The Optimal Portfolio and the Efficient Frontier

Module Author

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Math level and Science level

The module covers topics in finance, economics, statistics, calculus, and matrix algebra. It is suggested that students have taken one year of calculus and know the concept of a derivative and optimization from first order conditions. A course in introductory statistics and matrix algebra will be very helpful. Courses in micro-economics or introductory finance would be an added advantage but not necessary. A course in computational science (Computational Science I) would be beneficial in giving them the background in modeling techniques prior to beginning this module.

Overview

This module is designed for undergraduate students in mathematics, statistics, physics, computer science, engineering and geology who wish to take an introductory course in computational finance. This module would also cater to undergraduate students in economics and finance having strong quantitative background and aptitude. The module can be used as an integrated part of a "Computational Finance" course or a stand-alone component within the typical "Investments" or "security analysis and portfolio management" course in finance. The course can be taught in the class room or an excel lab or both. The module is intended to be taught at an intermediate point in the course. Prior to reaching the teaching module, the students should have a basic knowledge of financial markets and financial assets like stocks. The students should also be familiar with using formulas in excel and have some knowledge of charting in excel.

Learning goals:

- Understand the concept of Expected value and random variables.
- Understand the concept of risk in portfolio context and able to measure risk using real world financial data.
- Understand the benefits of diversification and able to quantify the benefits in the form of reduced portfolio risk.
- Apply Modern portfolio theory techniques to trace out a Portfolio frontier in the risk-return space.
- Understand how the portfolio frontier changes as the degree of correlation between stocks change.
- Apply the techniques to create a real world investment portfolio of stocks.
- To be able to select points (or optimal portfolios) on the efficient frontier that corresponds to different investment objectives and preferences.

This module is implemented using Microsoft Excel which is the most widely used spreadsheet application. [1]

[1] Students not having access to Microsoft Excel can use the spreadsheets with "OpenOffice Calc" which is part of an open source office suite available for free download at www.OpenOffice.org. However, students

Keywords

Optimal Portfolio, Efficient Frontier, Risk, Expected Return and Risk-free asset.

Introduction

Finance theory has become increasingly mathematical and in recent years there is an increasing trend of mathematics and physics Ph.Ds heading towards financial services industry rather than the ivory tower. There is a lack of computational finance materials in undergraduate level which stems from the fact that undergraduate finance and economics are tend to be taught in business schools or under social sciences and more mathematical oriented courses are in Math and natural science departments. In recent years we have seen some increase in academic programs that embrace computational finance but mostly at the graduate level. This module can be treated as a small step towards integrating computational finance at undergraduate level. There is perhaps no better way to start than introducing the "Modern Portfolio Theory", introduced for the first time by Harry Markowitz, a Nobel laureate, in his 1952 paper, "Portfolio Selection", which laid the groundwork for mathematical theorization in finance. He developed a notion of mean return and covariances for stocks which allowed him to quantify what is called "not putting all your eggs in one basket" or more formally the concept of "diversification".

At the end of the module, the students would be able to build real world investment portfolio of risky assets using modern portfolio theory techniques. The students should also be equipped to make modifications to the portfolio if some real world variable changes or if some assumptions of the model are relaxed.

Problem Statement

Investors and portfolio managers concentrate their efforts on achieving the best possible trade-off between risk and return. For portfolios constructed from a fixed set of assets, the risk/return profile varies with the portfolio composition. Portfolios that maximize the return, given the risk, or, conversely, minimize the risk for the given return, are called optimal. Optimal portfolios define a line in the risk/return plane called the efficient frontier. In the module we address questions like - How do you determine optimal investment allocations in a portfolio? Where is the best investment portfolio given investor's objectives and preferences? What would influence different allocations?

But first things first, we need to know some basic background material before we can start answering the questions.

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may have to download some add-ins in order to use the formulas properly.

Background Information

Returns

Stocks prices are the most widely available financial data in US. When investors invest in a company stock they expect return in the form of dividends[2] and capital gains. The stock return in any time period *t*, is simply the ratio of the sum of the dividends, D_t, plus the capital gains[3] to the stock price at time *t-1*. Algebraically, return r_t is given as

$$r_t = \frac{D_t + (P_t - P_{t-1})}{P_{t-1}}$$

There are several good websites from where students can download price information. Some of them are MSN money, NASDAQ.com, Bloomberg.com. I particularly prefer Yahoo! Finance and it has a wealth of financial information in the website. It is well structured and it automatically adjusts the price data to account for dividends and stock splits[4]. So, the return formula using Yahoo! Finance data reduces to:

$$r_t = \frac{P_t}{P_{t-1}} - 1$$

For details on how to download stock price data from Yahoo and computing the monthly stock returns, refer to the sub-section "*Downloading Stock data from Yahoo! Finance and calculating the monthly stock returns using Excel*" under the section "Data".

Expected Values

One of the most important statistical concepts used in finance is that of expected value, which is very similar to a mean or the arithmetic average. The crucial difference is that mean or arithmetical average is used if we work with past outcomes, while the expected value is applied when we work with future outcomes. When the future outcomes of a variable is uncertain, the variable is called stochastic or a random variable. For example

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[2] Some companies like CSCO have never paid a dividend. Investors buy non-dividend paying stocks because they tend to compensate the investors with higher stock prices.

[3] Capital gains or loss from a stock at time t is the difference between the stock price at time *t* and price at time *t-1* i.e., $(P_t - P_{t-1})$.

[4] Companies may decide to split their stock in to two or more parts. For example, if Microsoft price is $40, and the company announces a 2:1 stock split, then each stock of $40 would be now counted as 2 stocks of $20 each. A company may also wish to do a reverse split usually when they think their stock value is too low. A 1:2 reverse split would bundle two unit stocks into a single unit. If Microsoft goes for a reverse split in our example, then 2 stocks of $40 would be bundled into a single stock of $80. Yahoo! Finance takes into account the effect of any such stock splits using a split adjustment factor.

say, you toss a coin, which can either come up heads or tails with equal chance or probability, you receive \$6.00 if the coin comes up heads and \$0 if the coin comes up tails. We can define a random variable *r* (for coin-toss dollar return) that takes \$6.00 with 50% probability (for 'heads') and \$0 with 50% probability (for 'tails'). The expected value of '*r*' is given by

$$E(r) = 50\% \times \$6.00 \ + \ 50\% \times \$0 = \$3.00$$

Where the notation E denotes expected value

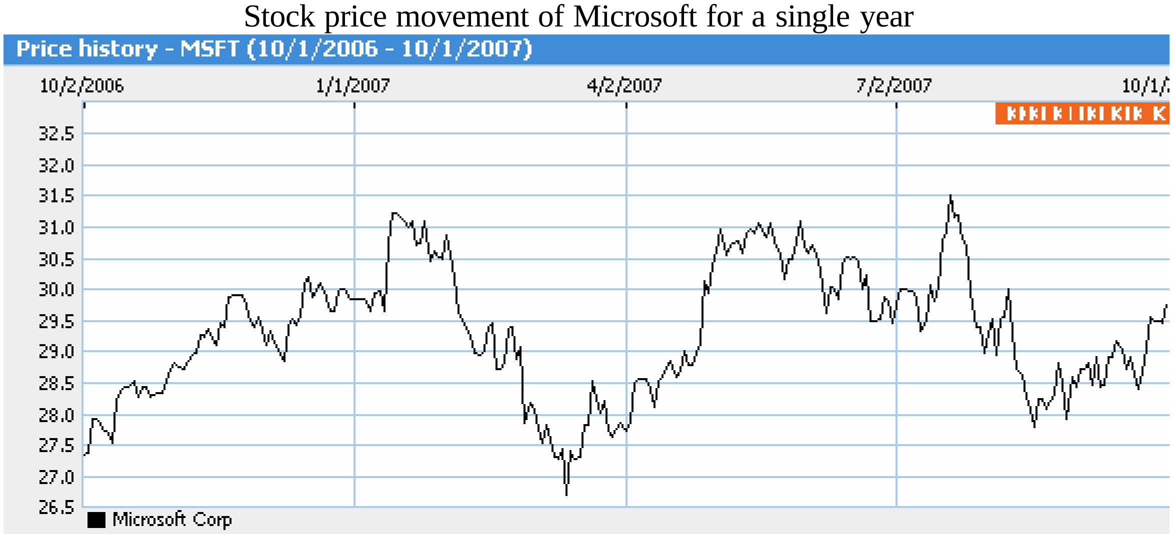
In any one throw, \$3.00 will never come up, but if you repeated (and recorded) this experiment hundreds of times, the mean of actual outcomes would converge to \$3.00.

After the coin toss, *r* is no longer a random variable as you have an actual outcome. In finance we also call it as an ex-post return as opposed to ex-ante return.

Stock price changes or returns are random variables as the future returns on a stock are uncertain and unpredictable. A stock can have significant 'up' and 'down' movement even within a small time-frame like a single day. Figure 1 shows how the stock price of r Microsoft stock fluctuated in a single year.

Figure 1

Stock price movement of Microsoft for a single year



Source:*http://moneycentral.msn.com/investor/charts/chartdl.aspx?symbol=MSFT*

For finding an estimate of future returns on Microsoft, we need to estimate the expected value of the Microsoft stock returns. Ideally, we would try to come up with an expected value of a stock by associating the returns with a probability distribution (very much like the coin-toss example where the probability of getting 'heads' and 'tails' is 50% each).

But in reality, no model of finance is likely to claim that investors can find great bets "+$1 million with 99% probability" and "-$100 with 1% probability." Such an expected return would be way out of line. In financial markets, no one knows the correct model of expected stock returns well enough to know if the stock market can set the price of Microsoft stock so as to offer an expected rate of return on Microsoft of 7% a year or 12% a year. If it so difficult to estimate true expected returns, the strategy often used to estimate the expected returns is based on some form of past historical return averages. In this module, we employ the most widely used measure- expected return on a stock is the mean return of its past historical returns. For, e.g., the expected mean monthly return of Microsoft would be simply

$$E(r_{msft}) = \frac{1}{N}\sum_{i=1}^{N} \bar{r}_{i,msft}$$

Where $\bar{r}_{i,msft}$ is the historical monthly return for i^{th} month and N is the number of months over which the returns are averaged.

In the portfolio context, the expected return of a portfolio, m_p , is the weighted average of the expected returns on the individual assets in the portfolio, with the weights being the percentage of the total portfolio invested in each asset.

$$E(r_p) = s_A E(r_A) + s_B E(r_B) + \ldots\ldots + s_n E(r_n)$$

$$m_p = \sum_{i=1}^{n} s_i m_i$$

Where $E(r_p) = m_i$ are the expected returns on the individual stocks, the s_i 's are the weights, and there are n stocks in the portfolio.

Risk

Statistically, risk measures how spread out are the outcomes from the center (mean or expected return). If your mean is $3, an outcome of $2 would be closer to the mean than an outcome of $0. The former is only $1 away from the mean; the later would be away from the mean. It therefore makes sense to think in terms of such deviations from mean. The table below gives us the outcomes of the coin toss example and the deviation of payoffs from the expected value or mean.

	Heads	**Tails**
Coin toss payoff	$6	$0
Deviation from the expected value (mean)	+$3	-$3

You cannot simply compute risk as the average deviation from the mean. It is always zero (in our example it is $(+\$3 - \$3) = 0$). By taking positive and negative deviations it cancels each other out. So the alternative is to calculate the squared deviations and add them up. This is called the variance, s^{2} or *Sigma-Squared*, and for our example

$$Var(r) = s^{2} = \frac{(\$6 - \$3)^{2} + (\$0 - \$3)^{2}}{2} = \frac{(\$3)^{2} + (-\$3)^{2}}{2} = \$\$9$$

The unit of variance here is "\$\$" which is extremely difficult to interpret. If we consider the payoffs in %', then the variance unit would be in "%%" which also makes little sense except for the fact that higher variance means more risk. A popular measure is to scale the variance (by taking its square root) back to the units of mean, in "\$"s or %'s which would make risk more meaningful (no pun intended). This measure is called Standard Deviation. In our example, standard deviation is simply *Sigma*, $s = \sqrt{s^{2}} = \sqrt{\$\$9} = \$3$. Standard deviation is the most common measure of portfolio risk[5]. The higher the level of standard deviation, the more variability between the pay-offs or returns.

Looking back at our example, you can deduce that the variance can be expressed as

$$s^{2} = \frac{\Sigma(r - m)^{2}}{N} \qquad \text{for } N \text{ observations}^{6}$$

And the standard deviation is expressed as the square-root of variance

$$s = \sqrt{\frac{\Sigma(r - m)^{2}}{N}}$$

[5] The use of standard deviation as a measure of risk is related to the theory of utility functions widely used in economics and finance theory. The concept of expected utility was however developed by the famous Swiss mathematician Daniel Bernoulli in the 1700s, to explain the St. Petersburg Paradox. Bernoulli noted that a particular coin toss game led to an infinite expected payoff, but participants were willing to pay only a modest fee to play. He resolved the paradox by noting that participants do not assign the same value to each dollar of payoff. Larger payoffs resulting in more wealth are appreciated less and less, so that at the margin, players exhibit decreasing marginal utility as the payoff increases. The particular function that assigns a value to each level of payoff is referred to as the investor's utility function. Von Neumann and Morgenstern applied this approach to investment theory in 1944 in a volume that formed the basis for Markowitz's article in 1952 on how to form an efficient portfolio of securities using expected return and variance.

[6] The expression we have used is for *Population Variance*. The formula for sample variance is $s^{2} = \frac{\Sigma(r - m)^{2}}{N - 1}$ and for sample standard deviation is $s = \sqrt{\frac{\Sigma(r - m)^{2}}{N - 1}}$. Throughout this module when we refer to variance and standard deviation, we will refer to as population variance and standard deviation.

When we say something like "this investment is risky" we refer generally refer to the downside risk only. That is, to an investor, the relevant risk of investing in a portfolio is the chance that he would end up with returns which are less than the expected returns or simply the extent to which he can lose money from his investment. There is a subtle difference between this and how we measure it statistically (by variance or standard deviation). When we measure it by, we measure the variability or dispersion around the mean. We put equal weights to variability of returns both above and below the mean. Though the portfolio expected return is simply the weighted average of the expected returns of the individual assets in the portfolio, the risky-ness (measured by the standard deviation, s_p) is not the weighted average of the individual assets' standard deviations. The portfolio risk is generally smaller than the average of the assets' standard deviations.

Risk Neutrality and Risk Aversion

If a bet costs as much as its expected value, it is called a fair bet. It is fair because, if repeated often, both the person offering the bet and the person taking the bet would expect to end up even. For the coin-toss game, if you are willing to pay $3 to play the game, you are called risk- neutral. To be more specific, you are considered risk-neutral if you are indifferent between getting $3 for sure and getting either $6 or $0, each with 50% probability.

Now my friend Jack is not too keen to play this game. He thinks it is too risky. He would prefer $3 to the unsafe $6 or $0 investment. Jack would not want to invest in the more risky alternative if the risky alternative offered the same expected return as the sure return on the safe alternative. In other words, Jack is risk-averse. In order to induce Jack to play the coin-toss game, I would either have to lower the price of the game, say, $2 instead of $3 or I might have to pay him a "risk premium", of say $1, out of my own pocket to make him take the bet.

In reality most people in the world (except probably Ph.D students surviving on Ramen noodles) would not worry much for a bet for either + $3 or -$3. For small bets, most people are probably close to risk-neutral. But what about a bet for + $100 or - $100? Or for plus or minus $1000? It is likely that most people (myself included) would be reluctant to accept these bets without being compensated for bearing this risk. For most people, larger the bet, the more risk averse they become. We will assume throughout the rest of the module that investors are risk-averse.

Diversification

Diversification is equivalent of not putting all eggs in one basket. It is akin to not putting all your money in one risky asset but allocating your money across a number of risky assets.[7] A well diversified portfolio will significantly lower the downside risk without

[7] In this module we are looking at stocks as the risky asset. In real world, the investor will have other choices like investing in bonds, Bank CDs, real estate, commodities and derivatives.

lowering your expected return. Take the coin-toss example. Suppose the return on $10,000 you have put in a risky stock depends on the flip of a coin. Heads, it quadruples in value (becomes $40,000); tails, you lose $10,000. The expected return is very good at 100%[8]. Unfortunately, the downside risk is extremely bad – there is a 50% chance that you would lose your $10,000.

Now, instead of just flipping a single coin, we bring 10 more coins. Suppose you now spread the $10,000 investment into 10 different stocks, each with the same payoff scheme – Heads, the stock quadruples in value; tails, it becomes worthless. Your expected return is still $20,000. On average, you will flip five heads and an identical number of tails. Five of your investments are now worth $5 \times \$4{,}000 = \$20{,}000$, and the other five is $0. That still works out to be a 100% return. Now consider what has happened to the downside risk. The only way you can lose your entire $10,000 is by flipping ten tails, which is highly unlikely.

Let us consider another example. Associated Umbrella makers and Beach Surfers Inc. are the companies. Every time there is a spate a good sunny weather the stock of Beach Surfers Inc. tend to do well while Associated Umbrella makers stock tend to go down. The pattern reverses when it is raining. If you buy stock in both companies, you can shield yourself from fluctuations, as one stock's losses are kind of offset by the other stock's gain.

To show the effect of combining assets, we should call the stocks A (for *A*ssociated Umbrella makers) and B (for *B*each Surfers Inc.), respectively. Table 1 gives data on expected rates of return on the individual stock and also for a portfolio invested 50% in each stock. The returns correspond to the three states of nature-sunny, rainy and average weather. The three graphs in Figure 2 show plots of the data for returns of stock A, B and the portfolio AB. As seen from the top two graphs the two stocks seemed quite risky if they were held in isolation, but when they are held in a portfolio, they are not risky at all.

Stocks A and B can be combined to form a risk less portfolio because their returns move counter cyclically to each other. The tendency of two variables to move together is called correlation, and the correlation coefficient, r (pronounced as "rho"), measures the degree of this association in a scale of -1 to +1. In our example, we say that the returns on stock A and B are perfectly negatively correlated, with $r = -1$.

In the next section, we would see that as long as the stocks are not perfectly positively correlated, i.e. $r = +1$, it is possible to get the benefits of diversification. The real world lies between the two examples ($r = -1$ and $r = +1$), so combining stocks into portfolio reduces, but does not eliminate, the risk inherent in individual stocks. Also, in the real world, it is impossible to find stocks like A and B, whose returns are expected to be perfectly negatively correlated. So it is somewhat impossible to form completely risk free portfolios with using only stocks.

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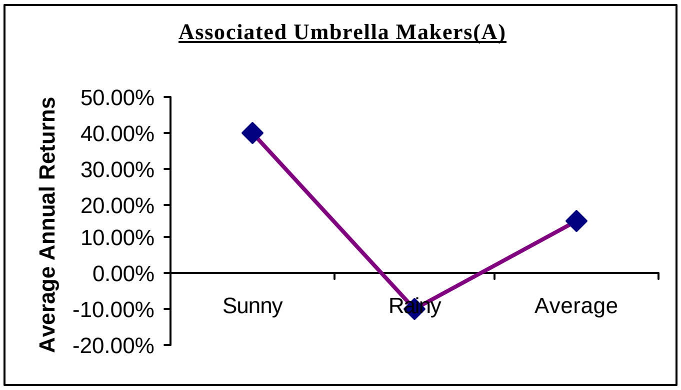
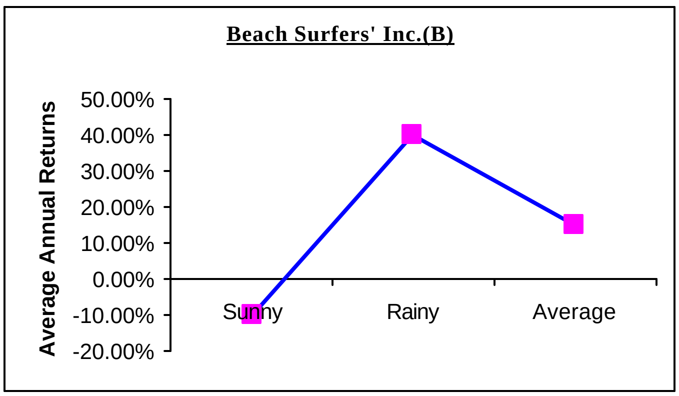
[8] it is $50\% \times \$40{,}000 + 50\% \times \$0 = \$20{,}000$ or 100% return on the initial $10,000 investment

Figure 2

Rate of Return Distributions for two perfectly negatively Correlated stocks and for a Portfolio AB invested 50% in each stock





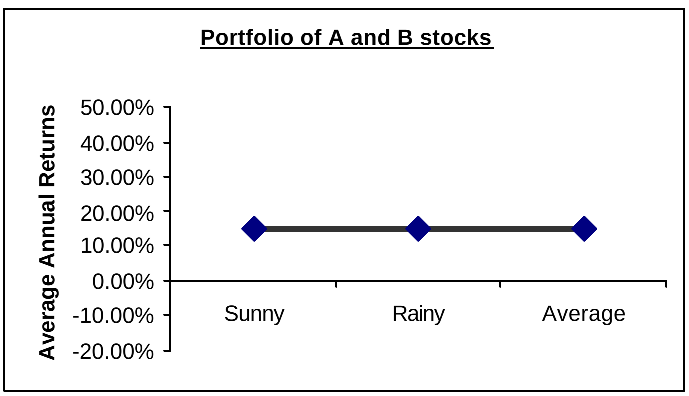


Table 1

Expected Return Distributions for Two perfectly negatively Correlated stocks and for Portfolio AB

	Stock A	Stock B	Portfolio of A B
Sunny	**40.00%**	**-10.00%**	**15.00%**
Rainy	**-10.00%**	**40.00%**	**15.00%**
Average	**15.00%**	**15.00%**	**15.00%**
Mean	**15.00%**	**15.00%**	**15.00%**
Standard Deviation	**20.41%**	**20.41%**	**0.00%**

What would happen if we included more than two stocks in the portfolio? There is a situation in states of nature where both A and B fall, like if there is a hurricane or an earthquake. In that case it would be good to have a stock like “Countrywide Home Repairs” whose value would go up if the nature’s fury strikes. As a rule, portfolio risk declines as the number of stocks in the portfolio increases[9]. Thus careful diversification can create a portfolio that is less risky and earns more on average for same degree of risk than any single company stock.

