UofT-MAT257-2014 Note

hysw, etc(will add later)...

January 26, 2015

Contents

1 KW

Euclidean n-space $\mathbb{R}^n$, norm, inner product, usual basis, distance, closed rectangle, open rectangle, open, closed, interior, exterior, boundary, open cover, compact(etc...), Heine-Borel Theorem, function, composition, component function, projection function π^i, continuous function, continuous & compact, oscillation, Jacobian matrix, chain rule, partial derivative, continuously differentiable, inverse functions, implicit functions,

2 TODO

Munkers 11.3, 13.*, 14.3, 14.4, 15.*

3 Basic knowledge

Note, this section is for something that does not fit anywhere

3.1 Abbreviations

cts Continuous

msr Measure

3.2 Multi-index Notation

- $\alpha = (\alpha_1, \ldots, \alpha_n)$
- $|\alpha| = \alpha_1 + \cdots + \alpha_n$
- $\alpha! = \alpha_1! \cdots \alpha_n!$
- $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$
- $\partial^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$

4 Linear Algebra

4.1 Definitions

norm A function $p : V \to \mathbb{R}$ such that
For all $a \in F$ and all $u, v \in V$

- $p(v) \geq 0 \wedge [p(v) = 0 \iff v = 0]$ (separates points)
- $p(av) = |a|p(v)$ (absolute homogeneity)
- $p(u + v) \leq p(u) + p(v)$ (triangle inequality)

inner product A function $\langle x, y\rangle : V \times V \to \mathbb{R}$ such that
For all $x, y, z \in V$ and $c \in \mathbb{F}$.

- $\langle x, y\rangle = \langle y, x\rangle$
- $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
- $\langle cx, y\rangle = c\langle x, y\rangle = \langle x, cy\rangle$
- $\langle x, x\rangle > 0$ if $x \neq 0$

5 Topology

5.1 Definitions

metric A function $d : X \times X \to \mathbb{R}$ such that
For all $x, y, z \in X$

- $d(x, y) \geq 0$
- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$

ϵ-neighborhood $\mathbf{U}(x; \epsilon) = \{y | d(x, y) < \epsilon\}$

open set in metric space A set $U \subseteq X$ is said to be open in X if $\forall x \in U \exists \epsilon > 0 [\mathbf{U}(x; \epsilon) \subseteq U]$ note that finite intersections and arbitrary unions of open set are open set

closed set in metric space A set contains all its limit point.
note that closed set is complement of open set in topology

5.2 Partition of unity

TODO

6 Measure Theory

6.1 Measure zero

Let $A \subseteq \mathbb{R}^n$. We say A has measure zero in $\mathbb{R}^n$ if for every $\epsilon > 0$, there is a covering $Q_1, Q_2, \dots$ of A by countably many rectangles such that $\sum_{i=1}^{\infty} v(Q_i) < \epsilon$. If this inequality holds, we often say that the total volume of the rectangles $Q_1, Q_2, \dots$ is less than ϵ.

6.2 Theorems

{Munkers-11.1}

1. If $B \subseteq A$ and A has measure zero in $\mathbb{R}^n$, the so does B.
2. Let A be the union of the collection of sets $A_1, A_2, \dots$ If each A_i has measure zero, so does A.
3. A set A has measure zero in $\mathbb{R}^n$ if and only if for every $\epsilon > 0$, there is a countable covering if A by open rectangles $IntQ_1, IntQ_2, \dots$ such that $\sum v(Q_i) < \epsilon$
4. If Q is a rectangles in R^n then BdQ has measure zero in R^n.

7 General Calculus

7.1 Definitions

oscillation

Given $a \in Q$ define $A_\delta = \{f(x) | x \in Q \wedge |x-a| < \delta\}$. Let $M_\delta(f) = \sup A_\delta$, and let $m_\delta(f) = \inf A_\delta$, define oscillation at f by $\operatorname{osc}(f;a) = \inf_{\delta>0}[M_\delta(f) - m_\delta(f)]$. f is cts at a iff $\operatorname{osc}(f;a) = 0$

7.2 Extreme Value Theorem

Suppose $f : X \to \mathbb{R}$ is continuous and X is compact, then $\exists x_0 \in X$ such that $\forall x \in X . f(x) \leq f(x_0)$.

7.3 Intermediate Value Theorem

Suppose $E \in \mathbb{R}$ is connected and $f : E \to \mathbb{R}$ is continuous.
Suppose $f(x) = a$ and $f(y) = b$ for some $x, y \in E$ and $a < b$.
Then $\forall a < c < b \exists$ some $z \in E$ such that $f(z) = c$.

7.4 Mean Value Theorem

Suppose $\phi : [a,b] \to \mathbb{R}$ is

- continuous at each point of **closed** interval $[a,b]$
- differentiable at each point of **open** interval (a,b)

Then there exists a point $c \in (a,b)$ such that $\phi(b) - \phi(a) = \phi'(c)(b-a)$.

