Why a Tilted Sensor Cannot Measure Direct Gravitational Pull

The Claim

> A sensor tilted relative to the local vertical can detect the direct gravitational pull of the Sun ($\sim 6 \times 10^{-3}$ m/s^2) or the Moon ($\sim 3.3 \times 10^{-5}$ m/s^2) as a time-varying signal, because the projection of the gravitational acceleration onto the sensor axis changes as these bodies move across the sky.

This is incorrect. The derivation below shows, in a point-mass model with no approximations beyond Newtonian mechanics, that the direct pull of *every* external body cancels exactly against the corresponding acceleration of the laboratory, leaving only the tidal residual — regardless of sensor orientation.

1 Setup

Four point masses in an inertial frame:

Body	Mass	Position
Sun	M_S	$\mathbf{R}_S(t)$
Moon	M_M	$\mathbf{R}_M(t)$
Earth	M_E	$\mathbf{R}_E(t)$
Test mass	m	$\mathbf{x}(t)$

The test mass sits on Earth's surface and is connected to the sensor by a spring (or equivalent restoring mechanism). The sensor frame is rigidly attached to the Earth.

The sensor reads the **spring force $\mathbf{F}_{\text{spring}}$** — the non-gravitational force required to keep the test mass co-moving with the lab. This is the only quantity accessible to experiment; gravity itself is not directly measurable (equivalence principle).

2 Equation of Motion in the Inertial Frame

Newton's second law for the test mass:

$$m\,\ddot{\mathbf{x}} = \underbrace{-\frac{GM_E\,m}{|\mathbf{x}-\mathbf{R}_E|^3}(\mathbf{x}-\mathbf{R}_E)}_{\text{Earth}}\ \underbrace{-\frac{GM_S\,m}{|\mathbf{x}-\mathbf{R}_S|^3}(\mathbf{x}-\mathbf{R}_S)}_{\text{Sun}}\ \underbrace{-\frac{GM_M\,m}{|\mathbf{x}-\mathbf{R}_M|^3}(\mathbf{x}-\mathbf{R}_M)}_{\text{Moon}} + \mathbf{F}_{\text{spring}} \tag{1}$$

3 Equation of Motion for Earth's Center

Earth's center of mass is accelerated by both external bodies:

$$\ddot{\mathbf{R}}_E = -\frac{GM_S}{|\mathbf{R}_E-\mathbf{R}_S|^3}(\mathbf{R}_E-\mathbf{R}_S) \;-\; \frac{GM_M}{|\mathbf{R}_E-\mathbf{R}_M|^3}(\mathbf{R}_E-\mathbf{R}_M) \tag{2}$$

This is Earth's acceleration toward the Sun and Moon combined. Every object attached to the Earth — including the sensor housing, the mounting bracket, and the reference frame of the measurement — shares this acceleration.

4 Transform to the Earth-Centered Frame

Define the position of the test mass relative to Earth's center:

$$\mathbf{r} \equiv \mathbf{x} - \mathbf{R}_E \tag{3}$$

and the geocentric positions of the external bodies:

$$\mathbf{R} \equiv \mathbf{R}_S - \mathbf{R}_E, \qquad \mathbf{D} \equiv \mathbf{R}_M - \mathbf{R}_E \tag{4}$$

The relative acceleration is $\ddot{\mathbf{r}} = \ddot{\mathbf{x}} - \ddot{\mathbf{R}}_E$. Substituting from Eqs. (1) and (2), each external body B produces a term of the form

$$-\frac{GM_B\, m}{|\mathbf{x} - \mathbf{R}_B|^3}(\mathbf{x} - \mathbf{R}_B) \; + \; \frac{GM_B\, m}{|\mathbf{R}_E - \mathbf{R}_B|^3}(\mathbf{R}_E - \mathbf{R}_B) \tag{5}$$

where $B \in \{S, M\}$. Rewriting in geocentric variables:

$$\boxed{m\,\ddot{\mathbf{r}} = -\frac{GM_E\, m}{|\mathbf{r}|^3}\,\mathbf{r} \; + \; GM_S\, m\left[\frac{\mathbf{R} - \mathbf{r}}{|\mathbf{R} - \mathbf{r}|^3} - \frac{\mathbf{R}}{|\mathbf{R}|^3}\right] \; + \; GM_M\, m\left[\frac{\mathbf{D} - \mathbf{r}}{|\mathbf{D} - \mathbf{r}|^3} - \frac{\mathbf{D}}{|\mathbf{D}|^3}\right] \; + \; \mathbf{F}_{\text{spring}}} \tag{6}$$

