Weinberg QFT Exercise

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2.5

A massive particle in $1+2$ spacetime has standard momentum:

$$k^\mu = (0, 0, M) \tag{0.0.1}$$

The little group is $SO(2)$. Under Lorentz transform Λ, an arbitrary state $\Psi_{p,\sigma}$ (could be a multi-particle state) transforms as:

$$U(\Lambda)\Psi_{p,\sigma} = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda, p))\Psi_{\Lambda p,\sigma'} \tag{0.0.2}$$

The matrices $D_{\sigma'\sigma}(W)$ are given by a unitary representation of $SO(2)$:

$$D : SO(2) \to SU(N) \tag{0.0.3}$$

We shall parametrize $SO(2)$ by the rotation angle θ. To find the spin states of a *single* particle, we should find the irreducible representations of $SO(2)$. Since $SO(2)$ is 1-dimensional, we can write D for the infinitesimal rotation $R(\theta)$ as:

$$D = 1 + i\theta t + \mathcal{O}(\theta^2) \tag{0.0.4}$$

Here, t is a Hermitian matrix, so we can use spectral decomposition to find an orthonormal basis in which it is diagonal. We can thus assume WLOG that t is diagonal: (by redefining $\Psi_{p,\sigma}$ using this orthonormal basis)

$$t = \mathrm{diag}(n_1, n_2, \ldots, n_N) \tag{0.0.5}$$

So:

$$D(R(\theta)) = \exp(i\theta \mathrm{diag}(n_1, n_2, \ldots, n_N)) = \mathrm{diag}(e^{in_1\theta}, e^{in_2\theta}, \ldots, e^{in_N\theta}) \tag{0.0.6}$$

The representation is decomposed into a direct sum of N 1-dimensional representations. An (1-dimensional) irreducible representation has the form:

$$D(R(\theta)) = e^{in\theta}, \tag{0.0.7}$$

where n is an integer. Under Lorentz transform, a single particle transforms as:

$$U(\Lambda)\Psi_{p,n} = \sqrt{\frac{(\Lambda p)^0}{p^0}} e^{in\theta(W(\Lambda,p))}\Psi_{\Lambda p,n} \tag{0.0.8}$$

Note that σ is replaced with n.

Now we consider the transformation of $\Psi_{k,n}$ under parity P and time-reversal T. We know:

$$\begin{aligned} HP\Psi_{k,n} &= PHP^{-1}P\Psi_{k,n} = PH\Psi_{k,n} = MP\Psi_{k,n} \\ \mathbf{P}P\Psi_{k,n} &= -P\mathbf{P}P^{-1}P\Psi_{k,n} = P\mathbf{P}\Psi_{k,n} = 0 \\ tP\Psi_{k,n} &= P(P^{-1}J^{12}P)\Psi_{k,n} = Pt\Psi_{k,n} = nP\Psi_{k,n} \end{aligned} \tag{0.0.9}$$

If the particle with $H = M$, $\mathbf{P} = 0$ and $t = n$ has no degeneracy, then $\Psi_{k,n}$ and $P\Psi_{k,n}$ are the same state, and:

$$\begin{aligned} &P\Psi_{k,n} = \eta\Psi_{k,n} \\ &P\Psi_{p,n} = \sqrt{M/p^0}(PU(L(p))P^{-1})P\Psi_{k,n} = \sqrt{M/p^0}U(L(\mathcal{P}p))P\Psi_{k,n} = \eta\Psi_{\mathcal{P}p,n} \end{aligned} \tag{0.0.10}$$

where $|\eta| = 1$ is some phase. Similarly:

$$\begin{aligned} HT\Psi_{k,n} &= MT\Psi_{k,n} \\ \mathbf{P}T\Psi_{k,n} &= 0 \\ tT\Psi k, n &= -nT\Psi_{k,n} \end{aligned} \tag{0.0.11}$$

