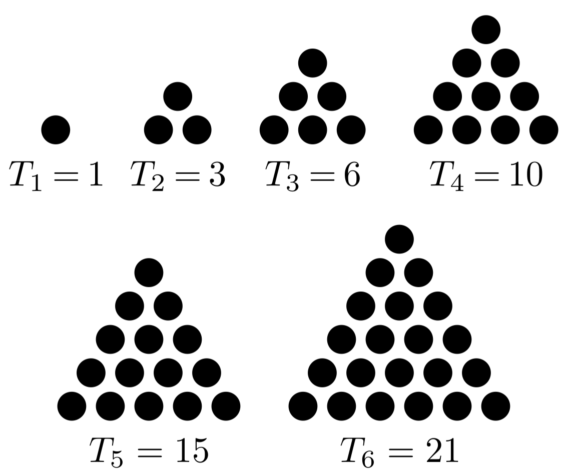
Ethan Simmonds (ID: 1423402)

House Numbers - solution report

To solve Gustav’s conundrum we begin with the definition of some terms.

Triangular numbers

Triangular numbers count sequences which can be arranged in a triangle. The first handful of these numbers can be shown as such:



The formula to determine triangular numbers is:

$$\frac{n \cdot (n+1)}{2}$$

An example from above may be a more pragmatic description:

$$n = 3 \rightarrow \frac{3 \cdot (3+1)}{2} \rightarrow T_3 = 6$$

The result of T_3 can be seen in the sample image.

Square-triangular numbers

Square-triangular numbers are the sequence of numbers which are both triangular numbers and perfect squares. The square root of a perfect square is a whole number, which is how we know they are perfect squares. We can prove a number is a square-triangular via a small example:

An example of a proven square-triangular number is **36.**

36 is triangular as determined by the triangular number formula:

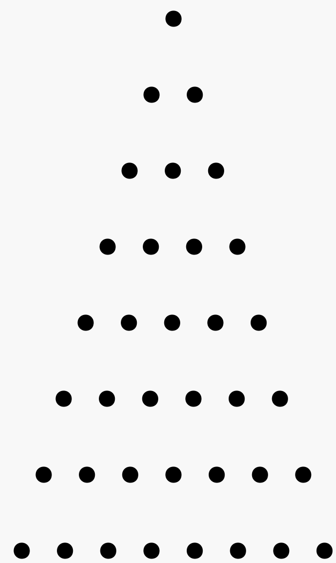
$$n = 8 \rightarrow \frac{8 \cdot (8+1)}{2} \rightarrow T_8 = 36$$

36 is also a perfect square as determined by the square root:

$$\sqrt{36} = 6$$

Therefore 36 is a square-triangular number. We can further demonstrate this with the following illustration.

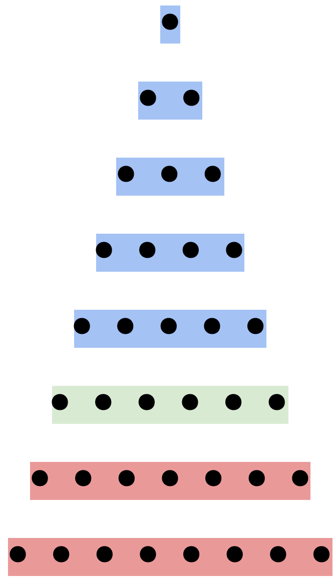
If T_8 the triangular number sequence will look like this:



As a triangular number sequence of order 8 (T_8) the sum of these dots is 36. The total number of rows is equal to this order.

In Gustav’s problem each house number would be represented by a dot in the triangular number sequence. The total amount of dots here (meaning the total number of houses) is 36. If we take the square root of this number (since we already know it is a perfect square) we get the number 6. If we examine the 6th order row of numbers we can see an interesting pattern to the rows above and below.

Ethan Simmonds (ID: 1423402)



The 6th row (highlighted in green) acts as the intersection point for the groups of house numbers above and below it to be the same sum. To better explain, the total amount of houses above the green line (highlighted in blue) is 15. The total amount of houses below the green line (highlighted in red) is also 15.

These two numbers being the same demonstrates that given a square-triangular number, it can be divided into two equilateral triangles which is the basis for:

$$1 + \ldots + (k-1) = (k+1) + \ldots + n$$

Where the lesser triangle (blue-highlighted numbers) represents the left side of the equation, and the the higher triangle (red-highlighted numbers) represents the right.

We have represented the numbers above in a triangular layout. It may be easier to immediately see the result when they are placed in square form (since they **square**-triangular numbers after all):

X	O	O	O	O	O
O	X	O	O	O	O
O	O	X	O	O	O
O	O	O	X	O	O
O	O	O	O	X	O
O	O	O	O	O	X

By laying out the numbers in square form we can see that when the line of crosses (highlighted in green) intersects the square of numbers straight through the middle, an equilateral right-angle triangle is formed on each side. This reinforces the fact that if numbers are square-triangular that division of the square number results in two equal triangular numbers.

The sum of dots (or noughts) that these two triangles form is equal to the sum of dots from rows $1 + ... + (k-1)$ and $(k + 1) + ... + n$ where *k* is the square root and n is the triangular root. These are the components that make up a square-triangular number.

Ethan Simmonds (ID: 1423402)

If we wanted to find all instances of *k* and *n* even outside the scope of this example, there are two recurrent formulas which can help find these:

For k:

$$k(x) = 6 * k(x - 1) - k(x - 2),\ with\ k(0) = 0,\ k(1) = 1$$

For n:

$$n(x) = 6 * k(x - 1) - k(x - 2) + 2,\ with\ k(0) = 0,\ k(1) = 1$$

Ethan Simmonds (ID: 1423402)

References

Sloane, N. J. A., A001108, *The Online Encyclopedia of Integer Sequences,* https://oeis.org/A001108

Sloane, N. J. A., A001109, *The Online Encyclopedia of Integer Sequences,* https://oeis.org/A001109

Sloane, N. J. A., A001110, *The Online Encyclopedia of Integer Sequences,* https://oeis.org/A001110

Moore, Thomas. (2013), *An Investigation Relating Square and Triangular Numbers*, (p. 1 - 4) https://vc.bridgew.edu/cgi/viewcontent.cgi?referer=https://www.google.com/&httpsredir=1&article=1036&context=math_fac

Wikipedia, https://en.wikipedia.org/wiki/Square_number

Wikipedia, https://en.wikipedia.org/wiki/Triangular_number

Wikipedia, https://en.wikipedia.org/wiki/Square_triangular_number