Conceptual Question

Q) Would you expect to find a higher correlation between Coke and Pepsi or between Microsoft and Pepsi? Explain the implications of creating a two stock portfolio.

Ans. Coke and Pepsi stock returns are highly positively correlated with one another because both are affected by similar forces. On the other hand, returns of Microsoft are likely to be less correlated with Pepsi because stocks in different industries are subjected to different factors. A two stock portfolio consisting of Coke and Pepsi would be less diversified than a two stock portfolio consisting of Pepsi and Microsoft or Coke and Microsoft. To minimize risk, portfolios ought to be diversified across the industries.

Model

A model is a simplified representation of reality that is used to better understand real-life situations. In economics and finance the use of mathematical models has a long history and has generated powerful insights[10] and lot of Nobel prizes. According to the famous economist Hal Varian all economics (and finance) models look quite similar. There is an individual or economic agent and they make choices in order to achieve objectives. The choices have to satisfy various constraints so there is something that would adjust to make all these choices consistent. So the model should address-Who are making the choices? What constraints do they face? What information they have? What adjusts if the choices are not mutually consistent?

In the previous sub-section we were glorifying the virtues of diversification. We know that it is essential to diversify but it still does not tell you how much of each security you

[9] In practice, investment managers and finance practitioners say that if you hold anywhere between 10 and 30 stocks you can capture all the diversification benefits. Adding assets beyond a certain number (like 30) does not generate much incremental benefit as far as reducing portfolio risk is concerned.

[10] Since the middle of the 20th century there has been a sharp escalation of the level of mathematical formalism leading many with the economics and finance academia to conclude that it has become excessive. Cornell University professor Robert Frank points out in his “The Economic Naturalist” that because of an increasingly stiff competition for academic jobs, candidates have obvious incentive to invest additional time and effort in honing their mathematical skills to appear more rigorous. The level of mathematical formalism may be thus too high for the same reason that people tends to raise their voices at cocktail parties. In a crowded space, one must speak up simply to be heard. But everyone speak in louder voices, the noise level rises, making it necessary to speak still more loudly.

should purchase. It may be better to purchase 60% in Associated Umbrella Makers and 40% in Beach Surfers Inc.; rather than 50% in both. We repeat the same questions we raised at the "Problem Statement" section and now it seems to make more sense; How do you determine investment allocations in a portfolio? Where is the best investment portfolio given investor's objectives and preferences? What would influence different allocations? To answer these questions we start with a modified version of the Nobel Laureate Harry Markowitz's (1952) portfolio theory model.

We begin with the simplest example – one period, two assets, and one investor, normally distributed returns (which makes mean and standard deviation all important) – whatever it takes to get something simple enough to see what is going on. Once we are comfortable with the simplest model we can add bits and pieces to it - relax couple of assumptions, increase number of assets and constraints. Einstein once said "Everything should be as simple as possible……but no simpler." A model is supposed to reveal the essence of what is going on.

The model assumes that investors are risk-averse, meaning that if there are two assets that offer identical expected return, the investor will prefer the less risky asset. An investor will undertake increased risk only if compensated by higher expected returns. The model also assumes that the investor's risk-reward preference can be completely explained by expected return and volatility (measured by standard deviation of historical returns). There are no transaction costs or taxes. A risk-free asset exits (in the form of US Treasury Bills) and it is possible to borrow and lend money at the risk-free rate. All stocks are perfectly divisible (e.g. it is possible to buy $1/1000^{th}$ of a share).

We use the following notations

$r_i =$ the return (sometimes called *rate* of return) on asset i

$n =$ number of available assets

$r_f =$ the return on the risk free asset

$m_i =$ mean of the return on asset i

$s_i =$ standard deviation of the return on asset i

$s_i^2 =$ variance of the return on asset i

$s_{ij} =$ covariance between the returns on assets i and j

$r_{ij} =$ correlation between the returns on assets i and j

$s_i =$ the share of asset i in the portfolio

$r \equiv r_p =$ the (rate of) return of a portfolio

$m \equiv m_p =$ the mean of the portfolio return

Two risky assets

We assume that there are only 2 risky assets, *A* and *B*, available for consideration in an investment portfolio. The portfolio return is given as:

(1) $$r = r_A s_A + r_B s_B$$

The portfolio shares need to add up to one:

(2) $$s_A + s_B = 1$$

Taking expectations of (1):

(3) $$E(r) = s_A E(r_A) + s_B E(r_B)$$

yields the mean portfolio return

(4) $$m = m_A s_A + m_B s_B$$

Based on equation (4), portfolio variance is (make sure to derive this yourself) [11]:

(5) $$s_p^2 = s_A^2 s_A^2 + 2 r_{AB} s_A s_B s_A s_B + s_B^2 s_B^2$$

which simplifies to:

(6) $$s_P = \sqrt{s_A^2 s_A^2 + (1 - s_A)^2 s_B^2 + 2 s_A (1 - s_A) r_{A,B} s_A s_B}$$

Combining equations (4) and (5), and using (2) to eliminate the portfolio shares, provides the feasible combinations of mean and standard deviation. The ***portfolio frontier*** is a plot of these feasible combinations of overall portfolio risk and returns.

Combining equations (4), (5), and (2) yields the portfolio frontier for two risky assets:

(7)

$$s_p = \sqrt{\frac{1}{(m_B - m_A)^2}\left[s_A^2 (m - m_B)^2 - 2 r_{AB} s_A s_B (m - m_A)(m - m_B) + s_B^2 (m - m_B)^2\right]}$$

should be μ A

The equation represents a ***hyperbola*** in mean-standard deviation space.

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[11] ***Note:*** Students should refer to the section on Covariance, Variance, Standard Deviation and the Correlation Coefficient under the Mathematical Appendix if they would like to derive some of these mathematical relationships from first principles.

Conceptual Question
(This may be given as a homework exercise or can be done in class as part of the class lecture)

Q. Using first order conditions what will be the portfolio weight of security *A*, that will **minimize** the portfolio standard deviation, s_p.

Ans: Minimizing the variance is equivalent to minimizing the standard deviation of the portfolio.

Take the first derivative of equation (6) with respect to s_A:

$$\frac{ds_P}{ds_A} = \frac{2s_A s_A^2 - 2(1-s_A)s_B^2 + 2\,(1-s_A)r_{A,B}s_A s_B - 2s_A r_{A,B} s_A s_B}{2\sqrt{s_A^2 s_A^2 + (1-s_A)^2 s_B^2 + 2\,s_A(1-s_A)r_{A,B}s_A s_B}}$$

To minimize set the first order derivative equal to zero; $\frac{d\,s_P}{ds_A} = 0$, this should give us:

When Cov of A, B = 0, sA is the form I've derived in school's text book.

(8) $$s_A = \frac{s_B^2 - r_{A,B}s_A s_B}{s_A^2 + s_B^2 - 2\,r_{A,B}s_A s_B} = \frac{s_B^2 - Cov(r_A, r_B)}{s_A^2 + s_B^2 - 2\,Cov(r_A, r_B)}$$

For sufficient conditions for global minimum, you may wish to verify that the second derivative $\frac{d^2 s_P}{d^2 s_A} \geq 0$ is non –negative.

The shape of the portfolio frontier for two risky assets, *A* and *B*, depends on the degree of correlation between the two assets. The correlation between the risky assets is a crucial aspect of any portfolio decision, so to get an idea of the general shapes of the portfolio frontier that are possible, we explicitly consider three extreme assumptions about the correlation between the returns of assets A and B.

(i) $r = 1$

The assets are perfectly correlated. Equation (5) now simplifies to:

$$s_p^2 = s_A^2 s_A^2 + s_B^2 s_B^2 + 2s_A s_B s_A s_B$$

(5a) $$s_p = s_A s_A + s_B s_B$$

Figure 3 depicts that the portfolio frontier becomes a straight line sloping up from the point where $s_A = 0$ to the point where $s_A = 1$. The dotted lines indicate again the opportunities when short sales are permitted. When the returns on the risky assets are perfectly correlated, no diversification benefits occur and combining the assets will just lead to a linear combination between the extreme positions of putting the whole portfolio in either of the assets.

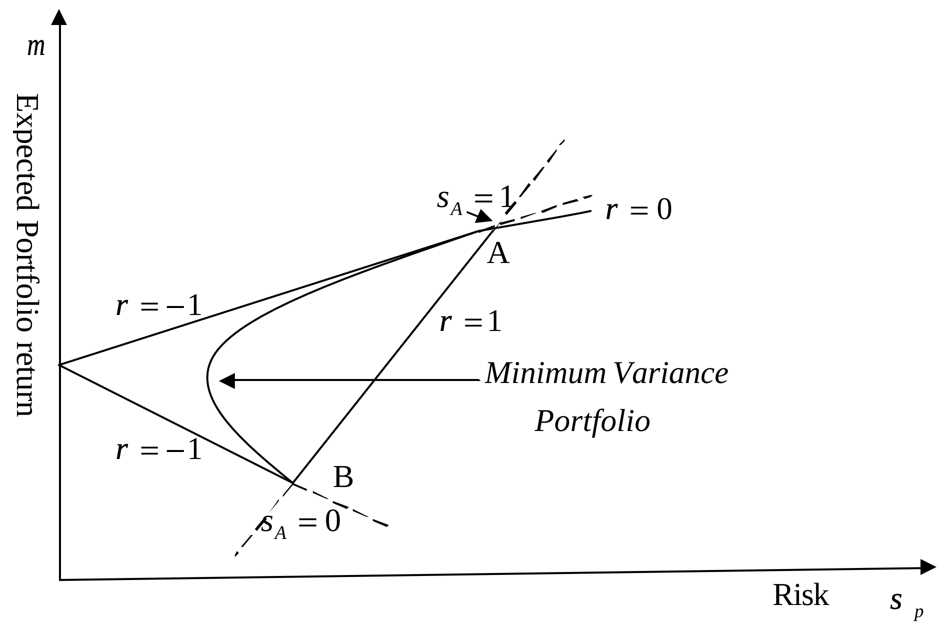

Figure 3
Portfolio Frontiers for 2 Risky Assets, A and B
The shape of the portfolio frontier depends on the value of the correlation coefficient between the two assets, A and B.

(ii) $r = -1$

The assets are perfectly negatively correlated. Equation (5) becomes:

$$s_p^2 = s_A^2 s_A^2 + s_B^2 s_B^2 - 2 s_A s_B s_A s_B$$

(5b) $$s_p = \left| s_A s_A - s_B s_B \right|$$

Note that, strictly speaking, the absolute value should also be taken in equation (5a) if short sales are allowed. Diversification benefits are maximal due to the negative correlation between the asset returns. Figure 3, shows that the portfolio can be diversified such that no risk is incurred; asset B can be a perfect hedge for asset A and vice versa.

(iii) $r = 0$

The assets are uncorrelated. One may think that this implies that diversification is not possible; in fact, the benefits of diversification are quite clear in this case. It is one of the basic insights necessary to understand portfolio choice. Even though the middle term in equation (5) drops out due to this assumption, the analysis in this case is substantially more complex than in the previous two cases. Equation (5) becomes:

(5c) $$s_p = \sqrt{\left(s_A^2 s_A^2 + s_B^2 s_B^2\right)}$$

Consider the mean/standard deviation tradeoff in this case derived from equations (4) and (5c). The slope of the frontier can be written as:

(9) $$\frac{dm}{ds_p} = \frac{dm / ds_A}{ds / ds_A} = \frac{m_A - m_B}{(s_A s_A^2 - s_B s_B^2)/s_p}$$

If we assume that: $m_A > m_B$ and $s_A > s_B$, the sign of the slope in equation (9) depends on the denominator. It is easy to see that at some point the slope is vertical. The portfolio that produces this point is called the minimum variance portfolio. Further, at the fully undiversified point where ~~s1~~ = 0, the slope must be negative as shown in Figure 3. Thus, starting from this undiversified point; more diversification is beneficial for every investor with mean-variance preferences: expected return rises while standard deviation falls.

It is clear that the portfolio frontier in case (iii) lies between the frontiers of cases (i) and (ii). It can be shown that this is true for the general case as well. For general correlation between assets 1 and 2, it is also true that the portfolio frontier has the same hyperbolic shape as in case (iii).

Q. Derive the portfolio frontier in the case of 1 risky asset and 1 risk-free asset. Does the frontier look like a hyperbola? What does the slope of the frontier signify?

Ans:
Assume that asset A_0 is the risk –free asset and A_1 the risky asset. We can start with the 2 risky assets case and take into account the fact that the risk free asset has zero risk and is not correlated with the risky asset.

(i) $s_0 + s_1 = 1$

(ii) $m = s_0 r_f + s_1 m_1$

With $s_0^2 = 0$ and $r = 0$, the portfolio variance is given by

(iii) $$s_p^2 = s_1^2 s_1^2$$

From the above equation we can easily derive the portfolio weight of the risky asset 1 as

(iv) $$s_1 = \sqrt{s_p^2 / s_1^2}$$

Substituting the value of s_1 in (2), we get

(v) $$m = r_f + \left({s_p}/{s_1} \right)(m_1 - r_f)$$

Equation (v) gives the opportunity set or the portfolio frontier for one risky asset and the risk-free asset.

Figure 4 shows the opportunity set available to any investor. The dotted line indicates opportunities that are only possible if short-sales are allowed $(s_0 < 0 \; or \; s_1 < 0)$. The intercept of the opportunity line is at the risk-free rate, since the standard deviation of the portfolio return is only zero when the whole portfolio is the risk-free asset $(s_0 = 1 ; s_1 = 0)$. Note that the risk or standard deviation cannot fall below zero. If short sales occur at this point (short-selling the risky asset and investing the proceeds in the risk-free asset), positive risk occurs due to the investor's obligation in servicing the risky asset; this explains the kink in the opportunity line. The slope of the opportunity cost line is given as

$$\frac{dm}{ds} = \frac{m_1 - r_f}{s_1}$$

The slope is positive under the reasonable assumption that $m_1 > r_f$, which is realistic as a risk –averse investor you would not undertake to purchase a risky asset which gives a lower expected return than the certain return on the risk –free asset.
The slope is also called "*Sharpe Ratio*" (named after William Sharpe, Nobel Prize winner in economics) and it gives the excess return per unit of risk. It is useful in measuring and comparing performance of different portfolios.
Conversely, the slope can also be looked as the "price of risk reduction" that each investor faces. It shows by how much expected portfolio return rises if the standard deviation (chosen by the investor) increases by one unit

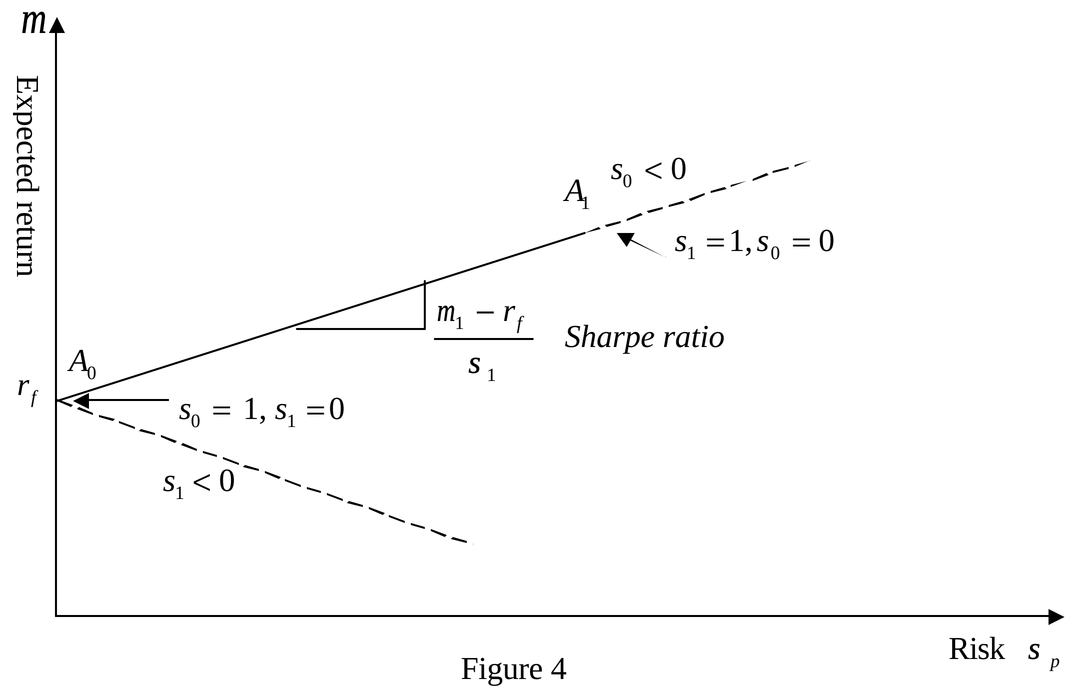

Figure 4

Portfolio Frontier for One risky and One Risk-free Asset

A_1 is the risky asset and A_0 the risk-free asset, respectively. Portfolio shares range from zero to one along the solid line. Short-sales of either the risk-free asset or the risky asset extend the frontier along dashed lines.

Three Assets

(This may be given as a homework exercise or can be done in class as part of the class lecture)

Derive the expression for portfolio Expected return and portfolio variance for a portfolio consisting of three risky assets *A, B and C*

The return for the portfolio is simply

$$r = s_A r_A + s_B r_B + s_C r_C$$

Taking expectations of both sides of the above equation we get

$$E(r) = s_A E(r_A) + s_B E(r_B) + s_C E(r_C)$$

This can be re-written as

$$m = s_A m_A + s_B m_B + s_C m_C$$

The notations have their usual significance.

The variance of the portfolio can be written as

$$Var(r) = Var\,(s_A r_A + s_B r_B + s_C r_C)$$

Let X = $s_A r_A + s_B r_B$, then we can re-write the above equation using relation (A6) from Mathematical Appendix:

$$Var\,(r) = Var\,(X,\, s_C r_C) = Var\,(X) + 2\,Cov(X,\, s_C r_C) + Var(s_C r_C) \qquad \textbf{\textit{(J)}}$$

Take the first term on the Left hand side and using the relation (A6) again

$$\begin{aligned} Var\,(X) &= Var(s_A r_A + s_B r_B) = Var(s_A r_A) + 2Cov(s_A r_A,\, s_B r_B) + Var\,(s_B r_B) \\ &= s_A^2\; Var(r_A) + 2s_A s_B\; Cov(r_A,\, r_B) + s_B^2\; Var\,(r_B) \qquad \textbf{\textit{(L)}} \end{aligned}$$

Now, take the second term on the Left hand side of ***(J)***

$$2\,Cov(X,\, s_C r_C) = 2\,Cov\,(s_A r_A + s_B r_B\,,\, s_C r_C)$$

Using the relation (A4)

$$= 2s_A s_C\; Cov(r_A,\, r_C) + 2s_B s_C\; Cov(r_B, r_C) \qquad \textbf{\textit{(M)}}$$

and using (A5) we get the Third term of ***(J)*** as

$$Var\,(s_C r_C) = s_C^2\; Var\,(r_C) \qquad \textbf{\textit{(N)}}$$

Let $s_A^2 = Var\,(r_A)$, $s_B^2 = Var\,(r_B)$ and $s_C^2 = Var\,(r_C)$.