So:

$$\begin{aligned} &T\Psi_{k,n} = \zeta\Psi_{k,-n} \\ &T\Psi_{p,n} = \sqrt{M/p^0}(TU(L(p))T^{-1})T\Psi_{k,n} = \sqrt{M/p^0}U(L(\mathcal{P}p))T\Psi_{k,n} = \zeta\Psi_{\mathcal{P}p,-n} \end{aligned} \tag{0.0.12}$$

where $|\zeta| = 1$ is some phase. However, this phase has no physical significance since it can be eliminated by redefining $\Psi_{k,n}$:

$$T\zeta^{1/2}\Psi_{k,n} = \zeta^{*1/2}T\Psi_{k,n} = \zeta^{1/2}\Psi_{k,n} \tag{0.0.13}$$

3.4

To show (3.5.8) is equivalent to (3.5.3), we need to evaluate $S_{\beta\alpha}$ from (3.5.3):

$$
\begin{aligned}
S_{\beta\alpha} =&(\Phi_\beta, S\Phi_\alpha) \\
=&(\Phi_\beta, \Phi_\alpha) + (-i)\int_{-\infty}^{\infty} \mathrm{d}t_1\, (\Phi_\beta, V(t_1)\Phi_\alpha) + (-i)^2 \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2\, (\Phi_\beta, V(t_1)V(t_2)\Phi_\alpha) + ...
\end{aligned}
\tag{0.0.14}
$$

The term 0-th order in V is simply:

$$
(\Phi_\beta, \Phi_\alpha) = \delta(\beta - \alpha) \tag{0.0.15}
$$

To evaluate the 1-st order term, we first show:

$$
(\Phi_\beta, V(t)\Phi_\alpha) = (\Phi_\beta, e^{iH_0t}Ve^{-iH_0t}\Phi_\alpha) = (e^{-iH_0t}\Phi_\beta, Ve^{-iH_0t}\Phi_\alpha) = e^{i(E_\beta - E_\alpha)t}V_{\beta\alpha} \tag{0.0.16}
$$

So:

$$
(-i)\int_{-\infty}^{\infty} \mathrm{d}t_1\, (\Phi_\beta, V(t_1)\Phi_\alpha) = (-i)\int_{-\infty}^{\infty} \mathrm{d}t_1\, e^{i(E_\beta - E_\alpha)t_1}V_{\beta\alpha} = (-2\pi i)\delta(E_\beta - E_\alpha)V_{\beta\alpha} \tag{0.0.17}
$$

For the 2-nd order term, we can use the completeness of the Φ_α's to show:

$$
(\Phi_\beta, V(t_1)V(t_2)\Phi_\alpha) = \int \mathrm{d}\gamma\, (\Phi_\beta, V(t_1)\Phi_\gamma)(\Phi_\gamma, V(t_2)\Phi_\alpha) \tag{0.0.18}
$$

So:

$$
\begin{aligned}
&(-i)^2 \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2\, (\Phi_\beta, V(t_1)V(t_2)\Phi_\alpha) \\
=&(-i)^2 \int \mathrm{d}\gamma \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2\, e^{i(E_\beta - E_\gamma)t_1} e^{i(E_\gamma - E_\alpha)t_2} V_{\beta\gamma}V_{\gamma\alpha}
\end{aligned}
\tag{0.0.19}
$$

Now we do a change of integration variables in the form of:

$$
\tau = t_1 - t_2, \quad t_2' = t_2 \tag{0.0.20}
$$

Then the integral becomes:

$$\begin{aligned}
&(-i)^2 \int \mathrm{d}\gamma \int_0^\infty \mathrm{d}\tau \int_{-\infty}^\infty \mathrm{d}t_2\, e^{i(E_\beta - E_\gamma)(\tau + t_2)} e^{i(E_\gamma - E_\alpha)t_2} V_{\beta\gamma} V_{\gamma\alpha} \\
=&(-i)^2 \int \mathrm{d}\gamma \int_0^\infty \mathrm{d}\tau\, 2\pi\delta(E_\beta - E_\alpha) e^{i(E_\beta - E_\gamma)\tau} V_{\beta\gamma} V_{\gamma\alpha} \\
=&(-2\pi i)\delta(E_\beta - E_\alpha) \int \mathrm{d}\gamma\, (-i) \int_0^\infty \mathrm{d}\tau\, e^{i(E_\alpha - E_\gamma)\tau} V_{\beta\gamma} V_{\gamma\alpha} \\
=&(-2\pi i)\delta(E_\beta - E_\alpha) \int \mathrm{d}\gamma\, \frac{V_{\beta\gamma} V_{\gamma\alpha}}{E_\alpha - E_\gamma + i\epsilon}
\end{aligned} \tag{0.0.21}$$

