Predicting Knot Invariants using Machine Learning

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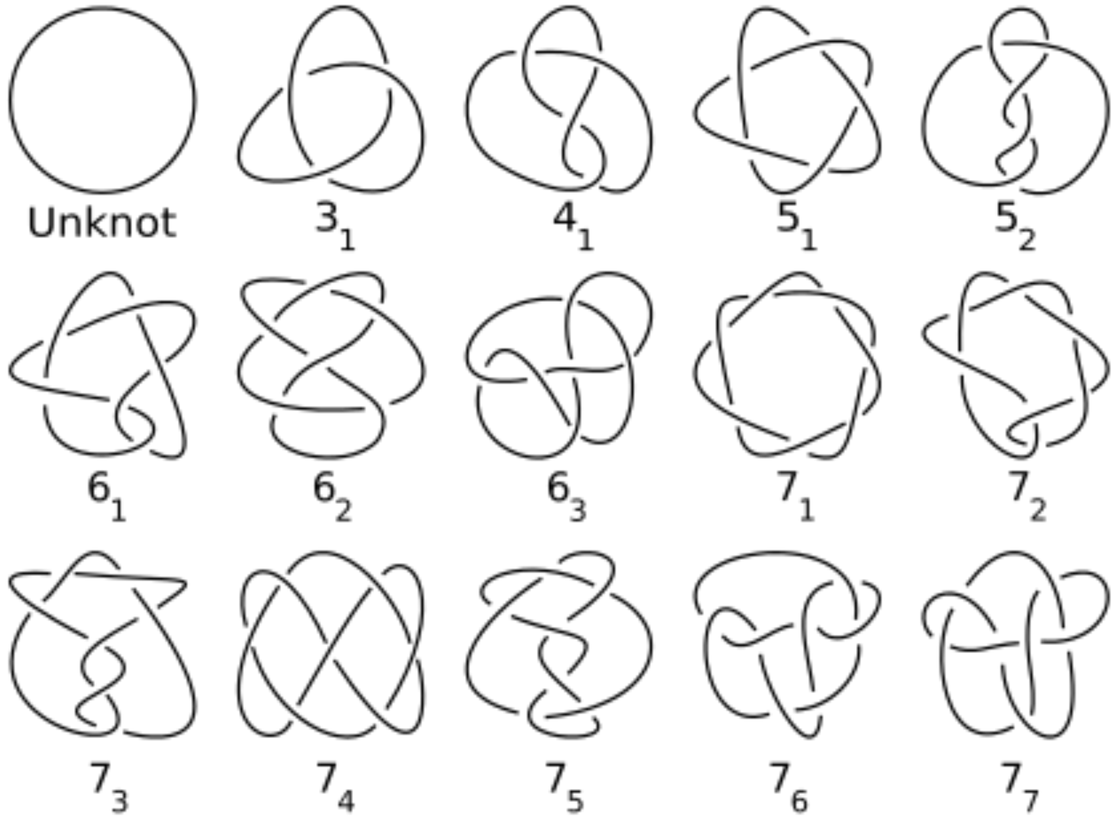
Knot Theory

