

Falling Together: Why Springs Can't See the Sun

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A common claim holds that a spring-based sensor can detect the direct gravitational pull of the Sun ($\sim 6 \times 10^{-3} \text{ m/s}^2$) or the Moon ($\sim 3.3 \times 10^{-5} \text{ m/s}^2$) as a time-varying signal. This is incorrect. We prove the following: *The spring force on a test mass held at rest near the Earth contains no contribution from the direct gravitational pull GM/R^2 of any external body; the direct pull cancels exactly against the free-fall acceleration of the laboratory, leaving only the tidal residual of order $GM r_\oplus/R^3$ — regardless of sensor orientation.* The proof uses only Newtonian point-mass mechanics in an inertial frame where no terms are hidden. The result is then transformed to the rotating laboratory frame, where the only surviving terms — Earth's gravity, centrifugal correction, and tidal residuals — are fully consistent with standard terrestrial gravimetry.

I. INTRODUCTION

Can a spring-based accelerometer measure the Sun's direct gravitational pull on its test mass? This note shows that the answer is **no** — not because the pull is too small, but because the equivalence principle forbids it. The Sun pulls the sensor housing and the test mass by exactly the same amount; the spring connecting them cannot register a force that acts identically on both ends.

The difficulty in seeing this lies in the choice of reference frame. A laboratory on Earth's surface is non-inertial for two reasons:

1. **Earth rotates**, producing centrifugal and Coriolis pseudo-accelerations ($\sim 0.03 \text{ m/s}^2$ at the equator).
2. **Earth freely falls toward the Sun and Moon.** The sensor housing is rigidly attached to the Earth's crust; it shares the Earth's free-fall acceleration toward the Sun ($GM_S/R^2 \approx 5.93 \times 10^{-3} \text{ m/s}^2$) and Moon ($GM_M/D^2 \approx 3.32 \times 10^{-5} \text{ m/s}^2$) exactly. The test mass, connected only by a spring, is the one body free to deviate — but gravity accelerates it by the same amount, so the spring registers no difference.

The second point is the origin of the cancellation. When one writes equations of motion in the lab frame, the free-fall acceleration toward the Sun has already been subtracted from every term. A spring-based sensor physically implements this subtraction: both ends share the same gravitational acceleration, so the spring reads zero from that field. Only the tiny *gradient* of the field across the sensor — the tidal term — survives.

To make this cancellation explicit rather than hidden, we begin in a true inertial frame and carry every term (Sec. II). We then transform to the rotating laboratory (Sec. III), project onto an arbitrary sensor axis (Sec. IV), and compare the magnitudes of all surviving terms (Sec. V). Section VI places the result in the context of the equivalence principle.

II. THE PROOF

A. Setup

Four point masses in an inertial frame (Fig. 1). The test mass is connected to the sensor by a spring (or equivalent restoring mechanism). The sensor housing is rigidly attached to the Earth. The sensor reads the spring force $\mathbf{F}_{\text{spring}}$ — the non-gravitational force required to keep the test mass co-moving with the lab.

B. Test mass equation of motion

In the inertial frame, Newton's second law for the test mass:

$$m \ddot{\mathbf{x}} = -\frac{GM_E m}{|\mathbf{x} - \mathbf{R}_E|^3} (\mathbf{x} - \mathbf{R}_E) - \frac{GM_S m}{|\mathbf{x} - \mathbf{R}_S|^3} (\mathbf{x} - \mathbf{R}_S) - \frac{GM_M m}{|\mathbf{x} - \mathbf{R}_M|^3} (\mathbf{x} - \mathbf{R}_M) + \mathbf{F}_{\text{spring}} \quad (1)$$

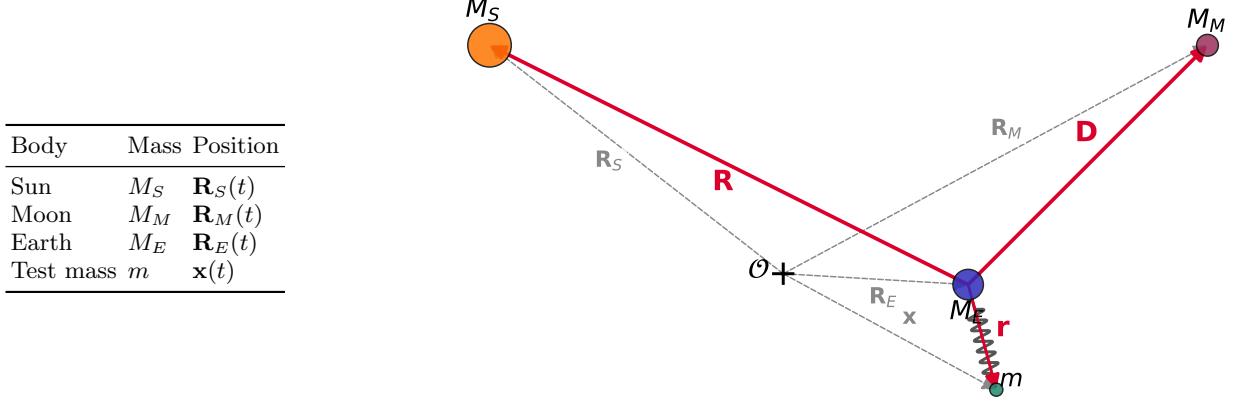


FIG. 1. Four-body setup. *Left:* masses and inertial-frame positions. *Right:* schematic geometry. Gray dashed arrows: position vectors \mathbf{R}_S , \mathbf{R}_E , \mathbf{R}_M , and \mathbf{x} from the inertial origin. Red solid arrows: geocentric vectors $\mathbf{R} = \mathbf{R}_S - \mathbf{R}_E$, $\mathbf{D} = \mathbf{R}_M - \mathbf{R}_E$, and $\mathbf{r} = \mathbf{x} - \mathbf{R}_E$. The spring connects the test mass m to the Earth-fixed sensor housing.

C. Earth's equation of motion

Earth's center accelerates toward the Sun and Moon:

$$\ddot{\mathbf{R}}_E = -\frac{GM_S}{|\mathbf{R}_E - \mathbf{R}_S|^3}(\mathbf{R}_E - \mathbf{R}_S) - \frac{GM_M}{|\mathbf{R}_E - \mathbf{R}_M|^3}(\mathbf{R}_E - \mathbf{R}_M) \quad (2)$$

Every object rigidly attached to the Earth — including the sensor housing — shares this acceleration.

D. Subtract to get the relative motion

Define $\mathbf{r} \equiv \mathbf{x} - \mathbf{R}_E$ and the geocentric positions $\mathbf{R} \equiv \mathbf{R}_S - \mathbf{R}_E$, $\mathbf{D} \equiv \mathbf{R}_M - \mathbf{R}_E$. Then $\ddot{\mathbf{r}} = \ddot{\mathbf{x}} - \ddot{\mathbf{R}}_E$ gives:

$$m\ddot{\mathbf{r}} = -\frac{GM_E m}{|\mathbf{r}|^3}\mathbf{r} + GM_S m \left[\frac{\mathbf{R} - \mathbf{r}}{|\mathbf{R} - \mathbf{r}|^3} - \frac{\mathbf{R}}{R^3} \right] + GM_M m \left[\frac{\mathbf{D} - \mathbf{r}}{|\mathbf{D} - \mathbf{r}|^3} - \frac{\mathbf{D}}{D^3} \right] + \mathbf{F}_{\text{spring}} \quad (3)$$

E. The cancellation

Each bracketed term has the form

$$\mathbf{f}(\mathbf{r}) = \frac{\mathbf{R} - \mathbf{r}}{|\mathbf{R} - \mathbf{r}|^3} - \frac{\mathbf{R}}{R^3} \quad (4)$$

This is the Sun's gravitational field at the test mass *minus* the field at Earth's center. At $\mathbf{r} = \mathbf{0}$:

$$\mathbf{f}(\mathbf{0}) = \frac{\mathbf{R}}{R^3} - \frac{\mathbf{R}}{R^3} = \mathbf{0} \quad (\text{exactly}) \quad (5)$$

The direct pull — $GM_S \mathbf{R}/R^3$, the uniform field that accelerates the entire Earth at $5.93 \times 10^{-3} \text{ m/s}^2$ — appears with equal magnitude and opposite sign in Eqs. (1) and (2), and cancels identically. No approximation is involved. The same holds for the Moon with \mathbf{D} replacing \mathbf{R} .