8 Differential Calculus

8.1 Definitions

differentiable f is differentiable at a if there is an n by m matrix B such that

$$\frac{f(a+h)-f(a)-B\cdot h}{|h|} \to 0 \quad \text{as} \quad h \to 0$$

The matrix B is unique.

Directional derivative Given $u \in \mathbb{R}^m$ which $u \neq 0$ define

$$f'(a;u) = \lim_{t\to 0}\frac{f(a+tu)-f(a)}{t}$$

Provide the limit exists.

Partial derivative Define the j^{th} partial derivative of f at a to be the directional derivative of f at a with respect to the vector e_j, provide derivative exists.

$$D_jf(a) = \lim_{t\to 0}\frac{f(a+te_j)-f(a)}{t}$$

8.2 Notations

$Df(a)$: derivative of f at a

$f'(a;u)$: directional derivative of f at a respect to vector u.

$D_jf(a)$: j^{th}partial derivative of f at a.

f_i : i^{th} component function of f.

∇g : gradient of g, $\nabla g = \mathbf{grad} g = \sum_i (D_i g)e_i$

Jf : Jacobian matrix, $J_{ij} = D_jf_i(a)$

8.3 Differentiability Theorems

Theorems Munkers.5.1
If f is differentiable at a then all directional derivative of f at a exist and $f'(a;u) = Df(a)\cdot u$

Theorems Munkers.5.2
If f is differentiable at a then f is continuous at a.

Theorems Munkers.5.3
If f is differentiable at a then $Df(a) = [D_1f(a) \quad D_2f(a) \quad \cdots \quad D_mf(a)]$.

Theorems Munkers.5.4

a. $[f$ is differentiable at $a]$ $\Leftrightarrow$ $\forall i[f_i$ is differentiable at $a]$.

b. If f is differentiable at a, then its derivative is the n by m matrix whose i^{th} row is the derivative of the function f_i. $(Df(a))_i = Df_i(a)$

8.4 Continuously Differentiable Functions

A function is C^1 if all of its partial derivatives are continous. A function is C^r if all of its partial derivatives are C^{r-1}.

Munkers 6.1
If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ such that $f(b)-f(a) = f'(c)(b-a)$.

Munkers 7.3
Let A be open in $\mathbb{R}^m$; let $f : A \to \mathbb{R}$ be differentiable on A. If A contains the line segment with end points a and $a+h$, then there is a point $c = a+th$ with $0<t<1$ of this line segment such that $f(a+h) - f(a) = (Df(c))h$.

Munkers 6.2
Let A be open in $\mathbb{R}^m$. Suppose that the partial derivative $D_if_i(x)$ of the component function of f exists at each point x of A and are continuous on A. Then f is differentiable at each point of A.

Munkers 6.3
Let A be open in $\mathbb{R}^m$, let $f : A \to \mathbb{R}$ be a function of class $\mathbf{C}^2$. Then for each $a \in A$: $D_kD_jf(a) = D_jD_kf(a)$.

8.5 Chain Rule

Let $A \subset \mathbb{R}^m$. Let $B \subset \mathbb{R}^n$. Let $f : A \to \mathbb{R}^n$ and $g : B \to \mathbb{R}^p$, with $f(A) \subset B$. Suppose $f(a) = b$. If f is diferentiable at a and g is differentiable at b, then the composite function $g \circ f$ is differentiable at a. Furthermore,

$$D(g\circ f)(a) = Dg(b)\cdot Df(a)$$

8.6 Inverse Function Theorem

Let A be open in $\mathbb{R}^n$. Let $f : A \to \mathbb{R}^n$ be of class C^r.

IF $Df(x)$ is invertible at $a \in A$.

THEN There exists a neighborhood of a such that

- $f|_U$ is injective AND $f(U) = V$ open in $\mathbb{R}^n$
- the inverse function is of class C^r
- $f^{-1}(y) = [f'(f^{-1}(y))]^{-1}$

8.7 Implicit Function Theorem

Suppose $f : A \to \mathbb{R}^n$ be of class C^r.

Write f in the form $f(x, y)$, for $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$.

IF $(a, b) \in A$ AND $f(a, b) = 0$ AND det $\frac{\partial f}{\partial y}(a, b) \neq 0$

THEN There exists $B \in \mathbb{R}^k, a \in B$ and a unique $g : B \to \mathbb{R}^n$ such that $g(a) = b$ AND $\forall x \in B. f(x, g(x)) = 0$ AND g is C^r

Munkers 9.1

Let A be open in $\mathbb{R}^{k+n}$, B be open in $\mathbb{R}^k$.

Let $f : A \to \mathbb{R}^n$, $g : B \to \mathbb{R}^n$ be differentiable.

Write f in the form $f(x, y)$, for $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$.

