Macroeconomic Theory

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OLG

Introduction

Allais (1947), Samuelson (1958), Diamond (1965) introduce OLG models.

- Second basic model used in micro-based macroeconomics
- Life-cycle saving behavior
- Particularity: generations yet unborn whose preferences may not be registered in current market transactions $\rightarrow$ pecuniary externalities
- Competitive equilibrium might **not** be Pareto optimal (overaccumulation of capital)

OLG

Two-period lives

Time structure:

- Time is discrete
- Every period 2 cohorts are alive: **young** and **old**
- Individual born at time t consumes c_{1t} in period t and c_{2t+1} in period $t+1$

Utility function:

$$\boxed{u(c_{1t}) + (1+\rho)^{-1} u(c_{2t+1})}$$

where ρ is the discount rate and u is a strictly concave felicity function.

OLG

Demographics and production

Demographics:

- Individuals work only when **young**, supplying one unit of labor and receiving wage w
- They save for retirement consumption
- Number of individuals born at time t is N_t, population grows at rate n

Production:

- Saving of period t generates capital stock of period $t + 1$
- Firms produce using CRS neoclassical production function $y = f(k)$
- k is the capital-labor ratio (net output)

Decentralized Equilibrium

Individual's problem

An individual born at time t solves:

$$\boxed{\max_{c_{1t}, c_{2t+1}, s_t} \; u(c_{1t}) + (1+\rho)^{-1} u(c_{2t+1})}$$

subject to budget constraints:

$$\begin{aligned} c_{1t} + s_t &= w_t \\ c_{2t+1} &= (1 + r_{t+1}) s_t \end{aligned}$$

Decentralized Equilibrium

Optimal saving

The first order condition for a maximum is:

$$u'(c_{1t}) = (1+\rho)^{-1}(1+r_{t+1})u'(c_{2t+1})$$

Using the budget constraints, we obtain the saving function:

$$s_t = s(w_t, r_{t+1}), \quad 0 < s_w < 1, \quad s_r \lesseqgtr 0$$

where $s_w > 0$ (separability and concavity) and s_r has ambiguous sign (substitution vs. income effects).

Decentralized Equilibrium

Firms

Firms maximize profits. The first order conditions are:

$$\boxed{f'(k_t) = r_t}$$

$$\boxed{f(k_t) - k_t f'(k_t) = w_t}$$

Decentralized Equilibrium

Market equilibrium

Goods market equilibrium:

Equilibrium requires total savings $S_t = s_t N_t$ equal investment K_{t+1}:

$$\boxed{K_{t+1} = N_t s(w_t, r_{t+1})}$$

which gives in per capita terms:

$$\boxed{(1+n)k_{t+1} = s(w_t, r_{t+1})}$$

Factor market equilibrium:

- Supply of labor is inelastic

Decentralized Equilibrium

Definition

A **competitive equilibrium** is the sequence of aggregate capital stock, consumption and prices:

$$\{K_t, c_{1t}, c_{2t}, r_t, w_t\}_{t=0}^{\infty}$$

such that:

- First order conditions of firms and households are respected
- All markets clear

Decentralized Equilibrium

Dynamics

Capital accumulation plus equilibrium conditions imply:

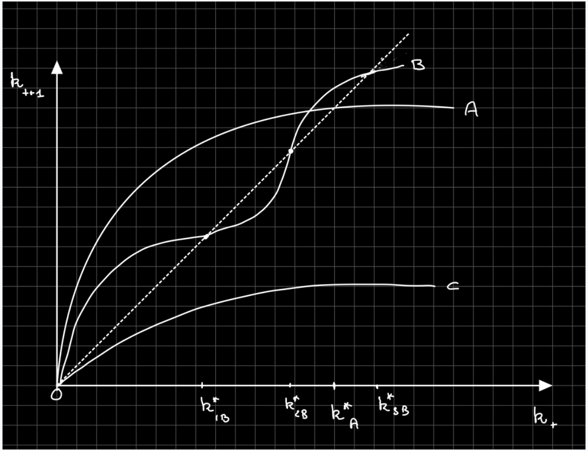
$$\boxed{k_{t+1} = \frac{s[f(k_t) - k_t f'(k_t), f'(k_{t+1})]}{1+n}}$$

The derivative is:

$$\frac{dk_{t+1}}{dk_t} = \frac{-s_w(k_t)k_t f''(k_t)}{1 + n - s_r(k_{t+1})f''(k_{t+1})}$$

Competitive Equilibrium

Steady state



CRRA and Cobb-Douglas Case

Utility specification

Suppose the utility function is CRRA:

$$\boxed{\frac{(c_{1t})^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} + (1+\rho)^{-1}\frac{(c_{2t+1})^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}}$$

subject to:

$$c_{1t} + s_t = w_t$$
$$c_{2t+1} = (1 + r_{t+1})s_t$$

CRRA and Cobb-Douglas Case

Saving function

The first order condition is:

$$\boxed{(c_{1t})^{-\frac{1}{\sigma}} = (1+\rho)^{-1}(1+r_{t+1})(c_{2t+1})^{-\frac{1}{\sigma}}}$$

Using budget constraints, the saving function is:

$$\boxed{s_t = \frac{w_t}{1+(1+\rho)^{\sigma}(1+r_{t+1})^{1-\sigma}} = \frac{w_t}{\psi(t+1)}}$$

where $\psi(t+1) > 1$.