What survives is $\mathbf{f}(\mathbf{r}) - \mathbf{f}(\mathbf{0})$: the *variation* of the gravitational field across the baseline \mathbf{r} — the **tidal acceleration**. To leading order in r/R :

$$\mathbf{a}_{\text{tidal}} = \frac{GM}{R^3} \left[3(\mathbf{e}_R \cdot \mathbf{r}) \mathbf{e}_R - \mathbf{r} \right] + O\left(\frac{GM r^2}{R^4}\right) \quad (6)$$

This is suppressed by a factor r/R relative to the direct pull.

F. Derivation of Eq. (6)

Expand $|\mathbf{R} - \mathbf{r}|^{-3}$ for $r \ll R$. Write

$$|\mathbf{R} - \mathbf{r}|^2 = R^2 \left(1 - 2 \frac{\mathbf{e}_\mathbf{R} \cdot \mathbf{r}}{R} + \frac{r^2}{R^2} \right) \equiv R^2(1 - \epsilon) \quad (7)$$

with $\epsilon = 2 \mathbf{e}_\mathbf{R} \cdot \mathbf{r}/R - r^2/R^2 = O(r/R)$. Then

$$\frac{1}{|\mathbf{R} - \mathbf{r}|^3} = \frac{1}{R^3}(1 - \epsilon)^{-3/2} = \frac{1}{R^3} \left(1 + \frac{3 \mathbf{e}_\mathbf{R} \cdot \mathbf{r}}{R} + O\left(\frac{r^2}{R^2}\right) \right) \quad (8)$$

Multiply by $(\mathbf{R} - \mathbf{r})$ and keep terms through first order:

$$\frac{\mathbf{R} - \mathbf{r}}{|\mathbf{R} - \mathbf{r}|^3} = \frac{1}{R^3} [\mathbf{R} + 3(\mathbf{e}_\mathbf{R} \cdot \mathbf{r}) \mathbf{e}_\mathbf{R} - \mathbf{r}] + O\left(\frac{r^2}{R^4}\right) \quad (9)$$

Subtract \mathbf{R}/R^3 :

$$\mathbf{f}(\mathbf{r}) = \frac{\mathbf{R} - \mathbf{r}}{|\mathbf{R} - \mathbf{r}|^3} - \frac{\mathbf{R}}{R^3} = \frac{1}{R^3} [3(\mathbf{e}_\mathbf{R} \cdot \mathbf{r}) \mathbf{e}_\mathbf{R} - \mathbf{r}] + O\left(\frac{r^2}{R^4}\right) \quad (10)$$

The \mathbf{R}/R^3 terms cancel identically — confirming that the direct pull drops out — and the leading surviving term is the tidal quadrupole, Eq. (6). \square

We summarize the result of this section as a general theorem.

Theorem 1 (Cancellation of direct gravitational pull). *Let a test mass at position \mathbf{x} be subject to the gravitational fields of the Earth (M_E) and an external body (M , at geocentric distance $R \gg r$) together with a non-gravitational spring force $\mathbf{F}_{\text{spring}}$. In the geocentric frame, the spring force required to keep the test mass at rest is*

$$\frac{\mathbf{F}_{\text{spring}}}{m} = \frac{GM_E}{r^2} \hat{\mathbf{r}} - \mathbf{a}_{\text{tidal}}$$

where $\mathbf{a}_{\text{tidal}} = (GM/R^3)[3(\hat{\mathbf{R}} \cdot \mathbf{r})\hat{\mathbf{R}} - \mathbf{r}] + O(r^2/R^4)$ is the tidal acceleration. The direct pull GM/R^2 of the external body does not appear: it cancels exactly against the free-fall acceleration of the laboratory. This holds for any measurement axis $\hat{\mathbf{n}}$, any number of external bodies, and to all orders in r/R .

The remainder of this paper verifies that rotating-frame corrections, sensor orientation, and all other terrestrial effects leave this conclusion intact.

III. TRANSFORMATION TO THE LABORATORY FRAME

Equation (3) is written in the non-rotating geocentric frame. The laboratory rotates with the Earth at angular velocity $\boldsymbol{\omega}$. This is a second source of non-inertiality (see Sec. I).

A. Rotating-frame equation of motion

Let \mathbf{r}' be the test mass position in the lab frame, related to its geocentric position by $\mathbf{r} = \mathcal{R}(t)\mathbf{r}'$ where $\mathcal{R}(t)$ is the time-dependent rotation matrix. For uniform rotation about axis $\hat{\mathbf{n}} = \boldsymbol{\omega}/\omega$ at angular rate $\omega = |\boldsymbol{\omega}|$, it takes the Rodrigues form

$$\mathcal{R}(t) = \mathbf{I} \cos \omega t + (1 - \cos \omega t) \hat{\mathbf{n}} \hat{\mathbf{n}}^T + \sin \omega t [\hat{\mathbf{n}}]_\times \quad (11)$$

where $[\hat{\mathbf{n}}]_\times$ is the skew-symmetric matrix whose action on any vector \mathbf{v} is $[\hat{\mathbf{n}}]_\times \mathbf{v} = \hat{\mathbf{n}} \times \mathbf{v}$ — i.e. the matrix representation of the cross product. Its time derivative satisfies

$$\dot{\mathcal{R}}(t) = [\boldsymbol{\omega}]_\times \mathcal{R}(t) \quad (12)$$

so that $\dot{\mathcal{R}} \mathbf{v} = \boldsymbol{\omega} \times (\mathcal{R} \mathbf{v})$ for every vector \mathbf{v} . Each time derivative of \mathcal{R} thus generates a cross product with $\boldsymbol{\omega}$.

Differentiating $\mathbf{r} = \mathcal{R} \mathbf{r}'$ twice using Eq. (12):

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}' + 2\boldsymbol{\omega} \times \dot{\mathbf{r}}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') \quad (13)$$

(the Euler term $\dot{\boldsymbol{\omega}} \times \mathbf{r}'$ vanishes for uniform rotation). Note that $\boldsymbol{\omega}$ carries no prime: $\mathcal{R}(t)$ is a rotation *about* $\boldsymbol{\omega}$, so $\mathcal{R}^{-1}\boldsymbol{\omega} = \boldsymbol{\omega}$ — the rotation vector is invariant under the rotation it generates. Substituting into Eq. (3) and expressing all vectors in the lab basis:

$$\ddot{\mathbf{r}}' = \underbrace{-\frac{GM_E}{|\mathbf{r}'|^3} \mathbf{r}'}_{\text{Earth's gravity}} + \underbrace{\mathbf{a}'_{\text{tidal}}}_{\text{tidal}} - \underbrace{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')}_{\text{centrifugal}} - \underbrace{2\boldsymbol{\omega} \times \dot{\mathbf{r}}'}_{\text{Coriolis}} + \frac{\mathbf{F}'_{\text{spring}}}{m} \quad (14)$$

The direct solar and lunar pulls do **not** reappear — they canceled in Eq. (3) before the frame change, and rotating the coordinate basis cannot restore a term that is already zero.

B. What the sensor reads

The sensor housing is held in place by the laboratory floor, which provides whatever normal force \mathbf{N} is required to prevent acceleration. This constraint force acts on the housing, not on the test mass, and therefore does not appear in Eq. (14). Its effect enters through the spring: the housing end of the spring follows the Earth’s motion, so the spring force $\mathbf{F}'_{\text{spring}}$ adjusts to keep the test mass co-moving.

The spring force is thus entirely *caused by* the constraint. The floor holds the housing fixed; the spring transmits that constraint to the test mass. This is the equivalence principle in action: if the floor were removed and the entire apparatus fell freely, housing and test mass would share the same gravitational acceleration. The spring would relax and $\mathbf{F}'_{\text{spring}} \rightarrow \mathbf{0}$ — precisely because no constraint force distinguishes the housing from the test mass. A spring-based sensor can only measure accelerations that *differ* between its two ends, i.e. tidal effects and non-gravitational forces.

The test mass is thus at rest in the lab: $\dot{\mathbf{r}}' = \ddot{\mathbf{r}}' = \mathbf{0}$. The Coriolis term vanishes. Setting the left-hand side of Eq. (14) to zero determines the spring force — the quantity the sensor reads:

$$\frac{\mathbf{F}'_{\text{spring}}}{m} = \frac{GM_E}{|\mathbf{r}'|^3} \mathbf{r}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - \mathbf{a}'_{\text{tidal}} \quad (15)$$

The sensor projects this onto its measurement axis \mathbf{e}_n :

$$g_{\text{measured}} = \frac{\mathbf{F}'_{\text{spring}}}{m} \cdot \mathbf{e}_n \quad (16)$$

The three contributions are:

- $GM_E \mathbf{r}' / |\mathbf{r}'|^3$: Earth’s gravity (the dominant, static term).
- $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') = -\omega^2 \mathbf{r}'_\perp$: the centrifugal reduction (objects weigh less at the equator by $\sim 0.3\%$).
- $-\mathbf{a}'_{\text{tidal}}$: the tidal perturbation from Sun and Moon, of order 10^{-7} m/s^2 .

