Macroeconomic Theory

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Infinite Horizon

Foundations of dynamic macroeconomics

- Economic aggregates dynamics are determined by decisions at the microeconomic level, Ramsey (1928), Cass (1965), Koopmans (1965)
- Centralized versus decentralized economy
- How interest rates affect savings
- How does the choice between tax and debt financing affects capital accumulation

Infinite Horizon

Key assumptions

- Individuals have infinite horizon and are homogenous
- Markets are competitive
- There are CRS in production

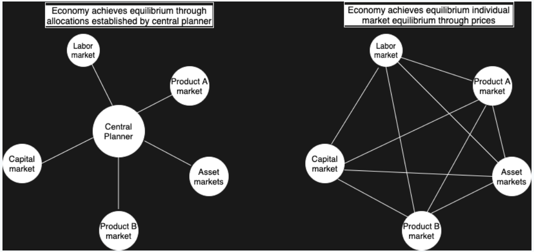
These assumptions imply that the decentralized equilibrium is the same as the one chosen by a benevolent social planner that maximizes the RA utility

Competitive Markets

Market structure

- Many small agents that are price takers
- Free entry
- True for every market (Labor, Assets, Goods...)
- Constant return to scale is a key assumption

Centralized versus Decentralized



We start with the planner and then move to decentralized economy organized with markets. In this model they coincide.

Resources Constraints

Output allocation

Output is either consumed or invested (depreciation is zero):

$$\boxed{Y(t) = F(K(t), L(t)) = C(t) + \dot{K}(t)}$$

F is a neoclassical production function, namely it respects:

- $F_K > 0$, $F_L > 0$ (positive marginal products)
- $F_{KK} < 0$, $F_{LL} < 0$ (diminishing returns)
- Exhibits CRS (constant returns to scale)

Constant Returns to Scale

Linear homogeneity

F exhibits CRS in K and L. i.e., it is *linearly homogenous* in these two variables:

$$\boxed{F(\lambda K, \lambda L) = \lambda F(K, L)}$$

Homogeneity of degree 1 allows us to use Euler's Theorem and write:

$$\boxed{F(K, L) = F_K(K, L)K + F_L(K, L)L}$$

Namely that output equals the sum of the input's marginal product times that input

Intensive Form

Per capita production function

Homogeneity of degree 1 also allows us to write output per capita $\frac{F}{L}$ as the production function in intensive form $f(k)$.

Use $\lambda = \frac{1}{L}$ and plug it into $F(\lambda K, \lambda L) = \lambda F(K, L)$, to obtain:

$$\boxed{f(k) \equiv F\left(\frac{K}{L}, 1\right)}$$

Boundary Conditions

Inada conditions

$f(.)$ satisfies the Inada conditions:

- $f(0) = 0$
- $f'(x) > 0$ and $f''(x) < 0$
- $\lim_{x \to \infty} f'(x) = 0$
- $\lim_{x \to 0} f'(x) = \infty$

Resource Constraint in Per Capita

We can write in per capita terms the resource constraint:

$$\boxed{f(k(t)) = c(t) + \dot{k}(t) + nk(t)}$$

where n is the growth rate of population.

Ramsey

Representative household

- Many identical households grow in size at rate n resulting in total population $L(t)$
- Each member of household supplies inelastically 1 unit of labor per period
- The lifetime utility of the households:

$$\boxed{U(0) = \int_0^\infty e^{-\rho t} u(c(t))dt}$$

where ρ is the discount rate of time preference.

Preferences

Utility specification

The instantaneous (Bernoulli) utility function (or felicity function) $u(c(t))$ is increasing and concave.

The agent preferences are obtained by combining this utility with the exponential discounting $e^{-\rho t}$.

Also notice that preferences are stationary and time separable, namely u does not depend on t and on past or future consumption levels.

Solution

Time consistency

In this context a solution will be a function $c(t)$ for every t.

The assumptions of time separability and stationarity will imply that the solution is time consistent.

> Time consistency implies that a solution starting at t will continue to be a solution at $t' > t$

Ramsey

Command economy setup

The central planner wants at time $t = 0$ to maximize the economy welfare. The only choice to be made is how many resources to allocate to consumption and how many resources to allocate to savings:

$$\boxed{U(0) = \int_0^{\infty} e^{-\rho t} u(c(t)) dt}$$

subject to:

- $f(k(t)) = c(t) + \dot{k}(t) + nk(t)$
- k_0 given
- $c(t), k(t) \geq 0$

Ramsey

Command economy solution

The solution is found applying the maximum principle. Let H be the present value Hamiltonian function:

$$\boxed{H(t) = u(c(t))e^{-\rho t} + \mu(t)[f(k(t)) - nk(t) - c(t)]}$$

The variable μ is a costate variable associated with the state variable k and corresponds to the value as of time zero of an additional unit of capital at time t.