What happened

The **direct pull** of each external body canceled independently. To see this explicitly, expand the bracket for a generic external body at geocentric position $\mathbf{R}$ (with $R = |\mathbf{R}|$ and $\mathbf{e}_\mathbf{R} = \mathbf{R}/R$) in powers of the small parameter $|\mathbf{r}|/R$:

$$\frac{\mathbf{R} - \mathbf{r}}{|\mathbf{R} - \mathbf{r}|^3} - \frac{\mathbf{R}}{R^3} \tag{7}$$

Start with the inverse cube:

$$|\mathbf{R} - \mathbf{r}|^2 = R^2\left(1 - 2\,\frac{\mathbf{e}_\mathbf{R} \cdot \mathbf{r}}{R} + \frac{r^2}{R^2}\right) \tag{8}$$

so, defining $\epsilon = 2\,\mathbf{e}_\mathbf{R} \cdot \mathbf{r}/R - r^2/R^2 = O(r/R)$:

$$\frac{1}{|\mathbf{R} - \mathbf{r}|^3} = \frac{1}{R^3}(1 - \epsilon)^{-3/2} = \frac{1}{R^3}\Big(1 + \tfrac{3}{2}\epsilon + O(\epsilon^2)\Big) = \frac{1}{R^3}\left(1 + \frac{3\,\mathbf{e}_\mathbf{R} \cdot \mathbf{r}}{R} + O\!\left(\frac{r^2}{R^2}\right)\right) \tag{9}$$

Multiply by $(\mathbf{R} - \mathbf{r})$ and keep terms through first order in r/R:

$$\begin{aligned}
\frac{\mathbf{R} - \mathbf{r}}{|\mathbf{R} - \mathbf{r}|^3} &= \frac{1}{R^3}\left(1 + \frac{3\,\mathbf{e}_\mathbf{R} \cdot \mathbf{r}}{R}\right)(\mathbf{R} - \mathbf{r}) + O\!\left(\frac{r^2}{R^4}\right) \\
&= \frac{1}{R^3}\Big[\mathbf{R} + 3(\mathbf{e}_\mathbf{R} \cdot \mathbf{r})\,\mathbf{e}_\mathbf{R} - \mathbf{r}\Big] + O\!\left(\frac{r^2}{R^4}\right)
\end{aligned} \tag{10}$$

where the cross term $-3(\mathbf{e}_\mathbf{R} \cdot \mathbf{r})\,\mathbf{r}/R$ is $O(r^2/R)$ and has been absorbed into the remainder.

Now subtract $\mathbf{R}/R^3$:

$$\underbrace{\frac{\mathbf{R}-\mathbf{r}}{|\mathbf{R}-\mathbf{r}|^3}-\frac{\mathbf{R}}{R^3}}_{\text{Eq. (7)}}=\frac{1}{R^3}\left[\cancel{\mathbf{R}}+3(\mathbf{e_R}\cdot\mathbf{r})\,\mathbf{e_R}-\mathbf{r}-\cancel{\mathbf{R}}\right]+O\left(\frac{r^2}{R^4}\right) \tag{11}$$

The $\mathbf{R}/R^3$ terms — which represent the **direct gravitational pull** GM/R^2 in the direction $\mathbf{e_R}$ — cancel identically. What survives is the quadrupole (tidal) term:

$$\boxed{\mathbf{a}_{\text{tidal}}=\frac{GM}{R^3}\left[3\,(\mathbf{e_R}\cdot\mathbf{r})\,\mathbf{e_R}-\mathbf{r}\right]+O\left(\frac{GM\,r^2}{R^4}\right)} \tag{12}$$

This is suppressed by a factor of r/R relative to the direct pull — the ratio of Earth's radius to the distance to the external body. For the Sun, $r/R\approx 4.3\times10^{-5}$; for the Moon, $r/D\approx 1.7\times10^{-2}$.