CRRA and Cobb-Douglas Case

Saving function properties

Marginal propensity to save:

$$s_w = \frac{1}{\psi(t+1)} \in (0,1)$$

Interest rate effect:

$$s_r = (\sigma - 1)\left(\frac{1+\rho}{1+r_{t+1}}\right)^{\sigma} \frac{s_t}{\psi(t+1)}$$

where:

- $s_r > 0$ if $\sigma > 1$ (substitution effect dominates)
- $s_r < 0$ if $\sigma < 1$ (income effect dominates)
- $s_r = 0$ if $\sigma = 1$ (log utility)

CRRA and Cobb-Douglas Case

Permanent expected shocks

Note that there could be another solution: with very high intertemporal substitution, the saddle path would be very steep causing consumption to fall initially.

The adjustment would also be much faster.

CRRA and Cobb-Douglas Case

Production function

Using the Cobb-Douglas production function $f(k) = k^{\alpha}$:

$$\boxed{\alpha k_t^{(\alpha-1)} = r_t}$$

$$\boxed{(1-\alpha)k_t^{\alpha} = w_t}$$

CRRA and Cobb-Douglas Case

Equilibrium dynamics

The equilibrium simplifies to:

$$\boxed{k_{t+1} = \frac{s_t}{1+n}}$$

$$\boxed{k_{t+1} = \frac{(1-\alpha)k_t^{\alpha}}{\left(1+(1+\rho)^{\sigma}(1+\alpha k_{t+1}^{(\alpha-1)})^{1-\sigma}\right)(1+n)}}$$

CRRA and Cobb-Douglas Case

Steady state

The steady state is unique and satisfies:

$$\boxed{k^* = \frac{(1-\alpha)(k^*)^\alpha}{\left(1+(1+\rho)^\sigma(1+\alpha(k^*)^{(\alpha-1)})^{1-\sigma}\right)(1+n)}}$$

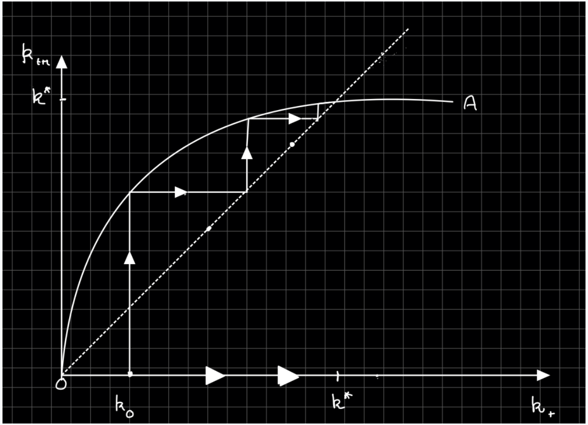
Using the definition of the interest rate:

$$\boxed{\left(1+(1+\rho)^\sigma(1+r^*)^{1-\sigma}\right)(1+n) = \frac{(1-\alpha)}{\alpha}r^*}$$

(depends on more parameters than in the Ramsey model)

CES-Cobb-Douglas

Steady state



Government

Taxes and public spending

Suppose households pay taxes when they work to finance government spending g:

$$\boxed{c_{1t} + s_t = w_t - \tau_{1,t}}$$

Government budget constraint with debt:

$$\boxed{b_{t+1}(1+n) = g_t - \tau_{1,t} + (1+r_t)b_t}$$

Now changes in g financed through future τ have effects. **Failure of Ricardian Equivalence.**

Government

Intertemporal budget constraint

The government's intertemporal budget constraint:

$$\sum_{i=0}^{\infty}(1+n)^i \prod_{j=1}^{i} \frac{1}{1+r_{t+j}}(\tau_{1,t+i} - g_{t+i}) = b_t$$

Capital market equilibrium:
Saving equals investment plus government debt:

$$K_{t+1} + B_{t+1} = N_t s(w_t, r_{t+1})$$

which gives:

Government

Effects on capital accumulation

The government budget implies:

$$\boxed{b(1+r) = \tfrac{\tau_1 - g}{r-n}}$$

Equilibrium in asset markets, with household and firm FOCs:

$$(1+n)(k+b) = \frac{w - \tau_1}{1 + (1+\rho)^{\sigma}(1+r)^{1-\sigma}}$$

$$f'(k) = r$$

$$f(k) - kf'(k) = w$$

Social Planner Solution

Welfare function

When individuals have infinite horizons, we use their utility function. Here, each generation cares only about itself. What is the Social Planner objective?

