

Dominance Matrices

Text Reference: Section 2.1, p. 114

The purpose of this set of exercises is to apply matrices and their powers to questions concerning various forms of competition between individuals and groups.

In Section 2.1 powers of a square matrix A are defined:

$$A^k = \underbrace{A \cdots A}_k$$

To see one area in which matrix powers are useful, consider the following situation.

A football conference consists of 5 teams: Teams A, B, C, D, and E. Each team plays the other 4 once, and the results this season were:

A	defeated	C and D
B	defeated	A, C, and E
C	defeated	D
D	defeated	B
E	defeated	A, C, and D

The question is how to rank these teams. Teams B and E both won 3 games while losing 1. Which team is more powerful? How shall the tie between these teams be broken? One way is to note that B defeated E, and thus should be more powerful. In this case B has **one-step dominance** over E. There is a problem with this method of tie-breaking: in larger conferences each team may not play every other team. Another way to handle the question of tie-breaking is to investigate **two-step dominance**. For example, A has two-step dominance over B because A defeated D and D defeated B. Notice that one team can have two-step dominance over another in multiple ways: team B has two-step dominance over team C in two ways: B defeated A who defeated C, and B defeated E who defeated C.

Question:

1. Find all of the two-step dominances in the example above, keeping track of the number of ways each dominance occurs.

Luckily, there is an easier way to calculate two-step dominances. Create a 5×5 matrix by first associating each team with a corresponding row and column of the matrix: team A corresponds to row 1 and column 1, team B to row and column 2, etc. Place a 1 at the (i, j) entry of the matrix if the team corresponding to row i defeated the team corresponding to column j . Otherwise a 0 is recorded. The example generates the following **dominance matrix**:

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Thus each win for team A, for example, is encoded in the matrix A as a 1 in the appropriate place in the first row of the matrix A . If the entries in the first row of the matrix are added, the total number of wins (or one-step dominances) which Team A has will result. To compute two-step dominances, consider the matrix

$$A^2 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix}$$

By the row-column rule (p. 103 of the text), the (i, j) entry in A^2 is

$$a_{i1}a_{1j} + a_{i2}a_{2j} + a_{i3}a_{3j} + a_{i4}a_{4j} + a_{i5}a_{5j}$$

Each term in this sum is either a 0 or a 1, since the terms in A are either 0's or 1's. If a term of the form $a_{ik}a_{kj} = 1$, then $a_{ik} = 1$ and $a_{kj} = 1$. In terms of the athletic competition, $a_{ik} = 1$ and $a_{kj} = 1$ implies that the i^{th} team defeated the k^{th} team and the k^{th} team defeated the j^{th} team. Thus there is one way in which the i^{th} team has two-step dominance over the j^{th} team. Adding up all terms of the form $a_{ik}a_{kj}$ will thus count how many two-step dominances the i^{th} team has over the j^{th} team.

Question:

2. By comparing the entries in A^2 with your answer to Question 1, confirm that the (i, j) entry in A^2 is the number of ways in which the team corresponding to row i has two-step dominance over the team corresponding to column j .

Now the entries in each row of A^2 may be added to see how many two-step dominances each team has. Team A has 2, team B has 6, team C has 1, team D has 3, and team E has 4. So since team B has 6 two-step dominances and team E has only 4, two-step dominances can be used to break their tie and declare B the champion. Alternatively, one could count both one-step and two-step dominances by computing

$$A + A^2 = \begin{bmatrix} 0 & 1 & 1 & 2 & 0 \\ 2 & 0 & 3 & 3 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 3 & 0 \end{bmatrix}$$

then adding the entries in each row of this matrix. Some authors in the fields of sociology and psychology call this sum the **power** of the team associated with row i . The powers of the example teams are:

- A 4
- B 9
- C 2
- D 4
- E 7

By this measure, B is the most powerful (or dominant) team, followed by E, A and D (tied), then C. Of course, this measure of power treats two-step dominances to be as important as one-step dominances, and this can create odd results.

Now consider a real-world example of a power calculation. Here is the dominance matrix for the 2001 Big 12 Conference football season. Rows and columns are keyed to the teams listed. This matrix is also the matrix B that accompanies this exercise set.

	Bay	Col	ISU	Kan	KSU	Mis	Neb	Okl	OSU	Tex	TAM	TTU
Baylor	0	0	0	0	0	0	0	0	0	0	0	0
Colorado	0	0	1	1	1	1	1	0	1	0	1	0
Iowa State	1	0	0	1	0	1	0	0	1	0	0	0
Kansas	0	0	0	0	0	0	0	0	0	0	0	1
Kansas State	0	0	1	1	0	1	0	0	0	0	0	0
Missouri	1	0	0	1	0	0	0	0	1	0	0	0
Nebraska	1	0	1	1	1	1	0	1	0	0	0	1
Oklahoma	1	0	0	1	1	0	0	0	0	1	1	1
Oklahoma State	1	0	0	0	0	0	0	1	0	0	0	0
Texas	1	1	0	1	0	1	0	0	1	0	1	1
Texas A& M	1	0	1	0	1	0	0	0	1	0	0	0
Texas Tech	1	0	0	0	1	0	0	0	1	0	1	0

Questions:

3. Compute the number of victories for each team and the power of each team. Rank-order the teams by number of victories and power. **Note:** You can easily add up each row of an $n \times n$ matrix by multiplying the matrix by a column vector containing n 1's.
4. Consider the power rankings of Oklahoma State and Missouri. How is it possible that Oklahoma State has a higher power ranking than Missouri while Oklahoma State has fewer victories than Missouri **and** Missouri defeated Oklahoma State?
5. The dominance matrix for the 2001 Southeastern Conference football season accompanies this exercise set as matrix S . The rows represent, respectively: Alabama, Arkansas, Auburn, Florida, Georgia, Kentucky, Louisiana State, Mississippi, Mississippi State, South Carolina, Tennessee, and Vanderbilt. Compute the number of victories for each team and the power of each team. Rank-order the teams in number of victories and in power.

$$S = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

6. How might the analysis be altered to allow teams to play each other more than once? Consider what two-step dominance means in such a situation; could a team have two-step dominance over itself? To help with this investigation, you may want to look at the matrix N which accompanies this set, which is a dominance matrix for the 2001-2002 NFL season. Since NFL teams may play each other twice in a season, the (i, j) entry in N is the number of times the team for row i defeated the team for column j . The rows of this matrix correspond to the following teams: Buffalo, Indianapolis, Miami, New England, New York (Jets), Baltimore, Cincinnati, Cleveland, Jacksonville, Pittsburgh, Tennessee, Denver, Kansas City, Oakland, San Diego, Seattle, Arizona, Dallas, New York (Giants), Philadelphia, Washington, Chicago, Detroit, Green Bay, Minnesota, Tampa Bay, Atlanta, Carolina, New Orleans, St. Louis, and San Francisco.