where we used (3.5.9) in the last equality. To evaluate the higher order terms, we first use the completeness relation for Φ_α's (as in the case for the 2-nd order term) to expand $(\Phi_\beta, V(t_1)...V(t_n)\Phi_\alpha)$:

$$(\Phi_\beta, V(t_1)...V(t_n)\Phi_\alpha) = \int \mathrm{d}\gamma_1 ... \int \mathrm{d}\gamma_{n-1}\, e^{i(E_\beta - E_{\gamma_1})t_1} ... e^{i(E_{\gamma_{n-1}} - E_\alpha)t_n} V_{\beta\gamma_1} ... V_{\gamma_{n-1}\alpha} \tag{0.0.22}$$

And then evaluate the $t_1...t_n$ integral by doing a change of variables:

$$\begin{aligned}
&\tau_i = t_i - t_{i+1} \quad \text{for} \quad i \leq n-1 \\
&t'_n = t_n
\end{aligned} \tag{0.0.23}$$

The procedure is similar to that for the 2-nd order term. Combining the results, we get:

$$S_{\beta\alpha} = \delta(\beta - \alpha) - 2\pi i \delta(E_\beta - E_\alpha) \Big[V_{\beta\alpha} + \int \mathrm{d}\gamma\, \frac{V_{\beta\gamma} V_{\gamma\alpha}}{E_\alpha - E_\gamma + i\epsilon} + ... \Big] \tag{0.0.24}$$

The term in the bracket is simply $T^+_{\beta\alpha}$. We can see that it is consistent with (3.5.3).

4.1

We can modify (4.3.2) to write:

$$\begin{aligned}
S_{\beta\alpha} &= \sum_{\text{PART}} S^C_{\beta_1\alpha_1} S^C_{\beta_2\alpha_2} ... \\
&= \sum_{n=1}^{\infty} \sum_{\substack{\text{PART} \\ \text{in n}}} S^C_{\beta_1\alpha_1} ... S^C_{\beta_n\alpha_n}
\end{aligned} \tag{0.0.25}$$

Here, PART in n is the sum over different partitions with n parts. (see (4.3.4), (4.3.5), (4.3.6)) Now we have:

$$
\begin{aligned}
& F[v] \\
= & 1+\sum_{N=1}^{\infty}\sum_{M=1}^{\infty}\frac{1}{N!M!}\int v^*(q_1')...v^*(q_N')v(q_1)...v(q_M) \\
& S_{q_1'...q_N',q_1...q_M}\,\mathrm{d}q_1'\,...\,\mathrm{d}q_N'\,\mathrm{d}q_1\,...\,\mathrm{d}q_M \\
= & 1+\sum_{N=1}^{\infty}\sum_{M=1}^{\infty}\frac{1}{N!M!}\int v^*(q_1')...v^*(q_N')v(q_1)...v(q_M) \\
& \sum_{n=1}^{\infty}\sum_{\substack{\text{PART}\\ \text{in n}}} S^C_{\beta_1\alpha_1}...S^C_{\beta_n\alpha_n}\,\mathrm{d}q_1'\,...\,\mathrm{d}q_N'\,\mathrm{d}q_1\,...\,\mathrm{d}q_M
\end{aligned} \tag{0.0.26}
$$

Here, we have expanded $S_{q_1'...q_N',q_1...q_M}$ in terms of $S^C_{\beta_1\alpha_1}...S^C_{\beta_n\alpha_n}$. The β_i's give a partition for $q_1'...q_N'$, and similarly for the α_i's. If the size of β_i is N_i and the size of α_i is M_i, then we can write the sum over partitioning as a sum over different sizes N_i and M_i, satisfying:

$$N_1+\ldots+N_n=N, \quad M_1+\ldots+M_n=M. \tag{0.0.27}$$

The number of partitionings of q_i''s and q_i's of the a particular size N_i and M_i is given by:

$$\frac{N!}{N_1!...N_n!}, \quad \frac{M!}{M_1!...M_n!}, \tag{0.0.28}$$

respectively. Since we're integrating over all the q_i''s and q_i's, the contribution from all terms of a particular partitioning size is the same, so we must add the above factors inside the summation of the N_i and M_i's.

