拓扑学讲义

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红色 = 强调　　蓝色 = 补充说明　　紫色 = 习题

主要参考文献：[Mun00, 第2、3、4章].

References

[Mun00] James R. Munkres. *Topology*. Prentice Hall, Inc., Upper Saddle River, NJ, second edition, 2000.

目录

1. General topology

1.1. Metric spaces.

Definition 1 (**Metric space and balls**). *A metric on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying the following conditions, for any $x, y, z \in X$:*

- $d(x,y) = d(y,x)$;
- $d(x,y) \geq 0$, *with "=" if and only if* $x = y$;
- $d(x,y) \leq d(x,z) + d(z,y)$ (Triangle Inequality).

The pair (X,d) is called a metric space. An open ball (resp. closed ball) in the metric space is a subset of the form

$$B(x,r) := \{y \in X \mid d(x,y) < r\} \quad \left(\text{resp. } \overline{B}(x,r) := \{y \in X \mid d(x,y) \leq r\}\right),$$

where $x \in X$ and $r > 0$ are the center and radius of the ball, respectively.

Example 1. The standard metric on $\mathbb{R}^N$ is defined as

$$d(x,y) := \sqrt{|x_1 - y_1|^2 + \cdots + |x_n - y_n|^2} = \|x - y\|_{\ell^2}, \quad x, y \in \mathbb{R}^N,$$

where $\|\cdot\|_{\ell^2}$ is the ℓ^2-norm[1]. The resulting metric space $(\mathbb{R}^N, d)$ is the N-dimensional Euclidean space, where balls are the familiar round ones. More generally, given a vector space[2] X, a norm on X is a function $\|\cdot\| : X \to \mathbb{R}$ such that for any $x, y \in X$ and $a \in \mathbb{R}$,

- $\|x\| \geq 0$, with "=" if and only if $x = 0$;
- $\|ax\| = |a| \cdot \|x\|$;
- $\|x + y\| \leq \|x\| + \|y\|$.

Any norm $\|\cdot\|$ induces a metric d on X defined by $d(x,y) := \|x - y\|$.

Example 2. Given a connected graph $G = (V, E)$ (i.e. some vertices V connected by edges E), the graph metric is the metric on the set of vertices V defined by

$$d(x,y) := \min\{n \mid \text{there exists a path of } n \text{ edges connecting } x \text{ and } y\}, \ \ x, y \in V.$$

In particular, if any two vertices are joint by an edge, this gives the discrete metric $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$.

The following lemma and exercise are typical applications of Triangle Inequality.

Lemma 1. *For any point x in an open ball $B(y,r)$, there exists an open ball $B(x,\epsilon)$ centered at x such that $B(x,\epsilon) \subset B(y,r)$.*

Proof. The hypothesis $x \in B(y,r)$ means $d(x,y) < r$, so we may set $\epsilon := r - d(x,y) > 0$. Let us check that $B(x,\epsilon)$ is contained in $B(y,r)$. For any point z in the former ball, by Triangle Inequality we have $d(y,z) \leq d(y,x) + d(x,z) < d(y,x) + \epsilon = r$, which means that z is contained in the latter ball, as required. $\square$

Exercise 1. Let E be a subset in a metric space (X,d). Show that if there exists a ball containing E, then for any $x \in X$ there exists a ball centered at x containing E. Also show that it does not matter whether the ball is open or closed (i.e. $\exists$ open ball containing E $\Leftrightarrow$ $\exists$ closed ball containing E). Such a set E is said to be bounded.

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[1]For $p \in [1, +\infty]$, the ℓ^p-norm $\|\cdot\|_{\ell^p}$ on $\mathbb{R}^N$ is defined as follows (where x^i denotes the ith coordinate of x):

$$\|x\|_{\ell^p} := (|x^1|^p + \cdots + |x^N|^p)^{\frac{1}{p}} \ \text{ for } \ p \in [1, +\infty), \qquad \|x\|_{\ell^\infty} := \max\{|x^1|, \cdots, |x^N|\}.$$

We have $\lim_{p \to +\infty} \|x\|_{\ell^p} = \|x\|_{\ell^\infty}$. See Example 9 below for more about this norm.

[2]All vector spaces that we consider in these notes are defined over $\mathbb{R}$, but can be infinite dimensional unless otherwise specified.

Definition 2 (**Topological notions in metric space**). *In a metric space* (X,d),

(1) *An open set is a subset which is a union of open balls* ("a union of ×××'s" means a set of the form $\bigcup_{i\in I} A_i$, where $(A_i)_{i\in I}$ is a family of ×××'s. We allow empty family ($I=\varnothing$), in which case the union is empty).

- $U\subset X$ *is open* $\Leftrightarrow$ $\forall x\in U$, $\exists \epsilon>0$ *such that* $B(x,\epsilon)\subset U$.

(2) *Given a point* $x\in X$, *any open set containing* x *is called a neighborhood of* x.

(3) *A sequence* $(x_n)_{n=1,2,\dots}$ *is said to converge to a point* x *if* $\lim_{n\to\infty} d(x_n,x)=0$. *Denote this by* $x_n\to x$. *Such an* x *is unique if it exists and is called the limit of* (x_n).

- $x_n\to x$ $\Leftrightarrow$ $\forall$*neighborhood* U *of* x, $\exists N>0$ *s.t.* $x_n\in U$ *when* $n>N$ ("(x_n) *eventually enters any neighborhood of* x").

(4) *A closed set is a subset whose complement is open.*

- *The empty set* $\varnothing$ *is both open and closed. So is the whole* X.
- $C\subset X$ *is closed* $\Leftrightarrow$ $\forall x\notin C$, $\exists r>0$ *s.t.* $B(x,r)\cap C=\varnothing$
  $\Leftrightarrow$ *the limit of any convergent sequence in* C *is again in* C;
- *a union (resp. intersection) of open (resp. closed) sets is still open (resp. closed);*
- *an intersection (resp. union) of finitely many open (resp. closed) sets is still open (resp. closed);*
- *the closed ball* $\overline{B}(x,r)$ *is a closed set.*

(5) *The closure of* $E\subset X$, *denoted by* $\overline{E}$, *is the intersection of all closed sets containing* E. E *is said to be dense if* $\overline{E}=X$. By the 3rd point in (4), $\overline{E}$ is closed, hence it is the smallest closed set containing E. In particular, we have $\overline{E}=E$ iff. E is closed.

- $x\notin\overline{E}$ $\Leftrightarrow$ $\exists r>0$ *s.t.* $B(x,r)\cap E=\varnothing$ $\Leftrightarrow$ x *has a neighborhood* U *s.t.* $E\cap U=\varnothing$;
- $x\in\overline{E}$ $\Leftrightarrow$ *any neighborhood of* x *intersects* E $\Leftrightarrow$ *there exists a sequence* (x_n) *in* E *s.t.* $x_n\to x$;
- $\overline{B(x,r)}\subset\overline{B}(x,r)$. Can be strict inclusion (e.g. ball of radius 1 under discrete metric). See also Exercise 3;
- E *is dense* $\Leftrightarrow$ *every open ball intersects* E $\Leftrightarrow$ *every nonempty open set intersects* E.

(6) *A map* f *from* (X,d) *to another metric space* (X',d') *is continuous at a point* $x\in X$ *if* $\forall\epsilon>0$, $\exists\,\delta>0$ *such that* $f(B(x,\delta))\subset B(f(x),\epsilon)$ (in other words: $\forall\epsilon>0$, $\exists\delta>0$ s.t. when $y\in X$ satisfies $d(x,y)<\delta$ we have $d'(f(x),f(y))<\epsilon$).
If f *is continuous at every point of* X, *it is simply said to be continuous.*

- f *is continuous at* x $\Leftrightarrow$ *for any neighborhood* V *of* $f(x)$, *there exists a neighborhood* U *of* x *s.t.* $f(U)\subset V$;
  $\Leftrightarrow$ *for any sequence* $(x_n)_{n=1,2,\dots}$ *in* X *converging to* x, *we have* $f(x_n)\to f(x)$ *in* (X',d');
- f *is continuous* $\Leftrightarrow$ *the preimage of any open subset of* X' *is open in* X
  $\Leftrightarrow$ *the preimage of any closed subset of* X' *is closed in* X.

(7) *A set* $E\subset X$ *is compact if any sequence in* E *has a subsequence converging to a point in* E.

- $E\subset X$ *is compact* $\Rightarrow$ E *is closed and bounded.*

  The converse "$\Leftarrow$" holds when X is a finite dimensional vector space and d is induced by a norm, as will be shown in §1.2, but does not hold in general. Typical counterexamples include infinite set with discrete metric (Example 2) and infinite dimensional vector space with metric induced by norm (see Exercise 7).

When we want to emphasize which metric is being used, we say something like

$$\left.\begin{array}{r} U\subset X \text{ is open} \\ (x_n) \text{ converges to } x \\ \cdots \end{array}\right\} \text{with respect to } d \text{ (or under } d\text{)}.$$

For a norm $\|\cdot\|$ on a vector space, "w.r.t. the metric induced by $\|\cdot\|$" is abbreviated to "w.r.t. $\|\cdot\|$".

Example 3. On the real line $\mathbb{R}$, we always consider the standard metric $d(x,y)=|x-y|$. Open balls w.r.t. d are finite open intervals $(a,b)\subset\mathbb{R}$. Open sets are unions of such intervals, and examples include the half-infinite open intervals $(-\infty,a)$ and $(a,+\infty)$. Examples of closed sets include the one-point set $\{a\}$ and closed intervals $[a,b]$, $[a,+\infty)$ and $(-\infty,a]$, as well as finite unions of such sets. Note that

- As a nontrivial fact (exercise below), no subset of $\mathbb{R}$, except the empty set $\varnothing$ and the whole $\mathbb{R}$, can be both open and closed. In particular, $[a,b]$ is not open and (a,b) is not closed. This can also be shown directly (a is contained in $[a,b]$ but no open ball centered at a is contained in $[a,b]$; (a,b) contains a sequence converging to the point a which lies outside of (a,b)). Similarly, $[a,b)$ and $(a,b]$ are neither open nor closed.
- An infinite union of closed sets can be either closed or not. For example, as countable unions of one-point sets, $\mathbb{Z}$ is closed in $\mathbb{R}$, while $\left\{\frac{1}{n}\,\middle|\, n\in\mathbb{Z}_+\right\}$ is not (since it contains a sequence converging to a point outside of it).
- By taking the complement of the last example, we get an example of a countable intersection of open sets which is not open. Another classical example is $[a,b]=\bigcap\limits_{n=1}^{\infty}\left(a-\frac{1}{n},b+\frac{1}{n}\right)$.

The set $\mathbb{Q}$ of rational numbers and the set $\mathbb{R}\setminus\mathbb{Q}$ of irrational numbers are examples of dense sets in $\mathbb{R}$. The notion of limit for a sequence in $\mathbb{R}$ w.r.t. d coincides with the familiar one from Calculus, and so does the notion of continuity for a function $\mathbb{R}\to\mathbb{R}$. Compact subsets of $\mathbb{R}$ are just bounded closed subsets, as will be shown in §1.2.

Exercise 2. Show that

(1) Under the discrete metric (Example 2), every subset is both open and closed.

(2) Suppose $E\subset\mathbb{R}$ is bounded from above (i.e. $\sup E<+\infty$). If E is open (resp. closed) in $\mathbb{R}$, then we have $\sup E\notin E$ (resp. $\sup E\in E$). The same statement, only with $\sup E$ replaced by $\inf E$, holds when E is bounded from below.

The closed case can be stated as: *a closed subset of $\mathbb{R}$ bounded from above (resp. below) has a largest (resp. smallest) element.*

(3) No subset of $\mathbb{R}$, except $\varnothing$ and the whole $\mathbb{R}$, can be both open and closed.

The lines in purple color in Def. 2 are basic statements that we will frequently use. Here are proofs for some of them.

Proof of the "⇔" in (1). "⇒": Suppose $U\subset X$ is open and $x\in U$. Since U is a union of open balls, there is some $B(y,r)$ contained in U and containing x. Then Lemma 1 gives a ball $B(x,\epsilon)$ such that $B(x,\epsilon)\subset B(y,r)\subset U$, as required.

"⇐": If for any $x\in U$ there is $\epsilon_x>0$ s.t. $B(x,\epsilon_x)\subset U$, then U is just the union $\bigcup_{x\in U}B(x,\epsilon_x)$, hence open. □

Proof of "C is closed ⇔ the limit of any convergent sequence in C is again in C".

"⇒": Let C be closed and $(x_n)_{n=1,2,\cdots}$ be a sequence in C converging to some $x\in X$. Suppose by contradiction that $x\notin C$. Since $X\setminus C$ is open, it is a neighborhood of x. Therefore, the assumption $x_n\to x$ implies that $x_n\in X\setminus C$ when n is large enough. This contradicts the assumption that (x_n) is in C.

"⇐": Suppose the limit of any convergent sequence in C is again in C. In order to show that C is closed, we pick $x\notin C$ and only need to find $r>0$ such that $B(x,r)\cap C=\varnothing$. Assume by contradiction that there is no such r. Then $B(x,\frac{1}{n})\cap C$ is nonempty for $n=1,2,\cdots$. By picking a point x_n in this intersection, we get a sequence $(x_n)_{n=1,2,\cdots}$ in C converging to x, a contradiction. □

Proof of "f is continuous ⇔ the preimage of any open subset of X' is open in X.

"⇒": Letting $V\subset X'$ be open, we need to show that $f^{-1}(V)$ is open, or equivalently, that any $x\in f^{-1}(V)$ has a ball $B(x,r)$ contained in $f^{-1}(V)$. Since V is open, we may find a ball $B(f(x),\epsilon)$ at $f(x)$ contained in V. The continuity of f at x then provides us with a ball $B(x,\delta)$ at x s.t. $f(B(x,\delta))\subset B(f(x),\epsilon)$. This inclusion implies $B(x,\delta)\subset f^{-1}(B(f(x),\epsilon))\subset f^{-1}(V)$, as required.

"$\Leftarrow$": Assume that the preimage of any open set is open. Fix $x \in X$ and $\epsilon > 0$. We need to find $\delta > 0$ s.t. $f(B(x,\delta)) \subset B(f(x),\epsilon)$. To this end, note that the set $f^{-1}(B(f(x),\epsilon))$ is open in X because $B(f(x),\epsilon)$ is open in X'. Since this open set contains x, it also contains some open ball $B(x,\delta)$ centered as x. Thus, we have $B(x,\delta) \subset f^{-1}(B(f(x),\epsilon))$, which implies $f(B(x,\delta)) \subset B(f(x),\epsilon)$, as required. □

Proof of "$E \subset X$ is compact $\Rightarrow E$ is closed and bounded". Suppose E is not closed, then there exists a sequence $(x_n)_{n=1,2,\cdots}$ in E converging to some $x \notin E$. Any subsequence of (x_n) must also converge to x, so there is no subsequence converging to a point of E, which means that E is not compact.

Suppose E is not bounded. Pick a point $x \in X$. Since E is not contained in any ball centered at x, there exists a sequence $(x_n)_{n=1,2,\cdots}$ in E such that $\lim_{n\to\infty} d(x_n,x) = +\infty$. Let us show that (x_n) does not have any convergent subsequence, which would imply that E is not compact. If there exists a subsequence $(x_{n_i})_{i=1,2,\cdots}$ converging to some $y \in X$, then $\lim_{i\to\infty} d(x_{n_i},y) = 0$. By Triangle Inequality we have $d(x_{n_i},x) \le d(x_{n_i},y) + d(y,x)$, where the right-hand side tends to $d(y,x)$, while the left-hand side tends to $+\infty$, a contradiction. □

A key message from Def. 2 is that notions such as limit, continuity, closure, compactness, etc. only depends on *which sets are open*, rather than on the metric itself. For example, by the following lemma, if two metrics d_1, d_2 satisfy

$$\exists C > 1 \text{ such that } C^{-1} d_1(x,y) \le d_2(x,y) \le C d_1(x,y),\ \forall x,y \in X, \tag{1.1}$$

then (X,d_1) and (X,d_2) have exactly the same open sets, hence the above notions are the same under d_1 and d_2. The motivation of defining a topological space (see §1.3) is to single out this information of open sets.

Lemma 2. *Let d_1, d_2 be two metrics on a set X. Suppose there is a constant $C > 0$ such that $d_1(x,y) \le C d_2(x,y)$ for all $x,y \in X$. Then every open set with respect to d_1 is also open with respect to d_2.*

Proof. Let $B_i(x,r)$ denote the open ball w.r.t. d_i $(i = 1,2)$ of radius $r > 0$ centered at $x \in X$. We only need to show that every open ball w.r.t. d_1, namely $B_1(x,r)$, is also an open set w.r.t. d_2. Equivalently, for any $y \in B_1(x,r)$, we need to find $\epsilon > 0$ such that $B_2(y,\epsilon) \subset B_1(x,r)$. To this end, we may use Lemma 1 to obtain $\epsilon_0 > 0$ such that

$$B_1(y,\epsilon_0) \subset B_1(x,r). \tag{1.2}$$

On the other hand, we may deduce from the assumption $d_1 \le C d_2$ that

$$B_2(y, C^{-1}\epsilon_0) \subset B_1(y,\epsilon_0). \tag{1.3}$$

This is because

$$z \in B_2(y, C^{-1}\epsilon_0) \ \Leftrightarrow\ d_2(y,z) < C^{-1}\epsilon_0 \ \overset{\text{assumption}}{\Rightarrow}\ d_1(y,z) < \epsilon_0 \ \Leftrightarrow z \in B_1(y,\epsilon_0).$$

By (1.2) and (1.3), we may put $\epsilon := C^{-1}\epsilon_0$, so that $B_2(y,\epsilon) \subset B_1(y,\epsilon_0) \subset B_1(x,r)$, as required. □

Exercise 3. Show that the following conditions are equivalent for a metric space (X,d):

(a) the closure of any open ball is the corresponding closed ball, namely $\overline{B(x,r)} = \overline{B}(x,r)$, $\forall x \in X$, $r > 0$;

(b) for any two distinct points $x,y \in X$ and any neighborhood U of y, there exists $z \in U$ s.t. $d(x,z) < d(x,y)$.

Also show that when X is a vector space and d is induced by a norm, these conditions hold. *Hint:* Given $x,y \in X$, consider a point of the form $z_t := (1-t)x + ty$ $(t \in [0,1])$ and compute its distance to x and y.

Exercise 4. Let (X,d) be a metric space. A function $f : X \to \mathbb{R}$ is said to be 1-Lipschitz if

$$|f(x) - f(y)| \le d(x,y) \text{ for all } x,y \in X.$$

Define the distance from a point $x \in E$ to a set $E \subset X$ as $d(x,E) := \inf_{y \in E} d(x,y)$. Show that

(1) Any 1-Lipschitz function is continuous.
(2) $|d(x,z)-d(y,z)| \le d(x,y)$ for any $x,y,z \in X$. In order words: for fixed $z \in X$, the function $x \mapsto d(x,z)$ is 1-Lipschitz.
(3) For any $E \subset X$, the function $x \mapsto d(x,E)$ is 1-Lipschitz.
(4) $d(x,E)=0$ iff. $x \in \overline{E}$. As a consequence, if E is closed, then $d(x,E)>0$ for any $x \notin E$.
(5) $d(x,E)=d(x,\overline{E})$ for any $x \in X$, $E \subset X$.

Exercise 5. Let X be a vector space and $\|\cdot\|$ be a norm on X. Show that

(1) $\big|\|x\|-\|y\|\big| \le \|x-y\|$ for any $x,y \in X$.
(2) The function $X \to \mathbb{R}$, $x \mapsto \|x\|$ is continuous (w.r.t. $\|\cdot\|$ itself).
(3) If X' is another vector space, $\|\cdot\|'$ is a norm on X' and $A : X \to X'$ is a linear map, then the followings are equivalent:
   (a) there exists $C>0$ such that $\|A(x)\|' \le C\|x\|$ for any $x \in X$;
   (b) A is continuous at $0 \in X$;
   (c) A is continuous.
   (Here continuity is defined w.r.t. $\|\cdot\|$ and $\|\cdot\|'$.)
   Recall that A being *linear* means $A(x+y)=A(x)+A(y)$ and $A(\lambda x)=\lambda A(x)$ for any $x,y \in X$ and $\lambda \in \mathbb{R}$.

1.2. **Finite dimensional vector spaces.** Any finite dimensional vector space (over $\mathbb{R}$) is isomorphic to $\mathbb{R}^N$ for some $N \in \mathbb{Z}_+$. Metrics induced by norms on $\mathbb{R}^N$ are simple to understand: it is a fundamental fact that any two such metrics satisfy (1.1). As a consequence, they all give the same open sets and hence the same notions of limit, continuity, etc. This notion of limit is just the familiar coordinate-wise limit. Precise statements are given as the next theorem and corollary.

