Chapter 7
Applications of Hypergraph Theory: A Brief Overview

LIKE in most fruitful mathematical theories, the theory of hypergraphs has many applications. Hypergraphs model many practical problems in many different sciences. it makes very little time (20 years) that the theory of hypergraphs is used to model situations in the applied sciences. We find this theory in psychology, genetics, ... but also in various human activities. Hypergraphs have shown their power as a tool to understand problems in a wide variety of scientific field.
Moreover it well known now that hypergraph theory is a very useful tool to resolve optimization problems such as scheduling problems, location problems and so on. This chapter shows some possible uses of hypergraphs in Applied Sciences.

7.1 Hypergraph Theory and System Modeling for Engineering

Modeling is a particularly important aspect in apprehending the continuous or discrete physical systems. The mathematical foundations of the modeling come from:

- Algebraic theory
- The concepts of duality
- Complex and real analysis
- And many others

Since combinatorics is the common denominator of these mathematical areas, combinatorial paradigms are suited to express the mathematical properties of physical objects. Thus, it is natural to develop the hypergraph theory as a modeling concept. In this section, we are going to briefly present some applications of hypergraphs in science and engineering. It turns out that hypergraph theory can be used in many areas of sciences. We does not claim to be exhaustive. We limit ourselves to present some aspects of the application of hypergraphs in order to prove the relevance of this theory in science and engineering.

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7.1.1 Chemical Hypergraph Theory

The graph theory is very useful in chemistry. The representation of molecular structures by graphs is widely used in computational chemistry. But the main drawback of the graph theory is the lack of convenient tools to represent organometallic compounds, benzenoid systems and so on.

A hypergraph $\mathcal{H} = (V, E)$ is a *molecular hypergraph* if it represents molecular structure, where $x \in V$ corresponds to an individual atom, hyperedges with degrees greater than 2 correspond to polycentric bonds and hyperedges with $deg(x) = 2$ correspond to simple covalent bonds.

Hypergraphs appear to be more convenient to describe some chemical structures. Hence the concept of molecular hypergraph may be seen as a generalization of the concept of molecular graph. More informations can be found in [KS01] . Hypergraphs have also shown their interest in biology [KHT09].

7.1.2 Hypergraph Theory for Telecomunmications

A hypergraph theory can be used to model cellular mobile communication systems. A cellular system is a set of cells where two cells can use the same channel if the distance between them is at least some predefined value D. This situation can be represented by a graph where:

(a) Each vertex represents a cell.
(b) An edge exists between two vertices if and only if the distance between the corresponding cells is less than the distance called *reuse distance* and denoted by D.

A *forbidden set* is a group of cells all of which cannot use a channel simultaneously.

A *minimal forbidden set* is a forbidden set which is minimal with respect to this property, i.e. no proper subset of a minimal forbidden set is forbidden.

From these definitions it is possible to derive a better modelization using hypergraphs. We proceed in the following way:

(a) Each vertex represents a cell.
(b) A hyperedge is minimal forbidden set.

7.1.3 Hypergraph Theory and Parallel Data Structures

Hypergraphs provide an effective mean of modeling parallel data structures. A shared memory multiprocessor system consists of a number of processors and memory modules. We define a template as a set of data elements that need to be processed in parallel. Hence the data elements from a template should be stored in different memory modules. So we define a hypergraph in the following way:

(a) A data is represented by a vertex.
(b) The hyperedges are the templates.

From this model and by using the properties of hypergraphs one can resolve various problems such as the conflict-free access to data in parallel memory systems. Some informations can be found in [HK00] .

7.1.4 Hypergraphs and Constraint Satisfaction Problems

A *constraint satisfaction problem*, $\mathcal{P}$ is defined as a tuple:

$$\mathcal{P} = (V,\ D,\ R_1(S_1),\ \ldots,\ R_k(S_k))$$

where:

- V is a finite set of variables.
- D is a finite set of values which is called the *domain* of $\mathcal{P}$.
- Each $R_i(S_i)$ is a constraint.
  - S_i is an ordered list of n_i variables, called the constraint scope.
  - R_i is a relation over D of arity n_i, called the *constraint relation*.

To a constraint satisfaction problem one can associate a hypergraph in the following way:

(a) The vertices of the hypergraph are the variables of the problem.
(b) There is a hyperedge containing the vertices $v_1,\ v_2,\ \ldots, v_t$ when there is some constraint $R_i(S_i)$ with scope $S_i = \{v_1,\ v_2,\ \ldots v_t\}$.

7.1.5 Hypergraphs and Database Schemes

Hypergraphs have been introduced in database theory in order to model relational database schemes [Fag83]. The classes of acyclic hypergraphs defined in Sect. 4.5.0.3 play an important part in the modeling of relational database schemes.
A database can be viewed in the following way:

- A set of attributes.
- A set of relations between these attributes.