Applying Eq. (12) to each body:

$$\mathbf{a}^{\text{Sun}}_{\text{tidal}}=GM_S\left[\frac{\mathbf{R}-\mathbf{r}}{|\mathbf{R}-\mathbf{r}|^3}-\frac{\mathbf{R}}{|\mathbf{R}|^3}\right]\approx\frac{GM_S}{R^3}\left[3\,(\mathbf{e_R}\cdot\mathbf{r})\,\mathbf{e_R}-\mathbf{r}\right] \tag{13}$$

$$\mathbf{a}^{\text{Moon}}_{\text{tidal}}=GM_M\left[\frac{\mathbf{D}-\mathbf{r}}{|\mathbf{D}-\mathbf{r}|^3}-\frac{\mathbf{D}}{|\mathbf{D}|^3}\right]\approx\frac{GM_M}{D^3}\left[3\,(\mathbf{e_D}\cdot\mathbf{r})\,\mathbf{e_D}-\mathbf{r}\right] \tag{14}$$

The structure of the cancellation is now transparent:

- The **zeroth-order** term ($\mathbf{R}/R^3$) is the direct pull — the GM/R^2 acceleration toward the external body. It appears identically in the expansion of $(\mathbf{R}-\mathbf{r})/|\mathbf{R}-\mathbf{r}|^3$ and in the subtracted $\mathbf{R}/R^3$, so it cancels exactly (Eq. 11).
- The **first-order** term ($\propto r/R^3$) is the tidal acceleration. It has the quadrupolar structure $3(\mathbf{e_R}\cdot\mathbf{r})\,\mathbf{e_R}-\mathbf{r}$: stretching along the Earth–body axis, compression perpendicular to it, trace-free.
- The cancellation is **exact** in the full (unexpanded) bracket. The Taylor expansion is shown only to identify which term cancels and which survives — the boxed equation in (6) requires no expansion and is used in the code.

5 What the Sensor Reads

The sensor reads $\mathbf{F}_{\text{spring}}$, which already appears in Eq. (6). To isolate it, we solve that equation for $\mathbf{F}_{\text{spring}}$ by specifying the acceleration $\ddot{\mathbf{r}}$ of the test mass.

The test mass is stationary in the lab frame, which co-rotates with Earth at angular velocity $\boldsymbol{\omega}$. In the non-rotating geocentric frame used in Eq. (6), this corresponds to circular motion with centripetal acceleration:

$$\ddot{\mathbf{r}}=\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})=-\omega^2\,\mathbf{r}_\perp \tag{15}$$

where $\mathbf{r}_\perp$ is the component of $\mathbf{r}$ perpendicular to the rotation axis (pointing outward from that axis).

Substituting Eq. (15) into the boxed equation (6) and solving for $\mathbf{F}_{\text{spring}}$:

$$\cancel{m}\,\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})=-\frac{GM_E\cancel{m}}{|\mathbf{r}|^3}\mathbf{r}+\cancel{m}\,\mathbf{a}^{\text{Sun}}_{\text{tidal}}+\cancel{m}\,\mathbf{a}^{\text{Moon}}_{\text{tidal}}+\frac{\mathbf{F}_{\text{spring}}}{\cancel{m}}$$

$$\mathbf{F}_{\text{spring}}=m\left[\frac{GM_E}{|\mathbf{r}|^3}\mathbf{r}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})-\mathbf{a}^{\text{Sun}}_{\text{tidal}}-\mathbf{a}^{\text{Moon}}_{\text{tidal}}\right] \tag{16}$$

The three contributions have clear physical meaning:

- $GM_E\,\mathbf{r}/|\mathbf{r}|^3$: Earth's gravitational acceleration (the spring supports the test mass against this).
- $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = -\omega^2 \mathbf{r}_\perp$: centripetal acceleration from Earth's rotation, which *reduces* the required spring force (objects weigh less at the equator).
- $-\mathbf{a}_{\text{tidal}}$: the tidal perturbations from Eqs. (13)–(14), acting as small corrections.