Before proving the theorem, we will temporarily use the notions of limit, closed set, continuity, boundedness and compactness in $\mathbb{R}^N$ only with respect to the norm

$$\|x\|_{\ell^\infty} := \max\{|x^1|, \cdots, |x^N|\},$$

where x^i denotes the ith coordinate of x. Thus, for a sequence $(x_n)_{n=1,2,\cdots}$ in $\mathbb{R}^N$ and a point $x \in \mathbb{R}^N$, we have $x_n \to x$ iff. $\|x_n - x\|_{\ell^\infty} = \max\{|x_n^1 - x^1|, \cdots, |x_n^N - x^N|\} \to 0$, which is clearly equivalent to the coordinate-wise convergence $x_n^i \to x^i$, $i = 1, \cdots, N$. Meanwhile, a set $E \subset \mathbb{R}^N$ is bounded iff. E is contained in some $[-a,a]^N$, $a>0$.

Lemma 3 (**Bolzano-Weierstrass/Heine-Borel Theorem**)**.** *Any bounded closed subset of $\mathbb{R}^N$ is compact.*

Proof. We need to show that if $E \subset \mathbb{R}^N$ is bounded and closed (w.r.t. $\|\cdot\|_{\ell^\infty}$), then any sequence $(x_n)_{n=1,2,\cdots}$ in E has a subsequence converging to some point of E. Since the closedness of E means that the limit of any convergent sequence in E is again in E, it suffices to show that *any bounded sequence in $\mathbb{R}^N$ has a convergent subsequence.*

The $N=1$ case of the last statement is a fundamental result in Calculus and can be proved in different ways: roughly speaking, we can either

- find a monotone subsequence and use the *monotone convergence principle*, or
- find a nested sequence of closed intervals with length shrinking to 0, such that each interval contains infinitely many members of the sequence, then use the *principle of nested intervals*.

The case of general N is deduced from the $N=1$ case as follows. Let $(x_n)_{n=1,2,\cdots}$ be a sequence in $\mathbb{R}^N$ contained in some $[-a,a]^N$, $a>0$. Since the sequence of 1st coordinates (x_n^1) is bounded in $\mathbb{R}$, it has a convergent subsequence by the $N=1$ case. We may then take the corresponding subsequence of (x_n), consider its 2nd coordinates and obtain a further subsequence of (x_n) whose 2nd coordinate also converges. Doing this step by step for each of the N coordinates, we get a subsequence of (x_n) converging to some $x \in \mathbb{R}^N$ □

The following classical lemma is one of the most useful consequences of compactness:

Lemma 4 (**Extreme Value Theorem**). *Let (X, d) be a metric space, $f : X \to \mathbb{R}$ be a continuous function and $E \subset X$ be a compact subset. Then f attains its maximum and minimum on E, namely, there exists $x_{max}, x_{min} \in E$ such that*

$$f(x_{max}) = \sup_{x\in E} f(x), \quad f(x_{min}) = \inf_{x\in E} f(x).$$

As a consequence, we have $-\infty < \inf_{x\in E} f(x) \le \sup_{x\in E} f(x) < +\infty$. If moreover f is positive, then we also have $\inf_{x\in E} f(x) > 0$.

Proof. By definition of supremum, there exists a sequence $(x_n)_{n=1,2,\cdots}$ in E such that $\lim_{n\to\infty} f(x_n) = \sup_{x\in E} f(x)$. Since E is compact, we may pick a subsequence $(x_{n_i})_{i=1,2,\cdots}$ which converges to some $x_{\max} \in E$. By continuity of f, we have

$$f(x_{\max}) = \lim_{i\to\infty} f(x_{n_i}) = \sup_{x\in E} f(x),$$

as required. The proof for $x_{\min}$ is the same. □

Theorem 1 (**Equivalence of norms on** $\mathbb{R}^N$). *For any two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on the vector space $\mathbb{R}^N$ $(N \in \mathbb{Z}_+)$, it holds that*

$$\exists C > 1 \ \text{ such that } \ C^{-1}\|x\|_1 \le \|x\|_2 \le C\|x\|_1 \ \text{ for any } x \in \mathbb{R}^N. \tag{1.4}$$

Proof. Call two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ equivalent if (1.4) holds. It is easy to see that if $\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$ and $\|\cdot\|_2$ is equivalent to $\|\cdot\|_3$, then $\|\cdot\|_1$ and $\|\cdot\|_3$ are equivalent. Therefore, we only need to show that any norm $\|\cdot\|_2$ on $\mathbb{R}^N$ is equivalent to a fixed norm, say $\|\cdot\|_1 := \|\cdot\|_{\ell^\infty}$.

Pick any $\|\cdot\|_2$. Let us first show that the function $\mathbb{R}^N \to \mathbb{R}$, $x \mapsto \|x\|_2$ is continuous w.r.t $\|\cdot\|_{\ell^\infty}$. Consider the standard basis $(e_1, \cdots, e_N)$ of $\mathbb{R}^N$, where

$$e_1 := (1, 0, \cdots, 0), \ \ e_2 := (0, 1, 0, \cdots, 0), \cdots, e_N := (0, 0, \cdots, 0, 1).$$

Then we have

$$\|x\|_2 = \left\|x^1 e_1 + \cdots + x^N e_N\right\|_2 \le \left\|x^1 e_1\right\|_2 + \cdots + \left\|x^N e_N\right\|_2 = |x^1| \cdot \|e_1\|_2 + \cdots + |x^N| \cdot \|e_N\|_2.$$

This easily implies that $x \mapsto \|x\|_2$ is continuous at $x = 0$. The continuity at an arbitrary point $x_0 \in \mathbb{R}^N$ essentially follows from the continuity at 0 because of the inequality $\big|\|x\|_2 - \|x_0\|_2\big| \le \|x - x_0\|_2$ (see Exercise 5).

We proceed to show the required condition (1.4) with $\|\cdot\|_1 = \|\cdot\|_{\ell^\infty}$. Dividing by $\|x\|_{\ell^\infty}$ and using $\|ax\|_2 = |a| \cdot \|x\|_2$, we see that (1.4) is equivalent to

$$\exists C > 1 \ \text{ such that } \ C^{-1} \le \|x\|_2 \le C \ \text{ for any } x \in \mathbb{R}^N \text{ with } \|x\|_{\ell^\infty} = 1,$$

which just means that the function $x \mapsto \|x\|_2$, restricted to the set $S := \{x \in \mathbb{R}^N \,|\, \|x\|_{\ell^\infty} = 1\}$, has positive lower and upper bounds. But this follows from Lemma 4, because S is compact by Lemma 3 and $x \mapsto \|x\|_2$ is continuous. □

Corollary 1. *For all norms on $\mathbb{R}^N$, the induced metrics define the same system of open sets, hence the same notion of limit, continuity, compactness, etc. In particular, with respect to such a norm,*

- *a sequence $(x_n)_{n=1,2,\cdots}$ in $\mathbb{R}^N$ converges to a point x iff. for each $i = 1, \cdots, N$, the ith coordinate (x_n^i) converges to x^i in $\mathbb{R}$;*
- *a subset $E \subset \mathbb{R}^N$ is compact iff. it is bounded and closed.*

Proof. Since any two norms $\|\cdot\|_1, \|\cdot\|_2$ on $\mathbb{R}^N$ satisfy (1.4) by Theorem 1, their induced metrics d_1, d_2 satisfy (1.1). By Lemma 2, they define the same open sets. We have already seen that under the norm $\|\cdot\|_{\ell^\infty}$, the convergence $x_n \to x$ is characterized as the coordinate-wise convergence $x_n^i \to x^i$, $i = 1, \cdots, N$, while compact sets are characterized as bounded closed sets (Lemma 3). It follows that same characterizations holds for any other norm. □

For infinite dimensional vector spaces, the situation is completely different, see Example 9 below. The study of infinite dimensional normed vector spaces, notably spaces of various types of functions, as well as continuous linear maps between them, is the main theme of the theory of Functional Analysis. Exercises 7 and 15 give some taste of this theory.

Exercise 6. Given a vector space X, a function $\|\cdot\|: X \to \mathbb{R}$ is said to be a semi-norm if for any $x, y \in X$ and $a \in \mathbb{R}$,

- $\|x\| \geq 0$;
- $\|ax\| = |a| \cdot \|x\|$;
- $\|x+y\| \leq \|x\| + \|y\|$.

(The only difference with norm is that we do not require "$\|x\| = 0 \Rightarrow x = 0$", although "$\Leftarrow$" still holds by the 2nd property.)
Show that

(1) If X is finite dimensional, $\|\cdot\|$ is a norm on X and $\|\cdot\|'$ is a semi-norm, then there exists $C > 0$ such that $\|x\|' \leq C\|x\|$ for all $x \in X$. *Hint:* Re-examine the proof of Theorem 1.

(2) If X is a finite dimensional vector space and Y is any vector space, then any linear map $A: X \to Y$ is continuous w.r.t. any norms on X and Y. *Hint:* Show that if $\|\cdot\|_Y$ is a norm on Y, then $\|x\|' := \|Ax\|_Y$ is a semi-norm on X; then use Part (1) and Exercise 5 (3).

Exercise 7. Let X be a vector space endowed with a norm $\|\cdot\|$ and d be the induced metric. Prove the following statements.

(1) Any finite dimensional linear subspace $Y \subset X$ is closed[3].
Hint. First prove the case where X is finite dimensional. Then, in the general case, assuming that $(x_n)_{n=1,2,\cdots}$ is a sequence in Y converging to $x \notin Y$, we may restrict to the linear subspace spanned by Y and x.

(2) (Riesz's Lemma) For any closed linear subspace $Y \subset X$ and any $\lambda < 1$, there exists $v \in X$ with $\|v\| = 1$ such that $d(v, Y) \geq \lambda$ (we are using the notion of distance to a set defined in Exercise 4).
Hint. Pick any $x \in X \setminus Y$. Since Y is closed, by Exercise 4 (4) we have $a := d(x, Y) > 0$, while by the definition of $d(x, Y)$ as an infimum, there exists $y \in Y$ such that $a \leq d(x, y) < \lambda^{-1}a$. Show that $v := \frac{x-y}{\|x-y\|}$ fulfills the requirement.

(3) If X is infinite dimensional, then the unit closed ball $\overline{B}(0,1) \subset X$ is not compact.
Hint. Use (1) and (2) to construct a sequence $(v_n)_{n=1,2,\cdots}$ of unit vectors in X such that every v_{n+1} is at distance at least λ away from the linear subspace spanned by $\{v_1, \cdots, v_n\}$, and in particular $d(v_i, v_{n+1}) \geq \lambda$ for $i = 1, \cdots, n$.

1.3. Topologies and bases.

Definition 3 (Topology). *A topology on a set X is a collection $\mathscr{U}$ of subsets of X such that*

- *the empty subset $\varnothing$ and the whole X both belong to $\mathscr{U}$;*
- *a union of elements in $\mathscr{U}$ is still in $\mathscr{U}$;*
- *the intersection of two elements in $\mathscr{U}$ is still in $\mathscr{U}$.*

The pair $(X, \mathscr{U})$ is called a topological space. We also call X a topological space if it is understood as equipped with a topology.

Example 4. The simplest examples of topologies are the trivial topology, formed only by $\varnothing$ and X, and the discrete topology, formed by all subsets of X:

$$\mathscr{U}_{\text{trivial}} := \{\varnothing, X\}, \quad \mathscr{U}_{\text{discrete}} := \{U \mid U \subset X\}.$$

Furthermore, Def. 2 implies that a metric d on X induces a topology whose elements are unions of open balls. This is called the metric topology. $\mathscr{U}_{\text{discrete}}$ is also the metric topology given by the discrete metric (Example 2). The standard topology on $\mathbb{R}^N$ is the metric topology induced by any norm, which does not depend on the choice of the norm by Corollary 1. In what follows, metric spaces (including normed vector spaces) and $\mathbb{R}^N$ are always equipped with these topologies unless otherwise specified.

[3]Exercise 15 below give examples of infinite dimensional linear subspaces $Y \subsetneq X$ which are dense. Such a Y is not closed.

Equipping a set with a topology should be understood as prescribing which sets are open. Similarly as in Def. 2, this induces other notions such as limit, closure, compactness, etc., which we restate in Def. 4 below along with a few extra definitions (again, the purple lines are statements that we will use frequently and are easy to prove). However, be ware that in a topological space which is not metric, sequential limit might behave badly, namely

- Limit might not be unique: a sequence (x_n) can converge to two different points $x, y \in X$ at the same time. Exercises 8 and 17 below contain examples. So we speak of "a limit" or "any limit" of a sequence, rather than "the limit".
- Closedness, closure and continuity cannot be characterized in terms of sequential limit. More precisely, for the "$\Rightarrow$" and "$\Leftarrow$" in Def. 4 below, the converse is not true. Exercise 9 gives an example. Nevertheless, we will show in §1.8 that a simple extra assumption fixes this issue.
- Thus, the definition of compactness in Def. 2 (8) based on sequential limits is not quite handy. We rename it as *sequential compactness* and will introduce a more convenient notion of compactness in §1.7. The two notions are the same for metric spaces (see §1.8), so statements involving compactness in §1.1 and §1.2 are still true. However, for a general topology space, neither of them imply the other.

Definition 4 (**Basic notions associated to a topology**). *In a topological space $(X, \mathscr{U})$,*

(1) *An open set is a subset of X belonging to $\mathscr{U}$; a neighborhood of a point x is an open set containing x; a sequence $(x_n)_{n=1,2,\cdots}$ is said to converge to a point x if for any neighborhood U of x there is $N > 0$ such that $x_n \in U$ when $n > N$ (denote this by $x_n \to x$ and call x a limit of the sequence); a closed set is a subset whose complement is open.*
- *$\varnothing$ and X are both open and closed;*
- *an intersection of closed sets is closed; a union of finitely many closed sets is closed;*
- *$C \subset X$ is closed $\Rightarrow$ any limit of any convergent sequence in C is again in C.*

(2) *Given $E \subset X$, the closure $\overline{E}$ is the intersection of all closed sets containing E. E is said to be dense if $\overline{E} = X$.*
- *$x \in \overline{E}$ $\Leftrightarrow$ any neighborhood of x intersects E $\Leftarrow$ there exists a sequence $(x_n)_{n=1,2,\cdots}$ in E s.t. $x_n \to x$;*
- *E is dense $\Leftrightarrow$ every nonempty open set intersects E.*

(3) *Given a point $x \in X$, a map f from $(X, \mathscr{U})$ to another topological space $(Y, \mathscr{V})$ is continuous at x if for any neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subset V$. If f is continuous at every point of X, it is simply said to be continuous. If f is bijective, continuous and the inverse f^{-1} is continuous as well, then f is called a homeomorphism. Two topological spaces are said to be homeomorphic if there exists a homeomorphism between them.*
- *f is continuous at x $\Rightarrow$ for any sequence $(x_n)_{n=1,2,\cdots}$ in X converging to x, we have $f(x_n) \to f(x)$ in Y.*
- *f is continuous $\Leftrightarrow$ for any open set V in Y, the preimage $f^{-1}(V)$ is open in X*
  $\Leftrightarrow$ for any closed set C in Y, the preimage $f^{-1}(C)$ is closed in X.
- *any constant map is continuous; the composition of two continuous maps is continuous.*

(4) *A set $E \subset X$ is sequentially compact if any sequence in E has a subsequence converging to a point in E.*

Similarly as for metrics, we say something like "U is open/(x_n) converges to x w.r.t. $\mathscr{U}$ (or under $\mathscr{U}$)" when we want to emphasize which topology is being used.

Example 5 (**Finite complement topology**). On any set X, the finite complement topology $\mathscr{U}_{\text{f.c.}}$ (also known as cofinite topology) is defined in such a way that its nonempty open sets are exactly the complements of finite sets, namely

$$\mathscr{U}_{\text{f.c.}} := \{X \setminus F \,\big|\, F \subset X \text{ is finite}\} \cup \{\varnothing\}.$$

Equivalently, the closed sets under $\mathscr{U}_{\text{f.c.}}$ are finite sets or the whole X. When $X = \mathbb{C}$ (or any algebraically closed field), $\mathscr{U}_{\text{f.c.}}$ is a particular instance of Zariski topology on affine algebraic varieties. This topology takes an instrumental role in

Algebraic Geometry, and is by far the most important topology under which sequential limit behaves badly (or in the terminology from §1.8: the most important topology which is not *Hausdorff* or not *first countable*).

Exercise 8. Let X be a set, $x \in X$ be a point and $(x_n)_{n=1,2,\cdots}$ be a sequence in X. Show that

(1) Under the discrete topology on X, we have $x_n \to x$ iff. $x_n = x$ for all sufficiently large n.

(2) Under the trivial topology on X, we always have $x_n \to x$. In other words: any sequence converges to any point.

(3) Under the finite complement topology on X, the sequence (x_n) is convergent iff. there exists at most one $y \in X$ such that $x_n = y$ for infinitely many n's. Moreover,
   - if there is no such y, then $x_n \to x$ for any $x \in X$;
   - if there is one such y, then $x_n \to y$.

Exercise 9. On any set X, define the countable complement topology as

$$\mathscr{U}_{\text{c.c.}} := \{X \setminus C \,\big|\, C \subset X \text{ is countable}\} \cup \{\varnothing\}.$$

Check that $\mathscr{U}_{\text{c.c.}}$ is indeed a topology and show that convergent sequences under $\mathscr{U}_{\text{c.c.}}$ are exactly the same as under $\mathscr{U}_{\text{discrete}}$ (see Exercise 8 (1)). Use this to give counterexamples for the converse of the "⇒"s and "⇐" in Def. 4.

Exercise 10. Show that if a topological space X contains a countable dense subset, then any disjoint collection of open subsets of X has at most countably many elements. Also show that the converse does not hold when X is uncountable with the countable complement topology (Exercise 9).

Exercise 11. A map between topological spaces $f : X \to Y$ is said to be open (resp. closed) if f sends every open (resp. closed) set in X to an open (resp. closed) set in Y. If f is a bijection, then being open is equivalent to being closed (in this case, if furthermore f is continuous, then it is a homeomorphism), but there is no such equivalence for general f.
Find examples of continuous maps $f : \mathbb{R} \to \mathbb{R}$ fulfilling each of the following requirements:

(1) Closed but not open.

(2) Open but not closed.

(3) Both open and closed.

(4) Neither open nor closed.

Definition 5 (**Topological basis**)**.** *Given a set X, a collection $\mathscr{B}$ of nonempty subsets of X is called a topological basis if*
- $\bigcup_{B \in \mathscr{B}} B = X$ *($\Leftrightarrow$ any $x \in X$ is contained in some $B \in \mathscr{B}$);*
- *The intersection of any $B_1, B_2 \in \mathscr{B}$ is a union of elements in $\mathscr{B}$*
  ($\Leftrightarrow$ for any $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$, there exists $B \in \mathscr{B}$ s.t. $x \in B \subset B_1 \cap B_2$).

Given such a $\mathscr{B}$, the topology $\mathscr{U}$ generated by $\mathscr{B}$ consists, by definition, of all subsets of X which are unions of elements in $\mathscr{B}$ (in other words: $U \in \mathscr{U} \Leftrightarrow \forall x \in U, \exists B \in \mathscr{B}$ s.t. $x \in B \subset U$). **Exercise.** Check that $\mathscr{U}$ is indeed a topology.

The prototype example of a topological basis is the set of open balls in a metric space. It generates the metric topology.

As an obvious and frequently used fact, two topological bases $\mathscr{B}$ and $\mathscr{B}'$ generate the same topology iff. the following conditions are both satisfied:

(i) every $B \in \mathscr{B}$ is a union of elements of $\mathscr{B}'$ ($\Leftrightarrow$ for any $B \in \mathscr{B}$ and $x \in B$, there exists $B' \in \mathscr{B}'$ s.t. $x \in B' \subset B$),

(ii) every $B' \in \mathscr{B}'$ is a union of elements of $\mathscr{B}$ ($\Leftrightarrow$ for any $B' \in \mathscr{B}'$ and $x \in B'$, there exists $B \in \mathscr{B}$ s.t. $x \in B \subset B'$).

This will often be applied to the case $\mathscr{B}' \subset \mathscr{B}$ (we may call $\mathscr{B}'$ a sub-basis of $\mathscr{B}$ in this case), where (ii) is trivial and one only needs to check (i).

