

# Formally proving equivalence between abstract and concrete specifications of HMAC

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The OpenSSL implementation of HMAC has been proven to correctly implement a functional specification. HMAC uses SHA-256 as its hash function, and the SHA-256 program has also been proven to correctly implement a functional specification. At a higher level, HMAC has been proven secure: an different functional specification of HMAC has been proven to be a pseudo-random function given that its internal compression function is one as well. We bridge the gap between the two HMAC specs by formally proving their equivalence. This proof transfers the desirable and necessary property of being a pseudo-random function (with some caveats) to both the first spec and the C implementation of HMAC, guaranteeing that the OpenSSL code is secure.

## I. INTRODUCTION

*In our opinion, many proofs in cryptography have become essentially unverifiable. Our field may be approaching a crisis of rigor.* Bellare and Rogaway (2004)

There exists a gap between mathematical cryptography (rigorous paper proofs of correctness of an algorithm) and applied cryptography (implementing algorithms and studying their use). First, the paper proofs may be flawed, leading us to believe that algorithms are secure when they are not. Second, even if the proofs are right, they are accompanied by a specification of the algorithm only on paper. The practical specification may differ from the academic version, introducing theoretical security flaws. Worse, concrete code implementing this specification may contain bugs, allowing adversaries to exploit memory leaks and buffer overflows, as they did with recent OpenSSL vulnerabilities.

Recent work aims to close this gap. As a follow-up to Bellare and Rogaway (2004)’s probabilistic game framework for cryptographic proofs, Halevi (2005) advocates creating an “automated tool to help us with the mundane parts of writing and checking common arguments in [game-based] proofs.” Barthe et al. (2009) present such a tool in the form of CertiCrypt, a framework that “enables the machine-checked construction and verification” of proofs using the same game-based techniques, written in code.

This paper continues in the spirit of formal verification. Our higher-level goal is to prove that the widely-used OpenSSL implementation of HMAC is cryptographically secure. HMAC, which is constructed from a cryptographic hash function, is a natural extension of our existing work on SHA-256. The Verified Software Toolchain (VST) project has built the framework to apply the Verifiable C program logic to C programs, and Appel (2015) has completed a formal, machine-checked verification of the OpenSSL SHA-256 implementation.

Petcher (2015) has formalized a proof of security on one specification of HMAC. This proof is a combination of Bellare, Canetti, and Krawczyk’s 1996 proof (referred to as Bellare’s 1996 proof for brevity) and Bellare’s 2006 proof. Another collaborator, Beringer (2015), has formalized

a proof that the OpenSSL implementation of HMAC correctly fulfills a different specification. This paper bridges the gap between the two HMAC specs by formally proving their equivalence. We do this by gradually reconciling the specs in stages, proving equality or equivalence between each stage. The final proof transfers the desirable and necessary property of being a pseudo-random function (with some caveats) to both the concrete spec and the C implementation of HMAC, guaranteeing that the OpenSSL code is secure.

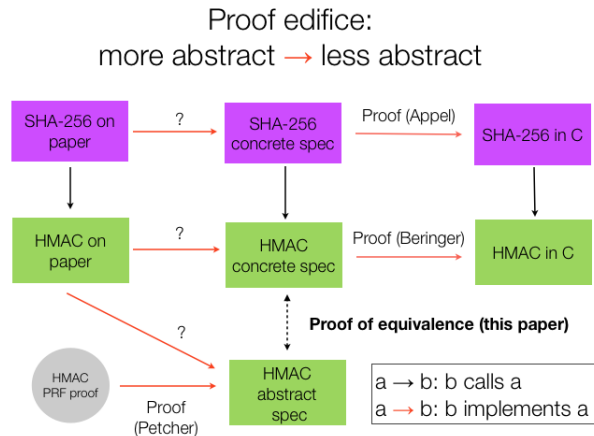


Figure 1. Mind the gap (the dotted arrow).

Mechanizing proofs of correctness lends the cryptographer (and the end-user confidence) that the proofs of correctness are indeed correct. However, this process cannot stop the cryptographer from proving the wrong theorem, for example, with respect to an impractical definition of security. Indeed, in a controversy termed “the HMAC brawl” (Bernstein (2012)), Koblitz and Menezes (2013) have pointed out several security problems related to practical use of HMAC. We believe that Bellare’s proofs themselves are valuable; thus, Petcher (2015)’s mechanization of Bellare’s proofs is still valuable to cryptographers interested in computer-checked proofs of correctness. Additionally, by proving the security of HMAC in its standardized form, we address Koblitz and Menezes’s separation result between the “idealized” and “standard-

ized” versions of HMAC.<sup>1</sup>

All specifications and theorems may be found at <https://github.com/hypotext/vst-hmac>. They can be compiled and run with Coq 8.4pl4.

### A. Formal verification in Coq

Coq is a proof assistant in which every proof step is checked by the small, trusted Coq kernel. It has been used to complete a computer-checked proof of the four-color theorem and to develop CompCert, a formally verified and “unbreakable” C compiler (Yang et al. (2011)), among other things.

Coq has two internal languages, Gallina and the tactic language. Gallina is a purely functional language similar to OCaml. The tactic language is used for doing proofs and defining new proof strategies.

As used here, formal verification of a piece of code means proving that this implementation fulfills some kind of high-level specification. The code will usually be written in a low-level language such as C and may contain optimizations and other tricks. The specification, or “spec,” will usually be written in a high-level language such as OCaml or Gallina and is typically more mathematical and abstract. A program without a specification *cannot be incorrect*, it can only be surprising.<sup>2</sup>

For example, one may specify that the English description “to be sorted” is equal to the conjunction of two mathematical properties, perhaps specified mathematically in Coq:

1. The output is a permutation of the input (meaning that no items have been changed, dropped, or added).
2. Each element in the output is less than or equal to all elements to the right, for some definition of  $\leq$ .

Appel (2014)’s exercises demonstrate this process of formal verification by asking the reader to prove that an implementation of insertion sort in Gallina always satisfies these properties. In the real world, formal verification might entail proving that an optimized implementation of quicksort in C satisfies these properties.

### B. The Merkle-Damgård construction

Say we have a strong one-way cryptographic compression function that has been proved to be collision resistant. The problem is that it only operates on an input of a fixed length.

The Merkle-Damgård construction (referred to as “M-D” from now on) is a way to extend this function to inputs of any length by iterating the compression function on identically-sized, adjacent blocks of the input. It has been proven to preserve the collision-resistance property of its internal compression function. M-D has been used to design many popular hash algorithms such as MD5, SHA-1, and SHA-2.

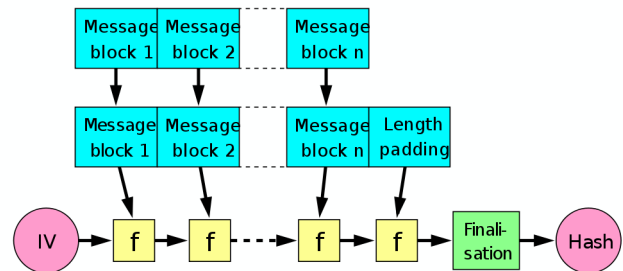


Figure 2. The Merkle-Damgård construction.  $IV$  denotes the initialization vector and  $f$  denotes the compression function. (Wikipedia)

Both of Bellare’s HMAC proofs (1996, 2006) require that the underlying hash function is a Merkle-Damgård construction on a compression function  $f$  such that  $f$  is a pseudo-random function. (Another assumption on  $f$  is discussed in Section III.)

