UofT-MAT257-2014 Note

hysw, etc(will add later)...

January 18, 2015

Contents

1 KW

Euclidean n-space $\mathbb{R}^n$, norm, inner product, usual basis, distance, closed rectangle, open rectangle, open, closed, interior, exterior, boundary, open cover, compact(etc...), Heine-Borel Theorem, function, composition, component function, projection function π^i, continuous function, continuous & compact, oscillation, Jacobian matrix, chain rule, partial derivative, continuously differentiable, inverse functions, implicit functions,

2 TODO

Munkers 11.3, 13.*, 14.3, 14.4, 15.*

3 Basic knowledge

Note, this section is for something that does not fit anywhere

3.1 Abbreviations

cts Continuous

msr Measure

3.2 Multi-index Notation

- $\alpha = (\alpha_1, \dots, \alpha_n)$
- $|\alpha| = \alpha_1 + \cdots + \alpha_n$
- $\alpha! = \alpha_1! \cdots \alpha_n!$
- $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$
- $\partial^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$

4 Linear Algebra

4.1 Definitions

norm A function $p : V \to \mathbb{R}$ such that
For all $a \in F$ and all $u, v \in V$

- $p(v) \geq 0 \land [p(v) = 0 \iff v = 0]$ (separates points)
- $p(av) = |a|p(v)$ (absolute homogeneity)
- $p(u + v) \leq p(u) + p(v)$ (triangle inequality)

inner product A function $\langle x, y \rangle : V \times V \to \mathbb{R}$ such that
For all $x, y, z \in V$ and $c \in \mathbb{F}$.

- $\langle x, y \rangle = \langle y, x \rangle$
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- $\langle cx, y \rangle = c\langle x, y \rangle = \langle x, cy \rangle$
- $\langle x, x \rangle > 0$ if $x \neq 0$

5 Topology

5.1 Definitions

metric A function $d : X \times X \to \mathbb{R}$ such that
For all $x, y, z \in X$

- $d(x, y) \geq 0$
- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$

ϵ-neighborhood $\mathbf{U}(x; \epsilon) = \{y | d(x, y) < \epsilon\}$

open set in metric space A set $U \subseteq X$ is said to be open in X if $\forall x \in U \exists \epsilon > 0 [\mathbf{U}(x; \epsilon) \subseteq U]$ note that finite intersections and arbitrary unions of open set are open set

closed set in metric space A set contains all its limit point.
note that closed set is complement of open set in topology

5.2 Partition of unity

TODO

6 Measure Theory

6.1 Measure zero

Let $A \subseteq \mathbb{R}^n$. We say A has measure zero in $\mathbb{R}^n$ if for every $\epsilon > 0$, there is a covering $Q_1, Q_2, \ldots$ of A by countably many rectangles such that $\sum_{i=1}^{\infty} v(Q_i) < \epsilon$. If this inequality holds, we often say that the total volume of the rectangles $Q_1, Q_2, \ldots$ is less than ϵ.

6.2 Theorems

{Munkers-11.1}

1. If $B \subseteq A$ and A has measure zero in $\mathbb{R}^n$, the so does B.
2. Let A be the union of the collection of sets $A_1, A_2, \ldots$ If each A_i has measure zero, so does A.
3. A set A has measure zero in $\mathbb{R}^n$ if and only i

7 General Calculus

7.1 Definitions

oscillation

Given $a \in Q$ define $A_\delta = \{f(x) | x \in Q \wedge |x - a| < \delta\}$. Let $M_\delta(f) = \sup A_\delta$, and let $m_\delta(f) = \inf A_\delta$, define oscillation at f by $\operatorname{osc}(f; a) = \inf_{\delta>0}[M_\delta(f) - m_\delta(f)]$. f is cts at a iff $\operatorname{osc}(f; a) = 0$

7.2 Extreme Value Theorem

Suppose $f : X \to \mathbb{R}$ is continuous and X is compact, then $\exists x_0 \in X$ such that $\forall x \in X. f(x) \leq f(x_0)$.

