

# UofT-MAT257-2014 Note

hysw, etc(will add later)...

Wednesday 26<sup>th</sup> November, 2014  
at 15:45

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# 1 Basic knowledge

Note, this section is for something that does not fit anywhere

## 1.1 Abbreviations

**cts** Continuous

**msr** Measure

## 1.2 Multi-index Notation

- $\alpha = (\alpha_1, \dots, \alpha_n)$
- $|\alpha| = \alpha_1 + \dots + \alpha_n$
- $\alpha! = \alpha_1! \dots \alpha_n!$
- $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$
- $\partial^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n} f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$

## 2 Linear Algebra

### 2.1 Definitions

**norm** A function  $p : V \rightarrow \mathbb{R}$  such that

For all  $a \in F$  and all  $u, v \in V$

- $p(v) \geq 0 \wedge [p(v) = 0 \iff v = 0]$  (separates points)
- $p(av) = |a|p(v)$  (absolute homogeneity)
- $p(u + v) \leq p(u) + p(v)$  (triangle inequality)

**inner product** A function  $\langle x, y \rangle : V \times V \rightarrow \mathbb{R}$  such that

For all  $x, y, z \in V$  and  $c \in \mathbb{F}$ .

- $\langle x, y \rangle = \langle y, x \rangle$
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- $\langle cx, y \rangle = c\langle x, y \rangle = \langle x, cy \rangle$
- $\langle x, x \rangle > 0$  if  $x \neq 0$

## 3 Topology

### 3.1 Definitions

**metric** A function  $d : X \times X \rightarrow \mathbb{R}$  such that

For all  $x, y, z \in X$

- $d(x, y) \geq 0$
- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$

**$\epsilon$ -neighborhood**  $U(x; \epsilon) = \{y | d(x, y) < \epsilon\}$

**open set in metric space** A set  $U \subseteq X$  is said to be open in  $X$  if

$$\forall x \in U \exists \epsilon > 0 [U(x; \epsilon) \subseteq U]$$

note that finite intersections and arbitrary unions of open set are open set

**closed set in metric space** A set contains all its limit point.

note that closed set is complement of open set in topology

## 4 Measure Theory

### 4.1 Measure zero

Let  $A \subseteq \mathbb{R}^n$ . We say  $A$  has measure zero in  $\mathbb{R}^n$  if for every  $\epsilon > 0$ , there is a covering  $Q_1, Q_2, \dots$  of  $A$  by countably many rectangles such that  $\sum_{i=1}^{\infty} v(Q_i) < \epsilon$ . If this inequality holds, we often say that the total volume of the rectangles  $Q_1, Q_2, \dots$  is less than  $\epsilon$ .

### 4.2 Theorems

{Munkers-11.1}

1. If  $B \subseteq A$  and  $A$  has measure zero in  $\mathbb{R}^n$ , then so does  $B$ .
2. Let  $A$  be the union of the collection of sets  $A_1, A_2, \dots$ . If each  $A_i$  has measure zero, so does  $A$ .
3. A set  $A$  has measure zero in  $\mathbb{R}^n$  if and only if

## 5 General Calculus

### 5.1 Definitions

#### oscillation

Given  $a \in Q$  define  $A_\delta = \{f(x) | x \in Q \wedge |x - a| < \delta\}$ . Let  $M_\delta(f) = \sup A_\delta$ , and let  $m_\delta(f) = \inf A_\delta$ , define oscillation at  $f$  by  $\text{osc}(f; a) = \inf_{\delta > 0} [M_\delta(f) - m_\delta(f)]$ .  $f$  is cts at  $a$  iff  $\text{osc}(f; a) = 0$

### 5.2 Extreme Value Theorem

Suppose  $f : X \rightarrow \mathbb{R}$  is continuous and  $X$  is compact, then  $\exists x_0 \in X$  such that  $\forall x \in X. f(x) \leq f(x_0)$ .

### 5.3 Intermediate Value Theorem

Suppose  $E \in \mathbb{R}$  is connected and  $f : E \rightarrow \mathbb{R}$  is continuous. Suppose  $f(x) = a$  and  $f(y) = b$  for some  $x, y \in E$  and  $a < b$ . Then  $\forall a < c < b \exists$  some  $z \in E$  such that  $f(z) = c$ .