The direct pull GM_S/R^2 and GM_M/D^2 appear nowhere. This holds for any \mathbf{e}_n .

C. Size of each term

Table I lists every acceleration that enters Eq. (15), together with the two direct pulls that cancel before reaching it.

The hierarchy is striking. Earth’s gravity dominates at $\sim 10 \text{ m/s}^2$. The centrifugal correction is four orders of magnitude smaller ($\sim 10^{-2} \text{ m/s}^2$), yet still measurable — it is the reason objects weigh less at the equator than at the poles. The tidal accelerations are smaller still, at 10^{-6} – 10^{-7} m/s^2 , but are routinely detected by superconducting gravimeters and satellite missions such as GRACE.

The canceled direct pulls occupy a revealing intermediate scale. The Sun’s direct pull ($5.93 \times 10^{-3} \text{ m/s}^2$) is $180\times$ stronger than the Moon’s ($3.32 \times 10^{-5} \text{ m/s}^2$), yet its tidal effect is $2.2\times$ *weaker* ($1/R^3$ vs. $1/R^2$ scaling). This

TABLE I. Magnitude of each acceleration term at Earth's surface, ordered by size. The grayed-out rows — the direct gravitational pulls — cancel exactly (Sec. II) and do not appear in the sensor reading [Eq. (15)]. The last column gives the ratio of each term to the solar tidal residual ($5.0 \times 10^{-7} \text{ m/s}^2$); for the canceled rows it is the ratio of direct pull to corresponding tidal residual (11,700× for the Sun, 30× for the Moon).

| TERM | EXPRESSION | MAGNITUDE (m/s^2) | DAILY VARIATION (m/s^2) | RATIO |
|--------------------------------|------------------------------------|-------------------------------------|------------------------------------|--------------------------------|
| Earth's gravity | GM_E/r_\oplus^2 | 9.82 | static | 2×10^7 |
| Centrifugal ($\lambda=48.4$) | $\omega^2 r_\oplus \cos^2 \lambda$ | $3.4 \times 10^{-2} \cos^2 \lambda$ | static | $7 \times 10^4 \cos^2 \lambda$ |
| Solar direct pull (canceled) | GM_S/R^2 | 5.93×10^{-3} | — | 11,700× |
| Lunar direct pull (canceled) | GM_M/D^2 | 3.32×10^{-5} | — | 30× |
| Lunar tide | $GM_M r_\oplus / D^3$ | 1.1×10^{-6} | $\lesssim 2 \times 10^{-6}$ | — |
| Solar tide | $GM_S r_\oplus / R^3$ | 5.0×10^{-7} | $\lesssim 8 \times 10^{-7}$ | — |

inversion — the Sun pulls harder but tidally disturbs less — is a direct signature of the cancellation. If direct pulls were observable, the Sun would dominate the signal by two orders of magnitude. Instead, the Moon dominates the tidal signal, exactly as observed.

The sensor reads a superposition of the four non-grayed terms in Table I. The direct pulls have been subtracted out by the physics of the measurement: the lab accelerates with the Earth, and the spring cannot see it.

IV. WHY SENSOR ORIENTATION IS IRRELEVANT

The cancellation in Eq. (5) is **vectorial**: the direct pull vanishes as a three-component vector, not as a particular scalar projection. Projecting onto any measurement axis \mathbf{e}_n :

$$\mathbf{e}_n \cdot \mathbf{f}(\mathbf{0}) = \mathbf{e}_n \cdot \mathbf{0} = 0 \quad \text{for all } \mathbf{e}_n \quad (17)$$

Tilting the sensor changes \mathbf{e}_n but cannot make zero non-zero. The spring force along any axis contains only the tidal terms from Eq. (6).

Physically: the sensor housing and the test mass both accelerate at GM_S/R^2 toward the Sun. The spring connecting them measures only the *difference* between the accelerations of its two endpoints. A uniform field produces no difference, regardless of the spring's orientation.

A. Sweeping the sensor orientation

Parameterise the measurement axis by zenith angle θ and azimuth φ :

$$\mathbf{e}_n(\theta, \varphi) = \sin \theta \cos \varphi \mathbf{e}_E + \sin \theta \sin \varphi \mathbf{e}_N + \cos \theta \mathbf{e}_U \quad (18)$$

where \mathbf{e}_E , \mathbf{e}_N , \mathbf{e}_U are the local East–North–Up unit vectors defined relative to the *geodetic* vertical. Projecting Eq. (15) onto $\mathbf{e}_n(\theta, \varphi)$ and writing each term explicitly:

$$\begin{aligned} g(\theta, \varphi) &= \underbrace{\frac{GM_E}{r_\oplus^2} \cos \theta}_{\text{gravity}} + \underbrace{[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')] \cdot \mathbf{e}_n}_{\text{centrifugal}} + \underbrace{(-2\boldsymbol{\omega} \times \dot{\mathbf{r}}') \cdot \mathbf{e}_n}_{=0 \text{ (static)}} - \mathbf{a}'_{\text{tidal}} \cdot \mathbf{e}_n \\ &= g_0 \cos \theta - a'_E \sin \theta \cos \varphi - a'_N \sin \theta \sin \varphi - a'_U \cos \theta \end{aligned} \quad (19)$$

In the first line, the gravitational and centrifugal projections combine into a single $\cos \theta$ term because both point along the geodetic vertical \mathbf{e}_U (see Appendix B for a detailed discussion of why the centrifugal term has no azimuthal component in this frame), and the Coriolis term vanishes for a stationary test mass ($\dot{\mathbf{r}}' = \mathbf{0}$). The result defines the effective gravity $g_0 \equiv GM_E/r_\oplus^2 - \omega^2 r_\oplus \cos^2 \lambda$ (where λ is the geodetic latitude), absorbing the centrifugal correction into the amplitude.

We now trace each contribution through this projection.

Earth's gravity and centrifugal ($g_0 \cos \theta$). The effective gravity $\mathbf{g}_{\text{eff}} = g_0 \mathbf{e}_U$ points purely along the geodetic vertical by definition. Its projection onto the sensor axis is $g_0 \cos \theta$: it reads the full $g_0 \approx 9.81 \text{ m/s}^2$ when the

