MATLAB Project: Visualizing Linear Transformations of the Plane   Name_____

Purpose: To understand the standard matrix of a linear transformation. In particular, to see the geometric effect of how a 2x2 matrix transforms R^2; and conversely, to learn how to write 2x2 matrices that will transform R^2 in specific ways.

Prerequisites: Section 1.9. A theorem from Section 1.4 is needed for the extra credit problem.

MATLAB functions used: **visdat** and **drawpoly** from Laydata Toolbox

Background.   As shown in Theorem 10 in Section 1.9, when a linear transformation $T: R^n \to R^n$ is given, it can be identified with a matrix, and this is an easy way to get a formula for the function. Let $T: R^n \to R^n$ be a linear transformation and let $e_1, e_2, \ldots, e_n$ denote the columns of the nxn identity matrix. Figure out what each $T(e_i)$ should be and write each $T(e_i)$ as a column vector. If you then define the matrix A = [$T(e_1)$ $T(e_2)$ .. $T(e_n)$], then it will be true that $T(x) = Ax$ for all **x**, and **Ax** gives is a formula for the function.   In other words, given a linear transformation $T: R^n \to R^n$, if you know its values at just the n independent vectors $e_1 , e_2 , \ldots , e_n$, then its value at every point **x** is determined!

1. Example.   The 2x2 linear transformation that maps e_1 to $e_1 + e_2$ and e_2 to e_1 - e_2 is $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

The 3x3 matrix transformation that maps e_1 to $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$, e_2 to $\begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix}$ and e_3 to $\begin{bmatrix} 5 \\ 4 \\ -1 \end{bmatrix}$ is $\begin{bmatrix} 3 & 6 & 5 \\ -2 & 0 & 4 \\ 1 & 7 & -1 \end{bmatrix}$.

2. Example. The function that reflects R^2 across the line y = -x is a linear transformation.   Notice it must map e_1

to -e_2 and e_2 to -e_1, so its matrix is $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.   See the sketch in Table 1 of Section 1.9.

3. Exercise. (hand) (a) Write a 2x2 matrix that maps e_1 to $4e_2$ and e_2 to -e_1 :

   (b) Write a 2x2 matrix that reflects R^2 across the line y = x :

More background. A matrix transformation always maps a line onto a line or a point, and maps parallel lines onto parallel lines or onto points. (See exercises 25-28 in Section 1.8.) In the following question, you will verify these things for a particular matrix.

4. Exercise. (hand) Let $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

(a) Explain why the function $T(x) = Mx$ maps the x-axis onto the line y = x, and why it maps the line y = 2 onto the line y = x+2. (Hints: A general point on the x-axis is of the form $\begin{bmatrix} t \\ 0 \end{bmatrix}$; calculate $M\begin{bmatrix} t \\ 0 \end{bmatrix}$ and interpret where those image points lie. Similarly, calculate $M\begin{bmatrix} t \\ 2 \end{bmatrix}$ and interpret.)

(b) Sketch here what you showed algebraically in 4(a). That is, sketch the x-axis and the line $y = 2$ on the left axes below, and sketch and label their images on the right:

T
----->
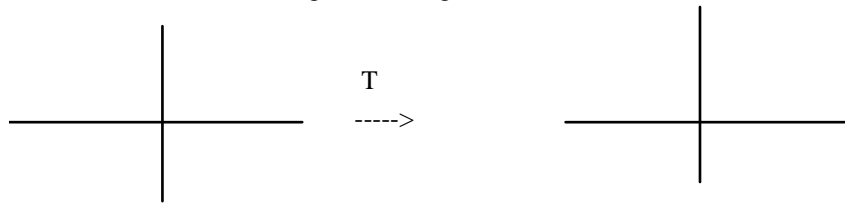
<u>Still more background</u>. Because a matrix transformation maps parallel lines to parallel lines, it will map any parallelogram to another parallelogram (which could be degenerate – one line segment or a single point). So when a linear transformation and parallelogram are given, the easy way to draw the image of the parallelogram is to plot the images of its four vertices and connect those points to make a parallelogram.

Define the *standard unit square* to be the square in R^2 whose vertices are (0,0), (1,0), (1,1) and (0,1). When you want to visualize what a 2x2 matrix transformation does geometrically, it is particularly useful to sketch the image of this standard square. Seeing how this square gets moved or distorted shows what the transformation does to the x-axis and y-axis and thus gives a good idea what the transformation does geometrically to the whole plane. Recall that any linear transformation maps the origin to itself (why?), so you only need to figure out where the transformation maps the other three vertices.

