Macroeconomic Theory

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The DSGE Project

Building macroeconomic models from microeconomic foundations

Foundation
Ramsey Model

Intertemporal Choice

Extensions
Imperfect Competition

Nominal Rigidities

Result
New Keynesian

Fluctuations Model

Unlike Ramsey and OLG models, we study the **linear approximation** around steady state — simple output requires many analytical steps.

The Classical Model

Household problem with labor choice

We simplify by abstracting from capital (no investment) but add labor choice:

Household maximizes:

$$\boxed{\max E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t)}$$

Subject to budget constraint:

$$\boxed{P_t C_t + Q_t A_t \leq A_{t-1} + W_t N_t + D_t}$$

The Classical Model

Discrete time formulation

- Timing convention: could write $P_t C_t + Q_t A_{t+1} \leq A_t + W_t N_t + D_t$
- Key insight: understand what is chosen at time t
- Expectation operator E_0 represents mathematical expectation
- Asset pricing: $Q_t = \frac{1}{1+i_t}$ (nominal price of future goods)

Discrete time facilitates stochastic analysis and empirical mapping.

Continuous Time Classical Model

Optimization problem

Household maximizes:

$$\boxed{\max E_0 \int_0^\infty e^{-\rho t} U\left(C_t, N_t\right) dt}$$

Budget constraint:

$$\boxed{\dot{A}(t) = i(t)A(t) + W(t)N(t) + D(t) - P(t)C(t)}$$

Using utility function: $U\left(C_t, N_t\right) = \frac{C(t)^{\mathbf{1}-\sigma}}{1-\sigma} - \frac{N(t)^{\mathbf{1}+\phi}}{1+\phi}$

Continuous Time Classical Model

First-order conditions

The Hamiltonian yields the following optimality conditions:

Labor supply:

$$\boxed{\frac{W(t)}{P(t)} = C(t)^{\sigma} N(t)^{\phi}}$$

Euler equation:

$$\boxed{\frac{\dot{C}(t)}{C(t)} = \frac{1}{\sigma}\left(i(t) - \frac{\dot{P}(t)}{P(t)} - \rho\right)}$$

Continuous Time Classical Model

Hamiltonian and optimality conditions

The non-stochastic Hamiltonian:

$$H = e^{-\rho t}\left[\frac{C(t)^{1-\sigma}}{1-\sigma} - \frac{N(t)^{1+\phi}}{1+\phi} + \lambda(t)[i(t)A(t) + W(t)N(t) + D(t) - P(t)C(t)]\right]$$

Implies first-order conditions:

- $H_C(t) = 0$ and $H_N(t) = 0$
- $\dot{\lambda}(t) - \rho\lambda(t) = -H_A(t)$
- $\lim_{t\to\infty} A(t)\lambda(t)e^{-\rho t} = 0$

Technology and Firms

Production and profit maximization

Production technology:

$$\boxed{Y(t) = Z(t)\, N(t)^{1-\alpha}}$$

where Z_t is total factor productivity (TFP).

Firm profit maximization:

$$\boxed{\max \;\; P(t)\, Y(t) - W(t)\, N(t)}$$

Labor demand (FOC):

Continuous Time Equilibrium

Market clearing conditions

Define inflation $\Pi(t) \equiv \frac{\dot{P}(t)}{P(t)}$ and real variables $w(t) = \frac{W(t)}{P(t)}$, $r(t) = i(t) - \Pi(t)$.

Labor market equilibrium:

$$\boxed{(1-\alpha) Z(t) N(t)^{-\alpha} = C(t)^{\sigma} N(t)^{\phi}}$$

Goods market equilibrium:

$$\boxed{Y(t) = Z(t) N(t)^{1-\alpha} = C(t)}$$

Continuous Time Equilibrium

Closed-form solution

Combining equilibrium conditions:

$$\boxed{N(t)^{\sigma(1-\alpha)+\alpha+\phi} = (1-\alpha)\, Z(t)^{1-\sigma}}$$

Therefore:

$$\boxed{N(t) = (1-\alpha)^{\psi}\, Z(t)^{(1-\sigma)\psi}}$$

where $\psi = \frac{1}{\sigma(1-\alpha)+\alpha+\phi}$.