IF $f(x, g(x)) = 0$ AND $\frac{\partial f}{\partial y}$ is invertible

THEN $Dg(x) = -\left[\frac{\partial f}{\partial y}(x, g(x))\right]^{-1} \cdot \frac{\partial f}{\partial x}(x, g(x))$

8.8 Taylor's theorem

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is of class C^k on an open convex set S. If $a \in S$ and $a + h \in S$, then

$$f(a + h) = \sum_{|\alpha| \leq k} \frac{\partial^\alpha f(a)}{\alpha!} h^\alpha + R_{a,k}(h),$$

If f is of class C^{k+1} on S, for some $c \in (0, 1)$ we have

$$R_{a,k}(h) = \sum_{|\alpha| = k+1} \frac{\partial^\alpha f(a + ch)}{\alpha!} h^\alpha$$

9 Integral Calculus

Note, Riemann Integral was taught in this class.

Measure Zero

9.1 Definitions

rectangle (in $\mathbb{R}^n$) $Q = [a_1, b_1] \times [a_2, b_2] \times [a_n, b_n]$

component interval of Q $[a_i, b_i]$

volume of Q $v(Q) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$

partition (of $[a, b]$) a finite collection of points of $[a, b]$ includes the points a, b

partition (of Q) n-tuple $(P_1, \ldots, P_n)$

subinterval(determined by P) *OMIT*

subrectangle(determined by P) *OMIT*

mech of P *OMIT*

refinement *OMIT*

common refinement *OMIT*

lower sum $L(f, P) = \sum_R m_R(f) \cdot v(R)$ where $m_R(f) = \inf\{f(x)|x \in R\}$

upper sum $U(f, P) = \sum_R M_R(f) \cdot v(R)$ where $M_R(f) = \inf\{f(x)|x \in R\}$

lower integral $\underline{\int_Q} f = \sup_P\{L(f, P)\}$

upper integral $\overline{\int_Q} f = \inf_P\{L(f, P)\}$

oscillation *TODO*

rectifiable set A bounded set $S \in \mathbb{R}^n$ is rectifiable if the constant function 1 is integrable over S. S is rectifiable iff S is bounded and BdS has measure zero

volume of a rectifiable set $v(S) = \int_S 1$

simple region Let C be a compact rectifiable in R^{n-1}; Let $\phi, \psi : C \to R$ be continuous function such that $\phi(x) \leq \psi(x)$ for $x \in C$. The subset $\{(x, t)|x \in C \wedge \phi(x) \leq t \leq \psi(x)\}$ is called a simple region in R^n.

9.2 Ignored

- Lemma 10.1: Munker p83.
- Lemma 10.2: Munker p84.
- Corollary 10.5: Munker p84.
- Lemma 13.1: Munker p104.
- Lemma 13.2: Munker p105.
- Lemma 14.3: Munker p114. a simple region is compact and rectifiable.

9.3 Riemann condition

Given: Q a rectangle, $f : Q \to \mathbb{R}$ a bounded function. Then $\underline{\int_Q} f = \overline{\int_Q} f$ iff given $\epsilon > 0$ there exist a partion P of Q for which $\forall \epsilon_{>0} \exists P[U(f, P) - L(f, P) \leq \epsilon]$

Corollary/Theorem: every constant function is integrable.

9.4 Riemann-Lebesgue theorem

A function on a compact interval [a, b] is Riemann integrable if and only if it is bounded and continuous almost everywhere (the set of its points of discontinuity has measure zero, in the sense of Lebesgue measure). [wiki] [Munkers 11.2]

9.5 Munkers 11.3

Let Q be a rectangle in R^n; let $f : Q \to \mathbb{R}$; assume f is integrable over Q.

- If f vanishes except on a set of measure zero, then $\int_Q f = 0$.
- If f is non-negative and if $\int_Q f = 0$, then f vanishes except on a set of measure zero.

9.6 Fundamental theorem of Calculus

- If f is continuous on $[a, b]$, and if $F(X) = \int_a^x f$ for $x \in [a, b]$, then $F'(x)$ exists and equals $f(x)$.
- If f is continious on $[a, b]$, and if g is a function such that $g'(x) = f(x)$ for $x \in [a, b]$ then $\int_a^b f = g(b) - g(a)$

9.7 Fubini's theorem

Let $Q = A \times B$, where A is a rectangle in $\mathbb{R}^k$ and B is a rectangle in $\mathbb{R}^n$. If f is bounded function and integrable over Q, then $\underline{\int_{y\in B}} f(x,y)$ and $\overline{\int_{y\in B}} f(x,y)$ are integrable over A and

$$\int_Q f = \int_{x\in A} \underline{\int_{y\in B}} f(x,y) = \int_{x\in A} \overline{\int_{y\in B}} f(x,y)$$

9.7.1 For simple regions

TODO[Munkers 14.4]

$$\int_S f = \int_{x\in C} \int_{t=\phi(x)}^{t=\psi(x)} f(x,t)$$

9.8 Properties of integral

TODO[Munkers 13.3]

9.9 Properties of integral

*TODO*A set S is rectifiable iff S is bounded and has measure zero boundary.[Munkers 14.1]

9.10 Properties of rectifiable set

TODO[Munkers 14.2]

10 Change of Variables

`Diffeomorphism` [wiki]

$f : A \to B$ is a diffeomorphism if f is a bijection AND
f and f^{-1} are of class C^r.