### 5.4 Mean Value Theorem

Suppose  $\phi : [a, b] \rightarrow \mathbb{R}$  is

- continuous at each point of **closed** interval  $[a, b]$
- differentiable at each point of **open** interval  $(a, b)$

Then there exists a point  $c \in (a, b)$  such that  $\phi(b) - \phi(a) = \phi'(c)(b - a)$ .

## 6 Differential Calculus

### 6.1 Definitions

**differentiable**  $f$  is differentiable at  $a$  if there is an  $n$  by  $m$  matrix  $B$  such that

$$\frac{f(a+h) - f(a) - B \cdot h}{|h|} \rightarrow 0 \quad \text{as } h \rightarrow 0$$

The matrix  $B$  is unique.

**Directional derivative** Given  $u \in \mathbb{R}^m$  which  $u \neq 0$  define

$$f'(a; u) = \lim_{t \rightarrow 0} \frac{f(a + tu) - f(a)}{t}$$

Provide the limit exists.

**Partial derivative** Define the  $j^{\text{th}}$  partial derivative of  $f$  at  $a$  to be the directional derivative of  $f$  at  $a$  with respect to the vector  $e_j$ , provide derivative exists.

$$D_j f(a) = \lim_{t \rightarrow 0} \frac{f(a + te_j) - f(a)}{t}$$

### 6.2 Notations

$Df(a)$  : derivative of  $f$  at  $a$

$f'(a; u)$  : directional derivative of  $f$  at  $a$  respect to vector  $u$ .

$D_j f(a)$  :  $j^{\text{th}}$  partial derivative of  $f$  at  $a$ .

$f_i$  :  $i^{\text{th}}$  component function of  $f$ .

$\nabla g$  : gradient of  $g$ ,  $\nabla g = \mathbf{grad} g = \sum_i (D_i g) e_i$

$Jf$  : Jacobian matrix,  $J_{ij} = D_j f_i(a)$

### 6.3 Differentiability Theorems

#### Theorems Munkers.5.1

If  $f$  is differentiable at  $a$  then all directional derivative of  $f$  at  $a$  exist and  $f'(a; u) = Df(a) \cdot u$



**Theorems Munkers.5.2**

If  $f$  is differentiable at  $a$  then  $f$  is continuous at  $a$ .

**Theorems Munkers.5.3**

If  $f$  is differentiable at  $a$  then  $Df(a) = [D_1f(a) \quad D_2f(a) \quad \cdots \quad D_nf(a)]$ .

**Theorems Munkers.5.4**

- a.  $[f \text{ is differentiable at } a] \Leftrightarrow \forall i [f_i \text{ is differentiable at } a]$ .
- b. If  $f$  is differentiable at  $a$ , then its derivative is the  $n$  by  $m$  matrix whose  $i^{\text{th}}$  row is the derivative of the function  $f_i$ .  $(Df(a))_i = Df_i(a)$

**6.4 Continuously Differentiable Functions**

A function is  $C^1$  if all of its partial derivatives are continuous. A function is  $C^r$  if all of its partial derivatives are  $C^{r-1}$ .

**Munkers 6.1**

If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then there exists  $c \in (a, b)$  such that  $f(b) - f(a) = f'(c)(b - a)$ .

**Munkers 7.3**

Let  $A$  be open in  $\mathbb{R}^m$ ; let  $f : A \rightarrow \mathbb{R}$  be differentiable on  $A$ . If  $A$  contains the line segment with end points  $a$  and  $a + h$ , then there is a point  $c = a + th$  with  $0 < t < 1$  of this line segment such that  $f(a + h) - f(a) = (Df(c))h$ .

**Munkers 6.2**

Let  $A$  be open in  $\mathbb{R}^m$ . Suppose that the partial derivative  $D_i f_i(x)$  of the component function of  $f$  exists at each point  $x$  of  $A$  and are continuous on  $A$ . Then  $f$  is differentiable at each point of  $A$ .

**Munkers 6.3**

Let  $A$  be open in  $\mathbb{R}^m$ , let  $f : A \rightarrow \mathbb{R}$  be a function of class  $\mathbf{C}^2$ . Then for each  $a \in A$ :  $D_k D_j f(a) = D_j D_k f(a)$ .

**6.5 Chain Rule**

Let  $A \subset \mathbb{R}^m$ . Let  $B \subset \mathbb{R}^n$ . Let  $f : A \rightarrow \mathbb{R}^n$  and  $g : B \rightarrow \mathbb{R}^p$ , with  $f(A) \subset B$ . Suppose  $f(a) = b$ . If  $f$  is differentiable at  $a$  and  $g$  is differentiable at  $b$ , then the composite function

$g \circ f$  is differentiable at  $a$ . Furthermore,

$$D(g \circ f)(a) = Dg(b) \cdot Df(a)$$

## 6.6 Inverse Function Theorem

Let  $A$  be open in  $\mathbb{R}^n$ . Let  $f : A \rightarrow \mathbb{R}^n$  be of class  $C^r$ .

