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Integration by Parts

Text Reference: Section 5.4, p. 330

The purpose of this set of exercises is to show how the matrix of a linear transformation relative to a basis $\mathcal{B}$ may be used to find antiderivatives usually found using integration by parts.

To find $\int t^2e^t\,dt$, the normal approach would be to integrate by parts twice, and find that

$$\int t^2e^t\,dt = t^2e^t - 2te^t + 2e^t + C$$

However, linear algebra can be used to solve this problem. Look at the set $\mathcal{B} = \{t^2e^t, te^t, e^t\}$. This set may be shown to be linearly independent by the method used in Exercise 37 in Section 4.3: start by assuming that

$$c_1t^2e^t + c_2te^t + c_3e^t = 0$$

This equation must hold true for all real t. Choose three specific values for t: 0,1, and 2. This generates the following system of equations:

$$\begin{array}{rcrcrcl} & & & & c_3 & = & 0 \\ ec_1 & + & ec_2 & + & ec_3 & = & 0 \\ 4e^2c_1 & + & 2e^2c_2 & + & e^2c_3 & = & 0 \end{array}$$

Question:

1. Show that this system has only the trivial solution, and thus that the set $\mathcal{B} = \{t^2e^t, te^t, e^t\}$ is linearly independent.

Since $\mathcal{B} = \{t^2e^t, te^t, e^t\}$ is linearly independent, it is a basis for $V = \mathrm{Span}\{t^2e^t, te^t, e^t\}$. Now let D be the differentiation operator; that is $D(\mathbf{f}) = \mathbf{f}'$ for all functions $\mathbf{f}$ in V. Recall from Section 4.2 that D is a linear transformation, and notice that D maps V into V; that is, $D(\mathbf{f})$ is a member of V for all functions $\mathbf{f}$ in V. Thus the matrix for D relative to $\mathcal{B}$, which is denoted $[D]_{\mathcal{B}}$, exists. By Equation 4 on page 328, this matrix may be calculated by computing

$$[D]_{\mathcal{B}} = \left[\begin{array}{ccc} [D(t^2e^t)]_{\mathcal{B}} & [D(te^t)]_{\mathcal{B}} & [D(e^t)]_{\mathcal{B}} \end{array}\right]$$

Since $D(t^2e^t) = t^2e^t + 2te^t$, $D(te^t) = te^t + e^t$, and $D(e^t) = e^t$, it follows that

$$[D]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

The matrix $[D]_{\mathcal{B}}$ may be used to differentiate any member of V:

Example: Find the derivative of $f(x) = 5t^2e^t - 3te^t + 2e^t$.

Solution: Since $[\mathbf{f}]_{\mathcal{B}} = (5, -3, 2)$,

$$[D\mathbf{f}]_{\mathcal{B}} = [D]_{\mathcal{B}}[\mathbf{f}]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ -1 \end{bmatrix}$$

and $f'(x) = 5t^2e^t + 7te^t - e^t$.

Question:

2. Use this method to find the derivative of $f(x) = 5e^t + te^t - 2t^2e^t$.

Of course, antidifferentiation rather than differentiation is the issue of this problem set. To work on that problem, notice that $[D]_{\mathcal{B}}$ is invertible because its determinant (which in this case is just the product of its diagonal entries) is nonzero. It can be shown that in this case the operator D is an invertible linear transformation on $V = \text{Span}\{t^2e^t, te^t, e^t\}$ and the inverse of $[D]_{\mathcal{B}}$ is the $\mathcal{B}$-matrix of D^{-1}. (See the Theoretical Exercises at the end of this set.) Thus the inverse of $[D]_{\mathcal{B}}$ should be the $\mathcal{B}$-matrix for the **antidifferentiation** operator on V. That is, if a function $\mathbf{f}$ is in V, then $[D]_{\mathcal{B}}^{-1}[\mathbf{f}]_{\mathcal{B}}$ should be the $\mathcal{B}$-coordinate vector for a function in V whose derivative is $\mathbf{f}$. In the example, technology or the algorithm in Section 2.2 of the text may be used to show that

$$[D]_{\mathcal{B}}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

so this matrix should be the matrix for the antidifferentiation operator relative to the basis $\mathcal{B}$. Thus to find $\int t^2e^t\,dt$, first find the coordinate vector of t^2e^t relative to $\mathcal{B}$: $[t^2e^t]_{\mathcal{B}} = (1, 0, 0)$. Then multiply to find

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix},$$

so the antiderivative of t^2e^t in the vector space V is $t^2e^t - 2te^t + 2e^t$. From calculus, it is known that **all** antiderivatives of t^2e^t have the form $t^2e^t - 2te^t + 2e^t + C$ for some constant C.

Question:

3. Use this method to find the following antiderivatives:

   a) $\int te^t\,dt$
   b) $\int 5t^2e^t - 3e^t\,dt$
   c) $\int 5e^t + te^t - 2t^2e^t\,dt$

Before other examples of this method are studied, note that the previous example was fortunate because it turned out that the matrix D was invertible. This is not always the case.

Example: Consider the space $V = \text{Span}\{1, t, t^2\}$. This space has the basis $\mathcal{B} = \{1, t, t^2\}$. Computing the $\mathcal{B}$-matrix for the differentiation operator D on this space, it is found that

$$[D]_{\mathcal{B}} = \left[\; [D(1)]_{\mathcal{B}} \quad [D(t)]_{\mathcal{B}} \quad [D(t^2)]_{\mathcal{B}} \; \right] = \left[\; [0]_{\mathcal{B}} \quad [1]_{\mathcal{B}} \quad [2t]_{\mathcal{B}} \; \right] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Since $[D]_{\mathcal{B}}$ is not an invertible matrix, the differentiation operator D is not invertible on $V = \text{Span}\{1, t, t^2\}$.

