Contents

1 Topics for First Midterm

1.1 Basic topology of metric spaces

norm $p : V \to \mathbb{R}$

For all $a \in F$ and all $u, v \in V$

- $p(v) \geq 0 \land [p(v) = 0 \iff v = 0]$ (separates points)
- $p(av) = |a|p(v)$ (absolute homogeneity)
- $p(u + v) \leq p(u) + p(v)$ (triangle inequality)

inner product $\langle x, y\rangle : V \times V \to \mathbb{R}$

For all $x, y, z \in V$ and $c \in \mathbb{F}$.

- $\langle x, y\rangle = \langle y, x\rangle$
- $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
- $\langle cx, y\rangle = c\langle x, y\rangle = \langle x, cy\rangle$
- $\langle x, x\rangle > 0$ if $x \neq 0$

distance functions $d : X \times X \to \mathbb{R}$

For all $x, y, z \in X$

- $d(x, y) \geq 0$
- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$

open and closed subsets of metric spaces

open: every point is a interior point
closed: iff contains all its limit point

closure the set with all its limit points

interior union of all open set contained in A

exterior union of all open set disjoint from A

boundary points that are neither interior nor exterior

limits $f : A \subseteq X \to Y$

$f(x) \to y_0$ as $x \to x_0$ if $\forall$ open $V \ni y_0$ $\exists$ open $U \ni x_0$ $[x \in U \cap A \land x \neq x_0 \to f(x) \in V]$

continuity f is cts at x_0 if x_0 is isolated point or $(\lim_{x\to x_0} f(x)) = f(x_0)$

Cauchy sequences A sequences $\langle x_i\rangle$ is Cauchy if

$\forall \varepsilon \exists N[n, m > N \implies |x_m - x_n| < \varepsilon]$

completeness A metric space X is complete if every Cauchy sequences converge(to some point in X).

compact sets every open cover of X has a finite subcover

connected sets X cannot be divided into two disjoint nonempty closed/open/clopen sets.

relatively open sets p26 A is relatively open in $Y \subseteq X$ if $\exists$ open $U \subseteq X$ such that $A = U \cap Y$

Note finite intersections and arbitrary unions of open set are open set

Proposition $f : X \to Y$ is cts iff $\forall$ open $V \in Y$, $F^{-1}(V)$ is open in X.

BW property A subset $E \in \mathbb{R}^n$ satisfies the BW property if every suquence has a convergent subsequence.

BW theorem $E \in \mathbb{R}^n$ satisfies the BW property iff E is closed and bounded.

Heine-Borel theorem $E \in \mathbb{R}^n$ is compact iff E is closed and bounded.

Application Suppose $f : X \to Y$ is continuous and X is compact then $f(X)$ is compact

Extreme value theorem Suppose $f : X \to \mathbb{R}$ is continuous and X is compact then $\exists x_0 \in X$ such that $f(x) \leq f(x_0) \forall x \in X$.

Path connected A set E is path connected if $\forall x, y \in E$, $\exists$ continuous map $f : [a, b] \to E$ such that $f(a) = x$ and $f(b) = y$.

Proposition If E is connected, and $f : E \to Y$ is continuous then $f(E)$ is connected

Proposition If E is path connected then E is connected.

Intermediate Value Theorem Suppose $E \in \mathbb{R}$ is connected and $f : E \to \mathbb{R}$ is continuous. Suppose $f(x) = a$ and $f(y) = b$ for some $x, y \in E$ and $a < b$. Then $\forall a < c < b \exists$ some $z \in E$ such that $f(z) = c$.

Cauchy-Schwarz inequality; all norms on a finite-dimensional vector space are equivalent; Bolzano Weierstrass theorem; Heine-Borel theorem; the con- tinuous image of a compact set is compact; the continuous image of a connected set is connected; intermediate value theorem; extreme value theorem. minima and maxima of continuous functions on compact sets

1.2 Differentiation

Derivative

- definition of the derivative
- partial derivatives
- directional derivatives

chain rule

- $(f \circ g) = (f' \circ g) \cdot g'$
- $\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$

Mean Value Theorem see note

Inverse Function Theorem see note

Implicit Function Theorem see note

continuity and differentiability

- differentiable implies continuity
- C^1 implies differentiable
- C^2 implies equality of mixed partial derivatives

Jacobian matrix

$$Jf = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \dots & \ddots & \dots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

continuously differentiable functions TODO

higher order derivatives TODO

gradient TODO

geometry of the Jacobian, the rows, the columns TODO

1.3 Max-min problems

Multi-index notation; Taylor's theorem with remainder; basic facts about the gra- dient; critical points; the Hessian; quadratic forms; classification of critical points; max-min problems with constraints; Lagrange multipliers. General Comments: Should you memorize proofs of theorems? It is very hard to memorize all proofs of all theorems. In the long run, it is much more efficient, as well as useful and interesting, to first try to understand the proofs, and internalize the methods of proof, as well as possible; then to remember just an outline of the proof, or some key idea; roughly speaking, the minimum you would need to allow yourself to reconstruct the proof out of your base of general knowledge/understanding. Remember: it is important to know not simply whether something is true, but why it is true.

2 Abbreviation

cts Continuous

msr Measure

3 Notation and Terminology

3.1 Derivative

$Df(a)$
derivative of f at a

$f'(a;u)$
directional derivative of f at a respect to vector u.

$D_j f(a)$
j^{th}partial derivative of f at a.

f_i
i^{th} component function of f.