5. <u>Example</u>. Continue to use $T(\mathbf{x}) = M\mathbf{x}$ where $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Sketch the image of the standard unit square:

$\mathbf{e_2}$ (1,1)   T $T(\mathbf{e_2})=(0,1)$ $T(1,1)=(1,2)$
        $\mathbf{e_1}$ ----> $T(\mathbf{e_1})=(1,1)$
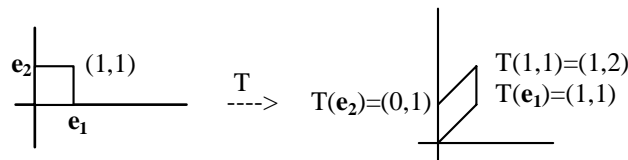
So you can see that the y-axis stays fixed and the x-axis is mapped onto the line $y=x$, causing a vertical shear of the plane. See more examples like this in Table 3 in Section 1.9.

6. <u>Example.</u> Find a matrix M which maps the standard unit square to the parallelogram with vertices (0,0), (3,1), (2,2), (-1,1). To do this, sketch the parallelogram and recognize that $M\mathbf{e_1}$ and $M\mathbf{e_2}$ must be (3,1) and (-1,1), or vice versa. (Why?) So either of the matrices $\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ or $\begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$ will work. <u>Be sure you understand why these are the only two matrices that will work here.</u> Calculate the image of (1,1) under each of these matrices, to verify that it is (2,2).

$\mathbf{e_2}$ (1,1)   M   (-1,1)   (2,2)
        $\mathbf{e_1}$ ---->   (3,1)
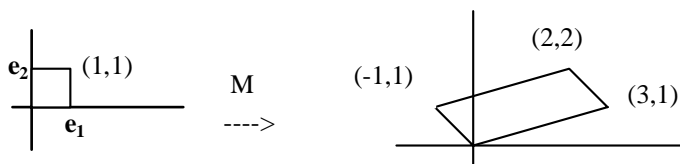
7. <u>Exercise</u>. (hand) Each of the seven matrices below is one of the special, simple types described in Sections 1.8 and 1.9. Each determines a linear transformation of R^2. For each, <u>sketch the image of the standard unit square</u>, <u>label the vertices of the image, and describe how the matrix is transforming the plane</u>. To get you started, answers are given for the first matrix.

$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ Description: a vertical shear. It leaves the y-axis fixed and increases the slope of all other lines through origin.
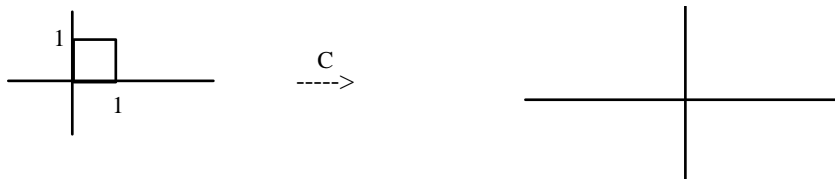
1   A   (1,3)
        1   ---->   (1,2)
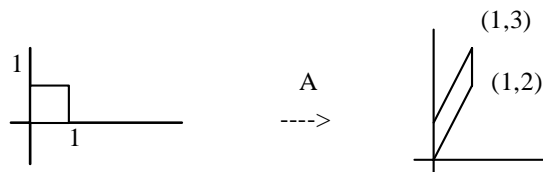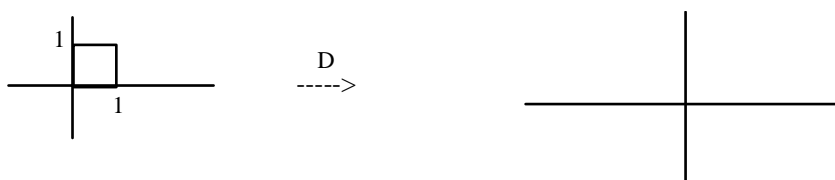
$B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ Description: _____

$$\underset{\text{----->}}{B}$$

$C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ Description: _____

$$\underset{\text{----->}}{C}$$

$D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ Description: _____

$$\underset{\text{----->}}{D}$$

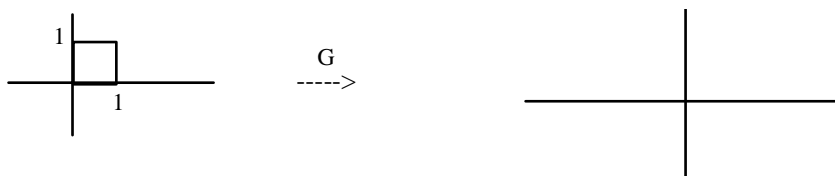$E = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ Description: _____

