

# UofT-MAT257-2014 Note

hysw, etc(will add later)...

January 11, 2015

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## 1 KW

Euclidean n-space  $\mathbb{R}^n$ , norm, inner product, usual basis, distance, closed rectangle, open rectangle, open, closed, interior, exterior, boundary, open cover, compact(etc...), Heine-Borel Theorem, function, composition, component function, projection function  $\pi^i$ , continuous function, continuous & compact, oscillation, Jacobian matrix, chain rule, partial derivative, continuously differentiable, inverse functions, implicit functions,

## 2 TODO

Munkers 11.3, 13.\*, 14.3, 14.4, 15.\*

### 3 Basic knowledge

Note, this section is for something that does not fit anywhere

#### 3.1 Abbreviations

**cts** Continuous

**msr** Measure

#### 3.2 Multi-index Notation

- $\alpha = (\alpha_1, \dots, \alpha_n)$
- $|\alpha| = \alpha_1 + \dots + \alpha_n$
- $\alpha! = \alpha_1! \dots \alpha_n!$
- $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$
- $\partial^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n} f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$

## 4 Linear Algebra

### 4.1 Definitions

**norm** A function  $p : V \rightarrow \mathbb{R}$  such that

For all  $a \in F$  and all  $u, v \in V$

- $p(v) \geq 0 \wedge [p(v) = 0 \iff v = 0]$  (separates points)
- $p(av) = |a|p(v)$  (absolute homogeneity)
- $p(u + v) \leq p(u) + p(v)$  (triangle inequality)

**inner product** A function  $\langle x, y \rangle : V \times V \rightarrow \mathbb{R}$  such that

For all  $x, y, z \in V$  and  $c \in \mathbb{F}$ .

- $\langle x, y \rangle = \langle y, x \rangle$
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- $\langle cx, y \rangle = c\langle x, y \rangle = \langle x, cy \rangle$
- $\langle x, x \rangle > 0$  if  $x \neq 0$

## 5 Topology

### 5.1 Definitions

**metric** A function  $d : X \times X \rightarrow \mathbb{R}$  such that

For all  $x, y, z \in X$

- $d(x, y) \geq 0$
- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$

**$\epsilon$ -neighborhood**  $U(x; \epsilon) = \{y | d(x, y) < \epsilon\}$

**open set in metric space** A set  $U \subseteq X$  is said to be open in  $X$  if  $\forall x \in U \exists \epsilon > 0 [U(x; \epsilon) \subseteq U]$  note that finite intersections and arbitrary unions of open set are open set

**closed set in metric space** A set contains all its limit point.  
note that closed set is complement of open set in topology

### 5.2 Partition of unity

*TODO*

## 6 Measure Theory

### 6.1 Measure zero

Let  $A \subseteq \mathbb{R}^n$ . We say  $A$  has measure zero in  $\mathbb{R}^n$  if for every  $\epsilon > 0$ , there is a covering  $Q_1, Q_2, \dots$  of  $A$  by countably many rectangles such that  $\sum_{i=1}^{\infty} v(Q_i) < \epsilon$ . If this inequality holds, we often say that the total volume of the rectangles  $Q_1, Q_2, \dots$  is less than  $\epsilon$ .

### 6.2 Theorems

{Munkers-11.1}

1. If  $B \subseteq A$  and  $A$  has measure zero in  $\mathbb{R}^n$ , then so does  $B$ .
2. Let  $A$  be the union of the collection of sets  $A_1, A_2, \dots$ . If each  $A_i$  has measure zero, so does  $A$ .
3. A set  $A$  has measure zero in  $\mathbb{R}^n$  if and only if

## 7 General Calculus

### 7.1 Definitions

#### oscillation

Given  $a \in Q$  define  $A_\delta = \{f(x) | x \in Q \wedge |x - a| < \delta\}$ . Let  $M_\delta(f) = \sup A_\delta$ , and let  $m_\delta(f) = \inf A_\delta$ , define oscillation at  $f$  by  $\text{osc}(f; a) = \inf_{\delta > 0} [M_\delta(f) - m_\delta(f)]$ .  $f$  is cts at  $a$  iff  $\text{osc}(f; a) = 0$

### 7.2 Extreme Value Theorem

Suppose  $f : X \rightarrow \mathbb{R}$  is continuous and  $X$  is compact, then  $\exists x_0 \in X$  such that  $\forall x \in X. f(x) \leq f(x_0)$ .

