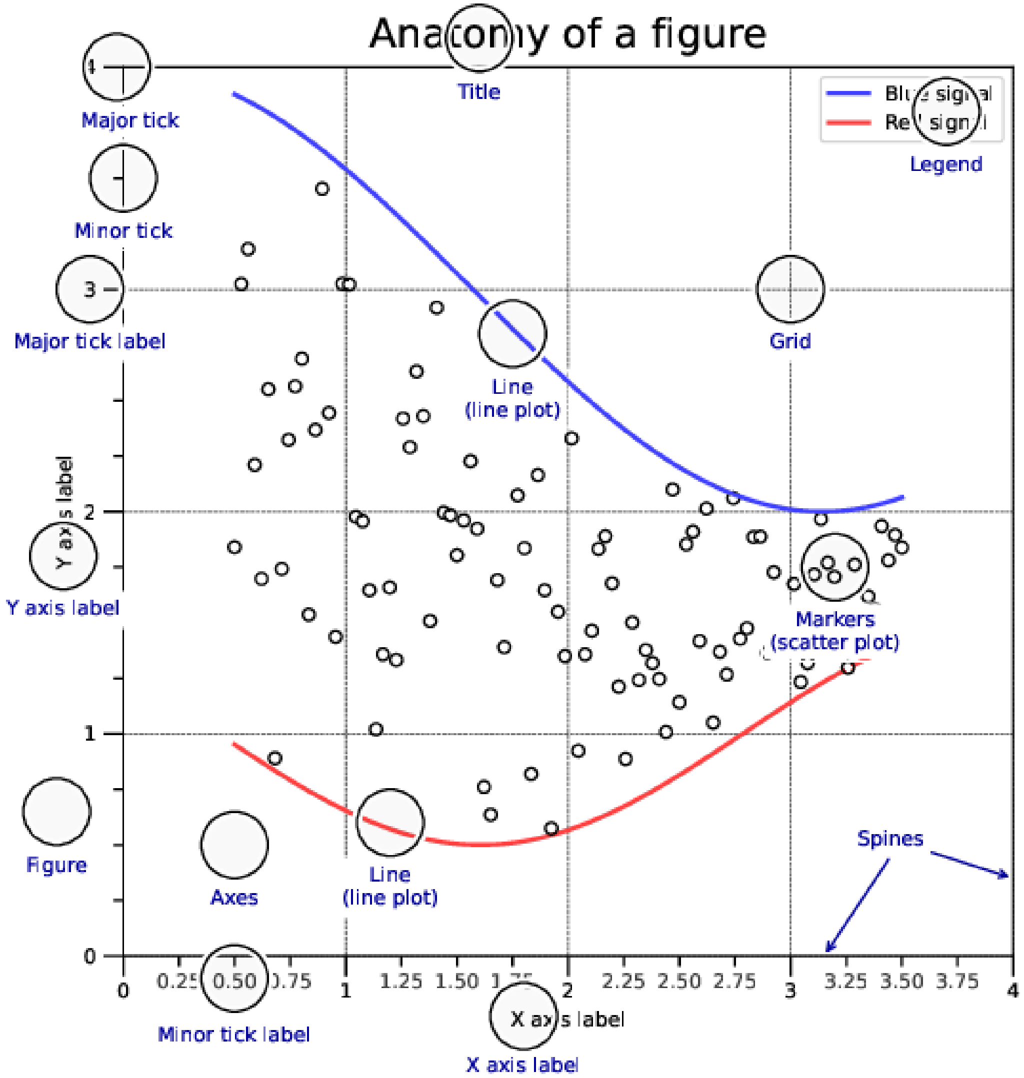

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  - References: Documentation, Examples (Gallery), Tutorials, and **Cheatsheets** (made by Matplotlib)
- API examples @ **`matplotlib.pyplot` module** (a state-based interface; ~ MATLAB)
  - `plot([x], y, [fmt], …)`: Plot `y` versus `x` as lines and/or markers
  - `hist(x, bins=None, range=None, …)`: Plot a histogram
  - `contour([X, Y,] Z, [levels], …)`: Plot contour lines
  - `imshow(X, cmap=None, …)`: Display data as an image (a 2D regular raster)
  - `title(label, …)`: Set a title for the `Axes`
  - `axis(…)`: Get or set axis properties (e.g. `'on'`/`'off'`, `'equal'`, and range of X/Y axes)
  - `legend(…)`: Place a legend on the `Axes`
  - `grid(…)`: Configure the grid lines
  - `xlabel(label, …)`, `xlim(left, right)`, …: Set the label and limit for the x-axis
  - `figure(num=None, …)`: Create a new figure or activate an existing figure
  - `show(…)`: Display all open figures
  - `savefig(filename, …)`: Save the current figure