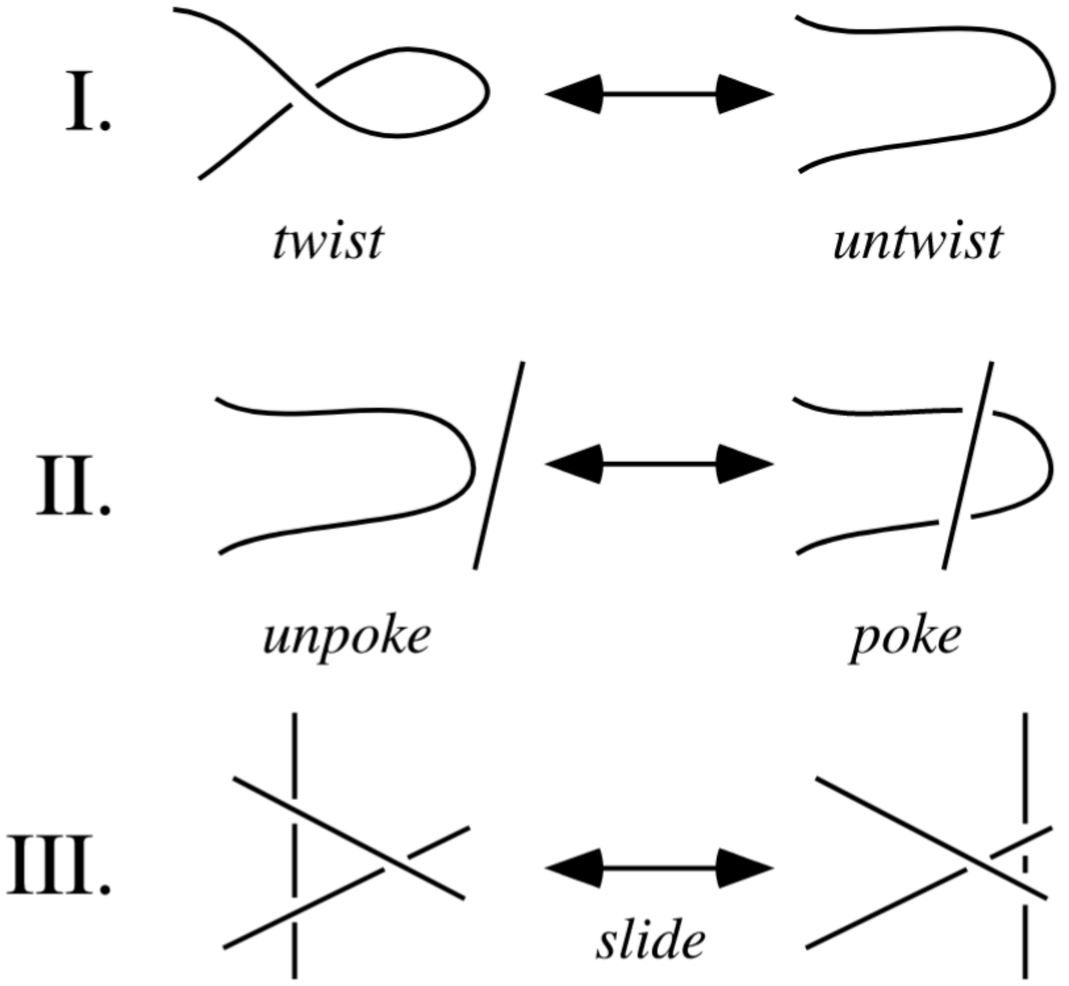
Mathematical Knots

- Smoothly embedded circle in $\mathbb{R}^3$
- No self-intersections
- Can smoothly map space between equivalent knots
- Equivalence is difficult to determine



Redemeister Moves

- Moves that preserve equivalence
- Can describe all knot deformations
- Can be done computationally



Knot Invariants

- Properties that don’t change under Redemeister moves
- Not always unique
- Jones Polynomial
  - Recursive combinatorial definition
- Hyperbolic Volume
  - Knot complement with hyperbolic structure