### C. SHA-256

SHA-256, a version of the “Secure Hash Algorithm,” is a popular cryptographic hash algorithm constructed via Merkle-Damgård.

It operates on a message of any length by padding the message and breaking the result into 512-bit blocks. After iterating its internal compression function on each block, it outputs a 256-bit digest.

Like all such hash functions, the version of SHA-256 specified in English in FIPS 180-4 comes with guarantees of pre-image resistance, second pre-image resistance, and collision resistance. (Code that implements FIPS 180-4 does not necessarily come with such guarantees unless they are formally proven about the code.) Thus, it is very difficult for an adversary to change the input message without changing the digest. It will be assumed for the HMAC proof that the functional spec of SHA-256 is a pseudo-random function on its inputs; however, nobody knows how to prove that.

### D. HMAC

SHA-256 provides only a guarantee of integrity; that is, a guarantee the message has not been tampered with. A message authentication code (MAC) is used to guarantee

<sup>1</sup> Koblit (2007) critiques the very approach of applying automated theorem proving to cryptography. Addressing this is out of the scope of this paper.

<sup>2</sup> Paraphrase of J. J. Horning (1982) by Appel (2015).

both integrity and authenticity, the latter meaning that the message has been sent by the expected sender. To ensure this, sender and receiver need only exchange a secret key before beginning their communication.

To accomplish this, Bellare et al. (1996) designed HMAC, or the “Hash-based Message Authentication Code.” To compute the authentication code of a message, RFC 2104 defines HMAC as the following action:

$$\text{HMAC}_{H,K}(m) = H((K \oplus \text{opad}) \parallel H(K \oplus \text{ipad} \parallel m))$$

where

- its block length is 512 bits, or 64 bytes,
- $H$  is a cryptographic hash function (here, *SHA-256*),
- $K$  is a secret key padded to the right with extra zeros to the input block size of the hash function, or the hash of the original key if it’s longer than that block size,
- $m$  is the message to be authenticated,
- $\parallel$  denotes concatenation,
- $\oplus$  denotes bitwise exclusive or (XOR),
- *opad* is the outer padding (the byte `0x5c` repeated 64 times to be the length of one block),
- and *ipad* is the inner padding (`0x36` repeated as above).

Note that formalizing HMAC is a natural extension of our work on *SHA-256*, since HMAC is not much more complicated than applying a cryptographic hash function such as *SHA-256* twice.

Bellare et al. (1996) includes a proof that the HMAC protocol (described below) is a pseudo-random function (PRF) on its inputs given that the underlying hash primitive is a PRF, a proof which was improved by Bellare (2006). In *SHA-256*, the underlying hash primitive would be its compression function. In addition, whereas *SHA-256* is vulnerable to length-extension attacks, HMAC is not.

OpenSSL includes an implementation of HMAC in C which calls the OpenSSL implementation of *SHA-256*.

## II. PROOFS OF EQUALITY AND EQUIVALENCE

### A. The concrete HMAC spec

Beringer (2015) wrote a specification of HMAC to conform to RFC 2104 (IETF’s English description of the algorithm). Thus, it contains runnable code, operates on byte lists, and returns a byte list. Because the code contains no abstract parameters and can be run as-is, we call it the “concrete” spec. The spec distinguishes between

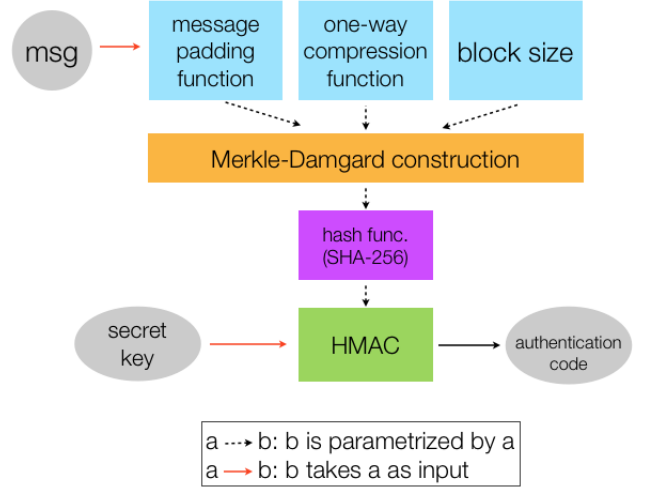


Figure 3. The entire system. It is slightly more correct to think of a “black box” HMAC taking message and key as input, and outputting the authentication code.

the  $Z$  type and the *byte* type (values of type  $Z$  in range  $[0, 255]$ ); however, in this paper, we will use  $Z$  synonymously with *byte* with the understanding that all values are in range.

Again, Beringer (2015) has formalized a proof that the OpenSSL implementation of HMAC correctly fulfills this spec.

The spec is constructed to work with generic cryptographic hash functions. We will instantiate it with the *SHA-256* functional program, which the spec treats as a black box that takes care of the message padding and compression function iteration.

The code for this spec and the next is in the Appendix.

### B. The abstract HMAC spec

Petcher (2015) wrote a specification of HMAC to conform to the protocol defined by Bellare (1996, 2006). Because the code cannot be run until several abstract parameters are instantiated, we call it the “abstract” spec.

To be as general as possible, it operates on bits instead of bytes (as specified by Bellare). Instead of the byte lists used in the concrete spec, it uses bit vectors. (A bit list, or *Blist*, is a list of bits of any length. A bit vector *Bvector*  $n$  is a type dependent on a natural number value  $n$ , the length of the vector. *Bvector*  $n$  is essentially a *Blist* constrained to be of length  $n$ , which is checked at compile-time.) The abstract spec uses vectors because Bellare’s proof often requires that a list of bits possess a certain length. For example, we translate “*opad* and *ipad* are exactly one block long” into code as `(opad ipad : Bvector b)`.

The abstract spec defines HMAC via the two-keyed HMAC (*HMAC\_2K*) and generalized HMAC (*GHMAC\_2K*) structures, rather than straightforwardly

as the concrete spec does. The spec also includes an implementation of generalized NMAC (*GNMAC*), another structure used in the proof. Section III explains the purpose of these intermediate structures.

Here are the parameters that the spec leaves abstract:

```
(* c is the output size, b is the block size
   (larger than the output size),
   and p is the difference between them *)
Variable c p : nat.
Definition b := c + p.

(* compression function *)
Variable h : Bvector c -> Bvector b -> Bvector c.

(* initialization vector *)
Variable iv : Bvector c.

Variable splitAndPad : Blist -> list (Bvector b).

Variable fpad : Bvector c -> Bvector p.
Definition app_fpad (x : Bvector c) : Bvector b :=
  (Vector.append x (fpad x)).

Variable opad ipad : Bvector b.
```

Bellare’s proofs (1996, 2006), and hence Petcher (2015)’s proof, depend on the following assumptions that are explicit in the spec:

1. the key is of the right length (one block)
2. *opad*  $\neq$  *ipad* (they differ in at least one bit)
3. the padding function *splitAndPad* is one-to-one
4. the hash function (e.g. *SHA-256*) is an iterated version of the compression function; that is, it is constructed via Merkle-Damgård.

as well as other implicit assumptions explained in Section III.