Moreover f is called C^r-diffeomorphism

`Change of Variables Theorem` [17.2]

LET opensets $U, V \subseteq \mathbb{R}$
$g : U \to V$ a diffeomorphism
$f : V \to \mathbb{R}$ be a continuous function
THEN f is intergrable over B
IFF $(f \circ g)|\det Dg|$ is integrable over A
NOTE $\int_B f = \int_A (f \circ g)|\det Dg|$

`Substitution rule` [17.1]

LET $I = [a, b], J = [c, d] \subseteq \mathbb{R}$
$g : I \to J$ of class C^1
$f : J \to \mathbb{R}$ is continuous
IF $\forall x \in (a, b)\, [g'(x) \neq 0]$
THEN $\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g) g'$
equivalently $\int_J f = \int_I (f \circ g)|g'|$

11 Manifolds

12 Lecture Notes

12.1 2015-01-05 Monday

12.1.1 Review

THM : `rectifiable`
LET : S $\subset \mathbb{R}^n$ be bounded
THEN : S is rectifiable
IFF ∂S has measure zero, equivalently χ_s is integrable
IDEA : rectifiable is a property of the boundary of a set. If the boundary has measure zero then the set is rectifiable.

DEFN : `volume`
LET : S $\subset \mathbb{R}^n$ be bounded and *rectifiable*
THEN : define the volume of S as:
$V(S) := \int_{\mathrm{S}} 1 := \int_{\mathrm{Q}} \chi_x$

THM : `partition of unity`
LET : A $\subset \mathbb{R}^n$, $\{\mathrm{U}_\alpha\}_{\alpha\in\mathcal{A}}$ be an open cover of A
THEN : $\exists$ a collection of C^∞ functions $\{\psi_\beta\}_{\beta\in\mathcal{B}}$ s.t

i $\forall x \in$ A $\quad 0 < \psi_\beta \leq 1 \quad \forall\beta\in\mathcal{B}$
ii $\forall x \in$ A $\quad \exists$ open neighbourhood V of x such that: all but finitely many ψ_β vanish on V (*locally finite*)
iii $\forall x \in$ A $\quad \sum_{\beta\in\mathcal{B}} \psi_\beta(x) \equiv 1$
iv $\forall x \in$ A $\quad \exists\, \alpha$ such that supp$(\psi_\beta) \subset \mathrm{U}_\alpha$, ie $\{x|\psi_\beta(x) \neq 0\} \subset \mathrm{U}_\alpha$

A collection of functions satisfying i, ii, iii is called a *partition of unity*
It is *subordinate* to the open cover $\{\mathrm{U}_\alpha\}_{\alpha\in\mathcal{A}}$ if it satisfies condition iv

12.1.2 Open Sets

DEFN : `extended integral`
LET : $\mathrm{U}^{open} \subset \mathbb{R}^n$, $f : \mathrm{U} \to \mathbb{R}$ is continuous
IF $f \geq 0$
THEN : define then *extended integral* of f over U as:
$\int_{\mathrm{u}} f := \sup\{\int_{\mathrm{D}} f \mid \mathrm{D} \subset \mathrm{U}$ where D is compact, rectifiable$\}$
IF f is arbitrary and $\int_{\mathrm{u}} f_+$, $\int_{\mathrm{u}} f_-$ exist
THEN : define then *extended integral* of f over U as:
$\int_{\mathrm{u}} f := \int_{\mathrm{u}} f_+ - \int_{\mathrm{u}} f_-$ where
$f_+(x) = \max\{f(x), 0\}$ and $f_-(x) = \min\{-f(x), 0\}$

THM : `mnk 15.2`
LET : $\mathrm{U}^{open} \subset \mathbb{R}^n$, $f : \mathrm{U} \to \mathbb{R}$ is continuous
Choose an *exhaustion* of U by compact K_i such that $K_1 \subset K_2^\circ \subset K_2 \subset K_3^\circ \ldots$ and $\mathrm{U} = \cup_{i=1}^\infty \mathrm{K}_i$
f has an *extended integral*
IFF the sequence $\int_{\mathrm{k}_i} |f_i|$ is bounded
THEN : $\int_{\mathrm{u}} f = \lim_{i\to\infty} \int_{\mathrm{k}_i} f$

RMK : If $\mathrm{U}^{open} \subset \mathbb{R}^n$, then $\int_{\mathrm{u}} f$ refers to the *extended integral*