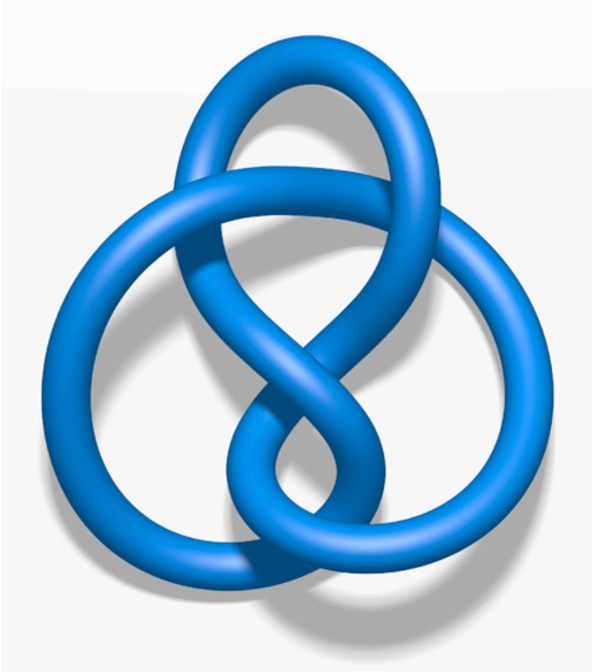
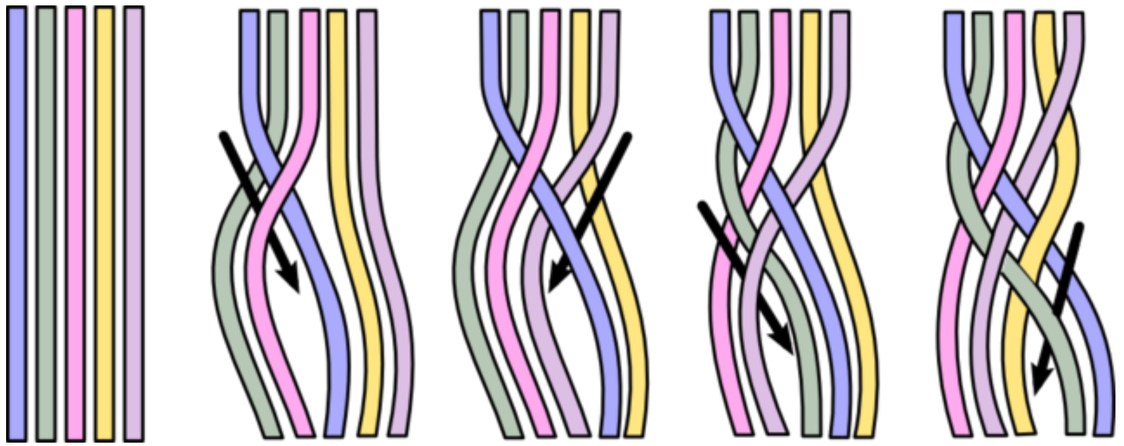
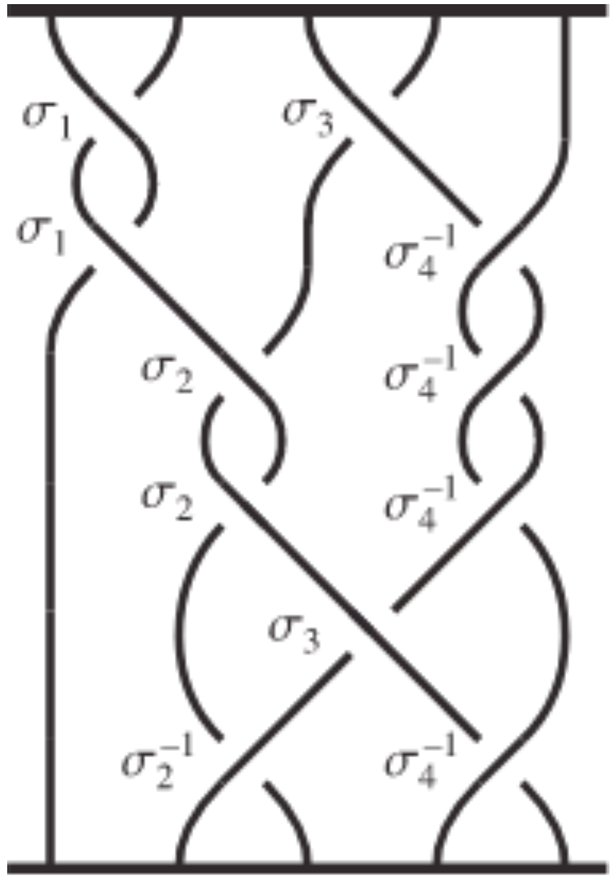


Figure-Eight Knot
Hyperbolic Volume: 2.0298832
Jones Polynomial:
$V(t) = t^2 - t + 1 - t^{-1} + t^{-2}$

Braid Words

- Can represent every knot
- List of crossings between strands
- Connect ends of strands to form knot