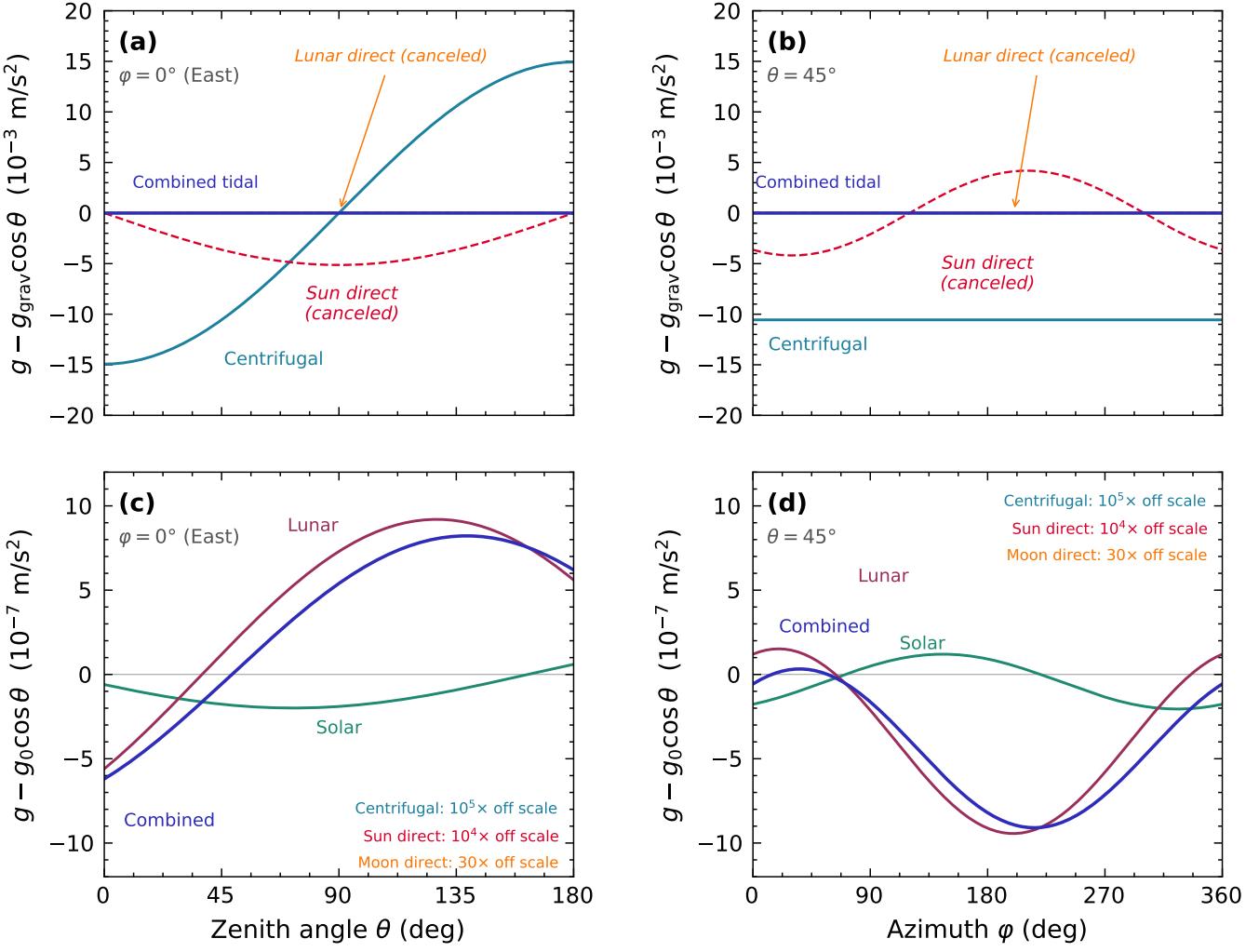


FIG. 2. Residual acceleration $g - g_{\text{grav}} \cos \theta$ as a function of sensor orientation, where g_{grav} is the pure gravitational acceleration (without centrifugal). Left column: sweep over zenith angle θ at fixed azimuth $\varphi = 0$ (East). Right column: sweep over azimuth φ at fixed $\theta = 45^\circ$. **Top row (a, b):** 10^{-3} m/s^2 scale. Solid blue: centrifugal correction ($\omega^2 r_\oplus \cos^2 \lambda \approx 1.7 \times 10^{-2} \text{ m/s}^2$ at Ulm, $\lambda = 48.4^\circ$); red dashed: hypothetical Sun direct pull ($5.93 \times 10^{-3} \text{ m/s}^2$); orange dashed: hypothetical Moon direct pull ($3.3 \times 10^{-5} \text{ m/s}^2$); blue solid: combined tidal residual (indistinguishable from zero at this scale). The centrifugal term is the largest non-gravitational contribution and exceeds even the (canceled) Sun direct pull. **Bottom row (c, d):** 10^{-7} m/s^2 scale, zoomed to the actual tidal signals. Teal: solar tidal; magenta: lunar tidal; blue: combined. The centrifugal ($10^5 \times$), canceled direct Sun ($10^4 \times$), and Moon ($30 \times$) pulls are all off this scale. No orientation reveals the claimed 10^{-3} m/s^2 signal.

sensor is vertical ($\theta = 0$), vanishes when horizontal ($\theta = 90^\circ$), and reverses when inverted ($\theta = 180^\circ$). The centrifugal correction modifies g_0 by at most 0.3% (latitude-dependent) but introduces no separate angular signature — it is entirely absorbed into the amplitude of the $\cos \theta$ term. There is no “rotational residual” in the sweep.

Solar and lunar tides (a'_E , a'_N , a'_U). The tidal accelerations from Eq. (6) decompose into East, North, and Up components. The horizontal components a'_E and a'_N enter through $\sin \theta \cos \varphi$ and $\sin \theta \sin \varphi$: they vanish for a vertical sensor and are maximized for a horizontal one pointed in the appropriate direction. The vertical component a'_U enters as $\cos \theta$, superimposed on the much larger $g_0 \cos \theta$. Representative combined amplitudes (from Sec. III): solar tidal $\sim 5.0 \times 10^{-7} \text{ m/s}^2$, lunar tidal $\sim 1.1 \times 10^{-6} \text{ m/s}^2$. These are the only time-varying signals in the sweep.

Direct solar pull (canceled). Were the direct pull GM_S/R^2 not canceled by Earth’s free fall, it would project as $a_\odot \sin \theta \cos(\varphi - \varphi_\odot)$ with amplitude $a_\odot \approx 5.93 \times 10^{-3} \text{ m/s}^2$ — four orders of magnitude above the tidal terms. As shown in Fig. 2, no such signal exists at any orientation. An angular scan therefore confirms the cancellation directly: one measures $g_0 \cos \theta$ plus tidal corrections of order 10^{-7} m/s^2 , with no 10^{-3} m/s^2 sinusoidal component.

V. WHAT IS OBSERVABLE

Given that the direct pulls cancel, what *does* a tilted sensor actually measure? From Eq. (19), the sensor reading decomposes into three physically distinct contributions. Each responds differently to changes in orientation and time.

A. Static baseline: gravity and centrifugal

The dominant term $g_0 \cos \theta$ is the projection of the effective gravity onto the measurement axis. It is entirely static — independent of time, the positions of Sun and Moon, or anything else in the sky. Tilting the sensor from vertical ($\theta = 0$) to horizontal ($\theta = 90$) sweeps this term from $g_0 \approx 9.81 \text{ m/s}^2$ through zero.

Embedded within g_0 is the centrifugal correction, which at Ulm amounts to $\omega^2 r_{\oplus} \cos^2 \lambda \approx 1.5 \times 10^{-2} \text{ m/s}^2$. It projects as $1.5 \times 10^{-2} \cos \theta \text{ m/s}^2$ and is visible in Fig. 2 (panels a, b) as the solid blue curve. This is the largest non-gravitational contribution the sensor can detect — it exceeds even the (canceled) solar direct pull by a factor of 2.5. Unlike the tidal terms, however, it carries no time dependence: it is a fixed geometric projection that shifts the baseline reading as the sensor is tilted.

B. Time-varying signals: tidal components

The only orientation-dependent terms that vary in time are the tidal projections $a'_E \sin \theta \cos \varphi$, $a'_N \sin \theta \sin \varphi$, and $a'_U \cos \theta$. These encode the three components of the tidal acceleration vector in the geodetic frame:

- A vertical sensor ($\theta = 0$) measures a'_U , the vertical tidal component ($\sim 5.0 \times 10^{-7} \text{ m/s}^2$, semi-diurnal with ~ 12 h period).
- A horizontal sensor ($\theta = 90$) pointing East ($\varphi = 0$) measures a'_E ; pointing North ($\varphi = 90$) it measures a'_N . These horizontal components are diurnal (~ 24 h period).
- Three non-coplanar sensors reconstruct the full tidal acceleration vector and, in principle, the tidal tensor.

The lunar tidal signal ($\sim 1.1 \times 10^{-6} \text{ m/s}^2$) is roughly twice the solar ($\sim 5.0 \times 10^{-7} \text{ m/s}^2$), consistent with the $1/R^3$ scaling that favors the nearby Moon over the distant Sun. Superconducting gravimeters routinely resolve these signals; they are the standard observable in terrestrial tidal gravimetry.

C. What is absent

Conspicuously absent from Eq. (19) are two terms that would dominate the reading if direct gravitational pulls were not canceled:

- A **solar** term of order $5.93 \times 10^{-3} \text{ m/s}^2$, projecting as $a_{\odot} \sin \theta \cos(\varphi - \varphi_{\odot})$ with a diurnal period (~ 24 h). This would exceed the actual tidal signals by four orders of magnitude.
- A **lunar** term of order $3.32 \times 10^{-5} \text{ m/s}^2$, projecting as $a_M \sin \theta \cos(\varphi - \varphi_M)$ with a period of ~ 24.8 h. Smaller than the solar term by a factor of 180, but still 30× larger than the lunar tidal signal it would accompany.

Figure 2 confirms their absence directly: the top panels show both hypothetical direct pulls as dashed curves that dwarf the actual tidal residual (blue, indistinguishable from zero at this scale). The bottom panels zoom into the 10^{-7} m/s^2 scale where the real tidal signals live — orders of magnitude below either claimed effect.