Ramsey

Optimality conditions

Define $\lambda(t) \equiv \mu(t)e^{\rho t}$ and rewrite:

$$\boxed{H(t) = \left(u(c(t)) + \lambda(t)[f(k(t)) - nk(t) - c(t)]\right) e^{-\rho t}}$$

Necessary and sufficient conditions for optimality are:

- $H_c(t) = 0$
- $\dot{\mu}(t) = -H_k(t)$
- $\lim_{t\to\infty} k(t)\mu(t) = 0$

Ramsey

Necessary conditions

We have the necessary conditions:

$$\boxed{u'(c(t)) = \lambda(t)}$$

$$\boxed{\dot{\lambda}(t) = \lambda(t)[\rho + n - f'(k(t))]}$$

$$\boxed{\lim_{t\to\infty} k(t)u'(c(t))e^{-\rho t} = 0}$$

Ramsey

Euler equation

Consolidate the first two conditions:

$$\boxed{\left[\frac{c(t)u''(c(t))}{u'(c(t))}\right]\left(\frac{\dot{c}(t)}{c(t)}\right) = \rho + n - f'(k(t))}$$

where the term in brackets is the negative of the risk aversion or the negative of the inverse of the elasticity of intertemporal substitution $\sigma(c(t))$. Therefore:

$$\boxed{\frac{\dot{c}(t)}{c(t)} = \sigma(c(t))[f'(k(t)) - \rho - n]}$$

Ramsey

The transversality condition

$$\boxed{\lim_{t\to\infty} k(t)\mu(t) = 0}$$

Best understood by considering a finite horizon: think of $k(T)u'(c(T))e^{-\rho T}$.

It would not be optimal to end life with a finite amount of capital

Ramsey

Utility functions

CRRA:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \text{ for } \gamma > 0, \gamma \neq 1$$
$$= \ln(c), \text{ for } \gamma = 1$$

implies σ is constant

CARA:

$$u(c) = -\left(\frac{1}{\alpha}\right) e^{-\alpha c}, \alpha > 0$$

implies $\sigma = (\alpha c)^{-1}$

Ramsey

Steady state

Denote the steady state values with *. We have that when consumption is constant:

$$\boxed{f'(k^*) = \rho + n}$$

which is the modified Golden Rule. And when capital is constant:

$$\boxed{c^* = f(k^*) - nk^*}$$

Discuss MGR.

Ramsey

Dynamics

The two equations in the space (k, c):

$$\boxed{\dot{k}(t) = f(k(t)) - c(t) - nk(t)}$$

$$\boxed{\frac{\dot{c}(t)}{c(t)} = \sigma(c(t))[f'(k(t)) - \rho - n]}$$

The Phase Diagram on the White Board.

Heuristic Derivation of the Hamiltonian*

Finite time version

Consider the finite time version of the planner problem:

$$\boxed{\max_{k(t),c(t),k_1} U \equiv \int_0^{t_1} e^{-\rho t} u(c(t))dt}$$

subject to:

- $\dot{k}(t) = f(k(t)) - c(t) - nk(t)$
- $c(t) \in C$, $k(t) \in K$, $k(0) = k_0$ and $k(t_1) = k_1$

Heuristic Derivation of the Hamiltonian*

Optimization challenges

Notice that there is also a terminal value constraint $k(t_1) = k_1$, but k_1 is included as an additional choice variable. Thus the terminal value of the state variable k is free.

The challenge in characterizing the (optimal) solution to this problem lies in two features:

- We are choosing a function $c : [0, t_1] \to C$ rather than a vector or a finite-dimensional object
- The constraint takes the form of a differential equation rather than a set of inequalities or equalities

Heuristic Derivation of the Hamiltonian*

Variation approach

Without entering in too many details assume that an interior continuous solution $\hat{c}(t), \hat{k}(t)$ exists.

Take an arbitrary fixed continuous function $\eta(t)$ and let $\varepsilon \in \mathbb{R}$ be a real number. Then a variation of the function $\hat{c}(t)$ is defined by:

$$\boxed{c(t, \varepsilon) = \hat{c}(t) + \eta(t)\varepsilon}$$

for all $t \in [0, t_1]$ and for all $\varepsilon \in [-\varepsilon_\eta, \varepsilon_\eta]$, so that $c(t, \varepsilon)$ constitutes a feasible variation.

Heuristic Derivation of the Hamiltonian*

Lagrangian of the variation

Now we can write the Lagrangian of the variation:

$$L(\varepsilon) = \int_0^{t_1} \{e^{-\rho t} u(c(t,\varepsilon)) + \lambda(t)[f(k(t,\varepsilon)) - c(t,\varepsilon) - nk(t,\varepsilon) - \dot{k}(t,\varepsilon)]\} dt$$

Take care of $\dot{k}$. Take the last part of the integral and integrate by parts:

$$\int_0^{t_1} \lambda(t)\dot{k}(t,\varepsilon)dt = \lambda(t_1)k(t_1,\varepsilon) - \lambda(0)k_0 - \int_0^{t_1} \dot{\lambda}(t)k(t,\varepsilon)dt$$

Heuristic Derivation of the Hamiltonian*

First-order conditions

Optimality of $\hat{c}(t), \hat{k}(t)$ requires that $L'(0) = 0$ for all $\eta(t)$, which implies:

- $e^{-\rho t} u_c(\hat{c}(t)) = \lambda(t)$ for all t
- $\dot{\lambda}(t) = \lambda(t)[n - f_k(\hat{k}(t))]$ for all t
- $\lambda(t_1) = 0$