

UofT-MAT257-2014 Note

hysw, etc(will add later)...

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1 KW

Euclidean n-space \mathbb{R}^n , norm, inner product, usual basis, distance, closed rectangle, open rectangle, open, closed, interior, exterior, boundary, open cover, compact(etc...), Heine-Borel Theorem, function, composition, component function, projection function π^i , continuous function, continuous & compact, oscillation, Jacobian matrix, chain rule, partial derivative, continuously differentiable, inverse functions, implicit functions,

2 TODO

Munkers 11.3, 13.*, 14.3, 14.4, 15.*

3 Basic knowledge

Note, this section is for something that does not fit anywhere

3.1 Abbreviations

cts Continuous

msr Measure

3.2 Multi-index Notation

- $\alpha = (\alpha_1, \dots, \alpha_n)$
- $|\alpha| = \alpha_1 + \dots + \alpha_n$
- $\alpha! = \alpha_1! \dots \alpha_n!$
- $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$
- $\partial^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n} f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$

4 Linear Algebra

4.1 Definitions

norm A function $p : V \rightarrow \mathbb{R}$ such that

For all $a \in F$ and all $u, v \in V$

- $p(v) \geq 0 \wedge [p(v) = 0 \iff v = 0]$ (separates points)
- $p(av) = |a|p(v)$ (absolute homogeneity)
- $p(u + v) \leq p(u) + p(v)$ (triangle inequality)

inner product A function $\langle x, y \rangle : V \times V \rightarrow \mathbb{R}$ such that

For all $x, y, z \in V$ and $c \in \mathbb{F}$.

- $\langle x, y \rangle = \langle y, x \rangle$
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- $\langle cx, y \rangle = c\langle x, y \rangle = \langle x, cy \rangle$
- $\langle x, x \rangle > 0$ if $x \neq 0$

5 Topology

5.1 Definitions

metric A function $d : X \times X \rightarrow \mathbb{R}$ such that

For all $x, y, z \in X$

- $d(x, y) \geq 0$
- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$

ϵ -neighborhood $U(x; \epsilon) = \{y | d(x, y) < \epsilon\}$

open set in metric space A set $U \subseteq X$ is said to be open in X if $\forall x \in U \exists \epsilon > 0 [U(x; \epsilon) \subseteq U]$ note that finite intersections and arbitrary unions of open set are open set

closed set in metric space A set contains all its limit point.
note that closed set is complement of open set in topology

5.2 Partition of unity

TODO

6 Measure Theory

6.1 Measure zero

Let $A \subseteq \mathbb{R}^n$. We say A has measure zero in \mathbb{R}^n if for every $\epsilon > 0$, there is a covering Q_1, Q_2, \dots of A by countably many rectangles such that $\sum_{i=1}^{\infty} v(Q_i) < \epsilon$. If this inequality holds, we often say that the total volume of the rectangles Q_1, Q_2, \dots is less than ϵ .

6.2 Theorems

{Munkers-11.1}

1. If $B \subseteq A$ and A has measure zero in \mathbb{R}^n , then so does B .
2. Let A be the union of the collection of sets A_1, A_2, \dots . If each A_i has measure zero, so does A .
3. A set A has measure zero in \mathbb{R}^n if and only if

7 General Calculus

7.1 Definitions

oscillation

Given $a \in Q$ define $A_\delta = \{f(x) | x \in Q \wedge |x - a| < \delta\}$. Let $M_\delta(f) = \sup A_\delta$, and let $m_\delta(f) = \inf A_\delta$, define oscillation at f by $\text{osc}(f; a) = \inf_{\delta > 0} [M_\delta(f) - m_\delta(f)]$. f is cts at a iff $\text{osc}(f; a) = 0$

7.2 Extreme Value Theorem

Suppose $f : X \rightarrow \mathbb{R}$ is continuous and X is compact, then $\exists x_0 \in X$ such that $\forall x \in X. f(x) \leq f(x_0)$.

7.3 Intermediate Value Theorem

Suppose $E \in \mathbb{R}$ is connected and $f : E \rightarrow \mathbb{R}$ is continuous.

Suppose $f(x) = a$ and $f(y) = b$ for some $x, y \in E$ and $a < b$.

