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Mathematica Maple MATLAB HP 48G

Equilibrium Temperature Distributions

Text Reference: Section 2.5, p. 150

The purpose of this set of exercises is to discuss a physical situation in which solving a system of linear equations becomes necessary: that of determining the equilibrium temperature of a thin plate. Methods for solving these systems will be compared, and some will be shown to be more efficient than others.

Consider a thin square plate whose faces are insulated from heat. Suppose that the temperature along the four edges of the plate is known, and further suppose that those temperatures are held constant. After some time has passed, the temperature inside the plate will reach an equilibrium. Finding this equilibrium temperature distibution at the points on the plate is desirable, given only the temperature data from the edges of the plate. Unfortunately, the exact determination of this temperature distribution is a difficult problem. An appproximation to the exact distribution may be found by **discretizing the problem**; that is, by only considering a few points on the plate and approximating the temperature at those points.

A property from thermodynamics helps with the discretization of the problem. The property says that if equilibrium has been acheived, then the temperature at a point is the average value of the temperature at surrounding points.

The Mean Value Property: If a plate has reached thermal equilibrium, and P is a point on the plate, and C is a circle centered at P fully contained in the plate, then the temperature at P is the average value of the temperature function over C.

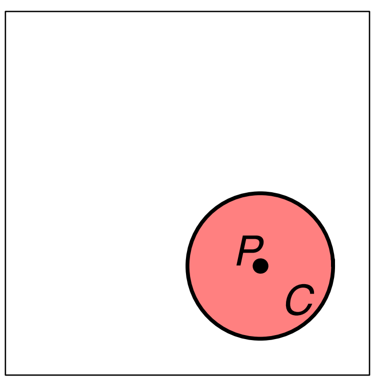

Figure 1: The Mean Value Property

See Figure 1 for a picture of this situation. In order to discretize the problem, place a grid over the plate and concentrate only on the points where the grid lines cross; the temperature at only those points will be considered. The grid is fashioned so that some grid points lie on the boundary of the plate; assume that the temperature at these points equals the external temperature. At grid points inside the plate, assume the following version of the Mean Value Property.

The Discretized Mean Value Property: If a plate has reached thermal equilibrium, and P is a grid point not on the boundary of the plate, then the temperature at P is the average of the temperatures at the four closest grid points to P.

Example: Consider placing the following grid on the square plate; see Figure 2. There are four points inside the plate to consider; the temperatures at these points are labelled x_1, x_2, x_3, and x_4. Assume that the exterior temperatures are as labelled in Figure 2. The Discretized Mean Value Property gives rise to the following four equations:

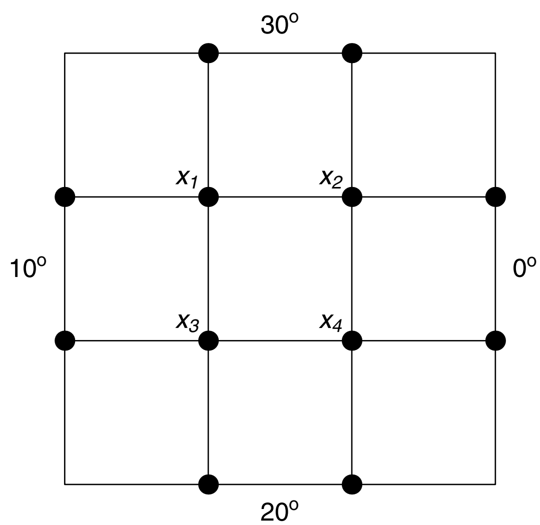

Figure 2: Grid on a Square Plate

$$x_1 = \frac{40 + x_2 + x_3}{4}, x_2 = \frac{30 + x_1 + x_4}{4}, x_3 = \frac{30 + x_1 + x_4}{4}, x_4 = \frac{20 + x_2 + x_3}{4}$$

which is equivalent to the following system of linear equations:

$$\begin{array}{rcrcrcrcr} 4x_1 & - & x_2 & - & x_3 & & & = & 40 \\ -x_1 & + & 4x_2 & & & - & x_4 & = & 30 \\ -x_1 & & & + & 4x_3 & - & x_4 & = & 30 \\ & - & x_2 & - & x_3 & + & 4x_4 & = & 20 \end{array}$$

What if the external temperatures change? A new system of equations would need to be solved, but it would be very similar to the one previously considered. To handle this difficulty more easily, the system may be rewritten in the form

$$
\begin{array}{rcrcrcr}
4x_1 & = & x_2 & + & x_3 & + & 40 \\
4x_2 & = & x_1 & + & x_4 & + & 30 \\
4x_3 & = & x_1 & + & x_4 & + & 30 \\
4x_4 & = & x_2 & + & x_3 & + & 20
\end{array}
$$