7.3 Intermediate Value Theorem

Suppose $E \in \mathbb{R}$ is connected and $f : E \to \mathbb{R}$ is continuous.
Suppose $f(x) = a$ and $f(y) = b$ for some $x, y \in E$ and $a < b$.
Then $\forall a < c < b \exists$ some $z \in E$ such that $f(z) = c$.

7.4 Mean Value Theorem

Suppose $\phi : [a, b] \to \mathbb{R}$ is

- continuous at each point of **closed** interval $[a, b]$
- differentiable at each point of **open** interval (a, b)

Then there exists a point $c \in (a, b)$ such that $\phi(b) - \phi(a) = \phi'(c)(b - a)$.

8 Differential Calculus

8.1 Definitions

differentiable f is differentiable at a if there is an n by m matrix B such that

$$\frac{f(a+h)-f(a)-B\cdot h}{|h|} \to 0 \quad \text{as} \quad h \to 0$$

The matrix B is unique.

Directional derivative Given $u \in \mathbb{R}^m$ which $u \neq 0$ define

$$f'(a;u) = \lim_{t\to 0} \frac{f(a+tu)-f(a)}{t}$$

Provide the limit exists.

Partial derivative Define the j^{th} partial derivative of f at a to be the directional derivative of f at a with respect to the vector e_j, provide derivative exists.

$$D_j f(a) = \lim_{t\to 0} \frac{f(a+te_j)-f(a)}{t}$$

8.2 Notations

$Df(a)$: derivative of f at a

$f'(a;u)$: directional derivative of f at a respect to vector u.

$D_j f(a)$: j^{th}partial derivative of f at a.

f_i : i^{th} component function of f.

∇g : gradient of g, $\nabla g = \mathbf{grad} g = \sum_i (D_i g) e_i$

Jf : Jacobian matrix, $J_{ij} = D_j f_i(a)$

8.3 Differentiability Theorems

Theorems Munkers.5.1
If f is differentiable at a then all directional derivative of f at a exist and $f'(a;u) = Df(a)\cdot u$

Theorems Munkers.5.2
If f is differentiable at a then f is continuous at a.

Theorems Munkers.5.3
If f is differentiable at a then $Df(a) = [D_1 f(a) \quad D_2 f(a) \quad \cdots \quad D_m f(a)]$.

Theorems Munkers.5.4

a. $[f$ is differentiable at $a]$ $\Leftrightarrow$ $\forall i [f_i$ is differentiable at $a]$.

b. If f is differentiable at a, then its derivative is the n by m matrix whose i^{th} row is the derivative of the function f_i. $(Df(a))_i = Df_i(a)$

8.4 Continuously Differentiable Functions

A function is C^1 if all of its partial derivatives are continous. A function is C^r if all of its partial derivatives are C^{r-1}.

Munkers 6.1
If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$.

Munkers 7.3
Let A be open in $\mathbb{R}^m$; let $f : A \to \mathbb{R}$ be differentiable on A. If A contains the line segment with end points a and $a+h$, then there is a point $c = a + th$ with $0 < t < 1$ of this line segment such that $f(a+h) - f(a) = (Df(c))h$.

Munkers 6.2
Let A be open in $\mathbb{R}^m$. Suppose that the partial derivative $D_i f_i(x)$ of the component function of f exists at each point x of A and are continuous on A. Then f is differentiable at each point of A.

Munkers 6.3
Let A be open in $\mathbb{R}^m$, let $f : A \to \mathbb{R}$ be a function of class $\mathbf{C}^2$. Then for each $a \in A$: $D_k D_j f(a) = D_j D_k f(a)$.