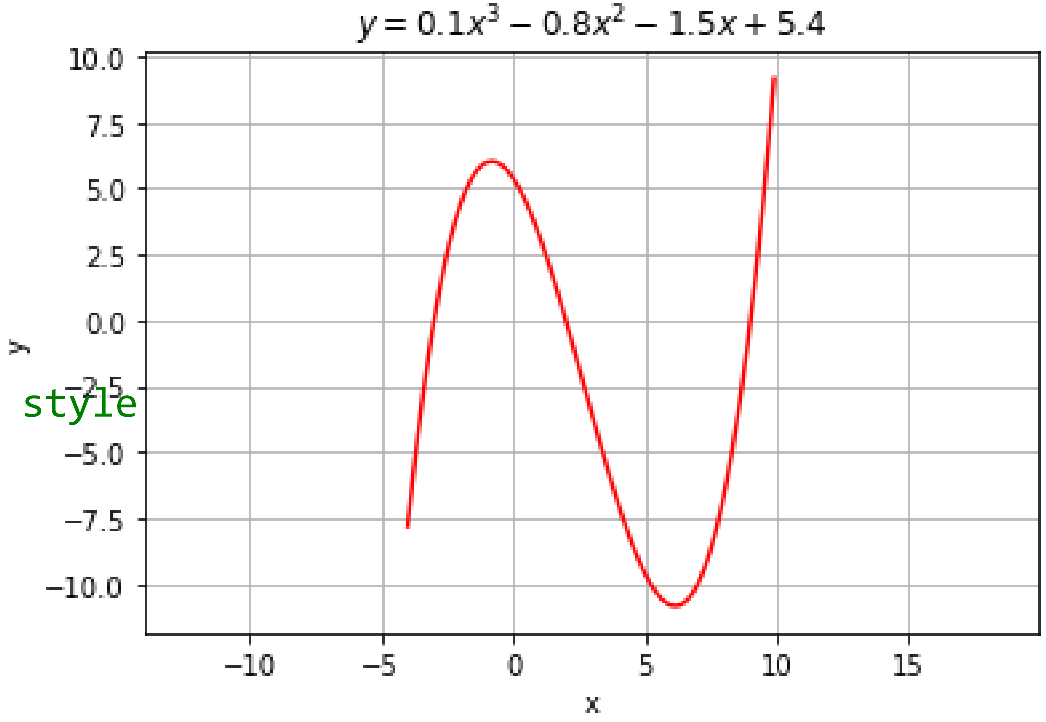
Matplotlib: A Plotting Library in Python

- Usage example) Drawing $y = 0.1x^3 - 0.8x^2 - 1.5x + 5.4$

```
import matplotlib.pyplot as plt

scale = 10
xs = [x/scale for x in range(-4*scale, 10*scale)]
ys = [0.1*x**3 - 0.8*x**2 - 1.5*x + 5.4 for x in xs]

plt.title('$y = 0.1x^3 - 0.8x^2 - 1.5x + 5.4$') # LaTeX style
plt.plot(xs, ys, 'r-')
plt.xlabel('x')
plt.ylabel('y')
plt.grid()
plt.axis('equal')
plt.show()
```



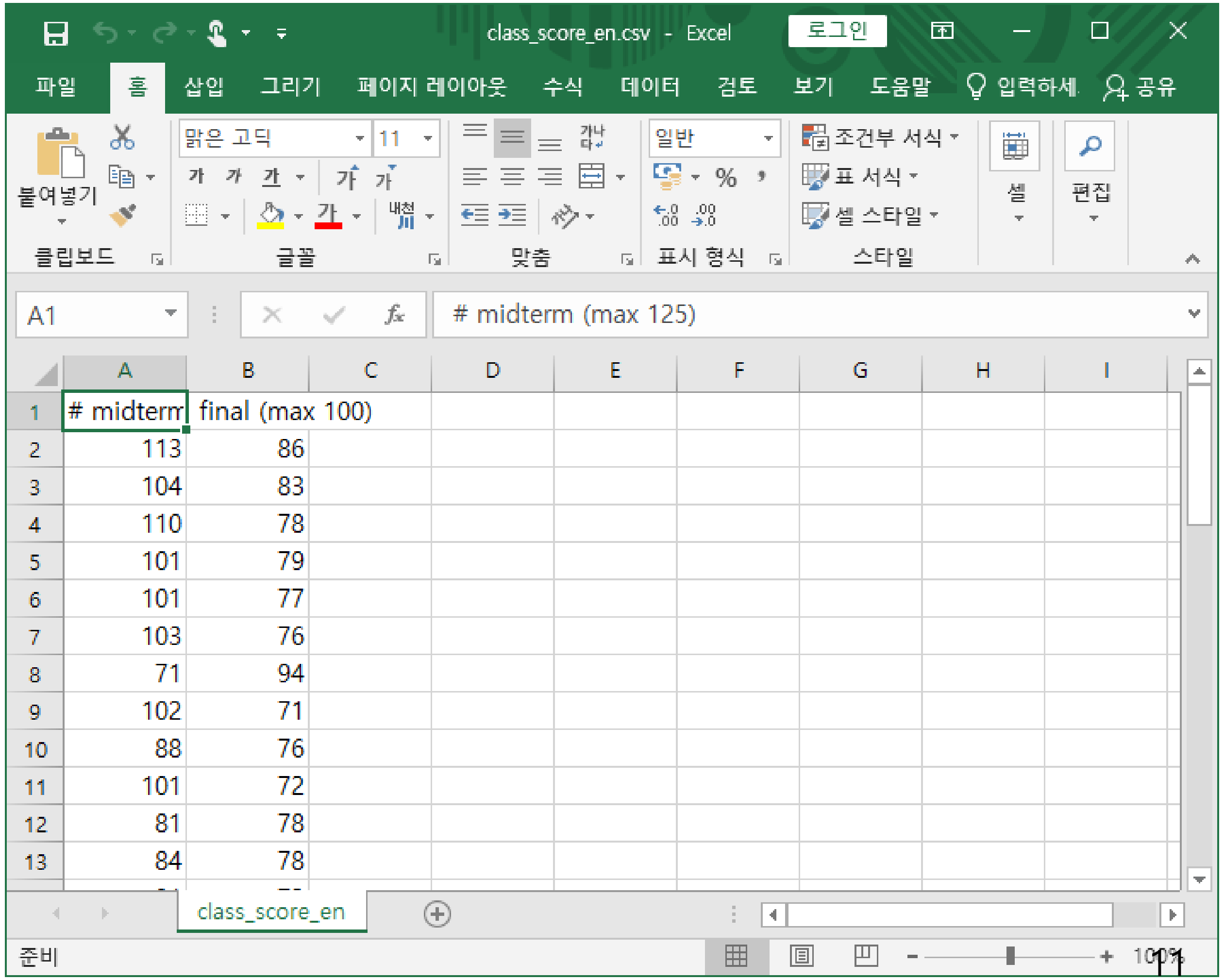
Further improvement)

- ✓ How to make the graph smooth?
- ✓ How to draw the grid?
- ✓ How to display labels on X and Y axes?
- ✓ How to make its aspect ratio equal?