Explaining the fourth assumption will reveal the purpose of the various padding constructs left abstract above. The assumption that the hash function is an iterated version of the compression function can be seen in this definition of the abstract spec’s analogue to *SHA-256*, *hash\_words*. It explicitly constructs the hash function via iteration.

```
(* The iteration of the compression function gives
   a keyed hash function on lists of words. *)
Definition h_star k (m : list (Bvector b)) :=
  fold_left h m k.

(* The composition of the keyed hash function with
   the IV gives a hash function on lists of
   words. *)
Definition hash_words := h_star iv.
```

Rather than treating the hash function as a black box, the spec defines it via a left fold.

However, *SHA-256* includes a padding function for the message, while this spec’s use of dependent types

(*Bvector n*) forces the use of two types of ad-hoc padding around *hash\_words*, the functions *app\_fpad* and *splitAndPad*.

```
Definition GHMAC_2K (k : Bvector (b + b)) m :=
  let (k_Out, k_In) := splitVector b b k in
  let h_in := (hash_words (k_In :: m)) in
  hash_words (k_Out :: (app_fpad h_in) :: nil)
  .
```

```
Definition HMAC_2K (k : Bvector (b + b)) (m : Blist)
:=
  GHMAC_2K k (splitAndPad m).
```

```
Definition HMAC (k : Bvector b) :=
  HMAC_2K (Vector.append (BVxor _ k opad) (BVxor _
    k ipad)).
```

*fpad* is used to pad the output size *c* to the block size *b*. *splitAndPad* is used to split the variable-length message (of type *Blist*) into a list of blocks, each size *b*, padding it along the way.

### C. Proof outline

There are six main differences between the concrete and abstract specs:

1. the abstract spec operates on bit lists, whereas the concrete spec operates on byte lists.
2. the abstract spec uses the dependent type *Bvector n*, which is a bit list of length *n*, whereas the concrete spec uses byte lists and int lists.
3. due to its use of dependent types, the abstract spec pads its input twice in an ad-hoc manner, whereas the concrete spec uses the *SHA-256* padding function consistently.
4. the concrete spec treats the hash function (*SHA-256*) as a black box, whereas the abstract spec exposes various parts of its functionality, such as its initialization vector, internal compression function, and manner of iteration. (It does this because the Bellare proofs require that the hash function is constructed via Merkle-Damgård.)
5. the abstract spec does an explicit fold over the message, which is now a list of blocks of bits, not a list of pure bits.
6. the abstract spec defines HMAC via the *HMAC\_2K* and *GHMAC\_2K* structures, not directly.

They are ranked in rough order of most difficult to least difficult.

The last, 6, is easily resolved by unfolding the definitions of *HMAC\_2K* and *GHMAC\_2K*. We solve the other five by changing definitions and massaging the two specs toward each other, proving equality each time.

5, the fold over a list of blocks, is resolved by concatenating the *list* (*Bvector* *b*) in *hash\_words* below. Then it can be seen that iterating *firstn* *b* and *splitn* *b* over the concatenated list is equal to left-folding over the list of blocks.

```
Function hash_blocks_bits
  (hash_block_bit : Blist -> Blist -> Blist)
  (r: Blist)
  (msg: Blist) {measure length msg} : Blist
  :=
  match msg with
  | nil => r
  | _ => hash_blocks_bits
    hash_block_bit
    (hash_block_bit r (firstn 512 msg)) (skipn
    512 msg)
end.
```

```
Definition h_star (k : Bvector c)
  (m : list (Bvector b)) : Bvector c :=
  fold_left h m k.
```

```
Definition hash_words : list (Bvector b) ->
  Bvector c := h_star iv.
```

4, the explicit iteration, is not too difficult. Its explicit fold (as iteration method) can be proven equal to the iteration method using *firstn* and *splitn* in the concrete SHA-256 spec, as seen in 3 above. It suffices to convert the SHA-256 initialization vector and wrap its internal compression function in conversion functions as well. See 1 and the next section for an elaboration on conversion and wrapping.

Before iterating the compression function on the message, SHA-256 pads it in a standard, one-to-one fashion so its length is a multiple of the block size. It pads it as such:

$$msg \mid [1] \mid [0, 0, \dots, 0] \mid [l_1, l_2]$$

where  $\mid$  denotes concatenation and  $[l_1, l_2]$  denote the two digits of the length of the message as a 64-bit integer. The number of 0s is calculated such that the length of the entire padded message is a multiple of the block size.

Thus, 3, the two ad-hoc padding methods, is resolved by rewriting the abstract spec to incorporate *fpad* and *splitAndPad* into a single padding function included in the hash function, much like SHA-256 does.

$$\begin{aligned} hash\_words\_padded &:= \\ hash\_words \circ split\_and\_pad \end{aligned}$$

As summarized in the previous section, *fpad* is used to pad the output size *c* to the block size *b*. *splitAndPad* is used to split the variable-length message (of type *Blist*) into a list of blocks, each size *b*, padding it along the way. *fpad* is instantiated as a constant, since we know that the length of the message is  $c < b$ . *splitAndPad* is instantiated as the normal SHA padding function, but tweaked to add one block size to the length appended

in  $[l_1, l_2]$ , since  $k_{in}$  (with a length of one block) will be pre-pended to it later.<sup>3</sup>

2, the use of bit vectors instead of bit lists, is resolved by changing all *Bvector* *n* to *Blist*, then proving that all functions preserve the length of the list when needed. This maintains the *Bvector* *n* invariant that its length is always *n*.

1, the use of bits instead of bytes, is the most difficult; since the types of each HMAC function differ, bridging this gap requires an equivalence proof rather than an equality proof. It is discussed in the next section.

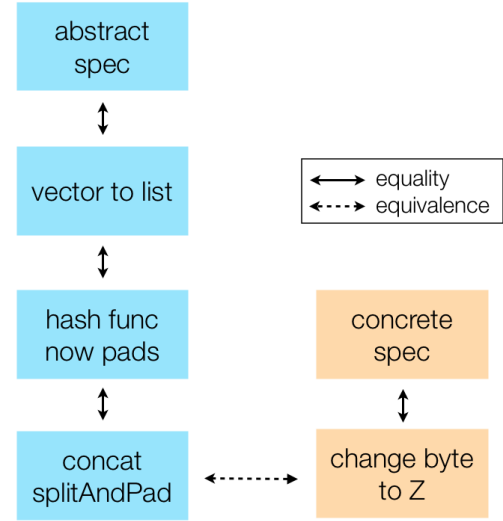


Figure 4. Stages of the equivalence/equality proofs.

