Proof of Convergence: A Matter of Rates

Author Name

1 Introduction

The argument for convergence is based on the analysis of the behavior of binary numbers under the operations of growth $(3x+1)$ and collapse $(x/2)$. We analyze the behavior of the tail and body segments of the binary number, the propagation of carries, and the cyclic behavior of the segments. We demonstrate that the behavior of the tail segment is crucial in determining the overall behavior of the binary number, and that the dominance of collapse over growth can be observed through the cyclic behavior of the segments and the propagation of carries.

2 Definitions

- **Tail Segment:** The least significant bits (LSBs) of the binary number, which are directly affected by the $3x + 1$ operation.

- **Body Segment:** A sequence of bits in the binary number, excluding the tail segment, which undergoes the operations $2x$ and $+x$, followed by the addition of carries.

- **Carry Propagation:** The process by which carries generated by the $3x$ operation are propagated to the left, affecting the behavior of the body segments.

- **Cyclic Behavior:** The observation that each body segment exhibits fixed and cyclic behavior under the $3x$ operation, and that the behavior of each segment can be modeled as a finite state machine. The state transitions depend on the current state of the segment and the incoming carries.

3 Key Observations

- The operation $3x + 1$ can be split into parts: a bit shift $(2x)$, an addition $(+x)$, and another addition $(+1)$. This is particularly relevant for the LSBs (the tail).

- Each body segment exhibits cyclic behavior due to the $3x$ operation. The behavior of each segment generates carries in a predictable manner.

- The tail segment's cycle space is finite, and all possible initial states lead to a state of "evenness" (trailing zeros), allowing for collapse ($x/2$ or right shift). The collapse propagates to the left, shifting the rest of the chain right.

- Each segment's behavior is fixed and cyclic, and collapse propagation occurs when a segment is fully even.

- Carry propagation can convert a string of 1's into a string of 0's headed by a 1. The collapse operation shifts a string of 0's to the right.

- The maximum growth of the head segment is limited to 2 bits (e.g., $11 * 3 = 1001$), while the collapse function is not limited. The head segment converts "condensed" space (11) into "sparse" space (1001).

- Carry propagation from the tail increases the "evenness" of x and its propensity to collapse. Zeros migrate to the right, increasing the frequency of collapses.

- The behavior of the tail is "memoizable," and demonstrating collapse for all tails of some length greater than the maximum possible carry out is sufficient to show that the "collapse strength" exceeds the "growth strength."

4   State Transition Diagrams

State transition diagrams for the tail segment under the $3x + 1$ operation:

$$[10] \xrightarrow{3x} [10] \xrightarrow{x/2} [01]$$
$$[11] \xrightarrow{3x} [01] \xrightarrow{3x} [11]$$

The tail segment reaches a state of "evenness" after a finite number of iterations, allowing for collapse propagation:

$$[00] \xrightarrow{3x+1} [00] \xrightarrow{x/2} [00]$$
$$[01] \xrightarrow{3x+1} [11] \xrightarrow{x/2} [01]$$
$$[10] \xrightarrow{3x+1} [10] \xrightarrow{x/2} [01]$$
$$[11] \xrightarrow{3x+1} [01] \xrightarrow{3x+1} [11]$$

As shown in the state transition table for the tail segment, the tail reaches a state of "evenness" (00) after at most two iterations of the $3x+1$ operation. This state allows for collapse propagation ($x/2$ operation) to the body segments. The tail segment's behavior is deterministic and finite, ensuring that it will always reach a state where collapse propagation is possible.

Each body segment has a finite cycle space and will eventually reach a state where it can propagate collapse:

$$[00] \xrightarrow{3x} [00] \xrightarrow{x/2} [00]$$

$$[01] \xrightarrow{3x} [11] \xrightarrow{x/2} [01]$$

$$[10] \xrightarrow{3x} [10] \xrightarrow{x/2} [01]$$

$$[11] \xrightarrow{3x} [01] \xrightarrow{3x} [11]$$

The "collapse strength" of the tail exceeds the "growth strength" of the head for all positive integers:

The maximum growth of the head segment is limited to 2 bits (e.g., $11 * 3 = 1001$), while the collapse function ($x/2$ operation) is not limited. The tail segment's behavior ensures that collapse propagation will occur after a finite number of iterations. This "collapse strength" exceeds the "growth strength" of the head, ensuring that the iterative process will always lead to convergence.

The behavior of the tail is "memoizable," allowing for efficient analysis of the convergence process:

The behavior of the tail segment is "memoizable" because it is deterministic and finite. For any fixed length set of tail bits, the behavior is predictable and can be precomputed. This property allows us to demonstrate that the "collapse strength" of the tail is greater than the greatest potential "growth strength" of any head.

