The Jacobian and Change of Variables

Text Reference: Section 3.3, p. 209

This set of exercises examines how a particular determinant called the Jacobian may be used to allow us to change variables in double and triple integrals.

First, recall how a change of variables is done in a single integral: this change is usually called a substitution. To integrate $f(x)$ over the interval $a \leq x \leq b$, let $x = g(u)$ and substitute u for x, giving

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))g'(u)\,du$$

where $a = g(c)$ and $b = g(d)$. Notice the presence of the factor $g'(u) = \frac{dx}{du}$. A similar factor will be introduced when substitutions in double and triple integrals are studied, and this factor will involve the Jacobian.

Now consider changing variables in double and triple integrals. These intergrals are typically studied in third or fourth semester calculus classes. For more information on these integrals, consult your calculus text; e.g., Reference 2, Chapter 12, "Multiple Integrals." It is assumed that the reader has a passing familiarity with these concepts.

To change variables in double integrals, points with coordinates (u, v) must be converted to points with coordinates (x, y). That is, there will be a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(u, v) = (x, y)$. Notice that x and y are functions of u and v; that is, $x = x(u, v)$ and $y = y(u, v)$. This transformation T may or may not be linear.

Example 1: A well-known change of variables is the change from rectangular to polar coordinates, which is accomplished by $x = r\cos\theta$, $y = r\sin\theta$. Here u may be identified with r and v with θ, and the transformation may be written as $T(r, \theta) = (r\cos\theta, r\sin\theta)$.

Example 2: The transformation $T(u, v) = (3u - 2v, u + v)$ is a linear transformation, and describes the change of variables $x = 3u - 2v$, $y = u + v$.

Example 3: The transformation $T(u, v) = (u^2 - v^2, 2uv)$ is accomplished with the changes $x = u^2 - v^2$, $y = 2uv$. It is not a linear transformation.

It is helpful to consider how transformations change regions in the uv plane to regions in the xy plane.

Example 4: Let T be the transformation in Example 3, and consider the region S in the $r\theta$ plane given by $S = \{(u, v)|0 \leq u \leq 1, 0 \leq v \leq 1\}$. See Figure 1. The goal is to find its image $R = T(S)$ in the xy plane. Begin by finding the images of the sides of S. The side S_1 is given by $v = 0$ and $0 \leq u \leq 1$. When $v = 0$, the equations $x = u^2 - v^2$ and $y = 2uv$ become

$$x = u^2, y = 0, \quad \text{with} \quad 0 \leq x \leq 1.$$

Thus the image of S_1 under T is the line segment from $(0,0)$ to $(1,0)$ in the xy plane. The side S_2 is given by $u = 1$ and $0 \leq v \leq 1$. When $u = 1$, the equations $x = u^2 - v^2$ and $y = 2uv$ become $x = 1 - v^2$ and $y = 2v$, with $0 \leq v \leq 1$. After eliminating v,

$$x = 1 - \frac{y^2}{4}, \quad \text{with} \quad 0 \leq x \leq 1.$$

Thus the image of S_2 a portion of a parabola. In like manner, the image of S_3 is

$$x = \frac{y^2}{4} - 1, \quad \text{and} \quad -1 \leq x \leq 0,$$

and the image of S_4 is the line segment from $(-1,0)$ to $(0,0)$ in the xy plane. The complete image of S (which is called $T(S)$) is also shown in Figure 1.