#### D. Bridging bytes and bits

We would like to prove that the concrete and abstract HMAC specs ( $HMAC_c$  and  $HMAC_a$ ) are extensionally equal. That is, equal inputs always result in equal outputs: for all keys *k* and messages *m*,

$$\begin{aligned} k_c &= k_a \rightarrow \\ m_c &= m_a \rightarrow \\ HMAC_c(k_c, m_c) &= HMAC_a(k_a, m_a). \end{aligned}$$

However,  $HMAC_c$  operates on bits (in fact, lists of bits, since we have replaced the vectors by lists) and  $HMAC_a$  operates on bytes (lists of bytes). So the statement  $k_c = k_a$  (for example) does not have meaning because the types are different. To solve this, we generalize equality to

<sup>3</sup> Bellare (1996) glosses over the padding methods: “Notice that a way to pad messages to an exact multiple of *b* bits needs to be defined (...) SHA pad[s] inputs to always include an encoding of their length” (8). The process of formalizing proofs in code dispels the vagueness that hides in natural language.

equivalence. Given that the inputs are *equivalent*, the outputs will be *equivalent*.

$$\begin{aligned} k_c &\approx k_a \rightarrow \\ m_c &\approx m_a \rightarrow \\ \text{HMAC}_c(k_c, m_c) &\approx \text{HMAC}_a(k_a, m_a). \end{aligned}$$

The equivalence relation  $\approx$  can be defined either computationally or inductively. Both definitions will turn out to be useful. From now on, we will use  $b$  or lowercase letters to name a value of type *Blist* (list of booleans, or bits) and  $B$  or uppercase letters to name a value of type *list Z* (list of bytes).

The computational relation is

$$\begin{aligned} b &\approx_c B := \\ b &= \text{bytesToBits } B, \end{aligned}$$

where  $\text{bytesToBits} : \text{list } Z \rightarrow \text{Blist}$  is a conversion function.

The inductive relation is

$$\begin{aligned} b &\approx_i B := \\ \text{bytes\_bits\_lists } b B. \end{aligned}$$

where  $\text{bytes\_bits\_lists} : \text{Blist} \rightarrow \text{list } Z \rightarrow \text{Prop}$  creates an “assertion” that  $b$  and  $B$  are related in this way, together with two constructors used to provide evidence to prove the assertion. For an explanation of inductively defined propositions, see the *Prop* chapter in the Coq textbook “Software Foundations” (Pierce et al. (2014)).

```
Inductive bytes_bits_lists : Blist -> list Z -> Prop
:=
| eq_empty : bytes_bits_lists nil nil
| eq_cons : forall (bits : Blist) (bytes : list Z)
  (b0 b1 b2 b3 b4 b5 b6 b7 : bool)
  (byte : Z),
  bytes_bits_lists bits bytes ->
  convertByteBits [b0; b1; b2; b3; b4;
    b5; b6; b7] byte ->
  bytes_bits_lists (b0 :: b1 :: b2 ::
    b3 :: b4 :: b5 :: b6 :: b7 ::
    bits)
    (byte :: bytes).
```

$\text{eq\_empty}$  is the axiom that empty lists of bits and empty lists of bytes are related by conversion, which is trivially true.  $\text{eq\_cons}$  is the axiom that, if two lists are already related by conversion in this way, they continue to be related after pre-pending a byte and a list of eight bits to their respective lists. The byte and list of eight bits also need to be related by conversion. See the Appendix for the definition.

The inductive and computational definitions have been proven equivalent in the sense that

$$\text{bytes\_bits\_lists } b B \leftrightarrow b = \text{bytesToBits } B.$$

For several other ways to convert between the definitions, see *ByteBitRelations.v* in the *vst-hmac* distribution linked in the Introduction.

## E. Instantiating the abstract specification

To instantiate the abstract spec’s parameters, we instantiate the block sizes to the byte block sizes times 8, and wrap the concrete functions in *byteToBit* and/or *intlist\_to\_Zlist* conversion functions. The latter is necessary because portions of the SHA-256 spec operate on lists of *Integers*, as specified in FIPS 180-4.<sup>4</sup> ( $Z$  in Coq denotes arbitrary-precision mathematical integers. The SHA-256 spec uses  $Z$  to denote bytes. An *Integer* is four  $Z$  packed big-endian into a 32-bit integer.)

```
Definition c := (SHA256.DigestLength * 8)%nat.
Definition p := (32 * 8)%nat.
```

```
Definition intsToBits (l : list int) : list bool
:=
  bytesToBits (SHA256.intlist_to_Zlist l).
```

```
Definition bitsToInts (l : Blist) : list int :=
  SHA256.Zlist_to_intlist (bitsToBytes l).
```

```
Definition sha_iv : Blist :=
  intsToBits SHA256.init_registers.
```

```
Definition sha_h (regs : Blist) (block : Blist) :
  Blist :=
  intsToBits (SHA256.hash_block (bitsToInts regs)
    (bitsToInts block)).
```

```
Definition sha_splitandpad (msg : Blist) : Blist
:=
  bytesToBits (sha_padding_lemmas.pad (bitsToBytes
    msg)).
```

Note that we are essentially converting the type of the values from  $\text{intlist} \rightarrow \dots \rightarrow \text{intlist}$  to  $\text{Blist} \dots \rightarrow \dots \text{Blist}$  by converting their inputs and outputs.

We can use the computational equivalence relation defined earlier ( $\approx_c$ ), instantiated with a generic conversion function, to reason abstractly about the behavior of such wrapped functions. Let’s define the framework as such (letting  $\cdot = \circ$ , function composition):

```
(* B corresponds to bytes, A corresponds to bits *)
Parameter A B : Type.
Parameter convert_BA : B -> A.
Parameter convert_AB : A -> B.
```

```
Definition wrap (F : B -> B) : A -> A :=
  convert_BA . F . convert_AB.
```

Note that the types  $B$  and  $A$  should not be symmetric, in the sense that one cannot convert equally between *byte list* and *bit list* and vice versa. We will formalize this intuition.

<sup>4</sup> FIPS 180-4 actually defines SHA-256 on a bitstring of any length, similar to the Bellare proof. But OpenSSL’s implementation permits only sequences of bytes; that is, multiples of 8 bits.



Define two relations as such:

$$\begin{aligned} x &\approx_c X := \\ x &= \text{convert\_BA } X. \\ f &\approx_w F := \\ f &= \text{wrap } F. \end{aligned}$$

We ask: what assumptions are needed such that application of equivalent (via wrapping) functions to equivalent (via conversion) inputs result in equivalent (via conversion) outputs?

*Lemma once\_eq :*

$$\begin{aligned} \forall (x : A) (X : B) (f : A \rightarrow A) (F : B \rightarrow B), \\ x &\approx_c X \rightarrow \\ f &\approx_w F \rightarrow \\ f \ x &\approx_c F \ X. \end{aligned}$$

The necessary and sufficient assumption (together with the other two assumptions) is that we have output equivalence exactly when a “round-trip” of composing the two conversion functions results in the identity function. (The reader is invited to finish this short proof.)

*Lemma roundtrip : convert\_AB  $\circ$  convert\_BA = id.*

Indeed, in the *BytesBitsRelations* library, we have proven that *bitsToBytes*  $\circ$  *bytesToBits* = *id*. Note that it does not hold the other way around. *bytesToBits*  $\circ$  *bitsToBytes* is false if the length of the bit list input is not a multiple of 8.

A natural extension is to prove output equivalence on repeated application of the wrapped functions, or iteration.

```
Fixpoint iterate {A : Type} (n : nat) (f : A -> A) (
  x : A) :=
  match n with
  | 0 => x
  | S n' => f (iterate n' f x)
end.
```

*Lemma iterate\_equiv :*

$$\begin{aligned} \forall (x : A) (X : B) \\ (f : A \rightarrow A) (F : B \rightarrow B) (n : nat), \\ x &\approx_c X \rightarrow \\ f &\approx_w F \rightarrow \end{aligned}$$

$$\text{iterate } n \ f \ x = \text{convert\_BA } (\text{iterate } n \ F \ X).$$

Here we need both *once\_eq* and our newly admitted assumption *roundtrip*, which can be rephrased as  $\forall(X : B), X = \text{roundtrip } X$ . The proof is completed by induction on *n*.