All orientation-dependent signals the sensor can detect — centrifugal and tidal — are real and well understood. None is a direct gravitational pull.

VI. THE EQUIVALENCE PRINCIPLE

The cancellation derived in Sec. II is not a calculational coincidence. It is the equivalence principle.

Einstein's key insight was that a uniform gravitational field is locally indistinguishable from an accelerating reference frame. No experiment confined to a sufficiently small laboratory can determine whether the laboratory is at rest in

a gravitational field or accelerating through empty space. This means that a freely falling observer cannot detect the uniform component of any gravitational field — only its *gradient* (the tidal part) is locally measurable.

The Earth, together with the sensor, the laboratory, and the observer, is in free fall in the Sun's gravitational field. The entire system accelerates at GM_S/R^2 toward the Sun. A spring-based sensor measures the force difference between its two endpoints. Because both endpoints share the same free-fall acceleration, the spring is blind to the Sun's uniform pull. Only the *non-uniformity* of the field across the sensor — the tidal term, of order $GM_S r_\oplus/R^3$ — can produce a differential signal.

In the language of general relativity, the Earth follows a geodesic in the Sun's spacetime. Along a geodesic, no proper acceleration is felt — the “force of gravity” is replaced by the geometry of spacetime. What remains observable is geodesic deviation: nearby geodesics converge or diverge due to spacetime curvature. This is the tidal acceleration of Eq. (6), encoded in the Riemann tensor. The direct pull GM_S/R^2 has no counterpart in this description; it is an artifact of the Newtonian decomposition into “gravitational force” and “inertial motion,” a decomposition that the equivalence principle declares unphysical.

This is why no adjustment of sensor orientation, no change of measurement axis, and no increase in sensitivity can reveal a 5.93×10^{-3} m/s² signal from the Sun. The signal does not exist at the sensor because the equivalence principle forbids it. What the sensor can measure — and what is measured routinely — are tidal effects at 10^{-7} – 10^{-6} m/s², four orders of magnitude below the direct pull. The proof in Sec. II is the Newtonian statement of this principle; the equivalence principle elevates it from a property of gravity to a law of nature.

Appendix A: Three-point formulation: housing as a separate body

The main text treats the sensor housing as coincident with Earth's center. Here we promote the housing to a separate coordinate and show that no new physics arises.

1. Setup

A real sensor does not sit at Earth's center; it is bolted to the crust at some location on the surface. One might therefore ask whether the cancellation of Sec. II relies on the idealisation of a test mass at \mathbf{R}_E . To address this, we introduce three distinct positions in the inertial frame:

- Earth's center: $\mathbf{R}_E(t)$, whose motion is governed by the combined gravitational attraction of all external bodies.
- Housing (spring attachment point): $\mathbf{R}_H(t) = \mathbf{R}_E(t) + \mathbf{d}(t)$, where $|\mathbf{d}| = r_\oplus$ and \mathbf{d} is fixed in the rotating frame. The housing is rigidly attached to the crust (Born rigidity): it co-rotates with the Earth and is held in place by internal stresses rather than orbiting freely.
- Test mass: $\mathbf{x}(t)$, connected to the housing by a spring of rest length ℓ .

The key dynamical variable is the *spring extension*,

$$\mathbf{s} \equiv \mathbf{x} - \mathbf{R}_H, \quad (\text{A1})$$

with $|\mathbf{s}| \sim \text{cm}$. This is the vector that the spring force acts along, and it is the quantity that the sensor ultimately transduces into a reading. Everything the sensor can measure must be encoded in \mathbf{s} and its time derivatives.

2. Equations of motion

We now write Newton's second law for both the test mass and the housing separately. This is the crucial difference from the main text, where both were lumped at Earth's center. We denote the external body's mass by M and its inertial position by $\mathbf{R}_B(t)$.

Test mass. The test mass is subject to gravitational attraction from the Earth and the external body, plus the spring force. Its equation of motion is identical to Eq. (1):

$$m \ddot{\mathbf{x}} = -\frac{GM_E m}{|\mathbf{x} - \mathbf{R}_E|^3}(\mathbf{x} - \mathbf{R}_E) - \frac{GM m}{|\mathbf{x} - \mathbf{R}_B|^3}(\mathbf{x} - \mathbf{R}_B) + \mathbf{F}_{\text{spring}} \quad (\text{A2})$$

(the lunar term is omitted for brevity; it enters identically).

Housing. Unlike the test mass, the housing is not free: it is a rigid extension of the Earth’s crust. It feels gravity from both the Earth and the external body, the reaction force of the spring (Newton’s third law), and a constraint (normal) force \mathbf{N} exerted by the crust that prevents it from falling:

$$m_H \ddot{\mathbf{R}}_H = -\frac{GM_E m_H}{|\mathbf{d}|^3} \mathbf{d} - \frac{GM m_H}{|\mathbf{R}_H - \mathbf{R}_B|^3} (\mathbf{R}_H - \mathbf{R}_B) + \mathbf{N} - \mathbf{F}_{\text{spring}} \quad (\text{A3})$$

The constraint force \mathbf{N} is whatever is required to enforce $\mathbf{d} = \text{const}$ in the rotating frame — it is the mechanical reaction of the ground on the sensor mount. Note that \mathbf{N} encodes the full weight of the housing in Earth’s gravity field, including the tidal gradient across r_{\oplus} . This will become important in Sec. A 6.

3. Subtraction: equation for the spring extension

The sensor measures the spring extension \mathbf{s} , not the absolute position of either endpoint. To obtain the equation governing \mathbf{s} , we subtract the housing equation (A3) (divided by m_H) from the test-mass equation (A2) (divided by m). Using $\ddot{\mathbf{s}} = \ddot{\mathbf{x}} - \dot{\mathbf{R}}_H$ and writing $\mathbf{R} = \mathbf{R}_B - \mathbf{R}_E$:

$$\begin{aligned} m \ddot{\mathbf{s}} = & -\frac{GM_E m}{|\mathbf{d} + \mathbf{s}|^3} (\mathbf{d} + \mathbf{s}) + \frac{GM_E m}{d^3} \mathbf{d} \\ & + GM m \left[\frac{\mathbf{R} - \mathbf{d} - \mathbf{s}}{|\mathbf{R} - \mathbf{d} - \mathbf{s}|^3} - \frac{\mathbf{R} - \mathbf{d}}{|\mathbf{R} - \mathbf{d}|^3} \right] \\ & + \left(1 + \frac{m}{m_H} \right) \mathbf{F}_{\text{spring}} + \frac{m}{m_H} \mathbf{N} \end{aligned} \quad (\text{A4})$$

4. The cancellation

The key question is whether the external body’s direct gravitational pull GM/R'^2 survives in the equation for \mathbf{s} . Examine the external-body term in Eq. (A4). It has the same structure as Eq. (3) of the main text:

$$\mathbf{g}(\mathbf{s}) = \frac{\mathbf{R}' - \mathbf{s}}{|\mathbf{R}' - \mathbf{s}|^3} - \frac{\mathbf{R}'}{R'^3} \quad (\text{A5})$$

where $\mathbf{R}' \equiv \mathbf{R} - \mathbf{d}$ is the external body’s position relative to the *housing* (not Earth’s center). Evaluating at $\mathbf{s} = \mathbf{0}$ (i.e. when the test mass sits exactly at the housing):

$$\mathbf{g}(\mathbf{0}) = \mathbf{0} \quad (\text{exactly}) \quad (\text{A6})$$

This is the same cancellation as in Sec. II, but now its physical meaning is sharper: the direct pull GM/R'^2 on the test mass cancels against the direct pull on the housing — not against Earth’s center, but against the other end of the spring. Both spring endpoints are pulled toward the external body by the same acceleration (to leading order in s/R'), so the spring itself cannot detect it. The cancellation is local: it occurs at the sensor, between its two endpoints.