8.5 Chain Rule

Let $A \subset \mathbb{R}^m$. Let $B \subset \mathbb{R}^n$. Let $f : A \to \mathbb{R}^n$ and $g : B \to \mathbb{R}^p$, with $f(A) \subset B$. Suppose $f(a) = b$. If f is diferentiable at a and g is differentiable at b, then the composite function $g \circ f$ is differentiable at a. Furthermore,

$$D(g \circ f)(a) = Dg(b) \cdot Df(a)$$

8.6 Inverse Function Theorem

Let A be open in $\mathbb{R}^n$. Let $f : A \to \mathbb{R}^n$ be of class C^r.

IF $Df(x)$ is invertible at $a \in A$.

THEN There exists a neighborhood of a such that

- $f|_U$ is injective AND $f(U) = V$ open in $\mathbb{R}^n$
- the inverse function is of class C^r
- $f^{-1}(y) = [f'(f^{-1}(y))]^{-1}$

8.7 Implicit Function Theorem

Suppose $f : A \to \mathbb{R}^n$ be of class C^r.

Write f in the form $f(x, y)$, for $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$.

IF $(a, b) \in A$ AND $f(a, b) = 0$ AND $\det \frac{\partial f}{\partial y}(a, b) \neq 0$

THEN There exists $B \in \mathbb{R}^k, a \in B$ and a unique $g : B \to \mathbb{R}^n$ such that $g(a) = b$ AND $\forall x \in B. f(x, g(x)) = 0$ AND g is C^r

Munkers 9.1

Let A be open in $\mathbb{R}^{k+n}$, B be open in $\mathbb{R}^k$.

Let $f : A \to \mathbb{R}^n$, $g : B \to \mathbb{R}^n$ be differentiable.

Write f in the form $f(x, y)$, for $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$.

IF $f(x, g(x)) = 0$ AND $\frac{\partial f}{\partial y}$ is invertible

THEN $Dg(x) = -\left[\frac{\partial f}{\partial y}(x, g(x))\right]^{-1} \cdot \frac{\partial f}{\partial x}(x, g(x))$

8.8 Taylor's theorem

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is of class C^k on an open convex set S. If $a \in S$ and $a + h \in S$, then

$$f(a + h) = \sum_{|\alpha| \leq k} \frac{\partial^\alpha f(a)}{\alpha!} h^\alpha + R_{a,k}(h),$$

If f is of class C^{k+1} on S, for some $c \in (0, 1)$ we have

$$R_{a,k}(h) = \sum_{|\alpha| = k+1} \frac{\partial^\alpha f(a + ch)}{\alpha!} h^\alpha$$

9 Integral Calculus

Note, Riemann Integral was taught in this class.

9.1 Definitions

rectangle (in $\mathbb{R}^n$) $Q = [a_1, b_1] \times [a_2, b_2] \times [a_n, b_n]$

component interval of Q $[a_i, b_i]$

volume of Q $v(Q) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$

partition *TODO*

subinterval(determined by P**)** *OMIT*

subrectangle(determined by P**)** *OMIT*

mech of P *OMIT*

refinement *OMIT*

common refinement *OMIT*

?- $m_R(f) = \inf\{f(x) | x \in R\}$

?- $M_R(f) = \inf\{f(x) | x \in R\}$

lower sum $L(f, P) = \sum_R m_R(f) \cdot v(R)$

upper sum $U(f, P) = \sum_R M_R(f) \cdot v(R)$

lower integral $\underline{\int_Q} f = \sup_P\{L(f, P)\}$

upper integral $\overline{\int_Q} f = \inf_P\{L(f, P)\}$

oscillation *TODO*

rectifiable set A bounded set $S \in \mathbb{R}^n$ is rectifiable if the constant function 1 is integrable over S. S is rectifiable iff S is bounded and BdS has measure zero

volume of a rectifiable set $v(S) = \int_S 1$

9.2 Riemann condition

Given: Q a rectangle, $f : Q \to \mathbb{R}$ a bounded function.

$\boxed{\underline{\int_Q} f = \overline{\int_Q} f \text{ iff } \forall \epsilon_{>0} \exists P[U(f, P) - L(f, P) \leq \epsilon]}$

P is a partion of Q

Corollary/Theorem: every constant function is integrable.

