UofT-MAT257-2014 Note

hysw, etc(will add later)...

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Contents

1 KW

Euclidean n-space $\mathbb{R}^n$, norm, inner product, usual basis, distance, closed rectangle, open rectangle, open, closed, interior, exterior, boundary, open cover, compact(etc...), Heine-Borel Theorem, function, composition, component function, projection function π^i, continuous function, continuous & compact, oscillation, Jacobian matrix, chain rule, partial derivative, continuously differentiable, inverse functions, implicit functions,

2 TODO

Munkers 11.3, 13.*, 14.3, 14.4, 15.*

3 Basic knowledge

Note, this section is for something that does not fit anywhere

3.1 Abbreviations

cts Continuous

msr Measure

3.2 Multi-index Notation

- $\alpha = (\alpha_1, \dots, \alpha_n)$
- $|\alpha| = \alpha_1 + \cdots + \alpha_n$
- $\alpha! = \alpha_1! \cdots \alpha_n!$
- $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$
- $\partial^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$

4 Linear Algebra

4.1 Definitions

norm A function $p : V \to \mathbb{R}$ such that
For all $a \in F$ and all $u, v \in V$

- $p(v) \geq 0 \wedge [p(v) = 0 \iff v = 0]$ (separates points)
- $p(av) = |a|p(v)$ (absolute homogeneity)
- $p(u + v) \leq p(u) + p(v)$ (triangle inequality)

inner product A function $\langle x, y \rangle : V \times V \to \mathbb{R}$ such that
For all $x, y, z \in V$ and $c \in \mathbb{F}$.

- $\langle x, y \rangle = \langle y, x \rangle$
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- $\langle cx, y \rangle = c\langle x, y \rangle = \langle x, cy \rangle$
- $\langle x, x \rangle > 0$ if $x \neq 0$

5 Topology

5.1 Definitions

metric A function $d : X \times X \to \mathbb{R}$ such that
For all $x, y, z \in X$

- $d(x, y) \geq 0$
- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$

ϵ-neighborhood $\mathbf{U}(x; \epsilon) = \{y | d(x, y) < \epsilon\}$

open set in metric space A set $U \subseteq X$ is said to be open in X if $\forall x \in U \exists \epsilon > 0 [\mathbf{U}(x; \epsilon) \subseteq U]$ note that finite intersections and arbitrary unions of open set are open set

closed set in metric space A set contains all its limit point.
note that closed set is complement of open set in topology

5.2 Partition of unity

TODO

6 Measure Theory

6.1 Measure zero

Let $A \subseteq \mathbb{R}^n$. We say A has measure zero in $\mathbb{R}^n$ if for every $\epsilon > 0$, there is a covering $Q_1, Q_2, \dots$ of A by countably many rectangles such that $\sum_{i=1}^{\infty} v(Q_i) < \epsilon$. If this inequality holds, we often say that the total volume of the rectangles $Q_1, Q_2, \dots$ is less than ϵ.

6.2 Theorems

{Munkers-11.1}

1. If $B \subseteq A$ and A has measure zero in $\mathbb{R}^n$, the so does B.
2. Let A be the union of the collection of sets $A_1, A_2, \dots$ If each A_i has measure zero, so does A.
3. A set A has measure zero in $\mathbb{R}^n$ if and only i

7 General Calculus

7.1 Definitions

oscillation

Given $a \in Q$ define $A_\delta = \{f(x)|x \in Q \wedge |x-a| < \delta\}$. Let $M_\delta(f) = \sup A_\delta$, and let $m_\delta(f) = \inf A_\delta$, define oscillation at f by $\operatorname{osc}(f;a) = \inf_{\delta>0}[M_\delta(f) - m_\delta(f)]$. f is cts at a iff $\operatorname{osc}(f;a) = 0$

7.2 Extreme Value Theorem

Suppose $f : X \to \mathbb{R}$ is continuous and X is compact, then $\exists x_0 \in X$ such that $\forall x \in X. f(x) \leq f(x_0)$.

7.3 Intermediate Value Theorem

Suppose $E \in \mathbb{R}$ is connected and $f : E \to \mathbb{R}$ is continuous.
Suppose $f(x) = a$ and $f(y) = b$ for some $x, y \in E$ and $a < b$.
Then $\forall a < c < b \exists$ some $z \in E$ such that $f(z) = c$.