Definition 6 (**Neighborhood basis**). *Given a topological space $(X,\mathscr{U})$, a point $x \in X$ and a set $\mathscr{N} \subset \mathscr{U}$ of neighborhoods of x, we call $\mathscr{N}$ a neighborhood basis of x if for any neighborhood U of x, there exists $U' \in \mathscr{N}$ such that $U' \subset U$.*

Example 6 (**Prototype examples of neighborhood bases**).

- If $\mathscr{U}$ is generated by a basis $\mathscr{B}$, then all elements of $\mathscr{B}$ containing x form a neighborhood basis of x.
- In a metric space, all open balls centered at a point x, namely $\big(B(x,r)\big)_{r>0}$, are a neighborhood basis of x. Furthermore, for any sequence of positive numbers $(r_n)_{n=1,2,\cdots}$ converging to 0, the sequence of balls $\big(B(x,r_n)\big)_{n=1,2,\cdots}$ is still a neighborhood basis of x.

A simple fact which we will use freely is that in Def. 4, for everything concerned with "any neighborhood of x", it does not matter if we change it into "any neighborhood from a neighborhood basis of x". More precisely, let $\mathscr{N}_x$ be a neighborhood basis of $x \in X$, then

- a sequence $(x_n)_{n=1,2,\cdots}$ converges to $x \overset{\text{def.}}{\Leftrightarrow}$ for any neighborhood U of x, $\exists N > 0$ s.t. $x_n \in U$ when $n > N$.
  $\Leftrightarrow$ for any $U \in \mathscr{N}_x$, $\exists N > 0$ s.t. $x_n \in U$ when $n > N$.
- x is contained in the closure $\overline{E}$ of $E \Leftrightarrow$ any neighborhood of x intersects E
  $\Leftrightarrow$ any $U \in \mathscr{N}_x$ intersects E.

Similarly, if $f : X \to Y$ is a map and $\mathscr{N}_{f(x)}$ is a neighborhood basis of $f(x)$ in Y, then

- f is continuous at $x \overset{\text{def.}}{\Leftrightarrow} \forall$neighborhood V of $f(x)$, $\exists$ neighborhood U of x s.t. $f(U) \subset V$
  $\Leftrightarrow \forall V \in \mathscr{N}_{f(x)}$, $\exists$ neighborhood U of x s.t. $f(U) \subset V$

Example 7 (**Order topology**). A total order on a set X is a partial order (namely a binary relation $\le$ which is reflexive ($a \le a$), transitive ($a \le b,\ b \le c \Rightarrow a \le c$) and antisymmetric ($a \le b,\ b \le a \Rightarrow a = b$)) such that any two elements are comparable. Given a total order $\le$ on X, all open intervals, namely subsets of the forms

$$(a,b) := \{x \in X \,|\, a < x < b\}, \text{ and } (a,+\infty) := \{x \in X \,|\, a < x\}, \text{ and } (-\infty,b) := \{x \in X \,|\, x < b\}, \quad a,b \in X$$

(where "$x < y$" means "$x \le y$ and $x \ne y$"), form a topological basis. The topology generated is called the order topology. For $X = \mathbb{R}$ with the usual order, the order topology coincides with the standard topology. In general, totally ordered sets can be quite complicated and provide somewhat strange topological spaces. For example, the lexicographic order on $\mathbb{R}\times\mathbb{R}$ (i.e. the order $\le$ such that $(a_1,a_2) < (b_1,b_2)$ iff. either $a_1 < b_1$, or $a_1 = b_1$ and $a_2 < b_2$) yields a topological space which locally looks like $\mathbb{R}$ but is not homeomorphic to $\mathbb{R}$ because one can find in it uncountably many open subsets disjoint from each other, which is impossible in $\mathbb{R}$ (see Exercise 10).

Example 8 (**A topology on $\mathbb{Z}$ that proves the the infinitude of primes**). In $X = \mathbb{Z}$, the arithmetic sequences (i.e. subsets of the form $a\mathbb{Z} + b$, $a > 0$, $b \in \mathbb{Z}$) form a topological basis. Let $\mathscr{A}$ denote the topology generated by it. This topology was devised by H. Furstenberg to give a new proof of the fact that *there are infinitely many primes*.

Furstenberg's proof. $\mathscr{A}$ has the following properties:

(a) Any nonempty open set is infinite. Equivalently, any closed set other than the whole $\mathbb{Z}$ has infinite complement.

(b) Any arithmetic sequence is also closed (since its complement is a union of arithmetic sequences).

Now, note that we can write

$$\mathbb{Z} \setminus \{\pm 1\} = \bigcup_{p:\text{ primes}} p\mathbb{Z}$$

(because any integer other than ± 1 is a multiple of some prime). If there were only finitely many primes, the right-hand side would be closed by property (b). But the left-hand side is not closed by (a), a contradiction. □

Essentially, the proof deduces the required fact from properties of arithmetic sequence. It is an interesting coincidence that the argument can be written down cleanly and nicely in the language of topology.

Exercise 12. Consider the discrete topology $\mathscr{U}_{\mathrm{discrete}}$ on a set X and put $\mathscr{B}_{\mathrm{pt}} := \{\text{one-point subsets of } X\} \subset \mathscr{U}_{\mathrm{discrete}}$. Show that any $\mathscr{B}$ satisfying $\mathscr{B}_{\mathrm{pt}} \subset \mathscr{B} \subset \mathscr{U}_{\mathrm{discrete}}$ is a topological basis generating $\mathscr{U}_{\mathrm{discrete}}$, and these are all the topological bases generating $\mathscr{U}_{\mathrm{discrete}}$.

Exercise 13. Let (X,d) be a metric space, $E \subset X$ be a dense subset, and pick a sequence $(r_n(x))_{n=1,2,\cdots}$ of positive numbers converging to 0 for every $x \in X$. By definition, the metric topology is generated by the basis

$$\mathscr{B} := \{B(x,r) \,|\, x \in X, r > 0\}.$$

Show that the following subsets of $\mathscr{B}$ are both topological bases and generate the metric topology as well:

$$\mathscr{B}' := \{B(x,r) \,|\, x \in E, r > 0\}, \quad \mathscr{B}'' := \{B(x,r_n(x)) \,|\, x \in X, n \in \mathbb{Z}_+\}.$$

Hint. For any $y \in B(x,\frac{r}{3})$, we have $B(x,\frac{r}{3}) \subset B(y,\frac{2r}{3}) \subset B(x,r)$.

Exercise 14. Given a norm $\|\cdot\|$ on $\mathbb{R}^N$, consider the following unusual "post office metric" on $\mathbb{R}^N$:

$$d(x,y) := \begin{cases} \|x\| + \|y\| & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Show that d is indeed a metric, and that a set $E \subset \mathbb{R}^N$ is open w.r.t the metric topology of d iff. either E does not contain 0 or E contains a neighborhood of 0 w.r.t the standard topology.

1.4. Comparison between topologies.

Definition 7 (**Comparion and pullback**).

(1) *If two topologies $\mathscr{U}_1$ and $\mathscr{U}_2$ on a set X satisfy $\mathscr{U}_1 \subset \mathscr{U}_2$, namely every open set under $\mathscr{U}_1$ is also open under $\mathscr{U}_2$, then we say that "$\mathscr{U}_1$ is weaker than $\mathscr{U}_2$" or "$\mathscr{U}_2$ is stronger than $\mathscr{U}_2$".*

(2) *Given a map $f: X \to Y$ and a topology $\mathscr{V}$ on Y, define the pullback topology as $f^{-1}(\mathscr{V}) := \{f^{-1}(V) \,|\, V \in \mathscr{V}\}$.*

It follows immediately from the definitions that continuity of maps can be interpreted as a comparison property between topologies:

Proposition 1. *Given two topological spaces $(X,\mathscr{U})$ and $(Y,\mathscr{V})$, a map $f: X \to Y$ is continuous iff. $\mathscr{U}$ is stronger than $f^{-1}(\mathscr{V})$. In particular, $\mathscr{U}$ is stronger than another topology $\mathscr{U}'$ on X iff. the identity map $(X,\mathscr{U}) \to (X,\mathscr{U}')$ is continuous.*

The following basic facts are also obvious:

- On a set X, the trivial and discrete topologies are the weakest and strongest topologies , respectively.
- If a sequence $(x_n)_{n=1,2,\cdots}$ converges to a point x w.r.t. a topology $\mathscr{U}$, then the convergence $x_n \to x$ also holds w.r.t. any topology weaker than $\mathscr{U}$.
- If a map $f: X \to Y$ is continuous w.r.t a topology $\mathscr{U}$ on X and a topology $\mathscr{V}$ on Y, then f is also continuous w.r.t. any topology on X stronger than $\mathscr{U}$ and any topology on Y weaker than $\mathscr{V}$.
- By Lemma 2, if two metrics d_1 and d_2 on a set X satisfies $d_1(x,y) \le C d_2(x,y)$ for all $x,y \in X$, then the metric topology given by d_1 is weaker than the one given by d_2. In particular, this happens when X is a vector space and d_1, d_2 are induced by norms $\|\cdot\|_1, \|\cdot\|_2$ such that $\|x\|_1 \le C\|x\|_2$ for all $x \in X$. Let us say that "$\|\cdot\|_1$ is weaker than $\|\cdot\|_2$" in this case.

Example 9 (**ℓ^p-spaces**)**.** The most basic example of infinite dimensional normed vector spaces, the ℓ^p-space, is defined as follows. Given $p \in [1,+\infty)$, we first generalize the definition of the norm $\|\cdot\|_{\ell^p}$ in Example 1 from $\mathbb{R}^N$ to the infinite dimensional vector space $\mathbb{R}^{\mathbb{Z}_+}$ by setting

$$\|x\|_{\ell^p} := \left(\sum_{i=1}^{\infty} \left|x^i\right|^p\right)^{\frac{1}{p}} \in \mathbb{R} \cup \{+\infty\} \text{ for } x = (x^1, x^2, \cdots) \in \mathbb{R}^{\mathbb{Z}_+}.$$

Now $\|\cdot\|_{\ell^p}$ can take the value $+\infty$ and hence is no longer a norm on $\mathbb{R}^{\mathbb{Z}_+}$, but it is indeed a norm on the linear subspace

$$\ell^p := \left\{x \in \mathbb{R}^{\mathbb{Z}_+} \,\middle|\, \|x\|_{\ell^p} < +\infty\right\}.$$

Therefore, $(\ell^p, \|\cdot\|_{\ell^p})$ is a normed vector space and is called the ℓ^p-space. Like in the finite dimensional case, we may extend the definition to $p = +\infty$ by setting

$$\|x\|_{\ell^\infty} := \sup\left\{|x^i| \,\middle|\, i = 1, 2, \cdots\right\} \text{ for } x = (x^1, x^2, \cdots) \in \mathbb{R}^{\mathbb{Z}_+}, \quad \ell^\infty := \left\{x \in \mathbb{R}^{\mathbb{Z}_+} \,\middle|\, \|x\|_{\ell^\infty} < +\infty\right\}.$$

Then $\|\cdot\|_p$, $p \in [0,+\infty]$ is a family of norms which become weaker and weaker, in the sense that if $1 \le p_1 < p_2 \le +\infty$, then the space ℓ^{p_1} is strictly contained in ℓ^{p_2}, while the norm $\|\cdot\|_{\ell^{p_1}}$ is weaker than the restriction of $\|\cdot\|_{\ell^{p_2}}$ to ℓ^{p_1}.

It is elementary but nontrivial to prove that $\|\cdot\|_p$ satisfies Triangle Inequality and that it becomes weaker and weaker as p increases. The most common proof is to deduce them from the Hölder inequality

$$\|x \cdot y\|_{\ell^1} \le \|x\|_{\ell^p} \|x\|_{\ell^q} \text{ for } x, y \in \mathbb{R}^{\mathbb{Z}_+} \text{ and } p, q \in [1,+\infty] \text{ satisfying } \frac{1}{p} + \frac{1}{q} = 1,$$

where $x \cdot y := (x^1 \cdot y^1, x^2 \cdot y^2, \cdots)$. All these heavily rely on the assumption $p \ge 1$, and fails if $p < 1$.

Exercise 15. We say that "A is *strictly* weaker than B" if A is weaker than B but B is not weaker than A. Let X be a vector space, $\|\cdot\|_1, \|\cdot\|_2$ be norms on X, and $\mathscr{U}_1, \mathscr{U}_2$ be the metric topologies on X given by these norms. Show that

(1) The following conditions are equivalent:
   (i) $\mathscr{U}_1$ is strictly weaker than $\mathscr{U}_2$.
   (ii) $\|\cdot\|_1$ is strictly weaker than $\|\cdot\|_2$.
   (iii) There exists a sequence $(x_n)_{n=1,2,\cdots}$ in X which is bounded w.r.t. $\|\cdot\|_1$ but not bounded w.r.t. $\|\cdot\|_2$.
   (iv) There exists a sequence $(x_n)_{n=1,2,\cdots}$ in X which converges to 0 w.r.t. $\mathscr{U}_1$ but does not converge to 0 w.r.t. $\mathscr{U}_2$.

(2) When these conditions are satisfied, the sequence (x_n) of (iv) does not have any convergent subsequence w.r.t. $\mathscr{U}_2$.

(3) If $1 \le p_1 < p_2 \le +\infty$, then $\|\cdot\|_{\ell^{p_1}}$ is strictly weaker than the restriction of $\|\cdot\|_{\ell^{p_2}}$ to ℓ^{p_1}.
   Hint. Find a sequence satisfying (iii) from the family of points $x_q := (1^{-\frac{1}{q}}, 2^{-\frac{1}{q}}, 3^{-\frac{1}{q}}, \cdots) \in \mathbb{R}^{\mathbb{Z}_+}$, $q \ge 1$.

(4) In every ℓ^p, the following linear subspace is dense:

$$L := \left\{x = (x^1, x^2, \cdots) \in \mathbb{R}^{\mathbb{Z}_+} \,\middle|\, \text{only finitely many } x^i\text{s are nonzero}\right\}.$$

(5) If $1 \le p_1 < p_2 \le +\infty$, then ℓ^{p_1} is dense in ℓ^{p_2} (w.r.t. $\|\cdot\|_{\ell^{p_2}}$) *Hint.* Use (4).

Example 10 (**Toplogies on $\mathbb{R}$ related to left/right continuity and upper/lower semi-continuity**)**.** Consider the following 4 topologies on $\mathbb{R}$:

- $\mathscr{U}_{\text{left}} :=$ the topology generated by the basis $\mathscr{B}_{\text{left}} := \left\{(a,b] \,\middle|\, a,b \in \mathbb{R}, a < b\right\}$,
- $\mathscr{U}_{\text{right}} :=$ the topology generated by the basis $\mathscr{B}_{\text{right}} := \left\{[a,b) \,\middle|\, a,b \in \mathbb{R}, a < b\right\}$[4],
- $\mathscr{U}_{\text{lower}} := \left\{(a,+\infty) \,\middle|\, a \in \mathbb{R}\right\} \cup \{\varnothing, \mathbb{R}\}$,
- $\mathscr{U}_{\text{upper}} := \left\{(-\infty,a) \,\middle|\, a \in \mathbb{R}\right\} \cup \{\varnothing, \mathbb{R}\}$.

[4] In the literature, $\mathscr{U}_{\text{right}}$ is usually called the *lower limit topology* (because a sequence $(x_n)_{n=1,2,\cdots}$ converges to a point x under this topology iff. $x_n \to x^+$, namely x is the "lower limit" of (x_n), see Exercise 17 (4) below), while the topological space $(\mathbb{R}, \mathscr{U}_{\text{right}})$ is usually denoted by $\mathbb{R}_\ell$ and called the *Sorgenfrey line*. However, note that it has nothing to do with what we denote by $\mathscr{U}_{\text{lower}}$.

They are related to the classical notions of left/right continuity and upper/lower semi-continuity (see definition below) as follows: let $\mathscr{U}$ denote the standard topology of $\mathbb{R}$, then

- $f:(\mathbb{R},\mathscr{U}_{\text{left}})\to(\mathbb{R},\mathscr{U})$ is continuous at $x \Leftrightarrow f$ is left continuous at x.
- $f:(\mathbb{R},\mathscr{U}_{\text{right}})\to(\mathbb{R},\mathscr{U})$ is continuous at $x \Leftrightarrow f$ is right continuous at x.
- $f:(\mathbb{R},\mathscr{U})\to(\mathbb{R},\mathscr{U}_{\text{lower}})$ is continuous at $x\Leftrightarrow f$ is lower semi-continuous at x.
- $f:(\mathbb{R},\mathscr{U})\to(\mathbb{R},\mathscr{U}_{\text{upper}})$ is continuous at $x \Leftrightarrow f$ is upper semi-continuous at x at x.

Details are given in the next exercise. If we view these as the definition for each notion of continuity and replace $(\mathbb{R},\mathscr{U})$ by an arbitrary topological space X, we obtain the definition of left/right continuity for maps $\mathbb{R}\to X$, as well as the definition of lower/upper semi-continuity for functions $X\to\mathbb{R}$. Also, we can actually generalize $\mathbb{R}$ to any totally ordered set, since the definition of the 4 topologies generalizes to this case.

Reminder of definitions from Calculus. Given a sequence $(x_n)_{n=1,2,\cdots}$ in $\mathbb{R}$ and a point $x\in\mathbb{R}$, we write $x_n\to x^-$ (resp. $x_n\to x^+$) if we have $x_n\to x$ and furthermore $x_n\le x$ (resp. $x_n\ge x$) when n is large enough. Given a function $f:\mathbb{R}\to\mathbb{R}$ and a point $x\in\mathbb{R}$, f is said to be

- left continuous at x if for any sequence (x_n) with $x_n\to x^-$ we have $f(x_n)\to f(x)$.
- right continuous at x if for any sequence (x_n) with $x_n\to x^+$ we have $f(x_n)\to f(x)$.
- lower semi-continuous at x if for any $t<f(x)$ there is $\delta>0$ such that when y satisfies $|x-y|<\delta$ we have $f(y)>t$.
- upper semi-continuous at x if for any $t>f(x)$ there is $\delta>0$ such that when y satisfies $|x-y|<\delta$ we have $f(y)<t$.

See the figures below for examples of functions which are respectively left/right continuous and lower/upper semi-continuous at one point (and continuous everywhere else). Also recall the related notion of limit inferior/superior:

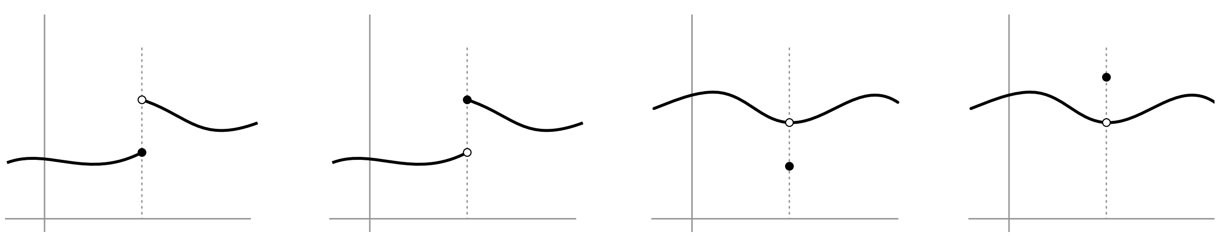
$$\varliminf_{n\to\infty} x_n := \lim_{n\to\infty} \underline{x}_n,\ \text{where } \underline{x}_n := \inf\{x_k \,|\, k\ge n\},$$

$$\varlimsup_{n\to\infty} x_n := \lim_{n\to\infty} \overline{x}_n,\ \text{where } \overline{x}_n := \sup\{x_k \,|\, k\ge n\}.$$

The following equivalences are useful:

$$\varliminf_{n\to\infty} x_n \ge x \ \Leftrightarrow\ \forall\epsilon>0,\ \exists N>0 \text{ such that } x_n > x-\epsilon \text{ when } n>N,$$

$$\varlimsup_{n\to\infty} x_n \ge x \ \Leftrightarrow\ \forall\epsilon>0,\ \exists N>0 \text{ such that } x_n < x+\epsilon \text{ when } n>N.$$



Exercise 16. Given two topologies $\mathscr{U}_1$ and $\mathscr{U}_2$ on a set X, show that

(1) $\mathscr{U}_1\cap\mathscr{U}_2$ is also a topology on X, and is the strongest topology which is weaker than both $\mathscr{U}_1$ and $\mathscr{U}_2$. Given another topological space $(Y,\mathscr{V})$, a map $f:X\to Y$ is continuous w.r.t. both $\mathscr{U}_1$ and $\mathscr{U}_2$ iff. it is continuous w.r.t. $\mathscr{U}_1\cap\mathscr{U}_2$.