The sensor projects this onto its measurement axis $\mathbf{e_n}$:

$$g_{\text{measured}} = \frac{\mathbf{F}_{\text{spring}}}{m} \cdot \mathbf{e_n} \tag{17}$$

Neither the direct solar acceleration GM_S/R^2 nor the direct lunar acceleration GM_M/D^2 appears. These terms canceled in Eq. (6) before we ever solved for $\mathbf{F}_{\text{spring}}$ — the spring force inherits only the tidal residuals. This holds for any $\mathbf{e_n}$.

6 Why Tilting Doesn't Help

The claim implicitly assumes that tilting the sensor changes the projection of a body's *direct* pull onto $\mathbf{e_n}$:

$$g^{\text{Sun}}_{\text{claimed}} = \frac{GM_S}{R^2}\,(\mathbf{e_R}\cdot\mathbf{e_n}), \qquad g^{\text{Moon}}_{\text{claimed}} = \frac{GM_M}{D^2}\,(\mathbf{e_D}\cdot\mathbf{e_n}) \quad \longleftarrow \quad \text{these terms do not exist} \tag{18}$$

But each was canceled by the identical projection of Earth's acceleration toward that body onto the same axis. Tilting $\mathbf{e_n}$ rotates all projections equally, because the cancellation is **vectorial** — it holds component by component, in every direction simultaneously.

To make the argument concrete: the sensor housing accelerates at $\ddot{\mathbf{R}}_E$ (toward both the Sun and Moon). The test mass, if released, would also accelerate at $\ddot{\mathbf{R}}_E$ (plus the tidal corrections). The spring connecting them sees only the difference — which is the sum of the two tidal terms. This is true regardless of the spring's orientation.

7 Numerical Comparison

Sun

Quantity	Formula	Value
Direct pull	GM_S/R^2	5.93×10^{-3} m/s^2
Earth's acceleration toward Sun (cancels)	same	5.93×10^{-3} m/s^2
Tidal residual	$\sim 2\,GM_S\,r_\oplus/R^3$	$\mathbf{5.1 \times 10^{-7}}$ **m/s^2**

$$\frac{a^{\text{Sun}}_{\text{tidal}}}{a^{\text{Sun}}_{\text{direct}}} = \frac{2\,r_\oplus}{R_{\text{Sun}}} = \frac{2 \times 6.371 \times 10^6}{1.496 \times 10^{11}} \approx 8.5 \times 10^{-5} \tag{19}$$

The direct pull is **11,700×** larger than the measurable tidal signal.

Moon

Quantity	Formula	Value
Direct pull	GM_M/D^2	3.32×10^{-5} m/s^2
Earth's acceleration toward Moon (cancels)	same	3.32×10^{-5} m/s^2
Tidal residual	$\sim 2\,GM_M\,r_\oplus/D^3$	$\mathbf{1.1 \times 10^{-6}}$ **m/s^2**

$$\frac{a_{\text{tidal}}^{\text{Moon}}}{a_{\text{direct}}^{\text{Moon}}} = \frac{2\,r_\oplus}{D_{\text{Moon}}} = \frac{2 \times 6.371 \times 10^6}{3.844 \times 10^8} \approx 3.3 \times 10^{-2} \tag{20}$$

The direct pull is **30×** larger than the measurable tidal signal. The Moon's ratio is much less extreme than the Sun's because the Moon is only ~ 60 Earth radii away, so the tidal approximation is coarser — but the cancellation is still exact.

Summary

Body	Direct pull	Tidal residual	Ratio	Suppression by
Sun	5.93×10^{-3} m/s^2	5.1×10^{-7} m/s^2	11,700×	$R_{\text{Sun}}/2r_\oplus$
Moon	3.32×10^{-5} m/s^2	1.1×10^{-6} m/s^2	30×	$D_{\text{Moon}}/2r_\oplus$

Note the inversion: the Sun's direct pull is 180× stronger than the Moon's, but its tidal effect is 2.2× *weaker*. This is the $1/R^3$ vs $1/R^2$ scaling — the hallmark of a tidal interaction, and a direct consequence of the cancellation derived above.

