Calibration of FX Ornstein-Uhlenbeck Process with Kalman Filter

Since current system cannot correctly calibrate parameters for FX evolution model, we compare several calibration approaches like least-square, autocorrelation, interest rate parity and Kalman Filter. Our test results show that only Kalman-filter approach can provide stable long-tem parameters and perform well in backtesting. The new calibration approach generates more conservative FX derivatives values and credit risk exposure.

1 FX Simulation Model

$$Z(t) = \ln(S(t)/S(0)) \quad (1)$$

$$dZ(t) = \alpha(\theta - Z(t))dt + \sigma dW(t) \quad (2)$$

where:

- S(t) is FX rate
- Z(t) is Ornstrein-Uhlenbeck process, Z(0)=0
- σ is constant volatility
- α is constant mean reversion speed
- θ is constant long-term mean
- W(t) is a Brownian-Motion, so dW(t)~N(0, $\varepsilon\sqrt{dt}$)

Solution of SDE (2):

$$Z(t) = Z(0)e^{-\alpha t} + \theta(1 - e^{-\alpha t}) + \int_0^t \sigma e^{\alpha(s-t)} dW_s \quad (3)$$

$$E(Z(t)) = Z(0)e^{-\alpha t} + \theta(1 - e^{-\alpha t}) \quad (4)$$

$$Var(Z(t)) = E[(Z(t) - E(Z(t)))^2] = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}) \quad (5)$$

$$E(S(t)) = S(0)e^{[\theta(1-e^{-\alpha t}) + \frac{\sigma^2}{4\alpha}(1-e^{-2\alpha t})]} \xrightarrow{t->\infty} S(0)e^{[\theta + \frac{\sigma^2}{4\alpha}]} \quad (6)$$

Proof (6):

$$E(e^X) = E(e^{\mu_X + \frac{1}{2}\sigma_X{}^2}) \quad (7)$$

Based on (4), (5), (7):

$$E(S(t)) = S(0)E(e^{Z(t)}) = S(0)e^{\mu_{Z(t)} + \frac{1}{2}\sigma_{Z(t)}{}^2} = S(0)e^{[\theta(1-e^{-\alpha t}) + \frac{\sigma^2}{4\alpha}(1-e^{-2\alpha t})]} \quad (8)$$

2 Calibration Approaches

2.1 Calibration with Least-square 1[4]

(2) rewritten as:

$$\Delta \ln S(t) = \alpha(\theta - \ln S(t)dt + \sigma dW(t) \quad (9)$$

$$\Delta \ln S(t) = \ln S(t) - \ln S(t-1)$$

$$\alpha = -\frac{Cov_{\Delta \ln S(t), \ln S(t)}}{\sigma_{\ln S(t)}{}^2} \tag{10}$$

$$\theta = \frac{mean_{\Delta \ln S(t)}}{\alpha} + mean_{\ln S(t)} \tag{11}$$

$$\sigma_{\Delta \ln S(t)}{}^2 = \alpha^2 \sigma_{\ln S(t)}{}^2 + \sigma^2 \tag{12}$$

2.2 Calibration with Least-square 2[4]

(3) rewritten as:

$$Z(t) = Z(t-1)e^{-\alpha\Delta t} + \theta(1-e^{-\alpha\Delta t}) + \sigma\sqrt{\frac{1-e^{-2\alpha\Delta t}}{2a}}dW_t \tag{13}$$

$$\alpha = -\ln\left(\frac{Cov_{Z(t), Z(t-1)}}{\sigma_{Z(t)}{}^2}\right) / \Delta t \tag{14}$$

$$\theta = \frac{mean_{Z(t)} - mean_{Z(t-1)e^{-\alpha\Delta t}}}{1-e^{-\alpha\Delta t}} \tag{15}$$

$$\sigma = \frac{\sigma_{\varepsilon t}}{\sqrt{\frac{1-e^{-2\alpha\Delta t}}{2\alpha}}} \tag{16}$$

2.3 Calibration with Autocorrelation [7]

Based on (13):

$$\mathrm{Corr(Z(t),Z(s))} = \sqrt{\frac{e^{2\alpha t}-1}{e^{2\alpha s}-1}} \tag{17}$$

Find α value which can match (17) best. Then follow (15) and (16) and get other parameters.