Continuous Time Equilibrium

Real interest rate determination

Since goods market clears every period:

$$\boxed{C\left(t\right) = Y\left(t\right) = Z\left(t\right)N\left(t\right)^{1-\alpha}}$$

The real interest rate is determined by:

$$\boxed{r\left(t\right) = \rho + \sigma\frac{\dot{C}(t)}{C(t)}}$$

Classical dichotomy: Real variables independent of nominal variables (i and P). Infinite combinations of i and Π consistent with required real rate.

Adding Stochasticity

From perfect foresight to rational expectations

- With Z_t as parameter, model always in steady state with $r = \rho$
- Continuous time inconvenient for stochastic Z processes
- Shift to discrete time for business cycle analysis
- Discrete time facilitates mapping to macro data (sampled at discrete intervals)

We model $z_t \equiv \log Z_t$ as AR(1) process and replace perfect foresight with rational expectations.

Discrete Time Classical Model

Stochastic TFP process

TFP follows AR(1):

$$\boxed{z_t = \rho_z z_{t-1} + \varepsilon_t^z}$$

where $z_t \equiv \log Z_t$.

- Agents shift from perfect foresight to rational expectations
- Probabilistic knowledge replaces deterministic knowledge
- Expectation operator E_t conditional on time t information

Discrete Time Classical Model

Lagrangian formulation

The household's Lagrangian with expectation operator E_t:

$$\boxed{L_t = E_0 \sum_{t=0}^{\infty} \left[\beta^t U\left(C_t, N_t\right) - \lambda_t \left(P_t C_t + Q_t A_t - A_{t-1} - W_t N_t - D_t\right)\right]}$$

Take first-order conditions with respect to A_t, N_t, and C_t.

Discrete Time First-Order Conditions

Household optimality

Using $U(C_t, N_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}$:

Labor supply:

$$\boxed{\frac{W_t}{P_t} = C_t^{\sigma} N_t^{\phi}}$$

Euler equation:

$$\boxed{1 = \beta E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} (1 + i_t) \frac{P_t}{P_{t+1}} \right]}$$

Log-Linear Approximation

Business cycle dynamics

Focus on dynamics around steady state using log-linear approximations.

Starting with Euler equation:

$$Q_t = \beta E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right]$$

Define $x_t = \ln X_t$, $\pi_t \equiv p_t - p_{t-1}$, $i_t \equiv -\log Q_t$, $\rho \equiv -\log \beta$.

In steady state: $i = \rho + \pi$ and $r = i - \pi = \rho$.

Log-Linear First-Order Conditions

Household behavior in deviations

Taking first-order Taylor expansion and using $\Delta c_{t+1} = c_{t+1} - c_t$:

Log-linear Euler equation:

$$\boxed{c_t = E_t[c_{t+1}] - \frac{1}{\sigma}(i_t - E_t[\pi_{t+1}] - \rho)}$$

Log-linear labor supply:

$$\boxed{w_t - p_t = \sigma c_t + \varphi n_t}$$

Technology and Firms

Log-linear production and labor demand

Production technology:

$$\boxed{Y_t = Z_t N_t^{1-\alpha}}$$

TFP process:

$$\boxed{z_t = \rho_z z_{t-1} + \varepsilon_t^z}$$

Log-linear labor demand:

$$w_t - p_t = z_t - \alpha n_t + \log(1-\alpha)$$

Log-Linear Equilibrium

Market clearing conditions

Goods market clearing:

$$\boxed{y_t = c_t}$$

Labor market clearing:

$$\boxed{\sigma c_t + \varphi n_t = z_t - \alpha n_t + \log(1-\alpha)}$$

Asset market clearing:

$$\boxed{A_t = 0}$$

Log-Linear Equilibrium Solution

Closed-form relationships

The system yields closed-form solutions:

Labor:

$$\boxed{n_t = \psi_{nz} z_t + \psi_n}$$

Output and consumption:

$$\boxed{y_t = c_t = \psi_{yz} z_t + \psi_y}$$

And for prices:

Classical Dichotomy

- ✓ Allocations **independent** of monetary fluctuations
- ✓ Monetary policy **neutral** for real variables
- ✓ Interest rate rules affect **only nominal** variables
- ✓ Foundation for **New Keynesian** extensions