5. What survives: tidal across the spring

With the direct pull gone, what remains? Expanding Eq. (A5) to leading order in s/R' (with $R' \approx R$ since $d \ll R$):

$$\mathbf{a}_{\text{tidal}}^{(\text{spring})} = \frac{GM}{R'^3} \left[3(\hat{\mathbf{R}}' \cdot \mathbf{s}) \hat{\mathbf{R}}' - \mathbf{s} \right] + O\left(\frac{s^2}{R'^4}\right) \quad (\text{A7})$$

This is the tidal acceleration across the *spring baseline* \mathbf{s} , not across r_{\oplus} . The tidal field is a gradient: it stretches space differentially, and the effect scales linearly with the baseline length. For a spring of length $\ell \sim 1 \text{ cm}$, the ratio to the Earth-radius tidal signal is:

$$\frac{a_{\text{tidal}}^{(\text{spring})}}{a_{\text{tidal}}^{(r_{\oplus})}} \sim \frac{\ell}{r_{\oplus}} \sim \frac{10^{-2}}{6.4 \times 10^6} \sim 1.6 \times 10^{-9} \quad (\text{A8})$$

The tidal acceleration across the spring is therefore $\sim 5 \times 10^{-7} \times 1.6 \times 10^{-9} \approx 10^{-15}$ m/s² — unmeasurably small by any current technology. This is the *only* external-body signal intrinsic to the spring in the three-point formulation. If a gravimeter observes tidal signals at 10⁻⁷ m/s², the signal must originate elsewhere. As the next subsection shows, it is transmitted through the constraint force \mathbf{N} .

6. Why the main text uses Earth's center

The reason gravimeters observe tidal signals at 10⁻⁷ m/s² and not at 10⁻¹⁵ m/s² is that the *constraint force* \mathbf{N} in Eq. (A4) transmits Earth's gravity gradient across the full radius r_{\oplus} to the housing. The spring then measures the difference between Earth's gravity at $\mathbf{d} + \mathbf{s}$ and at \mathbf{d} , which includes the r_{\oplus} -scale tidal contribution already encoded in \mathbf{N} .

Absorbing \mathbf{N} and the Earth gravity terms into the effective gravity g_0 at the housing reproduces Eq. (15) of the main text. The three-point formulation thus reduces to the two-point formulation: no information is lost and no new signal appears. The direct pull GM/R^2 cancels in both formulations, for the same reason — the housing and the test mass share the same free-fall acceleration.

The rotating-frame corrections of Sec. III carry over unchanged. Transforming to the co-rotating frame introduces centrifugal and Coriolis terms for both the test mass and the housing. In the equation for \mathbf{s} , these fictitious forces appear as the *difference* between their values at $\mathbf{R}_H + \mathbf{s}$ and at \mathbf{R}_H . Since both vary smoothly on the scale of r_{\oplus} , their difference across the spring baseline $|\mathbf{s}| \sim \ell$ is suppressed by the same factor $\ell/r_{\oplus} \sim 10^{-9}$ as the tidal terms — negligible. The Coriolis force $-2m\omega \times \dot{\mathbf{s}}$ vanishes identically for a static test mass ($\dot{\mathbf{s}} = \mathbf{0}$), just as in the main text. The full centrifugal correction at the housing location is, once again, absorbed into \mathbf{N} and hence into g_0 . No new rotational effect emerges in the three-point formulation.

Appendix B: Coordinate systems and the centrifugal projection

Section IV states that the centrifugal contribution to the sensor reading has no azimuthal dependence: it enters purely as a correction to $g_0 \cos \theta$, with no $\sin \theta \sin \varphi$ or $\sin \theta \cos \varphi$ term. This is not obvious *a priori* — the centrifugal acceleration has a horizontal component that could, in principle, project differently for different azimuths. The resolution lies in the choice of coordinate system: the ENU frame used in the main text is *geodetic*, not geocentric, and the geodetic vertical absorbs the horizontal centrifugal component by definition. This appendix makes the argument explicit.

1. The inertial (geocentric) frame

We begin in a non-rotating frame centered on the Earth, using spherical coordinates aligned with the rotation axis. At a point P on the surface at geocentric colatitude Θ (i.e. geocentric latitude $\lambda' = 90 - \Theta$), define a local geocentric triad:

$$\begin{aligned}\hat{\mathbf{U}}' &= \hat{\mathbf{r}} && \text{(radially outward from Earth's center)} \\ \hat{\mathbf{N}}' &= -\hat{\Theta} && \text{(northward along the meridian)} \\ \hat{\mathbf{E}}' &= \hat{\varphi}_{\text{long}} && \text{(eastward)}\end{aligned}\tag{B1}$$

In this frame, the gravitational acceleration of a spherically symmetric Earth is purely radial:

$$\mathbf{g}_{\text{grav}} = -g_{\text{grav}} \hat{\mathbf{U}}'\tag{B2}$$

with $g_{\text{grav}} = GM_E/r_{\oplus}^2 \approx 9.82$ m/s². There is no horizontal gravitational component: \mathbf{g}_{grav} has no $\hat{\mathbf{N}}'$ or $\hat{\mathbf{E}}'$ projection.

The centrifugal acceleration at P points radially outward from the rotation axis (not from Earth's center). The rotation axis lies in the local meridional plane, so the centrifugal vector has no eastward component but decomposes into radial and meridional parts:

$$\mathbf{a}_{\text{cf}} = \omega^2 r_{\oplus} \cos^2 \lambda' \hat{\mathbf{U}}' - \omega^2 r_{\oplus} \cos \lambda' \sin \lambda' \hat{\mathbf{N}}'\tag{B3}$$

The first term ($\hat{\mathbf{U}}'$) reduces the apparent weight; the second ($\hat{\mathbf{N}}'$) pushes equatorward. At $\lambda' = 48.4$ (Ulm), the horizontal component has magnitude $\omega^2 r_{\oplus} \cos \lambda' \sin \lambda' \approx 0.017$ m/s² — small compared to g_{grav} but three times larger than the Sun's direct pull $a_{\odot} \approx 5.93 \times 10^{-3}$ m/s².

Effective gravity in the geocentric frame. Combining Eqs. (B2) and (B3):

$$\mathbf{g}_{\text{eff}} = -(g_{\text{grav}} - \omega^2 r_{\oplus} \cos^2 \lambda') \hat{\mathbf{U}}' - \omega^2 r_{\oplus} \cos \lambda' \sin \lambda' \hat{\mathbf{N}}' \quad (\text{B4})$$

This vector does *not* point along $\hat{\mathbf{U}}'$: it is tilted toward the equator. If we were to project \mathbf{g}_{eff} onto a sensor axis parameterised in the geocentric triad, $\hat{\mathbf{n}}'(\theta, \varphi) = \sin \theta \cos \varphi \hat{\mathbf{E}}' + \sin \theta \sin \varphi \hat{\mathbf{N}}' + \cos \theta \hat{\mathbf{U}}'$, we would find:

$$\mathbf{g}_{\text{eff}} \cdot \hat{\mathbf{n}}' = -(g_{\text{grav}} - \omega^2 r_{\oplus} \cos^2 \lambda') \cos \theta - \omega^2 r_{\oplus} \cos \lambda' \sin \lambda' \sin \theta \sin \varphi \quad (\text{B5})$$

The $\sin \theta \sin \varphi$ term is the azimuthal centrifugal projection. It is largest when the sensor points North ($\varphi = 90^\circ$, $\theta = 90^\circ$) and vanishes when pointing East ($\varphi = 0^\circ$) or vertically ($\theta = 0^\circ$). Its amplitude ($\approx 0.017 \text{ m/s}^2$) would be clearly visible in any orientation sweep if the geocentric frame were used.

2. The rotating (geodetic) frame

A laboratory on Earth's surface does not use geocentric coordinates. The plumb line defines the local vertical, and instruments are levelled relative to it. The plumb line hangs along the direction of \mathbf{g}_{eff} , not along $\hat{\mathbf{r}}$. This motivates the geodetic triad:

$$\begin{aligned} \hat{\mathbf{U}} &= -\frac{\mathbf{g}_{\text{eff}}}{|\mathbf{g}_{\text{eff}}|} && \text{(along the plumb line, upward)} \\ \hat{\mathbf{N}} &\perp \hat{\mathbf{U}}, \text{ in the meridional plane} && \text{(geodetic northward)} \\ \hat{\mathbf{E}} &= \hat{\mathbf{E}}' && \text{(eastward, unchanged)} \end{aligned} \quad (\text{B6})$$

By construction, \mathbf{g}_{eff} is purely along $-\hat{\mathbf{U}}$:

$$\mathbf{g}_{\text{eff}} = -g_0 \hat{\mathbf{U}} \quad (\text{B7})$$

where $g_0 = |\mathbf{g}_{\text{eff}}|$. The horizontal centrifugal component has been absorbed into the *direction* of $\hat{\mathbf{U}}$ (tilted by the angle α from the geocentric radial) and into the *magnitude* g_0 .

The tilt angle between the geodetic and geocentric verticals is:

$$\alpha \approx \frac{\omega^2 r_{\oplus} \cos \lambda' \sin \lambda'}{g_{\text{grav}}} \approx 0.10 \quad \text{at } \lambda' = 45^\circ \quad (\text{B8})$$

This is the well-known *deflection of the vertical* due to Earth's rotation.