8 The Point-Mass Model Proves the Cancellation

The crucial point: the cancellation above used **only** Newton's laws applied to point masses. No general relativity, no equivalence principle, no appeal to "free fall" as a concept — just subtracting the equation of motion of Earth's center from the equation of motion of the test mass.

For each external body B, the term $GM_B/|\mathbf{R}'_B|^2$ appears with the same sign, same magnitude, and same direction in both equations, so it drops out identically.

Anyone who computes $GM_S/R^2 \approx 6 \times 10^{-3}$ m/s^2 or $GM_M/D^2 \approx 3.3 \times 10^{-5}$ m/s^2 and projects it onto a sensor axis is computing a quantity that is real (those bodies do pull the test mass that hard) but **unobservable** — because the sensor, the lab, and the entire Earth are being pulled equally hard in the same direction. The spring between the test mass and the sensor frame cannot detect a uniform acceleration field.

9 Why the Cancellation Works: Universality of Free Fall

One might object: the test mass m and the Earth M_E have very different masses, so why should the Sun's effect cancel between them?

The answer is that gravitational **force** is proportional to the accelerated mass ($F = GM_S\,m/R^2$), so the **acceleration** is mass-independent:

$$a = \frac{F}{m} = \frac{GM_S}{R^2} \tag{21}$$

This is the equality of inertial and gravitational mass ($m_{\text{inert}} = m_{\text{grav}}$), built into Newton's law of gravitation. The test mass and Earth's center experience the same gravitational acceleration from the Sun (to leading order in r/R), despite their mass ratio of $\sim 10^{-25}$.

This is **not** Newton's third law (action = reaction between two interacting bodies). It is a deeper property: two *different* bodies in the *same* external field accelerate identically. The spring connecting them therefore reads zero from the external field — only the spatial *variation* of the field (the tidal gradient) survives.

If any material violated this equality — if some substance fell faster than others in the Sun's field — the direct pull *would* be measurable as a composition-dependent residual. This is precisely what Eötvös-type experiments test, achieving constraints at the 10^{-15} level. Their null results confirm that the cancellation is exact to extraordinary precision, and that the tidal residual is all that remains.

10 Why Converting to Frequency Doesn't Help

One might attempt to circumvent the cancellation by converting the acceleration measurement into a frequency measurement: use an optomechanical oscillator (a mirror on a spring, coupled to a cavity) whose resonance frequency depends on g, and beat it against a reference cavity whose frequency depends only on its length. If the Sun's pull shifts ω_{mech} but not f_{cav}, the beat note should reveal the direct pull.

This fails because the cancellation occurs *before* the readout — at the level of what forces exist between the test mass and its mount.

The optomechanical oscillator

Consider a mirror of mass m on a spring of constant k, anchored to the lab ceiling. In an inertial frame, equilibrium requires:

$$k\,x_{\text{eq}} = m\,(g_{\text{mass}} - a_{\text{anchor}}) \tag{22}$$

where g_{mass} is the total gravitational acceleration at the mirror, and a_{anchor} is the acceleration of the anchor point (which co-moves with the Earth). From Section 4:

$$g_{\text{mass}} - a_{\text{anchor}} = g_{\text{Earth}} - \omega^2 r_\perp + a_{\text{tidal}}^{\text{Sun}} + a_{\text{tidal}}^{\text{Moon}} \tag{23}$$

The Sun's direct pull appears in both g_{mass} and a_{anchor} and cancels. The equilibrium position — and therefore the oscillation frequency $\omega_{\text{mech}} \propto \sqrt{g_{\text{eff}}/L}$ — contains only the tidal part of the solar and lunar fields.

The reference cavity

A Fabry–Pérot reference cavity has resonance frequency $f = mc/(2L)$. The spacer length L is set by electromagnetic bond forces. The Sun's uniform gravitational field accelerates every atom in the spacer equally — no differential stress, no deformation. Only the tidal gradient across the cavity length L would deform it:

$$\frac{\delta L}{L} \sim \frac{GM_S}{R^3}\,\frac{L}{E/\rho} \sim 10^{-26} \tag{24}$$

which is unmeasurable.

