9548

8.5

(a)

$V[x] = m g x$

(b)

$$-\frac{\hbar^2}{2m}\psi''[x] = (E - m g x)\,\psi$$

$$\frac{\hbar^2}{2m}\psi''[x] = \left(\frac{2 m^2 g x}{\hbar^2} - \frac{E}{m g}\right)\psi$$

Let $w = x - \frac{E}{mg}$ and $\alpha = \left(\frac{2 m^2 g}{\hbar^2}\right)^{1/3}$

$\psi''[y] = \alpha^3 y \psi$

Let $z = \alpha y$

$\psi''[z] = z \psi$ which is the Airy equation

$\psi = a\,Ai[z] + b\,Bi[z]$

$Bi[z]$ blows up as z gets big, so

$\psi[z] = a\,Ai[z]$

$$= a\,Ai\left[\alpha\left(x - \frac{E}{mg}\right)\right]$$

(c)

$\psi[0] = 0$

$$\Rightarrow a\,Ai\left[\frac{E}{mg}\right] = 0$$

```
a = {-2.338, -4.088, -5.521, -6.787};
```

m = .1; g = 9.8; ℏ = 1.054*^-34; $\boldsymbol{\alpha = \left(\frac{2 m^2 g}{\hbar^2}\right)^{1/3}}$**;**

$\mathbf{En = -\frac{m g}{\alpha} a}$

$\{8.80128\times10^{-23}, 1.53891\times10^{-22}, 2.07835\times10^{-22}, 2.55493\times10^{-22}\}$

(d)

$$2\langle T\rangle = \left\langle x\frac{dV}{dx}\right\rangle$$

$$\frac{dV}{dx} = m g$$

$$\langle T\rangle = \frac{1}{2} m g \langle x\rangle = \frac{1}{2}\langle V\rangle$$

$$E_n = \langle T\rangle + \langle V\rangle = \frac{3}{2}\langle V\rangle$$

$$\langle x\rangle = \frac{2 E_n}{3 m g}$$

```
m = 9.11*^-31;
```

$$\text{En} = -\left(\frac{1}{2}\,m\,g^2\,\hbar^2\right)^{1/3} a[[1]]$$

```
Print[% / 1.6*^-19, " eV"]
```

1.83817×10^{-32}

1.14885×10^{-13} eV

$$\text{Print}\left[\langle x\rangle = \frac{2\,\text{En}}{3\,m\,g}, \text{" m"}\right]$$

0.00137262 m

8.6

(a)

$$\int_0^{x2} p[x]\,dx = \left(n-\frac{1}{4}\right)\pi\hbar$$

$$p[x] = \sqrt{2\,m\,(E - m\,g\,x)}$$

$$\int_0^{E/mg} \sqrt{2\,m\,(E - m\,g\,x)}\,dx = \left(n-\frac{1}{4}\right)\pi\hbar$$

$$-\sqrt{2\,m}\;\frac{2}{3}\;\frac{1}{m\,g}\;(E - m\,g\,x)^{3/2} = \left(n-\frac{1}{4}\right)\pi\hbar$$

$$\text{Solve}\left[-\sqrt{2\,m}\;\frac{2}{3}\;\frac{1}{m\,g}\;(\mathbb{E})^{3/2} == \left(n-\frac{1}{4}\right)\pi\hbar,\ \mathbb{E}\right] \text{ // FullSimplify}$$

$$\left\{\left\{\mathbb{E}\to\frac{(3\pi)^{2/3}\left(g\sqrt{m}\,(1-4n)\,\hbar\right)^{2/3}}{4\times2^{1/3}}\right\}\right\}$$

(b)

```
m = .1; g = 9.8; ℏ = 1.054*^-34;
```

$$\text{En} = \left(\frac{9}{8}\pi^2\,m\,g^2\,\hbar^2\,(n-1/4)^2\right)^{1/3}$$

$$1.0581\times10^{-22}\left(\left(-\frac{1}{4}+n\right)^2\right)^{1/3}$$

```
Table[En, {n, 1, 4}]
```

$\{8.73446\times10^{-23}, 1.53658\times10^{-22}, 2.07691\times10^{-22}, 2.55397\times10^{-22}\}$