Then $\forall a < c < b \exists$ some $z \in E$ such that $f(z) = c$.

7.4 Mean Value Theorem

Suppose $\phi : [a, b] \rightarrow \mathbb{R}$ is

- continuous at each point of **closed** interval $[a, b]$
- differentiable at each point of **open** interval (a, b)

Then there exists a point $c \in (a, b)$ such that $\phi(b) - \phi(a) = \phi'(c)(b - a)$.

8 Differential Calculus

8.1 Definitions

differentiable f is differentiable at a if there is an n by m matrix B such that

$$\frac{f(a+h) - f(a) - B \cdot h}{|h|} \rightarrow 0 \quad \text{as } h \rightarrow 0$$

The matrix B is unique.

Directional derivative Given $u \in \mathbb{R}^m$ which $u \neq 0$ define

$$f'(a; u) = \lim_{t \rightarrow 0} \frac{f(a + tu) - f(a)}{t}$$

Provide the limit exists.

Partial derivative Define the j^{th} partial derivative of f at a to be the directional derivative of f at a with respect to the vector e_j , provide derivative exists.

$$D_j f(a) = \lim_{t \rightarrow 0} \frac{f(a + te_j) - f(a)}{t}$$

8.2 Notations

$Df(a)$: derivative of f at a

$f'(a; u)$: directional derivative of f at a respect to vector u .

$D_j f(a)$: j^{th} partial derivative of f at a .

f_i : i^{th} component function of f .

∇g : gradient of g , $\nabla g = \mathbf{grad} g = \sum_i (D_i g) e_i$

Jf : Jacobian matrix, $J_{ij} = D_j f_i(a)$

8.3 Differentiability Theorems

Theorems Munkers.5.1

If f is differentiable at a then all directional derivative of f at a exist and $f'(a; u) = Df(a) \cdot u$

Theorems Munkers.5.2

If f is differentiable at a then f is continuous at a .

Theorems Munkers.5.3

If f is differentiable at a then $Df(a) = [D_1 f(a) \ D_2 f(a) \ \cdots \ D_m f(a)]$.

Theorems Munkers.5.4

a. $[f \text{ is differentiable at } a] \Leftrightarrow \forall i [f_i \text{ is differentiable at } a]$.

b. If f is differentiable at a , then its derivative is the n by m matrix whose i^{th} row is the derivative of the function f_i . $(Df(a))_i = Df_i(a)$

8.4 Continuously Differentiable Functions

A function is C^1 if all of its partial derivatives are continuous. A function is C^r if all of its partial derivatives are C^{r-1} .

Munkers 6.1

If $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) , then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.

Munkers 7.3

Let A be open in \mathbb{R}^m ; let $f : A \rightarrow \mathbb{R}$ be differentiable on A . If A contains the line segment with end points a and $a + h$, then there is a point $c = a + th$ with $0 < t < 1$ of this line segment such that $f(a + h) - f(a) = (Df(c))h$.

Munkers 6.2

Let A be open in \mathbb{R}^m . Suppose that the partial derivative $D_i f_i(x)$ of the component function of f exists at each point x of A and are continuous on A . Then f is differentiable at each point of A .

Munkers 6.3

Let A be open in \mathbb{R}^m , let $f : A \rightarrow \mathbb{R}$ be a function of class C^2 . Then for each $a \in A$: $D_k D_j f(a) = D_j D_k f(a)$.

8.5 Chain Rule

Let $A \subset \mathbb{R}^m$. Let $B \subset \mathbb{R}^n$. Let $f : A \rightarrow \mathbb{R}^n$ and $g : B \rightarrow \mathbb{R}^p$, with $f(A) \subset B$. Suppose $f(a) = b$. If f is differentiable at a and g is differentiable at b , then the composite function $g \circ f$ is differentiable at a . Furthermore,

$$D(g \circ f)(a) = Dg(b) \cdot Df(a)$$

8.6 Inverse Function Theorem

Let A be open in \mathbb{R}^n . Let $f : A \rightarrow \mathbb{R}^n$ be of class C^r .

IF $Df(x)$ is invertible at $a \in A$.