(2) $\mathscr{B}:=\{U_1\cap U_2 \,|\, U_1\in\mathscr{U}_1, U_2\in\mathscr{U}_2\}$ is a topological basis on X. The topology generated by $\mathscr{B}$, denoted by $\mathscr{U}_1\vee\mathscr{U}_2$, is the weakest topology which is stronger than both $\mathscr{U}_1$ and $\mathscr{U}_2$. Given another topological space $(Y,\mathscr{V})$, a map $f:X\to Y$ is continuous w.r.t. both $\mathscr{U}_1$ and $\mathscr{U}_2$ iff. it is continuous w.r.t. $\mathscr{U}_1\cap\mathscr{U}_2$.

Exercise 17. Prove the followings. Here x is a point in $\mathbb{R}$ and $(x_n)_{n=1,2,\cdots}$ is a sequence .

(1) $\mathscr{B}_{\text{left}}$ and $\mathscr{B}_{\text{right}}$ are indeed topological basis on $\mathbb{R}$; $\mathscr{U}_{\text{lower}}$ and $\mathscr{U}_{\text{upper}}$ are indeed topologies.

(2) $\mathscr{U}_{\text{left}}$ and $\mathscr{U}_{\text{right}}$ are both stronger than the standard topology $\mathscr{U}$ on $\mathbb{R}$, and we have $\mathscr{U} = \mathscr{U}_{\text{left}} \cap \mathscr{U}_{\text{right}}$.
(cf. Exercise 16 (1).)

(3) $\mathscr{U}_{\text{lower}}$ and $\mathscr{U}_{\text{upper}}$ are both weaker than the standard topology $\mathscr{U}$ on $\mathbb{R}$, and we have $\mathscr{U} = \mathscr{U}_{\text{lower}} \vee \mathscr{U}_{\text{upper}}$.
(cf. Exercise 16 (2).)

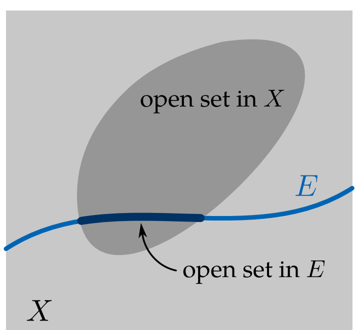
(4) $x_n \to x$ under the topology $\mathscr{U}_{\text{left}}$ (resp. $\mathscr{U}_{\text{right}}$) iff. $x_n \to x^-$ (resp. $x_n \to x^+$).

(5) $x_n \to x$ under the topology $\mathscr{U}_{\text{lower}}$ (resp. $\mathscr{U}_{\text{upper}}$) iff. $\underline{\lim}_{n\to\infty} x_n \geq x$ (resp. $\overline{\lim}_{n\to\infty} x_n \leq x$).
In particular, limit is not unique under $\mathscr{U}_{\text{lower}}$ (resp. $\mathscr{U}_{\text{upper}}$): every value which is $\leq \underline{\lim}_{n\to\infty} x_n$ (resp. every value which is $\geq \overline{\lim}_{n\to\infty} x_n$) is a limit of (x_n) under $\mathscr{U}_{\text{lower}}$ (resp. $\mathscr{U}_{\text{upper}}$).

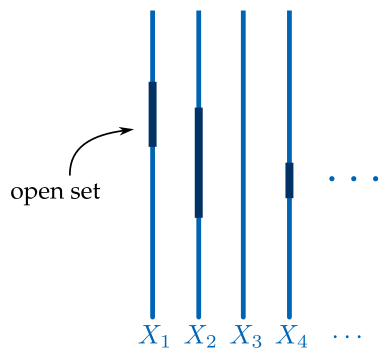
(6) The equivalences in Example 10.

(7) The following conditions are equivalent for any function $f : \mathbb{R} \to \mathbb{R}$:
- f is continuous at x;
- f is both left continuous and right continuous at x;
- f is both upper semi-continuous and lower semi-continuous at x.

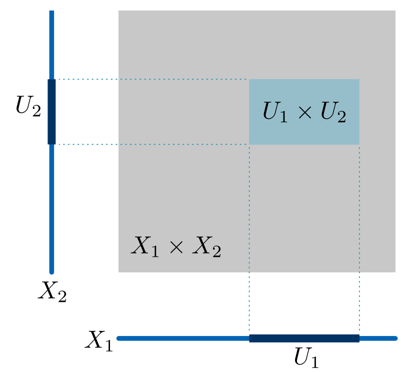
1.5. **Subspace, disjoint union and product.** One can construct new sets from old ones by taking subset, disjoint union, product or quotient. If the old sets are each equipped with a topology, one can also define a topology on the new set in a natural way. We discuss here the first 3 constructions. Their definitions are illustrated by the following figures.



open set under subspace topology



disjoint union space



basis element of product topology

Definition 8 (**Subspace**). *Given a topological space* $(X, \mathscr{U})$ *and a subset* $E \subset X$, *the subspace topology on* E *inherited from* X *is defined as*

$$\mathscr{U}|_E := \{U \cap E \,|\, U \in \mathscr{U}\}.$$

Namely, open subsets of E *under the subspace topology are intersections of open sets of* X *with* E. Check that $\mathscr{U}|_E$ is a topology.

Basic properties (prove them):

- Closed sets under the subspace topology are exactly subsets of E of the form $C \cap E$, where C is closed in X.
- An open (resp. closed) set in E under the subspace topology is not necessarily open (resp. closed) in X. For example, under the subspace topology on $[0,1) \cup (2,3] \subset \mathbb{R}$, the set $[0,1)$ is both open and closed, but it is neither open nor closed in $\mathbb{R}$. Nevertheless, if E itself is open (resp. closed) in X, then its open (resp. closed) sets under the subspace topology are also open (resp. closed) in X.
- If $\mathscr{U}$ is generated by a topological basis $\mathscr{B}$, then $\mathscr{B}|_E := \{B \cap E \,|\, B \in \mathscr{B}\}$ is a topological basis on E generating $\mathscr{U}|_E$.
- If $F \subset E \subset X$, then $(\mathscr{U}|_E)|_F = \mathscr{U}|_F$.
- The inclusion map $E \to X$ is continuous.
- Let $f : X \to Y$ be a continuous map between topological spaces. Then
  - For any $E \subset X$, the restricted map $f|_E : E \to Y$ is continuous.

– For any $F \subset Y$ containing the image $f(X)$, f is continuous when viewed as a map from X to F.
  This includes the case $F = f(X)$. In particular, if f is a homeomorphism from X to $f(X)$, it is called a homeomorphism to the image.

We always use the subspace topology to view subsets of $\mathbb{R}^N$, such as intervals in $\mathbb{R}$, the set of rational numbers $\mathbb{Q} \subset \mathbb{R}$, domains in $\mathbb{R}^2$, etc., as topological spaces.

We know that a metric d (resp. total order $\le$) on a set X induces a metric (resp. order) topology $\mathscr{U}$ on X. For any subset $Y \subset X$, d (resp. $\le$) naturally restricts to a metric $d|_Y$ (resp. total order $\le|_Y$) on Y, which in turn induces a metric (resp. order) topology on Y. It is natural to ask whether this topology coincides with the subspace topology $\mathscr{U}|_Y$. The answer is YES for metrics and NO for total orders in general (exercises below). For metrics, this in particular implies that on any linear subspace $\mathbb{R}^{N'} \cong V$ of $\mathbb{R}^N$, the subspace topology coincides with the standard topology.

Exercise 18. Let (X,d) be a metric space, $\mathscr{U}$ be its metric topology and $Y \subset X$ be a subset. Show that the subspace topology $\mathscr{U}|_Y$ on Y coincides with the metric topology given by the restricted metric $d|_Y$.
Hint: $\mathscr{U}|_Y$ is generated by $\mathscr{B}|_Y = \{B(x,r) \cap Y \mid x \in X, r > 0\}$, which is larger than the set of open balls under $d|_Y$; but by using Lemma 1, it can be shown that every element of $\mathscr{B}|_Y$ is a union of open balls under $d|_Y$.

Exercise 19. An interval in $\mathbb{R}$ is defined precisely as a subset $I \subset \mathbb{R}$ satisfying any of the following equivalent conditions:

(a) I has one of the following forms:

$$[a,b],\ (a,b),\ [a,b),\ (a,b],\ (a,+\infty),\ [a,+\infty),\ (-\infty,a),\ (-\infty,a],\ (-\infty,+\infty) \quad \text{where } a,b \in \mathbb{R},\ a \le b.$$

The last one is just the whole $\mathbb{R}$; when $a = b$ the 1st one is the one-point set $\{a\}$, while the 2nd and 3rd are empty; all these are considered as intervals.

(b) I contains $(\inf I, \sup I)$.

(c) I contains $[a,b]$ for any $a,b \in I$ with $a < b$.

Prove the following statements. Here the subspace topologies and total orders are inherited from $\mathbb{R}$.

(1) The subspace topology on any interval I in $\mathbb{R}$ coincides with its order topology.

(2) The subspace topology on $\mathbb{Q}$ coincides with its order topology.

(3) The subspace topology on $[0,1) \cup \{2\}$ does not coincides with its order topology.

Disjoint union and *product* are both constructions which produce a new topological space from a family of old ones. The former is quite simple:

Definition 9 (Disjoint union). *Given a family of topological spaces $(X_i, \mathscr{U}_i)_{i \in I}$, their disjoint union topological space is the disjoint union set $\bigsqcup_{i \in I} X_i$ equipped with the topology $\mathscr{U} = \{\bigcup_{i \in I} U_i \mid U_i \in \mathscr{U}_i, i \in I\}$* (in order words, its open sets are subsets of $\bigsqcup_{i \in I} X_i$ whose intersection with each X_i is open).

Note that every X_i is open in the disjoint union $\bigsqcup_{i \in I} X_i$, and its subspace topology inherited from $\bigsqcup_{i \in I} X_i$ is the same as its own topology. Therefore, we may understand the disjoint union as the construction of a new space by putting a family of old spaces together in such a way that every old one is an open subspace of the new one. Simple examples:

- A discrete topological space can be characterized as the disjoint union space of a family of one-point spaces.
- $[0,1] \cup [2,3]$ is the disjoint union space of $[0,1]$ and $[2,3]$.
- $[0,2]$ is not the disjoint union space of $[0,1]$ and $(1,2]$.

We proceed to discuss *finite* product. Given $n \ge 2$ sets[5] $X_1, \cdots, X_n$, recall that the Cartesian product $X_1 \times \cdots \times X_n$ is the set whose elements are n-tuples $(x_1, \cdots, x_n)$ with $x_i \in X_i$ $(i = 1, 2, \cdots, n)$. Given a subset A_i of each X_i, $A_1 \times \cdots \times A_n$ is a subset

[5] Although we treat here the product of n sets/spaces, the definitions and results below are already nontrivial when $n = 2$, while generalizing from $n = 2$ to all $n \ge 2$ is relatively easy. Therefore, we recommend the reader to first try to understand everything in the $n = 2$ case. On the other hand, everything actually also works for infinitely (even uncountably) many sets/spaces, but the notation will be heavier and there will be some subtlety when defining product topology. See [Mun00, §19] for details.

of $X_1 \times \cdots \times X_n$. The intersection of two subsets of this form is

$$(A_1 \times \cdots \times A_n) \cap (B_1 \times \cdots \times B_n) = (A_1 \cap B_1) \times \cdots \times (A_n \cap B_n). \tag{1.5}$$

If $(A^{(i)}_{j_i})_{j_i \in J_i}$ is a family of subsets of X_i, then we have

$$\left(\bigcup_{j_1 \in J_1} A^{(1)}_{j_1}\right) \times \left(\bigcup_{j_2 \in J_2} A^{(2)}_{j_2}\right) \times \cdots \times \left(\bigcup_{j_n \in J_n} A^{(n)}_{j_n}\right) = \bigcup_{j_1 \in J_1, \cdots, j_n \in J_n} A^{(1)}_{j_1} \times A^{(2)}_{j_2} \times \cdots \times A^{(n)}_{j_n}. \tag{1.6}$$

There is a natural projection map from the product to each factor X_i, defined as

$$\Pi_i : X_1 \times \cdots \times X_n \to X_i, \quad (x_1, \cdots, x_n) \to x_i.$$

On the other hand, given a point x_j in each X_j with $j \in \{1, \cdots, n\} \setminus \{i\}$, we define the slice map from X_i to the product as

$$S_i : X_i \to X_1 \times \cdots \times X_n, \quad x_i \mapsto (x_1, \cdots, x_n).$$

Definition 10 (**Finite product**). *Given finitely many topological spaces $(X_1, \mathscr{U}_1), \cdots, (X_n, \mathscr{U}_n)$, consider the set*

$$\mathscr{B} := \{U_1 \times \cdots \times U_n \,|\, U_i \in \mathscr{U}_i \text{ for } i = 1, \cdots, n\},$$

whose elements are products of open sets in each factor. Then $\mathscr{B}$ is a topological basis on $X_1 \times \cdots \times X_n$ (because by (1.5), the intersection of two elements of $\mathscr{B}$ still belongs to $\mathscr{B}$). The topology on $X_1 \times \cdots \times X_n$ generated $\mathscr{B}$ is called the product topology. $X_1 \times \cdots \times X_n$ endowed with this topology is called the product topological space of $(X, \mathscr{U}_1), \cdots, (X, \mathscr{U}_n)$.

In order words, open sets w.r.t. product topology are unions of *products of open sets in each factor*. Note that

- The above $\mathscr{B}$ is only a basis, not a topology.
  Because the union of two elements $U_1 \times \cdots \times U_n, V_1 \times \cdots \times V_n$ of $\mathscr{B}$ might not belong to $\mathscr{B}$.
- If each $\mathscr{U}_i$ is generated by a topological basis $\mathscr{B}_i$ on X_i $(i = 1, \cdots, n)$, then the subset
  $$\mathscr{B}' := \{U_1 \times \cdots \times U_n \,|\, U_i \in \mathscr{B}_i \text{ for } i = 1, \cdots, n\} \subset \mathscr{B}$$
  is still a topological basis and generates the same topology as $\mathscr{B}$. Check this (use (1.6)).

The last point allows us to establish the following simple facts:

Proposition 2 (**Basic facts about product spaces**).

(1) *Given topological spaces $X_1, \cdots, X_n$, the space $((X_1 \times X_2) \times X_3) \times \cdots) \times X_n$ given by successive binary product coincides with the product space $X_1 \times \cdots \times X_n$.*

(2) *Let E_i be a subset of X_i $(i = 1, \cdots, n)$. Then the following two topologies on $E_1 \times \cdots \times E_n$ are the same:*
   (a) *the subspace topology inherited from the product space $X_1 \times \cdots \times X_n$;*
   (b) *the topology given by first taking the subspace topology on each E_i inherited from X_i, then taking their product.*

(3) *The product topology on $\mathbb{R}^N = \mathbb{R} \times \cdots \times \mathbb{R}$ coincides with the standard topology.*

Proof. Parts (1) and (2) are easily deduced from the definitions and the last point above, so we omit the proof here. As for (3), since the topology on $\mathbb{R}$ is generated by the basis $\{(a, b) \,|\, a, b \in \mathbb{R}, a < b\}$, by the last point above, the product topology on $\mathbb{R}^N = \mathbb{R} \times \cdots \times \mathbb{R}$ can be generated by the basis

$$\mathscr{B}_1 := \{(a^1, b^1) \times \cdots \times (a^n, b^n) \,|\, a^i, b^i \in \mathbb{R}, a^i < b^i \text{ for } i = 1, \cdots, n\}.$$

On the other hand, consider the norm $\|\cdot\|_{\ell^\infty}$ on $\mathbb{R}^N$ as in §1.2 and let $\mathscr{B}_2$ denote the set of all open balls $B(x, r)$ w.r.t. the metric induced by $\|\cdot\|_{\ell^\infty}$, so that $\mathscr{B}_2$ is a basis generating the standard topology. It suffices to show that $\mathscr{B}_1$ and $\mathscr{B}_2$ generate the same topology.

By definition of $\|\cdot\|_{\ell^\infty}$, we may write

$$B(x, r) = \{y \in \mathbb{R}^N \,|\, \max\{|x^1 - y^1|, \cdots, |x^N - y^N|\} < r\} = (x^1 - r, x^1 + r) \times \cdots \times (x^N - r, x^N + r).$$

Thus, every element $B(x,r)$ of $\mathscr{B}_2$ also belongs to $\mathscr{B}_1$, so we have $\mathscr{B}_2 \subset \mathscr{B}_1$. Therefore, we only need to show that every element of $\mathscr{B}_1$ is a union of elements of $\mathscr{B}_2$, or equivalently, that

$$\text{for any } B_1 \in \mathscr{B}_1 \text{ and } x \in B_1, \text{ there exists } B_2 \in \mathscr{B}_2 \text{ such that } x \in B_2 \subset B_1.$$

Assume $B_1 = (a^1, b^1) \times \cdots \times (a^n, b^n)$ and pick $x \in B_1$. It is easy to see that if $\epsilon \le \min\{x^1 - a^1, b^1 - x^1, \cdots, x^N - a^N, b^N - x^N\}$, then $B_2 := B(x,\epsilon)$ fulfills the above condition. This completes the proof. □

Properties of the above projection and slice maps w.r.t. the product topology:

Lemma 5 (**Projection and slice maps**). *Let $X_1, \cdots, X_n$ be topological spaces. Then for each $i = 1, \cdots, n$,*

(1) *The projection map $\Pi_i : X_1 \times \cdots \times X_n \to X_i$ is a continuous open map.* See Exercise 11 for definition of "open map".

(2) *The slice map $S_i : X_i \to X_1 \times \cdots \times X_n$, defined w.r.t. given $x_j \in X_j$, $j \in \{1, \cdots, n\} \setminus \{i\}$, is a homeomorphism to the image.*

Proof. For simplicity, we only give here a proof for $i = 1$. Generalizing to all $i = 1, \cdots, n$ is just a matter of notation.

(1) Consider the topological basis $\mathscr{B} := \{U_1 \times \cdots \times U_n \,|\, U_i \text{ is open in } X_i \ (i = 1, \cdots, n)\}$ which generates the product topology on $X_1 \times \cdots \times X_n$. For any open set U in X_1, the preimage $\Pi_1^{-1}(U) = U \times X_2 \times \cdots \times X_n$ belongs to $\mathscr{B}$, hence is open. Therefore, Π_1 is continuous. In order to show that Π_1 is an open map, it suffices to show that $\Pi_1(B)$ is open for any $B \in \mathscr{B}$, but this is obvious since we have $\Pi_1(B) = U_1$ if $B = U_1 \times \cdots \times U_n$.

(2) Fix $x_2 \in X_2, \cdots, x_n \in X_n$. The slice map S_1, defined w.r.t. these given points, sends each $x_1 \in X_1$ to $(x_1, x_2, \cdots, x_n)$. This map is clearly injective. In order to prove that it is a homeomorphism to the image, what we need to show is that for any subset U of X_1,

$$U \text{ is open in } X_1 \Leftrightarrow S_1(U) \text{ is open in } S_1(X_1),$$

where we equip $S_1(X_1) \subset X_1 \times \cdots \times X_n$ with the subspace topology.

To show the implication "$\Rightarrow$", note that

$$S_1(U) = U \times \{x_2\} \times \cdots \times \{x_n\} = S_1(X_1) \cap (U \times X_2 \times \cdots \times X_n).$$

Therefore, if U is open in X_1, then $S_1(U)$ is the intersection of the open subset $U \times X_2 \times \cdots \times X_n$ of $X_1 \times \cdots \times X_n$ with $S_1(X_1)$, hence is open in $S_1(X_1)$ under the subspace topology. This proves "$\Rightarrow$".