Sensor projection in the geodetic frame. Parameterising the sensor axis in the geodetic triad, $\hat{\mathbf{n}}(\theta, \varphi) = \sin \theta \cos \varphi \hat{\mathbf{E}} + \sin \theta \sin \varphi \hat{\mathbf{N}} + \cos \theta \hat{\mathbf{U}}$, the projection is simply:

$$\mathbf{g}_{\text{eff}} \cdot \hat{\mathbf{n}} = -g_0 \cos \theta \quad (\text{B9})$$

No azimuthal dependence survives. The centrifugal contribution is entirely contained in g_0 and appears only through $\cos \theta$.

3. Reconciling the two frames

Equations (B5) and (B9) describe the same physics in different coordinates. They are related by a rotation of the local triad by the angle α about the East axis:

$$\begin{pmatrix} \hat{\mathbf{U}} \\ \hat{\mathbf{N}} \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \hat{\mathbf{U}}' \\ \hat{\mathbf{N}}' \end{pmatrix} \quad (\text{B10})$$

The angles θ and φ defined relative to the geodetic triad differ from those defined relative to the geocentric triad by corrections of order $\alpha \sim 10^{-3}$ rad ($\sim 0.1^\circ$). In the geodetic frame, the horizontal centrifugal component vanishes identically, not approximately: it is absorbed exactly into the definition of the vertical. In the geocentric frame, the same information appears as an explicit $\sin \theta \sin \varphi$ term. The physics is identical; only the bookkeeping differs.

4. Why the geodetic frame is natural

Every laboratory-based sensor defines its orientation relative to the plumb line. A tiltmeter measures deviation from the geodetic vertical; a gravimeter is levelled to the plumb line; an accelerometer’s “zero” is set by the local g_0 . The geodetic ENU frame is therefore the natural coordinate system for the orientation sweep of Sec. IV.

In this frame, the centrifugal acceleration produces no azimuthal signature. It is absorbed completely into the magnitude g_0 that multiplies $\cos \theta$. The only azimuth-dependent terms in the sensor reading, Eq. (19), are the tidal components a'_E and a'_N , at the 10^{-7} m/s² level. The centrifugal horizontal component (≈ 0.017 m/s²) is real, but it is not a separate signal: it defines the direction the sensor calls “up.”

Appendix C: Implications for atomic transitions and mass defect

The cancellation theorem of Sec. II was derived for a classical spring sensor. Here we show that the same cancellation extends to quantum systems — in particular to atomic transitions, where the mass–energy equivalence $E = mc^2$ ties internal energy directly to gravitational coupling.

1. Mass defect, recoil, and gravitational coupling

An atomic transition changes both the internal state and the external momentum of the atom. Consider an atom initially at rest in the lab frame that absorbs a photon of wave vector \mathbf{k} . After the transition:

Internal state. The atom’s rest mass increases by the mass defect

$$\Delta m = \frac{E_e - E_g}{c^2} = \frac{\hbar\nu_0}{c^2} \quad (\text{C1})$$

where ν_0 is the transition frequency.

External state. The atom acquires a recoil momentum $\Delta\mathbf{p} = \hbar\mathbf{k}$, producing a recoil velocity

$$\mathbf{v}_r = \frac{\hbar\mathbf{k}}{m_e} \quad (\text{C2})$$

where $m_e = m_g + \hbar\nu_0/c^2$ is the excited-state mass. The recoil velocity is state-dependent through m_e : compared to the ground-state value $\hbar k/m_g$, it differs by the fractional amount $\hbar\nu_0/(m_g c^2) \sim 10^{-10}$ for typical optical transitions.

Both changes couple to gravity. By the equivalence principle, the gravitational force on an atom in state n with total mass–energy $m_n = m_0 + E_n/c^2$ is

$$\mathbf{F}_{\text{grav}}^{(n)} = -m_n \nabla \Phi = - \left(m_0 + \frac{E_n}{c^2} \right) \nabla \Phi \quad (\text{C3})$$

where Φ is the gravitational potential. The internal energy E_n enters the equation of motion on exactly the same footing as the rest mass m_0 . Meanwhile, after the recoil, the atom follows a trajectory governed by the same gravitational acceleration $-\nabla\Phi$ regardless of its internal state — only the initial velocity \mathbf{v}_r differs. Both the mass defect (what the atom weighs) and the recoil (where the atom goes) are subject to the same gravitational field.

2. Cancellation for internal and external degrees of freedom

Consider an atom at position \mathbf{x} in the field of the Earth and a distant external body of mass M at position \mathbf{R}_B . By Eq. (C3), the gravitational acceleration is

$$\mathbf{a}_{\text{grav}} = -\nabla\Phi = -\frac{GM_E}{r^2} \hat{\mathbf{r}} - \frac{GM}{|\mathbf{x} - \mathbf{R}_B|^3} (\mathbf{x} - \mathbf{R}_B) + \dots \quad (\text{C4})$$

Crucially, this acceleration is *independent of m_n* : the factor m_n in the gravitational force (C3) cancels against the m_n in Newton’s second law, $\mathbf{F} = m_n \mathbf{a}$. This is the equivalence principle at work. An excited atom and a ground-state atom fall with the same acceleration — their different masses do not produce different trajectories.

The cancellation theorem of Sec. II therefore applies to both degrees of freedom changed by a transition:

Mass defect (internal). The direct pull GM/R^2 of the external body vanishes from the relative acceleration between the atom and its environment (trap, lattice, or mirror), regardless of the internal state. The mass defect Δm couples to the same gravitational field as the rest mass, and that field undergoes the same cancellation.

Recoil trajectory (external). After absorbing a photon, the atom follows a parabolic arc $\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{v}_r t + \frac{1}{2}\mathbf{g} t^2$ in the gravitational field. The acceleration \mathbf{g} governing this arc is $g_0 \hat{z} + \mathbf{a}_{\text{tidal}}$ — it does not include the direct pull GM/R^2 , because the laboratory (and hence the laser source that defined \mathbf{v}_r) shares the same free-fall acceleration. The recoil sets the initial velocity; gravity steers the subsequent trajectory; and the cancellation theorem ensures that both are referenced to g_0 and tidal corrections only.

3. Gravitational redshift as the surviving effect

Although the direct pull cancels, the gravitational *potential* does affect the transition frequency. An atom at height h above a reference point experiences a gravitational redshift:

$$\frac{\Delta\nu}{\nu_0} = \frac{\Phi(h) - \Phi(0)}{c^2} = \frac{g_0 h}{c^2} \quad (\text{C5})$$

This is the standard result confirmed by Pound and Rebka (1960) and measured to parts in 10^{18} by modern optical lattice clocks. The gravitational acceleration entering Eq. (C5) is g_0 — the effective gravity including the centrifugal correction, but *not* the direct pull of the external body. The external body's potential $\Phi_{\text{ext}} = -GM/R$ can be enormous (e.g. $\Phi_{\text{ext}}/c^2 \approx -10^{-8}$ for the Sun), but it is spatially uniform across any terrestrial laboratory and therefore contributes no measurable frequency difference between two clocks at different heights.

The only contribution of the external body to the redshift is tidal. At a fixed location on Earth's surface, the tidal potential of the external body oscillates semi-diurnally with amplitude

$$\frac{\delta\Phi_{\text{tidal}}}{c^2} \sim \frac{GM r_{\oplus}^2}{R^3 c^2} \sim 2 \times 10^{-17} \quad (\text{C6})$$

as the Earth rotates through the tidal bulge. This is within reach of the best optical lattice clocks ($\sim 10^{-18}$ fractional uncertainty) and must be corrected for in high-precision clock comparisons. A further contribution of similar magnitude arises from the solid Earth tide, which physically displaces the clock by ~ 30 cm vertically, shifting its position in Earth's own potential by $g_0 \times 0.3 \text{ m}/c^2 \sim 3 \times 10^{-17}$. Both effects are routinely modeled as systematic corrections in precision metrology.

For a lab-scale height difference h , the tidal redshift is far smaller: $GM r_{\oplus} h/(R^3 c^2) \sim 3 \times 10^{-24}(h/1 \text{ m})$ — well beyond current sensitivity.

In all cases, the tidal redshift is the spectroscopic counterpart of the tidal acceleration in Eq. (6); the direct pull GM/R^2 has no counterpart, for the same reason as in the classical case.