Matplotlib: A Plotting Library in Python

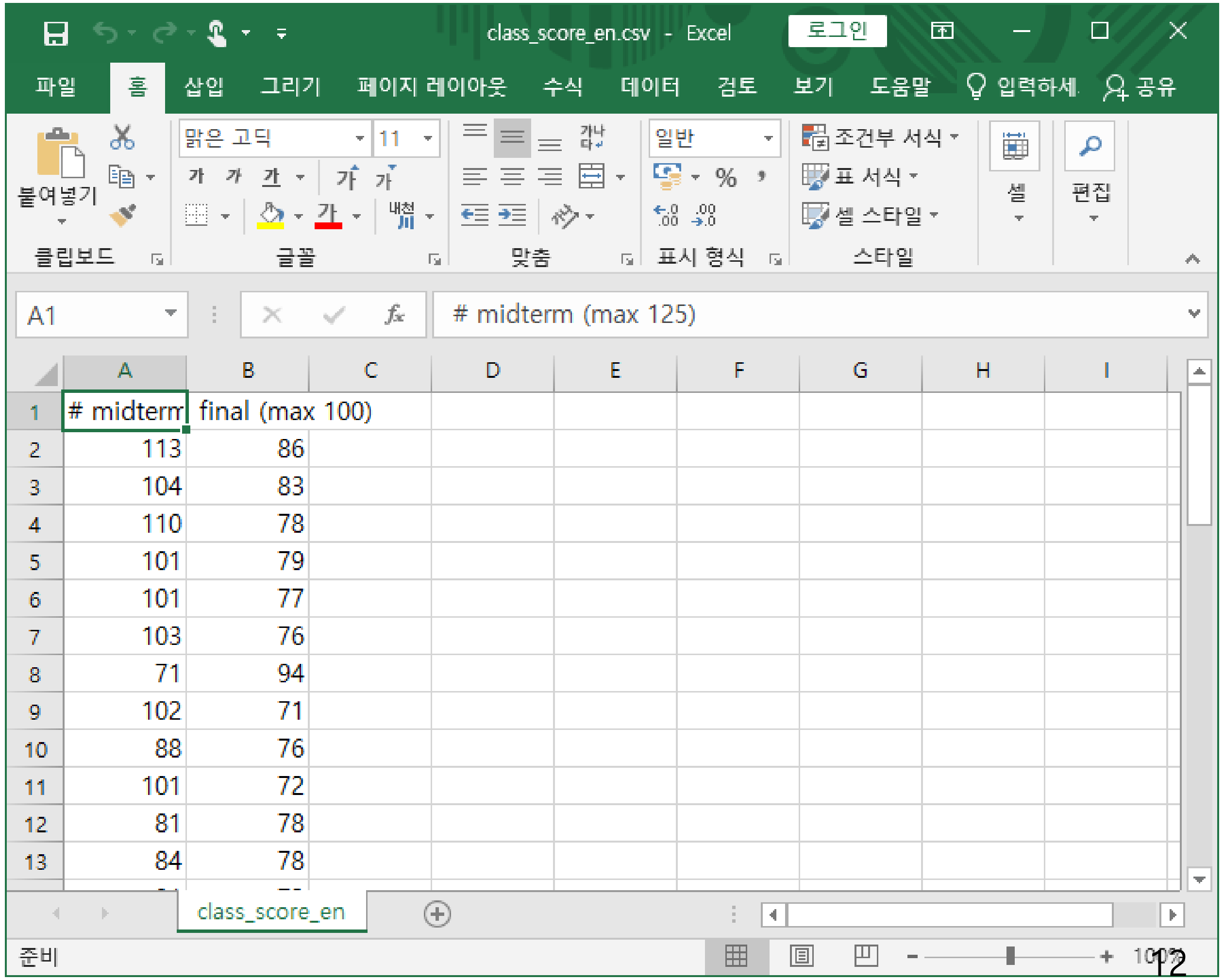
- Example) Plotting midterm and final scores
  - The given data (file: `data/class_score_en.csv`)

```
# midterm (max 125), final (max 100)
113, 86
104, 83
110, 78
101, 79
101, 77
103, 76
71, 94
102, 71
88, 76
101, 72
81, 78
84, 78
. . .
```