9.3 Riemann-Lebesgue theorem

A function on a compact interval [a, b] is Riemann integrable if and only if it is bounded and continuous almost everywhere (the set of its points of discontinuity has measure zero, in the sense of Lebesgue measure). [wiki] [11.2]

9.4 Fundamental theorem of Calculus

- If f is continuous on $[a, b]$, and if $F(X) = \int_a^x f$ for $x \in [a, b]$, then $F'(x)$ exists and equals $f(x)$.
- If f is contininous on $[a, b]$, and if g is a function such that $g'(x) = f(x)$ for $x \in [a, b]$ then $\int_a^b f = g(b) - g(a)$

9.5 Fubini's theorem

Let $Q = A \times B$, where A is a rectangle in $\mathbb{R}^k$ and B is a rectangle in $\mathbb{R}^n$. If f is bounded function and integrable over Q, then $\underline{\int_{y \in B}} f(x, y)$ and $\overline{\int_{y \in B}} f(x, y)$ are integrable over A and

$$\int_Q f = \int_{x \in A} \underline{\int_{y \in B}} f(x, y) = \int_{x \in A} \overline{\int_{y \in B}} f(x, y)$$

9.6 Properties of integral

TODO

9.7 Properties of rectifiable set

TODO

10 Change of Variables

`Diffeomorphism` [wiki]

$f : A \to B$ is a diffeomorphism if f is a bijection AND
f and f^{-1} are of class C^r.

Moreover f is called C^r-diffeomorphism

`Change of Variables Theorem` [17.2]

LET opensets $U, V \subseteq \mathbb{R}$
$g : U \to V$ a diffeomorphism
$f : V \to \mathbb{R}$ be a continuous function
THEN f is intergrable over B
IFF $(f \circ g)|\det Dg|$ is integrable over A
NOTE $\int_B f = \int_A (f \circ g)|\det Dg|$

`Substitution rule` [17.1]

LET $I = [a, b], J = [c, d] \subseteq \mathbb{R}$
$g : I \to J$ of class C^1
$f : J \to \mathbb{R}$ is continuous
IF $\forall x \in (a, b)\, [g'(x) \neq 0]$
THEN $\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g) g'$
equivalently $\int_J f = \int_I (f \circ g)|g'|$

11 Manifolds

12 Lecture Notes

12.1 2015-01-05 Monday

12.1.1 Review

THM : `rectifiable`
LET : S $\subset \mathbb{R}^n$ be bounded
THEN : S is rectifiable
IFF ∂S has measure zero, equivalently χ_s is integrable
IDEA : rectifiable is a property of the boundary of a set. If the boundary has measure zero then the set is rectifiable.

DEFN : `volume`
LET : S $\subset \mathbb{R}^n$ be bounded and *rectifiable*
THEN : define the volume of S as:
$V(S) := \int_{\mathrm{S}} 1 := \int_{\mathrm{Q}} \chi_x$

THM : `partition of unity`
LET : A $\subset \mathbb{R}^n$, $\{\mathrm{U}_\alpha\}_{\alpha\in\mathcal{A}}$ be an open cover of A
THEN : $\exists$ a collection of C^∞ functions $\{\psi_\beta\}_{\beta\in\mathcal{B}}$ s.t

i $\forall x \in$ A $\quad 0 < \psi_\beta \leq 1 \quad \forall \beta \in \mathcal{B}$
ii $\forall x \in$ A $\quad \exists$ open neighbourhood V of x such that: all but finitely many ψ_β vanish on V (*locally finite*)
iii $\forall x \in$ A $\quad \sum_{\beta\in\mathcal{B}} \psi_\beta(x) \equiv 1$
iv $\forall x \in$ A $\quad \exists\, \alpha$ such that supp$(\psi_\beta) \subset \mathrm{U}_\alpha$, ie $\{x|\psi_\beta(x) \neq 0\} \subset \mathrm{U}_\alpha$