If

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 40 \\ 30 \\ 30 \\ 20 \end{bmatrix}$$

then $4\mathbf{x} = C\mathbf{x} + \mathbf{b}$, and $\mathbf{x}$ is the vector of equilibrium temperatures. Notice that C functions as a type of adjacency matrix; the (i, j) entry is a 1 if the points corresponding to x_i and x_j are connected in the grid, and is 0 otherwise. To solve for $\mathbf{x}$, note that $(4I - C)\mathbf{x} = \mathbf{b}$, and if $A = 4I - C$, the solution to this system may be written as

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Even though this formula appears to simplify matters when the external temperatures change, difficulties with using A^{-1} abound, especially for large systems. See the Numerical Notes on pages 125 and 146 for more details. The LU decomposition may be used to alleviate these problems. If $A = LU$, the system $A\mathbf{x} = \mathbf{b}$ becomes $LU\mathbf{x} = \mathbf{b}$. As shown in Section 2.5, the systems $L\mathbf{y} = \mathbf{x}$ and $U\mathbf{x} = \mathbf{y}$ can be solved in turn to find $\mathbf{x}$.

Questions:

1. Solve the above system by row reduction.
2. Find A, A^{-1}, and $\mathbf{x}$ for the above system of equations. Confirm that your answer matches that of Question 1.
3. Suppose that the external temperature at the top of the plate changes to $50°$. Use results from Question 2 to find the new equilibrium temperature distribution.
4. Do an LU decomposition of A above. Use it to solve the original system and the system with altered external temperatures. Confirm your earlier results.
5. A finer grid should give a better idea of the equilibrium temperatures. Consider the grid in Figure 3. Now there are 25 grid points inside the plate, so now a system of 25 equations in 25 unknowns must be solved. The matrix C and vector $\mathbf{b}$ which accompany this set apply to this situation. Find the temperature distribution in this case with the original external temperatures by either finding $(4I - C)^{-1}$ or by using the LU decomposition.

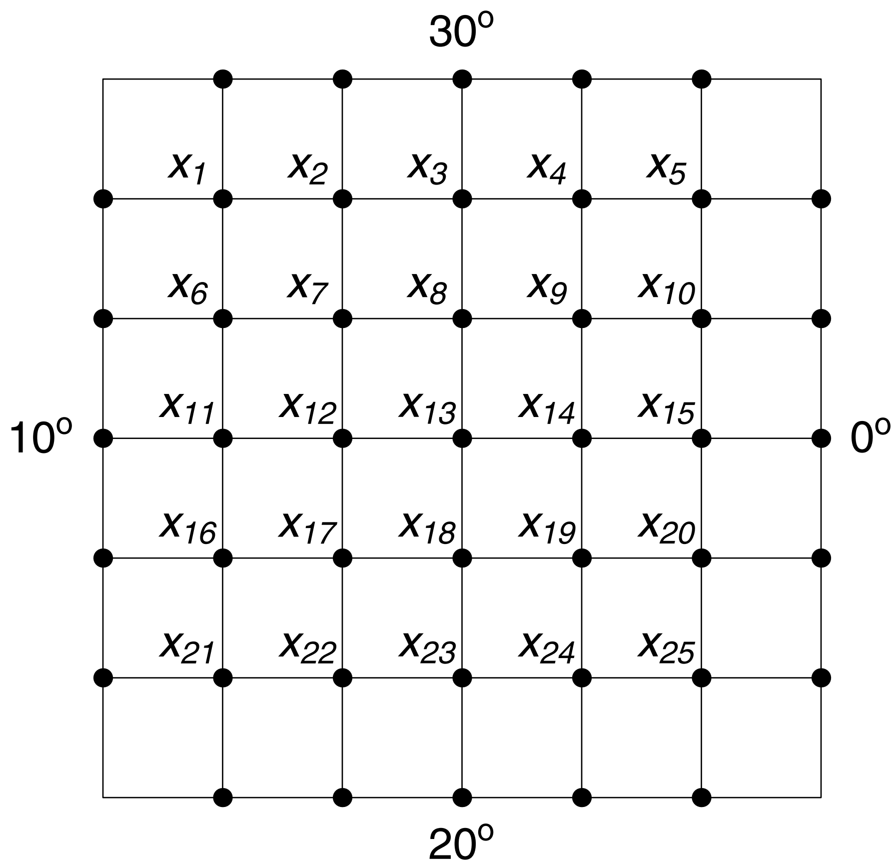

Figure 3: Finer Grid on a Square Plate