**IF**  $Df(x)$  is invertible at  $a \in A$ .

**THEN** There exists a neighborhood of  $a$  such that

- $f|_U$  is injective AND  $f(U) = V$  open in  $\mathbb{R}^n$
- the inverse function is of class  $C^r$
- $f^{-1}(y) = [f'(f^{-1}(y))]^{-1}$

## 6.7 Implicit Function Theorem

Suppose  $f : A \rightarrow \mathbb{R}^n$  be of class  $C^r$ .

Write  $f$  in the form  $f(x, y)$ , for  $x \in \mathbb{R}^k$  and  $y \in \mathbb{R}^n$ .

**IF**  $(a, b) \in A$  AND  $f(a, b) = 0$  AND  $\det \frac{\partial f}{\partial y}(a, b) \neq 0$

**THEN** There exists  $B \in \mathbb{R}^k$ ,  $a \in B$  and a unique  $g : B \rightarrow \mathbb{R}^n$  such that  $g(a) = b$  AND  $\forall x \in B. f(x, g(x)) = 0$  AND  $g$  is  $C^r$

### Munkers 9.1

Let  $A$  be open in  $\mathbb{R}^{k+n}$ ,  $B$  be open in  $\mathbb{R}^k$ .

Let  $f : A \rightarrow \mathbb{R}^n$ ,  $g : B \rightarrow \mathbb{R}^n$  be differentiable.

Write  $f$  in the form  $f(x, y)$ , for  $x \in \mathbb{R}^k$  and  $y \in \mathbb{R}^n$ .

**IF**  $f(x, g(x)) = 0$  AND  $\frac{\partial f}{\partial y}$  is invertible

**THEN**  $Dg(x) = - \left[ \frac{\partial f}{\partial y}(x, g(x)) \right]^{-1} \cdot \frac{\partial f}{\partial x}(x, g(x))$

## 6.8 Taylor's theorem

Suppose  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is of class  $C^k$  on an open convex set  $S$ . If  $a \in S$  and  $a + h \in S$ , then

$$f(a + h) = \sum_{|\alpha| \leq k} \frac{\partial^\alpha f(a)}{\alpha!} h^\alpha + R_{a,k}(h),$$

If  $f$  is of class  $C^{k+1}$  on  $S$ , for some  $c \in (0, 1)$  we have

$$R_{a,k}(h) = \sum_{|\alpha|=k+1} \frac{\partial^\alpha f(a + ch)}{\alpha!} h^\alpha$$

## 7 Integral Calculus

### 7.1 Fundamental theorem of Calculus

- If  $f$  is continuous on  $[a, b]$ , and if  $F(x) = \int_a^x f$  for  $x \in [a, b]$ , then  $F'(x)$  exists and equals  $f(x)$ .
- If  $f$  is continuous on  $[a, b]$ , and if  $g$  is a function such that  $g'(x) = f(x)$  for  $x \in [a, b]$  then  $\int_a^b f = g(b) - g(a)$

### 7.2 Fubini's theorem

Let  $Q = A \times B$ , where  $A$  is a rectangle in  $\mathbb{R}^k$  and  $B$  is a rectangle in  $\mathbb{R}^n$ . If  $f$  is bounded function and integrable over  $Q$ , then  $\int_{y \in B} f(x, y)$  and  $\int_{x \in A} f(x, y)$  are integrable over  $A$  and

$$\int_Q f = \int_{x \in A} \int_{y \in B} f(x, y) = \int_{x \in A} \overline{\int_{y \in B} f(x, y)}$$

## 8 Topics for First Midterm

### 8.1 Basic Definitions

**norm**

see note [basic:definitions:norm](#)

**inner product**

see note [basic:definitions:inner product](#)

**metric / distance functions**

see note [basic:definitions:metric](#)

**open and closed subset of metric space**

see note [basic:definitions:metricspace-openset](#) and [basic:definitions:metricspace-closedset](#)