THM : `mnk 15.4`
LET : $\mathrm{U}^{\mathrm{open}}$ is bounded, $f : \mathrm{U} \to \mathbb{R}^n$ is bounded and continuous
THEN :

i the extended integral exists
ii if the ordinary integral exists, then they are equal

THM : `mnk 16.5`
LET : $\mathrm{U}^{open} \subset \mathbb{R}^n$, $f : \mathrm{U} \to \mathbb{R}$ is continuous, $\{\psi_i\}$ be a partition of unity with compact support
THEN : $\int_{\mathrm{u}} f$ exists
IFF $\sum_{i=1}^\infty \int_{\mathrm{u}} \psi_i |f|$ converges to a finite number
In this case $\int_{\mathrm{u}} f = \sum\limits_{i=1}^{\infty} (\int_{\mathrm{u}} \psi_i f)$

12.2 2015-01-07 Wednesday

Note: this is not finished -_-.

- Diffeomorphism
- Change of Variables
- Substitution Rule

12.3 2015-01-09 Friday

`Change of Variables Theorem` [17.2]

LET opensets $U, V \subseteq \mathbb{R}$
$g : U \to V$ a diffeomorphism
$f : V \to \mathbb{R}$ be a continuous function
THEN f is intergrable over B
IFF $(f \circ g)|\det Dg|$ is integrable over A
NOTE $\int_B f = \int_A (f \circ g)|\det Dg|$

IDEA : The theorem is proved in 5 steps.

i Let $g : U \to V$ and $h : V \to W$ be diffeomorphisms of open sets in $\mathbb{R}^n$. Show that if the theorem holds for g and h, then it holds for $g \circ h$. The idea is to show that $\int_W f = \int_V (f \circ h)|\det Dh| = \int_U (f \circ h \circ g)|(\det Dh) \circ g||\det Dg| = \int_U (f \circ h \circ g)|\det(h \circ g)|$

ii Suppose that for each $x \in A$, there is a neighborhood U of x contained in A such that the theorem holds for the diffeomorphism $g : U \to V$ where $V = g(U)$ and all continues functions $f : V \to \mathbb{R}$ whose supports are compact subsets of V. Then show that the theorem holds for g. This basically means: show that if the theorem holds locally for g and fuctions f having compact support, then it holds for all g and all f. The idea is to find a partition of unity $\{\phi_i\}$ of B with compact support and show that the collection $\{\phi_i \circ g\}$ is a partition of unity on A having compact support. Then use the fact that $\int_B f = \sum_{i=1}^{\infty} \int_B (\phi_i)(f)$.

iii Show the the theorem holds for $n = 1$. This can be done by showing that the theorem holds for the interiors of all closed intervals contained in A then applying step 2.

iv We only need to prove the theorem for primitive diffeomorphisms. This is because, when restricted to a small enough neighbourhood, all diffeomorphisms are a composition of primitive diffeomorphisms. Step 1 shows that the validity of the theorem is preserved by composition. Step 2 shows that if the theorem holds on all the small neighbourhoods in A then it holds for all of A.

v Show that if the theorem holds in dimension $n - 1$, it holds in dimension n. Due to step 4, we only need to prove this for primitive diffeomorphisms. Assume without loss of generality that the primitive diffeomorphism preserves the last component. We can use Fuibini's theorem to reduce this to the $n - 1$ dimensional case.

12.4 2015-01-12 Monday

12.4.1 Torus

Problem: Find the volume of the solid torus given by

$$\{(x, y, z)|(\sqrt{x^2 + y^2} - b)^2 + 2^2 \le a\}$$

When $y = 0$, $(x - b)^2 + z^2 \le$ is the circle centered at $(b, 0, 0)$ in the (x, z) plane with radius a.

For angle $0 < \varphi < 2\pi$, parametrize the circle by

- $x = b + a\text{cos}\varphi$
- $z = a\text{sin}\varphi$

and for $0 < \rho < a$ and parametrize the disc by

- $x = b + \rho\text{cos}\varphi$
- $z = \rho\text{sin}\varphi$

This defines a diffeomorphism from $(0, a) \times (0, 2\pi) \to$ disc with slit removed. Now rotate around z axis. For $0 < \theta < 2\pi$:

- $x = (b + \rho\text{cos}\varphi)\text{cos}\theta$
- $y = (b + \rho\text{cos}\varphi)\text{sin}\theta$
- $z = \rho\text{sin}\varphi$

We want $G(\rho, \varphi, \theta) := (x, y, z)$, so $G(\rho, \varphi, \theta) \to \{(\sqrt{x^2 + y^2} - b)^2 + 2^2 < a\}\backslash(T \cap \{x > 0, y = 0\}) \cup (T \cap \{z = 0 \cap x^2 + y^2 \ge b^2\})$ is a diffeomorphism.

Notice that the set we have removed has measure 0 (First part $\subset \{y = 0\}$ which has measure 0 and second part $\subset \{z = 0\}$ which has measure 0).