$\vec{\nabla} g$
gradient of g, $\vec{\nabla} g = \mathbf{grad} g = \sum_i (D_i g) e_i$

Jf
Jacobian matrix, $J_{ij} = D_j f_i(a)$

3.2 Multi-index Notation

$\alpha = (\alpha_1, \ldots, \alpha_n)$

$|\alpha| = \alpha_1 + \cdots + \alpha_n$

$\alpha! = \alpha_1! \cdots \alpha_n!$

$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

4 Measure zero

measure zero

Let $A \subseteq \mathbb{R}^n$. We say A has measure zero in $\mathbb{R}^n$ if for every $\epsilon > 0$, there is a covering $Q_1, Q_2, \ldots$ of A by countably many rectangles such that $\sum_{i=1}^{\infty} v(Q_i) < \epsilon$. If this inequality holds, we often say that the total volume of the rectangles $Q_1, Q_2, \ldots$ is less than ϵ.

oscillation

Given $a \in Q$ define $A_\delta = \{f(x) | x \in Q \wedge |x - a| < \delta\}$. Let $M_\delta(f) = \sup A_\delta$, and let $m_\delta(f) = \inf A_\delta$, define oscillation at f by $\operatorname{osc}(f; a) = \inf_{\delta > 0}[M_\delta(f) - m_\delta(f)]$.

- f is cts at a iff $\operatorname{osc}(f; a) = 0$

4.1

4.1.1 Theorem {Munkers-11.1}

1. If $B \subseteq A$ and A has measure zero in $\mathbb{R}^n$, the so does B.
2. Let A be the union of the collection of sets $A_1, A_2, \ldots$ If each A_i has measure zero, so does A.
3. A set A has measure zero in $\mathbb{R}^n$ if and only i

4.1.2 Theorem {Munkers-11.2}

4.2 Mean Value Theorem

IF$\phi : [a, b] \to \mathbb{R}$

- ϕ is continuous at each point of **closed** interval $[a, b]$
- ϕ is differentiable at each point of **open** interval (a, b)

THENThere exists a point $c \in (a, b)$ such that $\phi(b) - \phi(a) = \phi'(c)(b - a)$.

5 Differentiation

5.1 Derivative

5.1.1 Differentiable

f is differentiable at a if there is an n by m matrix B such that

$$\frac{f(a+h)-f(a)-B\cdot h}{|h|} \to 0 \quad \text{as} \quad h \to 0$$

The matrix B is unique.

5.1.2 Directional derivative

Given $u \in \mathbb{R}^m$ which $u \neq 0$ define

$$f'(a;u) = \lim_{t\to 0} \frac{f(a+tu)-f(a)}{t}$$

Provide the limit exists.

5.1.3 Partial derivative

Define the j^{th} partial derivative of f at a to be the directional derivative of f at a with respect to the vector e_j, provide derivative exists.

$$D_j f(a) = \lim_{t\to 0} \frac{f(a+te_j)-f(a)}{t}$$

5.2 Theorems

Munkers.5.1
If f is differentiable at a then all directional derivative of f at a exist and $f'(a;u) = Df(a)\cdot u$

Munkers.5.2
If f is differentiable at a then f is continuous at a.

Munkers.5.3
If f is differentiable at a then $Df(a) = [D_1 f(a) \quad D_2 f(a) \quad \cdots \quad D_m f(a)]$.

Munkers.5.4

a. $[f$ is differentiable at $a] \Leftrightarrow \forall i[f_i$ is differentiable at $a]$.

b. If f is differentiable at a, then its derivative is the n by m matrix whose i^{th} row is the derivative of the function f_i. $(Df(a))_i = Df_i(a)$

5.3 Continuously Differentiable Functions

Munkers 6.1
Mean-value Theorem

Munkers 6.2
Let A be open in $\mathbb{R}^m$. Suppose that the partial derivative $D_i f_i(x)$ of the component function of f exists at each point x of A and are continuous on Av Then f is differentiable at each point of A.

Munkers 6.3
Let A be open in $\mathbb{R}^m$, let $f: A \to \mathbb{R}$ be a function of class $\mathbf{C}^2$. Then for each $a \in A$: $D_k D_j f(a) = D_j D_k f(a)$.

5.4 Inverse Function Theorem

Let A be open in $\mathbb{R}^n$. Let $f: A \to \mathbb{R}^n$ be of class C^r.

IF$Df(x)$ is invertible at $a \in A$.

THENThere exists a neighborhood of a such that

- $f|_U$ is injective AND $f(U) = V$ open in $\mathbb{R}^n$

- the inverse function is of class C^r
- $f^{-1}(y) = [f'(f^{-1}(y))]^{-1}$

5.5 Implicit Function Theorem

Munkers 9.1

Let A be open in $\mathbb{R}^{k+n}$, B be open in $\mathbb{R}^k$.

Let $f : A \to \mathbb{R}^n$, $g : B \to \mathbb{R}^n$ be differentiable.

Write f in the form $f(x, y)$, for $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$.

IF $f(x, g(x)) = 0$ AND $\frac{\partial f}{\partial y}$ is invertible

THEN $Dg(x) = -\left[\frac{\partial f}{\partial y}(x, g(x))\right]^{-1} \cdot \frac{\partial f}{\partial x}(x, g(x))$

Suppose $f : A \to \mathbb{R}^n$ be of class C^r.

Write f in the form $f(x, y)$, for $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$.

IF $(a, b) \in A$ AND $f(a, b) = 0$ AND $\det \frac{\partial f}{\partial y}(a, b) \neq 0$

THEN There exists $B \in \mathbb{R}^k, a \in B$ and a unique $g : B \to \mathbb{R}^n$ such that $g(a) = b$ AND $\forall x \in B. f(x, g(x)) = 0$ AND g is C^r

6 Integration

6.1 Fundamental theorem of Calculus

- If f is continuous on $[a, b]$, and if $F(X) = \int_a^x f$ for $x \in [a, b]$, then $F'(x)$ exists and equals $f(x)$.
- If ...