SOLUTIONS TO FIRST EXAM-A

1. $\begin{bmatrix} 1 & -3 & -9 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{bmatrix}$

$$\begin{array}{rcrcrcl} x_1 & & - & 3x_3 & + & 2x_4 & = 0 \\ & x_2 & + & 2x_3 & - & x_4 & = 0 \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_3 - 2x_4 \\ -2x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

An appropriate geometric picture for the solution set is a plane through the origin.

2. a. Linearly dependent because the vectors are multiples of each other.

b. Linearly dependent because the set contains the zero vector.

c. Study $A\mathbf{x} = \mathbf{0}$ to determine if the set is linearly independent.

$$\begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 1 & 1 & -1 & 9 & 0 \\ 0 & -3 & 2 & -4 & 0 \\ -1 & 5 & -3 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 3 & -2 & 9 & 0 \\ 0 & -3 & 2 & -4 & 0 \\ 0 & 3 & -2 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 3 & -2 & 9 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

At this point, the lack of a pivot position in column 3 shows that the equation $A\mathbf{x} = \mathbf{0}$ has a free variable and hence has a nontrivial solution. So the columns of A are linearly dependent.

3. a. Determine if the equation $A\mathbf{x} = \mathbf{y}$ has a solution.

$$\begin{bmatrix} 1 & 1 & 2 \\ -2 & -1 & -7 \\ -1 & -3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

The equation is consistent, so $\mathbf{y}$ is in the range of the linear transformation T.

b. The calculation in (a) shows that A does not have a pivot in each row. By a theorem (Theorem 4 in Section 1.4), the columns of A do not span $\mathbb{R}^3$. By another theorem (Theorem 12 in Section 1.8), T does **not** map $\mathbb{R}^2$ onto $\mathbb{R}^3$. (You do not have to mention the theorem numbers.)

4. a. ... a linear combination of the columns of A using the (corresponding) entries in $\mathbf{x}$ as weights.

b. $I - BAB^{-1} = C \Rightarrow I = C + BAB^{-1} \Rightarrow BAB^{-1} = I - C$. Left multiply by B^{-1} to obtain $\underbrace{B^{-1}B}_{I} AB^{-1} = B^{-1}(I-C) \Rightarrow$ $AB^{-1} = B^{-1}(I - C)$. Right-multiply by B: $A\underbrace{B^{-1}B}_{I} = B^{-1}(I - C)B \Rightarrow A = B^{-1}(I - C)B$. Other steps are possible.

An equivalent answer is $A = I - B^{-1}CB$.

c. $T(3\mathbf{u} + \mathbf{v}) = T(3\mathbf{u}) + T(\mathbf{v}) = 3T(\mathbf{u}) + T(\mathbf{v})$

$$= 3\begin{bmatrix} -2 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ -7 \end{bmatrix} = \begin{bmatrix} -6 \\ 15 \end{bmatrix} + \begin{bmatrix} 3 \\ -7 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \end{bmatrix}.$$

5. See the proof of Theorem 8 in Section 1.6.

6. $I_1 \begin{bmatrix} 7 \\ -3 \\ 0 \end{bmatrix} + I_2 \begin{bmatrix} -3 \\ 5 \\ -2 \end{bmatrix} + I_3 \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 10 \\ 15 \end{bmatrix}$. The second vector $\mathbf{r}_2$, for example, lists in its second entry the total resistance in loop 2; the other entries in $\mathbf{r}_2$ show the resistances that also belong to adjacent loops, with negative signs because the currents in those loops flow against the current in loop 2. The voltage in loop 1 is negative because the current flows from the negative (shorter) side of the battery to the positive side.

7. Migration matrix: $M = \begin{array}{cc} \text{City} & \text{Sub} \\ \left[\begin{array}{cc} .92 & .03 \\ .08 & .97 \end{array}\right] \end{array} \begin{array}{l} \text{To:} \end{array} \begin{array}{l} \text{City} \\ \text{Suburb} \end{array}.$

Initial vector: $\mathbf{x}_0 = \begin{bmatrix} 100{,}000 \\ 200{,}000 \end{bmatrix}$.

After one 10-year period, the population will be

$$\mathbf{x}_1 = M\mathbf{x}_0 = \begin{bmatrix} .92 & .03 \\ .08 & .97 \end{bmatrix} \begin{bmatrix} 100{,}000 \\ 200{,}000 \end{bmatrix} = \begin{bmatrix} 98{,}000 \\ 202{,}000 \end{bmatrix}$$

8. (i) the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution.
(ii) the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
or: the transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
(iii) the columns of A span $\mathbb{R}^n$.
(iv) A has n pivot positions.