ECS 122A: Algorithm Design and Analysis

Week 2 Discussion

Ji Wang

Fall 2020

A bit about logistics

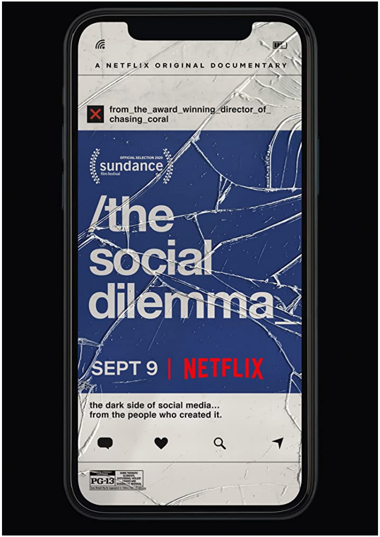
Discussion Schedule:

Ji Wang $\rightarrow$ Christopher Peterson $\rightarrow$ Terry Guan

- We will rotate to lead the discussion, with the exception of Veteran Day (Wed) and the entire Thanksgiving week.
- The first discussion of the week will be live and recorded. The rest will be in the form of live Q&A or extra office hours.
- Check Canvas homepage frequently for update on discussion notes and videos.

Office hours: Available every weekday
Ji: M 10:30am - 12:30pm, T 5 - 6pm, F 4 - 6pm

Contact us: For private questions, send direct message via Canvas.



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- Why the posts on Instagram not shown chronologically?
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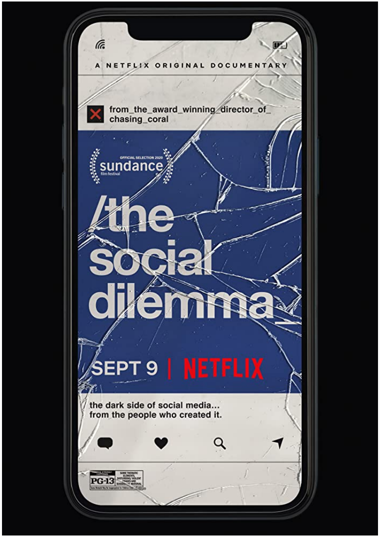

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It's the **algorithms** that work under the hood!

Outline

- Proof technique: Mathematical Induction
- Data structure: Heap
- Design description: Pseudocode
- Analysis: Asymptotic Notation
- Example: Insertion Sort

Mathematical Induction: Approach

When applicable: Prove that a property $P(n)$ holds for every natural number n.

How it works:

1. Show that a property $P(n)$ holds for the base case, usually when $n = 0$ or 1.
2. Assume that $P(n)$ is true for $n = k$ where k is greater than the base case.
3. Prove that $P(n)$ is true for $n = k + 1$. In this step we will use the assumption above.
4. Then, by the principle of induction, we can conclude that $P(n)$ is true for every natural number that is greater than or equal to the base case.

Mathematical Induction: Example

Problem Statement: Prove the following statement $P(n)$ is true

$$\sum_{i=1}^{n} i^3 = (\frac{n(n+1)}{2})^2$$

Build a skeleton:

- Base Case:
- Inductive Hypothesis:
- Inductive Step:
- Thus, by the principle of induction, we can conclude that the statement above is true for every natural number n.

Mathematical Induction: Example

$$1^3 + 2^3 + \cdots + n^3 = (\frac{n(n+1)}{2})^2$$

- Base Case:
  Verify $P(n)$ is true when $n = 1$. $1^3 = (\frac{1 \cdot (1+1)}{2})^2$.

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- Base Case:
  Verify $P(n)$ is true when $n = 1$. $1^3 = (\frac{1 \cdot (1+1)}{2})^2$.
- Inductive Hypothesis:
  Assume $P(n)$ is true when $n = k$. Then, we have
  $1^3 + 2^3 + \cdots + k^3 = (\frac{k(k+1)}{2})^2$.

Mathematical Induction: Example

$$1^3 + 2^3 + \cdots + n^3 = (\frac{n(n+1)}{2})^2$$

- Inductive Step: We need to show $P(n)$ is true when $n = k + 1$.

$$\begin{aligned}
1^3 + 2^3 + \cdots + (k+1)^3 &= 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 \\
&= (\frac{k(k+1)}{2})^2 + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= (\frac{(k+1)((k+1)+1)}{2})^2
\end{aligned}$$

Recap: Max-heap

Definition: Tree-based data structure which is an almost complete tree.

