UofT-MAT347-2015-Fall Note

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Monday 14th September, 2015
at 23:07

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2.1 Lecture 1

September 14, 2015

The website of the course: http://www.math.toronto.edu/~jkamnitz/courses/mat347/index.html

Lecture time: Monday, Wednesday 11-12 at BA1200 Friday 10-12 at UC52

Topics: group (symmetries).

2.1.1 Definitions

A binary operation on a set G is a map $\bullet$, $G \times G \to G$ (aka. $(a, b) \to a \cdot b$).

It's called **associative** if $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

It's called **commutative** if $a \cdot b = b \cdot a$.

It has an **identity** if $\exists e \in G$ st. $a \cdot e = a = e \cdot a$.

It has **inverses** if for all $a \in G$ there exists $a^{-1} \in G$ such that $a \cdot a^{-1} = e = a^{-1} \cdot a$.

A **group** is a set with a binary operation which is associative and has an identity and inverses.

A commutative group is called an **abelian group**.

A **permutation** of a set X is a bijection $\sigma : X \to X$.

2.1.2 Examples

Ex1 $\mathbb{Z}$ with addition, $\mathbb{R}$ with addition, and $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ with multiplication are all examples of abelian group.

Ex2 "The main example"

For any set X define $S_X = \{\text{permutation of the set } X\}$. The binary operation on S_X **composition**.

If $X = \{1, \ldots, n\}$, then define S_n to be $S_{\{1,\ldots,n\}}$.

Ex3 Let V be a vector space over any fields $(\mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{F}_2, \mathbb{F}_p)$.

$(V, +)$ is a group, S_V is also a group.

An more interesting example is

$$GL(V) := \{\text{all invertable linear operator } T : V \to V\}$$

note that $GL(V) \subseteq S_V$

Another interesting example is

$$GL_n(k) := \{\text{invertable nxn matrices with entries in } k\}$$

Ex4 If V catties a symmetric bilinear form $\langle , \rangle$, then we define $O(V, \langle , \rangle) = \{\text{invertable } T : V \to V \text{ st } \langle Tv, Tw\rangle = \langle v, w\rangle \text{ for all } v, w \in V\}$.

For example, $O(\mathbb{R}^n, \langle , \rangle) = O_n(\mathbb{R}) = \{A^{\text{invertable}}$ nxn matrices st $AA^{-1} = I$. When $n = 2$, $O_2(\mathbb{R})$ is rotation and reflection of $\mathbb{R}^2$.

Proposition 1. If G is a group and $a, b, c, d \in G$ and $ab = ac$ then $b = c$.

Proof. Start with $ab = ac$ multiply both side by a^-1, then

$$\begin{aligned} ab &= ac \\ a^{-1}(ab) &= a^{-1}(ac) \\ (a^{-1}a)b &= (a^{-1}a)c \\ b &= c \end{aligned}$$

□

Remark 1. Special case: every group has unique identity.