The same process may be used in other cases where integration by parts was used to find antiderivatives, if there is a basis with respect to which the differentiation operator is invertible.

Questions:

4. Consider finding the antiderivative $\int t^3 e^t \, dt$. Let $\mathcal{B} = \{t^3 e^t, t^2 e^t, t e^t, e^t\}$ and let V be the vector space of functions spanned by the functions in $\mathcal{B}$.

    a) Show that the set $\mathcal{B}$ is linearly independent.
    b) Show that the differentation operator D maps V into V.
    c) Find the matrix $[D]_{\mathcal{B}}$ for the differentiation operator D.
    d) Use your technology to find $[D]_{\mathcal{B}}^{-1}$.
    e) Use $[D]_{\mathcal{B}}^{-1}$ to find $\int t^3 e^t \, dt$.
    f) Find the antiderivative of $\int (t^3 - t^2 + t - 1) e^t \, dt$.

5. Find an appropriate basis for computing $\int t^2 e^{5t} \, dt$ and then find the antiderivative.

6. Consider the set $\mathcal{B} = \{t \sin t, t \cos t, \sin t, \cos t\}$. Let V be the vector space of functions spanned by the functions in $\mathcal{B}$.

    a) Show that the set $\mathcal{B}$ is linearly independent.
    b) Show that the differentation operator D maps V into V.
    c) Find the matrix $[D]_{\mathcal{B}}$ for the differentiation operator D.
    d) Use your technology to find $[D]_{\mathcal{B}}^{-1}$.
    e) Use $[D]_{\mathcal{B}}^{-1}$ to find $\int t \cos t \, dt$ and $\int t \sin t \, dt$

7. The antiderivatives $\int e^t \sin t \, dt$ and $\int e^t \cos t \, dt$ are rather tricky to compute using integration by parts. Using the set $\mathcal{B} = \{e^t \sin t, e^t \cos t\}$, find the matrix $[D]_{\mathcal{B}}$ for the differentiation operator D and use it to compute both antiderivatives.

Theoretical Exercises:

The goal of this set of exercises is to prove the following theorem, which is an analogue of Theorem 9 in Section 2.3 of the text.

Theorem: Let $T : V \longrightarrow V$ be a linear transformation, let $\mathcal{B}$ be a basis for V and let $[T]_{\mathcal{B}}$ be the $\mathcal{B}$-matrix for T. Then T is an invertible transformation if and only if $[T]_{\mathcal{B}}$ is an invertible matrix. In that case, the linear transformation S whose $\mathcal{B}$-matrix is $[T]_{\mathcal{B}}^{-1}$ is the unique function satisfying $T(S(\mathbf{v})) = S(T(\mathbf{v}))$ for all $\mathbf{v}$ in V.

This theorem will be proven by following these steps.

1. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for V and let $T : V \longrightarrow V$ and $S : V \longrightarrow V$ be linear transformations. It can be shown that the composite transformation TS is also linear.

   a) Write the $\mathcal{B}$-matrix for the transformation TS.
   b) Show that for each vector $\mathbf{b}_j$ in $\mathcal{B}$, the $\mathcal{B}$-coordinate vector of $(TS)(\mathbf{b}_j)$ is $[T]_{\mathcal{B}}[S(\mathbf{b}_j)]_{\mathcal{B}}$. Find an equation in Section 5.4 which justifies this.
   c) Show that $[TS]_{\mathcal{B}} = [T]_{\mathcal{B}}[S]_{\mathcal{B}}$. In words, the $\mathcal{B}$-matrix of the composite transformation TS is the product of the $\mathcal{B}$ matrices of T and S in the same order.

2. Suppose that T is an invertible linear transformation on V, in the sense that there is a linear transformation $S : V \longrightarrow V$ such that $T(S(\mathbf{v})) = S(T(\mathbf{v})) = \mathbf{v}$ for all $\mathbf{v}$ in V. (This generalizes the definition given in Section 2.3 for $V = \mathbb{R}^n$.) The transformation S is denoted by T^{-1}. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for V. Show that the $\mathcal{B}$-matrix for T is invertible and that the inverse of this matrix is $[S]_{\mathcal{B}} = [T^{-1}]_{\mathcal{B}}$.

This proves one implication in the theorem. To prove the other half, suppose that $T : V \longrightarrow V$ is a linear transformation with an invertible $\mathcal{B}$-matrix $[T]_{\mathcal{B}}$. Define $S : V \longrightarrow V$ in the following way. Let $\mathbf{v}$ be in V, and let $\mathbf{x} = [T]_{\mathcal{B}}^{-1}[\mathbf{v}]_{\mathcal{B}}$. If

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

define

$$S(\mathbf{v}) = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \ldots + x_n\mathbf{b}_n$$

3. Show that S is a linear transformation.

4. Show that $T(S(\mathbf{v})) = \mathbf{v}$ for all $\mathbf{v}$ in V by showing that $[T(S(\mathbf{v}))]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{B}}$.

5. Show that $S(T(\mathbf{v})) = \mathbf{v}$ for all $\mathbf{v}$ in V by showing that $[S(T(\mathbf{v}))]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{B}}$.

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