This framework is not just an academic exercise. *once\_equiv* and *iterate\_equiv* correspond directly to several lemmas in the bytes/bits equivalence proof. In particular, *iterate* corresponds directly to the SHA-256 operation of hashing blocks.

```
Function hash_blocks (r: registers) (msg: list int)
  {measure length msg} : registers :=
  match msg with
  | nil => r
  | _ => hash_blocks (hash_block r (firstn 16 msg))
    (skipn 16 msg)
end.
```

The proof of equivalence involving *SHA256.hash\_blocks* and *hash\_blocks\_bits* (where *hash\_blocks\_bits*  $\approx_w$  *SHA256.hash\_blocks*) works the same way. One can almost directly substitute the parameters into *iterate\_equiv*.

*Lemma fold\_equiv\_blocks :*

$$\begin{aligned} \forall (l : Blist) (acc : Blist) (L : list int) (ACC : list int), \\ \text{InBlocks } 16 \ L \rightarrow \\ l &\approx_c L \rightarrow \\ acc &\approx_c ACC \rightarrow \end{aligned}$$

$$\text{hash\_blocks\_bits } sha\_h \ acc \ l \approx_c \text{SHA256.hash\_blocks } ACC \ L.$$

The only differences are that the functions in question take multiple inputs, the length of *L* must be a multiple of 16 (the block size for the *Integer* type, 512 bits), and the method of iteration is slightly modified to be parametrized by the length of the input list (*firstn*, *skipn*). *sha\_h* contains the wrapped *SHA256.hash\_block* function. *hash\_blocks\_bits* is a version of *SHA256.hash\_blocks* modified to use 512 as its block size. The code for the proof is in *HMAC\_spec\_concat.v*.

In sum, the computational relation  $\approx_c$  makes abstract reasoning easy precisely because it is computational, so one can rewrite using equality in proofs. If no conversion function existed, one could use the inductive relation  $x \approx_i X$ , but it does not easily allow rewrites, since *x* and *X* are of different types. The strength of  $\approx_i$  will be discussed in the next section.

One wonders whether this “one-way roundtrip” property (*Lemma roundtrip*) has been formalized elsewhere, perhaps in a Coq abstract algebra library. We found that the concepts of Galois connections, setoids, and setoid rewrites were similar, but not useful.

## F. The proof of equivalence

The main equivalence theorem is as follows. The symbol  $\approx$  will be used to represent either relation ( $\approx_c, \approx_i$ )

interchangeably.

*Theorem HMAC\_spec\_equiv :*  
 $\forall(k\ m\ h : \text{Blist})(op\ ip : \text{Blist})$   
 $(K\ M\ H : \text{list } Z)(OP\ IP : \text{list } Z),$   
 $\text{length } K = \text{SHA256.BlockSize} \rightarrow$   
 $k \approx K \rightarrow$   
 $m \approx M \rightarrow$   
 $op \approx OP \rightarrow$   
 $ip \approx IP \rightarrow$

$\text{HMAC } c\ p\ sha.h\ sha.iv\ sha.splitandpad\ op\ ip\ k\ m = h \rightarrow$   
 $\text{HMAC\_SHA256.HMAC } IP\ OP\ M\ K = H \rightarrow$   
 $h \approx H.$

Note that the following are implicit:

$sha.iv \approx_c \text{SHA256.init\_registers}$   
 $sha.h \approx_w \text{SHA256.hash\_block}$   
 $sha.splitandpad \approx_w sha\_padding\_lemmas.pad$

$sha\_padding\_lemmas.pad$  is a version of  $\text{SHA256.generate\_and\_pad}$ , the padding function defined in FIPS 180-4, modified here to separate the padding from the  $Z \rightarrow \text{Integer}$  conversion.  $hash\_blocks\_bits$  is a version of  $\text{SHA256.hash\_blocks}$  modified to use 512 as its block size.

As a refresher, here is the HMAC code:

$$\text{HMAC}_{H,K}(m) = H((k \oplus opad) \parallel H(k \oplus ipad \parallel m))$$

Section IIc discussed the series of transformations applied to the abstract spec. Its structure was originally not close to the HMAC and SHA-256 code, but after some massaging, it and its internal hash function become structurally similar enough to the high-level code for the proof to be decomposed by function.

Three functions occur in the definition of HMAC:

$H$  (hashing),  $\parallel$  (concatenation), and  $\oplus$  (bitwise xor). Thus, the proof breaks up naturally and modularly into three theorems. For each function, its theorem states that the output is equivalent given that the input is equivalent.

*Lemma f\_equiv :*  $\forall(b : \text{bits})(B : \text{bytes}),$   
 $b \approx B \rightarrow$   
 $f \approx_w F \rightarrow$   
 $f\ b \approx F\ B.$

The proof is completed by backward-chaining these implications. The diagram makes it clear that in the end, the leaves of the tree are the sole propositions that need to be proved, and these are exactly the givens:  $k \approx K, m \approx M, op \approx OP$ , and  $ip \approx IP$ .

The theorem about  $\parallel$ , *concat\_equiv*, is easy to prove by induction.

The theorem about  $\oplus$ , *xor\_equiv\_Z*, is slightly harder.

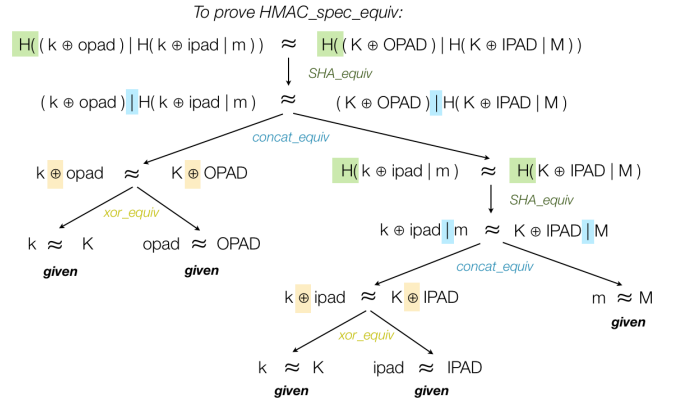


Figure 5. Lowercase denotes bits; uppercase denotes bytes.  $a \xrightarrow{T} b$  means “By  $T$ ,  $b$  implies  $a$ .” Note after HMAC has been unpacked, all that is left is our assumptions at the leaves of the tree.

It uses the inductive relation and depends on several lemmas in *XorCorrespondence*, including a large proof by brute force (written by Beringer). The proofs use the inductive relation  $\approx_i$  because it comes with a stronger induction principle that allows us to break up the input bit/byte lists correctly to perform induction, since eight bits correspond to one byte.

Lastly, the theorem about  $H$ , *SHA\_equiv\_pad*, depends on *fold\_equiv*. *fold\_equiv* requires a messy double induction that requires definition and use of the *InBlocks n* inductive proposition. This is necessary because one list is in blocks of 512 bits and the other is in blocks of 16 integers.

In addition to the main theorems, the equivalence proof involves about 45 more lemmas.

## G. Other proof techniques

The following techniques may be useful for future Coq developments.