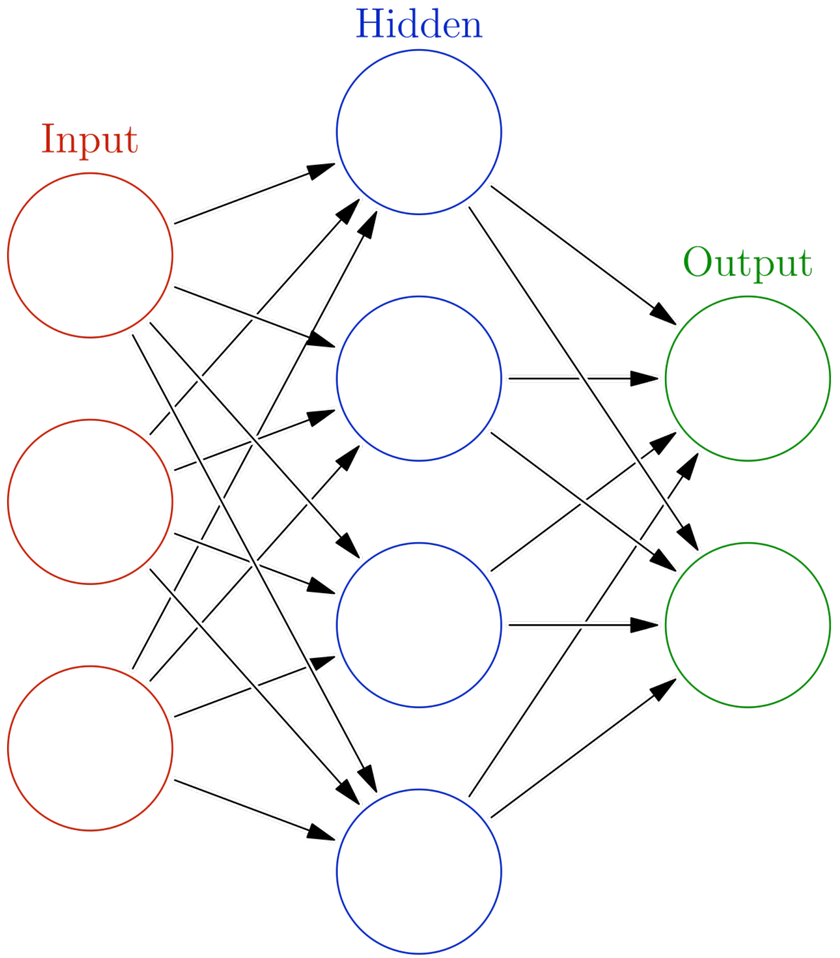


$$\sigma_1 \sigma_3 \sigma_1 \sigma_4^{-1} \sigma_2 \sigma_4^{-1} \sigma_2 \sigma_4^{-1} \sigma_3 \sigma_2^{-1} \sigma_4^{-1}$$

Machine Learning

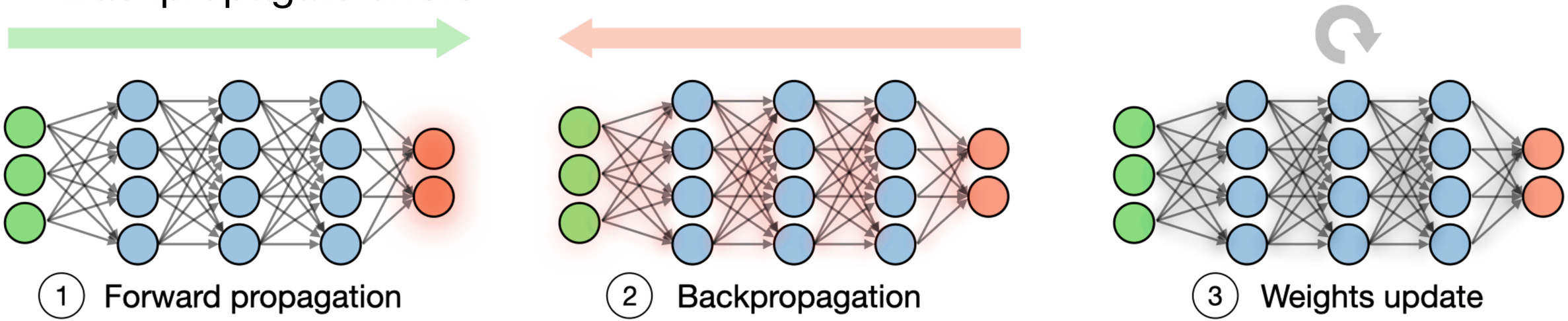
Neural Networks

- General function approximation
- Layers of neurons connected based on weights
- Weights updated based on network performance