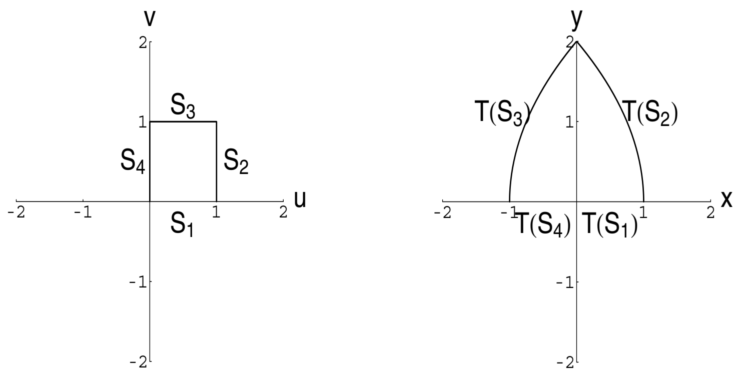

Figure 1: The region S and its image $T(S)$

Return now to Example 2 above and think about changing the variables of integration from x and y to u and v, respectively. To begin, consider the integral

$$\iint_R 1\,dA,$$

where R is a region in the xy plane and $dA = dx\,dy$ is the differential of area in the xy plane. Suppose that R is the image of the region S in the uv plane. Since the integral is just the area of the region R, Theorem 10 on p. 207 of *Linear Algebra and its Applications* and the observation on p. 209 lead to

$$\begin{aligned}
\iint_R 1\,dx\,dy &= \iint_R 1\,dA \\
&= \{\text{area of } T(S)\} \\
&= |\det B| \cdot \{\text{area of } S\} \\
&= |\det B| \iint_S 1\,du\,dv \\
&= \iint_S |\det B|\,du\,dv
\end{aligned}$$

In this case, then, $\iint_R 1\,dx\,dy = \iint_S 5\,du\,dv$. To integrate a function $f(x,y)$ over R, it now seems plausible to change variables to find

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v),y(u,v))|\det B|\,du\,dv$$

This is a ‘Change of Variables’ theorem when the change of variables is a linear transformation.

But what if change of variables transformation is not linear? Note that for the general linear transformation $x = au + bv$, $y = cu + dv$, the determinant of the standard matrix is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

It may be shown (see Reference 1, Sections 6.2 and 6.3) that the absolute value of this determinant is the factor to use in changing variables. This determinant is given a special name.

Definition: The **Jacobian** of the transformation T given by $x = x(u,v)$, $y = y(u,v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

This definition leads directly to the following theorem.

Change of Variables Theorem for Double Integrals: Suppose that T is a one-to-one transformation given by $x = x(u,v)$ and $y = y(u,v)$, and that the first partial derivatives of x and y with respect to u and v are continuous functions. If the Jacobian of T is nonzero and if T maps the region S in the uv plane onto the region $R = T(S)$ in the xy plane, if the function f is continuous on R and if the boundaries of R and S are sufficiently regular, then

$$\iint_R f(x,y)\,dx\,dy = \int_S\int f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv$$

Example 5: Integrate the function $x+y$ over the region R in the first quadrant of xy plane bounded by the x-axis, the y-axis, and the arc $x^2 + y^2 = 4$ (See Figure 2).

Solution: To integrate, change to polar coordinates: the region S so that $R = T(S)$ is seen to be $S = \{(r,\theta)|0 \le r \le 2, 0 \le \theta \le \frac{\pi}{2}\}$. Thus the desired integral is

$$\begin{aligned}\iint_R x + y\,dx\,dy &= \iint_S (r\cos\theta + r\sin\theta)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|\,dr\,d\theta \\ &= \int_0^{\frac{\pi}{2}}\int_0^2 (r\cos\theta + r\sin\theta)\ r\,dr\,d\theta\end{aligned}$$

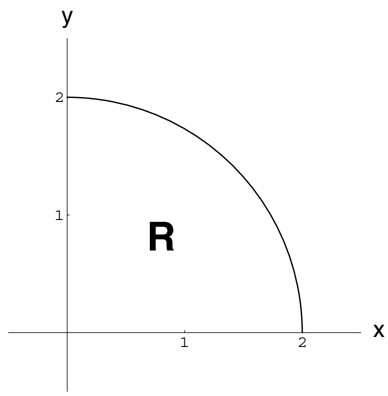

Figure 2: The region R

$$
\begin{aligned}
&= \int_0^{\frac{\pi}{2}} 2\,(\cos\theta + \sin\theta) \int_0^2 r^2\, dr\, d\theta \\
&= \frac{8}{3}\int_0^{\frac{\pi}{2}} 2\,(\cos\theta + \sin\theta)\; d\theta \\
&= \frac{16}{3}
\end{aligned}
$$