The beat note

The beat frequency $\Delta f = f_{\text{mech}} - f_{\text{cav}}$ therefore contains: tidal effects from the oscillator, minus negligible tidal deformation of the cavity. The direct pull GM_S/R^2 appears in neither channel.

Converting acceleration to frequency does not change *what physical quantity* is being measured. The spring (or pendulum, or optomechanical cavity) connects two objects — test mass and housing — that share the same orbital acceleration. It can only measure the differential force between its endpoints, regardless of whether the readout is a displacement, a voltage, or a frequency.

11 What Frequency Measurements *Can* Detect

While no local mechanical measurement can access the direct pull, the gravitational **potential** (as opposed to the force) can be detected through a fundamentally different channel: the gravitational redshift.

An atomic clock's tick rate depends on the spacetime metric:

$$\frac{d\tau}{dt} = \sqrt{-g_{00}} \approx 1 + \frac{\Phi}{c^2} \tag{25}$$

where Φ is the gravitational potential. This is not a force between two co-accelerating parts — it is a property of spacetime itself.

However, the equivalence principle guarantees that *all* local physics is shifted equally by the potential. A single clock in the Sun's field has nothing local to compare against. To detect the potential, one needs a **non-local** reference — a frequency source outside the freely-falling frame.

Clock vs. distant reference (pulsar)

The Sun's potential at Earth is $\Phi_\odot = -GM_S/R \approx -8.9 \times 10^8$ m^2/s^2. Because Earth's orbit is elliptical ($e \approx 0.0167$), this varies annually:

$$\frac{\Delta f}{f} = \frac{GM_S}{c^2}\left(\frac{1}{R_{\mathrm{peri}}} - \frac{1}{R_{\mathrm{aph}}}\right) \approx 3.3 \times 10^{-10} \tag{26}$$

This is the **full, direct** solar potential — not tidal, not suppressed by $r_\oplus/R$. Pulsar timing arrays detect exactly this effect as the "Einstein delay," with an annual amplitude of ~ 1.66 ms.

Two terrestrial clocks

Two clocks at different locations on Earth see different solar potentials, but the difference is only the tidal potential:

$$\frac{\delta f}{f} \sim \frac{GM_S\, r_\oplus^2}{R^3\, c^2} \sim 10^{-17} \tag{27}$$

Modern optical lattice clocks ($\Delta f/f \sim 10^{-18}$) are just reaching this regime.

Hierarchy of measurements

Method	Measures	Sun contribution	Magnitude
Accelerometer	$-\nabla\Phi$ (force)	tidal only	5×10^{-7} m/s^2
Optomech. vs. ref. cavity	$-\nabla\Phi$ (force as frequency)	tidal only	5×10^{-7} m/s^2
Two local clocks	$\Delta\Phi$ (potential difference)	tidal potential	$\Delta f/f \sim 10^{-17}$
Clock vs. pulsar	Φ (absolute potential)	**full direct**	$\Delta f/f \sim 3 \times 10^{-10}$

The dividing line: any instrument where two parts are co-accelerating sees the cancellation. To escape it, one end of the measurement must be outside the freely-falling frame — a non-local comparison of proper time flow rates at different locations in the potential.

12 What IS Observable by Tilting

Tilting the sensor does reveal genuinely interesting physics, but at the tidal scale ($\sim 10^{-7}$ m/s^2), not at the direct-pull scale ($\sim 10^{-3}$ m/s^2):

- **Horizontal tidal acceleration.** A vertical sensor sees only the vertical component of $\mathbf{a}_{\mathrm{tidal}}$. A tilted sensor picks up horizontal components, which have comparable magnitude but different time dependence.
- **Angular separation of static and tidal signals.** Normal gravity is purely vertical (projects as $g_0 \cos\theta$), while the tidal vector has East/North/Up components with independent angular signatures. Scanning the sensor azimuth at $\theta = 90$ř gives a pure tidal signal with zero static background.

- **Reconstruction of the full tidal vector.** Three non-coplanar measurements at one instant determine all four unknowns (g_0, a_E, a_N, a_U) without a time series.

These effects are real, measurable, and interesting — but they are four orders of magnitude smaller than the claimed signal.