2.4 Calibration with Interest Rate Parity

$$S(t) = S(0)\frac{D_f(t)}{D(t)} = S(0)e^{-\int_0^t (R_f(s)-R(s))ds} \tag{18}$$

(18) is Interest Parity under no-arbitrage condition.
Transform (18):

$$\ln S(t) = \ln S(0) - \int_0^t (R_f(s) - R(s))ds \tag{19}$$

$$d\ln S(t) = -\int_{t-dt}^t (R_f(s) - R(s))ds = (R(t) - R_f(t))dt \tag{20}$$

Input (20) into (2), the OU process under risk-neutral measure:

$$(R(t) - R_f(t))dt = \alpha(\theta - Z(t))dt + \sigma dW(t) \tag{21}$$

Based on equation (4) and (21):

$$E(\ln S(t)) = \theta(1-e^{-\alpha t}) = (R(t) - R_f(t))t \tag{22}$$

$$\theta = \frac{(R(t)-R_f(t))t}{(1-e^{-\alpha t})} \tag{23}$$

Observe 1 month to 5 year interest rates of local currency and foreign currency $R(t), R_f(t)$. Put into the formula (24). Change speed and mean until sum of difference on each terms (24) is minimized.

$$\arg\min\left(\sum_{t=1M}^{5Y}(\theta(1-e^{-\alpha t}) - (R(t)-R_f(t))t)\right) \tag{24}$$

Calibration using least-square regression or autocorrelation cannot generate correct and stable parameters for FX model. Both least-square minimization and maximum likelihood estimation techniques are known to be good at estimating volatility, but poor in estimating long-term mean and reversion speed [3,4,5].

Although Interest rate parity can make 50% percentile very close to daily forward FX curve, but it's still unstable due to daily interest rate changes. Please refer figure 8 and figure 9 for detail. Since credit risk expects long-term trend and quantiles, so its simulation quantiles and expectation should be stable and avoid daily change. Kalman Filter provides a good solution.

3 Calibration with Kalman Filter

3.1 Kalman Filter Process [8]

Kalman filter is mainly used to estimate system states that can only be observed indirectly or inaccurately by the system itself. In order to use a Kalman filter to remove noise from a time series, consider a linear discrete time system, the state space representation of such a system would be,

$$\begin{aligned} x_k &= Ax_{k-1} + C_{k-1} + w_{k-1} \\ y_k &= Bx_k + D_k + v_k \end{aligned} \qquad (25)$$

Where A, C, B, D are the System Matrices. x_k is estimated state. y_k is measurement and be observable. w,v are zero mean mutually uncorrelated white noises with covariances,

$$\begin{aligned} E[w_k w_k^T] &= Q_k \\ E[v_k v_k^T] &= R_k \end{aligned} \qquad (26)$$