7.4 Mean Value Theorem

Suppose $\phi : [a, b] \to \mathbb{R}$ is

- continuous at each point of **closed** interval $[a, b]$
- differentiable at each point of **open** interval (a, b)

Then there exists a point $c \in (a, b)$ such that $\phi(b) - \phi(a) = \phi'(c)(b - a)$.

8 Differential Calculus

8.1 Definitions

differentiable f is differentiable at a if there is an n by m matrix B such that

$$\frac{f(a+h)-f(a)-B\cdot h}{|h|}\to 0 \quad \text{as} \quad h\to 0$$

The matrix B is unique.

Directional derivative Given $u\in\mathbb{R}^m$ which $u\neq 0$ define

$$f'(a;u)=\lim_{t\to 0}\frac{f(a+tu)-f(a)}{t}$$

Provide the limit exists.

Partial derivative Define the j^{th} partial derivative of f at a to be the directional derivative of f at a with respect to the vector e_j, provide derivative exists.

$$D_jf(a)=\lim_{t\to 0}\frac{f(a+te_j)-f(a)}{t}$$

8.2 Notations

$Df(a)$: derivative of f at a

$f'(a;u)$: directional derivative of f at a respect to vector u.

$D_jf(a)$: j^{th}partial derivative of f at a.

f_i : i^{th} component function of f.

∇g : gradient of g, $\nabla g=\mathbf{grad}g=\sum_i(D_ig)e_i$

Jf : Jacobian matrix, $J_{ij}=D_jf_i(a)$

8.3 Differentiability Theorems

Theorems Munkers.5.1
If f is differentiable at a then all directional derivative of f at a exist and $f'(a;u)=Df(a)\cdot u$

Theorems Munkers.5.2
If f is differentiable at a then f is continuous at a.

Theorems Munkers.5.3
If f is differentiable at a then $Df(a)=[D_1f(a) \quad D_2f(a) \quad \cdots \quad D_mf(a)]$.

Theorems Munkers.5.4

a. [f is differentiable at a] $\Leftrightarrow$ $\forall i[f_i$ is differentiable at $a]$.

b. If f is differentiable at a, then its derivative is the n by m matrix whose i^{th} row is the derivative of the function f_i. $(Df(a))_i=Df_i(a)$

8.4 Continuously Differentiable Functions

A function is C^1 if all of its partial derivatives are continous. A function is C^r if all of its partial derivatives are C^{r-1}.

Munkers 6.1
If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then there exists $c\in(a,b)$ such that $f(b)-f(a)=f'(c)(b-a)$.

Munkers 7.3
Let A be open in $\mathbb{R}^m$; let $f:A\to\mathbb{R}$ be differentiable on A. If A contains the line segment with end points a and $a+h$, then there is a point $c=a+th$ with $0<t<1$ of this line segment such that $f(a+h)-f(a)=(Df(c))h$.

Munkers 6.2
Let A be open in $\mathbb{R}^m$. Suppose that the partial derivative $D_if_i(x)$ of the component function of f exists at each point x of A and are continuous on A. Then f is differentiable at each point of A.

Munkers 6.3
Let A be open in $\mathbb{R}^m$, let $f:A\to\mathbb{R}$ be a function of class $\mathbf{C}^2$. Then for each $a\in A$: $D_kD_jf(a)=D_jD_kf(a)$.

8.5 Chain Rule

Let $A\subset\mathbb{R}^m$. Let $B\subset\mathbb{R}^n$. Let $f:A\to\mathbb{R}^n$ and $g:B\to\mathbb{R}^p$, with $f(A)\subset B$. Suppose $f(a)=b$. If f is diferentiable at a and g is differentiable at b, then the composite function $g\circ f$ is differentiable at a. Furthermore,

$$D(g\circ f)(a)=Dg(b)\cdot Df(a)$$

8.6 Inverse Function Theorem

Let A be open in $\mathbb{R}^n$. Let $f:A\to\mathbb{R}^n$ be of class C^r.

IF $Df(x)$ is invertible at $a\in A$.