For the inverse implication "$\Leftarrow$", assume that $S_1(U)$ is open in $S_1(X_1)$. Given any $x_1 \in U$, we will find an open set U_1 in X_1 such that $x_1 \in U_1 \subset U$, which will imply that U is open, as required. The argument is illustrated by the figure below. First, since $S_1(U)$ is open in $S_1(X_1)$, we may write

$$S_1(U) = O \cap S_1(X_1) \tag{1.7}$$

for some open subset O of $X_1 \times \cdots \times X_n$. The point $S_1(x_1) = (x_1, x_2, \cdots, x_n)$ is contained in $S_1(U)$ and hence also in O, so there exists a member $B = U_1 \times \cdots \times U_n$ of the basis $\mathscr{B}$ such that

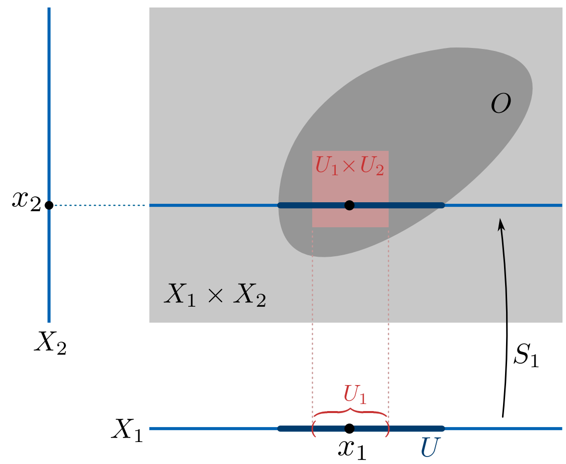
$$S_1(x_1) \in B \subset O.$$

Note that $S_1(x_1) \in B$ just means $x_1 \in U_1, \cdots, x_n \in U_2$, so it only remains to check $U_1 \subset U$. To this end, note that

$$B \cap S_1(X_1) = U_1 \times \{x_2\} \times \cdots \times \{x_n\} = S_1(U_1) \tag{1.8}$$

(the first equality is because $B = U_1 \times \cdots \times U_n$, $S_1(X_1) = X_1 \times \{x_2\} \times \cdots \{x_n\}$ and $x_2 \in U_2, \cdots, x_n \in U_n$). In view of (1.7) and (1.8), we deduce from the inclusion $B \subset O$ that $S_1(U_1) \subset S_1(U)$. Since S_1 is injective, it follows that $U_1 \subset U$, as required. □

We often need to consider continuous maps $f : X \to Y$ such that the domain X or the target Y is a product topological space. The case where Y is a product $Y_1 \times \cdots \times Y_n$ is simple: in this case, the information of f is captured by the *component*



$S_1(U)$ is open $\Rightarrow U$ is open

maps $f_i := \Pi_i \circ f : X \to Y_i$ $(i = 1, \cdots, n)$ because we may write $f(x) = (f_1(x), \cdots, f_n(x))$. The continuity of f is also equivalent to the continuity of these maps:

Proposition 3 (**Continuous map to a product**). *Let $X, Y_1, \cdots, Y_n$ be topological spaces and $f : X \to Y_1 \times \cdots \times Y_n$ be a map. Then f is continuous iff. the map $f_i := \Pi_i \circ f : X \to Y_i$ is continuous for every $i = 1, \cdots, n$.*

Proof. The "only if" part follows from the continuity of Π_i (Lemma 5) and the fact that a composition of continuous maps is continuous. For the "if" part, assume that every f_i is continuous. In order to prove the continuity of f, we only need to show that $f^{-1}(B)$ is open in X for any subset B of $Y_1 \times \cdots \times Y_n$ belonging to the topological basis $\mathscr{B} := \{U_1 \times \cdots \times U_n \,|\, U_i \text{ is open in } Y_i \ (i = 1, \cdots, n)\}$. But we may write

$$f^{-1}(B) = \{x \in X \,|\, f(x) \in B\} = \{x \in X \,|\, f_i(x) \in U_i \ (i = 1, \cdots, n)\} = \bigcap_{i=1}^{n} f_i^{-1}(U_i)$$

for $B = U_1 \times \cdots \times U_n \in \mathscr{B}$, so $f^{-1}(B)$ is a finite intersection of open sets and hence is open, as required. $\square$

The situation where the domain X is a product is different: if a map $f : X_1 \times \cdots \times X_n \to Y$ is continuous, then for any slice map $S_i : X_i \to X_1 \times \cdots \times X_n$ defined w.r.t. given $x_j \in X_j$, $j \in \{1, \cdots, n\} \setminus \{i\}$, the composition

$$f \circ S_i : X_i \to Y, \quad x_i \mapsto f(x_1, \cdots, x_n)$$

is continuous (in other words: if f is continuous, then $f(x_1, \cdots, x_n)$ is continuous *in the variable* x_i when the other variables are fixed); but conversely, continuity of every $f \circ S_i$ does not imply continuity of f. As a classical example, the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

is not continuous at 0, even though for any fixed $x_0, y_0 \in \mathbb{R}$, $x \mapsto f(x, y_0)$ and $y \mapsto f(x_0, y)$ are both continuous functions on $\mathbb{R}$.

Exercise 20. Let (X, d) be a metric space. Show that the metric itself, viewed as the function $d : X \times X \to \mathbb{R}$, $(x, y) \mapsto d(x, y)$, is continuous w.r.t. the product topology on $X \times X$.

Exercise 21. Let X be a vector space and $\|\cdot\|$ be a norm on X. Show that the vector space operations

$$X \times X \to X,\ (x, y) \mapsto x + y \quad \text{and} \quad \mathbb{R} \times X \to X,\ (a, x) \mapsto ax$$

are continuous (where $X \times X$ and $\mathbb{R} \times X$ are equipped with product topology).
As a consequence, for fixed $x_0 \in X$ and $a_0 \in \mathbb{R}$, the maps $x \mapsto x + x_0$, $x \mapsto a_0 x$ and $a \mapsto a x_0$ are continuous (the first two are maps from X to itself, the last one is from $\mathbb{R}$ to X).

Exercise 22. Given a topological space X and a function $f : X \to \mathbb{R}$, the *graph* and *epigraph* of f are defined as the following subsets of the product topological space $X \times \mathbb{R}$:

$$\mathrm{Gr}(f) := \{(x, f(x)) \in X \times \mathbb{R} \,|\, x \in X\}, \quad \mathrm{Epi}(f) := \{(x,t) \in X \times \mathbb{R} \,|\, x \in X,\ t \geq f(x)\}.$$

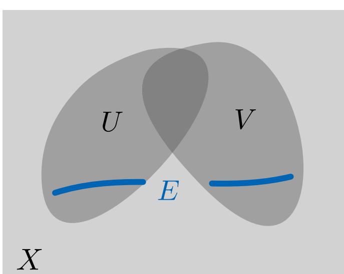
(1) Show that a function $f : \mathbb{R} \to \mathbb{R}$ is lower semi-continuous (see Example 10) iff. its epigraph is closed in $\mathbb{R}^2$.
(2) Generalize (1) to functions $f : X \to \mathbb{R}$ for any topological space X, and also to upper semi-continuity.
(3) Show that if $f : X \to \mathbb{R}$ is continuous, then $\mathrm{Gr}(f)$ is closed in $X \times \mathbb{R}$, but the converse does not hold.

1.6. **Connectedness.** A topological space being *connected* just means it is not a disjoint union in the sense of Def. 9. The precise definition is as follows. To simply the discussions, let us call a set clopen if it is open and closed at the same time.

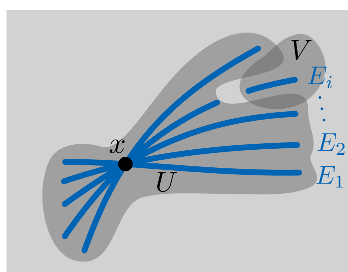
Definition 11 (**Connectedness**).

(1) *A topological space X is said to be disconnected if it is the disjoint union of at least two nonempty topological spaces*
($\Leftrightarrow$ X is the disjoint union of two disjoint nonempty open subsets
$\Leftrightarrow$ X has clopen subset which is neither $\varnothing$ nor X).
(2) *A subset $E \subset X$ is disconnected if E endowed with the subspace topology is a disconnect topological space*
($\Leftrightarrow$ there exists open sets $U, V \subset X$ whose union contains E, such that $U \cap E$ and $V \cap E$ are nonempty and disjoint; We call such (U, V) a separating pair for E, see the figure below).
(3) *A topological space or subset is connected if it is not disconnected.*

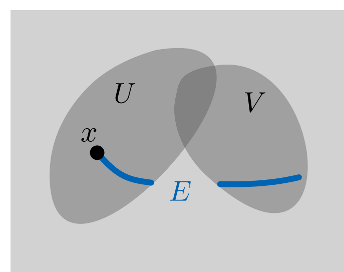
For example, $[0,1] \cup [2,3] \subset \mathbb{R}$ is disconnected because $U = (-\epsilon, 1+\epsilon)$ and $V = (2-\epsilon, 3+\epsilon)$ form a separating pair when $0 < \epsilon < \frac{1}{2}$. Exercise 2 (3) shows that $\mathbb{R}$ is connected, from which we may further deduced that every interval in $\mathbb{R}$ is connected (Corollary 2 (1) below). It is trivial that the empty set is connected.



Separating pair for E



Proof of Prop. 4 (1)



Proof of Prop. 4 (2)

Proposition 4 (**Properties of connected sets**).

(1) *If a family of connected subsets have nonempty intersection, then their union is connected.*
(2) *If E is a connected subset, then any subset F satisfying $E \subset F \subset \overline{E}$ is connected.*
(3) *The image of any connected subset by a continuous map is connected.*

Proof. (1) Let $(E_i)_{i \in I}$ be a family of connected subsets of a topological space X with $\bigcap_{i \in I} E_i \neq \varnothing$ and pick a point x in this intersection. If $E := \bigcup_{i \in I} E_i$ is disconnected, then E has a separating pair (U, V). Switching U and V if necessary, we may assume without loss of generality that $x \in U$. Since V intersects E, it intersects some E_i. It follows that (U, V) is a separating pair for E_i, contradicting the assumption that E_i is connected.

(2) Let E be a connected subset of a topological space X. If $E = \varnothing$, the required statement is trivial. When $E \neq \varnothing$, using Part (1), we only need to show that $E \cup \{x\}$ is connected for any $x \in \overline{E} \setminus E$, because any F satisfying $E \subset F \subset \overline{E}$ is the union of the family of subsets $(E \cup \{x\})_{x \in \overline{E} \setminus E}$, whose intersection is E.

Suppose by contradiction that $E\cup\{x\}$ is disconnected, so that it has a separating pair (U,V). By switching U and V if necessary, we may assume $x\in U$. Then,

- Since $x\in\overline{E}$, any neighborhood of x must intersect E. In particular, U intersects E.
- Since $E\cup\{x\}$ is the disjoint union of the nonempty sets $(E\cup\{x\})\cap U$ and $(E\cup\{x\})\cap V$, while x belongs to the former set, we infer that the latter set contains some element of E, or in other words, V intersects E.

It follows that (U,V) is a separating pair for E, a contradiction.

(3) Let $f: X\to Y$ be a continuous map between topological spaces and E be a connected subset of X. If $f(E)$ is disconnected, then it has a separating pair (U,V). It is easy to check that $(f^{-1}(U), f^{-1}(V))$ is a separating pair for E, a contradiction. □

Corollary 2.

(1) (**Connected sets in** $\mathbb{R}$) *A subset of* $\mathbb{R}$ *is connected if and only if it is an interval.*
See Exercise 19 for the precise meaning of "interval".

(2) (**Intermediate Value Theorem**) *Let X be a connected topological space and $f: X\to\mathbb{R}$ be a continuous function. Then the image $f(X)$ is an interval in* $\mathbb{R}$.

(3) (**Path-connected implies connected**) *If a topological space X has the property*

$$\forall x,y\in X,\ \exists \text{ continuous map } f \text{ from the interval } [0,1]\subset\mathbb{R} \text{ to } X \text{ with } f(0)=x,\ f(1)=y$$

then X is connected.

Such an f is called a path from x to y. A space X satisfying this property is said to be path-connected. This statement just says "path-connected ⇒ connected". The converse does not hold, see Exercise 24.

Proof. (1) Every interval $I\subset\mathbb{R}$ is the image of a certain continuous map $f:\mathbb{R}\to\mathbb{R}$, see ... Therefore, it follows from the connectedness of $\mathbb{R}$ and Prop. 4 (3) that I is connected. Conversely, if $E\subset\mathbb{R}$ is not an interval, then there exist $a,b\in E$ with $a<b$ and $c\in(a,b)$ such that $c\notin E$. It follows that $((-\infty,c),(c,+\infty))$ is a separating pair for E, so E is disconnected.

(2) This follows immediately from Part (1) and Prop. 4 (3).

(3) Suppose by contradiction that X is disconnected, so that X is the union of disjoint nonempty open sets U and V. Pick $x\in U$, $y\in V$ and let $f:[0,1]\to X$ be a path from x to y. Then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint nonempty open sets in $[0,1]$ whose union is $[0,1]$, contradicting the connectedness of $[0,1]$. □

Exercise 23. Show that any normed vector space is connected.

Exercise 24. Let $S\subset\mathbb{R}^2$ be the graph of the function $(0,\frac{1}{\pi}]\to\mathbb{R}$, $t\mapsto\sin\left(\frac{1}{t}\right)$, namely

$$S:=\left\{\left(t,\sin\left(\tfrac{1}{t}\right)\right)\in\mathbb{R}^2 \,\middle|\, t\in(0,\tfrac{1}{\pi}]\right\},$$

and $I:=\{0\}\times[-1,1]\subset\mathbb{R}^2$ be the vertical segment on the y-axis between -1 and 1 (figure below). Show that

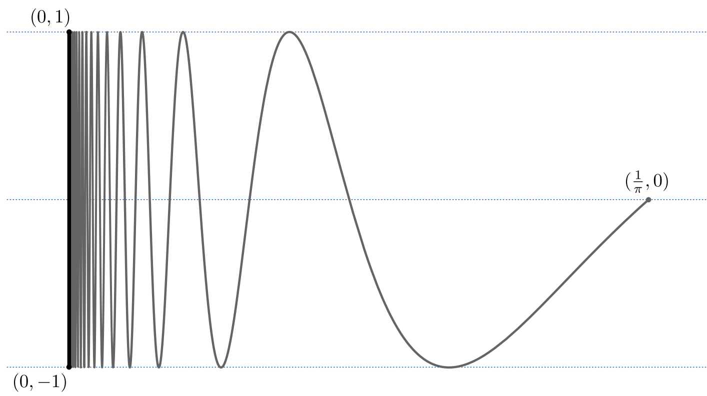
(1) $\overline{S}$ is connected.

(2) For any $x\in I$ and $y\in S$, there is no path in $\overline{S}$ from x to y. *Hint.* Suppose by contradiction that $f:[0,1]\to\overline{S}$ is a path joining x and y. Since I is closed in $\overline{S}$, $f^{-1}(I)$ is closed in $[0,1]$ and hence has a largest element $a\in[0,1)$. Use the fact that the composition $\Pi_1\circ f$ is continuous (where $\Pi_1:\mathbb{R}^2\to\mathbb{R}$ is the projection map to the x-axis) to show that for any $h\in[0,1]$, there exists a sequence $(s_n)_{n=1,2,\dots}$ in $[a,1]$ such that $f(s_n)$ converges to $(0,h)\in I$.

Exercise 25. Let X and Y be nonempty topological spaces. Show that the product space $X\times Y$ is connected iff. X and Y are both connected.

Hint. For the "if" part, show that a subset of $X\times Y$ of the form $(X\times\{y\})\cup(\{x\}\times Y)$ (where $x\in X$, $y\in Y$) is connected. Then find a family of such subsets, with nonempty intersection, such that the union is $X\times Y$.

A disconnected topological space has a canonical decomposition into connected pieces:



Proposition 5 (**Connected components**). *Given a topological space* $(X, \mathscr{U})$*, define the following binary relation* $\sim$ *on* X*:*

$$x \sim y \overset{\text{def.}}{\Longleftrightarrow} \text{there exists a connected subset of } X \text{ containing } x \text{ and } y.$$

Then $\sim$ *is an equivalence relation. The equivalence classes, called connected components of* X*, are connected closed subsets. If furthermore every point of* X *has a connected neighborhood, then every connected component is clopen* (so that X is the disjoint union topological space of its connected components).

Proof. As a binary relation, $\sim$ is clearly symmetric, and is reflexive because one-point subsets are connected. The transitivity follows from Prop. 4 (1) (here we only need the fact that the union of two connected subsets with nonempty intersection is connected). Thus, $\sim$ is an equivalence relation.

The equivalence class of $\sim$ containing $x \in X$, denoted by C_x, is the union of all connected subsets of X containing x. Again by Prop. 4 (1), C_x is connected. Therefore, C_x is the largest connected subset of X containing x, in the sense that any connected subset containing x must be contained in C_x. If C_x was not closed, then by Prop. 4 (2), $\overline{C_x}$ would be a larger connected subset, a contradiction.

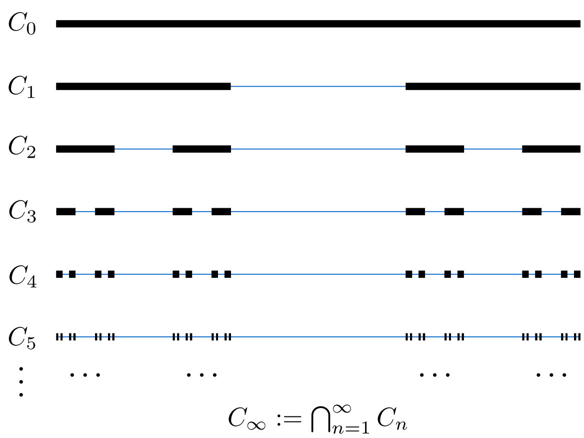
Now let us assume that any $y \in X$ has a connected neighborhood U_y and show that C_x is open. For any $y \in C_x$ we may write $C_x = C_y$, which is also the largest connected subset containing y. So we have $U_y \subset C_x$. This show that every point of C_x has a neighborhood contained in C_x, i.e. C_x is open. □

Simple examples:

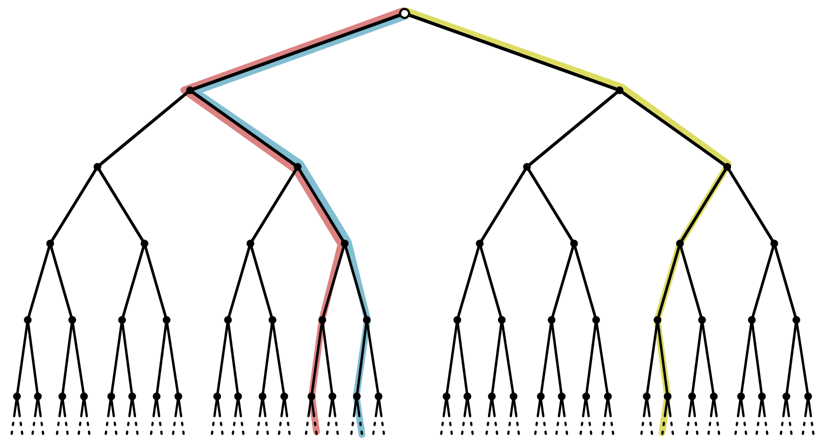
- If X is an open subset of $\mathbb{R}$, then each connected component of X is an open interval in $\mathbb{R}$, which is clopen in X (under subspace topology). By Exercise 10, X has at most countably many connected components.
- The connected components of $X = \bigcup_{n \in \mathbb{Z}} [2n, 2n+1]$ are just the intervals $[2n, 2n+1]$, $n \in \mathbb{Z}$, and each of them is clopen in X.
- A topological space is said to be totally disconnected if its connected components are all one-point sets. A discrete topological space is clearly totally disconnect. The rationals $\mathbb{Q}$ and irrationals $\mathbb{Q} \setminus \mathbb{R}$ are also totally disconnected (see Exercise 26). Another classical example is as follows.

Example 11 (**Cantor set**). The Cantor ternary set is the subset C_∞ of $\mathbb{R}$ shown below on the left. *Construction*: start with the interval $C_0 := [0, 1]$; to obtain C_{n+1} from C_n, we trisect every connected component of C_n (a closed interval), then remove the mid part and keep the two remaining closed intervals; finally, define $C_\infty := \bigcap_{n=1}^{\infty} C_n$. As usual, we consider C_∞ as a topology space by using the subspace topology inherited from $\mathbb{R}$, which is also the metric topology given by the restriction of the metric of $\mathbb{R}$ to C_∞ (Exercise 18). Note that C_∞ is compact since it is bounded and closed in $\mathbb{R}$.

Using the *principle of nested intervals*, it is easy to see that C_∞ has a combinatorial interpretation in terms of the binary tree T as shown on the right. In fact, there is a natural set bijection



Cantor ternary set



rays in the binary tree

$$C_\infty \cong R := \{\text{ rays in the tree } T\,\}. \tag{1.9}$$

Definitions of the binary tree T and its rays are self-explanatory from the figure, but here are the details: T is defined as the infinite graph such that there is a single vertex at generation 0, while every vertex at generation n is joint to two vertices at generation $n+1$, namely its "children". By a *ray* in the tree, we mean a half-infinite path which starts from the generation 0 vertex and goes down through every generation. Meanwhile, if we put, for each vertex v of T,

$$B_v := \{\text{ rays in } T \text{ passing through } v\,\} \subset R,$$

then $\mathscr{B} := \{B_v \,|\, v \text{ is a vertex of } T\}$ is a topological basis on R, and the topology generated by it makes (1.9) a homeomorphism. Therefore, we can identify C_∞ with R equipped with this topology.