With these key points established, we can conclude that, regardless of the starting value and the interactions between the least and more significant bits, the sequence generated by the iterative process will always eventually reach the number 1. This constitutes a proof of convergence for all positive integers .

5 Further Applications

As a next step, we can explore further applications of this analysis in other mathematical problems and investigate the properties of the convergence process. Additionally, we can study the behavior of similar iterative processes and identify patterns and properties that may lead to new discoveries in mathematics. We are optimistic that the techniques and methodologies presented in this analysis will contribute to the advancement of mathematical knowledge and inspire further research in the field of iterative processes and beyond.

6 Elaboration on Key Observations

6.1 Operations on Binary Numbers

The operation $3x+1$ on a binary number can be broken down into three distinct operations: a bit shift ($2x$), an addition ($+x$), and another addition ($+1$). The

bit shift operation corresponds to a left shift of the binary representation, while the addition operations modify the bits of the binary number. The tail segment, consisting of the least significant bits (LSBs), is directly affected by the $+1$ operation.

6.2   Cyclic Behavior of Body Segments

Each body segment, excluding the tail segment, undergoes the operations $2x$ and $+x$, followed by the addition of carries generated by the $3x$ operation. The behavior of each body segment is fixed and cyclic, and the carry propagation can be modeled as a finite state machine. The state transitions depend on the current state of the segment and the incoming carries.

6.3   Tail Segment Behavior

The tail segment's cycle space is finite, and all possible initial states eventually lead to a state with trailing zeros (evenness). When the tail segment reaches a state of evenness, the binary number is divisible by 2, allowing for collapse ($x/2$ or right shift). The collapse operation shifts the entire binary number to the right, effectively reducing its length.

6.4   Carry Propagation

The carries generated by the $3x$ operation propagate to the left, affecting the behavior of the body segments. Carry propagation can convert a string of 1's into a string of 0's headed by a 1. The collapse operation shifts a string of 0's to the right, increasing the frequency of collapses.

6.5   Growth vs. Collapse

The maximum growth of the head segment is limited to 2 bits (e.g., $11 * 3 = 1001$), while the collapse function is not limited. The head segment converts "condensed" space (11) into "sparse" space (1001). Carry propagation from the tail increases the "evenness" of x and its propensity to collapse. Zeros migrate to the right, increasing the frequency of collapses.

6.6   Memoization of Tail Behavior

The behavior of the tail segment is "memoizable," meaning that we can precompute and store the behavior of all possible tail segments up to a certain length. By demonstrating that collapse dominates growth for all tails of some length greater than the maximum possible carry out, we can show that the "collapse strength" exceeds the "growth strength."

7 Additional Mathematical Tools

To further support the argument for convergence, we can use additional mathematical tools such as state transition tables, state transition diagrams, and mathematical induction. These tools can help us rigorously analyze the behavior of the tail and body segments, the propagation of carries, and the cyclic behavior of the segments.

7.1 State Transition Tables and Diagrams

To analyze the behavior of the tail segment and the body segments, we can create state transition tables and diagrams. These tables and diagrams will show how the states of the segments change as a result of the $3x + 1$ operation and the carry propagation.

7.1.1 Tail Segment State Transition Table

The tail segment state transition table shows the possible transitions for the tail segment based on its current state and the incoming carry. The table can be constructed for tail segments of varying lengths.

Current State	Incoming Carry	Next State	Carry Out	Operation
000	0	000	0	Collapse
000	1	001	0	Growth
001	0	011	0	Growth
001	1	100	0	Growth + Carry
011	0	110	0	Growth
011	1	001	1	Growth + Carry
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

7.1.2 Body Segment State Transition Diagram

The body segment state transition diagram shows the possible transitions for each body segment based on its current state and the incoming carry. The diagram can be constructed for body segments of varying lengths.

$$[\text{Current State}] \xrightarrow{\text{Incoming Carry}} [\text{Next State}]$$
$$|$$
$$|$$
$$\vdots$$

7.2 Mathematical Induction

To prove that the "collapse strength" exceeds the "growth strength," we can use mathematical induction. We can show that for any binary number x, if

the property holds for x, then it also holds for $3x + 1$. The base case can be established by analyzing the behavior of the tail segment and showing that collapse dominates growth for all tails of some length greater than the maximum possible carry out.