THEN There exists a neighborhood of a such that

- $f|_U$ is injective AND $f(U) = V$ open in $\mathbb{R}^n$
- the inverse function is of class C^r
- $f^{-1}(y) = [f'(f^{-1}(y))]^{-1}$

8.7 Implicit Function Theorem

Suppose $f : A \to \mathbb{R}^n$ be of class C^r.

Write f in the form $f(x, y)$, for $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$.

IF $(a, b) \in A$ AND $f(a, b) = 0$ AND $\det \frac{\partial f}{\partial y}(a, b) \neq 0$

THEN There exists $B \in \mathbb{R}^k, a \in B$ and a unique $g : B \to \mathbb{R}^n$ such that $g(a) = b$ AND $\forall x \in B. f(x, g(x)) = 0$ AND g is C^r

Munkers 9.1

Let A be open in $\mathbb{R}^{k+n}$, B be open in $\mathbb{R}^k$.

Let $f : A \to \mathbb{R}^n$, $g : B \to \mathbb{R}^n$ be differentiable.

Write f in the form $f(x, y)$, for $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$.

IF $f(x, g(x)) = 0$ AND $\frac{\partial f}{\partial y}$ is invertible

THEN $Dg(x) = -\left[\frac{\partial f}{\partial y}(x, g(x))\right]^{-1} \cdot \frac{\partial f}{\partial x}(x, g(x))$

8.8 Taylor's theorem

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is of class C^k on an open convex set S. If $a \in S$ and $a + h \in S$, then

$$f(a + h) = \sum_{|\alpha| \leq k} \frac{\partial^\alpha f(a)}{\alpha!} h^\alpha + R_{a,k}(h),$$

If f is of class C^{k+1} on S, for some $c \in (0, 1)$ we have

$$R_{a,k}(h) = \sum_{|\alpha| = k+1} \frac{\partial^\alpha f(a + ch)}{\alpha!} h^\alpha$$

9 Integral Calculus

Note, Riemann Integral was taught in this class.

9.1 Definitions

rectangle (in $\mathbb{R}^n$) $Q = [a_1, b_1] \times [a_2, b_2] \times [a_n, b_n]$

component interval of Q $[a_i, b_i]$

volume of Q $v(Q) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$

partition *TODO*

subinterval(determined by P) *OMIT*

subrectangle(determined by P) *OMIT*

mech of P *OMIT*

refinement *OMIT*

common refinement *OMIT*

?- $m_R(f) = \inf\{f(x) | x \in R\}$

?- $M_R(f) = \inf\{f(x) | x \in R\}$

lower sum $L(f, P) = \sum_R m_R(f) \cdot v(R)$

upper sum $U(f, P) = \sum_R M_R(f) \cdot v(R)$

lower integral $\underline{\int_Q} f = \sup_P\{L(f, P)\}$

upper integral $\overline{\int_Q} f = \inf_P\{L(f, P)\}$

oscillation *TODO*

rectifiable set A bounded set $S \in \mathbb{R}^n$ is rectifiable if the constant function 1 is integrable over S. S is rectifiable iff S is bounded and BdS has measure zero

volume of a rectifiable set $v(S) = \int_S 1$

9.2 Riemann condition

Given: Q a rectangle, $f : Q \to \mathbb{R}$ a bounded function.

$$\boxed{\underline{\int_Q} f = \overline{\int_Q} f \text{ iff } \forall_{\epsilon > 0} \exists P[U(f, P) - L(f, P) \leq \epsilon]}$$

P is a partion of Q

Corollary/Theorem: every constant function is integrable.