Using this interpretation, it can be check that C_∞ is uncountable, totally disconnected and does not have any isolated point[6]. In fact, C_∞ is *zero dimensional*, which is a stronger property than being totally disconnected (see Exercise 26). Let us quote without proof the following *intrinsic* characterization of C_∞ (it involves the notions of *compactness* and *metrizability*, which will be discussed in detail later on):

Theorem 2 (**Brouwer's theorem on Cantor sets**). *A topological space X is homeomorphic to the Cantor ternary set C_∞ if and only if X is compact, metrizable, zero dimensional and does not have isolated points.*

In the literature, a Cantor set usually refers to a topological space homeomorphic to C_∞. As a nontrivial consequence of the theorem, if we modify the construction of C_∞ in such a way that each component of C_n is divided into more than 3 parts (the number of parts can even vary for different n's), by using the tree interpretation, it can still be shown that the resulting set satisfies the hypotheses of the theorem, hence is still a Cantor set.

Exercise 26. A topological space X is said to be totally separable if for any distinct points $x, y \in X$, there exists a clopen subset containing x but not y. X is said to be zero dimensional[7] if every one-point subset is closed and the topology is generated by a basis formed by clopen subsets. Show that

(1) The following implications hold:

$$\text{zero dimensional} \Rightarrow \text{totally separable} \Rightarrow \text{totally disconnected}.$$

(2) The rationals $\mathbb{Q}$, the irrationals $\mathbb{R}\setminus\mathbb{Q}$, the Cantor set C_∞ and the Sorgenfrey line $\mathbb{R}_\ell := (\mathbb{R}, \mathscr{U}_{\text{right}})$ (see Example 10) are zero dimensional.

[6] A point x in a topological space X is said to be isolated if the one-point set $\{x\}$ is open.

[7] One can define the *topological dimension* of any topological space (see [Mun00, §50]). The current definition is equivalent to the topological dimension being zero.

(3) Any metric space with countably many points is zero dimensional[8].
Hint. Given $x \in X$, show that $B(x,r) = \overline{B}(x,r)$ for all except countably many $r > 0$. Use Exercise 13.

Exercise 27. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that the following conditions are equivalent:

(a) f is strictly increasing or strictly decreasing.
(b) f is injective.
(c) f is an open map.

Hint. "(a) $\Leftrightarrow$ (b) $\Rightarrow$ (c)" only needs Intermediate Value Theorem. In order to show "(c) $\Rightarrow$ (b)", note that

- If f is zero at both endpoints of an interval $[a,b]$ and is positive on (a,b), then $f((a,b))$ is an interval of the form $(0,t]$ (because f attains its maximum on $[a,b]$, see Lemma 4).
- If f is zero at both endpoints of $[a,b]$ and is positive at some $x \in (a,b)$, by considering the connected component of $f^{-1}((0,+\infty))$ containing x, one can find $a_0, b_0 \in [a,b]$ such that $f(a_0) = f(b_0) = 0$ and f is positive on (a_0, b_0).

1.7. **Compactness.** For a general topological space, the following definition of compactness is more convenient than sequential compactness:

Definition 12 (**Compactness**).

(1) *Given a topological space X and a subset $E \subset X$, an open cover for E is a set of open sets $\mathscr{C}_E$ whose union $\bigcup_{U \in \mathscr{C}_E} U$ contains E. Given such a $\mathscr{C}_E$, if a subset $\mathscr{C}'_E \subset \mathscr{C}_E$ is still an open cover for E, then $\mathscr{C}'_E$ is called a sub-cover of $\mathscr{C}_E$.*
(2) *A topological space X is said to be compact if any open cover for the whole space has a finite sub-cover.*
(3) *A subset $E \subset X$ is said to be compact if E endowed with the subspace topology is a compact topological space ($\Leftrightarrow$ any open cover for E has a finite sub-cover).*

Finite sets (including $\varnothing$) are clearly compact. In general, it is difficult to show that a space is compact only using the definition, and we are obliged to invoke Corollary 1 (there, "compact" means "sequential compact") and the equivalence between compactness and sequential compactness in metric spaces (Prop. 10 below) so as to give the most fundamental example: compact subsets of $\mathbb{R}^N$ are exactly bounded closed subsets.

Proposition 6 (**Properties of compact sets**).

(1) *The union of finitely many compact sets is compact.*
(2) *Any closed subset of a compact topological space is compact.*
(3) *The image of a compact set by a continuous map is compact.*
(4) *The product of finitely many compact spaces is compact.*

Proof. (1) Suppose $E_1, \cdots, E_n$ are compact subsets of a topological space X and let $\mathscr{C}$ be an open cover for $E_1 \cup \cdots \cup E_n$. For each $i = 1, \cdots, n$, $\mathscr{C}$ is also an open cover for E_i, hence has a finite sub-cover $\mathscr{C}'_i \subset \mathscr{C}$ for E_i. It follows that $\mathscr{C}'_1 \cup \cdots \cup \mathscr{C}'_n$ is a finite sub-cover of $\mathscr{C}$ for $E_1 \cup \cdots \cup E_n$, which shows that $E_1 \cup \cdots \cup E_n$ is compact.

(2) Let X be a compact topological space, $E \subset X$ be closed, and $\mathscr{C}_E$ be an open cover for E. Then $\mathscr{C}_E \cup \{X \setminus E\}$ is an open cover for the whole X. Since X is compact, it has a finite sub-cover $\mathscr{C}' \subset \mathscr{C}_E \cup \{X \setminus E\}$. If $\mathscr{C}'$ does not contain $X \setminus E$, then it is already a sub-cover of $\mathscr{C}_E$ for E; otherwise, we may remove $X \setminus E$ from $\mathscr{C}'$ to get a sub-cover of $\mathscr{C}_E$. In any case, we have found a sub-cover of $\mathscr{C}_E$, which shows that E is compact.

(3) Let $f : X \to Y$ be a continuous map between topological spaces and $E \subset X$ be a compact set. Given an open cover $\mathscr{C}_{f(E)}$ for $f(E)$, the set of preimages $\mathscr{C}_E := \{f^{-1}(U) \,|\, U \in \mathscr{C}_{f(E)}\}$ is an open cover for E. Since E is compact, $\mathscr{C}_E$ has a

[8] There is a much stronger result due to Sierpiński: *any countable metrizable topological space without isolated point is homeomorphic to $\mathbb{Q}$.*

finite sub-cover, which can be written as $\{f^{-1}(U) \,|\, U \in \mathscr{C}'_{f(E)}\}$ for some finite subset $\mathscr{C}'_{f(E)}$ of $\mathscr{C}_{f(E)}$. Then $\mathscr{C}'_{f(E)}$ is a finite sub-cover of $\mathscr{C}_{f(E)}$. Therefore, $f(E)$ is compact.

(4) In view of Prop. 2 (1), we only need to consider binary products. Namely, it suffices to show that if X and Y are compact topological spaces, then the product space $X \times Y$ is compact. Let $\mathscr{C}_{X\times Y}$ be an open cover for $X \times Y$. In order to find a finite sub-cover, we proceed in the following steps.

Step 1. *For each $x \in X$, find a finite subset $\mathscr{C}_{\{x\}\times Y}$ of $\mathscr{C}_{X\times Y}$ which is an open cover for the slice $\{x\} \times Y$.*

This is possible because by Lemma 5, $\{x\}\times Y$ (with the subspace topology) is homeomorphic to Y, hence compact.

Step 2. *Prove the "Tube Lemma": If $O \subset X \times Y$ is open and contains the slice $\{x\} \times Y$, then O also contains $U_x \times Y$ for some neighborhood U_x of x.* This only needs Y to be compact, not X.

The proof is as follows (see the figure below). Consider the basis $\mathscr{B} := \{U \times V \,|\, U \subset X \text{ and } V \subset Y \text{ are open}\}$ which generates the product topology. For any $y \in Y$, since O contains $(x, y) \in \{x\} \times Y$, we can choose an element B_y of $\mathscr{B}$ such that $y \in B_y \subset O$. Write $B_y = U_y \times V_y$, where $U_y \subset X$ is a neighborhood of x and $V_y \subset Y$ is a neighborhood of y. As y runs over Y, all the latter neighborhoods form an open cover for Y. It has a finite sub-cover since Y is compact. Therefore, we get finitely many points $y_1, \cdots, y_m$ of Y such that

$$Y = V_{y_1} \cup \cdots \cup V_{y_m}. \tag{1.10}$$

Consider the union of the B_y's corresponding to these points, namely $B_{y_1} \cup \cdots \cup B_{y_m}$. On one hand, it is contained in O; on the other hand, we have

$$(U_{y_1} \cap \cdots \cap U_{y_m}) \times Y \subset B_{y_1} \cup \cdots \cup B_{y_m}$$

(because by (1.10), any point (x, y) belonging to the left-hand side satisfies $x \in U_{y_1} \cap \cdots \cap U_{y_m}$ and $y \in V_{y_i}$ for some $i = 1, \cdots, m$, which implies $(x, y) \in U_{y_i} \times V_{y_i} = B_{y_i}$). Therefore, we conclude that O contain $U_x \times Y$ with $U_x := U_{y_1} \cap \cdots \cap U_{y_m}$, as required.

Step 3. We shall find finitely many points $x_1, \cdots, x_k$ in X such that the finite subsets $\mathscr{C}_{\{x_1\}\times Y}, \cdots, \mathscr{C}_{\{x_k\}\times Y}$ from Step 1 together form an open cover for $X \times Y$. This will be the required finite sub-cover of $\mathscr{C}_{X\times Y}$ and will finish the proof. In order to find those points, we consider, for each $x \in X$, the union of all elements in $\mathscr{U}_{\{x\}\times Y}$ and denote it by

$$O_x := \bigcup_{P \in \mathscr{U}_{\{x\}\times Y}} P.$$

Since O_x is an open subset of $X \times Y$ containing $\{x\} \times Y$, by Step 2, it contains $U_x \times Y$ for some neighborhood U_x of x. All such neighborhoods form an open over $\{U_x \,|\, x \in X\}$ for X. It has a finite sub-cover by compactness of X, so we get finitely many points $x_1, \cdots, x_k \in X$ such that

$$X = U_{x_1} \cup \cdots \cup U_{x_k}.$$

We shall show these are the points which we are searching for. The above equality implies

$$X \times Y = (U_{x_1} \cup \cdots \cup U_{x_k}) \times Y = (U_{x_1} \times Y) \cup \cdots \cup (U_{x_k} \times Y).$$

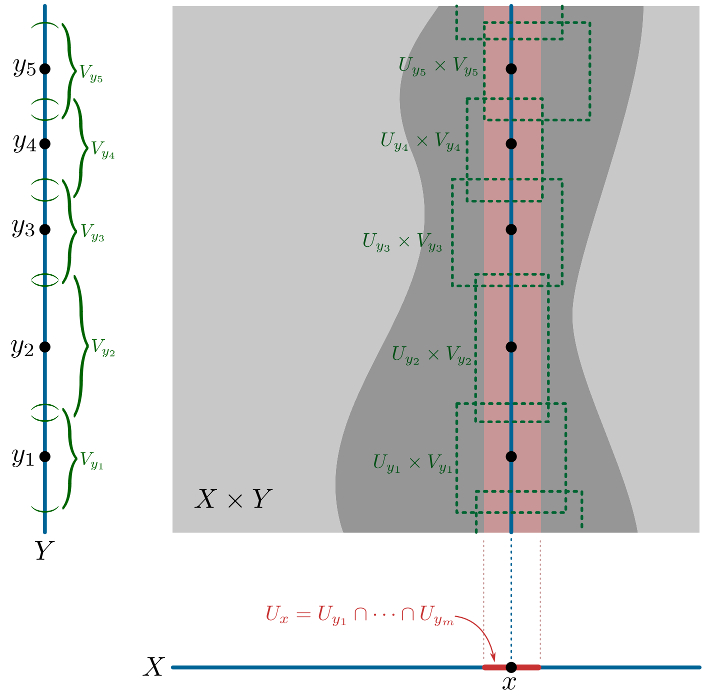
Since O_x contains $U_x \times Y$, we conclude that

$$X \times Y = O_{x_1} \cup \cdots \cup O_{x_k} = \bigcup_{i=1}^{k} \bigcup_{P \in \mathscr{U}_{\{x_i\}\times Y}} P,$$

namely, the union of $\mathscr{U}_{\{x_1\}\times Y}, \cdots, \mathscr{U}_{\{x_k\}\times Y}$ is an open cover for $X \times Y$, as required. □

The classical fact that any continuous function on a sequentially compact set attains its maximum and minimum (see Lemma 4; although we assume metric space there, the proof actually works for any topological space X and sequentially compact set $E \subset X$) also holds for compact sets:

Corollary 3 (Extreme Value Theorem, ver.2). *Let X be a topological space, $f : X \to \mathbb{R}$ be a continuous function and $E \subset X$ be a compact set. Then f attains its maximum and minimum on E* (the precise meaning is the same as in Lemma 4).



Proof of the Tube Lemma

Proof. By Prop. 6 (3), $f(E) \subset \mathbb{R}$ is compact, hence is closed and bounded. In view of Exercise 2, we conclude that $f(E)$ *has a largest and a smallest element*, which is equivalent to the required statement. □

Exercise 28. Let X be a topological space and $(x_n)_{n=1,2,\cdots}$ be a sequence in X converging to some $x \in X$. Show that the subset of X formed by x and all points of (x_n) is compact.

Exercise 29. Consider the trivial topology $\mathscr{U}_{\text{trivial}}$, the discrete topology $\mathscr{U}_{\text{discrete}}$, the finite complement topology $\mathscr{U}_{\text{f.c.}}$ and the countable complement topology $\mathscr{U}_{\text{c.c.}}$ on a set X (see §1.3). Show that

(1) Under $\mathscr{U}_{\text{trivial}}$ and $\mathscr{U}_{\text{f.c.}}$, every $E \subset X$ is compact.

(2) Under $\mathscr{U}_{\text{discrete}}$ and $\mathscr{U}_{\text{c.c.}}$, $E \subset X$ is compact iff. E is finite. *Hint.* If E is infinite, let $(x_n)_{n=1,2,\cdots}$ be a sequence of distinct points in E and consider the open cover $\{X \setminus (x_n)_{n \geq m} \,|\, m = 1, 2, \cdots\}$.

Exercise 30. Consider the topology $\mathscr{U}_{\text{lower}}$ on $\mathbb{R}$ defined in in Example 10. Show that

(1) A subset $E \subset \mathbb{R}$ is compact w.r.t. $\mathscr{U}_{\text{lower}}$ iff. E has a minimal element (i.e. E is bounded from below and $\inf E \in E$).

(2) Let X be a topological space, $f : X \to \mathbb{R}$ be a lower semi-continuous function and $E \subset X$ be a compact set. Then f attains its minimum on E, but not necessarily its maximum.

Compact sets behave particularly well when the topological space is *Hausdorff*[9]:

Proposition 7 (Compact sets in Hausdorff space). *Let $(X, \mathscr{U})$ be a Hausdorff topological space. Then the following statements hold.*

(1) *Every compact subset of X is closed. As a consequence, if X itself is compact, then $E \subset X$ is compact iff. E is closed.*

(2) *Suppose X is non-compact. We add to X an extra point ∞ to form the set*

$$\widehat{X} := X \cup \{\infty\},$$

[9]Readers are advised to take a quick look at the definition of Hausdorff property in Def. 14 below and come back. Also keep in mind that *metric spaces are Hausdorff* and *one-point sets are closed in a Hausdorff space*. We apologies for the non-chronological writing.

and consider the following collection of its subset:

$$\widehat{\mathscr{U}} := \mathscr{U} \cup \big\{\{\infty\} \cup X \setminus K \,\big|\, K \subset X \text{ is compact}\big\}$$

(namely, $\widehat{\mathscr{U}}$ consists of all open subsets of X along with all "compact-complements in X with ∞ attached"). *Then $\widehat{\mathscr{U}}$ is a topology on $\widehat{X}$. The topological space $(\widehat{X}, \widehat{\mathscr{U}})$ has the following properties:*

- *compact;*
- *contains X as a subspace* (i.e. the subspace topology $\widehat{\mathscr{U}}|_X$ on X equals the original topology $\mathscr{U}$);
- *X is dense in $\widehat{X}$.*

Proof. (1) Suppose $E \subset X$ is closed. In order to show that E is compact, we pick $x \in X \setminus E$ and only need to find a neighborhood U of x disjoint from E. To this end, for every $y \in E$, we use the Hausdorff property to pick a neighborhood U_y of x and a neighborhood V_y of y such that

$$U_y \cap V_y = \varnothing. \tag{1.11}$$

Then $\{V_y \,|\, y \in E\}$ is an open cover for E. Since E is compact, it has a finite sub-cover, namely there are finitely many points $y_1, \cdots, y_m$ in E such that

$$E \subset V_{y_1} \cup \cdots \cup V_{y_m}. \tag{1.12}$$

By (1.11), the neighborhood $U := U_{y_1} \cap \cdots \cap U_{y_m}$ of x does not intersect $V_{y_1} \cup \cdots \cup V_{y_m}$, so it does not intersect E by (1.12), as required.

(2) Denote $\mathscr{U}_\infty := \big\{\{\infty\} \cup X \setminus K \,\big|\, K \subset X \text{ is compact}\big\}$, so that $\widehat{\mathscr{U}} = \mathscr{U} \cup \mathscr{U}_\infty$ (once we have shown that $\widehat{\mathscr{U}}$ is a topology, $\mathscr{U}_\infty$ will be the set of neighborhoods of ∞). $\widehat{\mathscr{U}}$ clearly contains $\varnothing$ and the whole $\widehat{X}$, so we only need to show

(i) The intersection of two elements in $\widehat{\mathscr{U}}$ is still in $\widehat{\mathscr{U}}$.

(ii) A union of elements in $\widehat{\mathscr{U}}$ is still in $\widehat{\mathscr{U}}$.

For (i), the case where the two elements both belong to $\mathscr{U}$ is trivial. For two elements of $\mathscr{U}_\infty$, the intersection is

$$\big(\{\infty\} \cup X \setminus K_1\big) \cap \big(\{\infty\} \cup X \setminus K_2\big) = \{\infty\} \cup X \setminus (K_1 \cup K_2),$$

which still belongs to $\mathscr{U}_\infty$ since $K_1 \cup K_2$ is compact (Prop. 6 (1)). Finally, the intersection of an element of $\mathscr{U}$ with an element of $\mathscr{U}_\infty$ can be written as

$$U \cap \big(\{\infty\} \cup X \setminus K\big) = U \cap (X \setminus K).$$

Since K is closed by Part (1), this intersection belongs to $\mathscr{U}$. Thus, we have proven (i).

For (ii), first note that a union of elements in $\mathscr{U}$ is still in $\mathscr{U}$ because $\mathscr{U}$ is a topology. Let us show next that a union of elements in $\mathscr{U}_\infty$ is still in $\mathscr{U}_\infty$. Such a union can be written as

$$\bigcup_{i \in I} \big(\{\infty\} \cup X \setminus K_i\big) = \{\infty\} \cup X \setminus \bigcap_{i \in I} K_i,$$

where $(K_i)_{i \in I}$ is a family of compact subsets of X. Since every K_i is closed by Part (1), the intersection $\bigcap_{i \in I} K_i$ is closed as well, hence Prop. 6 (2) implies that it is compact. Therefore, the above union belongs to $\mathscr{U}_\infty$, as required.

Now, in order to show that a union of elements in $\widehat{\mathscr{U}}$ is still in $\widehat{\mathscr{U}}$, we only need to consider the case of two elements, one in $\mathscr{U}$ and the other in $\mathscr{U}_\infty$. Such a union can be written as

$$U \cup \big(\{\infty\} \cup X \setminus K\big) = \{\infty\} \cup X \setminus (K \setminus U).$$

It belongs to $\mathscr{U}_\infty$ because $K \setminus U$ is a closed subset of K and hence is compact by Prop. 6 (2). Thus, (ii) has been proven and we conclude that $\mathscr{U}_\infty$ is a topology.

We proceed to establish the three required properties. The last property is equivalent to the statement that any neighborhood of $\infty \in \widehat{X}$ intersects X. A neighborhood of ∞ is just an element of $\mathscr{U}_\infty$ and can be written as $\{\infty\} \cup X \setminus K$, where $K \subset X$ is compact. Since X itself is not compact by hypothesis, we have $K \subsetneq X$, hence the intersection of this neighborhood with X is $X \setminus K \neq \varnothing$, as required.