THEN There exists a neighborhood of a such that

- $f|_U$ is injective AND $f(U) = V$ open in \mathbb{R}^n
- the inverse function is of class C^r
- $f^{-1}(y) = [f'(f^{-1}(y))]^{-1}$

8.7 Implicit Function Theorem

Suppose $f : A \rightarrow \mathbb{R}^n$ be of class C^r .

Write f in the form $f(x, y)$, for $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$.

IF $(a, b) \in A$ AND $f(a, b) = 0$ AND $\det \frac{\partial f}{\partial y}(a, b) \neq 0$

THEN There exists $B \in \mathbb{R}^k, a \in B$ and a unique $g : B \rightarrow \mathbb{R}^n$ such that $g(a) = b$ AND $\forall x \in B. f(x, g(x)) = 0$ AND g is C^r

Munkers 9.1

Let A be open in \mathbb{R}^{k+n} , B be open in \mathbb{R}^k .

Let $f : A \rightarrow \mathbb{R}^n, g : B \rightarrow \mathbb{R}^n$ be differentiable.

Write f in the form $f(x, y)$, for $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$.

IF $f(x, g(x)) = 0$ AND $\frac{\partial f}{\partial y}$ is invertible

THEN $Dg(x) = - \left[\frac{\partial f}{\partial y}(x, g(x)) \right]^{-1} \cdot \frac{\partial f}{\partial x}(x, g(x))$

8.8 Taylor's theorem

Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is of class C^k on an open convex set S . If $a \in S$ and $a + h \in S$, then

$$f(a + h) = \sum_{|\alpha| \leq k} \frac{\partial^\alpha f(a)}{\alpha!} h^\alpha + R_{a,k}(h),$$

If f is of class C^{k+1} on S , for some $c \in (0, 1)$ we have

$$R_{a,k}(h) = \sum_{|\alpha|=k+1} \frac{\partial^\alpha f(a + ch)}{\alpha!} h^\alpha$$

9 Integral Calculus

Note, Riemann Integral was taught in this class.

9.1 Definitions

rectangle (in \mathbb{R}^n) $Q = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$

component interval of Q $[a_i, b_i]$

volume of Q $v(Q) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$

partition *TODO*

subinterval(determined by P) *OMIT*

subrectangle(determined by P) *OMIT*

mech of P *OMIT*

refinement *OMIT*

common refinement *OMIT*

?- $m_R(f) = \inf\{f(x) | x \in R\}$

?- $M_R(f) = \sup\{f(x) | x \in R\}$

lower sum $L(f, P) = \sum_R m_R(f) \cdot v(R)$

upper sum $U(f, P) = \sum_R M_R(f) \cdot v(R)$

lower integral $\int_Q f = \sup_P \{L(f, P)\}$

upper integral $\overline{\int_Q f} = \inf_P \{U(f, P)\}$

oscillation *TODO*

rectifiable set A bounded set $S \in \mathbb{R}^n$ is rectifiable if the constant function 1 is integrable over S . S is rectifiable iff S is bounded and $\text{Bd}S$ has measure zero

volume of a rectifiable set $v(S) = \int_S 1$

9.2 Riemann condition

Given: Q a rectangle, $f : Q \rightarrow \mathbb{R}$ a bounded function.

$$\boxed{\int_Q f = \overline{\int_Q f} \text{ iff } \forall \epsilon_{>0} \exists P [U(f, P) - L(f, P) \leq \epsilon]}$$

P is a partition of Q

Corollary/Theorem: every constant function is integrable.

9.3 Riemann-Lebesgue theorem

A function on a compact interval $[a, b]$ is Riemann integrable if and only if it is bounded and continuous almost everywhere (the set of its points of discontinuity has measure zero, in the sense of Lebesgue measure). [\[wiki\]](#) [\[11.2\]](#)

9.4 Fundamental theorem of Calculus

- If f is continuous on $[a, b]$, and if $F(x) = \int_a^x f$ for $x \in [a, b]$, then $F'(x)$ exists and equals $f(x)$.
- If f is continuous on $[a, b]$, and if g is a function such that $g'(x) = f(x)$ for $x \in [a, b]$ then $\int_a^b f = g(b) - g(a)$

9.5 Fubini's theorem

Let $Q = A \times B$, where A is a rectangle in \mathbb{R}^k and B is a rectangle in \mathbb{R}^n . If f is a bounded function and integrable over Q , then $\int_{y \in B} f(x, y)$ and $\overline{\int_{y \in B} f(x, y)}$ are integrable over A and

$$\int_Q f = \int_{x \in A} \int_{y \in B} f(x, y) = \int_{x \in A} \overline{\int_{y \in B} f(x, y)}$$

9.6 Properties of integral

TODO

9.7 Properties of rectifiable set

TODO

10 Change of Variables

Diffeomorphism [\[wiki\]](#)

$f : A \rightarrow B$ is a diffeomorphism if f is a bijection AND f and f^{-1} are of class C^r .

