Macroeconomic Theory

Francesco Franco

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The DSGE Project

Building macroeconomic models from microeconomic foundations

This course presents the Dynamic Stochastic General Equilibrium approach:

Foundation	Extensions	Dynamics
Intertemporal Model	Imperfect Competition	New Keynesian
Ramsey Core	Nominal Rigidities	Fluctuations Model

Simplified Ramsey model without capital but with labor choice. We study linear approximations around steady state rather than non-linear dynamics.

The Classical Model

Household optimization without capital

Household Problem:

$$\boxed{\max E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t)}$$

subject to budget constraint:

$$\boxed{P_t C_t + Q_t A_t \leq A_{t-1} + W_t N_t + D_t}$$

▷ Q_t is the price of asset A_t (delivers one unit next period)

The Classical Model

Discrete vs continuous time formulation

Key differences:

- Expectation operator E_0 for uncertainty (discrete time advantage)
- Think of $Q_t = \frac{1}{1+i_t}$ as nominal price of future consumption
- Timing: what is chosen at time t matters more than notation

Discrete time is more convenient for stochastic models and empirical mapping to macro data sampled at discrete intervals.

Continuous Time Version

Classical model in continuous time

Objective:

$$\max E_0 \int_0^{\infty} e^{-\rho t} U\left(C_t, N_t\right) dt$$

Budget constraint:

$$\dot{A}(t) = i(t)A(t) + W(t)N(t) + D(t) - P(t)C(t)$$

Use utility: $U\left(C_t, N_t\right) = \frac{C(t)^{1-\sigma}}{1-\sigma} - \frac{N(t)^{1+\phi}}{1+\phi}$

Continuous Time Version

Hamiltonian and first order conditions

Hamiltonian:

$$\boxed{H = e^{-\rho t}\left[\frac{C(t)^{1-\sigma}}{1-\sigma} - \frac{N(t)^{1+\phi}}{1+\phi} + \lambda(t)[i(t)A(t) + W(t)N(t) + D(t) - P(t)C(t)]\right]}$$

First order conditions:

- $H_C(t) = 0$
- $H_N(t) = 0$
- $\dot{\lambda}(t) - \rho\lambda(t) = -H_A(t)$
- $\lim_{t\to\infty} A(t)\lambda(t)e^{-\rho t} = 0$

Continuous Time Version

Simplified first order conditions

Using the functional form, the FOCs become:

Labor supply:

$$\boxed{\frac{W(t)}{P(t)} = C(t)^{\sigma} N(t)^{\phi}}$$

Euler equation:

$$\boxed{\frac{\dot{C}(t)}{C(t)} = \frac{1}{\sigma}\left(i(t) - \frac{\dot{P}(t)}{P(t)} - \rho\right)}$$

Firms and Technology

Production and profit maximization

Technology:

$$\boxed{Y\left(t\right)=Z\left(t\right)N\left(t\right)^{1-\alpha}}$$

where Z_t is total factor productivity.

Profit maximization:

$$\boxed{\max P\left(t\right)Y\left(t\right)-W\left(t\right)N\left(t\right)}$$

Equilibrium in Continuous Time

Market clearing conditions

Define: $\mathit{\Pi}(t) \equiv \frac{\dot{P}(t)}{P(t)}$, $w(t) = \frac{W(t)}{P(t)}$, $r(t) = i(t) - \mathit{\Pi}(t)$

Labor market clearing:

$$\boxed{(1-\alpha) Z(t) N(t)^{-\alpha} = C(t)^{\sigma} N(t)^{\phi}}$$

Goods market clearing:

$$\boxed{Y(t) = Z(t) N(t)^{1-\alpha} = C(t)}$$

Equilibrium Solution

Labor and output determination

Combining market clearing conditions:

$$\boxed{N(t)^{\sigma(1-\alpha)+\alpha+\phi} = (1-\alpha)\, Z(t)^{1-\sigma}}$$

Therefore:

$$\boxed{N(t) = (1-\alpha)^{\psi}\, Z(t)^{(1-\sigma)\psi}}$$

where $\psi = \frac{1}{\sigma(1-\alpha)+\alpha+\phi}$.