The property $\widehat{\mathscr{U}}|_X = \mathscr{U}$ follows immediately from the definitions once we notice the following fact: from any element $\{\infty\} \cup X \setminus K$ of $\mathscr{U}_\infty$, if we remove the point $\{\infty\}$, the remaining part $X \setminus K$ belongs to $\mathscr{U}$ (because K is closed by Part (1)).

Finally, to show that $(\widehat{X}, \widehat{\mathscr{U}})$ is compact, let $\mathscr{C} \subset \widehat{\mathscr{U}}$ be an open cover for $\widehat{X}$. The point ∞ is contained in some (possibly several) element of $\mathscr{C}$. Choose such an element $\{\infty\} \cup X \setminus K_0$. All the other elements of $\mathscr{C}$ form an open cover for the compact set K_0, so it has a finite sub-cover. This sub-cover for K_0, along with the element $\{\infty\} \cup X \setminus K_0$, form a finite sub-cover of $\mathscr{C}$ for $\widehat{X}$, proving that $\widehat{X}$ is compact. □

The above compact space $(\widehat{X}, \widehat{\mathscr{U}})$ is a *one-point compactification* of X in the following sense:

Definition 13 (Compactification). *Given a non-compact topological space X, a topological space X' is called a compactification of X if it satisfies the last three conditions in Prop. 7 (2), namely X' is compact and contains X as a dense subspace. If furthermore $X' \setminus X$ is a single point, then X' is called a one-point compactification of X.*

However, $(\widehat{X}, \widehat{\mathscr{U}})$ is not the unique one-compactification: for example, $\widehat{X} = X \cup \{\infty\}$ equipped with the topology $\mathscr{U} \cup \big\{\{\infty\} \cup X\big\}$ is another one. Nevertheless, the uniqueness holds if we also impose Hausdorff condition on the compactification[10]:

Proposition 8 (Hausdorff one-point compactification). *The following conditions are equivalent for any non-compact Hausdorff topological space X:*

(*a*) *$\forall x \in X$, $\exists$ compact $K \subset X$ containing a neighborhood of x.* X is said to be locally compact if it satisfies this condition.

(*b*) *The one-point compactification $(\widehat{X}, \widehat{\mathscr{U}})$ of X from Prop. 7 (2) is Hausdorff.*

(*c*) *X has a Hausdorff one-point compactification.*

Moreover, under these conditions, X has a unique Hausdorff one-point compactification.

Proof. "(a) ⇒ (b)": Assuming that X is locally compact, we need to show that any two distinct points $x, y \in \widehat{X}$ have respectively neighborhoods $U, V \subset \widehat{X}$ with $U \cap V = \varnothing$. When x, y both belong to the subspace X, this follows immediately from the Hausdorff property of X itself and the definition of $\widehat{\mathscr{U}}$, so we may assume $x \in X \subset \widehat{X}$ and $y = \infty \in \widehat{X}$. In this case, let $K \subset X$ be a compact set containing a neighborhood U of x, then U and $V = \{\infty\} \cup X \setminus K$ are the required neighborhoods of x and $y = \infty$.

"(b) ⇒ (c)": Trivial.

"(c) ⇒ (a)": Let X' be a Hausdorff one-point compactification of X and denote the only point of $X' \setminus X$ by ∞. Given any $x \in X \subset X'$, by the Hausdorff property of X', x and ∞ have respective neighborhoods U and V with $U \cap V = \varnothing$. Note that $X \subset X'$ is open because the one-point set $\{\infty\} \subset X'$ is closed, so U is not only open in X', but also open in X. The set $K := X' \setminus V$ is compact (since X' is compact and K is closed), contained in X and contains the neighborhood U of x in X, which shows that X is locally compact.

Uniqueness: Given Hausdorff one-point compactifications X' and X'' of X, let $p' \in X'$ and $p'' \in X''$ be the only point of $X' \setminus X$ and $X'' \setminus X$, respectively. We only need to show that the obvious bijection $f : X' \to X''$, which identifies the subspace X of X' with the subspace X of X'' and sends p' to p'', is a homeomorphism. In order to show that f is continuous, we let $U \subset X''$ be open and will show that $f^{-1}(U) \subset X'$ is open in the following two cases respectively.

Case 1. *$p'' \notin U$, or equivalently, $U \subset X \subset X''$.* Since one-point sets in a Hausdorff space are closed, X is open in both X' and X''. It follows that for any $U \subset X$ we have

[10] In the literature, sometimes the Hausdorff condition is included in the definition of compactifications.

$$U \text{ is open in } X \Leftrightarrow U \text{ is open in } X' \Leftrightarrow U \text{ is open in } X''.$$

The required openness of $f^{-1}(U)$ follows immediate.

Case 2. $p'' \in U$. Since $X' \setminus f^{-1}(U) = f^{-1}(X'' \setminus U)$, we only need to show that the preimage of $K := X'' \setminus U$ is closed. K is compact since it is a closed subset of the compact space X''. Therefore, K is a compact subset of X. The preimage $f^{-1}(K)$ is just K itself (but here we view X as a subspace of X'), hence still compact. Since X' is Hausdorff, we conclude that $f^{-1}(K) = K \subset X''$ is closed, as required. □

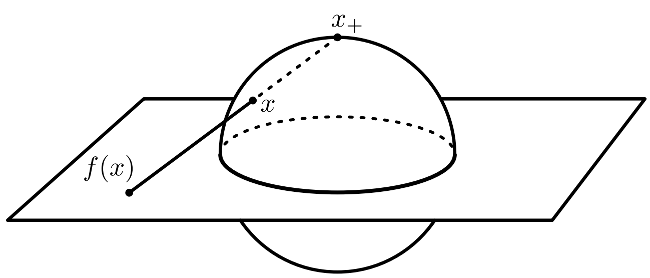
Example 12 (**The Hausdorff one-point compactification of $\mathbb{R}^n$ is $\mathbb{S}^n$**). The unit 2-sphere $\mathbb{S}^2 := \{x \in \mathbb{R}^3 \,|\, \|x\|_{\ell^2} = 1\}$ is a compact Hausdorff space. The stereographic projection from the north pole $x_+ := (0,0,1)$ is by definition the bijection

$$f : \mathbb{S}^2 \setminus \{x_+\} \to \mathbb{R}^2$$

such that $y = (y_1, y_2) \in \mathbb{R}^2$ is the image of $x \in \mathbb{S}^2 \setminus \{x_+\}$ iff. the line in $\mathbb{R}^3$ passing through x_+ and $(y_1, y_2, 0)$ intersects the sphere $\mathbb{S}^2$ at x. By elementary calculations, we obtain the following explicit expressions for f and its inverse f^{-1}:

$$f(x) = \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}\right) \quad \text{for } x = (x_1, x_2, x_3) \in \mathbb{S}^2 \setminus \{x_+\},$$
$$f^{-1}(y) = \left(\frac{2y_1}{|y|^2+1}, \frac{2y_2}{|y|^2+1}, \frac{|y|^2-1}{|y|^2+1}\right) \quad \text{for } y = (y_1, y_2) \in \mathbb{R}^2, \ \text{where } |y|^2 := y_1^2 + y_2^2.$$

If follows that f and f^{-1} are both continuous, hence f^{-1} is a homeomorphism from $\mathbb{R}^2$ to $\mathbb{S}^2 \setminus \{x_+\}$. Therefore, $\mathbb{S}^2$ is the Hausdorff one-point compactification of $\mathbb{R}^2$. The argument easily generalizes to any dimension and shows that $\mathbb{S}^n$ is the Hausdorff one-point compactification of $\mathbb{R}^n$.



Stereographic projection

Exercise 31. Show that

(1) If $f : X \to Y$ be a bijective continuous map, where X is a compact space and Y a Hausdorff space, then f is a homeomorphism.

(2) If $\mathscr{U}_1$ and $\mathscr{U}_2$ are topologies on a set X such that $(X, \mathscr{U}_1)$ and $(X, \mathscr{U}_2)$ are both compact Hausdorff, then $\mathscr{U}_1$ and $\mathscr{U}_2$ are either equal or not comparable. "Not comparable" means neither of them is strictly stronger than the other.

Exercise 32. Use Exercise 7 (3) to show that a normed vector space is locally compact iff. it is finite dimensional.

Exercise 33. In a non-compact topological space X, a sequence $(x_n)_{n=1,2,\cdots}$ is said to leave every compact set if for any compact $K \subset X$ there is $N > 0$ such that $x_n \notin K$ when $n > N$. A map $f : X \to Y$ is said to be proper if for any compact $K \subset Y$, the preimage $f^{-1}(K) \subset X$ is compact. Prove the following statements.

(1) If f is proper, then for any sequence $(x_n)_{n=1,2,\cdots}$ leaving every compact set in X, the image $(f(x_n))_{n=1,2,\cdots}$ leaves every compact set in Y.

(2) Assuming that X and Y are Hausdorff, we consider the one-point compactifications $\widehat{X}$ and $\widehat{Y}$ defined in Prop. 7 (2) and extend $f : X \to Y$ to a map $\widehat{f} : \widehat{X} \to \widehat{Y}$ sending $\infty \in \widehat{X}$ to $\infty \in \widehat{Y}$. Then f is continuous and proper if and only if $\widehat{f}$ is continuous.

1.8. **Separation, countability and metrizability.** As mentioned in §1.3, sequential limit does not behave well in a general topological space, and the notions of compactness and sequential compactness are not equivalent. These issues can be rectified by imposing separation and countability conditions on the space, as the next two propositions show. These conditions also enable us to give a criterion for when a topology is metric.

Definition 14 (**Separation and countability conditions**). *A topological space X is said to be*

(1) *T_1 if for any two distinct points x, y, there exists a neighborhood of x not containing y*
($\Leftrightarrow$ any one-point subset is closed $\Leftrightarrow$ any finite subset is closed $\Leftrightarrow$ any two-point set is disconnected);
By definition, y also has a neighborhood not containing x, although the two neighborhoods can intersect.

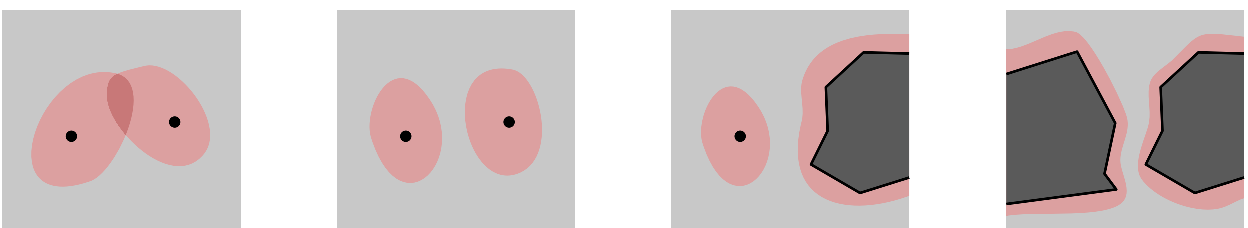
(2) *Hausdorff (or T_2) if any two distinct points x and y have respective neighborhoods U and V which are disjoint*
($\Leftrightarrow$ the diagonal $\{(x,x) \mid x \in X\}$ is closed in the product space $X \times X$);

(3) *regular (or T_3) if X is T_1 and for any point x and any closed set $C \subset X$ not containing x, there exist disjoint open sets $U, V \subset X$ which contain x and C, respectively;*

(4) *normal (or T_4) if X is T_1 and for any two disjoint closed sets $A, B \subset X$, there exist disjoint open sets $U, V \subset X$ containing A and B, respectively;*

(5) *first countable if any $x \in X$ has a countable neighborhood basis*
($\Leftrightarrow$ any $x \in X$ has a decreasing sequence of neighborhoods $U_1 \supset U_2 \supset \cdots$ which form a neighborhood basis);

(6) *second countable if there exists a countable topological basis on X generating $\mathscr{U}$.*



T_1 $\qquad$ T_2 $\qquad$ T_3 $\qquad$ T_4

$T_1 \sim T_4$ are called *separation conditions*[11] and are illustrated by the figures above. Another awkward terminology related to countability is: X is said to be separable if it contains a countable dense subset[12]. Listed below are basic facts about these definitions, partially summarized in the diagram.

normal (T4) $\Longrightarrow$ regular (T3) $\Longrightarrow$ Hausdorff (T2) $\Longrightarrow$ T1

$\underbrace{\text{separable} \quad \text{metric}}$

first countable

second countable

- Each of the 4 separation conditions is stronger than the previous one. An infinite set with finite complement topology is T_1 but not Hausdorff. See Munkres [Mun00, §31] for examples of a Hausdorff space which is not regular, and a regular space which is not normal,
- Any metric space is normal, see the proof below.

[11] "T" is from the German word *Trennungsaxiom* ("separation axiom").

[12] The most widespread usage of this term is in "separable Hilbert spaces" from Functional Analysis and Quantum Mechanics, where having a countable dense subset is equivalent to having a countable orthonormal basis. Do not confuse it with similar names in General Topology such as "separation conditions" or the "separating pair" from Def. 11.

- Second countability is stronger than first countability (by Example 6) and separability (since picking a point in each element of a countable basis yields a countable dense subset).
- Any metric space is first countable (by Example 6). A metric space is second countable iff. it is separable (by Exercise 13). A non-second-countable example: the discrete topology (= metric topology given by discrete metric) on an uncountable set (see Exercise 12).
- Let $(U_n)_{n=1,2,\cdots}$ be a decreasing sequence of neighborhoods of x which form a neighborhood basis. Then
  - Any subsequence $(U_{n_i})_{i=1,2,\cdots}$ is still a neighborhood basis.
  - If we pick a point x_n from each U_n, then the sequence $(x_n)_{n=1,2,\cdots}$ converges to x.

Proof of "any metric space is normal". Any metric space X is Hausdorff (hence T_1) because any distinct points $x, y \in X$ have respective neighborhoods $B(x, \frac{d(x,y)}{2})$ and $B(y, \frac{d(x,y)}{2})$ which are disjoint. In order to further show that X is normal, we pick disjoint closed sets $A, B \subset X$ and need to find disjoint open sets $U, V \subset X$ such that $A \subset U$ and $B \subset V$. At every $a \in A$ (resp. $b \in B$) there is a ball $B(a, \epsilon_a)$ (resp. $B(b, \epsilon_b)$) disjoint form B (resp. A). We claim that $B(a, \frac{\epsilon_a}{2}) \cap B(b, \frac{\epsilon_b}{2}) = \varnothing$ for all $a \in A, b \in B$. To prove this, suppose by contradiction that $z \in B(a, \frac{\epsilon_a}{2}) \cap B(b, \frac{\epsilon_b}{2})$ and assume $\epsilon_a \leq \epsilon_b$ (otherwise, switch the roles of a and b). Then $d(a,b) \leq d(a,z) + d(z,b) < \frac{\epsilon_a}{2} + \frac{\epsilon_b}{2} \leq \epsilon_b$, contradiction the fact that $B(b, \epsilon_b)$ is disjoint from A. Thus, the claim is proved, and it follows that we may take $U = \bigcup_{a \in A} B(a, \frac{\epsilon_a}{2})$ and $V = \bigcup_{b \in A} B(b, \frac{\epsilon_b}{2})$ to be the required open sets. $\square$

Proposition 9 (**Sequential limit behaves well in Hausdorff and first countable spaces**). *Let X be a topological space.*

(1) *If X is Hausdorff, then any convergent sequence has a unique limit.*

(2) *If X is first countable, then the following equivalences hold.*
  - *a subset $E \subset X$ is closed $\Leftrightarrow$ any limit of any convergent sequence in E is again in E;*
  - *a point $x \in X$ belongs to the closure $\overline{E}$ $\Leftrightarrow$ there exists a sequence $(x_n)_{n=1,2,\cdots}$ in E such that $x_n \to x$;*
  - *a map $f : X \to Y$ is continuous at a point $x \in X$ (where Y is any topological space)*
    $\Leftrightarrow$ for any sequence $(x_n)_{n=1,2,\cdots}$ in X converging to x, we have $f(x_n) \to f(x)$ in Y.

Proof. (1) Suppose by contradiction that a sequence $(x_n)_{n=1,2,\cdots}$ converges to two points $x, y \in X$ at the same time. Since X is Hausdorff, there is a neighborhood U of x and a neighborhood V of y such that $U \cap V = \varnothing$. But since the sequence converges to both x and y, the point x_n is contained in both U and V when n is large enough, a contradiction.

(2) We have seen in Def. 4 that for the 1st and 3rd point, the direction "$\Rightarrow$" holds in any topological space without extra assumption, and similarly for the direction "$\Leftarrow$" in the 2nd point. So we only prove below the inverse directions.

1st "$\Leftarrow$": If E is not closed, then there exists $x \notin E$ such that every neighborhood of x intersects E. Letting $(U_n)_{n=1,2\cdots}$ be a decreasing sequence of neighborhoods of x which form a neighborhood basis, we may pick $x_n \in E \cap U_n$ to obtain a sequence $(x_n)_{n=1,2,\cdots}$ in E converging to x.

2nd "$\Rightarrow$": If $x \in \overline{E}$, then every neighborhood of x intersects E. The above argument gives us a sequence $(x_n)_{n=1,2,\cdots}$ in E converging to x.

3rd "$\Leftarrow$": Suppose f is not continuous at x and let $(U_n)_{n=1,2,\cdots}$ be as above. Then there exists a neighborhood V of $f(x)$ in Y such that $f(U_n)$ is not contained in V for any n. Thus, we may pick $x_n \in U_n$ with $f(x_n) \notin V$. The resulting sequence $(x_n)_{n=1,2,\cdots}$ converges to x, while $(f(x_n))_{n=1,2,\cdots}$ does not converge to $f(x)$. $\square$

Proposition 10 (**Compactness vs. sequential compactness**).

(1) *In a second countable topological space, any sequential compact subset is compact.*

(2) *In a metric space, any sequentially compact subset is compact.*

(3) *In a first countable T_1 topological space, any compact subset is sequentially compact.*

As a consequence, compactness and sequential compactness are equivalent in metric spaces and second countable T_1 spaces.

Proof. (1) Given a second countable topological space $(X, \mathscr{U})$, let us first show that any open cover $\mathscr{C}_A \subset \mathscr{U}$ for a subset $A \subset X$ has a *countable* sub-cover.

Let $\mathscr{B} = \{B_1, B_2, \cdots\} \subset \mathscr{U}$ be a countable topological basis generating $\mathscr{U}$. We perform the following operation for each $n = 1, 2, \cdots$: if B_n is contained in some elements of $\mathscr{C}_A$, we pick one such element, and otherwise do nothing. All the elements picked form a countable subset $\mathscr{C}'_A$ of $\mathscr{C}_A$. Let us show that $\mathscr{C}'_A$ covers A, i.e. any $x \in A$ is contained in some element of $\mathscr{C}'_A$. Given x, there exists $U \in \mathscr{C}_A$ containing x (because $\mathscr{C}_A$ covers A) and some $B_n \in \mathscr{B}$ such that $x \in B_n \subset U$ (because $\mathscr{B}$ generates $\mathscr{U}$). So there exists some $U_n \in \mathscr{C}_A$ containing B_n which have been picked. Then this U_n is an element of $\mathscr{C}'_A$ containing x, as required. This shows that $\mathscr{C}_A$ has a countable sub-cover.

Now we assume that $A \subset X$ is sequentially compact and will show that A is compact, namely any open cover $\mathscr{C}_A \subset \mathscr{U}$ for A has a finite sub-cover. We have found above a countable sub-cover $\mathscr{C}'_A = \{U_1, U_2, \cdots\} \subset \mathscr{C}_A$. If $\mathscr{C}_A$ does not have any finite sub-cover, then $U_1 \cup \cdots \cup U_n$ does not cover A for $n = 1, 2, \cdots$, so we may pick $x_n \in A \setminus (U_1 \cup \cdots \cup U_n)$. The sequence $(x_n)_{n=1,2,\cdots}$ has a subsequence $(x_{n_i})_{i=1,2,\cdots}$ converging to some $x \in A$. Let U_k be an element of $\mathscr{C}'_A$ containing x. Then U_k contains x_{n_i} for any sufficiently large i, contradicting the fact that $x_n \notin U_1 \cap \cdots \cap U_n$.

(2) Let X be a metric space, $A \subset X$ be a sequentially compact subset and $\mathscr{C}_A$ be an open cover for A. We shall find a finite sub-cover $\mathscr{C}'_A \subset \mathscr{C}_A$ via the following two facts.