Thus, Vol(T) =volume of the image of G (they differ by a set of measure 0).

So $\chi_T \equiv \chi_{G(R)}$ except on a set of measure 0.

Apply change of variables!

First check if G is a diffeomorphism:

G is a bijection by construction.

$$DG = \begin{bmatrix} \text{cos}\varphi\text{cos}\theta & -\rho\text{sin}\varphi\text{cos}\theta & (b + \rho\text{cos}\varphi)\text{cos}\theta \\ \text{cos}\varphi\text{sin}\theta & -\rho\text{sin}\varphi\text{sin}\theta & (b + \rho\text{cos}\varphi)\text{sin}\theta \\ \text{sin}\varphi & \rho\text{cos}\varphi & 0 \end{bmatrix}$$

$$\begin{aligned} |DG| =& -(b + \rho\text{cos}\varphi)\text{sin}\theta(\rho\text{cos}^2\varphi\text{sin}\theta + \rho\text{sin}^2\varphi)\text{cos}\theta) \\ & -(b + \rho\text{cos}\varphi)\text{cos}\theta(\rho\text{cos}^2\varphi\text{cos}\theta + \rho\text{sin}^2\varphi)\text{cos}\theta) \\ =& -(b + \rho\text{cos}\varphi)\rho \\ \ne& 0 \end{aligned}$$

Thus G is a diffeomorphism of open sets.

Then by change of variables,

$$\int_{G(R)} 1 = \int_R 1|DG| = \int_{(0,a)\times(0,2\pi)\times(0,2\pi)} \rho(b + \rho\text{cos}\theta)$$

Then by Fubini's theorem:

$$\int_{\rho=0}^{a} \int_0^{2\pi} \int_0^{2\pi} \rho(b + \rho\text{cos}\varphi) = 4\pi^2 \int_0^a \rho b = 2\pi^2 a^2 b$$

That's the torus!

12.5 2015-01-14 Wednesday

Second test review

12.5.1 Existence of the Integral

DEFN : `integral`
LET : Q be a rectangle, $f : \mathrm{Q} \to \mathbb{R}$ be a bounded function, define:
$\overline{\int}_{\mathrm{Q}} f := \inf\{U(f, \mathrm{P})\}$ as the upper integral
$\underline{\int}_{\mathrm{Q}} f := \sup\{L(f, \mathrm{P})\}$ as the lower integral
IF then the upper and lower sums agree
THEN : f is *integrable* over Q

THM : `Riemann Condition`
LET : Q be a rectangle, $f : \mathrm{Q} \to \mathbb{R}$ be a bbd fn
THEN : upper and lower integral agree
IFF given $\epsilon > 0$ there is a partition P such that:
$U(f, P) - L(f, P) \leq \epsilon$
IDEA : necessary condition for the existence

THM : `mnk 11.2`
LET : $\mathrm{Q} \subset \mathbb{R}^n$, $f : \mathrm{Q} \to \mathbb{R}$ be a bounded function.
Define D to be the set of points for which f fails to be continuous
THEN : $\int_Q f$ exists
IFF D has measure zero (f is *almost continuous everywhere*)

THM : `mnk 11.3`
LET : $\mathrm{Q} \subset \mathbb{R}^n$, $f : \mathrm{Q} \to \mathbb{R}$. Assume f is *integrable*
IF f vanishes except on a set of measure zero
THEN : $\int_{\mathrm{Q}} f = 0$
IF f is non-negative and $\int_{\mathrm{Q}} f = 0$
THEN : $f = 0$ almost everywhere

12.5.2 Evaluation of the Integral

THM : `Fundamental Theory of Calculus`
LET : $f : [\mathrm{a}, \mathrm{b}] \to \mathbb{R}$ be continuous
IF $F(x) = \int_a^x f(x)$ for $x \in [a, b]$
THEN : $D \int_a^x f = f(x)$
IF g is a function such that $g'(x) = f(x)\ \forall x$
THEN : $\int_a^x Dg = g(x) - g(a)$

THM : `Fubini's Theorem`
LET : $\mathrm{Q} = \mathrm{A} \times \mathrm{B}$, where $\mathrm{A} \subset \mathbb{R}^k$ and $\mathrm{B} \subset \mathbb{R}^n$.
$f : \mathrm{Q} \to \mathbb{R}$ be bdd, write $f(x, y)$ for $x \in \mathrm{A}$ and $y \in \mathrm{B}$.
For each $x \in \mathrm{A}$ consider upper and lower integrals
$\underline{\int}_{y \in \mathrm{B}} f(x, y)$ and $\overline{\int}_{y \in \mathrm{B}} f(x, y)$
IF f is integrable over Q
THEN : these two functions are integrable over Q and
$\int_{\mathrm{Q}} f = \int_{x \in \mathrm{A}} \underline{\int}_{y \in \mathrm{B}} f(x, y) = \int_{x \in \mathrm{A}} \overline{\int}_{y \in \mathrm{B}} f(x, y)$