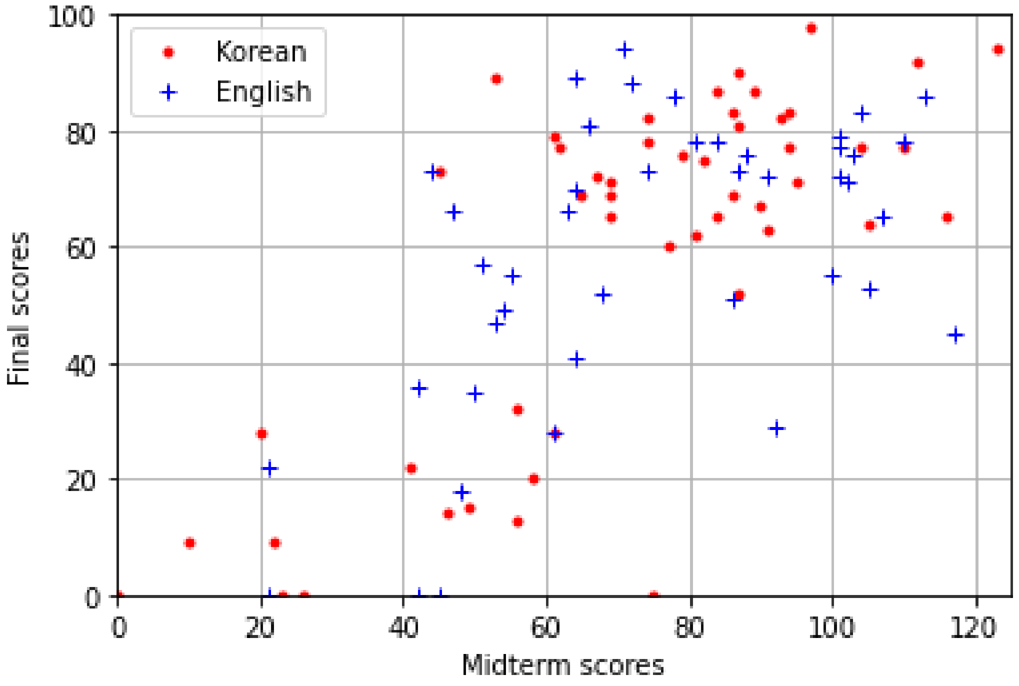
[Matplotlib](): A Plotting Library in Python

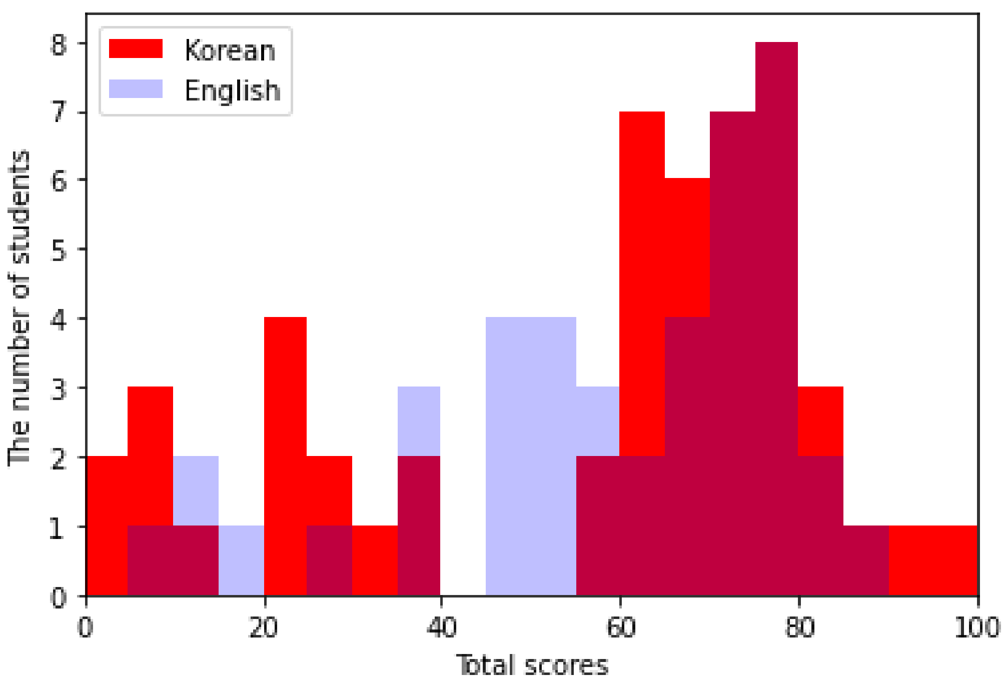
- Example) Plotting midterm and final scores
  - Analysis using representative values (e.g. mean, median, …; [대표값]() in Korean)
    - Midterm
      - Mean: **74.209**
      - Variance: 632.817
      - Median: **72.000**
      - Min/Max: (21.000, 117.000)
    - Final
      - Mean: **58.674**
      - Variance: 618.545
      - Median: **66.000**
      - Min/Max: (0.000, 94.000)
    - Total
      - Mean: **58.952**
      - Variance: 423.546
      - Median: **65.000**
      - Min/Max: (6.720, 87.760)



Matplotlib: A Plotting Library in Python

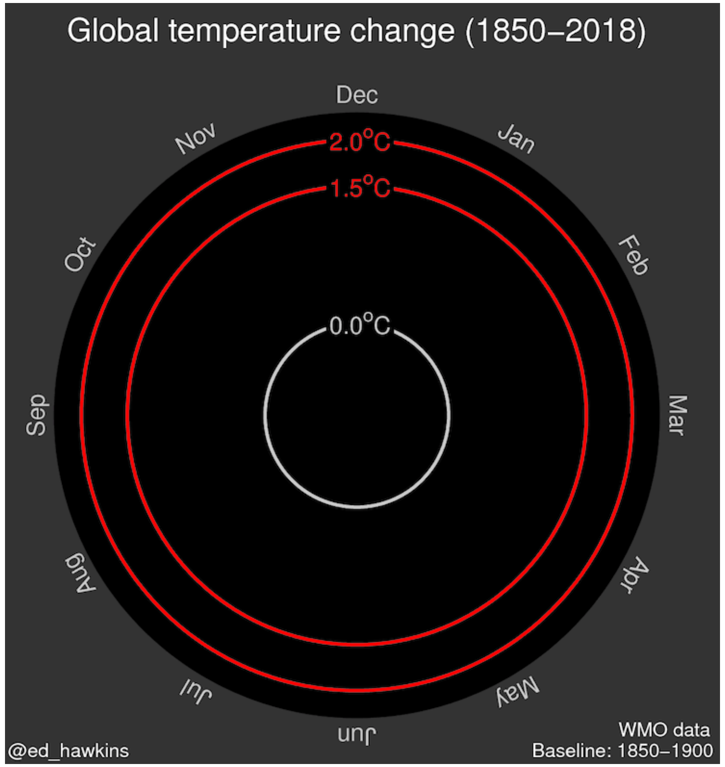
- Example) Plotting midterm and final scores
  - Practice) Plot the scatter plot of midterm and final scores
  - Practice) Plot the histogram of total scores
  - What can you discover from two graphs?





