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The Fibonacci Sequence and Generalizations

Text Reference: Section 5.3, p. 325

The purpose of this set of exercises is to introduce you to the much-studied Fibonacci sequence, which arises in number theory, applied mathematics, and biology. In the process you will see how useful eigenvalues and eigenvectors can be in understanding the dynamics of difference equations.

The Fibonacci sequence is the sequence of numbers

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots .$$

You can probably see the pattern: Each number is the sum of the two numbers immediately preceding it; if y_k is the kth number in the sequence (with $y_0 = 0$), then how can y_{100} be found without just computing the sequence term by term? The answer to this question involves matrix multiplication and eigenvalues. The Fibonacci sequence is governed by the equation

$$y_{k+2} = y_{k+1} + y_k,$$

or

$$y_{k+2} - y_{k+1} - y_k = 0.$$

If you have studied Section 4.8, you will recognize the last equation as a second-order linear difference equation. For reasons which will shortly become apparent, a trivial equation is added to get the following system of equations:

$$\begin{aligned} y_{k+1} &= y_{k+1} \\ y_{k+2} &= y_{k+1} + y_k \end{aligned}$$

To see how linear algebra applies to this problem, let

$$\mathbf{u}_k = \begin{bmatrix} y_k \\ y_{k+1} \end{bmatrix}$$

The above system of equations may then be written as

$$\mathbf{u}_{k+1} = A\mathbf{u}_k$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

To find y_k, just look at the bottom entry in $\mathbf{u}_k$. The vector $\mathbf{u}_k$ could be written in terms of $\mathbf{u}_0$ by noting that

$$\mathbf{u}_k = A\mathbf{u}_{k-1} = AA\mathbf{u}_{k-2} = \cdots = A^k\mathbf{u}_0$$

The first goal is to find an easy way to compute A^k. This is where eigenvalues and eigenvectors enter the picture.

Questions:

1. Using your technology, compute A^5 and use it to find $\mathbf{u}_5$ and y_5.

2. Show that the eigenvalues of A are

$$\lambda_1 = \frac{1+\sqrt{5}}{2}, \lambda_2 = \frac{1-\sqrt{5}}{2}$$

   by solving the characteristic equation of A.

3. Show that

$$\begin{bmatrix} -\lambda_2 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\lambda_1 \\ 1 \end{bmatrix}$$

   are eigenvectors of A corresponding to λ_1 and λ_2 respectively. You may find it helpful to note $\lambda_1 + \lambda_2 = 1$ and $\lambda_1\lambda_2 = -1$.

4. Explain why A is diagonalizable.

5. Find (by hand) a matrix P and a diagonal matrix D for which $A = PDP^{-1}$.

6. Use your technology to calculate D^{10}, and use it to find A^{10}, $\mathbf{u}_{10}$, and y_{10}. Confirm your result for y_{10} by writing out the Fibonacci sequence by hand.

7. A formula for y_k may be derived using the following two questions. Use the above expressions for P, D, and A^k to show that a general form for A^k is

$$A^k = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_2^k\lambda_1 - \lambda_1^k\lambda_2 & \lambda_2^{k+1}\lambda_1 - \lambda_1^{k+1}\lambda_2 \\ \lambda_1^k - \lambda_2^k & \lambda_1^{k+1} - \lambda_2^{k+1} \end{bmatrix}$$

8. Use the result of Question 7 to find $\mathbf{u}_k$ and y_k. Again make note of the fact that $\lambda_1\lambda_2 = -1$. If you've worked it out all right, you should have found that

$$y_k = \frac{1}{\sqrt{5}}\left(\lambda_1^k - \lambda_2^k\right) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$$

   Calculate y_{10} using this formula, and compare your result to that of Question 6.

9. Notice that the second of the two terms in parentheses is less than 1 in absolute value, so as higher and higher powers are taken, it will approach zero. The following equation results:

$$y_k \approx \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k\right]$$

   Use this approximation to approximate y_{k+1}/y_k.

The approximation for y_{k+1}/y_k which you found in the last question is called the golden ratio, or golden mean. The ancient Greek mathematicians thought that this ratio was the perfect proportion for the rectangle. That it appears in such a "remote" area as the limiting ratio for the Fibonacci sequence (which occurs in nature in sunflowers, nautilus shells, and in the branching behavior of plants) makes one wonder about the connection between nature, beauty, and number.

The above work on the Fibonacci sequence can be generalized to discuss any difference equation of the form

$$y_{k+2} = ay_{k+1} + by_k,$$

where a and b can be any real numbers. A sequence derived from this equation is often called a **Lucas sequence.**

Questions:

10. Consider the Lucas sequence generated by the difference equation

$$y_{k+2} = 3y_{k+1} - 2y_k,$$

with $y_0 = 0$ and $y_1 = 1$. Write out by hand the first seven terms of this sequence and see if you can find the pattern. Then repeat the above analysis on this sequence to find a formula for y_k.

11. Consider the Lucas sequence generated by the difference equation

$$y_{k+2} = 2y_{k+1} - y_k,$$

with $y_0 = 0$ and $y_1 = 1$. Find the pattern by writing out as many terms in the sequence as you need. Will an analysis like that for the Fibonacci sequence work in this case? Why or why not?