Fact 1. *There exists $\delta > 0$ such that any open ball $B(x, \delta)$ with $x \in A$ is contained in an element of $\mathscr{C}_A$.*

If this is not the case, then for any $n \in \mathbb{Z}_+$ there is a ball $B(x_n, \frac{1}{n})$ with $x_n \in A$ which is not contained in any element of $\mathscr{C}_A$. By sequentially compactness, the sequence $(x_n)_{n=1,2,\cdots}$ has a subsequence $(x_{n_i})_{i=1,2,\cdots}$ converging to some $x_0 \in A$. There exists some $U_0 \in \mathscr{C}_A$ containing x_0. Since U_0 is open, it also contains a ball $B(x_0, \epsilon)$ centered at x_0. But on the other hand, for all sufficiently large i, we have $x_{n_i} \in B(x_0, \frac{\epsilon}{2})$ and $\frac{1}{n_i} < \frac{\epsilon}{2}$, which implies

$$B(x_{n_i}, \tfrac{1}{n_i}) \subset B(x_{n_i}, \tfrac{\epsilon}{2}) \subset B(x_0, \epsilon) \subset U_0$$

(the 2nd "$\subset$" is because $y \in B(x, \frac{\epsilon}{2}) \Rightarrow B(y, \frac{\epsilon}{2}) \subset B(x, \epsilon)$). This contradicts the fact that $B(x_n, \frac{1}{n})$ is not contained in any element of $\mathscr{C}_A$. Thus, Fact 1 has been proven.

Fact 2. *Given any $r > 0$, there exist finitely many points $x_1, \cdots, x_m$ in A such that $A \subset B(x_1, r) \cup \cdots \cup B(x_m, r)$.*

If this is not the case, then we can find a sequence $(x_n)_{n=1,2,\cdots}$ in A such that $d(x_i, x_j) \geq r$ for any $i \neq j$ by the following induction procedure:

- Pick any point of A as x_1.
- Suppose $x_1, \cdots, x_i$ are chosen. Since A is not contained in $B(x_1, r) \cup \cdots \cup B(x_i, r)$ by assumption, we may pick a point of A not in this union as x_{i+1}.

It is easy to see that (x_n) does not have any convergent subsequence, contradicting the sequential compactness of A. This proves Fact 2.

Now we can find a finite sub-cover of $\mathscr{C}_A$ as follows. Consider the value δ given by Fact 1, and use Fact 2 to get $x_1, \cdots, x_m$ such that $A \subset B(x_1, \delta) \cup \cdots \cup B(x_m, \delta)$. For each x_i, there exists $U_i \in \mathscr{C}_A$ containing $B(x_i, \delta)$. Then $U_1, \cdots, U_m$ form the required sub-cover.

(3) Let $(X, \mathscr{U})$ be a T_1 topological space, $A \subset X$ be compact, and $(x_n)_{n=1,2,\cdots}$ be a sequence in A. We need to show that (x_n) has a subsequence converging to some $x \in A$.

If a point $x \in A$ appears in the sequence infinitely many times (i.e. $x_n = x$ for infinitely many n's), then the required conclusion is immediate. So we may assume that every $x \in A$ appears at most finitely many times. By taking a subsequence, this is further reduced to the case where every $x \in A$ appears at most once, namely (x_n) *is a sequence of distinct points*. Let us fix such a sequence (x_n) from now on. We claim that the followings are equivalent for any $y \in A$:

(a) y has a neighborhood U such that $U \setminus \{y\}$ does not intersect (x_n);

(b) (x_n) does not have any subsequence converging to y.

The implication "(a) ⇒ (b)" is obvious but won't be used later. To show "(b) ⇒ (a)", we assume by contradiction that $U \setminus \{y\}$ intersects (x_n) for any neighborhood U of y and will find a subsequence $(x_{n_i})_{i=1,2,\cdots}$ converging to y. To this end, let $(U_m)_{m=1,2,\cdots}$ be a decreasing sequence of neighborhoods of y which is a neighborhood basis. We construct (x_{n_i}) by the following procedure (which also yields a subsequence of neighborhoods $(U_{m_i})_{i=1,2,\cdots}$ at the same time):

- Take $m_1 = 1$, and take x_{n_1} to be any member of the sequence (x_n) contained in $U_1 \setminus \{y\} = U_{m_1} \setminus \{y\}$.
- By the T_1 assumption, $U_1 \setminus \{x_{n_1}\} = U_{m_1} \setminus \{x_{n_1}\}$ is open, hence is a neighborhood of y, so it contains some element of the neighborhood basis. Pick such an element as U_{m_2}. Then take x_{n_2} to be any member of (x_n) in $U_{m_2} \setminus \{y\}$.
- Continue the procedure: once $U_{m_1}, \cdots, U_{m_i}$ and $x_{n_1}, \cdots, x_{n_i}$ have been chosen, we take $U_{m_{i+1}}$ to be contained in $U_{m_i} \setminus \{x_{n_1}, \cdots, x_{n_i}\}$ and $x_{n_{i+1}}$ to be any member of (x_n) in $U_{m_i} \setminus \{y\}$.

By construction, we have $x_{n_i} \in U_{m_i}$. It follows that $(x_{n_i})_{i=1,2,\cdots}$ converges to y. Thus, the claim is proved

Now we can show that (x_n) has a subsequence converging to some $y \in A$. Suppose by contradiction that it is not the case, then by the above claim, every $y \in A$ has a neighborhood U_y such that $U \setminus \{y\}$ does not intersect (x_n). As y runs over A, the U_y's form an open cover for A. Since A is compact, it has a finite sub-cover. So there are finitely many points $y_1, \cdots, y_k \in A$ such that $A \subset U_{y_1} \cup \cdots \cup U_{y_k}$. Then it would be impossible that (x_n) is a sequence of distinct points in A and does not intersect $U \setminus \{y_1\}, \cdots, U \setminus \{y_k\}$. This contradiction finishes the proof. □

Exercise 34. Consider the discrete topology $\mathscr{U}_{\text{discrete}}$, the finite complement topology $\mathscr{U}_{\text{f.c.}}$ and the countable complement topology $\mathscr{U}_{\text{c.c.}}$ on an uncountable set X. Show that the topological spaces $(X, \mathscr{U}_{\text{f.c.}})$ and $(X, \mathscr{U}_{\text{c.c.}})$ are not first countable, while $(X, \mathscr{U}_{\text{discrete}})$ is first countable but not second countable.

Exercise 35. Let X be a non-compact topological space with the property that compact subsets of X are exactly sequentially compact ones (e.g. X can be a metric space or second countable T_1 space). Show that

(1) A sequence $(x_n)_{n=1,2,\cdots}$ in X leaves every compact set (Exercise 33) iff. it does not have any convergent subsequence.

(2) Let Y be another such space. Then the converse of Exercise 33 (1) holds, namely a map $f : X \to Y$ is proper iff. for any sequence $(x_n)_{n=1,2,\cdots}$ leaving every compact set in X, the image $(f(x_n))_{n=1,2,\cdots}$ leaves every compact set in Y.

Exercise 36. Let X be a T_1 topological space. Show that

(1) X is regular iff. for any neighborhood U of a point $x \in X$, there exists another neighborhood V of x such that $\overline{V} \subset U$.

(2) X is normal iff. for any open set U containing a closed set C, there exists another open set V containing C such that $\overline{V} \subset U$.

Exercise 37. Show that in a Hausdorff space X, for any disjoint compact sets $A, B \subset X$ there exist disjoint open sets $U, V \subset X$ containing A and B, respectively. Conclude that any compact Hausdorff space is normal.

A topological space $(X, \mathscr{U})$ is said to be metrizable if $\mathscr{U}$ is the metric topology of some metric on X. It is a natural but difficult question to find a criterion for metrizability. The first breakthrough is:

Theorem 3 (**Urysohn Metrization Theorem**)**.** *For any topological space, the following two conditions are equivalent:*

(a) metrizable and contains a countable dense subset;

(b) second countable and regular.

We have already seen "(a) ⇒ (b)" in the discussion after Def. 14, so the difficulty lies in the converse "(b) ⇒ (a)". Note that since metric spaces are norm, the theorem implies the nontrivial fact that a topological space satisfying (b) is norm. To prove "(b) ⇒ (a)", we actually have to establish this fact first.

Sketch of proof for "(b) ⇒ (a)". (The 3 steps are stated as Thm. 32.1, 33.1 and 34.1 in [Mun00] with detailed proofs.)

Step 1. *Prove "(b) ⇒ norm".*

Suppose X is second countable and regular, and let $A, B \subset X$ be disjoint closed subsets. By regularity, we may pick a neighborhood U_a for every $a \in A$ and a neighborhood V_b for every $b \in A$ such that

$$\overline{U}_a \cap B = \varnothing, \quad \overline{V}_b \cap A = \varnothing. \tag{1.13}$$

(cf. Exercise 36). In the proof of Prop. 10 (1), we have seen that in a second countable space, any open cover for a subset has a countable sub-cover. Therefore, there exist a sequence $(a_n)_{n=1,2,\cdots}$ in A and a sequence $(b_n)_{n=1,2,\cdots}$ in B such that

$$A \subset \bigcup_{n=1}^{\infty} U_{a_n}, \quad B \subset \bigcup_{n=1}^{\infty} V_{b_n} \tag{1.14}$$

Using (1.13) and (1.14), it can be checked that if we put

$$U'_n := U_{a_n} \setminus \bigcup_{i=1}^{n} \overline{V}_{b_i}, \quad U' := \bigcup_{n=1}^{\infty} U'_n, \quad V'_n := V_{b_n} \setminus \bigcup_{i=1}^{n} \overline{U}_{a_i}, \quad U' := \bigcup_{n=1}^{\infty} U'_n,$$

then U' and V' are disjoint open sets containing A and B, respectively. This shows that X is norm.

Step 2. *Prove the "Urysohn Lemma": If a topological space X is normal and $A, B \subset X$ are disjoint closed sets, then there exists a continuous function $f : X \to [0,1] \subset \mathbb{R}$ such that $f \equiv 0$ on A and $f \equiv 1$ on B.*

Let X be normal and $A, B \subset X$ be disjoint closed sets. Choose a countable dense subset P of the interval $[0,1]$ containing 0 and 1 (for example $P = \mathbb{Q} \cap [0,1]$). We shall construct an open set $U_p \subset X$ for every $p \in P$ such that

$$A \subset U_0, \quad U_1 = X \setminus B, \quad \overline{U}_p \subset U_q \text{ for all } p, q \in P \text{ with } p, q. \tag{1.15}$$

To this end, we label all points of P as $(p_n)_{n=1,2,\cdots}$ with $p_1 = 1$, $p_2 = 0$ (this sequence is neither increasing nor decreasing), and will construct $U_{p_1}, U_{p_2}, U_{p_3}, \cdots$ consecutively by the following induction procedure.

- Put $U_{p_1} = U_1 := X \setminus B$. By normality, there exists an open set containing A whose closure is contained in $X \setminus B$. Take this open set to be $U_{p_2} = U_0$.
- Assuming that $U_{p_1}, \cdots, U_{p_n}$ have been obtained and satisfy

$$\overline{U}_{p_i} \subset U_{p_j} \text{ for any } i, j \in \{1, \cdots, n\} \text{ such that } p_i < p_j, \tag{1.16}$$

  we construct $U_{p_{n+1}}$ as follows. The point p_{n+1} is located exactly in between two previous points p_i and p_j for some $i, j \in \{1, \cdots, n\}$ (in the sense that $p_i < p_{n+1} < p_j$ and there is no p_k with $k \in \{1, \cdots, n\}$ such that $p_i < p_k < p_j$). By (1.16) and normality, there exists an open set U such that $\overline{U}_{p_i} \subset U$ and $\overline{U} \subset U_{p_j}$, which we define to be $U_{p_{n+1}}$. $U_{p_1}, \cdots, U_{p_{n+1}}$ still satisfies (1.16) (with n replaced by $n+1$), so the induction can go on.

Once a family $(U_p)_{p \in P}$ of open sets satisfying (1.15) has been obtained, we may define

$$f : X \to [0,1], \quad f(x) := \inf\{p \in P \,|\, x \in U_p\}.$$

It can be check that f is continuous and satisfies $f^{-1}([0,p)) = U_p$, $f^{-1}([0,p]) = \overline{U}_p$. The required properties $f|_A \equiv 0$ and $f|_B \equiv 1$ follow immediately.

Step 3. *Given a second countable regular space X, construct a map f from X to the metric space $([0,1]^{\mathbb{Z}_+}, d_{uniform})$ which is a homeomorphism to the image.*

Here, an element of $[0,1]^{\mathbb{Z}_+}$ is a sequence $x = (x_n)_{n=1,2,\cdots}$ in $[0,1]$, while d_{uniform} is the *uniform metric*, defined by

$$d_{\text{uniform}}(x,y) := \sup\left\{|x_n - y_n| \,\middle|\, n \in \mathbb{Z}_+\right\} \ \text{ for } x,y \in [0,1]^{\mathbb{Z}_+}$$

(i.e., d_{uniform} is the restriction of the metric on the normed vector space ℓ^∞ (see Example 9) to the subset $[0,1]^{\mathbb{Z}_+} \subset \ell^\infty$). The required map $f : X \to [0,1]^{\mathbb{Z}_+}$ is given by a sequence of continuous functions $f_n : X \to [0,1]$, $n = 1,2,\cdots$ with the following properties:

(i) For any point $x \in X$ and any neighborhood U of x, there exists an index $n \in \mathbb{Z}_+$ such that f_n is positive at x and vanishes outside of U.

(ii) $f_n \le \frac{1}{n}$ for all n.

To achieve (i), we let $\mathscr{B}$ be a countable basis generating the topology of X and consider pairs $(B_1,B_2) \in \mathscr{B} \times \mathscr{B}$ such that $\overline{B}_1 \subset B_2$. The set of all such pairs is countable, hence can be indexed by $\mathbb{Z}_+$. Meanwhile, for each pair (B_1,B_2), by Steps 1 and 2, there is a continuous function $X \to [0,1]$ which is 0 on $\overline{B}_1$ and 1 on $X \setminus B_2$. This gives a sequence of continuous functions $(f_n)_{n=1,2,\cdots}$. Using Exercise 36 (2), we see that it satisfies (i). Furthermore, we may divide each f_n by n so that property (ii) is satisfied as well.

Once a sequence (f_n) with properties (i) and (ii) has been found, we put $f(x) := (f_1(x), f_2(x), \cdots)$. Using these properties, it can be check that f is a homeomorphism to the image, as required. □

Example 13 (**$\mathbb{R}_\ell$ is normal but not metrizable**). As discussed in Example 10, the Sorgenfrey line $\mathbb{R}_\ell$ is defined as $\mathbb{R}$ equipped with the topology $\mathscr{U}_{\text{right}}$ generated by the basis formed by all intervals of the form $[a,b)$. It holds that

- $\mathbb{R}_\ell$ is first countable and has a countable dense subset, but is not second countable, hence not metrizable.
- $\mathbb{R}_\ell$ is normal.

Proof of these properties. It is easy to see that $\left([x, x+\frac{1}{n})\right)_{n=1,2,\cdots}$ is a neighborhood basis of any $x \in \mathbb{R}_\ell$ and $\mathbb{Q}$ is dense in $\mathbb{R}_\ell$. Given a topological basis $\mathscr{B}$ generating $\mathscr{U}_{\text{right}}$, since $[a,+\infty)$ is open, there exists some $B \in \mathscr{B}$ such that $a \in B \subset [a,+\infty)$. Thus, for any $a \in \mathbb{R}$ there is an element of $\mathscr{B}$ whose smallest point is a, and it follows that $\mathscr{B}$ contains at least as many elements as $\mathbb{R}$. Finally, given disjoint closed sets $A, B \subset \mathbb{R}_\ell$, we may pick, for every $a \in A$ (resp. $b \in B$), a neighborhood of the form $[a, a+\epsilon_a)$ (resp. $[b, b+\epsilon_b)$) which does not intersect B (resp. A). Then $U_A := \bigcup_{a\in A}[a, a+\epsilon_a)$ and $U_B := \bigcup_{b\in A}[b, b+\epsilon_b)$ are open sets containing A and B, respectively, and are disjoint because if $U_A \cap U_B \neq \varnothing$ then there exist $a \in A$, $b \in B$ such that $[a,a+\epsilon_a) \cap [b, b+\epsilon_b) \neq \varnothing$, which implies that either $b \in [a, a+\epsilon_a)$ or $a \in [b, b+\epsilon_b)$, a contradiction. □

The *Nagata-Smirnov Metrization Theorem* improves on the techniques of the above theorem and gives a more complicated necessary and sufficient condition for any topological space to be metrizable. See [Mun00, §40] for details.

2. Further topics

2.1. **Quotient topology and gluing.** Recall that given an equivalence relation "$\sim$" on a set X, the quotient set, denoted by $X/\sim$, is the set whose elements are the equivalence classes. There is a natural projection $X \to X/\sim$ sending every $x \in X$ to the equivalence class $[x] \in X/\sim$ containing x. In fact, putting an equivalence relation on a set X is the same as giving a surjective map from X to another set Y, because such a map yields an equivalence relation "$\sim_f$" defined by

$$x_1 \sim_f x_2 \overset{\text{def.}}{\Leftrightarrow} f(x_1) = f(x_2),$$

while Y identifies with the quotient set $X/\sim_f$, and the map f identifies with the projection to the quotient set.

Definition 15 (**Quotient map/topology**).

(1) *Given a surjective map f from a topological space X to a set Y, there exists a unique topology on Y, called the quotient topology (w.r.t. the map $f : X \to Y$), such that*

$$V \subset Y \text{ is open} \Leftrightarrow f^{-1}(V) \subset X \text{ is open}. \tag{2.1}$$

(2) *Given topological spaces X and Y, a map $f : X \to Y$ is called a quotient map if it is surjective and satisfies (2.1)* (in other words: if f is surjective and the topology on Y is the quotient topology w.r.t. f).

(3) *Given an equivalence relation "$\sim$" on a topological space X, we view the quotient set $X/\sim$ as a topological space by using the quotient topology w.r.t. the projection map $X \to X/\sim$, and call it the quotient space.*

Proof of the statement in (1). Let $\mathscr{U}$ denote the topology on X. The only thing we need to prove is that the collection $\mathscr{V}$ of subsets of Y defined by

$$\mathscr{V} := \{V \subset Y \mid f^{-1}(V) \in \mathscr{U}\}$$

is indeed a topology. But this follows immediately from properties of preimages (namely $f^{-1}(\varnothing) = \varnothing$, $f^{-1}(Y) = X$, $f^{-1}(\bigcup_{i\in I} V_i) = \bigcup_{i\in I} f^{-1}(V_i)$ and $f^{-1}(\bigcap_{i\in I} V_i) = \bigcap_{i\in I} f^{-1}(V_i)$) and the fact that $\mathscr{U}$ is a topology. □

Basic observations about these definitions:

- The topology on $X/\sim$ in (3) can be described directly as follows: *$U \subset X/\sim$ is open iff. $\{x \in X \mid [x] \in U\} \subset X$ is open* (i.e., a set of equivalence classes is open in $X/\sim$ iff. the set formed by all elements of these classes is open in X).
- A quotient map must be continuous, since the direction "$\Rightarrow$" in (2.1) is exactly the condition for continuity.
- Not every surjective continuous map is a quotient map. For example, if X is a discrete topological space, then any map $f : X \to Y$ is continuous, but it is easy to see that f is a quotient map iff. f is surjective and Y is also discrete.
- Condition (2.1) is equivalent to

$$V \subset Y \text{ is closed } \Leftrightarrow f^{-1}(V) \subset X \text{ is closed.} \tag{2.2}$$

- Every continuous *open* map (see Exercise 11) satisfies (2.1), while every continuous *closed* map satisfies (2.2). It follows that open or closed continuous maps are quotient maps. In particular, since the projection from a product topological space $X_1 \times \cdots \times X_n$ to each factor X_i is continuous and open (Lemma 5 (1)), X_i is a quotient space of the product.
- In general, a quotient map can be neither open nor closed.

Two special types of quotient spaces, *gluing* and *quotient by group action*, are particularly useful.

Example 14 (Gluing). ...

Example 15 (Quotient by group action). ...

2.2. Manifolds and classification.

2.3. Homotopy equivalence and rudiments of Algebraic Topology. ...

Theorem 4 (Fundamental but highly nontrivial facts in Topology).

(1) *The spheres $\mathbb{S}^n$, $n = 1, 2, \cdots$ are not homotopy equivalent to each other and are all non-contractible.*

(2) *There is not continuous map $\overline{\mathbb{B}^n} \to \mathbb{S}^n$ whose restriction to $\mathbb{S}^n \subset \overline{\mathbb{B}^n}$ is the identity.*

(3) *The Euclidean spaces $\mathbb{R}^n$, $n = 1, 2, \cdots$ are not homeomorphic to each other.*

2.4. Brouwer's Fixed Point Theorem and Invariance of Domain.