(c)

$$\langle x\rangle = \frac{2\,\text{En}}{3\,m\,g}$$

$$\text{Solve}\left[1 == \frac{2\,\text{En}}{3\,m\,g}, n\right][[2]]$$

$\{n\to1.63751\times10^{33}\}$

8.7

$$\int_{x1}^{x2} p[x]\,dx == \left(n-\frac{1}{2}\right)\pi\hbar$$

$$x2 = -x1 = \frac{1}{\omega}\sqrt{\frac{2\,\mathrm{E}}{m}}$$

$$2\int_0^{x2} p[x]\,dx == \left(n-\frac{1}{2}\right)\pi\hbar$$

$$2\,m\,\omega\int_0^{x2}\sqrt{\frac{2\,E}{m\,\omega^2}-x^2}\,dx$$

$$2\,m\,\omega\int_0^{x2}\sqrt{x2^2-x^2}\,dx$$

$$m\,\omega\left(x\sqrt{x2^2-x^2}+x2^2\,\mathrm{ArcSin}[x/x2]\right)_0^{x2}$$

$$m\,\omega\,x2^2\,\mathrm{ArcSin}[1] == \left(n-\frac{1}{2}\right)\pi\hbar$$

$$m\,\omega\left(\frac{1}{\omega}\sqrt{\frac{2\,\mathrm{E}}{m}}\right)^2\frac{\pi}{2} == \left(n-\frac{1}{2}\right)\pi\hbar$$

Solve$\left[m\,\omega\left(\frac{1}{\omega}\sqrt{\frac{2\,\mathrm{E}}{m}}\right)^2\frac{\pi}{2} == \left(n-\frac{1}{2}\right)\pi\hbar,\ \mathrm{E}\right]$ **// FullSimplify**

$$\left\{\left\{\mathrm{E}\to\frac{1}{2}\,(-1+2\,n)\,\omega\,\hbar\right\}\right\}$$

$\mathrm{E} = \left(n-\frac{1}{2}\right)\hbar\,\omega$ which matches exactly the usual solution when you consider that n begins at 1 for the WKB instead of at 0

8.14

$$\left(n-\frac{1}{2}\right)\pi\hbar = \int_{r1}^{r2}\sqrt{2\,m\left(\mathrm{E}+\frac{e^2}{4\,\pi\,\epsilon_0}\,\frac{1}{r}-\frac{\hbar^2}{2\,m}\,\frac{l\,(l+1)}{r^2}\right)}\,dr$$

$$\left(n-\frac{1}{2}\right)\pi\hbar = \sqrt{-2\,m\,\mathrm{E}}\int_{r1}^{r2}\sqrt{-1+\frac{A}{r}-\frac{B}{r^2}}\,dr$$

$$\left(n-\frac{1}{2}\right)\pi\hbar = \sqrt{-2\,m\,\mathrm{E}}\int_{r1}^{r2}\frac{\sqrt{-r^2+A\,r-B}}{r}\,dr$$

Let r1 and r2 be the roots of the polynomial in the numerator of the integral

$$\left(n-\frac{1}{2}\right)\pi\hbar = \sqrt{-2\,m\,\mathrm{E}}\int_{r1}^{r2}\frac{\sqrt{(r-r1)\,(r2-r)}}{r}\,dr$$

$$\left(n-\frac{1}{2}\right)\pi\hbar = \sqrt{-2\,m\,\mathrm{E}}\,\frac{\pi}{2}\left(\sqrt{r1}-\sqrt{r2}\right)^2$$

$$\left(n-\frac{1}{2}\right)\pi\hbar = \sqrt{-2\,m\,\mathrm{E}}\,\frac{\pi}{2}\left(r1+r2-2\sqrt{r1\,r2}\right)$$

$$-r^2 + A\,r - B = (r - r1)\,(r2 - r) = -r^2 + (r1 + r2)\,r - r1\,r2$$

A = r1 + r2 and B = r1 r2

$$\left(n - \frac{1}{2}\right)\pi\hbar = \sqrt{-2\,m\,E}\;\frac{\pi}{2}\left(A - 2\sqrt{B}\right)$$

$$\left\{A \to \frac{-e^2}{4\,\pi\,\epsilon_0}\,\frac{1}{E},\; B \to -\frac{\hbar^2}{2\,m}\,\frac{l\,(l+1)}{E}\right\}$$

$$\left(n - \frac{1}{2}\right)\hbar = \sqrt{-2\,m\,E}\;\frac{\pi}{2}\left(\frac{-e^2}{4\,\pi\,\epsilon_0}\,\frac{1}{E} - 2\sqrt{-\frac{\hbar^2}{2\,m}\,\frac{l\,(l+1)}{E}}\right)$$

$$\left(n - \frac{1}{2}\right)2\,\hbar = \frac{-e^2}{4\,\pi\,\epsilon_0}\sqrt{-\frac{2\,m}{E}} - 2\,\hbar\sqrt{l\,(l+1)}$$

$$\frac{e^2}{4\,\pi\,\epsilon_0}\sqrt{-\frac{2\,m}{E}} = \left(n - \frac{1}{2}\right)\hbar + \hbar\sqrt{l\,(l+1)}$$

$$E = \frac{2\,m\left(\frac{e^2}{4\,\pi\,\epsilon_0}\right)^2}{2\,\hbar^2\left(n - \frac{1}{2} + \sqrt{l\,(l+1)}\right)^2} = \frac{-13.6\text{ eV}}{\left(n - \frac{1}{2} + \sqrt{l\,(l+1)}\right)^2}$$