

Contents

1 Topics for First Midterm

1.1 Basic Definations

norm

see note basic:definitions:norm

inner product

see note basic:definitions:inner product

metric / distance functions

see note basic:definitions:metric

open and closed subset of metric space

see note basic:definitions:metricspace-openset and basic:definitions:metricspace-closedset

1.2 Basic Theorems

EVT Extreme value theorem

IVT Intermediate value theorem

MVT Mean value theorem

IVFT ??

IPFT ??

1.3 Basic topology of metric spaces

closure

the set with all its limit points

interior

union of all open set contained in A

exterior

union of all open set disjoint from A

boundary

points that are neither interior nor exterior

limits $f : A \subseteq X \rightarrow Y$

$f(x) \rightarrow y_0$ as $x \rightarrow x_0$ if \forall open $V \ni y_0 \exists$ open $U \ni x_0 [x \in U \cap A \wedge x \neq x_0 \rightarrow f(x) \in V]$

continuity f is cts at x_0 if x_0 is isolated point or $(\lim_{x \rightarrow x_0} f(x)) = f(x_0)$

Cauchy sequences A sequences $\langle x_i \rangle$ is Cauchy if

$\forall \varepsilon \exists N [n, m > N \implies d(x_m, x_n) < \varepsilon]$

completeness A metric space X is complete if every Cauchy sequences converge(to some point in X).

compact sets every open cover of X has a finite subcover

connected sets X cannot be divided into two disjoint nonempty closed/open/clopen sets.

relatively open sets p26 A is relatively open in $Y \subseteq X$ if \exists open $U \subseteq X$ such that $A = U \cap Y$

Proposition $f : X \rightarrow Y$ is cts iff \forall open $V \in Y, F^{-1}(V)$ is open in X . Similarly for closed.

Bolzano–Weierstrass property

A subset $E \in \mathbb{R}^n$ satisfies the BW property if every suquence has a convergent subsequence.

Bolzano–Weierstrass theorem $E \in \mathbb{R}^n$ satisfies the BW property iff E is closed and bounded.

Heine-Borel theorem $E \in \mathbb{R}^n$ is compact iff E is closed and bounded.

Application/(topological invariant) Suppose $f : X \rightarrow Y$ is continuous and X is compact then $f(X)$ is compact

Extreme value theorem Suppose $f : X \rightarrow \mathbb{R}$ is continuous and X is compact then $\exists x_0 \in X$ such that $f(x) \leq f(x_0) \forall x \in X$.

Path connected A set E is path connected if $\forall x, y \in E, \exists$ continuous map $f : [a, b] \rightarrow E$ such that $f(a) = x$ and $f(b) = y$.

Proposition If E is connected, and $f : E \rightarrow Y$ is continuous then $f(E)$ is connected

Proposition If E is path connected then E is connected.

Intermediate Value Theorem Suppose $E \subset \mathbb{R}$ is connected and $f : E \rightarrow \mathbb{R}$ is continuous. Suppose $f(x) = a$ and $f(y) = b$ for some $x, y \in E$ and $a < b$. Then $\forall a < c < b \exists$ some $z \in E$ such that $f(z) = c$.

The ϵ -neighborhood theorem Let X be a compact subspace of \mathbb{R}^n ; Let U be an open set of \mathbb{R}^n containing X , Then there is an $\epsilon > 0$ such that the ϵ -neighborhood of X is contained in U .

Cauchy-Schwarz inequality; all norms on a finite-dimensional vector space are equivalent; Bolzano Weierstrass theorem; Heine-Borel theorem; the continuous image of a compact set is compact; the continuous image of a connected set is connected; intermediate value theorem; extreme value theorem. minima and maxima of continuous functions on compact sets

1.4 Differentiation

??

- definition of the derivative
- partial derivatives
- directional derivatives

chain rule

- $(f \circ g)' = (f' \circ g) \cdot g'$
- $\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$

continuity and differentiability

- differentiable implies continuity
- C^1 implies differentiable
- C^2 implies equality of mixed partial derivatives

Jacobian matrix

$$Jf = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

continuously differentiable functions TODO

higher order derivatives TODO

gradient TODO

geometry of the Jacobian, the rows, the columns TODO

1.5 Max-min problems

?? see note

?? see note

basic facts about the gradient

critical points

the Hessian

quadratic forms

classification of critical points

max-min problems with constraints

Lagrange multipliers

1.6 Instructor's comment

General Comments: Should you memorize proofs of theorems? It is very hard to memorize all proofs of all theorems. In the long run, it is much more efficient, as well as useful and interesting, to first try to understand the proofs, and internalize the methods of proof, as well as possible; then to remember just an outline of the proof, or some key idea; roughly speaking, the minimum you would need to allow yourself to reconstruct the proof out of your base of general knowledge/understanding. Remember: it is important to know not simply whether something is true, but why it is true.

