Theorem 1 (Bounded Growth Strength). *Let N_i be an odd positive integer with binary representation of the form $(b_1 b_2 \ldots b_k)_2$, where $b_1, b_2, \ldots, b_k$ are binary digits and $b_k = 1$ (since N_i is odd). Then, the growth strength $G(N_i)$ of the binary representation of N_i under the Collatz operation is bounded above by 2 bits per odd iteration.*

Proof. Let N_i be an odd positive integer with binary representation $(b_1 b_2 \ldots b_k)_2$. The Collatz operation for an odd integer is defined as $3N_i + 1$. Applying this operation to N_i, we have:

$$3N_i + 1 = 3(b_1 b_2 \ldots b_k)_2 + 1$$

In binary arithmetic, multiplying an odd number by 3 is equivalent to left-shifting the number by one position and adding the original number. Therefore, the binary representation of $3N_i$ is given by:

$$(b_1 b_2 \ldots b_k 0)_2 + (b_1 b_2 \ldots b_k)_2$$

Adding 1 to this sum, we obtain:

$$3N_i + 1 = (b_1 b_2 \ldots b_k 0)_2 + (b_1 b_2 \ldots b_k)_2 + (1)_2 = (c_1 c_2 \ldots c_m 1)_2$$

where $c_1, c_2, \ldots, c_m$ are binary digits, and the carry generated from the addition of the least significant bits may propagate to the left, potentially causing a change in the most significant bits.

The key observation is that the maximum growth in the number of bits occurs when the most significant bits of N_i are "11" (e.g., $(1101)_2$). In this case, the operation results in $(10001)_2$ after the carry propagates. This represents an increase of 2 bits compared to the original number of bits in N_i.

Therefore, we can formally conclude that the growth strength $G(N_i)$ is bounded above by 2 bits per odd iteration, as the maximum increase in the number of bits occurs when the head segment is "11" and the operation results in $(10001)_2$. This bound holds for all odd positive integers N_i in the sequence. $\square$