12.5.3 Integral Over a Bounded Set

THM : `mnk 13.5`
LET : $\mathrm{S} \subset \mathbb{R}^n$ be bounded, $f : \mathrm{S} \to \mathbb{R}$ be bounded and continuous function. Define E to be the set of all points $x_0 \in \partial S$ for which the condition $\lim_{x \to x_0} f(x) = 0$ fails
IF E has measure zero
THEN : f is integrable over S
Converse also holds

THM : `mnk 13.6`
LET : $\mathrm{S} \subset \mathbb{R}^n$ be bounded, $f : \mathrm{S} \to \mathbb{R}$ be bounded and continuous function, and $\mathrm{A} = \mathrm{S}^{\circ}$.
IF f is integrable over S
THEN : f is integrable over A and
$\int_{\mathrm{S}} f = \int_{\mathrm{A}} f$

12.5.4 Rectifiable Sets

THM : `mnk 14.1`
LET : $\mathrm{S} \subset \mathbb{R}^n$ **THEN** : S is *rectifiable*
IFF S bounded and ∂S has measure zero

THM : `mnk 14.2`
Properties of rectifiable sets

- i (Positivity). If S is rectifiable, $v(\mathrm{S}) \geq 0$
- ii (Monotonicity). If S_1 and S_2 are rectifiable with $\mathrm{S}_1 \subset \mathrm{S}_2$ then $v(\mathrm{S}_1) \leq v(\mathrm{S}_2)$
- iii (Additivity). If S_1 and S_2 are rectifiable so are, $\mathrm{S}_1 \cup \mathrm{S}_2$ and $\mathrm{S}_1 \cap \mathrm{S}_2$
- iv Suppose S is rectifiable. Then $v(\mathrm{S}) = 0$ iff S has measure zero
- v If S is rectifiable, so is S° and $v(\mathrm{S}) = v(\mathrm{S}^{\circ})$
- vi If S is rectifiable, and $f : \mathrm{S} \to \mathbb{R}$ is bounded continuous, then f is integrable over S.

DEFN : `simple region`
LET : C be a compact and rectifiable set in $\mathbb{R}^{n-1}$,
$\phi, \psi : \mathrm{C} \to \mathbb{R}$ be continuous functions such that

$\phi(x) \leq \psi(x)\ \forall x \in \mathrm{C}$.
THEN : $\mathrm{S} \subset \mathbb{R}^n$ defined as
$\mathrm{S} := \{(x,t)|x \in \mathrm{C} \text{ and } \phi(x) \leq t \leq \psi x\}$
is a simple region.

THM : `Fubini's Theorem for Simple Regions`
LET : $\mathrm{S} = \{(x,t)|x \in \mathrm{C} \text{ and } \phi(x) \leq t \leq \psi(s)\}$ be a simple region in $\mathbb{R}^n$ and let $f : \mathrm{S} \to \mathbb{R}$ be a continuous function
THEN : f is integrable over S and

$$\int_{\mathrm{S}} f = \int_{x \in \mathrm{c}} \int_{t=\phi(s)}^{t=\psi(x)} f(x,t)$$

12.5.5 Extended Integrals

Three definitions of the extended integral:

i Extended Integral 1
ii Extended Integral 2
iii Extended Integral 3

THM : `mnk 15.4`
LET : $\mathrm{U}^{\mathrm{open}}$ is bounded, $f : \mathrm{U} \to \mathbb{R}^n$ is bounded and continuous
THEN :

i the extended integral exists
ii if the ordinary integral exists, then they are equal

12.5.6 Change of Variables

DEFN : `diffeomorphism`
$f : A \to B$ is a diffeomorphism if f is a bijection AND f and f^{-1} are of class C^r.

THM : `Change of Variables`
LET : opensets $U, V \subseteq \mathbb{R}$
$g : U \to V$ a diffeomorphism
$f : V \to \mathbb{R}$ be a continuous function
THEN : f is intergrable over B
IFF $(f \circ g)|\det Dg|$ is integrable over A
RMK : $\int_B f = \int_A (f \circ g)|\det Dg|$

THM : `Substitution Rule`
LET : $I = [a,b], J = [c,d] \subseteq \mathbb{R}$
$g : I \to J$ of class C^1
$f : J \to \mathbb{R}$ is continuous
IF $\forall x \in (a,b)\ [g'(x) \neq 0]$
THEN : $\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g)g'$
equivalently $\int_J f = \int_I (f \circ g)|g'|$

