ECS 20: Discrete Mathematics for Computer Science

Winter 2021

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Outline

- Midterm 2 Recap
- Mathematical Induction
  1. Why does it work (handout)
  2. More examples
- Integer Algorithm Recap (if time permits)

Mathematical Induction: Example 1 (Homework problem)

Problem Statement: Prove the following statement $P(n)$ is true

$$\sum_{i=1}^{n} i^3 = (\frac{n(n+1)}{2})^2$$

Build a skeleton:

- Basis Step:
- Inductive Hypothesis:
- Inductive Step:
- Thus, by the principle of induction, we can conclude that the statement above is true for every natural number n.

Mathematical Induction: Example 1

$$1^3 + 2^3 + \cdots + n^3 = (\frac{n(n+1)}{2})^2$$

- Basis Step:
  Verify $P(n)$ is true when $n = 1$. $1^3 = (\frac{1 \cdot (1+1)}{2})^2$.

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- Basis Step:

  Verify $P(n)$ is true when $n = 1$. $1^3 = (\frac{1 \cdot (1+1)}{2})^2$.
- Inductive Hypothesis:

  Assume $P(n)$ is true when $n = k$. Then, we have $1^3 + 2^3 + \cdots + k^3 = (\frac{k(k+1)}{2})^2$.

Mathematical Induction: Example 1

$$1^3 + 2^3 + \cdots + n^3 = (\frac{n(n+1)}{2})^2$$

- Inductive Step: We need to show $P(n)$ is true when $n = k + 1$.

$$\begin{aligned}
1^3 + 2^3 + \cdots + (k+1)^3 &= 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 \\
&= (\frac{k(k+1)}{2})^2 + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= (\frac{(k+1)((k+1)+1)}{2})^2
\end{aligned}$$

Mathematical Induction: Example 2 (Last example on the slides)

Problem Statement: Show that if S is a finite set with n elements, then S has 2^n subsets.

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Rephrase: Proof by induction: Let $P(n)$ be the proposition that a set with n elements has 2^n subsets.

Proof

1. Basis step:

   $P(0)$ is true, a set with zero elements, namely the empty set, has exactly $2^0 = 1$ subset.

2. Inductive Hypothesis:

   Assume $P(k)$ is true for an arbitrary k. Thus, a set with k elements has 2^k subsets.

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Mathematical Induction: Example 2

3. Inductive Step:
   - Let S have $k+1$ elements.
   - Then $S = \{S'\} \cup \{w\}$, where w is an arbitrary element of S and $S' = S - \{w\}$. (*a partition*)
   - The subsets of S can be obtained in the following way: for each subset T of S', T and $T \cup \{w\}$ make up all the subsets of S, and are all distinct.
   - Since there are 2^k subsets of S', there are $2^k + 2^k = 2^{k+1}$ subsets of S.