Moreover f is called C^r -diffeomorphism

Change of Variables Theorem [\[17.2\]](#)

LET opensets $U, V \subseteq \mathbb{R}$

$g : U \rightarrow V$ a diffeomorphism

$f : V \rightarrow \mathbb{R}$ be a continuous function

THEN f is intergrable over B

IFF $(f \circ g)|\det Dg|$ is integrable over A

NOTE $\int_B f = \int_A (f \circ g)|\det Dg|$

Substitution rule [\[17.1\]](#)

LET $I = [a, b], J = [c, d] \subseteq \mathbb{R}$

$g : I \rightarrow J$ of class C^1

$f : J \rightarrow \mathbb{R}$ is continuous

IF $\forall x \in (a, b) [g'(x) \neq 0]$

THEN $\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g)g'$

equivalently $\int_J f = \int_I (f \circ g)|g'|$

11 Manifolds

12 Lecture Notes

12.1 2015-01-05 Monday

12.1.1 Review

THM : rectifiable

LET : $S \subset \mathbb{R}^n$ be bounded

THEN : S is rectifiable

IFF ∂S has measure zero, equivalently χ_S is integrable

IDEA : rectifiable is a property of the boundary of a set. If the boundary has measure zero then the set is rectifiable.

DEFN : volume

LET : $S \subset \mathbb{R}^n$ be bounded and *rectifiable*

THEN : define the volume of S as:

$$V(S) := \int_S 1 := \int_Q \chi_S$$

THM : partition of unity

LET : $A \subset \mathbb{R}^n$, $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of A

THEN : \exists a collection of C^∞ functions $\{\psi_\beta\}_{\beta \in \mathcal{B}}$ s.t

- i $\forall x \in A \quad 0 < \psi_\beta \leq 1 \quad \forall \beta \in \mathcal{B}$
- ii $\forall x \in A \quad \exists$ open neighbourhood V of x such that:
all but finitely many ψ_β vanish on V (*locally finite*)
- iii $\forall x \in A \quad \sum_{\beta \in \mathcal{B}} \psi_\beta(x) \equiv 1$
- iv $\forall x \in A \quad \exists \alpha$ such that
 $\text{supp}(\psi_\beta) \subset U_\alpha$, ie $\{x | \psi_\beta(x) \neq 0\} \subset U_\alpha$

A collection of functions satisfying i, ii, iii is called a *partition of unity*

It is *subordinate* to the open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ if it satisfies condition iv

12.1.2 Open Sets

DEFN : extended integral

LET : $U^{open} \subset \mathbb{R}^n$, $f : U \rightarrow \mathbb{R}$ is continuous

IF $f \geq 0$

THEN : define then *extended integral* of f over U as:

$$\int_U f := \sup \left\{ \int_D f \mid D \subset U \text{ where } D \text{ is compact, rectifiable} \right\}$$

IF f is arbitrary and $\int_U f_+$, $\int_U f_-$ exist

THEN : define then *extended integral* of f over U as:

$$\int_U f := \int_U f_+ - \int_U f_- \text{ where } f_+(x) = \max\{f(x), 0\} \text{ and } f_-(x) = \min\{-f(x), 0\}$$

THM : mnk 15.2

LET : $U^{open} \subset \mathbb{R}^n$, $f : U \rightarrow \mathbb{R}$ is continuous

Choose an *exhaustion* of U by compact K_i such that

$$K_1 \subset K_2^\circ \subset K_2 \subset K_3^\circ \dots \text{ and } U = \cup_{i=1}^\infty K_i$$

f has an *extended integral*

IFF the sequence $\int_{K_i} |f_i|$ is bounded

THEN : $\int_U f = \lim_{i \rightarrow \infty} \int_{K_i} f$

RMK : If $U^{open} \subset \mathbb{R}^n$, then $\int_U f$ refers to the *extended integral*