### 8.2 Basic Theorems

**EVT** [Extreme value theorem](#)

**IVT** [Intermediate value theorem](#)

**MVT** [Mean value theorem](#)

**IVFT** [Inverse Function Theorem](#)

**IPFT** [Implicit Function Theorem](#)

### 8.3 Basic topology of metric spaces

**closure**

the set with all its limit points

**interior**

union of all open set contained in A

**exterior**

union of all open set disjoint from A

**boundary**

points that are neither interior nor exterior

**limits**  $f : A \subseteq X \rightarrow Y$

$f(x) \rightarrow y_0$  as  $x \rightarrow x_0$  if  $\forall$  open  $V \ni y_0 \exists$  open  $U \ni x_0 [x \in U \cap A \wedge x \neq x_0 \rightarrow f(x) \in V]$

**continuity**  $f$  is cts at  $x_0$  if  $x_0$  is isolated point or  $(\lim_{x \rightarrow x_0} f(x)) = f(x_0)$

**Cauchy sequences** A sequences  $\langle x_i \rangle$  is Cauchy if

$\forall \varepsilon \exists N [n, m > N \implies d(x_m, x_n) < \varepsilon]$

**completeness** A metric space  $X$  is complete if every Cauchy sequences converge (to some point in  $X$ ).

**compact sets** every open cover of  $X$  has a finite subcover

**connected sets**  $X$  cannot be divided into two disjoint nonempty closed/open/clopen sets.

**relatively open sets** p26  $A$  is relatively open in  $Y \subseteq X$  if  $\exists$  open  $U \subseteq X$  such that  $A = U \cap Y$

**Proposition**  $f : X \rightarrow Y$  is cts iff  $\forall$  open  $V \in Y$ ,  $F^{-1}(V)$  is open in  $X$ . Similarly for closed.

**Bolzano–Weierstrass property**

A subset  $E \in \mathbb{R}^n$  satisfies the BW property if every suquence has a convergent subsequence.

**Bolzano–Weierstrass theorem**  $E \in \mathbb{R}^n$  satisfies the BW property iff  $E$  is closed and bounded.

**Heine-Borel theorem**  $E \in \mathbb{R}^n$  is compact iff  $E$  is closed and bounded.

**Application/(topological invariant)** Suppose  $f : X \rightarrow Y$  is continuous and  $X$  is compact then  $f(X)$  is compact

**Extreme value theorem** Suppose  $f : X \rightarrow \mathbb{R}$  is continuous and  $X$  is compact then  $\exists x_0 \in X$  such that  $f(x) \leq f(x_0) \forall x \in X$ .

**Path connected** A set  $E$  is path connected if  $\forall x, y \in E$ ,  $\exists$  continuous map  $f : [a, b] \rightarrow E$  such that  $f(a) = x$  and  $f(b) = y$ .

**Proposition** If  $E$  is connected, and  $f : E \rightarrow Y$  is continuous then  $f(E)$  is connected

**Proposition** If  $E$  is path connected then  $E$  is connected.

**Intermediate Value Theorem** Suppose  $E \in \mathbb{R}$  is connected and  $f : E \rightarrow \mathbb{R}$  is continuous. Suppose  $f(x) = a$  and  $f(y) = b$  for some  $x, y \in E$  and  $a < b$ . Then  $\forall a < c < b \exists$  some  $z \in E$  such that  $f(z) = c$ .

**The  $\epsilon$ -neighborhood theorem** Let  $X$  be a compact subspace of  $\mathbb{R}$ ; Let  $U$  be an open set of  $\mathbb{R}^n$  containing  $X$ , Then there is an  $\epsilon > 0$  such that the  $\epsilon$ -neighborhood of  $X$  is contained in  $U$ .

Cauchy-Schwarz inequality; all norms on a finite-dimensional vector space are equivalent; Bolzano Weierstrass theorem; Heine-Borel theorem; the continuous image of a compact set is compact; the continuous image of a connected set is connected; intermediate value theorem; extreme value theorem. minima and maxima of continuous functions on compact sets

## 8.4 Differentiation

### Derivative

- [definition of the differentiable](#)
- [partial derivatives](#)
- [directional derivatives](#)

### chain rule

- $(f \circ g)' = (f' \circ g) \cdot g'$
- $\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$

### continuity and differentiability

- differentiable implies continuity
- $C^1$  implies differentiable
- $C^2$  implies equality of mixed partial derivatives