Visualization

- **Visualization is very important** not only for your **data** but also for your **programs**, **systems**, and **society.**



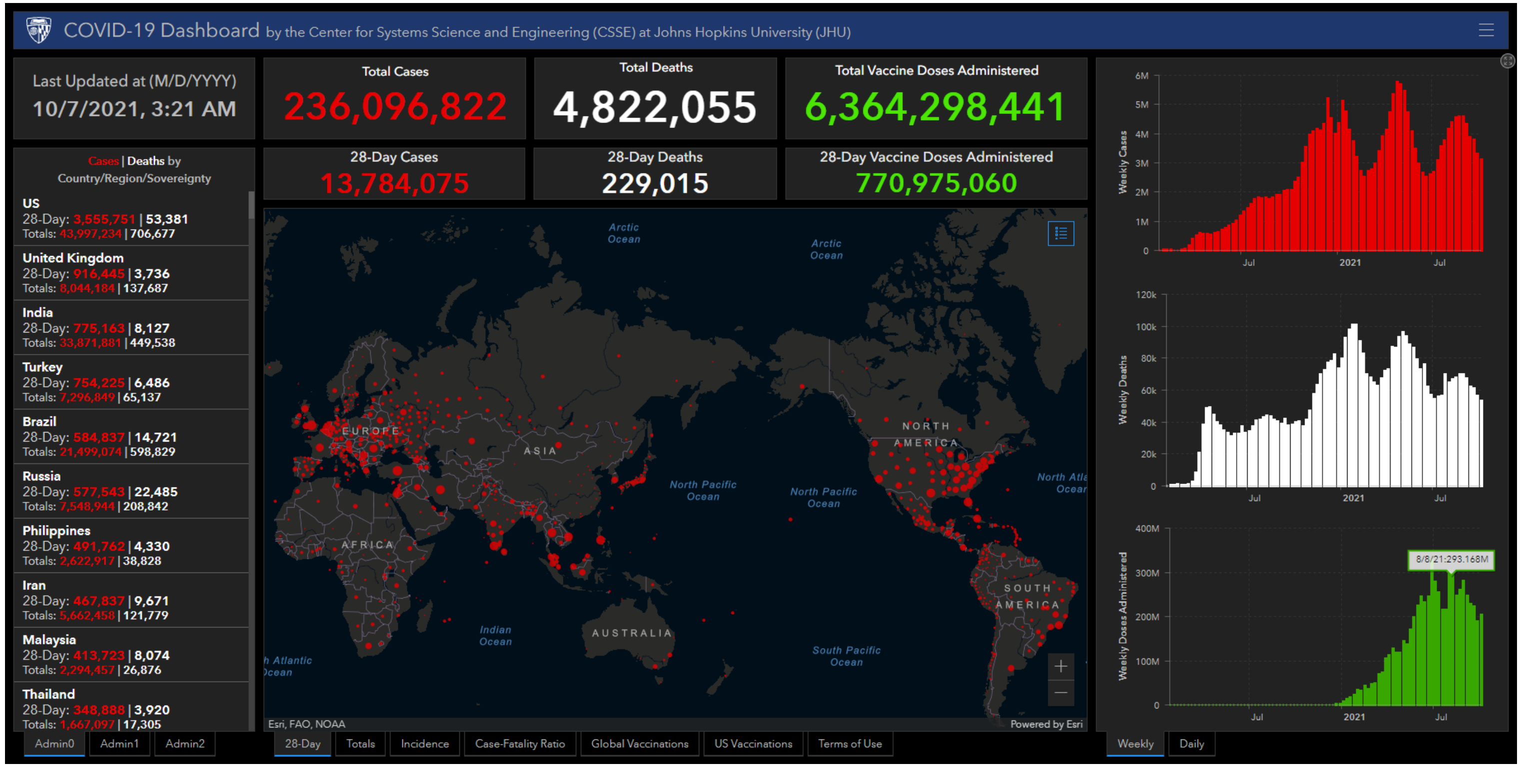


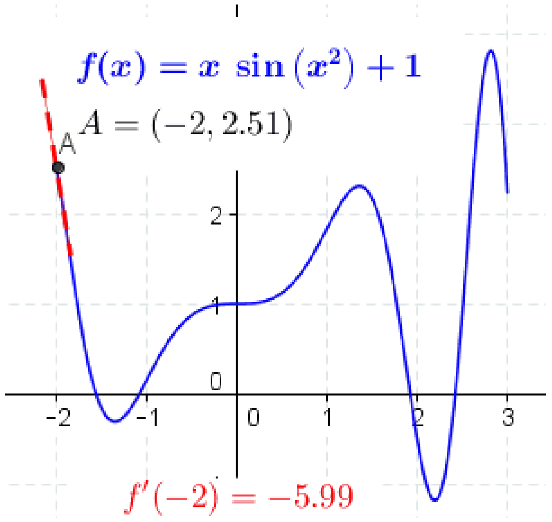
Image: Wikipedia, COVID-19 Dashboard (by JHU)

