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Case Study: Determinants in Analytic Geometry

In this case study, it is shown how determinants may be used to answer certain geometrical questions and to find equations for geometrical objects. For this work one must consider determinants of matrices whose entries are variables or algebraic expressions. In this case the determinant will itself be an algebraic expression. For example, consider the equation

$$\begin{vmatrix} x & y & 1 \\ 1 & 2 & 1 \\ 3 & -5 & 1 \end{vmatrix} = 0.$$

Since x and y are variables, this equation may be true for some points (x, y) in the x-y plane, but untrue for other points. Notice that this equation is clearly true for the points $(1, 2)$ and $(3, -5)$, since plugging those points in produces a matrix with two identical rows, whose determinant must be zero (See Question 1 below). Expanding the determinant along the first row gives

$$\begin{vmatrix} 2 & 1 \\ -5 & 1 \end{vmatrix} x - \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} y + \begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix} 1 = 0,$$

or $7x + 2y - 11 = 0$. This is clearly the equation for a line, and since the points $(1, 2)$ and $(3, -5)$ satisfy this equation, it is an equation for the line passing through $(1, 2)$ and $(3, -5)$. Thus there is a way to use a determinant to express the equation of a line through two given points.

The question remains of how to discover this "determinantal form" of an algebraic equation. Consider the problem of finding the equation of a circle through three given points. The standard equation for a circle is $(x - h)^2 + (y - k)^2 = r^2$, which may be expanded and regrouped into the form

$$a(x^2 + y^2) + bx + cy + d = 0.$$

Let x and y be variables, but assume that the point (x, y) lies on the circle. Assume also that three fixed points on the circle are given: (x_1, y_1), (x_2, y_2), and (x_3, y_3). Since these points as well as (x, y) must satisfy the above equation, the following system of linear equations is generated. Remember that a, b, c, and d are the unknowns in this system, while x_1, x_2, x_3, y_1, y_2, and y_3 are known constants.

$$\begin{array}{ccccccccc} a(x^2+y^2) & + & bx & + & cy & + & d & = & 0 \\ a(x_1^2+y_1^2) & + & bx_1 & + & cy_1 & + & d & = & 0 \\ a(x_2^2+y_2^2) & + & bx_2 & + & cy_2 & + & d & = & 0 \\ a(x_3^2+y_3^2) & + & bx_3 & + & cy_3 & + & d & = & 0 \end{array}$$

This system must have a nontrivial solution, and so Theorem 4 in Section 3.2 combined with the Invertible Matrix Theorem in Section 2.3 shows that

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

This should be the determinantal form for the equation of the circle through the three given points.

Example: The circle through $(6,3)$, $(1,-2)$, and $(4,7)$ has determinantal form

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 45 & 6 & 3 & 1 \\ 5 & 1 & -2 & 1 \\ 65 & 4 & 7 & 1 \end{vmatrix} = 0$$

Performing a cofactor expansion along the first row, the equation becomes

$$\begin{vmatrix} 6 & 3 & 1 \\ 1 & -2 & 1 \\ 4 & 7 & 1 \end{vmatrix}(x^2+y^2) - \begin{vmatrix} 45 & 3 & 1 \\ 5 & -2 & 1 \\ 65 & 7 & 1 \end{vmatrix} x + \begin{vmatrix} 45 & 6 & 1 \\ 5 & 1 & 1 \\ 65 & 4 & 1 \end{vmatrix} y - \begin{vmatrix} 45 & 6 & 3 \\ 5 & 1 & -2 \\ 65 & 4 & 7 \end{vmatrix} 1 = 0$$

or $-30(x^2+y^2)+60x+180y+450=0$. In standard form, this equation is $(x-1)^2+(y-3)^2=25$, so this circle has center $(1,3)$ and radius 5. Notice it can be immediately seen whether the points $(6,3)$, $(1,-2)$, and $(4,7)$ satisfy the determinantal equation: when those points are plugged into the first row of the determinant, the first row now matches another row in the matrix, so its determinant must be zero. Thus the determinant equation above gives us the promised result.

This same technique may be used to discover determinantal forms for equations of geometrical objects in two and three dimensions. In the questions below, you will be asked to find such forms for equations of a conic section through five given points, of a plane through three given points, and of a sphere through four given points.

Questions:

1. Let C be an $n \times n$ matrix with two identical rows. Explain why $\det C = 0$.

2. Explain why the equation of the line through (x_1, y_1) and (x_2, y_2) may be written as

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

   That is, use a cofactor expansion to show that the equation has the desired general form, and then explain why each specified point satisfies the equation.

3. Explain how to use a determinant to test whether three points lie on a single line. Then read Exercise 30 in Section 3.3 and discuss the connection between this result and your determinant test.

4. Find a determinantal formula for the equation of a line with slope m that passes through a point (x_1, y_1). The matrix involved in the formula should be a 3×3 matrix with variables x and y in the top row, and three of the remaining entries should be x_1, y_1 and m.

5. Find the standard equation of circle passing through $(.5, -2)$, $(4.5, 4)$, and $(-4.5, 10)$.

6. Find the point (x, y) equidistant from the points $(1, 1)$, $(4, -3)$, and $(-2, -4)$.

7. Find the determinantal form of the equation of the conic section with standard equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$ passing through the five points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), and (x_5, y_5). Apply a cofactor expansion to your determinant to show that the equation which you derive has the desired general form $ax^2+bxy+cy^2+dx+ey+f = 0$. Explain why each of the five specified points satisfies the equation.

8. Find the standard equation of the conic section passing through the points $(3, 0)$, $(-3, 0)$, $(5, 4)$, $(-5, 4)$, and $(5, -4)$.

9. Find the determinantal form of the equation of the plane with standard equation $ax + by + cz + d = 0$ passing through the points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3). Apply a cofactor expansion to your determinant to show that the equation which you derive has the desired general form $ax + by + cz + d = 0$. Explain why each of the three specified points satisfies this equation.

10. Find the standard equation of the plane passing through the points $(1, 1, 0)$, $(0, -2, 1)$, and $(2, 0, 1)$.

11. Find the determinantal form of the equation of the sphere with standard equation $a(x^2+y^2+z^2)+bx+cy+dz+e = 0$ passing through the four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4). Apply a cofactor expansion to your determinant to show that the equation which you derive has the desired general form $a(x^2 + y^2 + z^2) + bx + cy + dz + e = 0$. Explain why each of the four specified points satisfies this equation.

12. Find the standard equation of the sphere passing through the points $(6, 10, 0)$, $(13, 3, 0)$, $(1, 3, -12)$, and $(-4, -2, 12)$.

13. How could you use determinants to find the equation of the quadric surface $ax^2+by^2+cz^2+dxy + exz + fyz + gx + hy + iz + j = 0$? How many points on the surface would you need to have?

References:

1. Muir, Thomas. *The Theory of Determinants in the Historical Order of Development*. New York: Dover, 1960.

2. Muir, Thomas and Metzler, William. *A Treatise on the Theory of Determinants*. New York: Longmans, Green, and Co., 1933.

The first book is the four volume work referred to in the introduction to Chapter 3, and the second book is a revised edition of the first. They are both of interest not only for their geometrical content but also for their amazing classification and exploration of determinants.