### 7.3 Intermediate Value Theorem

Suppose  $E \in \mathbb{R}$  is connected and  $f : E \rightarrow \mathbb{R}$  is continuous.

Suppose  $f(x) = a$  and  $f(y) = b$  for some  $x, y \in E$  and  $a < b$ .

Then  $\forall a < c < b \exists$  some  $z \in E$  such that  $f(z) = c$ .

### 7.4 Mean Value Theorem

Suppose  $\phi : [a, b] \rightarrow \mathbb{R}$  is

- continuous at each point of **closed** interval  $[a, b]$
- differentiable at each point of **open** interval  $(a, b)$

Then there exists a point  $c \in (a, b)$  such that  $\phi(b) - \phi(a) = \phi'(c)(b - a)$ .

## 8 Differential Calculus

### 8.1 Definitions

**differentiable**  $f$  is differentiable at  $a$  if there is an  $n$  by  $m$  matrix  $B$  such that

$$\frac{f(a+h) - f(a) - B \cdot h}{|h|} \rightarrow 0 \quad \text{as } h \rightarrow 0$$

The matrix  $B$  is unique.

**Directional derivative** Given  $u \in \mathbb{R}^m$  which  $u \neq 0$  define

$$f'(a; u) = \lim_{t \rightarrow 0} \frac{f(a + tu) - f(a)}{t}$$

Provide the limit exists.

**Partial derivative** Define the  $j^{\text{th}}$  partial derivative of  $f$  at  $a$  to be the directional derivative of  $f$  at  $a$  with respect to the vector  $e_j$ , provide derivative exists.

$$D_j f(a) = \lim_{t \rightarrow 0} \frac{f(a + te_j) - f(a)}{t}$$

### 8.2 Notations

$Df(a)$  : derivative of  $f$  at  $a$

$f'(a; u)$  : directional derivative of  $f$  at  $a$  respect to vector  $u$ .

$D_j f(a)$  :  $j^{\text{th}}$  partial derivative of  $f$  at  $a$ .

$f_i$  :  $i^{\text{th}}$  component function of  $f$ .

$\nabla g$  : gradient of  $g$ ,  $\nabla g = \mathbf{grad} g = \sum_i (D_i g) e_i$

$Jf$  : Jacobian matrix,  $J_{ij} = D_j f_i(a)$

### 8.3 Differentiability Theorems

#### Theorems Munkers.5.1

If  $f$  is differentiable at  $a$  then all directional derivative of  $f$  at  $a$  exist and  $f'(a; u) = Df(a) \cdot u$

#### Theorems Munkers.5.2

If  $f$  is differentiable at  $a$  then  $f$  is continuous at  $a$ .

#### Theorems Munkers.5.3

If  $f$  is differentiable at  $a$  then  $Df(a) = [D_1 f(a) \ D_2 f(a) \ \cdots \ D_m f(a)]$ .

#### Theorems Munkers.5.4

a.  $[f \text{ is differentiable at } a] \Leftrightarrow \forall i [f_i \text{ is differentiable at } a]$ .

b. If  $f$  is differentiable at  $a$ , then its derivative is the  $n$  by  $m$  matrix whose  $i^{\text{th}}$  row is the derivative of the function  $f_i$ .  $(Df(a))_i = Df_i(a)$

### 8.4 Continuously Differentiable Functions

A function is  $C^1$  if all of its partial derivatives are continuous. A function is  $C^r$  if all of its partial derivatives are  $C^{r-1}$ .

#### Munkers 6.1

If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then there exists  $c \in (a, b)$  such that  $f(b) - f(a) = f'(c)(b - a)$ .

#### Munkers 7.3

Let  $A$  be open in  $\mathbb{R}^m$ ; let  $f : A \rightarrow \mathbb{R}$  be differentiable on  $A$ . If  $A$  contains the line segment with end points  $a$  and  $a + h$ , then there is a point  $c = a + th$  with  $0 < t < 1$  of this line segment such that  $f(a + h) - f(a) = (Df(c))h$ .

