ECS 122A: Algorithm Design and Analysis

Week 1 Discussion

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Outline

- Mathematical Induction
- Insertion Sort
- Merge Sort

Mathematical Induction: Approach

When applicable: Prove that a property $P(n)$ holds for every natural number n.

How it works:

1. Show that a property $P(n)$ holds for the basis case, usually when $n = 0$ or 1.
2. Assume that $P(n)$ is true for $n = k$ where k is greater than the basis case.
3. Prove that $P(n)$ is true for $n = k + 1$. In this step we will use the assumption above.
4. Then, by the principle of induction, we can conclude that $P(n)$ is true for every natural number that is greater than or equal to the basis case.

Mathematical Induction: Example

Problem Statement: Prove the following statement $P(n)$ is true

$$1^3 + 2^3 + \cdots + n^3 = (\frac{n(n+1)}{2})^2$$

- Basis Step:
- Inductive Hypothesis:
- Inductive Step:
- Thus, by the principle of induction, we can conclude that the statement above is true for every natural number n.

Mathematical Induction: Example

$$1^3 + 2^3 + \cdots + n^3 = (\frac{n(n+1)}{2})^2$$

- Basis Step:
  Verify $P(n)$ is true when $n = 1$. $1^3 = (\frac{1 \cdot (1+1)}{2})^2$.

Mathematical Induction: Example

$$1^3 + 2^3 + \cdots + n^3 = (\frac{n(n+1)}{2})^2$$

- Basis Step:
  Verify $P(n)$ is true when $n = 1$. $1^3 = (\frac{1 \cdot (1+1)}{2})^2$.
- Inductive Hypothesis:
  Assume $P(n)$ is true when $n = k$. Then, we have
  $1^3 + 2^3 + \cdots + k^3 = (\frac{k(k+1)}{2})^2$.

Mathematical Induction: Example

$$1^3 + 2^3 + \cdots + n^3 = (\frac{n(n+1)}{2})^2$$

- Inductive Step: We need to show $P(n)$ is true when $n = k + 1$.

$$\begin{aligned}
1^3 + 2^3 + \cdots + (k+1)^3 &= 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 \\
&= (\frac{k(k+1)}{2})^2 + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= (\frac{(k+1)((k+1)+1)}{2})^2
\end{aligned}$$

Insertion Sort: Example

Problem Statement: Given a list of numbers, sort them in non-decreasing order.

Algorithm: Partition the list into sorted part and unsorted part. Insert the head of unsorted one into the sorted one. Repeat until no unsorted part remaining.

Example: 8 3 2 7 4 1 6

$\boxed{8}$ 3 2 7 4 1 6
$\boxed{8\ 3}$ 2 7 4 1 6
$\boxed{3\ 8}$ 2 7 4 1 6
$\boxed{3\ 8\ 2}$ 7 4 1 6
$\boxed{2\ 3\ 8}$ 7 4 1 6

$\boxed{2\ 3\ 8\ 7}$ 4 1 6
$\boxed{2\ 3\ 7\ 8}$ 4 1 6
$\boxed{2\ 3\ 7\ 8\ 4}$ 1 6
$\boxed{2\ 3\ 4\ 7\ 8}$ 1 6
$\boxed{2\ 3\ 4\ 7\ 8\ 1}$ 6

$\boxed{1\ 2\ 3\ 4\ 7\ 8}$ 6
$\boxed{1\ 2\ 3\ 4\ 7\ 8\ 6}$
$\boxed{1\ 2\ 3\ 4\ 6\ 7\ 8}$

Insertion Sort: Pseudocode

```
INSERTION-SORT(A)
for j = 2 to A.length
    key = A[j]
    // Insert key into the sorted list A[1 ··· j − 1].
    i = j − 1
    while i > 0 and A[i] > key
        A[i + 1] = A[i]
        i = i − 1
    A[i + 1] = key
```

See Section 2.1 on textbook for pseudocode convention.

Merge Sort: Divide-and-Conquer Approach

1. **Divide** the problem into a number of subproblems that are smaller instances of the **same** problem.
2. **Conquer** by solving the subproblems **recursively**.
3. **Combine** the solutions to the subproblems to produce the solution to the original problem.

```
MERGE-SORT(A, p, r)
if p < r                          // Input list: A[p ··· r]
    q = ⌊(p + r)/2⌋               // Divide
    MERGE-SORT(A, p, q)           // Conquer
    MERGE-SORT(A, q + 1, r)       // Conquer
    MERGE(A, p, q, r)             // Combine
```

Merge Sort: How to combine

$\boxed{L_1 \;\; L_2 \;\; L_3 \;\; L_4 \;\; \infty}$ $\boxed{R_1 \;\; R_2 \;\; R_3 \;\; R_4 \;\; \infty}$

$$\boxed{A_1 \;\; A_2 \;\; A_3 \;\; A_4 \;\; A_5 \;\; A_6 \;\; A_7 \;\; A_8}$$

Comparison part of `Merge` function:

```
i = 1
j = 1
for k = 1 to A.length
    if L[i] ≤ R[j]
        A[k] = L[i]
        i = i + 1
    else
        A[k] = R[j]
        j = j + 1
```

For full version, refer to the textbook.

Next Week

- Asymptotic notation
- Recurrence relation and how to solve it
  - Substitution
  - Recurrence tree
  - Master theorem

Stay Safe!