Calculus

- Calculus (originally called infinitesimal calculus; 미적분학 in Korean) is the mathematical study of continuous change.
- Two major tools

Differentiation:

Finding a derivative of a function



Integration:

Assigning a number to a function

(by combining infinitesimal data)

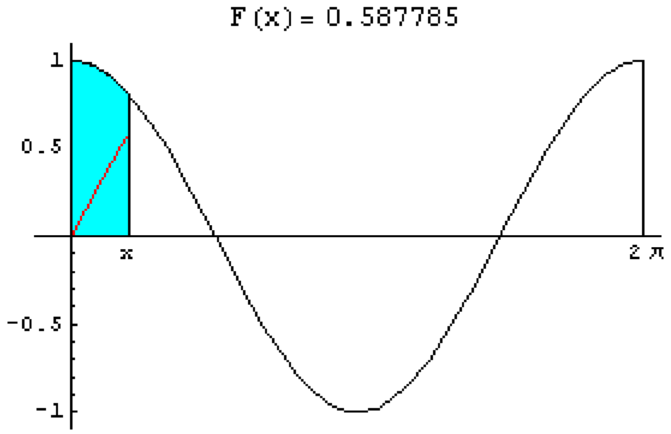
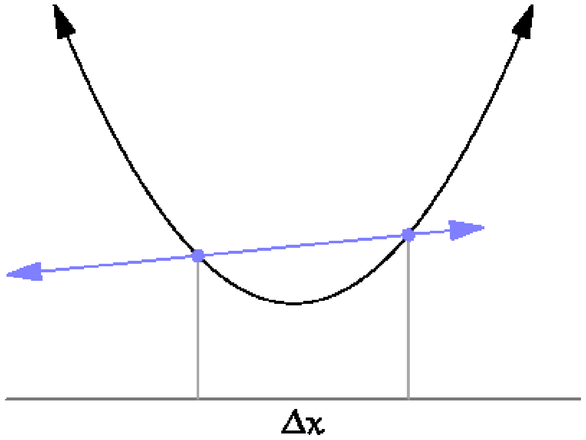


Image: Wikipedia, Docencia con Mathematica

~~Calculus~~ Differentiation

- **Differentiation**: The process of finding a **derivative**
- **Derivative** [Wikipedia]
  - Definition
    - Change of the function value (**output value**) with respect to a change in its argument (**input value**)

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$



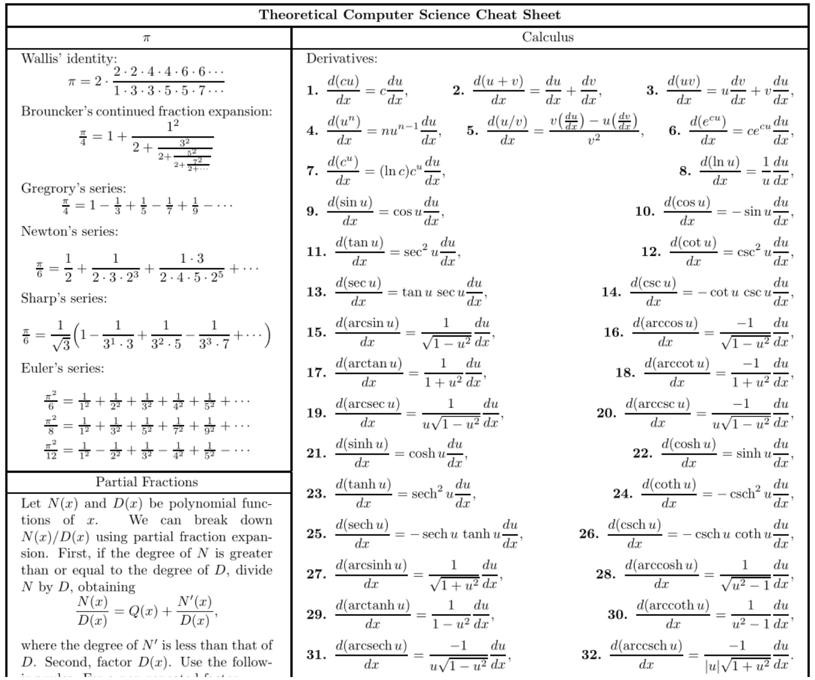
  - Meaning
    - The **slope** of the tangent line (plane) to the given function
    - The tangent line ~ **linear approximation** (cf. 1st-order Tayor series)
  - Notation
    - Leibniz's notation: $\frac{df}{dx}(x)$, $\frac{d^2 f}{dx^2}(x)$
    - Largrange's notation: $f'(x)$, $f''(x)$
    - Newton's notation: $\dot{y}$, $\ddot{y}$ where $y = f(x)$
    - Euler's notation: $D_x f(x)$, $D_x^2 f(x)$

Image: Wikipedia

Differentiation

- Rules of computing a derivative [Wikipedia]
  - The polynomial rule of $f(x) = x^n$
    - $f'(x) = nx^{n-1}$
  - **Logarithmic** and exponential functions
    - $\frac{d}{dx}\ln x = \frac{1}{x}$
    - $\frac{d}{dx}(e^x) = e^x$ (cf. e: Euler's number, 자연상수 in Korean)
  - The product rule of $h(x) = f(x)g(x)$
    - $h'(x) = f'(x)g(x) + f(x)g'(x)$
  - cf. The quotient rule of $h(x) = f(x)/g(x)$
    - $h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$
  - The **chain rule** of $h(x) = f(g(x))$
    - $h'(x) = f'\big(g(x)\big) \cdot g'(x) = \frac{df(g(x))}{dg(x)} \cdot \frac{dg(x)}{dx}$
  - Too many …