9.3 Riemann-Lebesgue theorem

A function on a compact interval [a, b] is Riemann integrable if and only if it is bounded and continuous almost everywhere (the set of its points of discontinuity has measure zero, in the sense of Lebesgue measure). [wiki] [11.2]

9.4 Fundamental theorem of Calculus

- If f is continuous on $[a, b]$, and if $F(X) = \int_a^x f$ for $x \in [a, b]$, then $F'(x)$ exists and equals $f(x)$.
- If f is contininous on $[a, b]$, and if g is a function such that $g'(x) = f(x)$ for $x \in [a, b]$ then $\int_a^b f = g(b) - g(a)$

9.5 Fubini's theorem

Let $Q = A \times B$, where A is a rectangle in $\mathbb{R}^k$ and A is a rectangle in $\mathbb{R}^n$. If f is bounded function and integrable over Q, then $\underline{\int_{y \in B}} f(x, y)$ and $\overline{\int_{y \in B}} f(x, y)$ are integrable over A and

$$\int_Q f = \int_{x \in A} \underline{\int_{y \in B}} f(x, y) = \int_{x \in A} \overline{\int_{y \in B}} f(x, y)$$

9.6 Properties of integral

TODO

9.7 Properties of rectifiable set

TODO

10 Change of Variables

TODO

11 Diffeomorphism

TODO

12 Manifolds

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13.1 Review

THM : `rectifiable`
LET : $S \subset \mathbb{R}^n$ be bounded
THEN : S is rectifiable
IFF ∂S has measure zero, equivalently χ_s is integrable
IDEA : rectifiable is a property of the boundary of a set. If the boundary has measure zero then the set is rectifiable.

DEFN : `volume`
LET : $S \subset \mathbb{R}^n$ be bounded and *rectifiable*
THEN : define the volume of S as:
$V(S) := \int_S 1 := \int_Q \chi_x$

THM : `partition of unity`
LET : $A \subset \mathbb{R}^n$, $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of A
THEN : $\exists$ a collection of C^∞ functions $\{\psi_\beta\}_{\beta \in \mathcal{B}}$ s.t

i $\forall x \in A \quad 0 < \psi_\beta \leq 1 \quad \forall \beta \in \mathcal{B}$
ii $\forall x \in A \quad \exists$ open neighbourhood V of x such that: all but finitely many ψ_β vanish on V (*locally finite*)
iii $\forall x \in A \quad \sum_{\beta \in \mathcal{B}} \psi_\beta(x) \equiv 1$
iv $\forall x \in A \quad \exists \alpha$ such that $\text{supp}(\psi_\beta) \subset U_\alpha$, ie $\{x | \psi_\beta(x) \neq 0\} \subset U_\alpha$

A collection of functions satisfying i, ii, iii is called a *partition of unity*
It is *subordinate* to the open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ if it satisfies condition iv

13.2 Open Sets

DEFN : `extended integral`
LET : $U^{open} \subset \mathbb{R}^n$, $f : U \to \mathbb{R}$ is continuous
IF $f \geq 0$
THEN : define then *extended integral* of f over U as:
$\int_u f := \sup\{\int_D f \mid D \subset U \text{ where D is compact, rectifiable}\}$
IF f is arbitrary and $\int_u f_+$, $\int_u f_-$ exist
THEN : define then *extended integral* of f over U as:
$\int_u f := \int_u f_+ - \int_u f_-$ where
$f_+(x) = \max\{f(x), 0\}$ and $f_-(x) = \min\{-f(x), 0\}$

THM : `mnk 15.2`
LET : $U^{open} \subset \mathbb{R}^n$, $f : U \to \mathbb{R}$ is continuous
Choose an exhaustion of U by compact K_i such that
$K_1 \subset K_2^\circ \subset K_2 \subset K_3^\circ \ldots$
THEN : f has an *extended integral* defined as:
$\int_u f := \sum_{i=1}^\infty (\int_u \psi_i f)$
IFF $\sum \int_u \psi_i |f|$ converges to a finite number

RMK : If $U^{open} \subset \mathbb{R}^n$, then $\int_u f$ refers to the *extended integral*

THM : `mnk 15.4`
LET : U^{open} is bounded, $f : U \to \mathbb{R}^n$ is bounded and continuous
THEN :

i the extended integral exists
ii if the ordinary integral exists, then they are equal

THM : `mnk 16.5`
LET : $U^{open} \subset \mathbb{R}^n$, $f : U \to \mathbb{R}$ is continuous, $\{\psi_i\}$ be a partition of unity with compact support
THEN : $\int_u f$ exists
IFF $\sum_{i=1}^\infty \int_u \psi_i |f|$ converges to a finite number
In this case $\int_u f = \sum\limits_{i=1}^{\infty} (\int_u \psi_i f)$