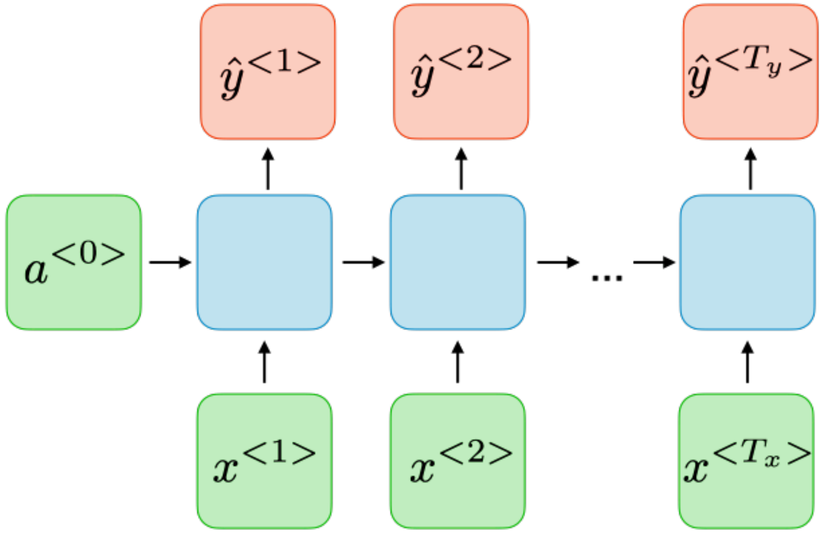
Training a Neural Network

- Basic form: supervised learning
- Start with labelled dataset
- Apply network to make prediction
- Backpropagate errors

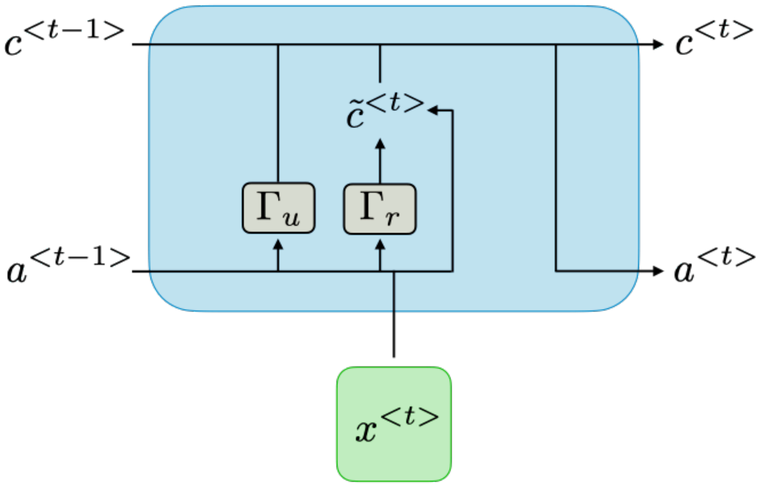


Recurrent Networks

- Used for sequences of data
- Activations depend on previous steps
- Simple recurrent net: activation from current and previous time step
- Can handle long-term dependencies
  - Gated Recurrent Unit (GRU)
  - Long Short-Term Memory (LSTM)