Moreover, since we are interested in distinct partitionings, a partitioning $\beta_i\alpha_i$ should be treated as the same as $\beta_{k_i}\alpha_{k_i}$, where k_i is some permutation of $\{1,...n\}$. To avoid over counting, we should divided the sum by $n!$.

Continuing with our previous calculation, we have:

$$
\begin{aligned}
& 1+\sum_{N=1}^{\infty}\sum_{M=1}^{\infty}\frac{1}{N!M!}\int v^*(q_1')...v^*(q_N')v(q_1)...v(q_M) \\
& \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{N_1+...N_n=N}\sum_{M_1+...M_n=M}\frac{N!}{N_1!...N_n!}\frac{M!}{M_1!...M_n!} \\
& S^C_{q_1'...q_{N_1}',q_1...q_{M_1}}...S^C_{q_{N-N_n+1}'...q_N',q_{M-M_n+1}...q_M}\,\mathrm{d}q_1'\,...\,\mathrm{d}q_N'\,\mathrm{d}q_1\,...\,\mathrm{d}q_M \\
= {} & 1+\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{N=n}^{\infty}\sum_{M=n}^{\infty}\sum_{N_1+...N_n=N}\sum_{M_1+...M_n=M}\frac{1}{N_1!...N_n!}\frac{1}{M_1!...M_n!} \\
& \int v^*(q_1')...v^*(q_N')v(q_1)...v(q_M)S^C_{q_1'...q_{N_1}',q_1...q_{M_1}}...S^C_{q_{N-N_n+1}'...q_N',q_{M-M_n+1}...q_M}\,\mathrm{d}q_1'\,...\,\mathrm{d}q_N'\,\mathrm{d}q_1\,...\,\mathrm{d}q_M \\
= {} & 1+\sum_{n=1}^{\infty}\frac{1}{n!}(\prod_{k=1}^{n}\sum_{N_k=1}^{\infty})(\prod_{k=1}^{n}\sum_{M_k=1}^{\infty})\frac{1}{N_1!...N_n!}\frac{1}{M_1!...M_n!} \\
& \int v^*(q_1')...v^*(q_N')v(q_1)...v(q_M)S^C_{q_1'...q_{N_1}',q_1...q_{M_1}}...S^C_{q_{N-N_n+1}'...q_N',q_{M-M_n+1}...q_M}\,\mathrm{d}q_1'\,...\,\mathrm{d}q_N'\,\mathrm{d}q_1\,...\,\mathrm{d}q_M \\
= {} & 1+\sum_{n=1}^{\infty}\frac{1}{n!}\prod_{k=1}^{n}\sum_{N_k=1}^{\infty}\sum_{M_k=1}^{\infty}\frac{1}{N_k!M_k!} \\
& \int v^*(q_1')...v^*(q_{N_k}')v(q_1)...v(q_{M_k})S^C_{q_1'...q_{N_k}',q_1...q_{M_k}}\,\mathrm{d}q_1'\,...\,\mathrm{d}q_{N_k}'\,\mathrm{d}q_1\,...\,\mathrm{d}q_{M_k} \\
= {} & 1+\sum_{n=1}^{\infty}\frac{1}{n!}(F^C[v])^n \\
= {} & \exp\big\{F^C[v]\big\}
\end{aligned}
\tag{0.0.29}
$$

So the desired formula is $F[v]=\exp\big\{F^C[v]\big\}$.