THM : mnk 15.4

LET : U^{open} is bounded, $f : U \rightarrow \mathbb{R}^n$ is bounded and continuous

THEN :

- i the extended integral exists
- ii if the ordinary integral exists, then they are equal

THM : mnk 16.5

LET : $U^{open} \subset \mathbb{R}^n$, $f : U \rightarrow \mathbb{R}$ is continuous, $\{\psi_i\}$ be a partition of unity with compact support

THEN : $\int_U f$ exists

IFF $\sum_{i=1}^\infty \int_U \psi_i |f|$ converges to a finite number

In this case $\int_U f = \sum_{i=1}^\infty (\int_U \psi_i f)$

12.2 2015-01-07 Wednesday

Note: this is not finished --.

- [Diffeomorphism](#)
- [Change of Variables](#)
- [Substitution Rule](#)

12.3 2015-01-12 Monday

12.3.1 Torus

Problem: Find the volume of the solid torus given by

$$\{(x, y, z) | (\sqrt{x^2 + y^2} - b)^2 + z^2 \leq a^2\}$$

When $y = 0$, $(x - b)^2 + z^2 \leq a^2$ is the circle centered at $(b, 0, 0)$ in the (x, z) plane with radius a .

For angle $0 < \varphi < 2\pi$, parametrize the circle by

- $x = b + a \cos \varphi$

- $z = a \sin \varphi$

and for $0 < \rho < a$ and parametrize the disc by

- $x = b + \rho \cos \varphi$
- $z = \rho \sin \varphi$

This defines a diffeomorphism from $(0, a) \times (0, 2\pi) \rightarrow$ disc with slit removed. Now rotate around z axis. For $0 < \theta < 2\pi$:

- $x = (b + \rho \cos \varphi) \cos \theta$
- $y = (b + \rho \cos \varphi) \sin \theta$
- $z = \rho \sin \varphi$

We want $G(\rho, \varphi, \theta) := (x, y, z)$, so $G(\rho, \varphi, \theta) \rightarrow \{(\sqrt{x^2 + y^2} - b)^2 + z^2 < a^2\} \setminus (T \cap \{x > 0, y = 0\}) \cup (T \cap \{z = 0 \cap x^2 + y^2 \geq b^2\})$ is a diffeomorphism.

Notice that the set we have removed has measure 0 (First part $\subset \{y = 0\}$ which has measure 0 and second part $\subset \{z = 0\}$ which has measure 0).

Thus, $\text{Vol}(T) = \text{volume of the image of } G$ (they differ by a set of measure 0).

So $\chi_T \equiv \chi_{G(R)}$ except on a set of measure 0.

Apply change of variables!

First check if G is a diffeomorphism:

G is a bijection by construction.

$$DG = \begin{bmatrix} \cos \varphi \cos \theta & -\rho \sin \varphi \cos \theta & (b + \rho \cos \varphi) \cos \theta \\ \cos \varphi \sin \theta & -\rho \sin \varphi \sin \theta & (b + \rho \cos \varphi) \sin \theta \\ \sin \varphi & \rho \cos \varphi & 0 \end{bmatrix}$$

$$|DG| = - (b + \rho \cos \varphi) \sin \theta (\rho \cos^2 \varphi \sin \theta + \rho \sin^2 \varphi) \cos \theta$$

$$- (b + \rho \cos \varphi) \cos \theta (\rho \cos^2 \varphi \cos \theta + \rho \sin^2 \varphi) \cos \theta$$

$$= - (b + \rho \cos \varphi) \rho$$

$$\neq 0$$

Thus G is a diffeomorphism of open sets.

Then by change of variables,

$$\int_{G(R)} 1 = \int_R 1 |DG| = \int_{(0,a) \times (0,2\pi) \times (0,2\pi)} \rho (b + \rho \cos \theta)$$

Then by Fubini's theorem:

$$\int_{\rho=0}^a \int_0^{2\pi} \int_0^{2\pi} \rho (b + \rho \cos \varphi) = 4\pi^2 \int_0^a \rho b = 2\pi^2 a^2 b$$

That's the torus!