"Unrolled" recurrent network



Gated Recurrent Unit

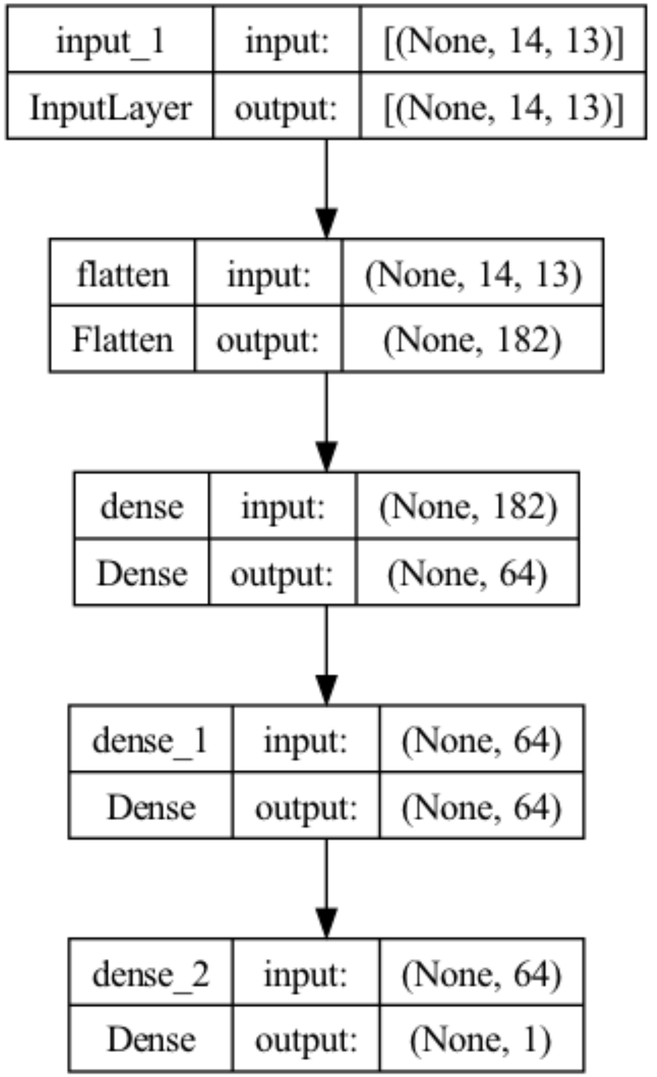
Applications to Knot Theory

- *A Neural Network Approach to Predicting and Computing Knot Invariants*
  - Predict knot quasipositivity from braid words
  - Also used other classification data
- *Deep Learning the Hyperbolic Volume of a Knot*
  - Predicted hyperbolic volume from Jones polynomial
  - Based on volume conjecture

Experiments

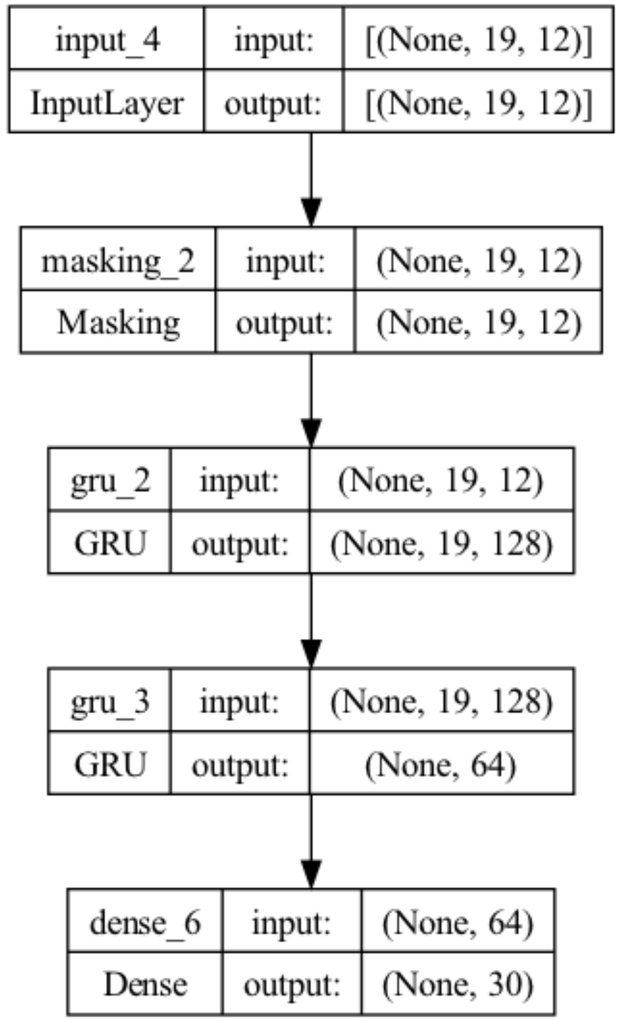
Predicting Knot Quasipositivity

- Recreated results from Knot Invariants paper
- Started with equivalent knots encoded as braid words
- Used a simple neural network with two linear layers
- Predicted knot quasipositivity (binary classification)
- 90% accuracy with just braid word
- 96% accuracy including braid positivity property