Many theorems are true for lists of any length (e.g. some involving map and zip). We found it difficult to do induction on *Bvector n*, dependent type induction, since its length is fixed. Instead, we found it easy to prove theorems by induction on a Blist, implying that the list may be any length, then specialize it to *Bvector n*, a Blist of length  $n$ .

Likewise, we found it easier to prove theorems about lists of any length (or a certain length given by an assumption), then prove that the functions involved preserve the length. This is equivalent to working with *Bvector n*.

When dealing with lists whose lengths must be a multiple of a block size (e.g. 512 bits or 64 bytes), we found it useful to define an inductive proposition *InBlocks n* that would allow one to do proofs by inducting in the block size, cons'ing elements to the front. This allows an easier



induction on bit/byte lists, since 512 bits correspond to 64 bytes. We then proved *InBlocks*  $n$  equivalent to the computational version (using the length function).

Likewise, we found it useful to define both inductive and computational definitions, as covered in the previous sections: a function to compute conversions between byte lists and bit lists, and an inductive proposition stating the existence of bits such that the two lists correspond.

When it comes to theorems that involve tricky math, we exploited the fact that range of a byte is  $[0, 255]$  and proved the theorems by brute force instead. The same technique also works in reverse. Proving something true for 8 booleans is as simple as checking that the statement is true for each of the 256 cases.

### H. Problems encountered

Initially, we attempted to do the proof on the abstract spec as-is. We found it difficult to work with dependent types, induction, and John Major equality (McBride (2002)), which is a generalization of equality necessary to deal with dependent types (Chlipala (2014)). One common problem was that there are no easy Coq facilities for type-level manipulation, e.g. Coq cannot automatically convince itself that  $\forall(n : \text{nat}), \text{Bvector } (n + 1) = \text{Bvector } (S \ n)$ . This crops up often in recursive functions. We avoided these difficulties by using the list techniques described above.

Additionally, we found it added extra work to have to convert between many machine representations of numbers: byte and bit (of course), byte and  $\mathbb{Z}$ , integer and  $\mathbb{Z}$ , and even little-endian and big-endian.

Lastly, we did not find much prior work on this sort of equivalence proof or the “one-way roundtrip” algebraic property, except for related conversion functions in *Coq.Strings.Ascii* (Thery).

## III. MECHANIZING THE PROOF OF HMAC’S SECURITY

Following in the tradition of CertiCrypt and EasyCrypt, Petcher and Morrisett (2015) present the Foundational Cryptography Framework (FCF), a general-purpose framework including a probabilistic programming language implemented in Coq.

Petcher (2015) uses FCF to mechanize a hybrid of Bellare’s 1996 and 2006 proofs. He mechanizes the result of Bellare’s 1996 proof, using the structure of the 2006 proof, simplified by using the stronger assumptions of the earlier proof. The two proofs also differ slightly in some definitions, such as resistance against related key attacks. The structure is summarized below and will be explained in greater detail in a future paper.

After composing all the proofs as in Figure 1, we may make the following argument:

If Petcher (2015)’s HMAC is a PRF, and this HMAC composed with some conversion functions is equal to Beringer (2015)’s HMAC, and these conversion functions are injections, then Beringer (2015)’s HMAC is a PRF.

Thus, we may state the following final theorem:

There exists a function  $f$  that is extensionally equivalent to the C program (the OpenSSL implementation of HMAC), and  $f$  is a pseudo-random function.

### A. The hybrid Bellare proof

Let  $f$  denote the compression function and  $f^*$  the iteration of  $f$ . That is,  $f^*$  is constructed via the Merkle-Damgård construction, like SHA-256. The “dual family” of  $f^*$  is  $f^*$  with its two arguments swapped. That is, for any key  $k$  (for example, the initialization vector) and message  $m$ , let  $f^*(k, m) = f_d^*(m, k)$ .

We start with the assumptions that  $f$  is a pseudo-random function (PRF) and that  $f^*$  is weakly collision-resistant. Later, we use the assumption that the dual family of  $f^*$  is resistant to related-key attacks (RKA). In a related-key attack, the adversary queries the function both with the target key and with other adversary-defined keys. The adversary may know that there is some mathematical relationship among the keys, but might not know the exact value of the keys.

The second Bellare proof uses weaker assumptions: Bellare (1996) uses weak collision-resistance, whereas Bellare (2006) weakens it to “computationally almost-universal” (*cAU*). Among other simplifications, Bellare (2006) proves that the  $f$  PRF assumption implies the  $f^*$  *cAU* assumption.

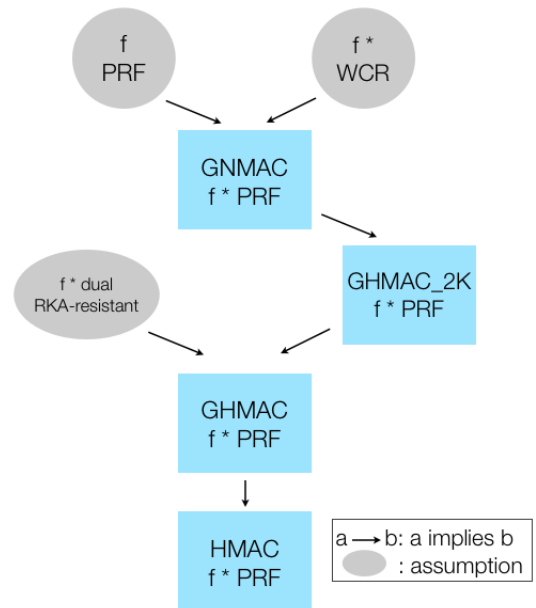


Figure 6. Petcher (2015)’s hybrid Bellare proof uses the stronger assumptions of 1996 and the intermediate structures of 2006.

First, the  $f$  PRF and  $f^*$  WCR assumptions are used to prove that generalized NMAC (GNMAC) using  $f^*$  is a PRF. The GNMAC property is used to prove that generalized HMAC with two keys (GHMAC\_2K) using  $f^*$  is a PRF.

Using that fact and the assumption that  $f^*$  is RKA-resistant, we prove that GHMAC (single-keyed) using  $f^*$  is a PRF. This, plus the fact that the padding function for  $f$  is one-to-one, leads finally to the proof that HMAC (single-keyed) using  $f^*$  as its hash function is a PRF.

Koblitz and Menezes (2013) give an accessible motivation for why Bellare introduces the generalized and two-key constructs. There already existed a “Nested Message Authentication Code,” or NMAC, with “no obvious security flaws.” However, it required two keys and required that the IV of its hash function be changed. Both were practical annoyances, the two keys because they created more work, and the IV because it precluded using an “off-the-shelf” hash function like SHA-256. Thus, Bellare et al. created HMAC. By *xoring* its key with *opad* and *ipad*, it created two keys from one key.

## B. Assumptions

Petcher (2015)’s proof that his abstract spec of HMAC is a pseudo-random function holds only under the following assumptions:

- the `splitAndPad` function is one-to-one (that is,  $\text{pad}(m_1) = \text{pad}(m_2) \rightarrow m_1 = m_2$ ).
- `opad` and `ipad` differ in at least one bit.
- $f$  is a pseudo-random function.
- $f^*$  is resistant to related-key attacks.
- Appel (2015)’s concrete spec of SHA-256 is a Merkle-Damgård construction.