2 Basic knowledge

2.1 Definations

norm A function $p : V \rightarrow \mathbb{R}$ such that

For all $a \in F$ and all $u, v \in V$

- $p(v) \geq 0 \wedge [p(v) = 0 \iff v = 0]$ (separates points)
- $p(av) = |a|p(v)$ (absolute homogeneity)
- $p(u + v) \leq p(u) + p(v)$ (triangle inequality)

inner product A function $\langle x, y \rangle : V \times V \rightarrow \mathbb{R}$ such that

For all $x, y, z \in V$ and $c \in \mathbb{F}$.

- $\langle x, y \rangle = \langle y, x \rangle$
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- $\langle cx, y \rangle = c\langle x, y \rangle = \langle x, cy \rangle$
- $\langle x, x \rangle > 0$ if $x \neq 0$

metric A function $d : X \times X \rightarrow \mathbb{R}$ such that

For all $x, y, z \in X$

- $d(x, y) \geq 0$
- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$

ϵ -neighborhood $U(x; \epsilon) = \{y | d(x, y) < \epsilon\}$

open set in metric space A set $U \subseteq X$ is said to be open in X if

$\forall x \in U \exists \epsilon > 0 [U(x; \epsilon) \subseteq U]$

note that finite intersections and arbitrary unions of open set are open set

closed set in metric space A set contains all its limit point.

note that closed set is complement of open set in topology

2.2 Theorems

Extreme value theorem

Suppose $f : X \rightarrow \mathbb{R}$ is continuous and X is compact,
then $\exists x_0 \in X$ such that $\forall x \in X. f(x) \leq f(x_0)$.

Intermediate Value Theorem

Suppose $E \in \mathbb{R}$ is connected and $f : E \rightarrow \mathbb{R}$ is continuous.
Suppose $f(x) = a$ and $f(y) = b$ for some $x, y \in E$ and $a < b$.
Then $\forall a < c < b \exists$ some $z \in E$ such that $f(z) = c$.

Mean Value Theorem

Suppose $\phi : [a, b] \rightarrow \mathbb{R}$ is

- continuous at each point of **closed** interval $[a, b]$
- differentiable at each point of **open** interval (a, b)

Then there exists a point $c \in (a, b)$ such that $\phi(b) - \phi(a) = \phi'(c)(b - a)$.

3 Abbreviation

cts Continuous

msr Measure

4 Notation and Terminology

4.1 Derivative

$Df(a)$
derivative of f at a

$f'(a; u)$
directional derivative of f at a respect to vector u .

$D_j f(a)$
 j^{th} partial derivative of f at a .

f_i
 i^{th} component function of f .

$\vec{\nabla} g$
gradient of g , $\vec{\nabla} g = \mathbf{grad} g = \sum_i (D_i g) e_i$

Jf
Jacobian matrix, $J_{ij} = D_j f_i(a)$

4.2 Multi-index Notation

$\alpha = (\alpha_1, \dots, \alpha_n)$

$|\alpha| = \alpha_1 + \dots + \alpha_n$

$\alpha! = \alpha_1! \dots \alpha_n!$

$x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$

$\partial^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n} f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$

5 Measure zero

measure zero

Let $A \subseteq \mathbb{R}^n$. We say A has measure zero in \mathbb{R}^n if for every $\epsilon > 0$, there is a covering Q_1, Q_2, \dots of A by countably many rectangles such that $\sum_{i=1}^{\infty} v(Q_i) < \epsilon$. If this inequality holds, we often say that the total volume of the rectangles Q_1, Q_2, \dots is less than ϵ .

oscillation

Given $a \in Q$ define $A_\delta = \{f(x) | x \in Q \wedge |x - a| < \delta\}$. Let $M_\delta(f) = \sup A_\delta$, and let $m_\delta(f) = \inf A_\delta$, define oscillation at f by $\text{osc}(f; a) = \inf_{\delta > 0} [M_\delta(f) - m_\delta(f)]$.

- f is cts at a iff $\text{osc}(f; a) = 0$

5.1

5.1.1 Theorem {Munkers-11.1}

1. If $B \subseteq A$ and A has measure zero in \mathbb{R}^n , then so does B .
2. Let A be the union of the collection of sets A_1, A_2, \dots . If each A_i has measure zero, so does A .
3. A set A has measure zero in \mathbb{R}^n if and only if

5.1.2 Theorem {Munkers-11.2}

5.2 Taylor's theorem

Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is of class C^k on an open convex set S . If $a \in S$ and $a + h \in S$, then

$$f(a + h) = \sum_{|\alpha| \leq k} \frac{\partial^\alpha f(a)}{\alpha!} h^\alpha + R_{a,k}(h),$$

If f is of class C^{k+1} on S , for some $c \in (0, 1)$ we have

$$R_{a,k}(h) = \sum_{|\alpha|=k+1} \frac{\partial^\alpha f(a + ch)}{\alpha!} h^\alpha$$

6 Differentiation

6.1 Derivative

6.1.1 Differentiable

f is differentiable at a if there is an n by m matrix B such that

$$\frac{f(a+h) - f(a) - B \cdot h}{|h|} \rightarrow 0 \quad \text{as } h \rightarrow 0$$

The matrix B is unique.