Adding (L), (M) and (N) to form (J) we have the portfolio variance for three risky assets

(O)

$$s_A^2 = s_A^2 s_A^2 + s_B^2 s_B^2 + s_C^2 s_C^2 + 2s_A s_B\; Cov(r_A, r_B) + 2s_A s_C\; Cov(r_A, r_C) + 2s_B s_C\; Cov(r_B, r_C)$$

<u>N Risky Assets</u>

As discussed in the background section, as we keep on adding more and more risky assets, the portfolio risk is expected to go down but at the same time the math also tends to become messier. We take resort to matrix algebra which can represent a lot of data by clubbing them into groups or cohorts which are called rectangular *arrays*. A brief refresher of matrix algebra is provided in the mathematic appendix.

We assume here that investors may invest in a total of *n* risky assets and that no risk-free asset exists. Short sales are not restricted. The portfolio frontier in this case was rigorously derived by another Nobel Laureate Robert Merton in his 1972 paper "An Analytic Derivation of the Efficient Frontier."[12]

Mathematically the portfolio frontier can be found by minimizing portfolio variance subject to a given expected return. The dual of this decision problem does not provide the same solution: maximizing expected return subject to a given portfolio variance only produces the upper half of the portfolio frontier. The lower half is dominated since a higher expected return can be found for any feasible variance. The upper half of the portfolio frontier obtained in this manner is called the ***efficient frontier*** for obvious reasons. Empirically, if the assumptions leading to mean-variance analysis are justified, we expect that no individual's complete portfolio lies below the efficient frontier.

Consider the following variable definitions:

$\Sigma = \lfloor s_{ij} \rfloor$, represents the *n x n* variance-covariance matrix of the *n* asset returns, where

m is a *1 x n* column vector of the expected returns m_i .

s represents a *1 x n* column vector of the portfolio shares s_i.

1 represents a *1 x n* column vector of 1's

Σ^{-1} represents the inverse of matrix Σ

The portfolio frontier is found by minimizing portfolio variance subject to a given portfolio mean:

(10) **Minimize** with respect to ***s***: $$\frac{1}{2} s^T \Sigma s$$

Subject to the constraints

(11) $$m^T s = m_p$$

(12) $$1^T s = 1$$

Thus portfolio variance is minimized subject to a given expected portfolio return m_p and given that all portfolio shares add up to 1.

Using the Lagrangian[13] method with multipliers l and k for constraints (11) and (12), respectively, produces the following first-order condition:

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[12] The model used in this module is a watered down to suite the level of advanced undergraduate students. I owe much of the material in this section to Dr. Ronald Balvers at West Virginia University who has allowed me to use his class notes from his graduate level asset pricing course.

[13] For students new to lagrangian multipliers it might be good idea to look at an undergraduate Finance and economics book like Alpha C. Chiang's "Fundamental Methods of Mathematical Economics." A brief

This formula has times both sides by inv(s),
so, the right multiply of s disappears.

(13) $$s^T \Sigma - l\, m^T - k\, 1^T = 0$$

Solving for the optimal portfolio shares yields:

(14) $$s^{T*} = l\; m^T\, \Sigma^{-1} + k\; 1^T\, \Sigma^{-1}$$

Based on the fact that Σ is positive definite[14] we can conclude that s^{T*} does minimize the variance and that the solution obtained here for the portfolio shares is unique. In this partial-equilibrium framework nothing guarantees that all portfolio shares are positive or below one.

Post-multiplying equation (13) by **s** and using constraints (11) and (12) gives:

(15) $$s_p^2 = l\, m_p + k$$

(14)

Post-multiplying equation (~~18~~) ~~by: and~~ separately by **1** yields the following two equations:

μ and

(16) $$m_p = l\; m^T\, \Sigma^{-1}\, m + k\; 1^T \Sigma^{-1}\, m$$ μ = λA + kC

(17) $$1 = l\; m^T\, \Sigma^{-1} 1 + k\, 1^T\, \Sigma^{-1}\, 1$$ 1 = λC' + kB

Define: it's transpose of C, but since C is a scalar, C' = C = constant !

(18) $$A = m^T \Sigma^{-1} m\,,\; B = 1^T\, \Sigma^{-1} 1\,,\; C = 1^T\, \Sigma^{-1} m\,, \text{ and } D = AB - C^2$$

Note that A, B, C, and D are scalars that depend only on the constant parameters of the set of available assets. It is now straightforward to solve for l and k from equations (16) and (17):

(19) $$l = \frac{(B m_p - C)}{D}$$

it's A*1 in fact

(20) $$k = \frac{(A - C m_p)}{D}$$

overview on Lagrangian multipliers is written by Steuard Jensen and is available at http://www.slimy.com/~steuard/teaching/tutorials/Lagrange.html

[14] For the variance-covariance matrix to have an inverse, it requires that no two assets are perfect substitutes; so that the matrix is not singular.

Plugging (19) and (20) into equation ~~(19)~~ (15) yields an explicit expression of the portfolio frontier:

$$s_p^2 = \frac{\left(Bm_p^2 - 2\,Cm_p + A\right)}{D} \tag{21}$$

The portfolio frontier is a hyperbola in mean-standard deviation space as in the case with 2 risky assets. The reason is intuitive: any two points on the frontier can be thought of as mutual funds that are individual assets. Taking different combinations of these two assets must trace out a hyperbola based on case with 2 risky assets ; but there is no way that this hyperbola can be different from the *n*-asset frontier as it can, at no point, lie to the left of the *n*-asset frontier (or the frontier wouldn't be a true frontier). Thus, once we understand the 2 risky assets case, we can deduce logically that the *n*-asset frontier must be a hyperbola as wellBased on the above formula it is now easy to find the ***minimum variance portfolio*** of risky assets. Differentiating equation (21) with respect to m_P and setting it equal to zero yields $m_P = C/B$ so that we obtain $s_p^2 = 1/B$ (after using the definition of *D* in (22).

Advanced Topic

Capital Market Line

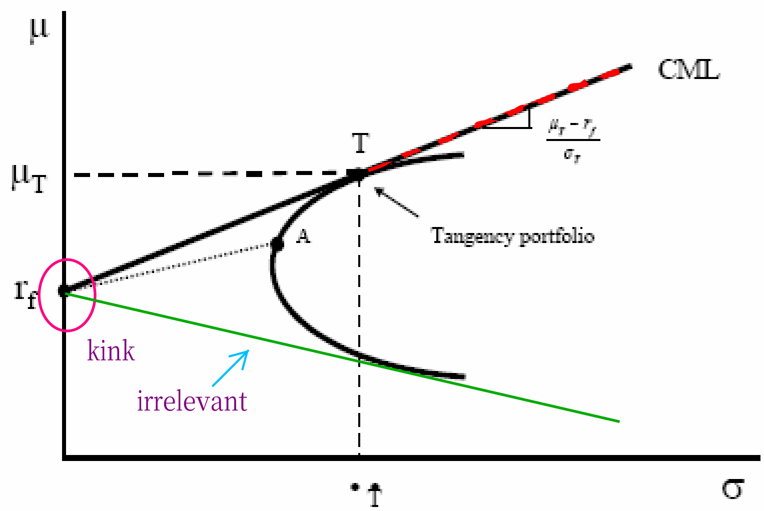
Recall the section where we considered one–risky asset and one-risk free asset. The single risky asset can be expanded to form a portfolio of assets. The opportunity frontier is then determined by the straight line going through the risk free rate on the vertical axis and the tangency point on the portfolio frontier (see figure A below). This is also called the Capital Market Line (or, sometimes, the Portfolio Market Line). Call the unique portfolio of risky assets that reaches the portfolio frontier at point T the **tangency portfolio**. Then point T on the CML implies full investment of all wealth in the tangency portfolio and no investment in the riskless asset. Any point on the CML to the right of T implies borrowing (going short) on the riskless asset. Any point on the CML below r_f (or the risk free rate) implies going short on the tangency portfolio and investing the proceeds plus initial wealth in the risk free asset.

Because portfolio frontier assume two "risky" assets, another words, every point on the frontier represents a portfolio full of risky assets. So, when the CML has one risk free asset with another risky asset, the tangent point indicates that the risk free part of CML must go down to zero to satisfy assumption of portfolio frontier.

As before, the CML here displays a kink since standard deviation cannot become negative: any further holdings of the risk free asset beyond initial wealth increase risk, even though expect portfolio return still falls. Clearly, the part below the kink is irrelevant and does not affect investment opportunities. The equation for the CML can easily be constructed see figure A

(22) $$m_p = r_f + \left(\frac{m_T - r_f}{s_T} \right) s_p$$

Figure A



The efficient frontier is the Capital Market Line (CML). The tangency portfolio is the unique portfolio of risky assets that reaches the portfolio frontier

The slope in equation (22) is the Sharpe ratio or the "price of risk reduction". i.e.,

(23) $$\boxed{Sharpe\ Ratio = \left(\frac{m_T - r_f}{s_T} \right)}$$

All investment opportunities for any risk averse investor are summarized by the CML. But the opportunities on the CML may be generated using only two "portfolios": the tangency portfolio and the risk free asset. Given our basic assumptions, every investor will be on the CML. To get there, the investor must split his wealth over the tangency portfolio and the risk free asset. Thus, every investor holds risky assets in the same proportion, as defined by the tangency portfolio. The result that all investor opportunities are summarized by a fixed set of portfolios or "mutual funds" is a portfolio separation result often referred to as the Mutual Fund Theorem.

Solution Methodology and Implementation

Given a potential set of assets, the efficient frontier can be created by portfolio optimization. Portfolio optimization involves a mathematical procedure called quadratic programming in which two objectives are considered: to maximize return and minimize risk. It is called a Quadratic Programming Problem (QPP) as the objective function consists of second degree terms. The two objectives are considered: (i) minimize risk given a specific return or (ii) maximize return for a given level of risk. The QPP for both objectives is generally subjected to the following constraints:

1. The weight of funds invested in different assets must add to unity;
2. There is no short sale provision.

The constraints imposed on the problem are neither exhaustive nor irrevocable. There might be a maximum limit on which an investor can purchase one stock. Similarly we can modify the short sales constraint to allow short selling.

Portfolios on the mean-standard deviation (or variance) efficient frontier are found by searching for the portfolio with the least variance given some minimum return. Repetition of this procedure for many return levels generates the efficient frontier.

The QPP can be solved using constrained optimization techniques involving calculus or by computational algorithms applicable to non-linear programming problems. Of the two approaches, the non-linear programming is more versatile as it is comfortable handling both equality and in-equality constraints.

We start with a simple excel exercise in tracing a portfolio frontier and then move to the constrained optimization techniques.

Tracing a Portfolio Frontier

Let's breathe some life into the formulas we derived in the last section. Consider three possible investments, say, JP Morgan Chase (JPM) and Oracle Corporation (ORCL). We have about six years or seventy two months worth of data[15] obtained from Yahoo! Finance.[16] The returns are arranged in ascending order of dates shown in figure 5.

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[15] There is no single consensus on how far back you should go to estimate expected returns and volatility. 5 to 6 years or 60 to 72 months is popular in empirical finance because that is the time frame used to calculate another important variable called the "beta" which is measure s a measure of a stock's volatility in relation to the entire market.

[16] For details on how to download stock price data from Yahoo and computing the monthly stock returns, refer to the sub-section "*Downloading Stock data from Yahoo! Finance and calculating the monthly stock returns using Excel*" under the section "Empirical Data".

Figure 5

Time-series returns of two risky assets, JPM and ORCL

	A	B	C	D	E	F
1	**Date**	JPM	ORCL			
2	6/1/2001	-0.09518	0.2418			
3	7/2/2001	-0.019	-0.0484			
4	8/1/2001	-0.08988	-0.3247			
5	9/4/2001	-0.13338	0.0303			
6	10/1/2001	0.045438	0.0779			
7	11/1/2001	0.066947	0.0347			
8	12/3/2001	-0.03647	-0.0157			
9	1/2/2002	-0.05421	0.2498			
10	2/1/2002	-0.14095	-0.0371			
11	3/1/2002	0.218632	-0.2298			
12	4/1/2002	-0.00585	-0.2156			
13	5/1/2002	0.024247	-0.2112			
14	6/3/2002	-0.05648	0.1957			
15	7/1/2002	-0.25627	0.0570			
16	8/1/2002	0.057831	-0.0420			
17	9/3/2002	-0.28064	-0.1804			
18	10/1/2002	0.112096	0.2964			
19	11/1/2002	0.212984	0.1923			
20	12/2/2002	-0.04648	-0.1111			
21	1/2/2003	-0.01379	0.1139			
22	2/3/2003	-0.02846	-0.0058			
23	3/3/2003	0.045735	-0.0936			
24	4/1/2003	0.255528	0.0959			
25	5/1/2003	0.119374	0.0951			

It is a good idea to start with defining your inputs into *named* Arrays. Look at the screen-shot in figure 6. For example, in excel worksheet with the stock returns data select “Insert” then “Name” and then “Define.” [17] Pick the stock you would like to consider, say, JP Morgan Chase (JPM). A box would appear and select the range of the data for the stock $B$2:$B$74 and name it JPM. The screen-shot below shows how it looks like in excel. The good thing is that you do not have to select the entire range every-time you want to calculate a formula using JPM. For mean return you can simply write =AVERAGE(JPM) , for standard deviation[18] it is =STDEVP(JPM), for variance it is =VARP(JPM) and so on.

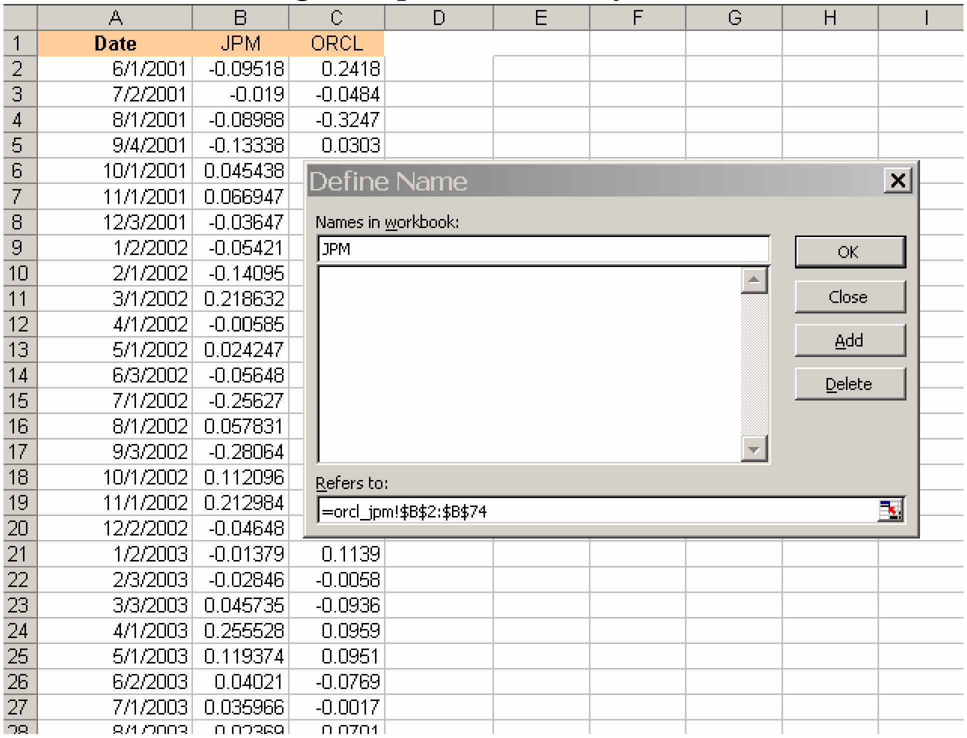
Our ***objective*** is to trace the portfolio frontier in mean – standard deviation space and identify the efficient frontier and the minimum variance portfolio. For simplicity, assume that both stocks can only have positive weights and there is no short-selling.

[17] Henceforward we would use hyphens for a string of commands. In this example it would be simply Insert-Name-Define

[18] Note that we use STDVEP instead of STDEV. The excel function STDEVP refers to the *population* standard deviation whereas STDEV measures the *sample* standard deviation. Similarly we use VARP instead of VAR.

<u>Figure 6</u>

Naming the input data as *Arrays* in Excel



<u>Step 1</u>

Using AVERAGE() , STDEVP() and VARP(), find the mean, standard deviations and the variance of JPM and ORCL. Also find the Correlation co-efficient and the Covariance between the two assets, JPM and ORCL using the CORREL() and COVAR() functions.

<u>Step 2</u>

Start with any portfolio weights; say your entire money is invested in ORCL. So, the weight of ORCL is 100% and JPM is 0% (remember the weights must add up to 1).

<u>Step3</u>

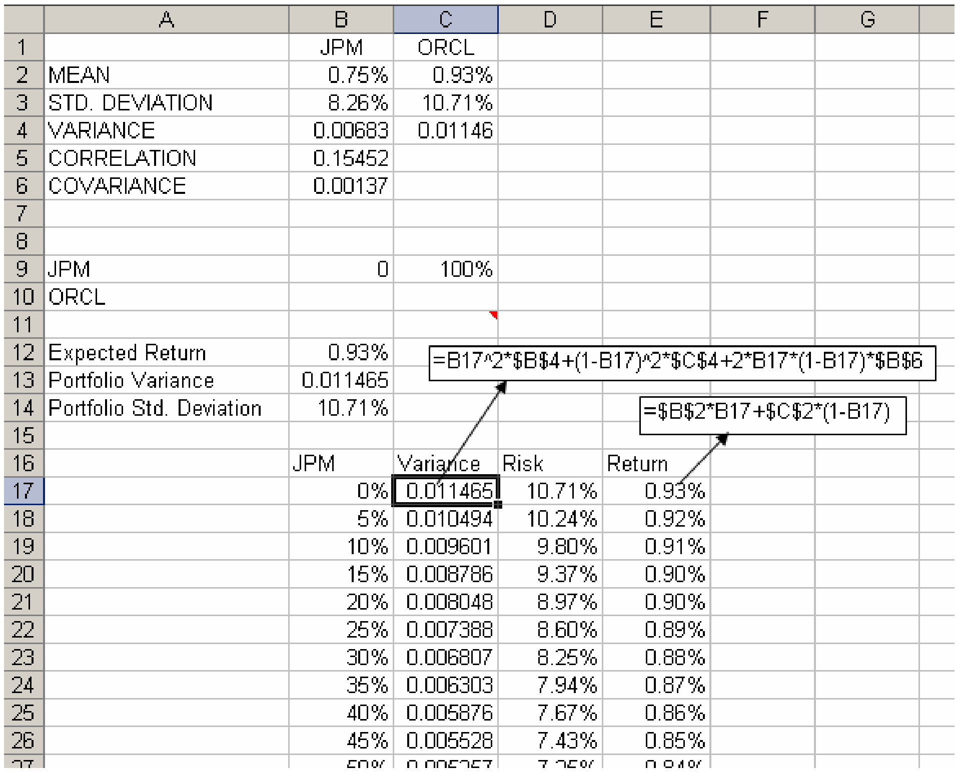
Use equations (4) and (5) from the previous section to compute the portfolio means and portfolio variance. To calculate portfolio standard deviation, simply take the square root of the portfolio variance. These portfolio values correspond to the portfolio weights of 0% in JPM is 100% in ORCL.