#### Munkers 6.2

Let  $A$  be open in  $\mathbb{R}^m$ . Suppose that the partial derivative  $D_i f_i(x)$  of the component function of  $f$  exists at each point  $x$  of  $A$  and are continuous on  $A$ . Then  $f$  is differentiable at each point of  $A$ .

#### Munkers 6.3

Let  $A$  be open in  $\mathbb{R}^m$ , let  $f : A \rightarrow \mathbb{R}$  be a function of class  $C^2$ . Then for each  $a \in A$ :  $D_k D_j f(a) = D_j D_k f(a)$ .

### 8.5 Chain Rule

Let  $A \subset \mathbb{R}^m$ . Let  $B \subset \mathbb{R}^n$ . Let  $f : A \rightarrow \mathbb{R}^n$  and  $g : B \rightarrow \mathbb{R}^p$ , with  $f(A) \subset B$ . Suppose  $f(a) = b$ . If  $f$  is differentiable at  $a$  and  $g$  is differentiable at  $b$ , then the composite function  $g \circ f$  is differentiable at  $a$ . Furthermore,

$$D(g \circ f)(a) = Dg(b) \cdot Df(a)$$

### 8.6 Inverse Function Theorem

Let  $A$  be open in  $\mathbb{R}^n$ . Let  $f : A \rightarrow \mathbb{R}^n$  be of class  $C^r$ .

**IF**  $Df(x)$  is invertible at  $a \in A$ .



**THEN** There exists a neighborhood of  $a$  such that

- $f|_U$  is injective AND  $f(U) = V$  open in  $\mathbb{R}^n$
- the inverse function is of class  $C^r$
- $f^{-1}(y) = [f'(f^{-1}(y))]^{-1}$

## 8.7 Implicit Function Theorem

Suppose  $f : A \rightarrow \mathbb{R}^n$  be of class  $C^r$ .

Write  $f$  in the form  $f(x, y)$ , for  $x \in \mathbb{R}^k$  and  $y \in \mathbb{R}^n$ .

**IF**  $(a, b) \in A$  AND  $f(a, b) = 0$  AND  $\det \frac{\partial f}{\partial y}(a, b) \neq 0$

**THEN** There exists  $B \in \mathbb{R}^k, a \in B$  and a unique  $g : B \rightarrow \mathbb{R}^n$  such that  $g(a) = b$  AND  $\forall x \in B. f(x, g(x)) = 0$  AND  $g$  is  $C^r$

### Munkers 9.1

Let  $A$  be open in  $\mathbb{R}^{k+n}$ ,  $B$  be open in  $\mathbb{R}^k$ .

Let  $f : A \rightarrow \mathbb{R}^n, g : B \rightarrow \mathbb{R}^n$  be differentiable.

Write  $f$  in the form  $f(x, y)$ , for  $x \in \mathbb{R}^k$  and  $y \in \mathbb{R}^n$ .

**IF**  $f(x, g(x)) = 0$  AND  $\frac{\partial f}{\partial y}$  is invertible

**THEN**  $Dg(x) = - \left[ \frac{\partial f}{\partial y}(x, g(x)) \right]^{-1} \cdot \frac{\partial f}{\partial x}(x, g(x))$

## 8.8 Taylor's theorem

Suppose  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is of class  $C^k$  on an open convex set  $S$ . If  $a \in S$  and  $a + h \in S$ , then

$$f(a + h) = \sum_{|\alpha| \leq k} \frac{\partial^\alpha f(a)}{\alpha!} h^\alpha + R_{a,k}(h),$$

If  $f$  is of class  $C^{k+1}$  on  $S$ , for some  $c \in (0, 1)$  we have

$$R_{a,k}(h) = \sum_{|\alpha|=k+1} \frac{\partial^\alpha f(a + ch)}{\alpha!} h^\alpha$$

## 9 Integral Calculus

Note, Riemann Integral was taught in this class.