We assume the planner maximizes future generations' utility at rate R:

$$\boxed{U = (1+\rho)^{-1} u(c_{20}) + \textstyle\sum_{t=0}^{T-1} (1+R)^{-t-1} [u(c_{1t}) + (1+\rho)^{-1} u(c_{2t+1})]}$$

where the Social Planner cares only about the next T generations.

Social Planner Solution

First order conditions

The FOCs of the planner are:

Intratemporal allocation:

$$\boxed{u'(c_{1t}) = (1+\rho)^{-1}(1+R)(1+n)u'(c_{2t})}$$

Intertemporal allocation:

$$\boxed{u'(c_{1t-1}) = (1+n)^{-1}(1+R)^{-1}(1+f'(k_t))u'(c_{1t})}$$

Combining the two FOCs:

Social Planner Solution

Economic system

The economy is described by:

Resource constraint:

$$k_t + f(k_t) = (1+n)k_{t+1} + c_{1t} + (1+n)^{-1}c_{2t}$$

Intratemporal condition:

$$u'(c_{1t}) = (1+\rho)^{-1}(1+R)(1+n)u'(c_{2t})$$

Intertemporal condition:

$$u'(c_{1t-1}) = (1+n)^{-1}(1+R)^{-1}(1+f'(k_t))u'(c_{1t})$$

with boundary conditions k_0 and k_{T+1}.

Social Planner Solution

Optimum and dynamic efficiency

The steady state (assuming convergence, $T \to \infty$, $R < 0$):

$$\boxed{1 + f'(k^*) = (1+n)(1+R)}$$

Given R is arbitrary, this can differ from the decentralized equilibrium. What about Pareto optimality?

Define $c_t = c_{1t} + (1+n)^{-1} c_{2t}$. In steady state:

$$c^* = f(k^*) - nk^*$$

and look at:

$$dc^* \quad f'(k^*) - n > 0$$

Social Planner Solution

Dynamic inefficiency

Suppose we decrease permanently k by dk starting from steady state:

$$\boxed{dc_t = -(1+n)dk > 0}$$

$$\boxed{dc_{t+i} = (f_k - n)dk > 0 \text{ if } f_k < n \text{ for } i > 0}$$

All generations can be made better off by reducing capital accumulation when the economy is dynamically inefficient.

Market Economy Inefficiency

Overaccumulation example

Can market economies overaccumulate capital?

Example: Log utility and Cobb-Douglas ($f(k) = k^\alpha - \delta k$)

$$\boxed{r^* = \frac{\alpha(1+n)(2+\rho)}{1-\alpha} - \delta}$$

Reasonable parameters are consistent with $r^* < n$.

Extensions:

- Extended to productivity growth
- Consider risky assets $\rightarrow$ which interest rate is relevant?

Altruism

Replicating command optimum

Bequests affect intertemporal allocations. With altruistic preferences:

$$\boxed{V_t = u(c_{1t}) + (1+\rho)^{-1} u(c_{2t+1}) + (1+R)^{-1} V_{t+1}}$$

and budget constraints:

$$c_{1t} + s_t = w_t + b_t$$
$$c_{2t+1} + (1+n) b_{t+1} = (1 + r_{t+1}) s_t$$

Steady state condition:

$$(1+r^*) \leq (1+n)(1+R) \text{ if } b = 0$$
$$(1+r^*) = (1+n)(1+R) \text{ if } b > 0$$

Social Security and Capital Accumulation

Two systems

Retirement programs affect capital accumulation. Let d_t be the contribution of a young person and b_t the benefit for an old.

Two systems:

1. **Fully funded:**

$$b_t = (1 + r_t)d_{t-1}$$

2. **Pay-as-you-go (PAYG):**

$$b_t = (1 + n)d_t$$

Social Security and Capital Accumulation

Effects on saving

1. **Fully Funded:** If $d_t < (1+n)k_{t+1}$, no effect on total savings:

$$\boxed{u'(w_t - s_t - d_t) = (1+\rho)^{-1}u'[(1+r_{t+1})(s_t+d_t)]}$$

2. **PAYG:** Rate of return is n on contribution:

$$\boxed{u'(w_t - s_t - d_t) = (1+\rho)^{-1}u'[(1+r_{t+1})s_t + (1+n)d_{t+1}]}$$

but only source of capital is private saving.

Social Security and Capital Accumulation

PAYG system effects

Effect of social security PAYG on private saving:

$$\boxed{\frac{\partial s_t}{\partial d_t} = -\frac{u_1'' + (1+\rho)^{-1}(1+n)u_2''}{u_1'' + (1+\rho)^{-1}(1+r_{t+1})u_2''} < 0}$$

PAYG decreases private saving, but whether more or less than one-to-one depends on n and r.

General equilibrium effect:

$$\frac{dk_{t+1}}{dd_t} = \frac{\frac{\partial s_t}{\partial d_t}}{1+n-s_r f''} < 0$$