<u>Step 4</u>

To trace the portfolio frontier we vary the weights of one of the assets, say JPM (the weight of ORCL will automatically vary as their sum has to add up to 1). In our example we create a column starting from cell B17 (see the screen-shot below) and put weights of JPM in increments of 5% (you can vary weights in smaller or larger increments). Cell C17 correspond to the portfolio variance formula, Cell D17 is the square root (=SQRT)

Figure 7

Solving for the Portfolio Frontier



	A	B	C	D	E	F	G
1		JPM	ORCL				
2	MEAN	0.75%	0.93%				
3	STD. DEVIATION	8.26%	10.71%				
4	VARIANCE	0.00683	0.01146				
5	CORRELATION	0.15452					
6	COVARIANCE	0.00137					
7							
8							
9	JPM	0	100%				
10	ORCL						
11							
12	Expected Return	0.93%					
13	Portfolio Variance	0.011465					
14	Portfolio Std. Deviation	10.71%					
15							
16		JPM	Variance	Risk	Return		
17		0%	0.011465	10.71%	0.93%		
18		5%	0.010494	10.24%	0.92%		
19		10%	0.009601	9.80%	0.91%		
20		15%	0.008786	9.37%	0.90%		
21		20%	0.008048	8.97%	0.90%		
22		25%	0.007388	8.60%	0.89%		
23		30%	0.006807	8.25%	0.88%		
24		35%	0.006303	7.94%	0.87%		
25		40%	0.005876	7.67%	0.86%		
26		45%	0.005528	7.43%	0.85%		
27		50%	0.005257	7.25%	0.84%		

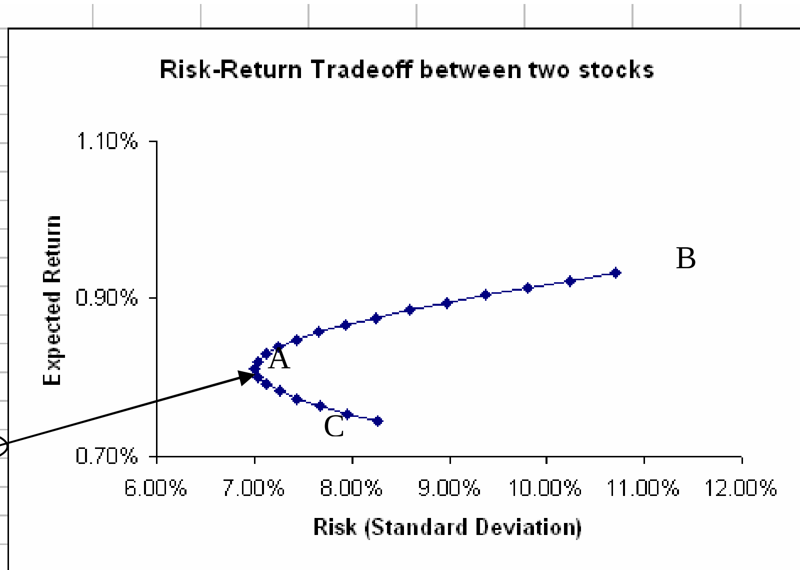
of variance in cell C17. Cell E17 gives the portfolio mean using the formula (4) as before. We copy the columns C17:E17 till the row corresponding to JPM's weight of 100%. The screen-shot in figure 7 shows the columns and the formulas used.

Step 5

We now have the data to trace out the frontier. Go to the Chart Wizard or "Insert-Chart." Select the XY (scatter) and then the second option on the right which is "Data points connected by smoothed lines." Select columns corresponding to the Portfolio Risk (standard deviation) and portfolio return as shown in the screen-shot in figure 8. It should generate a chart like this (to get a smooth-looking regular shaped frontier you may have to vary the units of the X and Y axis under chart options. I have compiled a video file "graphing_in_excel.avi" to give you head-start on the charting process. The Excel's help feature is also quite good.

Figure 8
Graphing of Efficient frontier

Risk	Return
10.71%	0.93%
10.24%	0.92%
9.80%	0.91%
9.37%	0.90%
8.97%	0.90%
8.60%	0.89%
8.25%	0.88%
7.94%	0.87%
7.67%	0.86%
7.43%	0.85%
7.25%	0.84%
7.12%	0.83%
7.03%	0.82%
7.01%	0.81%
7.04%	0.80%
7.12%	0.79%
7.26%	0.78%
7.44%	0.77%
7.68%	0.76%
7.95%	0.75%
8.26%	0.75%



We have now successfully created a real world portfolio frontier with two risky assets, JPM and ORCL. By looking at the table from the screen-shot above, we can say that the minimum variance portfolio is the one which corresponds to 7.01% risk and 0.81% monthly return or point "A" in the picture (we can be a little more specific and accurate when we use calculus optimization method and non-linear programming in the next sub-section but it is a good start). The efficient –frontier is traced by the locus of points from "A" to "B". Any portfolio like "C" which lies below the minimum variance portfolio in the picture (or to the south of the minimum variance portfolio) is *dominated* by the minimum variance portfolio and all portfolios which lie to the North East of it.

How would you interpret the finding? No risk-averse investor should buy portfolio in the region below A or anywhere else, except for points on the efficient frontier traced from A
B to ~~C~~. Depending on the risk taking ability the individual investor can choose between portfolio allocation A (more risk –averse investor) to allocation B (less risk averse investor). Remember, you can only expect higher return on the efficient frontier if you are willing to take more risk.

Finding the Minimum Variance Portfolio (more accurately)

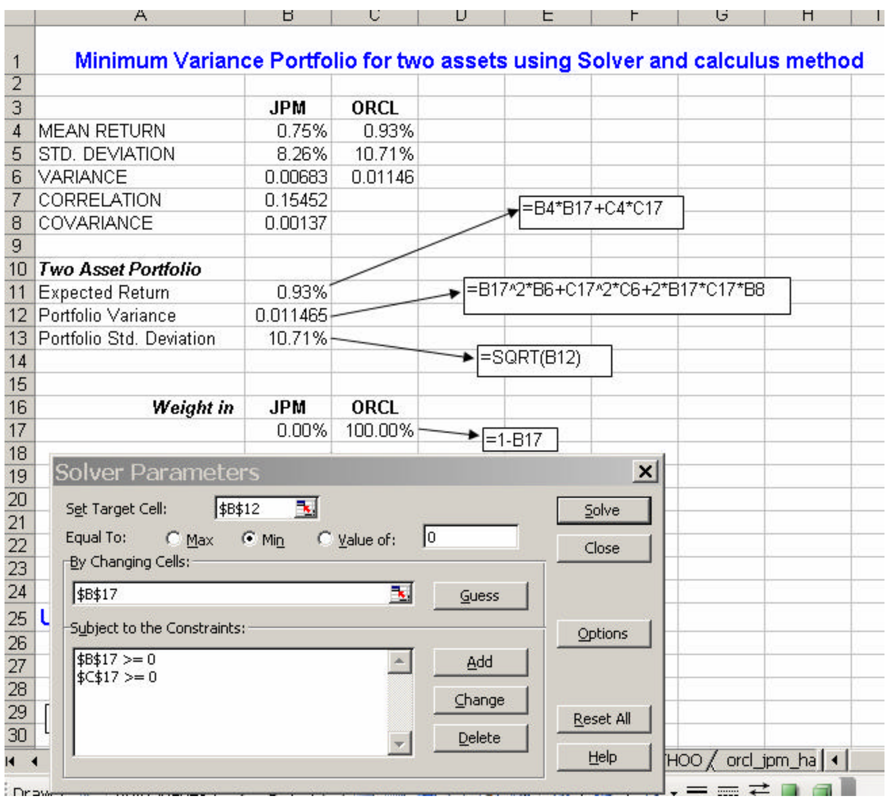
To find the exact location of the minimum variance portfolio we need now move to solving the QPP using constrained optimization techniques (we can very well do it after the pain we went through deriving all the formulas). We start with the non-linear programming method using solver and then move to the calculus approach.

Minimum Variance Portfolio using Solver

Excel Solver is a powerful tool for optimization[19] and produce targeted results for your models (the technical jargon is “calibrate your model”. [20] Microsoft solver can be added by selecting “Tools-Add-ins” and choosing the “Solver Add-in.” If you still have problems you can refer to the site http://hspm.sph.sc.edu/COURSES/J716/SolverInstall.html which shows you how to install solver when it is not available among add-ins.

By using solver, we can calculate the minimum variance portfolio. The screen-shot in figure 9 shows the solver dialog box. In this box, we have asked solver to minimize the variance in cell B12, by changing the weight of JPM (cell B17) in the portfolio. In order to ensure that the weights of JPM and ORCL are positive we put a non-negative constraint by ensuring the weight of JPM and ORCL’s weight are greater than equal to zero. The relation formula 1-B7 in cell C17 corresponding to ORCL’s weight ensures that the sum of the two weights do not exceed 1.

Figure 9



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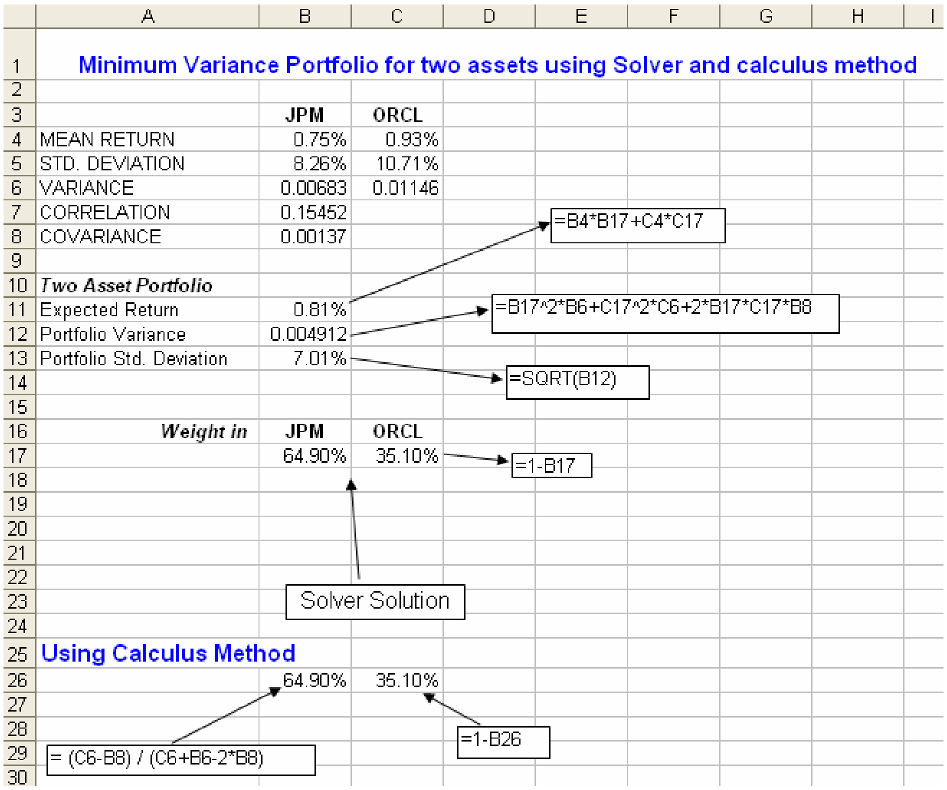
19 Solver tool uses the Generalized Reduced Gradient (GRG2) nonlinear optimization code developed by Leon Lasdon, University of Texas at Austin, and Allan Waren, Cleveland State University.

20 A nice book on using excel tools like Solver and Goal-seek and its application in Finance is by Simon Benninga, “Principles of Finance with Excel”.

Clicking on "Solve" in the solver dialog box gives (see screen-shot below), the minimum variance portfolio with 64.9% invested in JPM and 35.1% in Microsoft. The minimum v

Figure 10



	A	B	C
1	Minimum Variance Portfolio for two assets using Solver and calculus method		
2			
3		JPM	ORCL
4	MEAN RETURN	0.75%	0.93%
5	STD. DEVIATION	8.26%	10.71%
6	VARIANCE	0.00683	0.01146
7	CORRELATION	0.15452	
8	COVARIANCE	0.00137	
9			
10	*Two Asset Portfolio*		
11	Expected Return	0.81%	
12	Portfolio Variance	0.004912	
13	Portfolio Std. Deviation	7.01%	
14			
15			
16	*Weight in*	JPM	ORCL
17		64.90%	35.10%
18–24			
25	Using Calculus Method		
26		64.90%	35.10%
27–30			

Variance portfolio corresponding to 7.01% risk (standard deviation) and 0.81% returns matches our result obtained previously in "Tracing a Portfolio Frontier."

Minimum Variance Portfolio using Calculus Method

Recall the section where we derived the minimum variance portfolio for 2 risky assets (all those derivations coming to some use now). Equation (8) gave us the optimal portfolio weight as

$$s_A = \frac{s_B^2 - r_{A,B}\, s_A\, s_B}{s_A^2 + s_B^2 - 2\, r_{A,B}\, s_A\, s_B} = \frac{s_B^2 - Cov(r_A, r_B)}{s_A^2 + s_B^2 - 2\,Cov(r_A, r_B)}$$

We plug the values of Variance and Covariances in cell B26 to get the optimal weight of JPM. Implementing the formula in excel gives the same answer as that given by solver (see the screen-shot above).

Effect of Change in the Correlation Co-efficient in a portfolio.

Stock A and stock B are two risky assets which have the following characteristics.

	A	B
Mean	3.00%	2.00%
Standard Deviation	25.00%	15.00%
Variation	0.06	0.02
Correlation Co-efficient	***0.54***	

The correlation coefficient between the two assets is 0.54. Let us see what happens if the correlation coefficient changes to two extremes- perfectly negatively correlated and perfectly positively correlated.

We trace the portfolio frontier using the same methodology as in "Tracing a Portfolio Frontier" section. We do it for three different values of the correlation co-efficient r_{AB}. For $r_{AB} = 0.54$; $r_{AB} = -1$ and for $r_{AB} = +1$ as shown in figure 11 below.

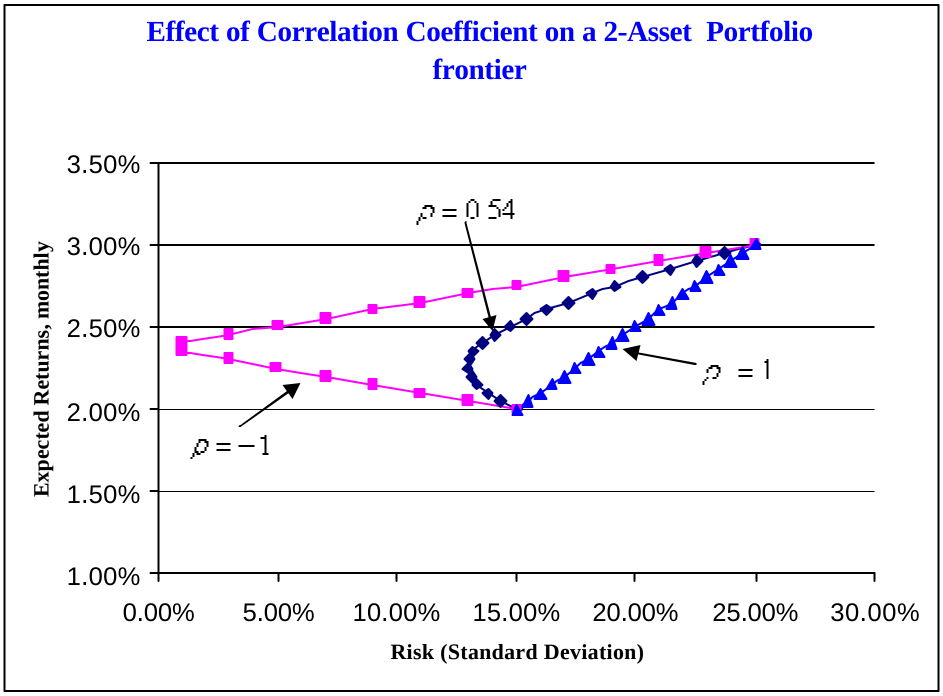
<u>Figure 11</u>

Effect of Correlation Co-efficient on Portfolio Frontier

	A	B	C	D	E	F	G	H	I	J	K	L	M
1		A	B		Effect of change in Correlation Co-efficient on 2-asset portfolio								
2	MEAN	3.00%	2.00%										
3	STD. DEVIATION	25.00%	15.00%										
4	VARIANCE	0.06	0.02										
5	CORRELATION	*0.54*			CORRELATION		-1				CORRELATION		1
6													
7				=SQRT(B17^2*$B$4+(1-B17)^2*$C$4+2*B17*(1-B17)*$B$4*$C$4*$B$5)									
8		A	B										
9	*Weights*	50%	50%										
10													
11													
12	Expected Return	2.50%										=$B17*$B$3+(1-$B17)*$C$3	
13	Portfolio Variance	14.71%				=ABS($B17*$B$3-(1-$B17)*$C$3)							
14													
15		*Weight*											
16		*of A*	Risk	Return			Risk	Return				Risk	Return
17		0%	15.00%	2.00%			15.00%	2.00%				15.00%	2.00%
18		5%	14.33%	2.05%			13.00%	2.05%				15.50%	2.05%
19		10%	13.78%	2.10%			11.00%	2.10%				16.00%	2.10%
20		15%	13.36%	2.15%			9.00%	2.15%				16.50%	2.15%
21		20%	13.09%	2.20%			7.00%	2.20%				17.00%	2.20%
22		25%	12.98%	2.25%			5.00%	2.25%				17.50%	2.25%
23		30%	13.03%	2.30%			3.00%	2.30%				18.00%	2.30%
24		35%	13.23%	2.35%			1.00%	2.35%				18.50%	2.35%

With the data we can generate three separate graphs for the three values of r_{AB}. There is a trick to superimpose all the three graphs in the same diagram. Select any graph, copy it and and paste it over the other graph, it should clearly superimpose if the size of the graph and the axis are identical. Do the same for the third graph and we have a graph very similar to figure 4 in the Model section. The screen-shot below (figure 12) of our example is shown below. We can observe, the best diversification can be obtained for $r_{AB} = -1$.

Figure 12



Minimum Variance Portfolio for Three Risky Assets

Let us pick a third asset, Haliburton (HAL), which would give us a three asset portfolio with ORCL and JPM. The primary objective is to find the minimum variance portfolio and the weights of the three asset which correspond to the minimum variance portfolio

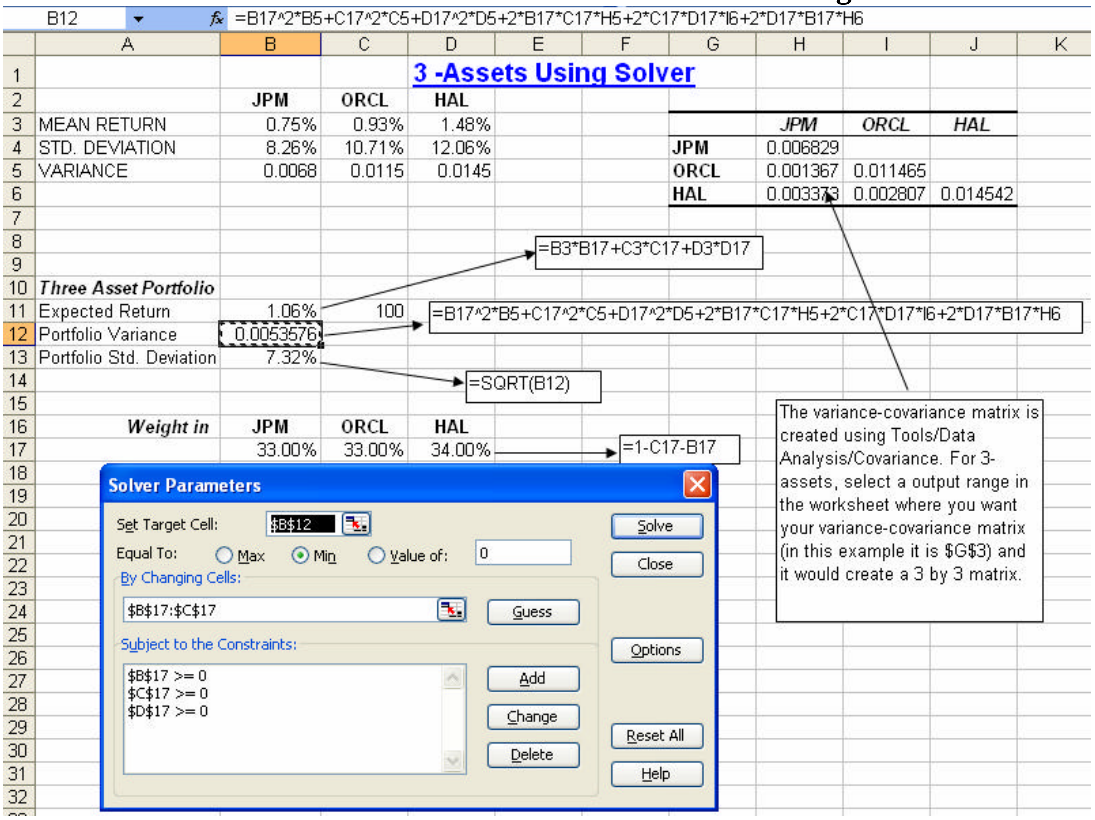
Solving the QPP: **Minimizing portfolio risk subjected to the constraints**

- The weight of funds invested in different assets must add to unity;
- There is no short sale provision.