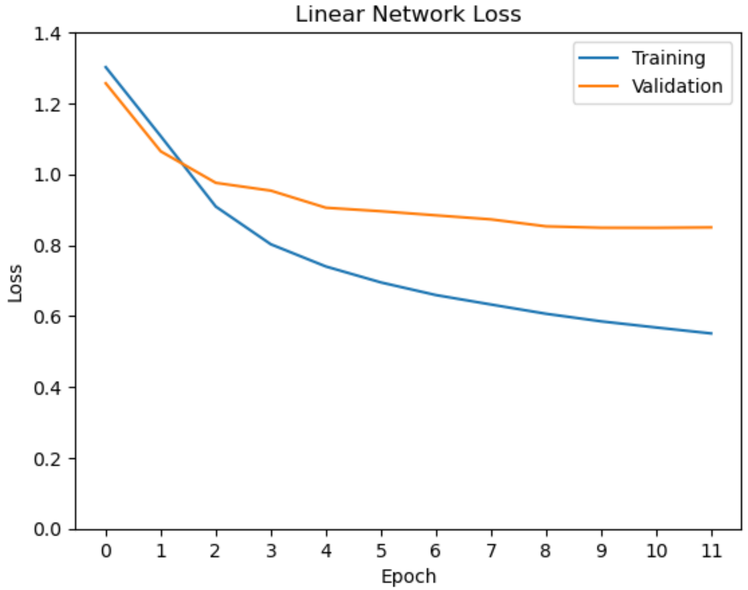


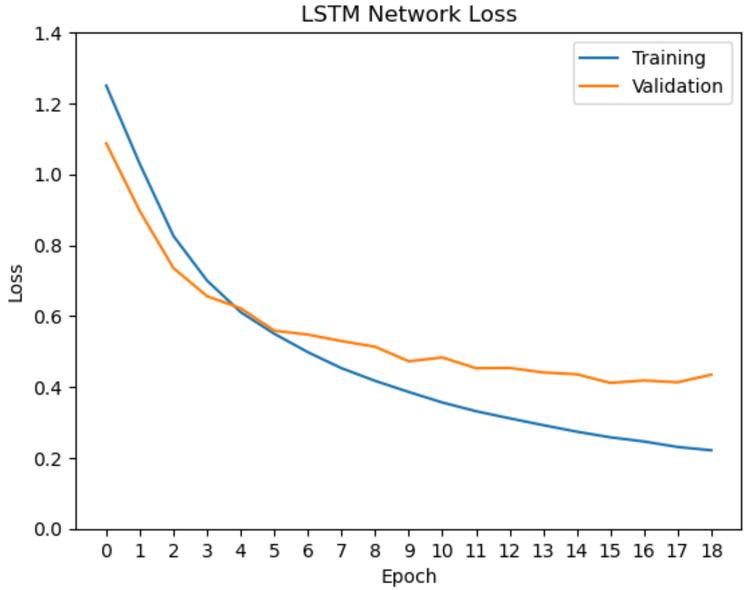
Predicting the Jones Polynomial

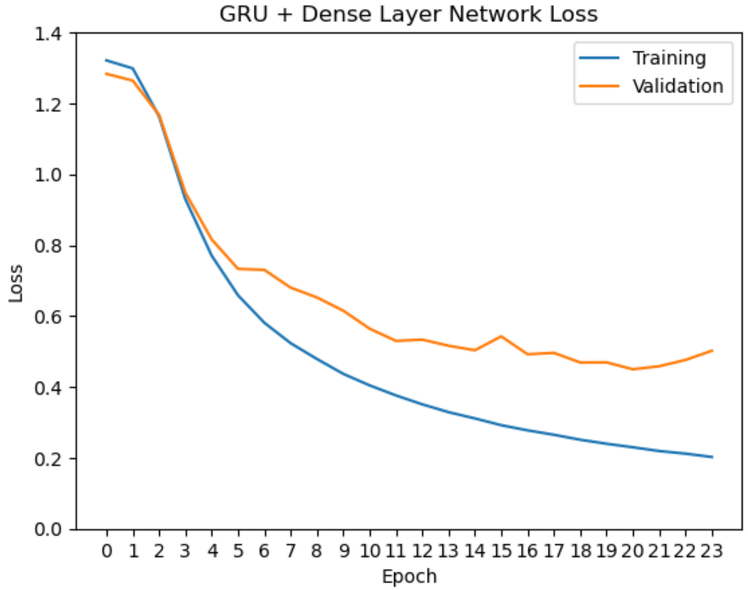
- Started from encoded braid words
- Predicted list of coefficients for polynomial
- Tested different architectures
  - Linear
  - Recurrent
  - LSTM
  - GRU and variants