$$\underset{\text{----->}}{E}$$
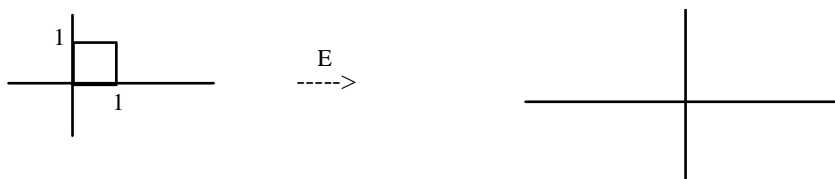
$F = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix}$ Description: _____

$$\underset{\text{----->}}{F}$$
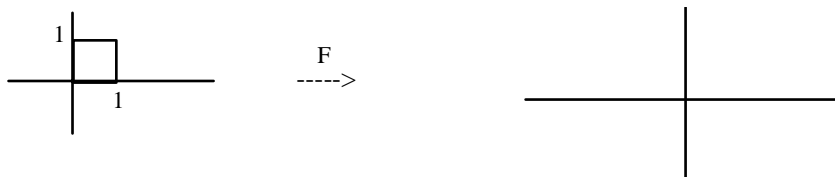
$G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ Description: _____

$$\underset{\text{----->}}{G}$$

8. Exercise. (MATLAB) In computer graphics, transformations are usually accomplished by applying a succession of simple matrix transformations – dilations, shears, reflections, rotations and projections. Here you will do a variety of such.

       To begin, type **visdat** to get the matrices A, B, ..., G used above. You will also get a matrix called **box** whose columns contain the coordinates of the vertices of the standard unit square.

       Type **A, box** and **A*box** and record these below. Notice the command **A*box** causes MATLAB to multiply A times each column of box – i.e., the columns of **A*box** are the images under A of the vertices of the standard unit square.

       A =   box =   A*box =


       When X is a matrix with two rows, each column represents a point in R^2, and the command **drawpoly(X)** will plot those points and draw a line segment from each to the next one. Try this: type **drawpoly(box)** to see the standard unit square. Then type **drawpoly(A*box)** to see the image of that square under A. To see both figures on the same axes, type **drawpoly(box, A*box)** -- first you will see the square, then press [Enter] and you will also see its image under A.

       (a) The commands below perform several successive transformations of the standard unit square. The first one does a shear using A; the second does the shear followed by rotation of the plane through π/4 using F; the third does the shear followed by the rotation and then reflects the plane across the line y = -x using E. Type these lines, and watch carefully to see the result of each successive transformation. Sketch the new figure obtained after each step:

       drawpoly(A*box)
       drawpoly(F*(A*box))
       drawpoly(E*(F*(A*box)))
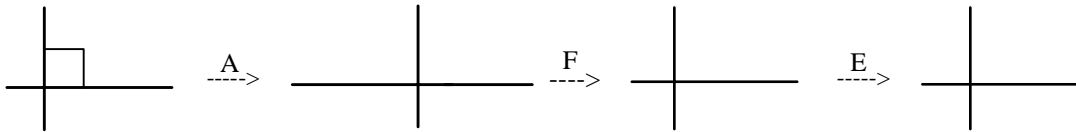

       (b) First shear using A, then reflect across the x-axis using C, then rotate through π/4 using F. Use **drawpoly** to sketch the result of each successive transformation, and sketch:
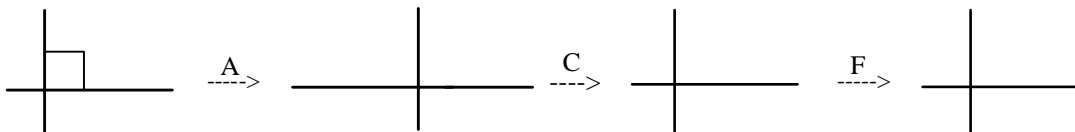

9. Exercise. (hand) Sketch the parallelogram with vertices (0,0), (4,2), (0,-4), (4,-2) and write two different 2x2 matrices X and Y which would transform the standard unit square into this parallelogram. Use **drawpoly** to verify that your matrices work:


       X =   Y =
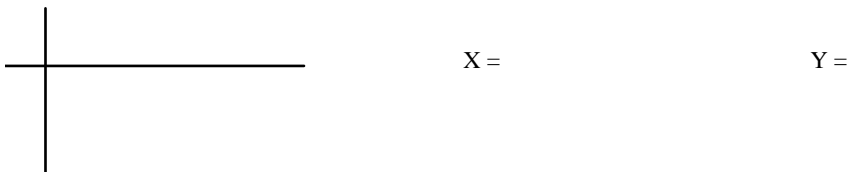
10. <u>Exercise</u>. (hand) Sketch the parallelogram with vertices (1,1), (1,2), (3,1), (3,2). Explain why no 2x2 matrix transformation could map the standard unit square onto this figure.
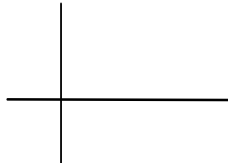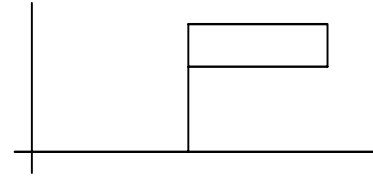
11. <u>Example</u>. In the figure below called flag1, the flagpole is from (1,0) to (1,3), and the vertices of the flag itself are (1,2), (1,3), (2,3), and (2,2). <u>Label these points</u>. We will find matrices which transform flag1 into the other figures, flagA and flagB.
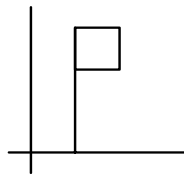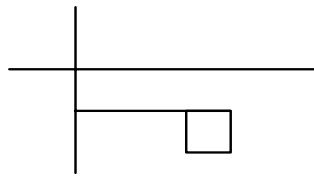
   flag1   flagA   flagB

        First consider flagA. One way to figure out the matrix would be to simply "see" that rotating the plane through -π/2 clearly takes flag1 into flagA, so the matrix must be $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Alternatively, inspect flag1 and flagA to see where $e_1 = (1,0)$ and $e_2 = (0,1)$ must be mapped, so as to find the columns of the desired matrix. Clearly (1,0) maps to (0,-1) – so the matrix we seek must look like $\begin{bmatrix} 0 & a \\ -1 & b \end{bmatrix}$. It is also easy to see by inspection that (1,1) must map to (1,-1), so solve the equation $\begin{bmatrix} 0 & a \\ -1 & b \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for a and b, and you will find this yields the same matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

      Now consider flagB. Clearly (1,0) must be mapped to (3,0). Since there appears to be no skewing or dilation in the vertical direction, you could guess that (0,1) maps to itself, hence the matrix is $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$. Or you could proceed as above, noticing that (1,1) must map to (3,1) and using that to solve for the entries of the second column of the matrix. This would also yield $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

        Whatever method you use to calculate the matrix, you should check that your matrix really does what you want. An easy way would be to use **drawpoly** to sketch flag1 and its image under A. The vertices of flag1 are already stored in the matrix **flag1**, so store your matrices and then use **drawpoly**. Try these commands to do that:

        MA = [0 1; -1 0], drawpoly(flag1, MA*flag1)
        MB = [3 0; 0 1], drawpoly(flag1, MB*flag1)

12. <u>Exercise</u>. The figure flag1 is shown again below. One at a time, consider each of the other three flags sketched below and find a 2x2 matrix which maps **flag1** onto it. Use **drawpoly** to verify that each of your matrices does what you want, and <u>record each matrix below the appropriate figure</u>.

flag1   flag2   flag3   flag4
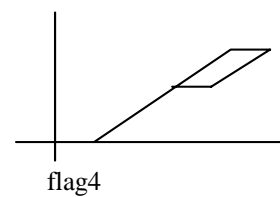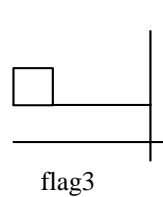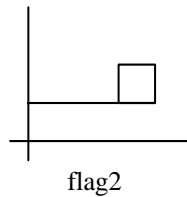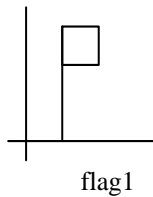
Matrices: _____   _____   _____

13. <u>Exercise</u>. (Extra credit, use your paper and attach.) Let $T: R^n \rightarrow R^n$ be a linear transformation. As pointed out in "Background" on page 1, T is completely determined by its values on the special vectors e_1 , e_2 , .. , e_n .

<u>Prove that T is completely determined by its values on any n independent vectors.</u>

Here is an outline. Assume $T: R^n \rightarrow R^n$ is a linear transformation, let v_1 , v_2 , .. , v_n be independent vectors in R^n and suppose $T(v_1)$, $T(v_2)$, .. , $T(v_n)$ are known. Let **x** be any element of R^n. Explain why span{ v_1 , v_2 , .. , v_n } = R^n, so **x** equals some linear combination of v_1 , v_2 , .. , v_n, then show $T(x)$ can be calculated using $T(v_1)$, $T(v_2)$, .. , $T(v_n)$. (For the spanning fact, let A = [v_1 v_2 .. v_n], explain why A has a pivot in each column, then why a pivot in each row, and apply Theorem 4 in Section 1.4.)