The first constraint can be specified by ensuring that the weight of any asset is one *minus* the sum of the weights of JPM and HAL. To incorporate the second constraint we specify that the weights are non-negative in the solver dialog box as shown in figure 13. We set the target cell as B12 which corresponds to the portfolio variance and allow the solver to change the portfolio weights (Cells B17 and C17) to solve for the minimum variance portfolio.

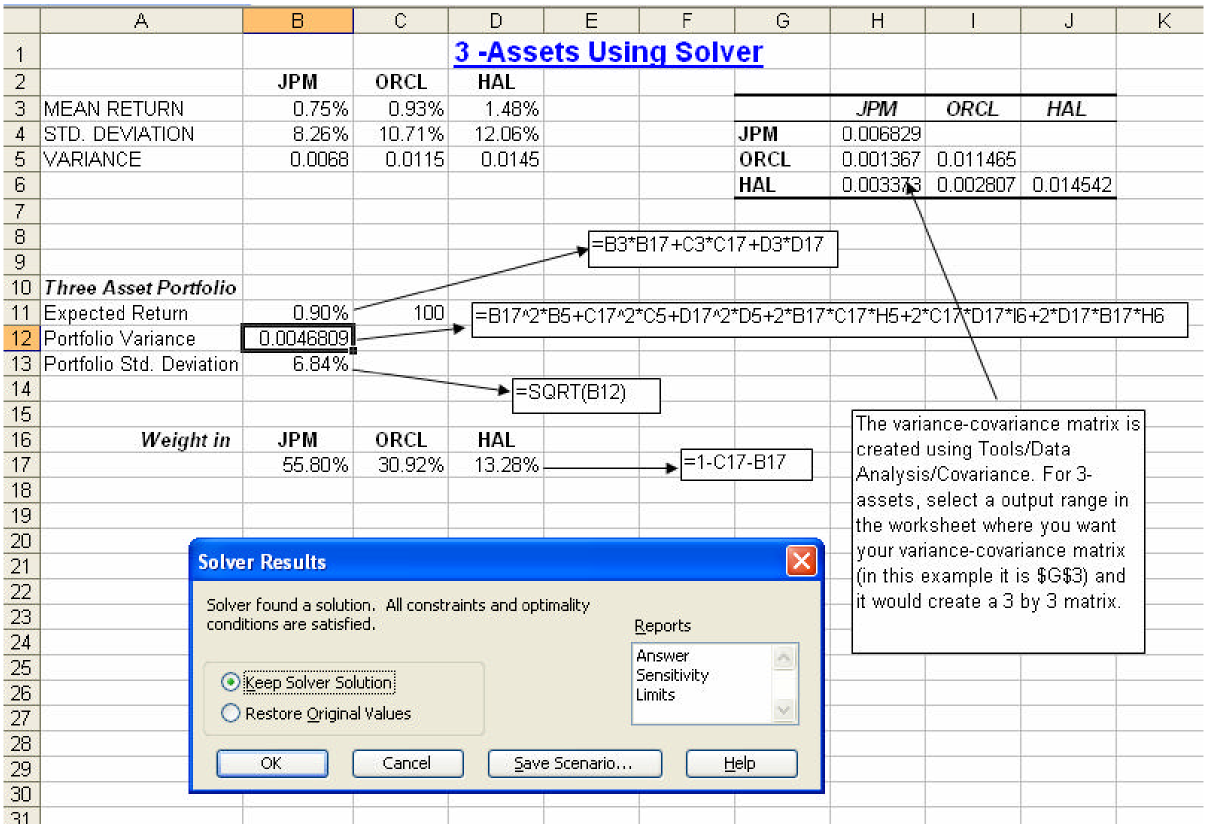
Figure 13

Minimum Variance Portfolio of Three Assets using Solver



The results (see figure 14) are reflected in cells B11 for expected return on the portfolio, cells B12 and B13 for minimum variance and the minimum standard deviation. The new portfolio weights are displayed in cells B17:D17. The minimum variance portfolio is characterized by an expected return of 0.90% and risk of 6.84% with portfolio weights of 55.8% for JPM, 30.92% for ORCL and 13.28% for HAL.

Figure 14



Efficient Frontier with 5 Risky Assets

Formulas become lengthy and complex as more and more assets are added to the portfolio. Excel has functions which allow you to do basic Matrix operations like addition, subtraction, matrix multiplication, inverse and transpose. We already started the section by naming the data inputs into *arrays*, which is the basic building block of matrix algebra. We name the following matrices as defined in the model section under "N Risky Assets";

Refer to the file "Portfolio_optimization_Matrix.xls" for complete solution and details

m is an 1×5 column vector of mean returns of 5 stocks. The array is named "**mu**".
s represents a 1×5 column vector of the portfolio shares , the array is named "**s**".
$\Sigma = \left[s_{ij}\right]$, represents the 5×5 variance-covariance matrix of the n asset returns, the array is named "**Sigma**".
1 represents a 1×5 column vector of 1's, named "**i**".
Σ^{-1} represent the inverse of matrix Σ , and can be represented by Excel's Matrix inverse function "MINVERSE(**Sigma**)", where "Sigma" is the named variance covariance matrix.

For Matrix multiplication we use "MMULT()" and for transposing matrices, we use the "TRANSPOSE()" function. For details on Matrix functions in excel refer to Excel's Help menu. When you enter a matrix algebra formula in Excel, remember to press CTRL + SHIFT+ ENTER together to execute the formula. Simply pressing ENTER would give an error.

Rewriting equations (10), (11) and (12) we have the corresponding equations (with *) in Excel notation the **QPP** is given as:

Minimize with respect to ***s***: $$\frac{1}{2} s^T \sum s$$

(10*) =MMULT(TRANSPOSE(s),MMULT(Sigma,s))

Subject to the constraints

$$m^T s = m_p$$

(11*) {=MMULT(TRANSPOSE(mu),s)}

$$1^T s = 1$$

(12*) {=MMULT(TRANSPOSE(I), s)}

Refer to figure 15. We set the target cell in the Solver dialog box equal to equation (10*) and specify the constraints as in equations (11*) and (12*) in the "Subject to Constraints" box. Select the range of the portfolio weights of the 5 risky assets (F4:F8) as the cells that would be changed by Solver to reach the optimization solution.

The solver solution is shown in figure 16. The optimal portfolio risk (as measured by standard deviation) is 4.79% and the corresponding portfolio expected monthly return is 0.6%. The weights in the optimal portfolio are also shown in cells F4 to F8.

Figure 15

Using Solver to solve for the Minimum Variance Portfolio for 5 -stocks

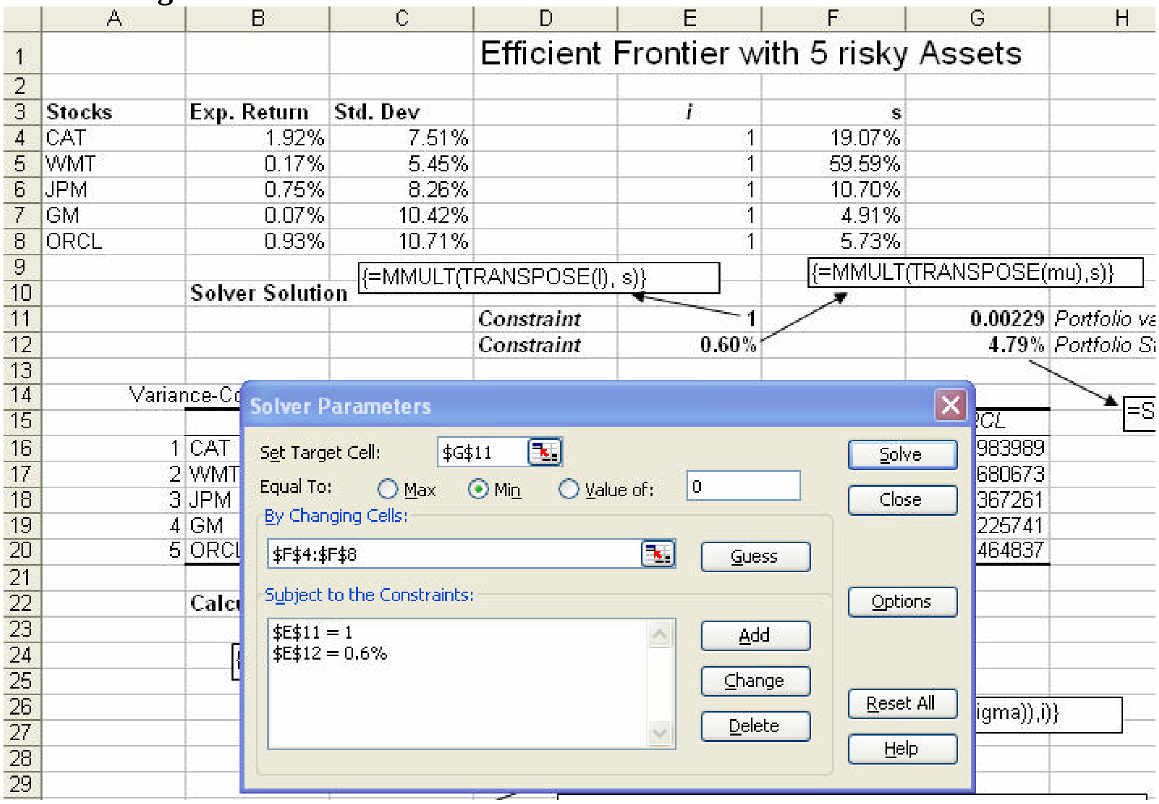
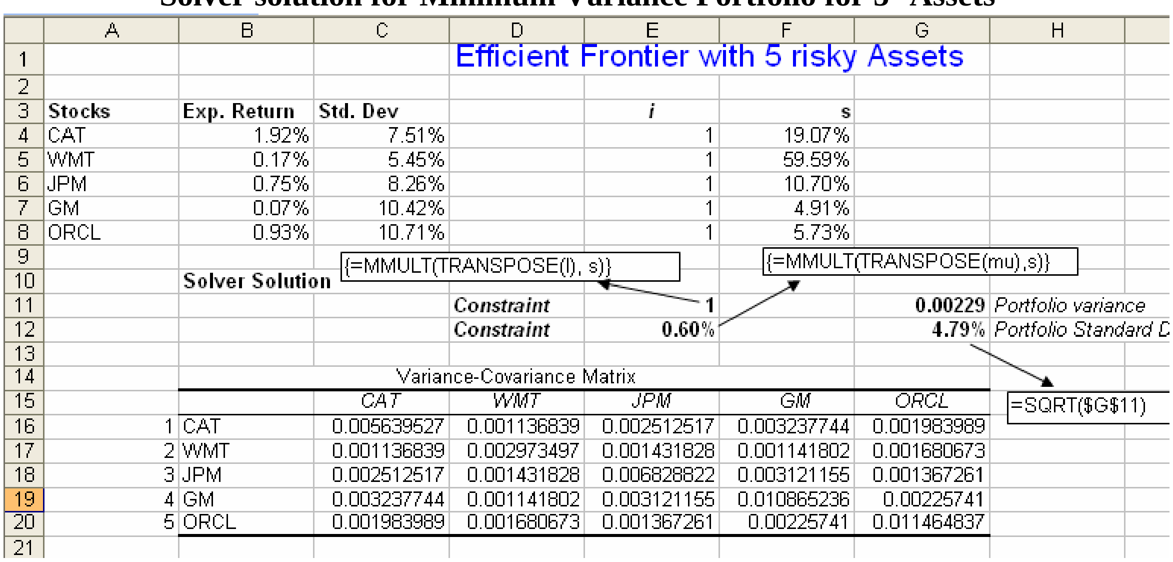


Figure 16

Solver solution for Minimum Variance Portfolio for 5 -Assets



Efficient Frontier with 5 risky Assets

Stocks	Exp. Return	Std. Dev		i	s
CAT	1.92%	7.51%		1	19.07%
WMT	0.17%	5.45%		1	59.59%
JPM	0.75%	8.26%		1	10.70%
GM	0.07%	10.42%		1	4.91%
ORCL	0.93%	10.71%		1	5.73%

{=MMULT(TRANSPOSE(I), s)} {=MMULT(TRANSPOSE(mu),s)}

Solver Solution

	Value		
Constraint	**1**	**0.00229**	*Portfolio variance*
Constraint	**0.60%**	**4.79%**	*Portfolio Standard D*

=SQRT($G$11)

Variance-Covariance Matrix

		CAT	*WMT*	*JPM*	*GM*	*ORCL*
1	CAT	0.005639527	0.001136839	0.002512517	0.003237744	0.001983989
2	WMT	0.001136839	0.002973497	0.001431828	0.001141802	0.001680673
3	JPM	0.002512517	0.001431828	0.006828822	0.003121155	0.001367261
4	GM	0.003237744	0.001141802	0.003121155	0.010865236	0.00225741
5	ORCL	0.001983989	0.001680673	0.001367261	0.00225741	0.011464837

Using Calculus Method

Refer to the case of "N Risky Assets" in the model section. We start by re-writing the formulas defined in the model section under "N Risky Assets" in Excel form. We continue with the same worksheet in "Portfolio_optimization_Matrix.xls".

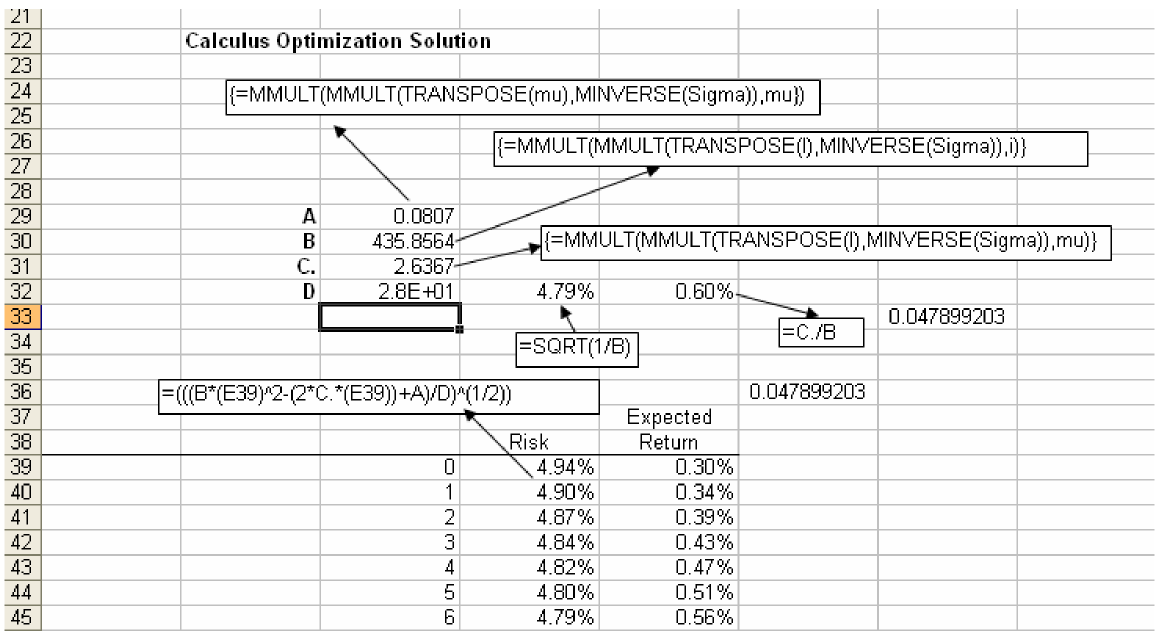
$A = m^T \Sigma^{-1} m$ as {=MMULT(MMULT(TRANSPOSE(mu),MINVERSE(Sigma)),mu})

$B = 1^T \Sigma^{-1} 1$ as {=MMULT(MMULT(TRANSPOSE(I),MINVERSE(Sigma)),i)}

[21] $C = 1^T \Sigma^{-1} m$ as {=MMULT(MMULT(TRANSPOSE(I),MINVERSE(Sigma)),mu)},

and, $D = AB - C^2$ as =A*B-(C.^2)

Figure 17
Calculus Method



For the return on the minimum risk portfolio we take $m_P = C/B$ as =C./B in excel notation and for minimum standard deviation we take the square-root of $s_p^2 = 1/B$ in excel notation as =SQRT(1/B). Figure 17 shows the solution using calculus method and it matches the solution by the solver method.

For the equation of the portfolio frontier in the mean-standard deviation space, we take the square-root of equation (21)

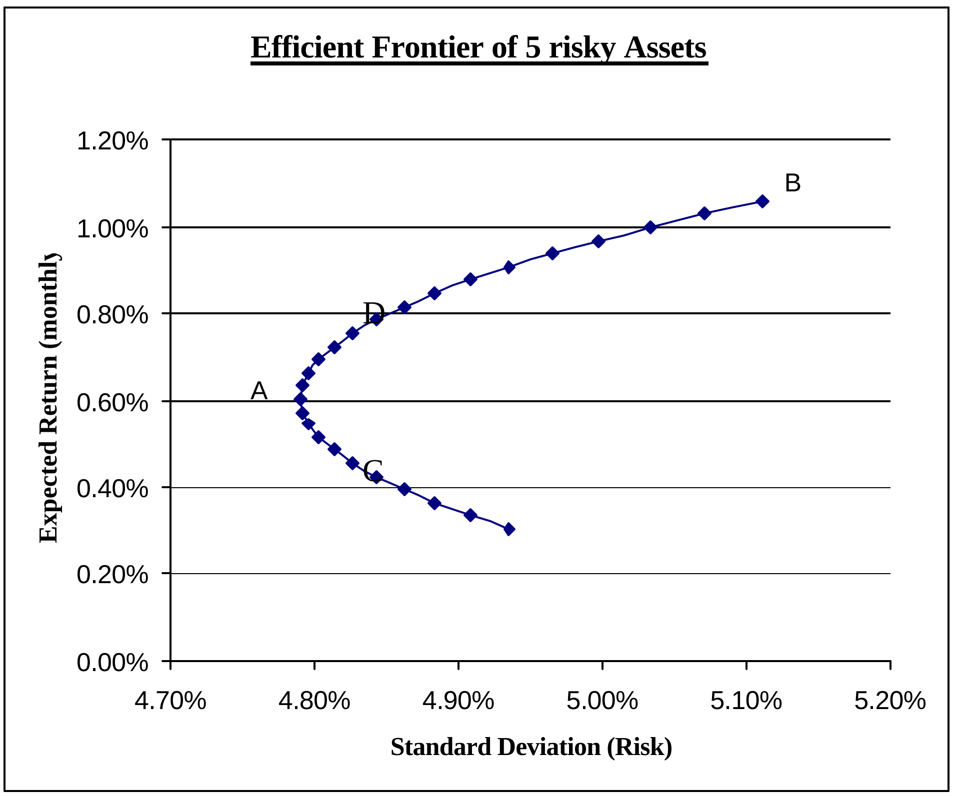
[21] Excel does not allow using "C" to name arrays. We use "C." in place of "C".

(21) $$s_p^2 = \frac{\left(Bm_p^2 - 2\,Cm_p + A\right)}{D}$$ as,

=(((B*(E39)^2-(2*C.*(E39))+A)/D)^(1/2)) in excel notation, where the cell E39 corresponds to a value of the portfolio expected return.

We can trace the Portfolio frontier by selecting different values of expected return and calculating the corresponding standard deviation using the above formula. **Note:** In solving by the calculus method we do not anymore consider the assumption that there are no short sale constraints. Figure 18 shows the graph of the frontier. Refer to the worksheet "Portfolio_optimization_Matrix.xls" for more description on how to select different values portfolio expected returns.