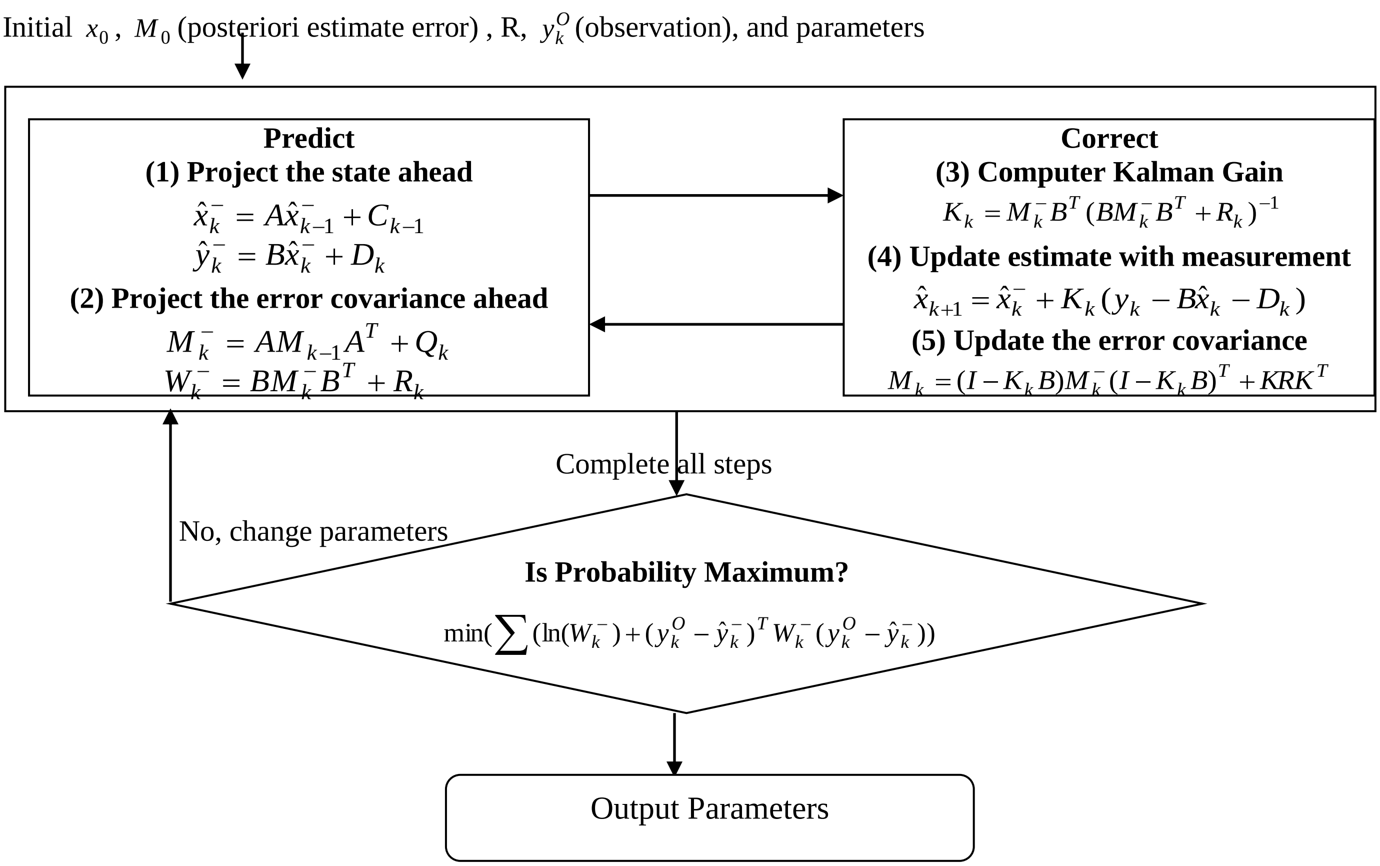

Figure 1 Kalman Filter Calibration Process

3.2 FX Rate Kalman Filter Process

Rewrite (13) as:

$$
\begin{aligned}
x(t) &= e^{-\alpha\Delta t}x(t-1)+\theta(1-e^{-\alpha\Delta t})+\sigma\sqrt{\frac{1-e^{-2\alpha\Delta t}}{2a}}dW_t \\
y(t) &= x(t)
\end{aligned}
\qquad (27)
$$

Since FX spot rate is observable, set measurement directly as Ln(FX spot).

$A = e^{-\alpha\Delta t}, C = \theta(1-e^{-\alpha\Delta t}), B = 1, D = 0$

$Q = \sigma^2 \dfrac{1-e^{-2\alpha\Delta t}}{2a} M_0, R = 1/25600, M_0 = 0.0001$

Implementation codes are attached.

Normrnd.m optimizeVol.m OUFXKalmanFilterWithTime.m Prctile.m Fxkalman.m FXKalmanSimu.m Norminv.m

3.3 FX Rate Kalman Filter Test Result

Use 15-year daily history data of AUD, CAD, EUR, GBP, JPY and CHF. Based on Kalman Filter calibration result, ten-year simulation percentiles are very stable compared the shorter history or other calibration methods. Since Kalman Filter focuses on likelihood of data, sometimes its volatility may be a little bit smaller than least-square. Based on historical data percentiles, volatility can be adjusted covering 97.5% or maximum values according to our risk appetite.

We also quickly verify Kalman Filter using simulation data for calibration. Under 3,900 time steps and 840 paths, mean reversion speed and mean level converge to true parameters faster and more accurate then least-square methods. Please refer table 1. We will make full equivalent test if conditions are available.