Examining the final spec, these are mostly true. It is obvious on inspection that the SHA padding function (described in Section Iic), and thus the wrapped SHA padding function, are one-to-one. Beringer (2015)’s concrete spec also defined `opad` and `ipad` to be different. However, it is unknown whether  $f$  (the internal SHA compression function) is a PRF and whether the “dual family” of SHA-256 is RKA-resistant. These are open conjectures that are generally accepted to be true. The last item, showing that Appel (2015)’s spec of SHA-256 is a Merkle-Damgård construction, is provable and is left as future work.

## IV. CONCLUSION

Again, we bridge the gap between the abstract and the concrete HMAC spec by formally proving their equivalence. This proof transfers the desirable and necessary

property of being a pseudo-random function (with some caveats) to both the concrete spec and the C implementation of HMAC, guaranteeing that the OpenSSL code provides cryptographic security (as specified by Bellare (1996, 2006)).

We also contribute a theory of reasoning about equivalence, especially between programs working with bytes and programs working with bits. We formalize the theory in a small library for reasoning about these correspondences and converting between inductive and computational definitions.

We have also added comments on our success or failure using general Coq proof techniques, including brute force, dependent types, and inductive propositions for length (*InBlocks n*).

Lastly, we have not entered the ring of the HMAC brawl, but we present our chain of successful formalizations in hopes that they are valuable to cryptographers interested in computer-checked proofs of correctness. Additionally, by proving the security of HMAC in its standardized form, we address Koblitz and Menezes’s separation result between the “idealized” and “standardized” versions of HMAC. We will state this result more precisely in a future paper.

## A. Future work

We would like to compose the various proofs (Figure 1) to leave an end-to-end proof of correctness. This requires more work: we need to enumerate and/or prove various assumptions made in the course of constructing the system, then state exactly which properties of security transfer to the C program. For example, it is still necessary to prove that Appel (2015)’s concrete spec of SHA-256 is a Merkle-Damgård construction.

In the fields of formal verification and programming languages, more libraries could be written to aid reasoning about equivalence and lengths, as well as facilitate type-level programming in Coq. We anticipate that future work bridging paper proofs with code will run into many of the same problems that we did.

In the field of cryptography, much interesting work is being done now on techniques, frameworks, and formalization of cryptographic protocols and algorithms. EasyCrypt (Backes (2012)), CertiCrypt (Barthe (2009)), and the Foundational Cryptography Framework (Petcher (2015)) are all inspired by Bellare, Rogaway, and Halevi’s vision of code-based, game-based, computer-checked proofs.

Lastly, for secure and practical use of HMAC (say, in a military setting) or any symmetric protocol, one needs a secure method of transmitting the secret key between two agents. To that end, it is desirable to formalize, check, and prove security of an implementation of an asymmetric cryptographic algorithm such as RSA.

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## V. APPENDIX

### A. The abstract HMAC spec

Set Implicit Arguments.

Require Import Bvector.  
Require Import List.  
Require Import Arith.

Definition Blist := list bool.

```
Fixpoint splitVector(A : Set)(n m : nat) : Vector.t
  A (n + m) -> (Vector.t A n * Vector.t A m) :=
  match n with
  | 0 =>
    fun (v : Vector.t A (0 + m)) => (@Vector.nil A
      , v)
  | S n' =>
    fun (v : Vector.t A (S n' + m)) =>
      let (v1, v2) := splitVector _ _ (Vector.tl v
        ) in
      (Vector.cons _ (Vector.hd v) _ v1, v2)
  end.
```

Section HMAC.

```

(* c is the output size, b is the block size (
   larger than the output size),
   and p is the difference between them *)
Variable c p : nat.
Definition b := c + p.

(* The compression function *)
Variable h : Bvector c -> Bvector b -> Bvector c.
(* The initialization vector is part of the spec
   of the hash function. *)
Variable iv : Bvector c.
(* The iteration of the compression function gives
   a keyed hash function on lists of words. *)
Definition h_star k (m : list (Bvector b)) :=
  fold_left h m k.
(* The composition of the keyed hash function with
   the IV gives a hash function on lists of
   words. *)
Definition hash_words := h_star iv.

Variable splitAndPad : Blist -> list (Bvector b).
Hypothesis splitAndPad_1_1 :
  forall b1 b2,
    splitAndPad b1 = splitAndPad b2 ->
      b1 = b2.

Variable fpad : Bvector c -> Bvector p.
Definition app_fpad (x : Bvector c) : Bvector b :=
  (Vector.append x (fpad x)).

Definition h_star_pad k x :=
  app_fpad (h_star k x).

Definition GNMAC k m :=
  let (k_Out, k_In) := splitVector c c k in
  h k_Out (app_fpad (h_star k_In m)).

(* The "two-key" version of GHMAC and HMAC. *)
Definition GHMAC_2K (k : Bvector (b + b)) m :=
  let (k_Out, k_In) := splitVector b b k in
  let h_in := (hash_words (k_In :: m)) in
  hash_words (k_Out :: (app_fpad h_in) :: nil)
  .

Definition HMAC_2K (k : Bvector (b + b)) (m :
  Blist) :=
  GHMAC_2K k (splitAndPad m).

(* opad and ipad are constants defined in the HMAC
   spec. *)
Variable opad ipad : Bvector b.
Hypothesis opad_ne_ipad : opad <> ipad.

Definition GHMAC (k : Bvector b) :=
  GHMAC_2K (Vector.append (BVxor _ k opad) (BVxor
    _ k ipad)).

Definition HMAC (k : Bvector b) :=
  HMAC_2K (Vector.append (BVxor _ k opad) (BVxor _
    k ipad)).

End HMAC.

```

## B. The concrete HMAC spec

```

Require Import Integers.
Require Import Coqlib.
Require Import Coq.Strings.String.
Require Import Coq.Strings.Ascii.
Require Import List. Import ListNotations.

(*SHA256: blocksize = 64bytes
   corresponds to
   #define SHA_LBLOCK 16
   #define SHA256_CBLOCK (SHA_LBLOCK*4) *)

Module Type HASH_FUNCTION.
  Parameter BlockSize:nat. (*measured in bytes; 64
    in SHA256*)
  Parameter DigestLength: nat. (*measured in bytes;
    32 in SHA256*)
  Parameter Hash : list Z -> list Z.
End HASH_FUNCTION.

Module Type HMAC_Module.
  Parameter HMAC: byte -> byte -> list Z -> list Z
    -> list Z.
End HMAC_Module.

Module HMAC_FUN (HF:HASH_FUNCTION) <: HMAC_Module.
  Fixpoint Nlist {A} (i:A) n: list A:=
    match n with 0 => nil
    | S m => i :: Nlist i m
  end.

Definition sixtyfour {A} (i:A): list A:= Nlist i HF.
  BlockSize.

(*Reading rfc4231 reveals that padding happens on
   the right*)
Definition zeroPad (k: list Z) : list Z :=
  k ++ Nlist ZO (HF.BlockSize-length k).

Definition mkKey (l:list Z) : list Z :=
  if Z.gtb (Zlength l) (Z.of_nat HF.BlockSize)
  then (zeroPad (HF.Hash l))
  else zeroPad l.

Definition mkArg (key:list byte) (pad:byte): list
  byte :=
  (map (fun p => Byte.xor (fst p) (snd p))
    (combine key (sixtyfour pad))).
Definition mkArgZ key (pad:byte): list Z :=
  map Byte.unsigned (mkArg key pad).

(*
   Definition Ipad := P.Ipad.
   Definition Opad := P.Opad.
   *)
(*innerArg to be applied to message, (map Byte.repr
   (mkKey password)))*)
Definition innerArg IP (text: list Z) key : list Z
  :=
  (mkArgZ key IP) ++ text.

Definition INNER IP k text := HF.Hash (innerArg IP

```

text k).