12.6 2015-01-14 Monday

12.6.1 Mapping Surfaces

DEFN : `volume of a parallelepiped`
LET : $x_1, \cdots x_k$ be k-independent vectors in $\mathbb{R}^n$ defining a parallelepiped $P(x_1, \cdots, x_k)$.
Define $V(x_1, \cdots, x_k) = (\det(X^T X))^{1/2}$ where

$$X = \left(\begin{array}{ccc} x_1 & \cdots & x_k \end{array} \right)$$

THEN : V is the volume of of P

DEFN : `parameterized manifold`
LET : $A^{open} \subset \mathbb{R}^k$ and $\alpha : \mathrm{A} \to \mathrm{Y} \subset \mathbb{R}^n$ be of class C^r with $r \geq 1$, $n \geq k$, α is injective, and $\mathrm{Y} = \alpha(\mathrm{A})$. Note, α is surjective by construction
THEN : Y together with α, denote the *parameterized manifold* Y_α

12.7 2015-01-23 Friday

12.7.1 Definition of Volume

Parametrize the manifold $Y = \alpha(A)$.

Define $v(Y) = \int_Y 1 = \int_A v(D_\alpha)$ where $v(D_\alpha) =$ the volume of the parallelopiped determined by columns of $D_\alpha = \sqrt{\det(D_\alpha)^T D_\alpha}$

We need to check if this is a reasonable definition: it must be independent of parametrization.

Suppose $h : \mathbb{R}^n \to \mathbb{R}^n$, h linear such that $(D_h)^T D_h) = I$ (in particular, h is an isometry, ie. it preserves distances, areas, ...).

Let $z = h(y) = h \circ \alpha(A)$.

Then $v(Z) = v(Y)$ since $D(h \circ \alpha) = D_h \cdot D_\alpha$.

Suppose $\alpha(x) = y = y + Ax$ for some matrix A (parametrized plane). Then it's true.

Eg.

$$y = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 3 \\ 1 & 1 \end{pmatrix} x$$

$$\alpha(0,0) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\alpha(s,0) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\alpha(0,t) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

Then the volume of the parallelopiped is $\sqrt{\det(D_\alpha)^T D_\alpha}$.

Triangulate the surface:

Eg. Finding the length of a curve $\gamma : [0,1] \to \mathbb{R}^2$. By our definition, $v(\Gamma) = L(\Gamma) = \int_0^1 \sqrt{\det(D_\gamma)^T D_\gamma} dt$

$$D_\gamma = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$$

$$\begin{aligned}(D_\gamma)^T D_\gamma &= \begin{pmatrix} \frac{dx}{dt} & \frac{dy}{dt} \end{pmatrix} \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} \\ &= (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 \\ &= \gamma' \cdot \gamma' \\ &= ||\gamma'||^2 \sqrt{\det(D_\gamma)^T D_\gamma} \qquad = ||\gamma'(t)||\end{aligned}$$

So our definition reduces to $L(\Gamma) = \int_0^1 ||\gamma'(t)|| dt$. Now to evaluate it.

By our definition of integrals, there exists a partition $t_0 = 0, ..., t_n = 1$ such that

$$\sum_{i=1}^{n} ((M_i ||\gamma'||) - m_i(||\gamma'||)) \cdot \Delta t_i < \epsilon$$

In this case, $L(\Gamma) - \sum_{i=1}^{n} ||\gamma'(t_i)|| \cdot \Delta t_i < \epsilon$.

On the other hand, consider the piecewise liner approximations to the image. Find the points $\gamma(t_i)$ and connect the dots.

The length of the piecewise linear approximation is $\sum_i \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$, where

$$\begin{aligned}\Delta x_i &= x(t_i) - x(t_{i-1}) \\ &= x'(c_i) \cdot \Delta t_i \text{ for some } c_i \in (t_i, t_{i-1}) \\ \Delta y_i &= y(t_i) - y(t_{i-1}) \\ &= y'(d_i) \cdot \Delta t_i \text{ for some } d_i \in (t_i, t_{i-1})\end{aligned}$$

Now in general, $c_i \neq d_i$. Then,

$$\begin{aligned}&\sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \\ =&\sum_{i=1}^{n} \sqrt{(x'(c_i))^2 (\Delta t_i)^2 + (y'(d_i))^2 (\Delta t_i)^2} \\ =&\sum_{i=1}^{n} \sqrt{(x'(c_i))^2 + (y'(d_i))^2} \cdot \Delta t_i\end{aligned}$$

Now use γ is continuous on $[0,1]$. Thus γ is uniformly continuous. Therefore there exists δ such that $|\Delta t_i| < \delta$ implies that $|c_i - t_i| < \delta$ and $|d_i - t_i| < \delta$ Thus, $|x'(c_i) - x'(t_i)| < \epsilon$ and $|y'(d_i) - y'(t_i)| < \epsilon$.

Thus,

$$|\sum_{i=1}^{n} \sqrt{(x'(c_i))^2 + (y'(d_i))^2} \cdot \Delta t_i - \sum_{i=1}^{n} ||\gamma'(t_i)|| \Delta t_i| < \epsilon$$

Thus,

$$|L(\gamma) - \text{length of piecewise approximations}| < \epsilon$$