### Jacobian matrix

$$Jf = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

**continuously differentiable functions** if the derivative exists and the derivative is continuous

**higher order derivatives** second derivative or higher

**gradient**  $\vec{\nabla} f = \sum_i (D_i f) e_i$  (aka,  $(\nabla f(x)) \cdot v = f'(x; v)$ )

**geometry of the Jacobian, the rows, the columns** TODO

## 8.5 Max-min problems

**Multi-index Notation** see note

**Taylor's theorem** The Taylor series is useful at critical points because if  $a$  is a critical point of  $f$  and  $f$  is  $C^2$  at  $a$  then

$$f(a + h) = f(a) + \frac{1}{2} \sum_{i,j=1}^n \partial_i \partial_j f(a) h^i h^j + R_{a,2}(h)$$

where

$$R_{a,2}(h) \rightarrow 0 \text{ as } h \rightarrow 0$$

**Basic facts about the gradient**  $\nabla f(a)$  points in the direction of maximal increase and  $|\nabla f(a)|$  is the rate of change of  $f$  in the direction of fastest increase.  $\nabla f(a)$  is orthogonal to the level set of  $f$  that passes through  $a$ .

**Critical points**  $a \in \mathbb{R}^n$  is said to be a critical point of  $f$  if  $Df = 0$ .

**Proposition** If  $f$  has a local maximum/minimum at  $a$  and  $f$  is differentiable at  $a$  then  $Df(a) = 0$ .

**the Hessian**  $H(f) = (D_i D_j f(a))$  which has  $n$  eigenvalues counting multiplicity.

**Hessian Matrix**

$$H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

**Minors** A  $k \times k$  principle minor of a matrix  $M$  is the matrix restricted to the first  $k$  rows and the first  $k$  columns.

**Quadratic forms** Completing the square.

**Proposition** A symmetric matrix  $h$  is positive definite if the determinant of all its principle  $k \times k$  minors are positive.

**Classification of critical points** Suppose  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is  $C^2$ ,  $a$  is a critical point. Then  $H(f)$  has  $n$  eigenvalues (counting multiplicity).

- $a$  is a local minimum if all eigenvalues are positive.
- $a$  is a local maximum if all eigenvalues are negative.
- $a$  is a saddle point if  $k$  eigenvalues are positive and  $n - k$  are negative.



- $a$  is a non-degenerate critical point if all eigenvalues are non-zero

**Folland 2.82** Suppose  $f$  is of class  $C^2$  on an open set in  $\mathbb{R}^2$  containing the point  $a$ , and suppose  $Df(a) = 0$ . Let  $\alpha = \partial_1^2 f(a)$ ,  $\beta = \partial_1 \partial_2 f(a)$ ,  $\gamma = \partial_2^2 f(a)$ . Then:

- If  $\alpha\gamma - \beta^2 < 0$ ,  $f$  has a saddle point at  $a$ .
- If  $\alpha\gamma - \beta^2 > 0$  and  $\alpha > 0$ ,  $f$  has a local minimum at  $a$ .
- If  $\alpha\gamma - \beta^2 > 0$  and  $\alpha < 0$ ,  $f$  has a local maximum at  $a$ .
- If  $\alpha\gamma - \beta^2 = 0$ , no conclusion can be drawn.

**Max-min problems with constraints** Apply the classification of critical points for points in the interior. On the boundary use lagrange multipliers.

**Lagrange multipliers** Consider the functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $g_1 : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $g_2 : \mathbb{R}^n \rightarrow \mathbb{R}$ . We want to find the extreme values for  $f$  subject to  $g_1 = c_1$  and  $g_2 = c_2$ . Then we just need find the values  $a \in \mathbb{R}^n$  that satisfy the system of equations:

- $\nabla f(a) = \mu \nabla g_1(a) + \lambda \nabla g_2(a)$
- $g_1(a) = c_1$
- $g_2(a) = c_2$

## 8.6 Instructor's comment

General Comments: Should you memorize proofs of theorems? It is very hard to memorize all proofs of all theorems. In the long run, it is much more efficient, as well as useful and interesting, to first try to understand the proofs, and internalize the methods of proof, as well as possible; then to remember just an outline of the proof, or some key idea; roughly speaking, the minimum you would need to allow yourself to reconstruct the proof out of your base of general knowledge/understanding. Remember: it is important to know not simply whether something is true, but why it is true.

## 9 **TODO**

## 10 Copyright

### 10.1 Reference

- Wikipedia - The free encyclopedia that anyone can edit
- *Calculus on Manifold* by Michael Spivak
- *Analysis on Manifold* by James R. Munkers
- *Advanced Calculus* by Gerald B. Folland

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