Definition outerArg OP (innerRes : list Z) key : list Z :=  
(mkArgZ key OP) ++ innerRes.

Definition OUTER OP k innerRes := HF.Hash (outerArg OP innerRes k).

Definition HMAC IP OP txt password : list Z :=  
let key := map Byte.repr (mkKey password) in  
OUTER OP key (INNER IP key txt).

End HMAC\_FUN.

Require Import SHA256.  
Require Import functional\_prog.

Module SHA256\_ <: HASH\_FUNCTION.  
Definition BlockSize := 64%nat.  
Definition DigestLength := 32%nat.  
Definition Hash : list Z -> list Z := SHA\_256'.  
End SHA256\_.

Module HMAC\_SHA256 := HMAC\_FUN SHA256\_.

Definition Ipad := Byte.repr 54. (\*0x36\*)  
Definition Opad := Byte.repr 92. (\*0x5c\*)

Definition HMAC256 := HMAC\_SHA256.HMAC Ipad Opad.

Definition HMACString (txt passwd:string) : list Z :=  
HMAC256 (str\_to\_Z txt) (str\_to\_Z passwd).

Definition HMACHex (text password:string) : list Z :=  
HMAC256 (hexstring\_to\_Zlist text) (  
hexstring\_to\_Zlist password).

Definition check password text digest :=  
listZ\_eq (HMACString text password) (  
hexstring\_to\_Zlist digest) = true.

### C. Definitions

Definition c:nat := (SHA256\_.DigestLength \* 8)%nat.  
Definition p:=(32 \* 8)%nat.

Definition sha\_iv : Blist :=  
bytesToBits (SHA256.intlist\_to\_Zlist SHA256.  
init\_registers).

Check SHA256.hash\_blocks. (\* SHA256.registers ->  
list int -> SHA256.registers \*)

Definition sha\_h (regs : Blist) (block : Blist) :  
Blist :=  
bytesToBits (SHA256.intlist\_to\_Zlist  
(SHA256.hash\_block (SHA256.  
Zlist\_to\_intlist (bitsToBytes  
regs))  
(SHA256.  
Zlist\_to\_intlist  
(bitsToBytes  
block)))

)).

Definition sha\_splitandpad (msg : Blist) : Blist :=  
bytesToBits (sha\_padding\_lemmas.pad (bitsToBytes  
msg)).

Definition convert (l : list int) : list bool :=  
bytesToBits (SHA256.intlist\_to\_Zlist l).

Definition convertByteBits (bits : Blist) (byte : Z)  
: Prop :=  
exists (b0 b1 b2 b3 b4 b5 b6 b7 : bool),  
bits = [b0; b1; b2; b3; b4; b5; b6; b7] /\  
byte = (1 \* (asZ b0) + 2 \* (asZ b1) + 4 \* (asZ b2  
) + 8 \* (asZ b3)  
+ 16 \* (asZ b4) + 32 \* (asZ b5) + 64 \* (asZ  
b6) + 128 \* (asZ b7)).

Inductive bytes\_bits\_lists : Blist -> list Z -> Prop  
:=  
| eq\_empty : bytes\_bits\_lists nil nil  
| eq\_cons : forall (bits : Blist) (bytes : list Z)  
(b0 b1 b2 b3 b4 b5 b6 b7 : bool)  
(byte : Z),  
bytes\_bits\_lists bits bytes ->  
convertByteBits [b0; b1; b2; b3; b4;  
b5; b6; b7] byte ->  
bytes\_bits\_lists (b0 :: b1 :: b2 ::  
b3 :: b4 :: b5 :: b6 :: b7 ::  
bits)  
(byte :: bytes).

### D. The equivalence proof

Theorem HMAC\_spec\_equiv : forall  
(K M H : list Z) (OP IP :  
Z)  
(k m h : Blist) (op ip :  
Blist),  
((length K) \* 8)%nat = (c + p)%nat ->  
Zlength K = Z.of\_nat SHA256\_.BlockSize ->  
(\* TODO: first implies this \*)  
bytes\_bits\_lists k K ->  
bytes\_bits\_lists m M ->  
bytes\_bits\_lists op (HMAC\_SHA256.sixtyfour OP) ->  
bytes\_bits\_lists ip (HMAC\_SHA256.sixtyfour IP) ->  
HMAC c p sha\_h sha\_iv sha\_splitandpad op ip k m =  
h ->  
HMAC\_SHA256.HMAC IP OP M K = H ->  
bytes\_bits\_lists h H.

Proof.

intros K M H OP IP k m h op ip.  
intros padded\_key\_len padded\_key\_len\_byte  
padded\_keys\_eq msgs\_eq ops\_eq ips\_eq.  
intros HMAC\_abstract HMAC\_concrete.

intros.  
unfold p, c in \*.  
simpl in \*.

rewrite <- HMAC\_abstract. rewrite <- HMAC\_concrete  
.

```

unfold HMAC. unfold HMAC_SHA256.HMAC. unfold
  HMAC_SHA256.OUTER. unfold HMAC_SHA256.INNER.
unfold HMAC_SHA256.outerArg. unfold HMAC_SHA256.
  innerArg.

unfold HMAC_2K. unfold GHMAC_2K. rewrite ->
  split_append_id.

simpl.

(* Major lemmas *)
Check SHA_equiv_pad.
apply SHA_equiv_pad.
Check concat_equiv.
apply concat_equiv.
SearchAbout bytes_bits_lists.
apply xor_equiv_Z; try assumption.

*
  apply SHA_equiv_pad.
  apply concat_equiv.

- apply xor_equiv_Z; try assumption.
- assumption.
(* xors preserve length *)
*
  unfold b in *. simpl. unfold BLxor. rewrite ->
    list_length_map.
  rewrite -> combine_length.
  pose proof bytes_bits_length op (HMAC_SHA256.
    sixtyfour OP) as ops_len.

```

```

rewrite -> ops_len.
pose proof bytes_bits_length k K as keys_len.
rewrite -> keys_len.
rewrite -> padded_key_len.
unfold HMAC_SHA256.sixtyfour.
rewrite -> length_list_repeat.
reflexivity.
apply padded_keys_eq.
apply ops_eq.

*
  unfold b in *. simpl. unfold BLxor. rewrite ->
    list_length_map.
  rewrite -> combine_length.
  pose proof bytes_bits_length ip (HMAC_SHA256.
    sixtyfour IP) as ips_len.
  rewrite -> ips_len.
  pose proof bytes_bits_length k K as keys_len.
  rewrite -> keys_len.
  rewrite -> padded_key_len.
  unfold HMAC_SHA256.sixtyfour.
  rewrite -> length_list_repeat.
  reflexivity.
  apply padded_keys_eq.
  apply ips_eq.

Qed.

```

## E. Other theorems and proofs

See the repository at  
[github.com/hypotext/vst-hmac](https://github.com/hypotext/vst-hmac).