AUD, CAD, CHF and EUR mean reversion speeds are very close to our exogenous speed 0.1. But GBP and JPY have very small speed. If we consider using 0.1 as the two currency's speed, Kalman Filter is also applied for calibration under exogenous speed.

		Speed	Mean level	Volatility
True Value		0.42	-0.28	0.143
Kalman Filter (840 paths)	mean	0.489	-0.236	0.153
	std	0.275	0.155	0.024
Least Square (10,000 paths)	mean	0.908	-0.284	0.143
	std	0.514	0.2	0.002

Table 1 Verification Least-square Regression and Kalman Filter Calibration

CCY15Y	Date	Speed	Mean	Volatility
AUD	11/23/1998-11/23/2013	0.12014	-0.002455193	0.148505
AUD	12/23/1998-12/23/2013	0.132954	-0.002439843	0.182247
AUD	1/22/1999-1/22/2014	0.123023	-0.002681599	0.155111
AUD	2/23/1999-2/23/2014	0.1244	-0.001987773	0.155102
AUD	3/23/1999-3/23/2014	0.12501	-0.001979903	0.155185

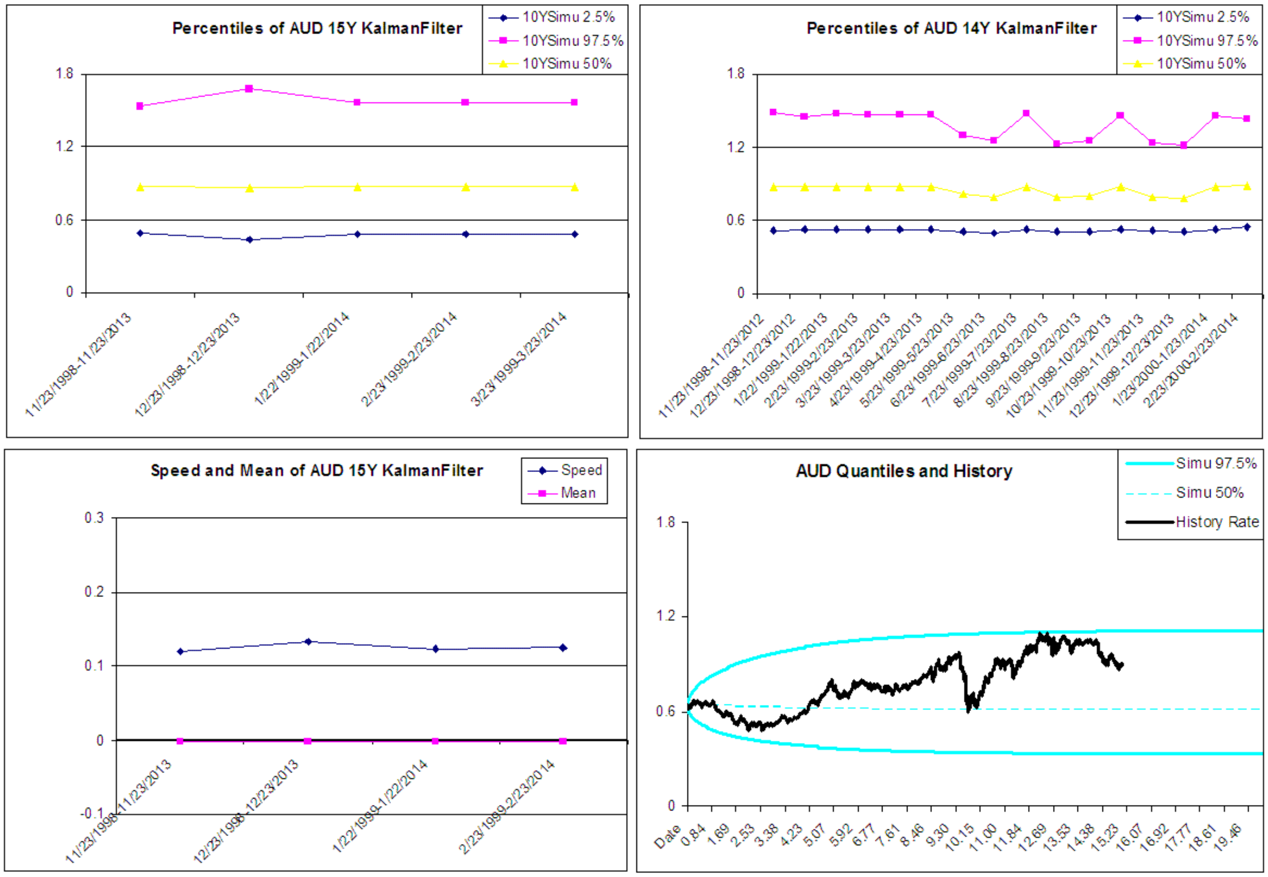

Figure 2 AUD Parameter Calibration and Percentiles

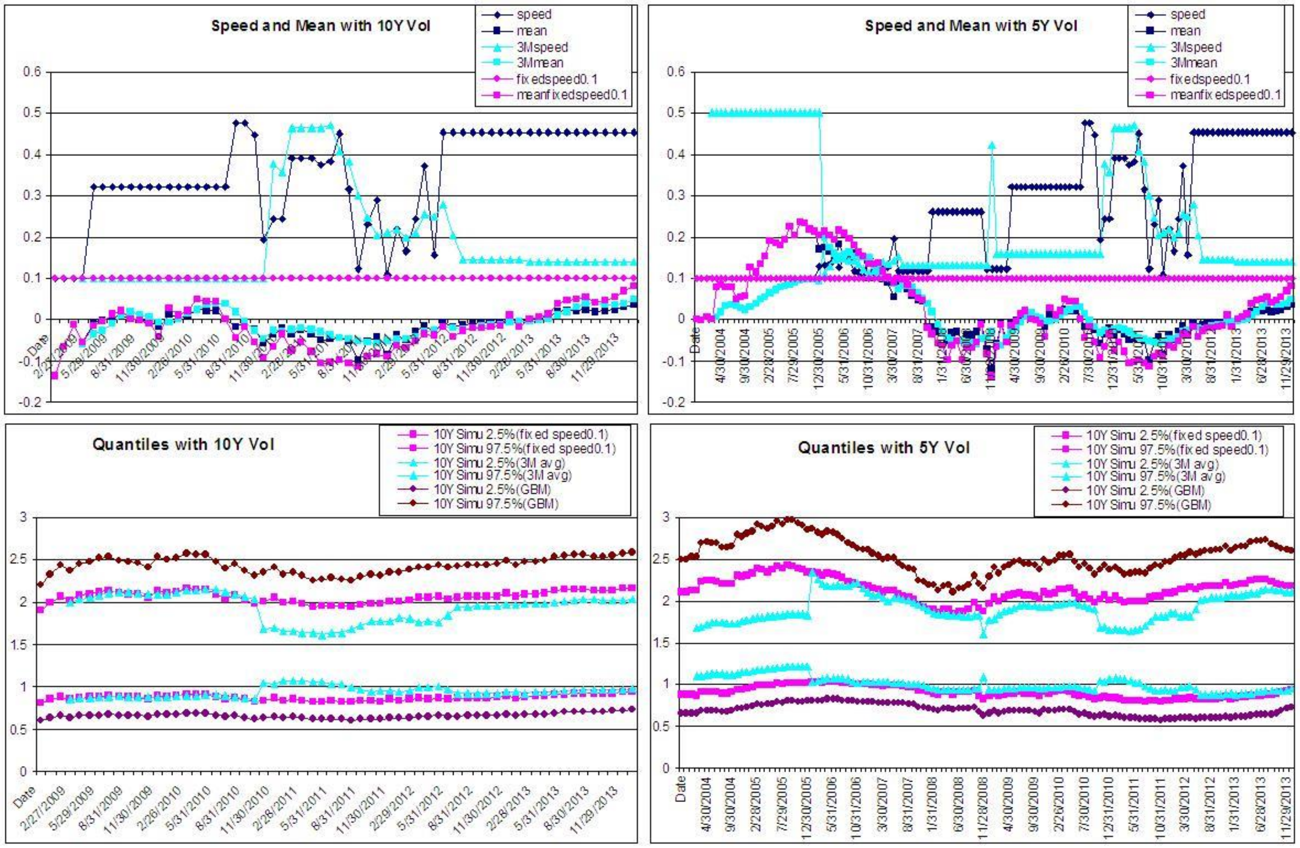

Figure 9 EUR FX Calibration under Interest Rate Parity and Simulation

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