Figure 18

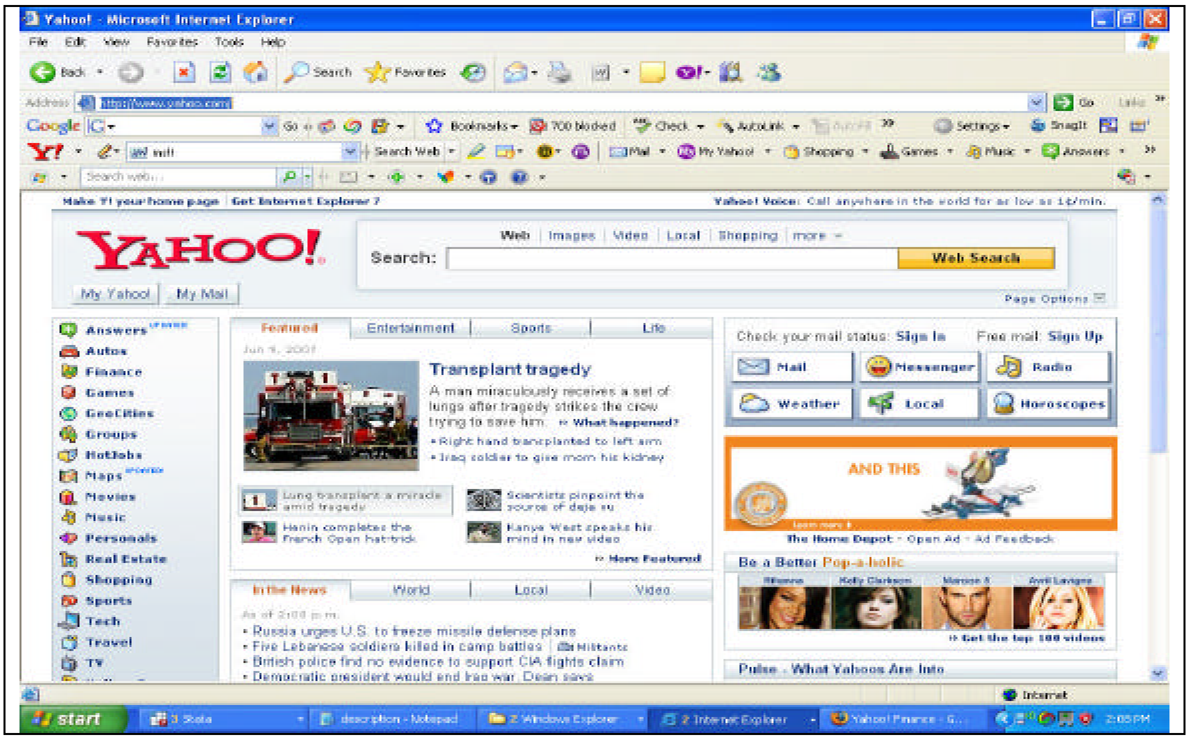


The line tracing the points from the minimum variance portfolio *A* to *B* is the efficient frontiers. A risk averse investor will never hold a portfolio which is to the south-east of point *A*. Any point to the south-east of *A* such as *C* would have a corresponding point like *D* which has a higher expected return than C with the same portfolio risk. Clearly, point *D* will ***dominate*** point C. The **optimal portfolio** is the portfolio allocation (or points on the efficient frontier) corresponding to the investor's risk preference. More risk averse investors would choose points on the efficient frontier that is closer to point *A*. Similarly, an investor who is less risk averse or can handle more risk would choose a portfolio closer to point *B*.

Empirical Data

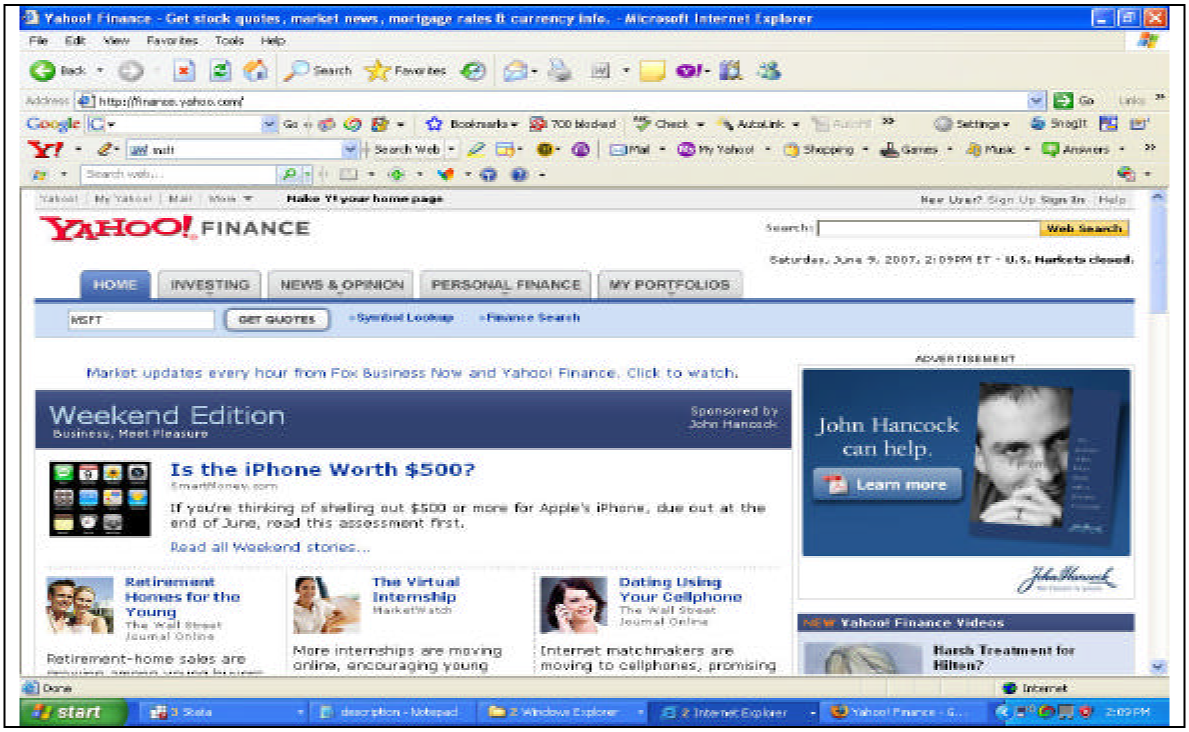
Downloading Stock data from Yahoo! Finance and calculating the monthly stock returns using Excel.[22]

Step 1: Type the url http://www.yahoo.com on the address bar of your web browser and click on ‘Finance’.

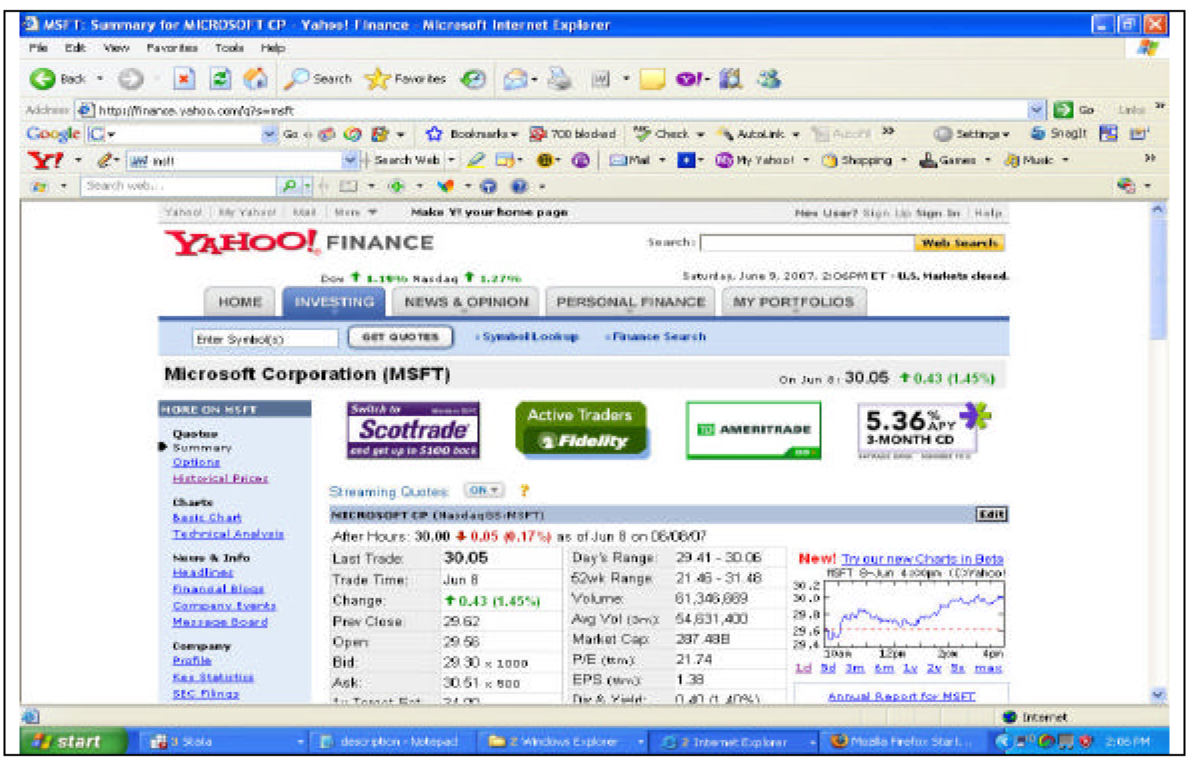


Step 2: It would open a page similar to as shown below. In the ‘Enter Symbol(s)’ box, type the ticker symbol for the stock you want to look up. If you do not know or remember the ticker symbol of a particular stock you can always click on the ‘Symbol Lookup’ button which would assist you finding the correct symbol. When you have entered your stock symbol (in our example it MSFT for Microsoft), press enter or click on “Get Quotes”.

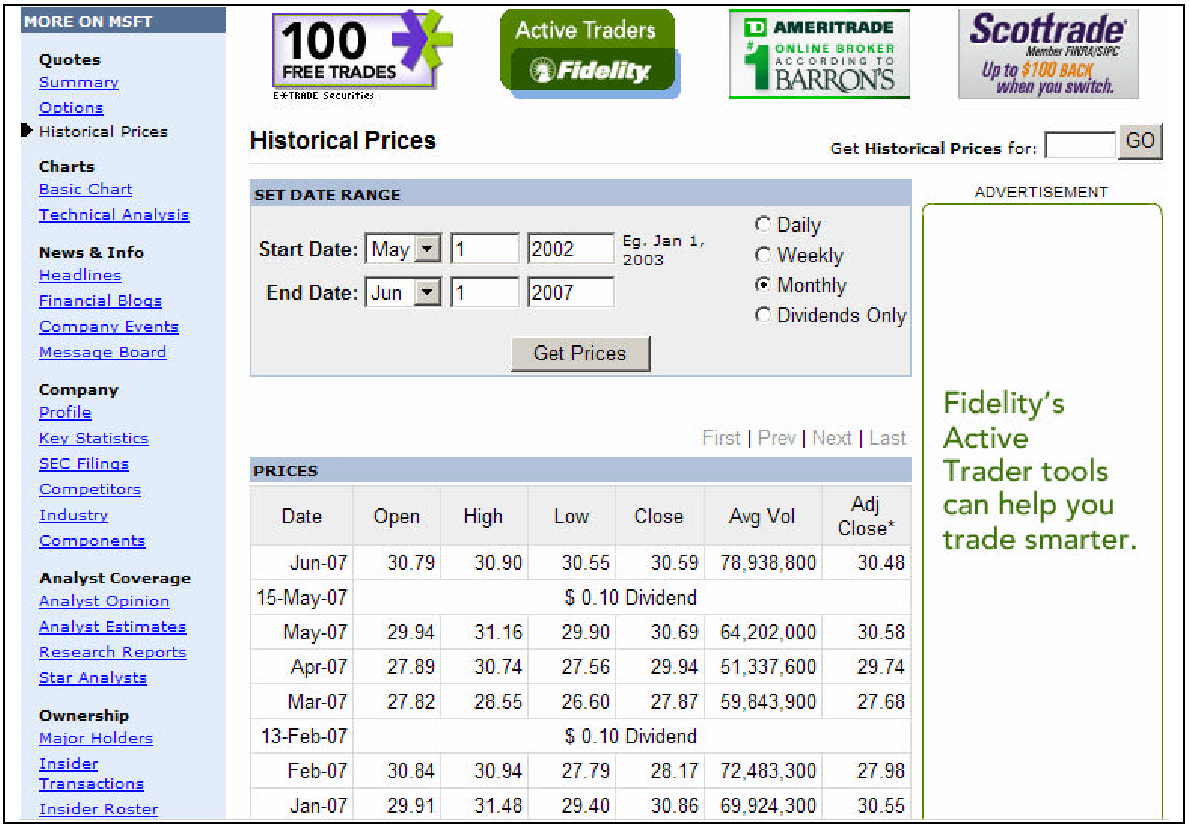
[22] For a detailed video demonstration of how to download data from yahoo and calculate monthly returns, lookup the video file “returns_from_yahoo.avi”.



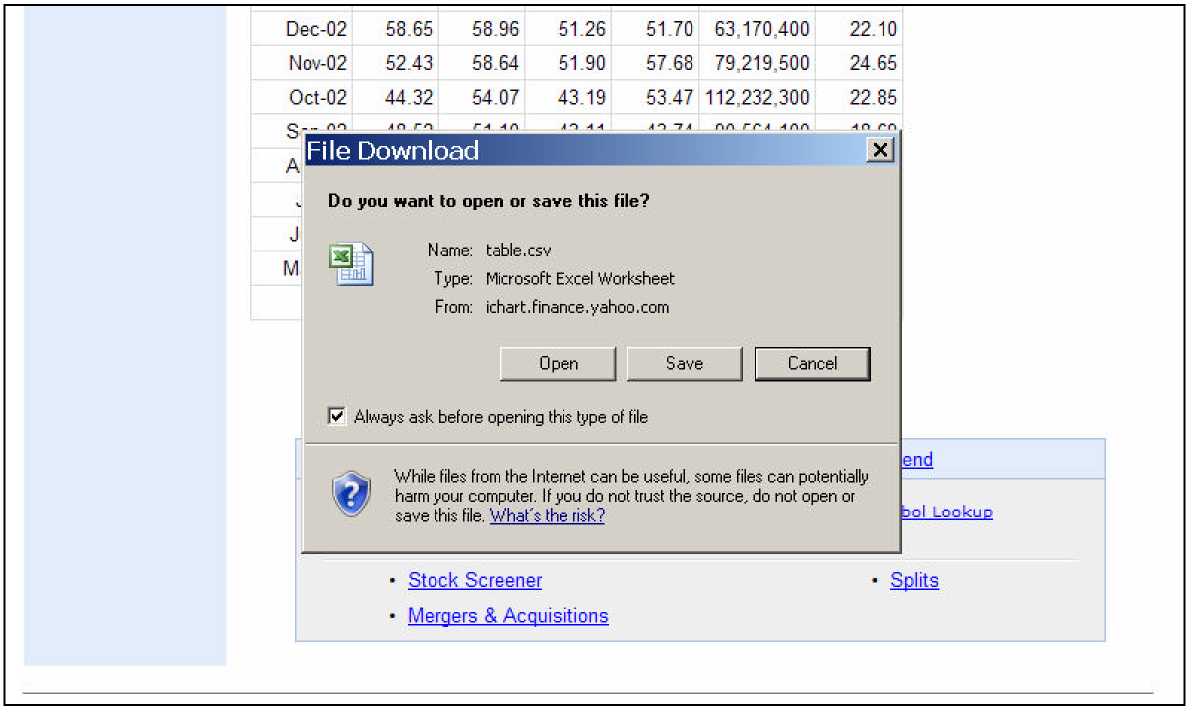
Step 3: It would take you to a page similar to as shown below. You can choose 'Historical Prices' to get Microsoft's price history or you can click on the picture of the price chart and then scroll down to arrive at 'Historical Prices'.



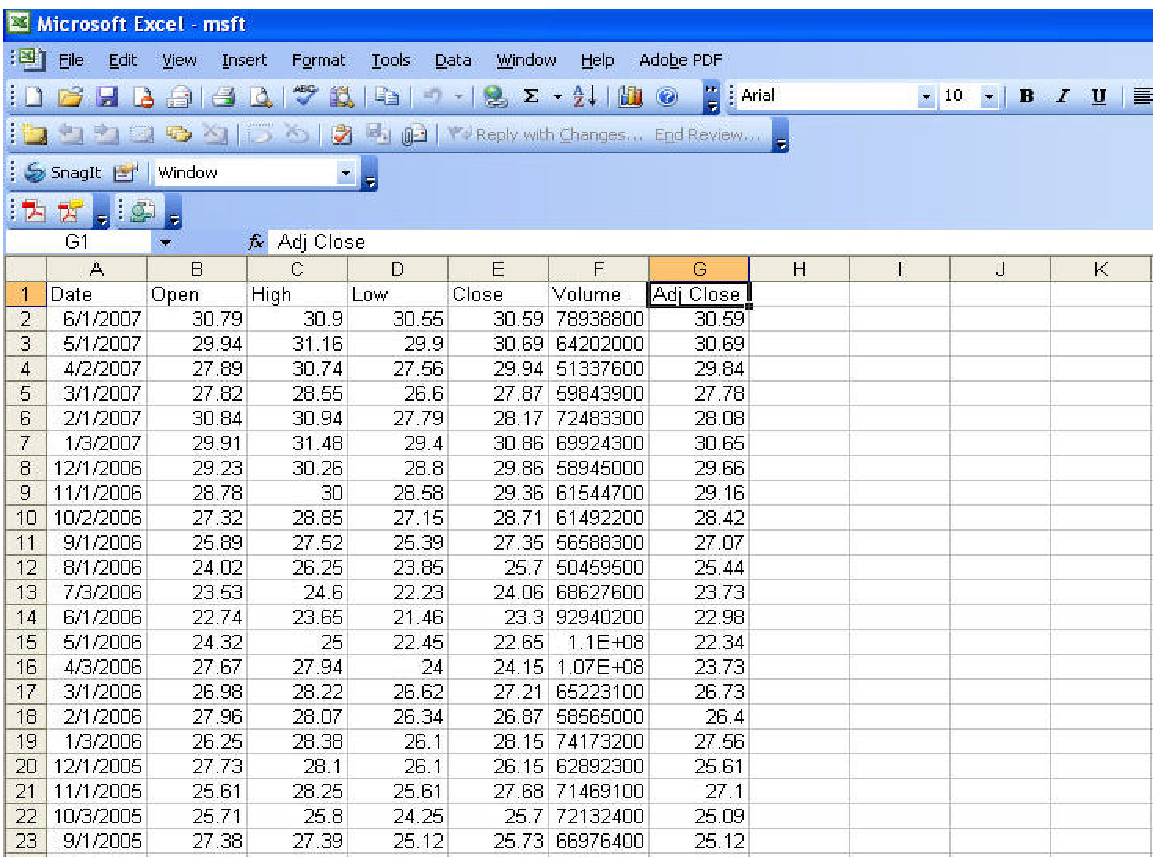
Step 4: Indicate the time period and frequency for the data we want. We have chosen a six year time period starting from, say, May 1, 2005 to June 1, 2007. We chose the 'Monthly' option to download monthly price data.



Step 5: If we scroll down the webpage it gives us an option to download the data in excel spreadsheet format. Yahoo allows you to save the spreadsheet file as 'Table.csv' which is a comma separated value file and can be opened directly by excel. We changed the name to 'msft.csv' to save it in our computer.