### 9.1 Definitions

**rectangle (in  $\mathbb{R}^n$ )**  $Q = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$

**component interval of  $Q$**   $[a_i, b_i]$

**volume of  $Q$**   $v(Q) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$

**partition** *TODO*

**subinterval(determined by  $P$ )** *OMIT*

**subrectangle(determined by  $P$ )** *OMIT*

**mech of  $P$**  *OMIT*

**refinement** *OMIT*

**common refinement** *OMIT*

?-  $m_R(f) = \inf\{f(x) | x \in R\}$

?-  $M_R(f) = \sup\{f(x) | x \in R\}$

**lower sum**  $L(f, P) = \sum_R m_R(f) \cdot v(R)$

**upper sum**  $U(f, P) = \sum_R M_R(f) \cdot v(R)$

**lower integral**  $\int_Q f = \sup_P \{L(f, P)\}$

**upper integral**  $\overline{\int_Q f} = \inf_P \{U(f, P)\}$

**oscillation** *TODO*

**rectifiable set** A bounded set  $S \in \mathbb{R}^n$  is rectifiable if the constant function 1 is integrable over  $S$ .  $S$  is rectifiable iff  $S$  is bounded and  $\text{Bd}S$  has measure zero

**volume of a rectifiable set**  $v(S) = \int_S 1$

### 9.2 Riemann condition

Given:  $Q$  a rectangle,  $f : Q \rightarrow \mathbb{R}$  a bounded function.

$$\boxed{\int_Q f = \overline{\int_Q f} \text{ iff } \forall \epsilon_{>0} \exists P [U(f, P) - L(f, P) \leq \epsilon]}$$

$P$  is a partition of  $Q$

Corollary/Theorem: every constant function is integrable.

### 9.3 Riemann-Lebesgue theorem

A function on a compact interval  $[a, b]$  is Riemann integrable if and only if it is bounded and continuous almost everywhere (the set of its points of discontinuity has measure zero, in the sense of Lebesgue measure). [\[wiki\]](#) [\[11.2\]](#)

### 9.4 Fundamental theorem of Calculus

- If  $f$  is continuous on  $[a, b]$ , and if  $F(x) = \int_a^x f$  for  $x \in [a, b]$ , then  $F'(x)$  exists and equals  $f(x)$ .
- If  $f$  is continuous on  $[a, b]$ , and if  $g$  is a function such that  $g'(x) = f(x)$  for  $x \in [a, b]$  then  $\int_a^b f = g(b) - g(a)$

### 9.5 Fubini's theorem

Let  $Q = A \times B$ , where  $A$  is a rectangle in  $\mathbb{R}^k$  and  $B$  is a rectangle in  $\mathbb{R}^n$ . If  $f$  is a bounded function and integrable over  $Q$ , then  $\int_{y \in B} f(x, y)$  and  $\overline{\int_{y \in B} f(x, y)}$  are integrable over  $A$  and

$$\int_Q f = \int_{x \in A} \int_{y \in B} f(x, y) = \int_{x \in A} \overline{\int_{y \in B} f(x, y)}$$

### 9.6 Properties of integral

*TODO*

### 9.7 Properties of rectifiable set

*TODO*

## 10 Change of Variables

Diffeomorphism [\[wiki\]](#)

$f : A \rightarrow B$  is a diffeomorphism if  $f$  is a bijection AND  $f$  and  $f^{-1}$  are of class  $C^r$ .

Moreover  $f$  is called  $C^r$ -diffeomorphism

Change of Variables Theorem [\[17.2\]](#)

**LET** opensets  $U, V \subseteq \mathbb{R}$

$g : U \rightarrow V$  a diffeomorphism

$f : V \rightarrow \mathbb{R}$  be a continuous function

**THEN**  $f$  is intergrable over  $B$

**IFF**  $(f \circ g)|\det Dg|$  is integrable over  $A$

**NOTE**  $\int_B f = \int_A (f \circ g)|\det Dg|$

Substitution rule [\[17.1\]](#)

**LET**  $I = [a, b], J = [c, d] \subseteq \mathbb{R}$

$g : I \rightarrow J$  of class  $C^1$

$f : J \rightarrow \mathbb{R}$  is continuous

**IF**  $\forall x \in (a, b) [g'(x) \neq 0]$

**THEN**  $\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g)g'$

equivalently  $\int_J f = \int_I (f \circ g)|g'|$

## 11 Manifolds

## 12 Lecture Notes

### 12.1 2015-01-05 Monday

#### 12.1.1 Review

**THM** : rectifiable

**LET** :  $S \subset \mathbb{R}^n$  be bounded

**THEN** :  $S$  is rectifiable

**IFF**  $\partial S$  has measure zero, equivalently  $\chi_S$  is integrable

**IDEA** : rectifiable is a property of the boundary of a set. If the boundary has measure zero then the set is rectifiable.