There is a similar change of variables theorem for triple integrals. Let T be a transformation that maps a region S in uvw space to a region $R = T(S)$ in xyz space, by the equations $x = x(u, v, w)$, $y = y(u, v, w)$, and $z = z(u, v, w)$. Then the Jacobian is defined as follows:

Definition: The **Jacobian** of the transformation T given by $x = x(u, v, w)$, $y = y(u, v, w)$, and $z = z(u, v, w)$ is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Then, under conditions similar to those the Change of Variables Theorem for Double Integrals, there is the following formula:

$$\iiint_R f(x, y, z)\, dx\, dy\, dz = \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| du\, dv\, dw$$

Questions:

1. Show that the transformations in Examples 1 and 3 are not linear by considering whether $T(u_1 + u_2, v_1 + v_2) = T(u_1, v_1) + T(u_2, v_2)$ for all choices of u_1, u_2, v_1, and v_2.

2. Show that the transformation in Example 2 is linear by the method suggested for Exercises 17-20 in Section 1.9 of the text.

3. In each part, draw a picture in $\mathbb{R}^2$ of the image set $T(S)$.

   a) $S = \{(r,\theta)|0 \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2}\}$, $T(r,\theta) = (r\cos\theta, r\sin\theta)$.
   b) $S = \{(r,\theta)|1 \leq r \leq 2, -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}\}$, $T(r,\theta) = (r\cos\theta, r\sin\theta)$.
   c) $S = \{(u,v)|0 \leq u \leq 1, 0 \leq v \leq 2\}$, $T(u,v) = (3u - 2v, u + v)$.
   d) $S = \{(u,v)| -1 \leq u \leq 1, 1 \leq v \leq 2\}$, $T(u,v) = (3u - 2v, u + v)$.

4. Use the transformation in Example 2 to evaluate the integral

$$\iint_R 2x - y\, dx\, dy$$

   where R is the region bounded by the lines $x + 2y = 0$, $x + 2y = 10$, $-x + 3y = 0$, and $-x + 3y = 5$.

5. Use polar coordinates to evaluate the integral

$$\iint_R e^{x^2+y^2}\, dx\, dy$$

   where R is the region enclosed by the unit circle $x^2 + y^2 = 1$.

6. Use the transformation in Example 3 to evaluate the integral

$$\iint_R y\, dx\, dy$$

   where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$.

7. Use the transformation $x = au$, $y = bv$, $z = cw$ to evaluate the integral

$$\iiint_R 1 dx\, dy\, dz$$

   where R is the region enclosed by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

   Thus the integral gives the volume of the ellipsoid.

8. Cylindrical coordinates are coordinates in $\mathbb{R}^3$ given by

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

   Use the change of variables formula for triple integrals and make appropriate calculations to show that

$$\iiint_R f(x,y,z)\, dx\, dy\, dz = \iiint_S f(r\cos\theta, r\sin\theta, z) r\, dr\, d\theta\, dz$$

   where S is the image of R under the transformation.

9. Spherical coordinates are coordinates in $\mathbb{R}^3$ given by

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

Use the change of variables formula for triple integrals and make appropriate calculations to show that

$$\iiint_R f(x, y, z)\, dx\, dy\, dz = \iiint_S f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

where S is the image of R under the transformation.

References:

1. Marsden, Jerrold and Tromba, Anthony. *Vector Calculus*. Third Edition. New York: W.H. Freeman, 1988.

2. Finney, Ross L., Weir, Maurice D., and Giordano, Frank R. *Thomas' Calculus*. Tenth Edition. Boston: Addison-Wesley, 2001.