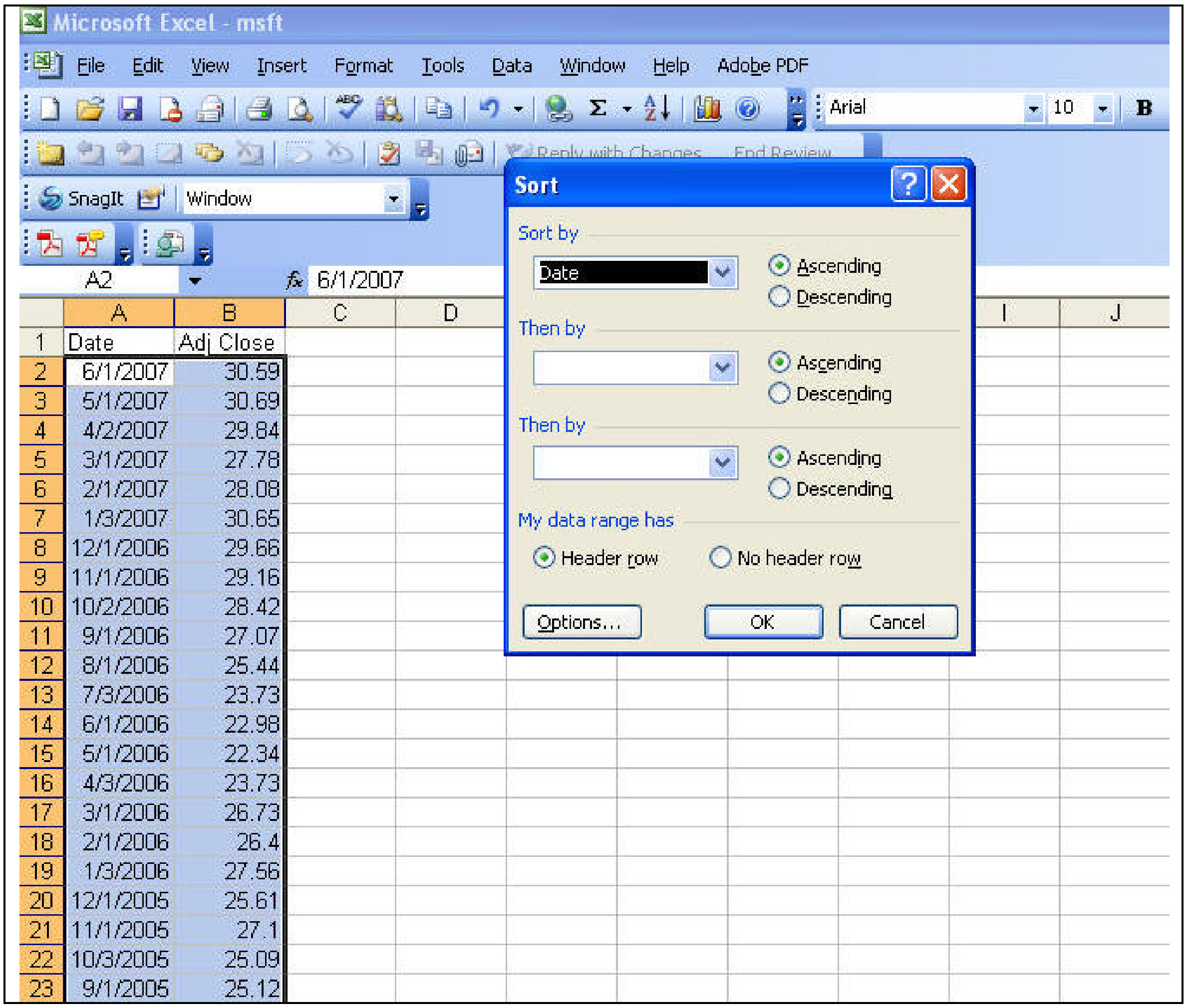


The file when opened through excel looks similar to the screen-shot below.



	Date	Open	High	Low	Close	Volume	Adj Close
2	6/1/2007	30.79	30.9	30.55	30.59	78938800	30.59
3	5/1/2007	29.94	31.16	29.9	30.69	64202000	30.69
4	4/2/2007	27.89	30.74	27.56	29.94	51337600	29.84
5	3/1/2007	27.82	28.55	26.6	27.87	59843900	27.78
6	2/1/2007	30.84	30.94	27.79	28.17	72483300	28.08
7	1/3/2007	29.91	31.48	29.4	30.86	69924300	30.65
8	12/1/2006	29.23	30.26	28.8	29.86	58945000	29.66
9	11/1/2006	28.78	30	28.58	29.36	61544700	29.16
10	10/2/2006	27.32	28.85	27.15	28.71	61492200	28.42
11	9/1/2006	25.89	27.52	25.39	27.35	56588300	27.07
12	8/1/2006	24.02	26.25	23.85	25.7	50459500	25.44
13	7/3/2006	23.53	24.6	22.23	24.06	68627600	23.73
14	6/1/2006	22.74	23.65	21.46	23.3	92940200	22.98
15	5/1/2006	24.32	25	22.45	22.65	1.1E+08	22.34
16	4/3/2006	27.67	27.94	24	24.15	1.07E+08	23.73
17	3/1/2006	26.98	28.22	26.62	27.21	65223100	26.73
18	2/1/2006	27.96	28.07	26.34	26.87	58565000	26.4
19	1/3/2006	26.25	28.38	26.1	28.15	74173200	27.56
20	12/1/2005	27.73	28.1	26.1	26.15	62892300	25.61
21	11/1/2005	25.61	28.25	25.61	27.68	71469100	27.1
22	10/3/2005	25.71	25.8	24.25	25.7	72132400	25.09
23	9/1/2005	27.38	27.39	25.12	25.73	66976400	25.12

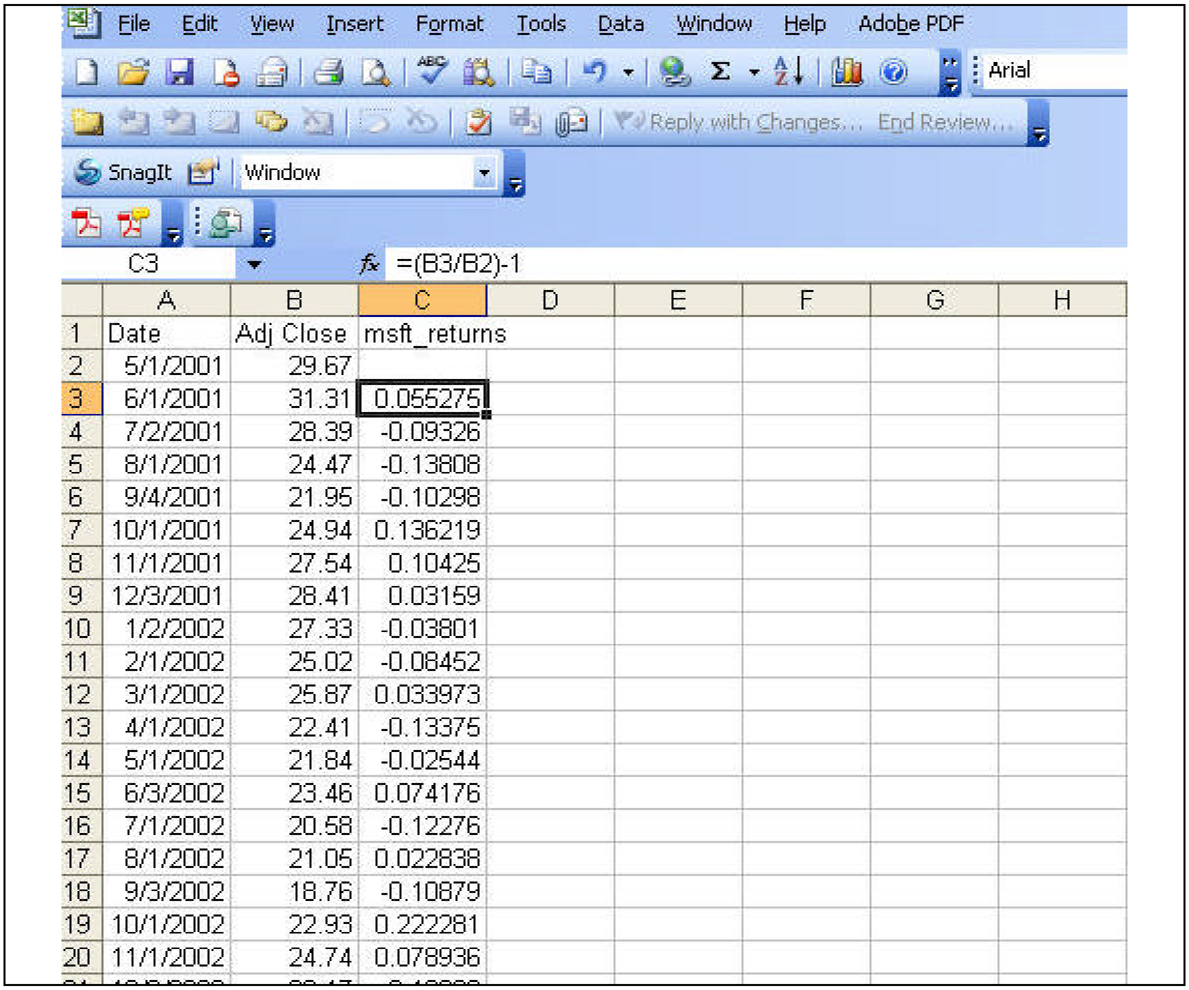
Step 6: To compute monthly returns we keep only the columns ‘date’ and ‘Adj. Close’. The adjusted closing stock price in yahoo accounts for any dividends and stock splits which may have occurred during our sample period. The date is tabulated in descending order and we need to sort it in ascending order in order to calculate returns. We use the Excel ‘Sort’ function to sort the date in ascending order. At this point it is preferable to save it as a ‘Microsoft excel workbook’ file instead of a ‘Comma separated value’ file as the excel workbook file preserves any formulas we work on while the .csv file does not.



Step 7: Once we have finished step 6 we are ready to calculate returns for Microsoft. The returns can be calculated from Yahoo! Finance by using the formula

$r_t = \frac{P_t}{P_{t-1}} - 1$. In our example Microsoft monthly stock returns corresponding to

6/1/2001 is given by typing the excel formula “$(B3/B2)-1$” in cell C3. Here P_t = Price on 6/1/2001 and P_{t-1} =Price on 5/1/2001. Copying the formula from cell C3 to the end of your date range will give you the time series data on Microsoft stock returns.



	A	B	C	D	E	F	G	H
1	Date	Adj Close	msft_returns					
2	5/1/2001	29.67						
3	6/1/2001	31.31	0.055275					
4	7/2/2001	28.39	-0.09326					
5	8/1/2001	24.47	-0.13808					
6	9/4/2001	21.95	-0.10298					
7	10/1/2001	24.94	0.136219					
8	11/1/2001	27.54	0.10425					
9	12/3/2001	28.41	0.03159					
10	1/2/2002	27.33	-0.03801					
11	2/1/2002	25.02	-0.08452					
12	3/1/2002	25.87	0.033973					
13	4/1/2002	22.41	-0.13375					
14	5/1/2002	21.84	-0.02544					
15	6/3/2002	23.46	0.074176					
16	7/1/2002	20.58	-0.12276					
17	8/1/2002	21.05	0.022838					
18	9/3/2002	18.76	-0.10879					
19	10/1/2002	22.93	0.222281					
20	11/1/2002	24.74	0.078936					

Problems and Projects

Q.1. Choose any risky asset (say ORCL) and a risk-free asset like a treasury bill which gives you a 0.43% monthly return.
Using Excel, graph the portfolio opportunity set and find the slope. Also find the *Sharpe Ratio.*

Q.2. Relax the assumption that no short selling is allowed. Now you are permitted to short sell the risk free asset (i.e., borrow at the risk-free rate) and also the risky. Using excel, graph the new opportunity set.

Q.3. Stocks of A and B have the following risk-return characteristics:

Stock	**A**	**B**
Expected return (%)	15	20
Variance of returns $(\%)^2$	9	16
Covariance $(\%)^2$		8

Show how the expected return and the risk of the portfolio changes with change in s_A , which is the proportion of funds invested in Asset A.

Q.4.
a. Consider the data provided in example 1. Calculate the composition of the minimum risk portfolio for the following values of r_{AB} :

i. $r_{AB} = 0.5$
ii. $r_{AB} = 0$
iii. $r_{AB} = (-)\ 0.5$
iv. $r_{AB} = (-)\ 1$

b. Calculate the measures of risk and return for the minimum risk portfolio determined in graph (a)

Q.5. Using Excel solver find the Dual of the QPP:-
Maximize expected returns subjected to a maximum level of risk (say 8% standard Deviation). The usual assumptions of no short sale constraints and the weights add-up to unity, still applies. Do it for the three stocks example, JPM, ORCL and HAL.

Q.6. Refer to the question number 5. Add an additional constraint that the maximum percentage of your portfolio you can invest in JPM is 30%. Did your answer change?

Q.7. Consider two stocks Yahoo and AT&T (you can use the data from Master_data.xls). Trace the portfolio frontier for Yahoo and AT&T keeping the weights of both stocks unchanged at 50% each. Now consider two extreme correlation coefficients of +1 and -1. Construct the new portfolio frontiers and show it in the graph.

The Following Questions can be used as a Class Project

Q.8. Using matrix algebra and solver, find the minimum variance portfolio for three risky assets, ORCL, HAL and JPM.

Q.9. Do the same exercise as in question 9 but using the Calculus method. Also draw trace the efficient frontier.

Q.10. A Wall Street Journal article of February 10, 1997 reports on a practical risk correction procedure employed by Morgan Stanley, one of the largest investment Banks in US. To apply the procedure "…..Morgan Stanley tweaks a fund's portfolio until the volatility exactly equals that of a benchmark like Standard & Poor's 500-stock index. ………Once that is done, the yield on the new hypothetical portfolio equals the risk adjusted return."

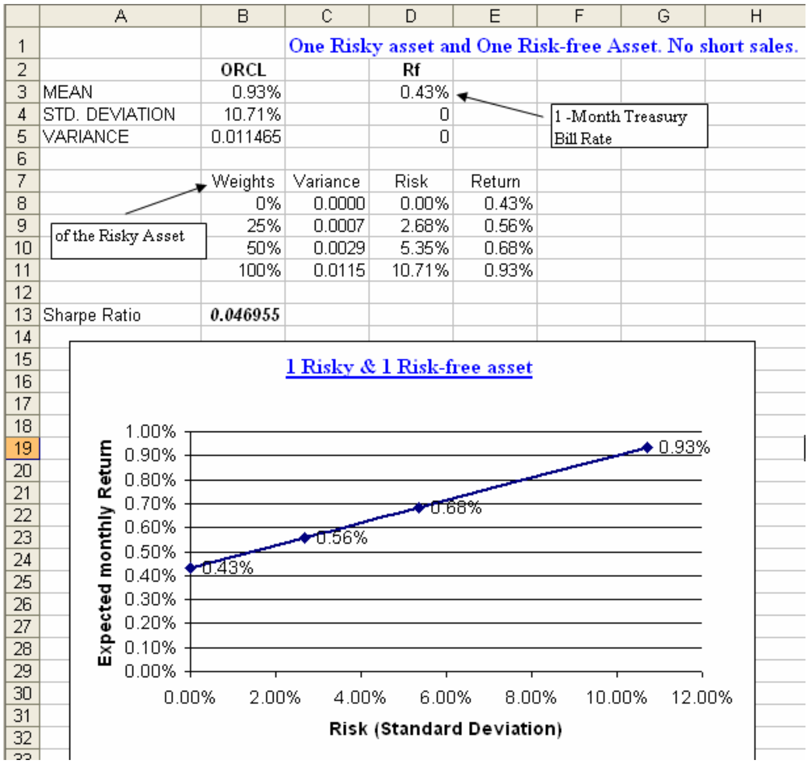
Use solver to construct a three-asset portfolio which mimics the volatility of the S&P 500 returns for a given period (say 5 years). You can download the S&P 500 data from Yahoo! Finance (the symbol is "^GSPC")
Is your portfolio expected return higher than the S&P 500 return? Why or Why not?

Advanced Topic (CML)

Q.11. Consider two risky assets, Infosys (INFY) and Yahoo (YHOO) and also a risk-free investment in the form of a treasury boll which gives you 0.40% monthly return.

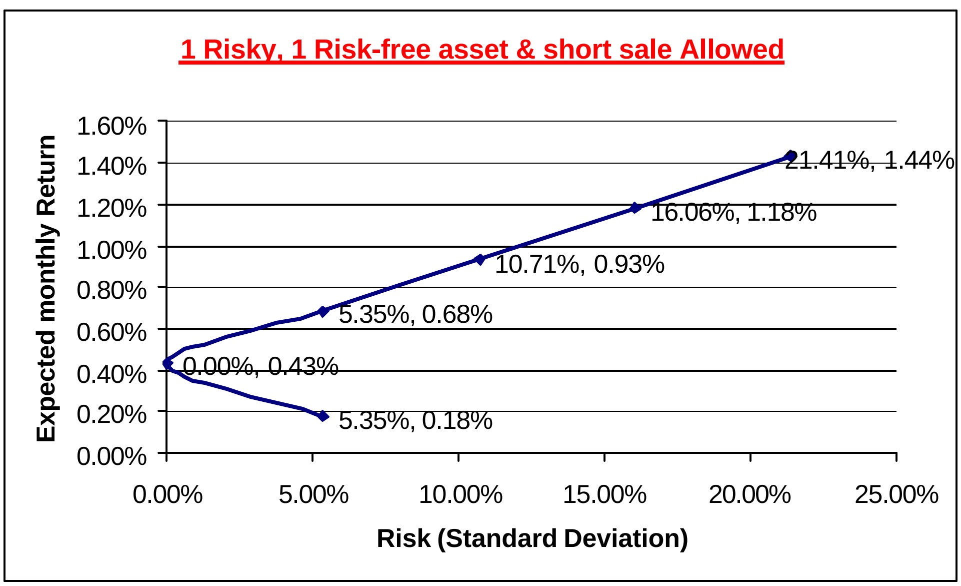
Solutions to Selected Problems

Solution 1: Refer to the tab "1 risky – 1 risk free" in the Assignment_solutions.xls file.



	A	B	C	D	E	F	G	H
1			One Risky asset and One Risk-free Asset. No short sales.					
2		ORCL		Rf				
3	MEAN	0.93%		0.43%		1 -Month Treasury Bill Rate		
4	STD. DEVIATION	10.71%		0				
5	VARIANCE	0.011465		0				
6								
7		Weights	Variance	Risk	Return			
8		0%	0.0000	0.00%	0.43%			
9	of the Risky Asset	25%	0.0007	2.68%	0.56%			
10		50%	0.0029	5.35%	0.68%			
11		100%	0.0115	10.71%	0.93%			
12								
13	Sharpe Ratio	*0.046955*						

Solution 2: Refer to the tab "Short Sale Allowed" in the Assignment_solutions.xls file.