A collection of functions satisfying i, ii, iii is called a *partition of unity*
It is *subordinate* to the open cover $\{\mathrm{U}_\alpha\}_{\alpha\in\mathcal{A}}$ if it satisfies condition iv

12.1.2 Open Sets

DEFN : `extended integral`
LET : $\mathrm{U}^{open} \subset \mathbb{R}^n$, $f : \mathrm{U} \to \mathbb{R}$ is continuous
IF $f \geq 0$
THEN : define then *extended integral* of f over U as:
$\int_{\mathrm{u}} f := \sup\{\int_{\mathrm{D}} f \mid \mathrm{D} \subset \mathrm{U}$ where D is compact, rectifiable$\}$
IF f is arbitrary and $\int_{\mathrm{u}} f_+$, $\int_{\mathrm{u}} f_-$ exist
THEN : define then *extended integral* of f over U as:
$\int_{\mathrm{u}} f := \int_{\mathrm{u}} f_+ - \int_{\mathrm{u}} f_-$ where
$f_+(x) = \max\{f(x), 0\}$ and $f_-(x) = \min\{-f(x), 0\}$

THM : `mnk 15.2`
LET : $\mathrm{U}^{open} \subset \mathbb{R}^n$, $f : \mathrm{U} \to \mathbb{R}$ is continuous
Choose an *exhaustion* of U by compact K_i such that
$K_1 \subset K_2^\circ \subset K_2 \subset K_3^\circ \ldots$ and $\mathrm{U} = \cup_{i=1}^\infty \mathrm{K}_i$
f has an *extended integral*
IFF the sequence $\int_{\mathrm{k}_i} |f_i|$ is bounded
THEN : $\int_{\mathrm{u}} f = \lim_{i\to\infty} \int_{\mathrm{k}_i} f$

RMK : If $\mathrm{U}^{open} \subset \mathbb{R}^n$, then $\int_{\mathrm{u}} f$ refers to the *extended integral*

THM : `mnk 15.4`
LET : $\mathrm{U}^{\mathrm{open}}$ is bounded, $f : \mathrm{U} \to \mathbb{R}^n$ is bounded and continuous
THEN :

i the extended integral exists
ii if the ordinary integral exists, then they are equal

THM : `mnk 16.5`
LET : $\mathrm{U}^{open} \subset \mathbb{R}^n$, $f : \mathrm{U} \to \mathbb{R}$ is continuous, $\{\psi_i\}$ be a partition of unity with compact support
THEN : $\int_{\mathrm{u}} f$ exists
IFF $\sum_{i=1}^\infty \int_{\mathrm{u}} \psi_i |f|$ converges to a finite number
In this case $\int_{\mathrm{u}} f = \sum\limits_{i=1}^{\infty} (\int_{\mathrm{u}} \psi_i f)$

12.2 2015-01-07 Wednesday

Note: this is not finished -_-.

- Diffeomorphism
- Change of Variables
- Substitution Rule

12.3 2015-01-09 Friday

`Change of Variables Theorem` [17.2]

LET opensets $U, V \subseteq \mathbb{R}$
$g : U \to V$ a diffeomorphism
$f : V \to \mathbb{R}$ be a continuous function
THEN f is intergrable over B
IFF $(f \circ g)|\det Dg|$ is integrable over A
NOTE $\int_B f = \int_A (f \circ g)|\det Dg|$

IDEA : The theorem is proved in 5 steps.

i Let $g : U \to V$ and $h : V \to W$ be diffeomorphisms of open sets in $\mathbb{R}^n$. Show that if the theorem holds for g and h, then it holds for $g \circ h$. The idea is to show that $\int_W f = \int_V (f \circ h)|\det Dh| = \int_U (f \circ h \circ g)|(\det Dh) \circ g||\det Dg| = \int_U (f \circ h \circ g)|\det(h \circ g)|$

ii Suppose that for each $x \in A$, there is a neighborhood U of x contained in A such that the lemma holds for the diffeomorphism $g : U \to V$ where $V = g(U)$ and all continues functions $f : V \to \mathbb{R}$ whose supports are compact subsets of V. Then show that the lemma holds for g. This basically means: show that if the lemma holds lically for g and fuctions f having compact support, then it holds for all g and all f. The idea is to find a partition of unity ϕ_i of B with compact support and show that the collection $\phi_i \circ g$ is a partition of unity on A having compact support. Then use the fact that $\int_B f = \sum_{i=1}^{infty} \int_B (\phi_i)(f)$.