Equilibrium Solution

Real interest rate determination

Since $C(t) = Y(t) = Z(t)\,N(t)^{1-\alpha}$ in every period:

Real interest rate:

$$\boxed{r(t) = \rho + \sigma \frac{\dot{C}(t)}{C(t)}}$$

Classical dichotomy:

- Real allocations independent of nominal variables
- Infinite combinations of i and Π consistent with required r
- Monetary neutrality holds

Adding Stochasticity

Moving to discrete time with shocks

TFP process:

$$\boxed{z_t = \rho_z z_{t-1} + \varepsilon_t^z}$$

where $z_t \equiv \log Z_t$ follows an AR(1).

Rational expectations:

- Perfect foresight $\rightarrow$ probabilistic knowledge
- Expectation operator E_t conditional on time t information
- Discrete time facilitates stochastic analysis

Discrete Time Model

Lagrangian formulation

Lagrangian:

$$\boxed{L_t = E_0 \sum_{t=0}^{\infty} \left[\beta^t U\left(C_t, N_t\right) - \lambda_t \left(P_t C_t + Q_t A_t - A_{t-1} - W_t N_t - D_t\right)\right]}$$

Take first order conditions with respect to A_t, N_t and C_t.

Using $U(C_t, N_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}$

Discrete Time Model

First order conditions

Labor supply:

$$\boxed{\frac{W_t}{P_t} = C_t^{\sigma} N_t^{\phi}}$$

Euler equation:

$$\boxed{Q_t = \beta E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right]}$$

Using $Q_t = \frac{1}{1+i_t}$:

Log-Linear Approximation

Focus on dynamics around steady state

Starting from the Euler equation:

$$\boxed{Q_t = \beta E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right]}$$

Define $x_t = \ln X_t$ and use:

- $\pi_t \equiv p_t - p_{t-1}$ (inflation)
- $i_t \equiv -\log Q_t$ (nominal interest rate)
- $\rho \equiv -\log \beta$ (discount rate)

Steady state: $i = \rho + \pi$ and $r = i - \pi = \rho$

Log-Linear First Order Conditions

Household behavior

After log-linearization and Taylor expansion:

Euler equation (IS curve):

$$\boxed{c_t = E_t[c_{t+1}] - \frac{1}{\sigma}(i_t - E_t[\pi_{t+1}] - \rho)}$$

Labor supply:

$$\boxed{w_t - p_t = \sigma c_t + \varphi n_t}$$

Firms in Log-Linear Form

Technology and labor demand

Production function:

$$\boxed{Y_t = Z_t N_t^{1-\alpha}}$$

TFP process:

$$\boxed{z_t = \rho_z z_{t-1} + \varepsilon_t^z}$$

Labor demand (log-linear):

Market Clearing Conditions

Equilibrium in log-linear form

Goods market: $y_t = c_t$

Labor market:

$$\boxed{\sigma c_t + \varphi n_t = z_t - \alpha n_t + \log(1-\alpha)}$$

Asset market: $A_t = 0$

Interest rate:

$$\boxed{i_t - E_t\{\pi_{t+1}\} = \rho + \sigma E_t\{\Delta c_{t+1}\}}$$

Equilibrium Solution

Closed-form solutions

The system can be solved directly:

Labor:

$$\boxed{n_t = \psi_{nz} z_t + \psi_n}$$

Output and consumption:

$$\boxed{y_t = c_t = \psi_{yz} z_t + \psi_y}$$

Real interest rate:

Classical Model: Key Insights

- ✓ Allocations depend **only on productivity** — not monetary policy
- ✓ **Classical dichotomy** holds: real and nominal sides decoupled
- ✓ Monetary policy is **neutral** — affects only nominal variables
- ✓ Foundation for **New Keynesian** extensions with rigidities
- ✓ Log-linearization enables **analytical solutions** for dynamics