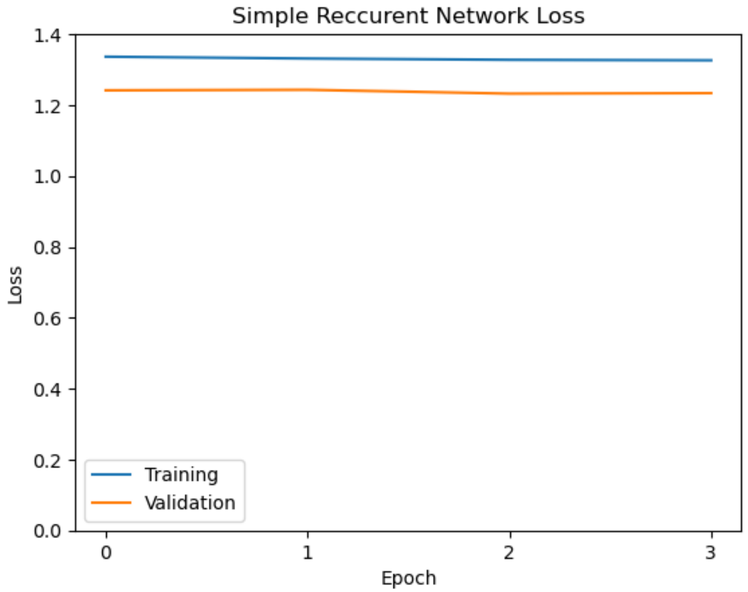


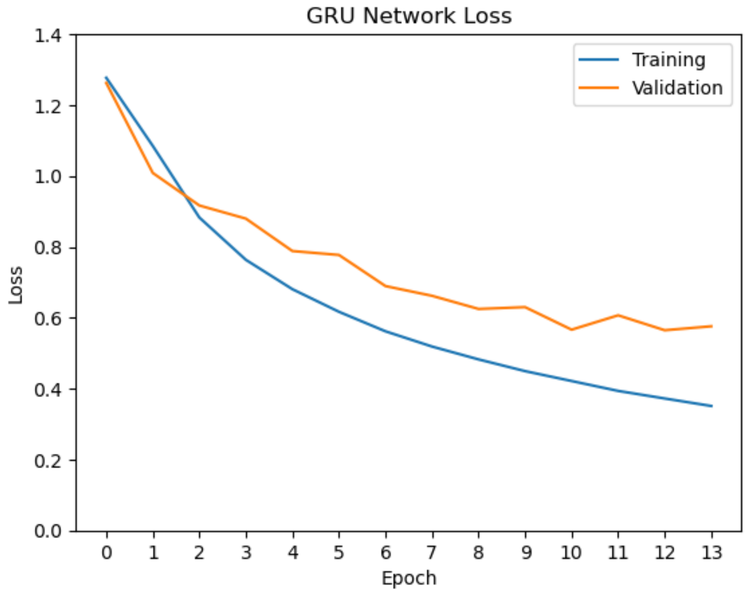
Experimental Results

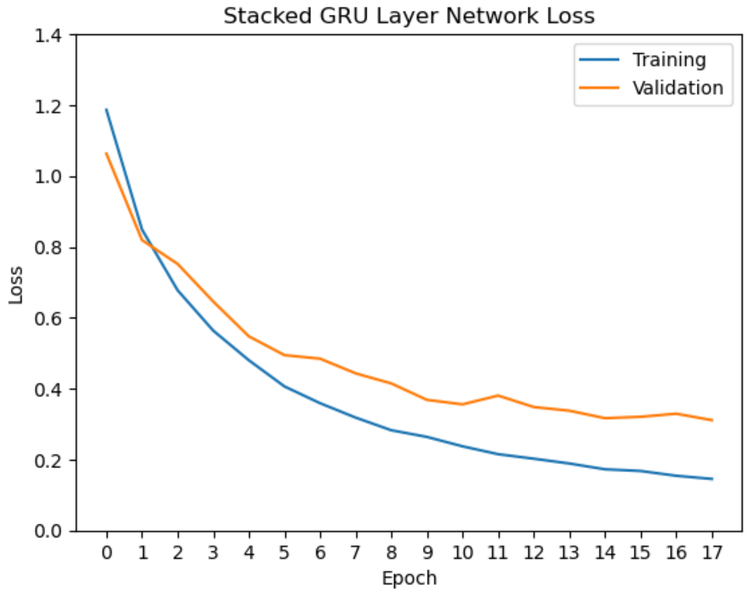












Future Research

- Test other network architectures for Jones polynomial
  - Custom recurrent layer
  - Attention / transformer network
- Predict hyperbolic volume
- Predict knot equivalence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