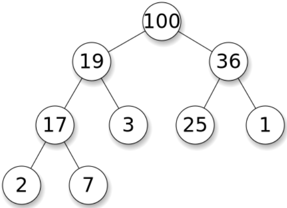
Property: For any given node, the key (value) of its parent node is greater than or equal to the that of itself.

Operations:

- find-max: find a maximum item of a max-heap. (peek)
- insert: add a new key to the heap. (push)
- extract-max: return the node of maximum value from a max-heap after removing it from the heap. (pop)
- increase-key or decrease-key: update a key within a max-heap.
- heapify: create a heap out of given array of elements.

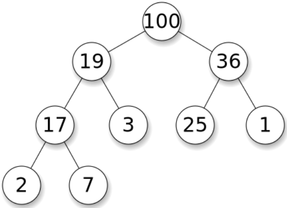
Applications: e.g. Prim's minimal-spanning-tree algorithm and Dijkstra's shortest-path algorithm

Max-heap: an example




```
push(10)
push(22)
pop()
update(7, 18)
```

Max-heap: an example

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The time complexity of each operation is left to you.

How to describe the design of an algorithm

Objective: Understandable (clear, precise) and concise if possible.

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   - Separate paragraphs if necessary, e.g. branches.
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   - Indentation indicates block structure, similar to Python.
   - Indices (often) start from 1, unlike most languages.
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3. in “real” code: whatever language you favor

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How to Analyze an algorithm: Asymptotic notation

Analyzing an algorithm means predicting the resources that the algorithm needs. The primary concern in this course is to measure computational time, a.k.a. running time.

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We need a definition to evaluate the order of growth to:

- characterize the efficiency of the algorithm when input sizes are large enough.
- compare the performance with other alternative algorithms.

Thus, we:

- study a way to describe the growth of functions in the limit.
- focus on what's important (leading factor) by ignoring lower-order terms and constant factors.

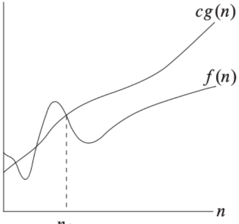
Asymptotic notation: Big O

Definition: $g(n)$ is an asymptotic upper bound for $f(n)$, denoted by

$$f(n) = O(g(n)),$$

if there exist constants c and n_0 such that:

$$0 \leq f(n) \leq cg(n) \quad \text{for } n \geq n_0$$



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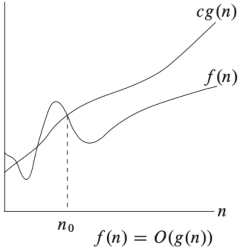

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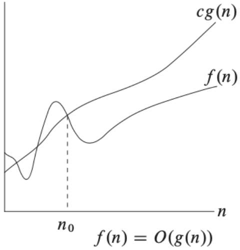

Example: Show $\lg n$ is $O(\ln n)$

$$\lg n = \log_2 n = \frac{\log_e n}{\log_e 2} = \frac{1}{\log_e 2} \ln n$$
$$\leq 2 \ln n \quad \text{for } n \geq 1$$

it is true for $c = 2$ and $n_0 = 1$. [2]

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Example: Insertion Sort

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Algorithm: We take an *incremental* approach. Split the list into sorted part and unsorted part. Insert the head of unsorted one into the appropriate position in the sorted one. Repeat until no unsorted part remaining.

Example: 8 3 2 7 4 1 6

$\boxed{8}$ 3 2 7 4 1 6
$\boxed{8\ 3}$ 2 7 4 1 6
$\boxed{3\ 8}$ 2 7 4 1 6
$\boxed{3\ 8\ 2}$ 7 4 1 6
$\boxed{2\ 3\ 8}$ 7 4 1 6

$\boxed{2\ 3\ 8\ 7}$ 4 1 6
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$\boxed{1\ 2\ 3\ 4\ 7\ 8}$ 6
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$\boxed{1\ 2\ 3\ 4\ 6\ 7\ 8}$

Insertion Sort: Pseudocode and Time Analysis

```
INSERTION-SORT(A)
for j = 2 to A.length
    key = A[j]
    // Insert key into the sorted list A[1 ··· j − 1].
    i = j − 1
    while i > 0 and A[i] > key
        A[i + 1] = A[i]
        i = i − 1
    A[i + 1] = key
```

- Worst case: reversed order, e.g. 5 4 3 2 1.

$$\sum_{j=2}^{n} \sum_{i=1}^{j-1} = O(n^2)$$

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- Best case: already sorted

$$\sum_{j=2}^{n} = O(n)$$