4. Implications for precision measurements

The cancellation has concrete consequences for precision experiments that use atoms as test masses. In each case, the sensor realises the three-point geometry of Appendix A, so the direct pull GM/R^2 of the external body drops out of the observable; only g_0 and tidal terms survive.

Optical lattice clocks. No state-dependent force from the external body perturbs the transition frequency. The lattice clock maps directly onto the three-point formulation: housing = lattice mirrors, spring = trapping potential, test mass = atom. The oscillation amplitude $\Delta x \sim 10\text{-}100 \text{ nm}$ suppresses tidal effects by $\Delta x/r_{\oplus} \sim 10^{-14}$ relative to the r_{\oplus} -scale signal, and the mass defect $\Delta m = \hbar\nu/c^2$ couples only to this already-negligible residual.

Atom interferometers. The measured acceleration g in $\Delta\phi = k_{\text{eff}} g T^2$ does not contain GM/R^2 . The atom is in free fall while the retroreflection mirror is fixed to the ground — a three-point geometry with no spring. Each beam-splitter pulse imparts recoil $\hbar k_{\text{eff}}$ and changes the internal state, creating two trajectories that share the same acceleration $g_0 + a_{\text{tidal}}$. The mass defect enters through the state-dependent recoil, Eq. (C2): the two arms carry momenta $\hbar k_{\text{eff}}/m_g$ vs. $\hbar k_{\text{eff}}/m_e$, producing a differential phase of order $\hbar\nu/(mc^2) \sim 10^{-10}$ per fringe that couples to g_0 alone and has been used to test $E = mc^2$ at the atomic scale.

Tests of the equivalence principle. An equivalence-principle violation at level η would couple internal energy to gravity differently from rest mass, producing a state-dependent frequency shift modulated at the orbital period. For the Sun, Earth's orbital eccentricity ($e \approx 0.017$) varies the external-body potential by $\delta\Phi_{\text{ext}}/c^2 \sim e GM/(Rc^2) \approx 1.7 \times 10^{-10}$, so a violation would appear as a fractional frequency modulation $\sim \eta \cdot 1.7 \times 10^{-10}$. Multi-year comparisons

of different atomic species (Rb/Cs microwave fountains, Sr and Yb optical lattice clocks) have searched for this annual modulation and currently constrain $\eta \lesssim 10^{-6}\text{--}10^{-7}$, confirming that the mass defect gravitates universally at this level. Next-generation optical clocks at 10^{-19} stability could push toward $\eta \sim 10^{-9}$.

5. Summary

The cancellation of the direct gravitational pull is not limited to classical springs. It extends to any local measurement of gravitational coupling, including atomic transitions. Both consequences of a transition — the mass defect $\Delta m = \hbar\nu/c^2$ (internal) and the recoil $\Delta\mathbf{p} = \hbar\mathbf{k}$ (external) — couple to gravity through the same field that undergoes cancellation. Both internal states fall identically in the external body's field, recoiling atoms follow arcs governed by g_0 alone, and no local experiment — spring, clock, or interferometer — can detect the direct pull GM/R^2 . What survives, as in the classical case, is the tidal field and the gravitational redshift due to the local potential gradient g_0 . The equivalence principle ensures that these are the only gravitational effects accessible to a terrestrial laboratory.

Appendix D: Frame hierarchy: what can be transformed away

The cancellation of the direct pull of external bodies (Theorem 1) removes GM/R^2 from the sensor reading. But it is natural to ask: is this the end of the story, or can further gravitational contributions be eliminated by an appropriate choice of reference frame? The answer reveals a hierarchy of physical content.

1. Three levels of elimination

Consider a test mass at position \mathbf{r} relative to Earth's center, subject to the local effective gravity g_0 and the tidal field $\mathbf{a}_{\text{tidal}}$. Different frame choices strip away different layers of the gravitational environment:

Level 1: Geocentric frame. Transforming from the inertial frame to one co-moving with Earth's center eliminates the direct pull GM/R^2 of any external body. This is the cancellation of Sec. II. What remains: g_0 (Earth's gravity plus centrifugal) and the tidal field from all external bodies.

Level 2: Freely falling frame at the test mass. Transforming to a frame in free fall at the test mass location further eliminates g_0 itself. In this frame, the “gravitational force” mg_0 disappears — it was a fictitious force arising from the non-inertial character of the laboratory. What remains: only the tidal field, i.e. the gradient of gravity across the finite extent of the apparatus.

Level 3: No further elimination. The tidal field cannot be removed by any local frame choice. It is encoded in the Riemann curvature tensor $R^\mu_{\nu\rho\sigma}$, which is a tensor: if it is nonzero in one frame, it is nonzero in all frames. Tidal effects are the irreducible gravitational observable.

This hierarchy is the equivalence principle expressed operationally. Each level eliminates a uniform contribution by recognizing it as indistinguishable from an acceleration. Only the non-uniform part — curvature — is frame-independent and therefore physical.

2. Spring-based sensors

A spring sensor holds the test mass at rest in the laboratory. The test mass is *not* freely falling: the spring force provides a proper acceleration g_0 that keeps it off its natural geodesic. The sensor reading is precisely this proper acceleration.

In the freely falling frame (Level 2), the physics is transparent: the laboratory accelerates upward at g_0 and drags the spring mount with it. The test mass, if released, would remain at rest (on a geodesic). The spring force is what prevents this. It measures the non-gravitational acceleration of the housing, not a “gravitational pull” on the mass. The only gravitational information accessible to the spring is the tidal correction — the difference in free-fall acceleration between the housing and the test mass location.

3. Optical lattice clocks

An atom trapped in a lattice potential is the quantum analog of the spring sensor. The atom is held off its geodesic by the trapping potential, which provides a proper acceleration g_0 . In the freely falling frame at the atom, g_0 vanishes

and the lattice potential simply confines the atom in a non-gravitational well.

The transition frequency is affected by gravity through the redshift $\Delta\nu/\nu = g_0 h/c^2$, where h is the height within the lattice. In the freely falling frame, this redshift is reinterpreted as a Doppler shift: the lattice (co-accelerating with the lab) moves relative to the local inertial frame, producing the same frequency difference. The physics is identical; only the description changes. The tidal contribution to the redshift ($\sim GM r_\oplus^2/R^3 c^2 \sim 10^{-17}$) survives in all frames.

4. Atom interferometers

Atom interferometers occupy a unique position in this hierarchy: the atom *is* in free fall between laser pulses. It follows a geodesic. In its rest frame (Level 2), the atom feels no gravitational force at all — neither GM/R^2 (eliminated at Level 1) nor g_0 (eliminated at Level 2).

The interferometer phase $\Delta\phi = k_{\text{eff}} g T^2$ does not arise from a force on the atom. It arises because the *mirror* is not in free fall: the mirror accelerates at g_0 upward (held by the Earth's crust), and each successive laser pulse is emitted from a different position in the atom's freely falling frame. The accumulated phase shift encodes the relative acceleration between the freely falling atom and the non-inertial mirror. In this picture, g is the proper acceleration of the mirror, not a force on the atom.

The tidal field enters as a correction: the atom's geodesic and the mirror's worldline are separated by a distance that grows during the interrogation time, and the tidal gradient across this baseline produces a small deviation from the ideal $k_{\text{eff}} g T^2$ scaling. For current atom interferometers with baselines of order meters, this tidal correction is of order $a_{\text{tidal}} \cdot T^2 \sim 10^{-7} \times T^2$ m — detectable in principle with long interrogation times but far below the dominant $g_0 T^2$ signal.

5. Summary: what is real

| Contribution | Eliminated by | Physical? |
|---|----------------------|-----------|
| Direct pull GM/R^2 | Geocentric frame | No |
| Local gravity g_0 | Freely falling frame | No* |
| Tidal field $\mathbf{a}_{\text{tidal}}$ | Nothing (curvature) | Yes |

*The reading g_0 of a spring sensor or lattice clock is real as a measurement of proper acceleration (the sensor's deviation from geodesic motion), but it is not a measurement of a gravitational field. A freely falling observer at the same location would measure zero. Only the tidal field is unambiguously gravitational: it cannot be mimicked or removed by any choice of motion.

This hierarchy underscores the central message: the direct pull of a distant body is not merely “canceled” by a fortunate subtraction. It is *unphysical* in the same sense that g_0 is unphysical — both are frame-dependent artifacts that can be transformed away. The only difference is that g_0 requires a freely falling frame to eliminate, while the direct pull is already absent in the geocentric frame. What remains in all frames and for all sensors — spring, clock, or interferometer — is the tidal field, the sole irreducible gravitational observable.