iii Show the the lemma holds for $n = 1$. This can be done by showing that the theorem holds for the interiors of all closed intervals contained in A then applying step 2.

iv We only need to prove the theorem for primitive diffeomorphisms. This is because, when restricted to a small enough neighbourhood, all diffeomorphisms are a composition of primitive diffeomorphisms. Step 1 shows that the validity of the theorem is preserved by composition. Step 2 shows that if the theorem holds on all the small neighbourhoods in A then it holds for all of A.

v Show that if the lemma holds in dimension $n-1$, it holds in dimension n. Due to step 4, we only need to prove this for primitive diffeomorphisms.

12.4 2015-01-12 Monday

12.4.1 Torus

Problem: Find the volume of the solid torus given by

$$\{(x, y, z)|(\sqrt{x^2 + y^2} - b)^2 + 2^2 \le a\}$$

When $y = 0$, $(x-b)^2 + z^2 \le$ is the circle centered at $(b, 0, 0)$ in the (x, z) plane with radius a.

For angle $0 < \varphi < 2\pi$, parametrize the circle by

- $x = b + a\cos\varphi$
- $z = a\sin\varphi$

and for $0 < \rho < a$ and parametrize the disc by

- $x = b + \rho\cos\varphi$
- $z = \rho\sin\varphi$

This defines a diffeomorphism from $(0, a) \times (0, 2\pi) \to$ disc with slit removed. Now rotate around z axis. For $0 < \theta < 2\pi$:

- $x = (b + \rho\cos\varphi)\cos\theta$
- $y = (b + \rho\cos\varphi)\sin\theta$
- $z = \rho\sin\varphi$

We want $G(\rho, \varphi, \theta) := (x, y, z)$, so $G(\rho, \varphi, \theta) \to \{(\sqrt{x^2 + y^2} - b)^2 + 2^2 < a\} \backslash (T \cap \{x > 0, y = 0\}) \cup (T \cap \{z = 0 \cap x^2 + y^2 \ge b^2\})$ is a diffeomorphism.

Notice that the set we have removed has measure 0 (First part $\subset \{y = 0\}$ which has measure 0 and second part $\subset \{z = 0\}$ which has measure 0).

Thus, Vol(T) =volume of the image of G (they differ by a set of measure 0).

So $\chi_T \equiv \chi_{G(R)}$ except on a set of measure 0.

Apply change of variables!

First check if G is a diffeomorphism:

G is a bijection by construction.

$$DG = \begin{bmatrix} \cos\varphi\cos\theta & -\rho\sin\varphi\cos\theta & (b + \rho\cos\varphi)\cos\theta \\ \cos\varphi\sin\theta & -\rho\sin\varphi\sin\theta & (b + \rho\cos\varphi)\sin\theta \\ \sin\varphi & \rho\cos\varphi & 0 \end{bmatrix}$$

$$\begin{aligned} |DG| =& -(b + \rho\cos\varphi)\sin\theta(\rho\cos^2\varphi\sin\theta + \rho\sin^2\varphi)\cos\theta) \\ & -(b + \rho\cos\varphi)\cos\theta(\rho\cos^2\varphi\cos\theta + \rho\sin^2\varphi)\cos\theta) \\ =& -(b + \rho\cos\varphi)\rho \\ \ne& 0 \end{aligned}$$

Thus G is a diffeomorphism of open sets.