Theoretical CS Mathematical Cheat Sheet



Theoretical Computer Science Cheat Sheet

π

Wallis' identity:
$$\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

Brouncker's continued fraction expansion:
$$\frac{\pi}{4} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}$$

Gregrory's series:
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

Newton's series:
$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \cdots$$

Sharp's series:
$$\frac{\pi}{6} = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3^1 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \cdots\right)$$

Euler's series:
$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$
$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots$$
$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots$$

Partial Fractions

Let $N(x)$ and $D(x)$ be polynomial functions of x. We can break down $N(x)/D(x)$ using partial fraction expansion. First, if the degree of N is greater than or equal to the degree of D, divide N by D, obtaining
$$\frac{N(x)}{D(x)} = Q(x) + \frac{N'(x)}{D(x)},$$
where the degree of N' is less than that of D. Second, factor $D(x)$. Use the follow-

Calculus

Derivatives:

1. $\frac{d(cu)}{dx} = c\frac{du}{dx}$,
2. $\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$,
3. $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$,
4. $\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}$,
5. $\frac{d(u/v)}{dx} = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$,
6. $\frac{d(e^{cu})}{dx} = ce^{cu}\frac{du}{dx}$,
7. $\frac{d(c^u)}{dx} = (\ln c)c^u\frac{du}{dx}$,
8. $\frac{d(\ln u)}{dx} = \frac{1}{u}\frac{du}{dx}$,
9. $\frac{d(\sin u)}{dx} = \cos u\frac{du}{dx}$,
10. $\frac{d(\cos u)}{dx} = -\sin u\frac{du}{dx}$,
11. $\frac{d(\tan u)}{dx} = \sec^2 u\frac{du}{dx}$,
12. $\frac{d(\cot u)}{dx} = \csc^2 u\frac{du}{dx}$,
13. $\frac{d(\sec u)}{dx} = \tan u \sec u\frac{du}{dx}$,
14. $\frac{d(\csc u)}{dx} = -\cot u \csc u\frac{du}{dx}$,
15. $\frac{d(\arcsin u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$,
16. $\frac{d(\arccos u)}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$,
17. $\frac{d(\arctan u)}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$,
18. $\frac{d(\mathrm{arccot}\, u)}{dx} = \frac{-1}{1+u^2}\frac{du}{dx}$,
19. $\frac{d(\mathrm{arcsec}\, u)}{dx} = \frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}$,
20. $\frac{d(\mathrm{arccsc}\, u)}{dx} = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}$,
21. $\frac{d(\sinh u)}{dx} = \cosh u\frac{du}{dx}$,
22. $\frac{d(\cosh u)}{dx} = \sinh u\frac{du}{dx}$,
23. $\frac{d(\tanh u)}{dx} = \mathrm{sech}^2 u\frac{du}{dx}$,
24. $\frac{d(\coth u)}{dx} = -\mathrm{csch}^2 u\frac{du}{dx}$,
25. $\frac{d(\mathrm{sech}\, u)}{dx} = -\mathrm{sech}\, u \tanh u\frac{du}{dx}$,
26. $\frac{d(\mathrm{csch}\, u)}{dx} = -\mathrm{csch}\, u \coth u\frac{du}{dx}$,
27. $\frac{d(\mathrm{arcsinh}\, u)}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$,
28. $\frac{d(\mathrm{arccosh}\, u)}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$,
29. $\frac{d(\mathrm{arctanh}\, u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}$,
30. $\frac{d(\mathrm{arccoth}\, u)}{dx} = \frac{1}{u^2-1}\frac{du}{dx}$,
31. $\frac{d(\mathrm{arcsech}\, u)}{dx} = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}$,
32. $\frac{d(\mathrm{arccsch}\, u)}{dx} = \frac{-1}{|u|\sqrt{1+u^2}}\frac{du}{dx}$.

Differentiation

- Example) Finding the slope and tangent line of $y = 0.1x^3 - 0.8x^2 - 1.5x + 5.4$ at $x = 2$
  - Derivative function: $\frac{dy}{dx} = 0.3x^2 - 1.6x - 1.5$
  - Derivative value at $x = 2$ $(y = 0)$: -3.5
  - Tangent line at $x = 2$ $(y = 0)$: $y - 0 = -3.5(x - 2) \rightarrow y = -3.5x + 7$