8   Conclusion

The argument for convergence is based on the analysis of the behavior of binary numbers under the operations of growth $(3x + 1)$ and collapse $(x/2)$. By analyzing the behavior of the tail and body segments, the propagation of carries, and the cyclic behavior of the segments, we can demonstrate that the behavior of the tail segment is crucial in determining the overall behavior of the binary number. The dominance of collapse over growth can be observed through the cyclic behavior of the segments, the propagation of carries, and the "memoizable" behavior of the tail segment. The maximum growth of the head segment is limited, while the collapse function is not limited. Carry propagation from the tail increases the "evenness" of x and its propensity to collapse, leading to the conclusion that the "collapse strength" exceeds the "growth strength" for the binary number under the operations of growth $(3x + 1)$ and collapse $(x/2)$.

9   Further Observations and Insights

9.1   Maximum Rate of Growth

The maximum rate of growth would be for the binary number 1111 with a constant supply of carries. The growth sequence would be as follows:

$$1$$
$$11$$
$$1001$$
$$11011$$
$$100101001$$

This sequence shows that the maximum growth of the head segment is limited to 2 bits (e.g., $11 * 3 = 1001$), while the collapse function is not limited. This further supports the argument that the "collapse strength" exceeds the "growth strength."

9.2   Carry Propagation and Collapse Propagation

Carry propagation and collapse propagation are two key mechanisms that drive the behavior of the binary number under the operations of growth $(3x + 1)$ and

collapse ($x/2$). Carry propagation can convert a string of 1's into a string of 0's headed by a 1. The collapse operation shifts a string of 0's to the right, increasing the frequency of collapses. These mechanisms ensure that the binary number will eventually reach a state where it is fully even, allowing for collapse propagation to the left.

9.3 Memoization of Tail Behavior

The behavior of the tail segment is "memoizable," meaning that we can pre-compute and store the behavior of all possible tail segments up to a certain length. This property allows us to demonstrate that the "collapse strength" of the tail is greater than the greatest potential "growth strength" of any head. By demonstrating collapse for all tails of some length greater than the maximum possible carry out, we can show that the "collapse strength" exceeds the "growth strength."

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11 Further Mathematical Tools

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To analyze the behavior of the tail segment and the body segments, we can create state transition tables and diagrams. These tables and diagrams will show how the states of the segments change as a result of the $3x + 1$ operation and the carry propagation.

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001	0	011	0	Growth
001	1	100	0	Growth + Carry
011	0	110	0	Growth
011	1	001	1	Growth + Carry

11.1.2 Body Segment State Transition Diagram

The body segment state transition diagram shows the possible transitions for each body segment based on its current state and the incoming carry. The diagram can be constructed for body segments of varying lengths.

[-¿,¿=stealth',shorten ¿=1pt,auto,node distance=2.8cm, semithick] every state=[fill=red,draw=none,text=white]
[initial,state] (A) $CurrentState$; [state] (B) [above right of=A] $NextState$; [state] (D) [below right of=A] $NextState$;
(A) edge node Incoming Carry (B) edge node Incoming Carry (D);

11.2 Mathematical Induction

To prove that the "collapse strength" exceeds the "growth strength," we can use mathematical induction. We can show that for any binary number x, if the property holds for x, then it also holds for $3x + 1$. The base case can be established by analyzing the behavior of the tail segment and showing that collapse dominates growth for all tails of some length greater than the maximum possible carry out.

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13 Future Work

As a next step, we can explore further applications of this analysis in other mathematical problems and investigate the properties of the convergence process. Additionally, we can study the behavior of similar iterative processes and identify patterns and properties that may lead to new discoveries in mathematics. We are optimistic that the techniques and methodologies presented in this analysis will contribute to the advancement of mathematical knowledge and inspire further research in the field of iterative processes and beyond.

14 Appendix: Additional Observations

14.1 Maximum Rate of Growth

The maximum rate of growth would be for the binary number 1111 with a constant supply of carries. The sequence of growth would be as follows:

$$1 \rightarrow 11$$
$$11 \rightarrow 1001$$
$$1001 \rightarrow 11011$$
$$11011 \rightarrow 100101001$$

This sequence demonstrates that the growth of the head segment is limited to 2 bits (e.g., $11 * 3 = 1001$), while the collapse function is not limited. This further supports the argument that the "collapse strength" exceeds the "growth strength."

14.2 Carry Propagation and Collapse Propagation

Carry propagation from the tail increases the "evenness" of x and its propensity to collapse. Zeros migrate to the right, increasing the frequency of collapses. This behavior can be observed in the state transition tables and diagrams for the tail and body segments.

14.3 Memoization of Tail Behavior

The behavior of the tail segment is "memoizable," meaning that we can precompute and store the behavior of all possible tail segments up to a certain length. By demonstrating that collapse dominates growth for all tails of some length greater than the maximum possible carry out, we can show that the "collapse strength" exceeds the "growth strength."