Then by change of variables,

$$\int_{G(R)} 1 = \int_R 1|DG| = \int_{(0,a)\times(0,2\pi)\times(0,2\pi)} \rho(b + \rho\cos\theta)$$

Then by Fubini's theorem:

$$\int_{\rho=0}^a \int_0^{2\pi} \int_0^{2\pi} \rho(b + \rho\cos\varphi) = 4\pi^2 \int_0^a \rho b = 2\pi^2 a^2 b$$

That's the torus!

12.5 2015-01-14 Wednesday

Second test review

12.5.1 Existence of the Integral

DEFN : `integral`
LET : Q be a rectangle, $f : \mathrm{Q} \to \mathbb{R}$ be a bounded function, define:
$\overline{\int}_{\mathrm{Q}} f := \inf\{L(f,\mathrm{P})\}$ as the upper integral
$\underline{\int}_{\mathrm{Q}} f := \sup\{L(f,\mathrm{P})\}$ as the lower integral
IF then the upper and lower sums agree
THEN : f is *integrable* over Q

THM : `Riemann Condition`
LET : Q be a rectangle, $f : \mathrm{Q} \to \mathbb{R}$ be a bbd fn
THEN : upper and lower integral agree
IFF given $\epsilon > 0$ there is a partition P such that:
$U(f,P) - L(f,P) \leq \epsilon$
IDEA : necessary condition for the existence

THM : `mnk 11.2`
LET : $\mathrm{Q} \subset \mathbb{R}^n$, $f : \mathrm{Q} \to \mathbb{R}$ be a bounded function.
Define D to be the set of points for which f fails to be continuous
THEN : $\int_Q f$ exists
IFF D has measure zero (f is *almost continuous everywhere*)

THM : `mnk 11.3`
LET : $\mathrm{Q} \subset \mathbb{R}^n$, $f : \mathrm{Q} \to \mathbb{R}$. Assume f is *integrable*
IF f vanishes except on a set of measure zero
THEN : $\int_{\mathrm{Q}} f = 0$
IF f is non-negative and $\int_{\mathrm{Q}} f = 0$
THEN : $f = 0$ almost everywhere

12.5.2 Evaluation of the Integral

THM : `Fundamental Theory of Calculus`
LET : $f : [\mathrm{a}, \mathrm{b}] \to \mathbb{R}$ be continuous
IF $F(x) = \int_a^x f(x)$ for $x \in [a,b]$
THEN : $D \int_a^x f = f(x)$
IF g is a function such that $g'(x) = f(x)\ \forall x$
THEN : $\int_a^x Dg = g(x) - g(a)$

THM : `Fubini's Theorem`
LET : $\mathrm{Q} = \mathrm{A} \times \mathrm{B}$, where $\mathrm{A} \subset \mathbb{R}^k$ and $\mathrm{B} \subset \mathbb{R}^n$.
$f : \mathrm{Q} \to \mathbb{R}$ be bdd, write $f(x,y)$ for $x \in \mathrm{A}$ and $y \in \mathrm{B}$.
For each $x \in \mathrm{A}$ consider upper and lower integrals
$\underline{\int}_{y \in \mathrm{B}} f(x,y)$ and $\overline{\int}_{y \in \mathrm{B}} f(x,y)$
IF f is integrable over Q
THEN : these two functions are integrable over Q and
$\int_{\mathrm{Q}} f = \int_{x \in \mathrm{A}} \underline{\int}_{y \in \mathrm{B}} f(x,y) = \int_{x \in \mathrm{A}} \overline{\int}_{y \in \mathrm{B}} f(x,y)$

12.5.3 Integral Over a Bounded Set

THM : `mnk 13.5`
LET : $\mathrm{S} \subset \mathbb{R}^n$ be bounded, $f : \mathrm{S} \to \mathbb{R}$ be bounded and continuous function. Define E to be the set of all points $x_0 \in \partial S$ for which the condition $\lim_{x \to x_0} f(x) = 0$ fails
IF E has measure zero
THEN : f is integrable over S
Converse also holds