Solution 3:

Proportion of funds invested in A	**0.0**	**0.2**	**0.4**	**0.5**	**0.8**	**1.0**
I. Expected Return of the Portfolio (%)	20	19.0	18.0	17.5	16.0	15.0
II. Variance of Portfolio Returns (%)	16	13.16	11.04	10.25	8.96	9.0
III. Standard Deviation of Portfolio Returns (%)	4	3.62	3.32	3.2	2.99	3.0
IV. Return per unit of Risk	5	5.25	5.42	5.47	5.35	5.0

Solution 4:

a. Plugging in the values of s_A^2, s_B^2 and r_{AB}, in equation (8) from the model section, We can arrive at the following compositions of the minimum risk portfolios:

r_{AB}	0.5	0	(-) 0.5	(-) 1
X_A	0.77	0.64	0.60	0.57
X_B	0.23	0.36	0.40	0.43

b. Given $r_{AB} = 0.5$, $X_A = 0.77$, $X_B = 0.23$

E (Rp) = (0.77 × 15) + (0.23 × 20) = 16.15%

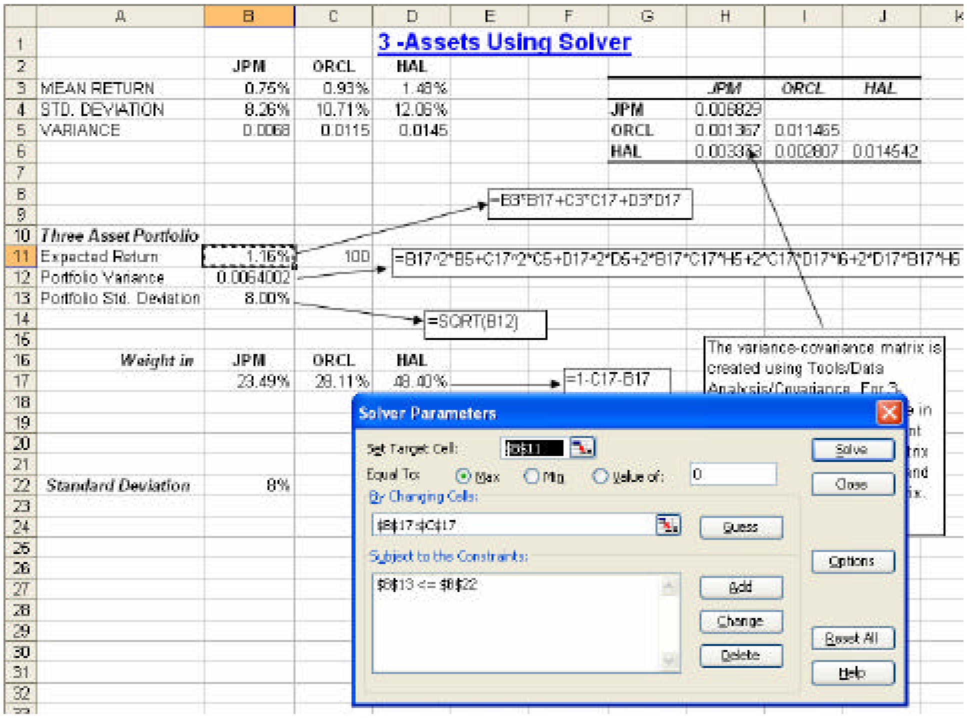
$$s_p = \sqrt{(0.77^2 \times 9) + (0.23^2 \times 16) + 2(0.5)(0.77)(0.23)(3)(4)}$$

= 2.88%

Repeating these calculations for the other combinations of r_{AB}, X_A and X_B, We can arrive at the following results:

r_{AB}	**0.5**	**0**	**(-) 0.5**	**(-) 1**
XA	0.77	0.64	0.60	0.57
XZ	0.23	0.36	0.40	0.40
E (RP) (%)	16.15	16.80	17.00	17.15
$s\,r$ (%)	2.88	2.40	1.71	0.01

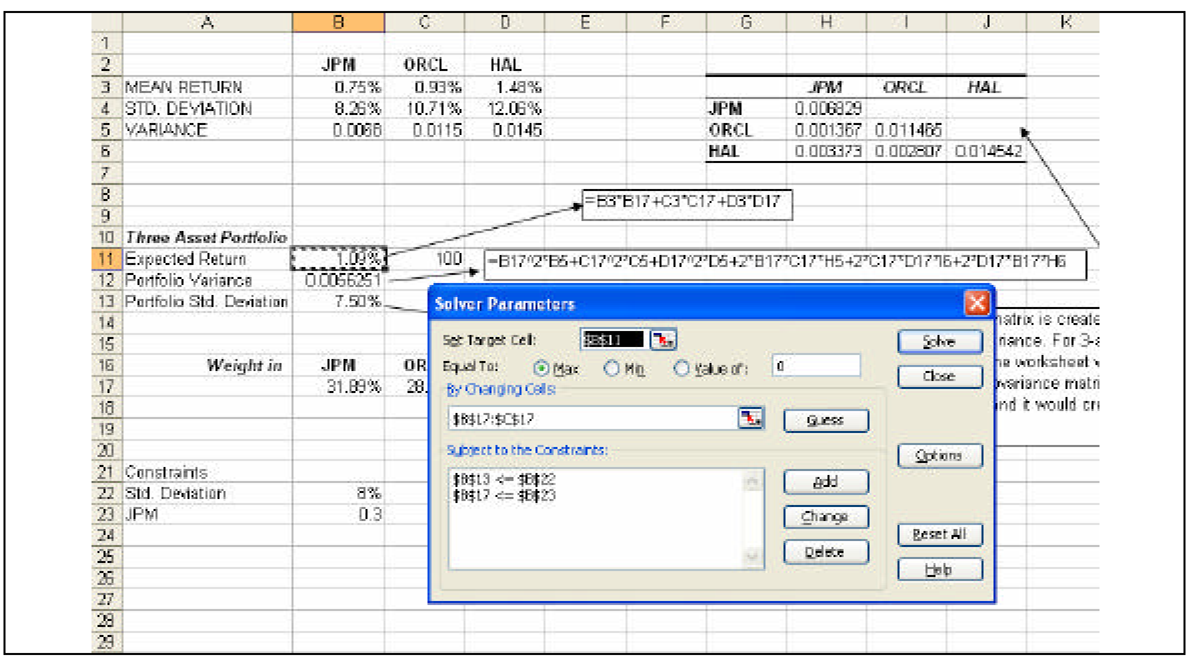
Solution 5: Solving the Dual of the QPP: Maximize return subjected to given risk. Refer to the Tab "Solver -3 assets" in Assignmnet_solutions.xls.



Solution 6:

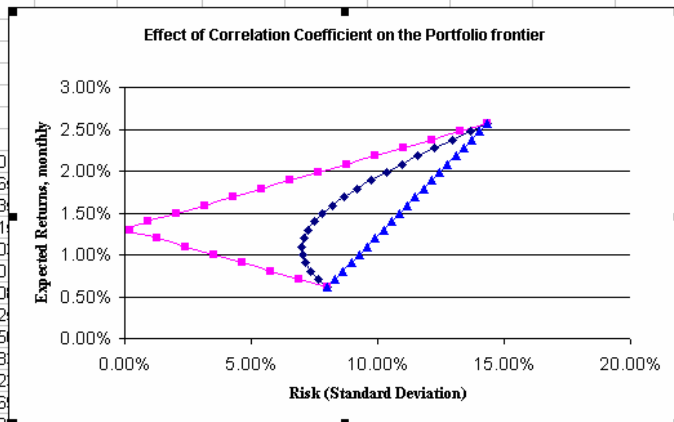
Adding an additional constraint to the above problem- Maximum weight of JPM is 30%

Refer to the Tab "Solver -3 assets" in Assignmnet_solutions.xls.



Solution 7: Refer to the tab “Yhoo-T” in the Assignment_solutions.xls

	A	B	C	D	E	F	G	H	I	J	K
1		YHOO	T								
2	MEAN	2.57%	0.61%								
3	STD. DEVIATION	14.36%	8.01%								
4	VARIANCE	0.02062	0.00642								
5	CORRELATION	***0.22253***				CORRELATION		*-1*			
6	COVARIANCE	0.00256									
7											
8		YHOO	T								
9	*Weights*	50%	50%								
10											
11											
12	Expected Return	1.59%									
13	Portfolio Variance	0.006759									
14	Portfolio Std. Deviation	8.22%									
15											
16		YHOO	Risk								
17		0%	8.0								
18		5%	7.6								
19		10%	7.3								
20		15%	7.1								
21		20%	7.0								
22		25%	7.0								
23		30%	7.0								
24		35%	7.2								
25		40%	7.5								
26		45%	7.8								
27		50%	8.2								
28		55%	8.6								
29		60%	9.2								
30		65%	9.75%	1.89%			6.53%	1.89%			
31		70%	10.34%	1.98%			7.65%	1.98%			

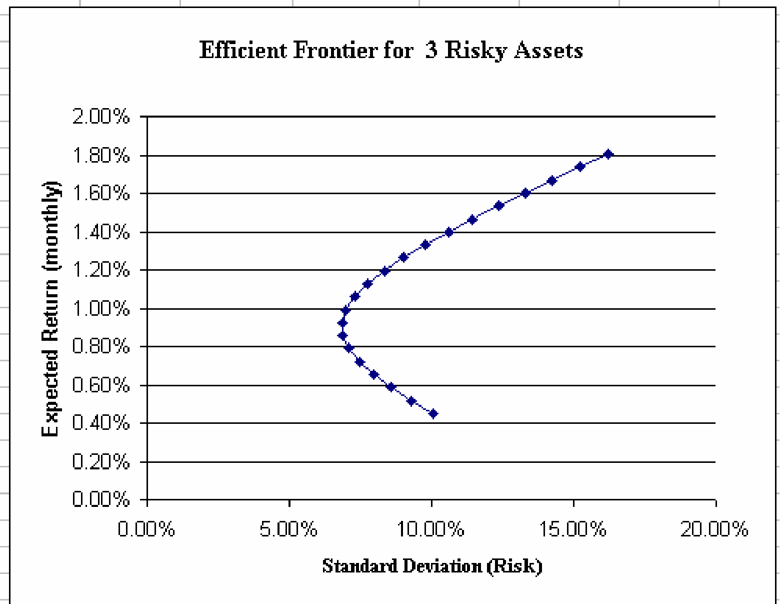


Solution 8: Refer to the tab “orcl_jpm_hal” in the master_data.xls

E	F	G	H	I	J	K	L
		Mean	Variance	Std. Dev			
	JPM	0.75%	0.00683	8.264%			
	ORCL	0.93%	0.01146	10.707%			
	HAL	1.48%	0.01454	12.059%			
		s					***i***
	JPM	55.796%					1
	ORCL	30.923%					1
	HAL	13.281%					1
	constraint	**1**		**0.004681**	Portfolio Variance		
	constraint	**0.90%**		**6.84%**	Portfolio Standard Deviation		
		JPM	*ORCL*	*HAL*			
	JPM	0.00682882	0.001367	0.003373			
	ORCL	0.00136726	0.011465	0.002807			
	HAL	0.0033726	0.002807	0.014542			

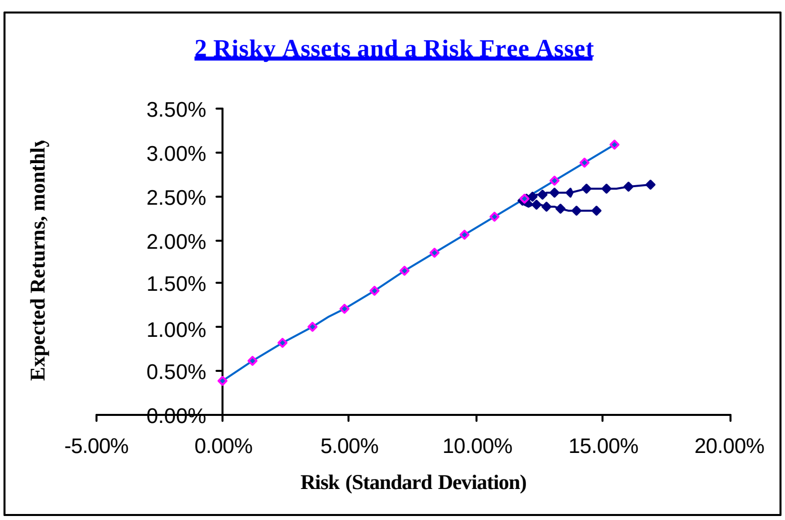
Solution 9: Refer to the tab “orcl_jpm_hal” in the master_xls

Calculus Solution		
A	0.02112574	
B	213.633332	
C	1.92562491	
D	0.8051	
Std. Dev	**6.84%**	**0.901%**
	Risk	Return
0	10.04%	0.45%
1	9.26%	0.52%
2	8.56%	0.59%
3	7.94%	0.65%
4	7.45%	0.72%
5	7.08%	0.79%
6	6.88%	0.86%
7	6.85%	0.92%
8	7.00%	0.99%
9	7.31%	1.06%
10	7.76%	1.13%
11	8.34%	1.19%
12	9.02%	1.26%



Solution 10: It is an open –ended solution. But typically the answer should be that the expected return of the 3-asset portfolio would be less than that of the S&P 500 portfolio for the same amount of risk.

Solution 11: Refer to the **CML** tab in the file Master_data.xls. Also refer to the equations (22) and (23) in the model section. The trick is to use the solver to maximize equation (23) which is the Sharpe Ratio. The slope is maximized at a point where the line from the risk free rate on the vertical axis is **tangent** to the portfolio frontier of two risky assets. The portfolio frontier is then traced by equation (22). The following two screen shots and the excel file would help you with the process.



	A	B	C	D	E	F
28						
29						
30						
31	**Standard Deviation**	14.36%	=SQRT(B28^2*$B$4+(1-B28)^2*$C$4+2*B28*(1-B28)*$B$6)			
32						
33						
34	**A**	***54.78%***		**0.1739**		
35	**B**	***45.22%***				**Solver Solution**
36	**Mean**	2.47%				
37	Standard eviation	11.88%	=B34*$B$3+(1-B34)*$C$3			
38						
39						
40						
41		**Risk**	**Return**	**Tangent Portfolio**		
42	-30.00%	16.87%	2.63%		=A42*$B$3+(1-A42)*$C$3	
43	-20.00%	15.97%	2.61%			
44	-10.00%	15.13%	2.59%	=SQRT(A42^2*$B$4+(1-A42)^2*$C$4+2*A42*(1-A42)*$B$7*$B$5*$C$5)		
45	0.00%	14.36%	2.57%			
46	10.00%	13.67%	2.55%			
47	20.00%	13.07%	2.53%			
48	30.00%	12.57%	2.51%			
49	40.00%	12.20%	2.49%			
50	50.00%	11.95%	2.48%			
51	60.00%	11.84%	2.46%			
52	70.00%	11.87%	2.44%			
53	80.00%	12.04%	2.42%	=(1-A59)*$D$3+A59*$B$36		
54	90.00%	12.35%	2.40%			
55	100.00%	12.78%	2.38%			
56	110.00%	13.32%	2.36%			
57	120.00%	13.96%	2.34%			
58	130.00%	14.69%	2.32%			
59	0.00%	0.00%		0.40%		
60	10.00%	1.19%		0.61%		
61	20.00%	2.38%		0.81%		
62	30.00%	3.56%		1.02%		
63	40.00%	4.75%		1.23%		

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Mathematical Appendix

Covariance, Variance, Standard Deviation and the Correlation Coefficient

Many of the asset pricing relationships involve the use of variance, standard deviation, covariance and correlation coefficient. We will look here at different ways of expressing the measures of variance and covariance, correlation coefficient. We will also derive a linearity rule of manipulating covariances and consider some other convenient properties of covariances.

Covariance and Related Definitions

Consider two random variables, *X* and *Y*, with means or expected values of μ*X* and μ*Y*. Then the *Covariance* between *X* and *Y* is given by:

(A1) $$Cov(X,Y) \equiv s_{XY} \equiv E[(X - m_X)(Y - m_Y)]$$

The ***Variance*** of a random variable *X*, Var (*X*), is a special case of the covariance, where the random variables, here *X* and *Y*, are identical. The ***Standard Deviation*** is simply the square root of the variance.

(A2) $$s_X \equiv [Var(X)]^{1/2} \equiv [E(X - m_X)^2]^{1/2}$$

To normalize the covariance such that its value must lie between -1 and +1, we define the ***Correlation Coefficient*** between *X* and *Y* as:

(A3) $$r_{XY} = \frac{s_{XY}}{s_X s_Y}$$

Some important properties of Covariances

Define $X = aX_1 + bX_2$. The linear property of Covariance ensures that

(A4) $$Cov(aX_1 + bX_2, Y) = a\,Cov(X_1, Y) + b\,Cov(X_2, Y)$$

As an application of (A4) we can also write

(A5) $$Var(aX) = Cov(aX, aX) = a^2 Var\ X$$

or $$Var(X + Y) = Cov(X + Y, X + Y) = Cov(X, X + Y) + Cov(Y, X + Y).$$ Thus

(A6) $Var(X+Y) \equiv Var(X) + 2Cov(X,Y) + Var(Y)$

Covariance and Matrix Notation

If we assume n equally likely possible outcomes then (A1) becomes

(A7) $$Cov(X,Y) = \frac{1}{n}\{[\sum_{i=1}^{n} x_i - (\sum_{i=1}^{n} x_i / n)][y_i - (\sum_{i=1}^{n} y_i / n)]\}$$

Consider n random variables X_i . Matrix $\sum$ is defined as the variance-covariance matrix of the X_i , having as its (i, j) element $Cov(X_i, X_j)$ and accordingly as its (i, i) element $Var(X_i)$. If we now define the vector of random variables X_i as **x** and define **X** as the weighted sum of the random variables X_i, with weights s_i in column vector notation **s,** then

$X = \mathbf{s}^T \mathbf{x}$ and

(A8) $$[Cov(X,X_1), Cov(X,X_2), \ldots Cov(X,X_n)] = s^T \sum$$

the Variance of X is given by

(A9) $$Var(X) = s^T \sum s$$

The above equation is the objective function for most of our optimization problems.

MATRIX ALGEBRA

A matrix is an ordered set of numbers listed in rectangular form.
Following are the couple of examples:

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 5 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 5 & 7 \\ -2 & 6 & 5 \end{bmatrix}$$

The size of a matrix is specified by rows and columns. Here matrix A has two rows and three columns. We say it is 2 x 3 matrix. Similarly, B is a 3x3 matrix with three rows and three columns.

DEFINITION:

A ***matrix*** is simply a rectangular array of numbers. For example,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

An m x n matrix is a rectangular array of real numbers with m rows and n columns. The dimensions of a matrix is denoted by m x n and the numbers that appear in the matrix is called its entries. Now, there is a systematic way of referring to particular entries in a matrix. If there are two numbers i and j, then the entry in the i^{th} row and j^{th} column of matrix A is written as a_{ij}.

In the above example, a_{11} denotes first row, first column; a_{12} denotes first row, second column and a_{1n} denotes first row, n^{th} column. It is to be noted that in a matrix, row number is specified first and the column number second.

There are different types of matrices such as: ***Row Matrix***, ***Column Matrix, Square Matrix, Diagonal Matrix and so on.***

A matrix with one row is called a row matrix i.e., $\begin{bmatrix} 0 & -2 & 3 \end{bmatrix}$

A matrix with one column is called a column matrix, i.e., $\begin{bmatrix} 3 \\ 6 \\ -1 \end{bmatrix}$

A square matrix is a matrix which has the same number of rows and columns. i.e.,

$$\begin{bmatrix} 5 & 0 & 1 \\ 0 & 3 & 2 \\ 6 & -2 & 7 \end{bmatrix}$$

A diagonal matrix is a square matrix which has zeros everywhere other than the main diagonal. Entries on the main diagonal may be any number including zero. For example,

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

<u>MATRIX ADDITION AND SUBTRACTION</u>

Two matrices can be added or subtracted if, and only if, they have the same dimensions. That is both the matrices should have same number of rows and columns. Matrices having different dimensions cannot be added or subtracted. A 3 x 3 matrix can be added to a 3 x 3 matrix, but it cannot be added to a 3 x 4 matrix.

In order to add or subtract two matrices, we simply add or subtract the corresponding entries. Let us take an example of 2 x 2 matrix.

$$\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1+1 & -2+2 \\ 0+3 & 3-1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1-1 & -2-2 \\ 0-3 & 3-(-1) \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ -3 & 4 \end{bmatrix}$$

TRANSPOSE OF A MATRIX

A transpose of a matrix is nothing but a matrix which is formed by turning all the rows of a given matrix into columns and vice versa. The transpose of matrix A is written as A^T. For example:

$$\text{If} \quad A = \begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix} \; then \quad A^T = \begin{bmatrix} 5 & 1 \\ 2 & 3 \end{bmatrix}$$

$$\text{If B} = \begin{bmatrix} 3 & 5 \\ 7 & 1 \\ 0 & 2 \end{bmatrix} \; then\ B^T = \begin{bmatrix} 3 & 7 & 0 \\ 5 & 1 & 2 \end{bmatrix}$$

TYPES OF MULTIPLACATION FOR MATRICES

There are two types of multiplication for matrices: ***scalar multiplication and Matrix multiplication.***

In Scalar multiplication, we take a number called a "scalar" and multiply it on every entry in the matrix. For example:

Let us take any number or scalar, say, 3 and let A = $\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ be a 2 x 2 matrix. Then,

$$3\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3\times 4 & 3\times 3 \\ 3\times 1 & 3\times 2 \end{bmatrix} = \begin{bmatrix} 12 & 9 \\ 3 & 6 \end{bmatrix}$$

The formula for scalar multiplication is written as kA = [ka_{ij}] where k is a scalar.

MATRIX MULTIPLICATION

In matrix multiplication, we can multiply two matrices if, and only if, the number of columns in the first matrix equals the number of rows in the second matrix. To illustrate this definition let us take two matrices A and B. Let

$$A = \begin{bmatrix} 4 & -1 & 2 \\ 3 & 5 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

.

Here matrix A has three columns and matrix B has three rows. So the product of A and B is defined. A matrix multiplication is done by multiplying corresponding entries together and then adding the results. In the above example, the product of matrix A and B is given by

$$3\begin{bmatrix} 4 \\ 3 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 5 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3\times 4 \\ 3\times 3 \end{bmatrix} + \begin{bmatrix} 1\times(-1) \\ 1\times 5 \end{bmatrix} + \begin{bmatrix} 2\times 2 \\ 2\times 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 16 \end{bmatrix}$$

IDENTITY MATRIX

An Identity matrix is a square matrix in which the quantities on the diagonal from top left to bottom right are all equal to 1 and all other entries are 0. For example let A is 3 x 3 Identity Matrix. Then A is written as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

INVERSE OF A MATRIX

Inverse of a matrix is possible only with square matrices. Non-Square matrices do not have inverse. For a square matrix A, the inverse is written as A^{-1}. When A is multiplied by A^{-1} the result is the identity matrix and is given by $AA^{-1} = A^{-1}A = I$. Example:

For matrix $A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$, its inverse is $A^{-1} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$ since

$$AA^{-1} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Every square matrix does not have inverse. A square matrix which has an inverse is called invertible or non-singular, while a square matrix without an inverse is called non-invertible or singular.