**DEFN** : volume

**LET** :  $S \subset \mathbb{R}^n$  be bounded and *rectifiable*

**THEN** : define the volume of  $S$  as:

$$V(S) := \int_S 1 := \int_Q \chi_S$$

**THM** : partition of unity

**LET** :  $A \subset \mathbb{R}^n$ ,  $\{U_\alpha\}_{\alpha \in \mathcal{A}}$  be an open cover of  $A$

**THEN** :  $\exists$  a collection of  $C^\infty$  functions  $\{\psi_\beta\}_{\beta \in \mathcal{B}}$  s.t

- i  $\forall x \in A \quad 0 < \psi_\beta \leq 1 \quad \forall \beta \in \mathcal{B}$
- ii  $\forall x \in A \quad \exists$  open neighbourhood  $V$  of  $x$  such that:  
all but finitely many  $\psi_\beta$  vanish on  $V$  (*locally finite*)
- iii  $\forall x \in A \quad \sum_{\beta \in \mathcal{B}} \psi_\beta(x) \equiv 1$
- iv  $\forall x \in A \quad \exists \alpha$  such that  
 $\text{supp}(\psi_\beta) \subset U_\alpha$ , ie  $\{x | \psi_\beta(x) \neq 0\} \subset U_\alpha$

A collection of functions satisfying i, ii, iii is called a *partition of unity*

It is *subordinate* to the open cover  $\{U_\alpha\}_{\alpha \in \mathcal{A}}$  if it satisfies condition iv

#### 12.1.2 Open Sets

**DEFN** : extended integral

**LET** :  $U^{open} \subset \mathbb{R}^n$ ,  $f : U \rightarrow \mathbb{R}$  is continuous

**IF**  $f \geq 0$

**THEN** : define then *extended integral* of  $f$  over  $U$  as:

$$\int_U f := \sup \left\{ \int_D f \mid D \subset U \text{ where } D \text{ is compact, rectifiable} \right\}$$

**IF**  $f$  is arbitrary and  $\int_U f_+$ ,  $\int_U f_-$  exist

**THEN** : define then *extended integral* of  $f$  over  $U$  as:

$$\int_U f := \int_U f_+ - \int_U f_- \text{ where } f_+(x) = \max\{f(x), 0\} \text{ and } f_-(x) = \min\{-f(x), 0\}$$

**THM** : mnk 15.2

**LET** :  $U^{open} \subset \mathbb{R}^n$ ,  $f : U \rightarrow \mathbb{R}$  is continuous

Choose an *exhaustion* of  $U$  by compact  $K_i$  such that

$$K_1 \subset K_2^\circ \subset K_2 \subset K_3^\circ \dots \text{ and } U = \cup_{i=1}^\infty K_i$$

$f$  has an *extended integral*

**IFF** the sequence  $\int_{K_i} |f_i|$  is bounded

**THEN** :  $\int_U f = \lim_{i \rightarrow \infty} \int_{K_i} f$

**RMK** : If  $U^{open} \subset \mathbb{R}^n$ , then  $\int_U f$  refers to the *extended integral*

**THM** : mnk 15.4

**LET** :  $U^{open}$  is bounded,  $f : U \rightarrow \mathbb{R}^n$  is bounded and continuous

**THEN** :

- i the extended integral exists
- ii if the ordinary integral exists, then they are equal

**THM** : mnk 16.5

**LET** :  $U^{open} \subset \mathbb{R}^n$ ,  $f : U \rightarrow \mathbb{R}$  is continuous,  $\{\psi_i\}$  be a partition of unity with compact support

**THEN** :  $\int_U f$  exists

**IFF**  $\sum_{i=1}^\infty \int_U \psi_i |f|$  converges to a finite number

In this case  $\int_U f = \sum_{i=1}^\infty (\int_U \psi_i f)$

### 12.2 2015-01-07 Wednesday

Note: this is not finished --.

- [Diffeomorphism](#)
- [Change of Variables](#)
- [Substitution Rule](#)