THM : `mnk 13.6`
LET : $\mathrm{S} \subset \mathbb{R}^n$ be bounded, $f : \mathrm{S} \to \mathbb{R}$ be bounded and continuous function, and $\mathrm{A} = \mathrm{S}^\circ$.
IF f is integrable over S
THEN : f is integrable over A and
$\int_{\mathrm{S}} f = \int_{\mathrm{A}} f$

12.5.4 Rectifiable Sets

THM : `mnk 14.1`
LET : $\mathrm{S} \subset \mathbb{R}^n$ **THEN** : S is *rectifiable*
IFF S bounded and ∂S has measure zero

THM : `mnk 14.2`
Properties of rectifiable sets

i (Positivity). If S is rectifiable, $v(\mathrm{S}) \geq 0$
ii (Monotonicity). If S_1 and S_2 are rectifiable with $\mathrm{S}_1 \subset \mathrm{S}_2$ then $v(\mathrm{S}_1) \leq v(\mathrm{S}_2)$
iii (Additivity). If S_1 and S_2 are rectifiable so are, $\mathrm{S}_1 \cup \mathrm{S}_2$ and $\mathrm{S}_1 \cap \mathrm{S}_2$
iv Suppose S is rectifiable. Then $v(\mathrm{S}) = 0$ iff S has measure zero
v If S is rectifiable, so is S° and $v(\mathrm{S}) = v(\mathrm{S}^\circ)$
vi If S is rectifiable, and $f : \mathrm{S} \to \mathbb{R}$ is bounded continuous, then f is integrable over S.

DEFN : `simple region`
LET : C be a compact and rectifiable set in $\mathbb{R}^{n-1}$,
$\phi, \psi : \mathrm{C} \to \mathbb{R}$ be continuous functions such that

$\phi(x) \leq \psi(x)\ \forall x \in \mathrm{C}$.
THEN : $\mathrm{S} \subset \mathbb{R}^n$ defined as
$\mathrm{S} := \{(x,t) | x \in \mathrm{C} \text{ and } \phi(x) \leq t \leq \psi x\}$
is a simple region.

THM : `Fubini's Theorem for Simple Regions`
LET : $\mathrm{S} = \{(x,t) | x \in \mathrm{C} \text{ and } \phi(x) \leq t \leq \psi(s)\}$ be a simple region in $\mathbb{R}^n$ and let $f : \mathrm{S} \to \mathbb{R}$ be a continuous function
THEN : f is integrable over S and

$$\int_{\mathrm{S}} f = \int_{x \in \mathrm{c}} \int_{t=\phi(s)}^{t=\psi(x)} f(x,t)$$

12.5.5 Extended Integrals

Three definitions of the extended integral:

i Extended Integral 1
ii Extended Integral 2
iii Extended Integral 3

THM : `mnk 15.4`
LET : U^{open} is bounded, $f : \mathrm{U} \to \mathbb{R}^n$ is bounded and continuous
THEN :

i the extended integral exists
ii if the ordinary integral exists, then they are equal

12.5.6 Change of Variables

DEFN : `diffeomorphism`
$f : A \to B$ is a diffeomorphism if f is a bijection AND f and f^{-1} are of class C^r.

THM : `Change of Variables`
LET : opensets $U, V \subseteq \mathbb{R}$
$g : U \to V$ a diffeomorphism
$f : V \to \mathbb{R}$ be a continuous function
THEN : f is intergrable over B
IFF $(f \circ g)|\det Dg|$ is integrable over A
RMK : $\int_B f = \int_A (f \circ g)|\det Dg|$

THM : `Substitution Rule`
LET : $I = [a,b], J = [c,d] \subseteq \mathbb{R}$
$g : I \to J$ of class C^1
$f : J \to \mathbb{R}$ is continuous
IF $\forall x \in (a,b)\, [g'(x) \neq 0]$
THEN : $\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g) g'$
equivalently $\int_J f = \int_I (f \circ g)|g'|$