ECS 122A: Algorithm Design and Analysis
Week 5 Discussion

Ji Wang

Fall 2020

FAQ on logistics

- ▶ Solutions will be posted ASAP (2 days after due time)
- ▶ Grades will be available in two weeks (might be released within a week)
- ▶ Gradescope: Start at your time, submit in time, then you're good!

Outline

- ▶ Divide and Conquer: Key Idea
- ▶ Variant of Maximum Subarray Problem: Stock Investment
- ▶ Another Example: Binary Integer Multiplication

Divide and Conquer: Key Idea

1. **Divide** the problem into a number of subproblems that are smaller instances of the same problem.

2. **Conquer** by solving the subproblems recursively.

3. **Combine** the solutions to the subproblems to produce the solution to the original problem.

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$$T(n) = aT(n/b) + f(n)$$

So far, we demonstrated two problems that use divide and conquer paradigm:

▶ Merge Sort

▶ Maximum Subarray

We will discuss matrix-matrix multiply this Tuesday, Oct 27th.

Stock Investment

Problem Statement:

We're doing a simulation in which we look at n consecutive days of a given stock, at some point in the past. Let's number the days $i = 1, 2, ..., n$ and $p(i)$ is the price per share for the stock on that day. We want to know: When should we have bought and sold in order to have made as much money as possible?

For example, $n = 4$, $p(1) = 9$, $p(2) = 1$, $p(3) = 5, p(4) = 3$. Then we should answer "buy on day 2, sell on day 3".

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Rephrase the problem:

Input: array p of length n

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Rephrase the problem:

Input: array p of length n

Output: `argmax`$\{p(j) - p(i)\}$ where $i \leq j$

Stock Investment

Revisit maximum subarray problem:

1. Divide $A[low \cdots high]$ into two subarrays of as equal size as possible by finding the midpoint mid
2. Conquer:
   a. finding maximum subarrays of $A[low \cdots mid]$ and $A[mid + 1 \cdots high]$
   b. finding a max-subarray that crosses the midpoint
3. Combine: returning the max of the three

Stock Investment

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Can we apply the same strategy on this problem?

Stock Investment

How does the conquer part work?

1. The optimal solution to $A[low \cdots mid]$
2. The optimal solution to $A[mid + 1 \cdots high]$
3. $\texttt{argmax}\{p(j) - p(i)\}$ where $low \leq i \leq mid$ and $mid + 1 \leq j \leq high$

Stock Investment

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1. The optimal solution to $A[low \cdots mid]$
2. The optimal solution to $A[mid + 1 \cdots high]$
3. $\text{argmax}\{p(j) - p(i)\}$ where $low \leq i \leq mid$ and $mid + 1 \leq j \leq high$

Time complexity:

$$T(n) = 2T(\frac{n}{2}) + \Theta(n) + \Theta(1)$$
$$= \Theta(n \log n)$$

1. $2T(\frac{n}{2})$: the first two optimal solutions
2. $\Theta(n)$: the third solution is equivalent to find the min of $A[low \cdots mid]$ and max of $A[mid + 1 \cdots high]$
3. $\Theta(1)$: compare the results of three solutions

Binary Integer Multiplication

Recall how we multiply two integers of equal length $= n$

$$
\begin{array}{r}
1100 \\
\times\, 1101 \\
\hline
1100 \\
0000 \\
1100 \\
1100 \\
\hline
10011100
\end{array}
$$

$$
\begin{array}{r}
12 \\
\times\, 13 \\
\hline
36 \\
12 \\
\hline
156
\end{array}
$$

(a)   **(b)**

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Time complexity: $O(n^2)$

1. bit multiplication: n^2
2. bit addition: $O(n)$

Binary Integer Multiplication: Speedup

First, how do we represent a decimal integer digit-wise?

$$1024 = 10^3 \times 1 + 10^2 \times 0 + 10^1 \times 1 + 10^0 \times 4$$

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Now, how about two n-bit binary integers?

$$x = \boxed{x_L}\ \boxed{x_R} = 2^{\frac{n}{2}}x_L + x_R$$
$$y = \boxed{y_L}\ \boxed{y_R} = 2^{\frac{n}{2}}y_L + y_R$$

For instance, $10110110 = \boxed{1011}\ \boxed{0110} = 2^4 \times 1011 + 0110$

$$xy = (2^{\frac{n}{2}}x_L + x_R)(2^{\frac{n}{2}}y_L + y_R) \quad = 2^n x_L y_L + 2^{\frac{n}{2}}(x_L y_R + x_R y_L) + x_R y_R$$

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Time complexity: [1]

$$T(n) = 4T(\frac{n}{2}) + \Theta(n) + \Theta(1)$$
$$= \Theta(n^2) \qquad \rightarrow \textbf{\textcolor{red}{No improvement!}}$$

---

[1] $2^n x$ is done by left shift with a constant time cost

Binary Integer Multiplication

Previously, we wrote:

$$xy = (2^{\frac{n}{2}}x_L + x_R)(2^{\frac{n}{2}}y_L + y_R)$$
$$= 2^n \underline{x_L y_L} + 2^{\frac{n}{2}}(x_L y_R + x_R y_L) + \underline{x_R y_R}$$

If we rewrite

$$x_L y_R + x_R y_L = \underline{(x_L + x_R)(y_L + y_R)} - x_L y_L - x_R y_R,$$

then the entire xy will only involve three multiplications, namely the underlined parts.

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Time complexity:

$$T(n) = 3T(\frac{n}{2}) + \Theta(n) + \Theta(1)$$
$$= \Theta(n^{\log_2 3})$$

Binary Integer Multiplication: Pseudocode

$\text{Multiply}(x, y)$

1   $// \ x, y$ are positive integers of n-bit
2   **if** $n = 1$
3   **return** xy
4   **else**
5       $x_L, x_R = $ leftmost $\lceil n/2 \rceil$, rightmost $\lfloor n/2 \rfloor$ bits of x
6       $y_L, y_R = $ leftmost $\lceil n/2 \rceil$, rightmost $\lfloor n/2 \rfloor$ bits of y
7       $p_1 = \text{Multiply}(x_L, y_L)$
8       $p_2 = \text{Multiply}(x_R, y_R)$
9       $p_3 = \text{Multiply}(x_L + x_R, y_L + y_R)$
10   **return** $p_1 \times 2^n + (p_3 - p_1 - p_2) \times 2^{\frac{n}{2}} + p_2$