Other Matrix Products

Text Reference: Section 2.1, p. 114

The purpose of this set of exercises is to introduce two new operations on square matrices called the Jordan product and the commutator product, and to explore some of their properties. The operations are defined using the ordinary product AB of two $n \times n$ matrices.

The Jordan Product

Definition: The *Jordan product* of two $n \times n$ matrices A and B is

$$A \overset{J}{\times} B = \frac{1}{2}(AB + BA).$$

Questions:

1. Why must both A and B be square matrices?

2. Calculate the Jordan product of the following pairs of matrices.

   a) $A = \begin{bmatrix} 0 & 2 \\ 7 & -8 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -2 \\ 4 & 5 \end{bmatrix}$

   b) $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -8 \\ 2 & 3 \end{bmatrix}$

   c) $A = \begin{bmatrix} 1 & -3 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix}$

   d) $A = \begin{bmatrix} 7 & -5 & 0 \\ 0 & 3 & 5 \\ -6 & -4 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -4 & 4 \\ 6 & -6 & -8 \\ -6 & -6 & 9 \end{bmatrix}$

3. Either prove that the following properties are true, or give a counterexample to show that the property does not always hold. All matrices below are assumed to be $n \times n$, I is the $n \times n$ identity matrix, and r is any scalar.

   a) $A \overset{J}{\times} B = B \overset{J}{\times} A$

   b) $(A \overset{J}{\times} B) \overset{J}{\times} C = A \overset{J}{\times} (B \overset{J}{\times} C)$

   c) $(A + B) \overset{J}{\times} C = (A \overset{J}{\times} C) + (B \overset{J}{\times} C)$

   d) $A \overset{J}{\times} (B + C) = (A \overset{J}{\times} B) + (A \overset{J}{\times} C)$

   e) If $A \overset{J}{\times} B = A \overset{J}{\times} C$, then $B = C$.

   f) $r(A \overset{J}{\times} B) = (rA) \overset{J}{\times} B = A \overset{J}{\times} (rB)$

g) $A \overset{J}{\times} I = I \overset{J}{\times} A = A$ *1/2

h) $(A \overset{J}{\times} B)^T = A^T \overset{J}{\times} B^T$

The facts shown in items a) and b) above (that the Jordan product is commutative but not associative) are used to define algebraic structures called Jordan algebras. The peculiar properties of the Jordan product make for interesting results in this area of mathematics.

Commutator Product

Definition: The *commutator product*, or *matrix cross product* of two n by n matrices A and B is

$$A \times B = AB - BA.$$

Questions:

4. Why must both A and B be square matrices?

5. Calculate the cross product of the following pairs of matrices.

   a) $\begin{bmatrix} 0 & 2 \\ 7 & -8 \end{bmatrix}$ and $\begin{bmatrix} 5 & -2 \\ 4 & 5 \end{bmatrix}$

   b) $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ and $\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$

   c) $\begin{bmatrix} 4 & 7 \\ -3 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix}$

   d) $\begin{bmatrix} 7 & -5 & 0 \\ 0 & 3 & 5 \\ -6 & -4 & -3 \end{bmatrix}$ and $\begin{bmatrix} 0 & -4 & 4 \\ 6 & -6 & -8 \\ -6 & -6 & 9 \end{bmatrix}$

6. Either prove that the following properties are true, or give a counterexample to show that the property does not always hold. All matrices below are assumed to be $n \times n$, I is the $n \times n$ identity matrix, O is the $n \times n$ zero matrix, and r is any scalar.

   a) $A \times B = B \times A$

   b) $A \times B = -(B \times A)$ I only checked this ~

   c) $A \times A = O$

   d) $A \times I = I \times A = O$

   e) $(A \times B) \times C = A \times (B \times C)$

   f) $A \times (B \times C) = B \times (A \times C) + C \times (B \times A)$

   g) $(A + B) \times C = (A \times C) + (B \times C)$

   h) $A \times (B + C) = (A \times B) + (A \times C)$

i) If $A \times B = A \times C$, then $B = C$.

j) $r(A \times B) = (rA) \times B = A \times (rB)$

k) $(A \times B)^T = A^T \times B^T$

l) $(A \times B)^T = B^T \times A^T$

As an anticommutative operator (item b) above), the commutator product has wide applicability in physics and geometry.

Reference:

1. Campbell, Hugh G. *Matrices with Applications*. New York: Appleton-Century-Crofts, 1968.
   This book contains the definitions of the matrix products studied in this exercise set.