Hence we have the following hypergraph:

(a) The set of vertices is the set of attributes.

(b) The set of hyperedges is the set of relations between these attributes.

We also find the theory of hypergraphs in data mining [HBC07] .

7.1.6 Hypergraphs and Image Processing

A *digital image* on a grid (4-connected grid, 8-connected grid, ...) is a two-dimensional discrete function that has been digitized both in spatial coordinates and in feature value. We may represent a digital image by an application

$$I : X \subseteq Z^m \to \mathcal{C} \subseteq Z^n,$$

with $n \geq 1$, $m = 2$ and we have a 2-dimensional image or $m = 3$ and we have a 3-dimensional image and where $\mathcal{C}$ is the set of the *feature intensity levels* and X represent a set of points called the *image points*. The couple $(x, I(x))$ is called a *pixel*.

Let d be a distance on $\mathcal{C}$, for given β there exists a neighborhood relation on an image I defined by:

$$\forall x \in X,\ \ \Gamma_{\alpha,\beta}(x) = \left\{x' \in X, x' \neq x \mid d(\mathcal{I}(x), \mathcal{I}(x')) < \alpha \text{ and } d'(x, x') \leq \beta\right\}$$

where d' is the distance on the grid and α is attribute on the image. The neighborhood of x on the grid is denoted by $\Gamma_\beta(x)$. So each image can be associated to a hypergraph:

$$H_{\alpha,\beta} = \left(X, (\{x\} \cup \Gamma_{\alpha,\beta}(x))_{x\in X}\right).$$

The attribute α can be computed in an adaptive way depending on local properties of the image.

- If α is constant, the hypergraph is called the *Image Neighborhood Hypergraph* (INH).
- If α is not constant, for instance α may be estimated by the standard deviation of the intensity levels of the pixels of $\{x\} \cup \Gamma_\beta(x)$, the hypergraph is called the *Image Adaptative Neighborhood Hypergraph* (IANH).

From this hypergraph we may developp some applications:

- we can do image segmentation,
- we use also Image (Adaptative) Neighborhood Hypergraph for the edge detection
- and thanks to our model we developed a noise cancellation algorithm.

Some others applications such as data compression can be also developed from our hypergraph model.

More informations can be found in [BG05, DBRL12] .

Algorithm 10: Image Adaptive Neighborhood Hypergraph.

Data: Image I, and neighborhood order β.
Result: hypergraph $H_{\alpha,\beta}$
begin
  $X := \emptyset$;
  foreach *For each pixel* $(x, I(x))$ *of* I **do**
    $\alpha =$ the standard deviation of the intensity levels of the pixels in $\{x\} \cup \Gamma_\beta(x)$;
    $\Gamma_{\alpha,\beta}(x) = \emptyset$;
    foreach y **of** $\Gamma_\beta(x)$, **do**
      if $d(I(x), I(y)) \leq \alpha$ **then**
        $\Gamma_{\alpha,\beta}(x) = \Gamma_{\alpha,\beta}(x) \cup \{y\}$;
      end
    end
    $X = X \cup \{x\}$; $E_{\alpha,\beta}(x) = \{\Gamma_{\alpha,\beta}(x) \cup \{x\}\}$;
  end
  $H_{\alpha,\beta} = (X, (E_{\alpha,\beta}(x))_{x \in X})$;
end

7.1.7 Other Applications

Hypergraph theory can lead to numerous other applications [HK00, HOS12, Rob39, STV04, Smo07, Zyk74]. Indeed we can find hypergraph models in machine learning, data mining, and so on [BP09, STV04, Sla78, Smo07, Rob39].
The properties of hypergraphs are equally important, for example hypergraph transversal computation has a large number of applications in many areas of computer science, such as distribued systems, databases, artificial intelligence, and so on. Hypergraph partitioning is also a very interesting property [BP09, HK00]. The *partitioning of a hypergraph* can be defined as follows:

(a) The set of vertices is partioned into k disjoint subsets $V_1, V_2, \ldots, V_k$.
(b) The partial subhypergraphs (or the set of hyperedges) generated by $V_1, V_2, \ldots, V_k$ verify the properties $P_1, P_2, \ldots, P_k$.

This property yields interesting results in many areas such as VLSI design, data mining, and so on.
Directed hypergraphs can be very useful in many areas of sciences. Indeed directed hypergraphs are used as models in:

- Formal languages.
- Relational data bases.
- Scheduling.

and many other applications. Numerous computational studies using hypergraphs have shown the importance of this field in many areas of science [Gol11, BP09, Bre04, HOS12, Hua08], and other fruitful applications should be developed in the future.

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