6.1.2 Directional derivative

Given $u \in \mathbb{R}^m$ which $u \neq 0$ define

$$f'(a; u) = \lim_{t \rightarrow 0} \frac{f(a + tu) - f(a)}{t}$$

Provide the limit exists.

6.1.3 Partial derivative

Define the j^{th} partial derivative of f at a to be the directional derivative of f at a with respect to the vector e_j , provide derivative exists.

$$D_j f(a) = \lim_{t \rightarrow 0} \frac{f(a + te_j) - f(a)}{t}$$

6.1.4 Continuously Differentiable

A function is C^1 if all of its partial derivatives are continuous. A function is C^r if all of its partial derivatives are C^{r-1} .

6.2 Theorems

Munkers.5.1

If f is differentiable at a then all directional derivative of f at a exist and $f'(a; u) = Df(a) \cdot u$

Munkers.5.2

If f is differentiable at a then f is continuous at a .

Munkers.5.3

If f is differentiable at a then $Df(a) = [D_1 f(a) \quad D_2 f(a) \quad \cdots \quad D_m f(a)]$.

Munkers.5.4

- $[f \text{ is differentiable at } a] \Leftrightarrow \forall i [f_i \text{ is differentiable at } a]$.
- If f is differentiable at a , then its derivative is the n by m matrix whose i^{th} row is the derivative of the function f_i . $(Df(a))_i = Df_i(a)$

6.3 Continuously Differentiable Functions

Munkers 6.1

If $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) , then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.

Munkers 7.3

Let A be open in \mathbb{R}^m ; let $f : A \rightarrow \mathbb{R}$ be differentiable on A . If A contains the line segment with end points a and $a + h$, then there is a point $c = a + th$ with $0 < t < 1$ of this line segment such that $f(a + h) - f(a) = (Df(c))h$.

Munkers 6.2

Let A be open in \mathbb{R}^m . Suppose that the partial derivative $D_i f_i(x)$ of the component function of f exists at each point x of A and are continuous on A . Then f is differentiable at each point of A .

Munkers 6.3

Let A be open in \mathbb{R}^m , let $f : A \rightarrow \mathbb{R}$ be a function of class C^2 . Then for each $a \in A$: $D_k D_j f(a) = D_j D_k f(a)$.

6.4 Inverse Function Theorem

Let A be open in \mathbb{R}^n . Let $f : A \rightarrow \mathbb{R}^n$ be of class C^r .

IF $Df(x)$ is invertible at $a \in A$.

THEN There exists a neighborhood of a such that

- $f|_U$ is injective AND $f(U) = V$ open in \mathbb{R}^n
- the inverse function is of class C^r
- $f^{-1}(y) = [f'(f^{-1}(y))]^{-1}$

6.5 Implicit Function Theorem

Munkers 9.1

Let A be open in \mathbb{R}^{k+n} , B be open in \mathbb{R}^k .

Let $f : A \rightarrow \mathbb{R}^n$, $g : B \rightarrow \mathbb{R}^n$ be differentiable.

Write f in the form $f(x, y)$, for $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$.

IF $f(x, g(x)) = 0$ AND $\frac{\partial f}{\partial y}$ is invertible

THEN $Dg(x) = - \left[\frac{\partial f}{\partial y}(x, g(x)) \right]^{-1} \cdot \frac{\partial f}{\partial x}(x, g(x))$

Suppose $f : A \rightarrow \mathbb{R}^n$ be of class C^r .

Write f in the form $f(x, y)$, for $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$.

IF $(a, b) \in A$ AND $f(a, b) = 0$ AND $\det \frac{\partial f}{\partial y}(a, b) \neq 0$

THEN There exists $B \in \mathbb{R}^k$, $a \in B$ and a unique $g : B \rightarrow \mathbb{R}^n$ such that $g(a) = b$ AND $\forall x \in B. f(x, g(x)) = 0$ AND g is C^r

7 Integration

7.1 Fundamental theorem of Calculus

- If f is continuous on $[a, b]$, and if $F(x) = \int_a^x f$ for $x \in [a, b]$, then $F'(x)$ exists and equals $f(x)$.
- If ...