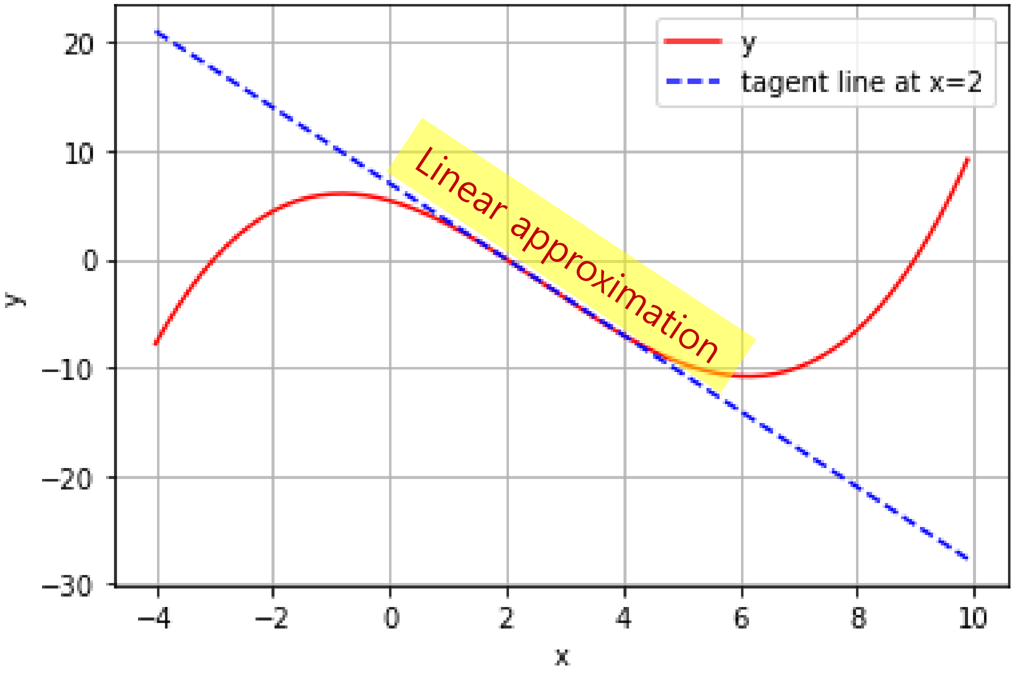
How to derive a derivative function of a complex function?

- Example) Plotting the above results

```
import matplotlib.pyplot as plt

scale = 10
xs = [x/scale for x in range(-4*scale, 10*scale)]
ys = [0.1*x**3 - 0.8*x**2 - 1.5*x + 5.4 for x in xs]
yt = [-3.5*x + 7 for x in xs]

plt.plot(xs, ys, 'r-',  label='y')
plt.plot(xs, yt, 'b--', label='tangent line at x=2')
plt.xlabel('x')
plt.ylabel('y')
plt.grid()
plt.legend()
plt.show()
```



Differentiation

- Example) Finding the derivative function of $y = 0.1x^3 - 0.8x^2 - 1.5x + 5.4$ using SymPy

```
import sympy as sp

x, y = sp.symbols('x y')
y = 0.1*x**3 - 0.8*x**2 - 1.5*x + 5.4

yd = sp.diff(y, x)
print(yd)                          # 0.3*x**2 - 1.6*x - 1.5
print(float(yd.subs({x: 2})))      # -3.5 / Casting from Float object to float
```

- Example) Solve the above equation (find x when $y = 0$) using SymPy

```
roots = sp.solveset(y, x)
print(roots)                       # FiniteSet(-3.0, 2.0, 9.0)
r0 = float(roots.args[0])          # -3.0 / Casting from Float object to float
```

Differentiation

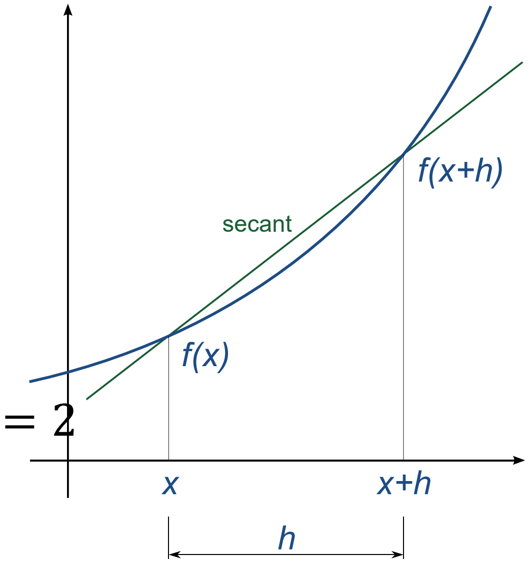
- Example) Factorize the above equation using SymPy

```
sp.pprint(sp.factor(y))              # 5.4·(0.11...·x - 1.0)·(0.33...·x + 1.0)·(0.5·x - 1.0)

y = (x**3 - 8*x**2 - 15*x + 54)/10 # Make the coefficients rational numbers
sp.pprint(sp.factor(y))              # (x - 9)·(x - 2)·(x + 3)
                                     # ───────────────────────
                                     #           10
```

Differentiation

- Numerical differentiation is an algorithm to approximate a derivative value of a function using values of the function. (cf. It does not derive a derivative function.)
  - The definition of derivative: $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$
  - Two-point estimation: $f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h}$ or $f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h}$



- Example) Finding the slope and tangent line of $y = 0.1x^3 - 0.8x^2 - 1.5x + 5.4$ at $x = 2$
  - Derivative at $x = 2$ $(y = 0)$: -3.5

```
y = lambda x: 0.1*x**3 - 0.8*x**2 - 1.5*x + 5.4
x0 = 2
h = 0.001

d1 = (y(x0+h) - y(x0)) / h       # -3.5002
d2 = (y(x0+h/2) - y(x0-h/2)) / h # -3.5000
```

Summary

- Building mathematical intuition: 1) Imagine, 2) Ask why
- Visualization
  - Matplotlib: A Plotting Library in Python
  - **Visualization is very important** not only for your **data** but also for your **programs**, **systems**, and **society.**
- ~~Calculus~~ Differentiation
  - Derivative definition: $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$
  - Meaning
    - The **slope** of the tangent line (plane) to the given function
    - The tangent line ~ **linear approximation**
  - How to get a derivative
    - ~~Your hand-on derivation using rules~~
    - Analytical differentiation using SymPy
    - Numerical differentiation