5.4

A type $(A, A+j)$ field for a massless particle of spin j' satisfy:

$$|A-(A+j)|\leq j'\leq 2A+j$$

We see that j' can go from j to $2A+j$. For each j', we know from (5.9.41) that the field can be formed only from annihilation operators of helicity σ, where

$$\sigma=(A+j)-A=j.$$

The total number of degrees of freedom here is:

$$(2A + j) - j + 1 = 2A + 1$$

Now consider the traceless $2A$-th derivatives of a massless type $(0, j)$ field ϕ_σ:

$$\{\partial_{\mu_1} \dots \partial_{\mu_{2A}}\}\phi_\sigma$$

This object tranforms with a representation $(A, A) \otimes (0, j)$, where the (A, A) representation comes from the traceless derivatives and the $(0, j)$ is the transformation of the field. Again, since σ is fixed by (5.9.41), for each $0 \leq j \leq 2A$ there is 1 degree of freedom, and there is 1 degree of freedom for $(0, j)$. The total number of degree of freedom is:

$$(2A + 1) \times 1 = 2A + 1$$

Since we have obtained the same number of dof for a general (and unique) type $(A, A + j)$ massless field, we know that all such fields must be formed from traceless derivatives of type $(0, j)$ field. The argument is similar for type $(B + j, B)$ fields.

5.5

For P, we can use result (5.7.43):

$$P\psi^{AB}_{ab}(x)P^{-1} = \eta^*(-1)^{A+B-j}\psi^{BA}_{ba}(-\mathbf{x}, x^0)$$

The $(j, 0)$ component of the field transforms like:

$$P\psi^{j0}_{ab}(x)P^{-1} = \eta^*\psi^{0j}_{ba}(-\mathbf{x}, x^0)$$

Since $\sigma = 0 - j = -j = a + b$, the only values for a and b are:

$$a = -j, \quad b = 0.$$

So the simplified transformation is:

$$P\psi^{j0}_{-j,0}(x)P^{-1} = \eta^*\psi^{0j}_{0,-j}(-\mathbf{x}, x^0)$$

Similarly, the $(0, j)$ component transforms like:

$$P\psi^{0j}_{0,j}(x)P^{-1} = \eta^*\psi^{j0}_{j,0}(-\mathbf{x}, x^0)$$

Note that the $(j,0)$ component is transformed into the $(0,j)$ component, and vice versa. The full transformation is thus:

$$P\psi(x)P^{-1} = \eta^*\beta\psi(-\mathbf{x},0)$$

where

$$\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

switches the two components.

Similarly, for C we use (5.7.52):

$$\begin{aligned} C\psi^{AB}_{ab}(x)C^{-1} =& \xi^*(-1)^{-2A-a-b-j}\psi^{BA\dagger}_{-b-a}(x) \\ C\psi^{j0}_{-j0}(x)C^{-1} =& \xi^*(-1)^{-2j-(-j)-0-j}\psi^{0j\dagger}_{0,j}(x) = \xi^*(-1)^{2j}\psi^{0j\dagger}_{0,j}(x) \\ C\psi^{0j}_{0j}(x)C^{-1} =& \xi^*(-1)^{-0-j-j}\psi^{j0\dagger}_{-j,0}(x) = \xi^*(-1)^{2j}\psi^{j0\dagger}_{-j,0}(x) \end{aligned}$$

For T, we use (5.7.60):

$$\begin{aligned} T\psi^{AB}_{ab}(x)T^{-1} =& (-1)^{a+b+A+B-2j}\zeta^*\psi^{AB}_{-a-b}(\mathbf{x},-x^0) \\ T\psi^{j0}_{-j,0}(x)T^{-1} =& (-1)^{-j+0+j+0-2j}\zeta^*\psi^{j0}_{j,0}(\mathbf{x},-x^0) = (-1)^{2j}\zeta^*\psi^{j0}_{j,0}(\mathbf{x},-x^0) \\ T\psi^{0j}_{0,j}(x)T^{-1} =& (-1)^{0+j+0+j-2j}\zeta^*\psi^{0j}_{0,-j}(\mathbf{x},-x^0) = \zeta^*\psi^{0j}_{0,-j}(\mathbf{x},-x^0) \end{aligned}$$