Relations, representation and composition

2.5. Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, and $C = \{x, y, z\}$. Consider the following relations R and S from A to B and from B to C, respectively.

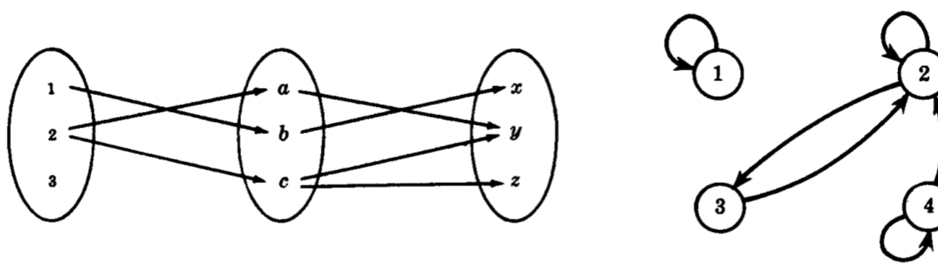
$$R = \{(1, b), (2, a), (2, c)\} \quad \text{and} \quad S = \{(a, y), (b, x), (c, y), (c, z)\}$$

(a) Find the composition relation $R \circ S$.

(b) Find the matrices M_R, M_S, and $M_{R\circ S}$ of the respective relations R, S, and $R \circ S$, and compare $M_{R\circ S}$ to the product $M_R M_S$.

(a) Draw the arrow diagram of the relations R and S as in Fig. 2-7(a). Observe that 1 in A is "connected" to x in C by the path $1 \to b \to x$; hence $(1, x)$ belongs to $R \circ S$. Similarly, $(2, y)$ and $(2, z)$ belong to $R \circ S$.
We have

$$R \circ S = \{(1, x), (2, y), (2, z)\}$$



(a) (b)

Fig. 2-7

(b) The matrices of M_R, M_S, and $M_{R\circ S}$ follow:

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \qquad M_S = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \qquad M_{R\circ S} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Multiplying M_R and M_S we obtain

$$M_R M_S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Observe that $M_{R\circ S}$ and $M_R M_S$ have the same zero entries.

Types of relations

2.9. Consider the following five relations on the set $A = \{1, 2, 3\}$:

$$R = \{(1,1), (1,2), (1,3), (3,3)\}, \qquad \emptyset = \text{empty relation}$$
$$S = \{(1,1)(1,2), (2,1)(2,2), (3,3)\}, \qquad A \times A = \text{universal relation}$$
$$T = \{(1,1), (1,2), (2,2), (2,3)\}$$

Determine whether or not each of the above relations on A is: (*a*) reflexive; (*b*) symmetric; (*c*) transitive; (*d*) antisymmetric.

$A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

	reflexive	symmetric	transitive	antisymmetric
R	✗	✗	✓	✓
$\emptyset$	✗	✓	✓	✓
S	✓	✓	✓	✗
$A \times A$	✓	✓	✓	✗
T	✗	✗	✗	✓

2.28. Let R and S be relations on a set A. Assuming A has at least three elements, state whether each of the following statements is true or false. If it is false, give a counterexample on the set $A = \{1, 2, 3\}$:

(a) If R and S are symmetric then $R \cap S$ is symmetric.

(b) If R and S are symmetric then $R \cup S$ is symmetric.

(c) If R and S are reflexive then $R \cap S$ is reflexive.

(d) If R and S are reflexive then $R \cup S$ is reflexive.

(e) If R and S are transitive then $R \cup S$ is transitive.

(f) If R and S are antisymmetric then $R \cup S$ is antisymmetric.

(g) If R is antisymmetric, then R^{-1} is antisymmetric.

(h) If R is reflexive then $R \cap R^{-1}$ is not empty.

(i) If R is symmetric then $R \cap R^{-1}$ is not empty. wrong answer in the textbook

(a) T

(b) T

(c) T

(d) T

(e) F

$R = \{(1,2)\}$

$S = \{(2,1)\}$

(f) F

$R = \{(1,2)\}$ $S = \{(2,1)\}$

(g) T (h) T (i) F $R = \emptyset$

Equivalence relations and partition

2.14. Consider the **Z** of integers and an integer $m > 1$. We say that x is congruent to y modulo m, written

$$x \equiv y \pmod{m}$$

if $x - y$ is divisible by m. Show that this defines an equivalence relation on **Z**.

We must show that the relation is reflexive, symmetric, and transitive.

(i) For any x in Z we have $x \equiv x \pmod{m}$ because $x - x = 0$ is divisible by m. Hence the relation is reflexive.

(ii) Suppose $x \equiv y \pmod{m}$, so $x - y$ is divisible by m. Then $-(x - y) = y - x$ is also divisible by m, so $y \equiv x \pmod{m}$. Thus the relation is symmetric.

(iii) Now suppose $x \equiv y \pmod{m}$ and $y \equiv z \pmod{m}$, so $x - y$ and $y - z$ are each divisible by m. Then the sum

$$(x - y) + (y - z) = x - z$$

is also divisible by m; hence $x \equiv z \pmod{m}$. Thus the relation is transitive.

Accordingly, the relation of congruence modulo m on **Z** is an equivalence relation.

2.16. Let R be the following equivalence relation on the set $A = \{1, 2, 3, 4, 5, 6\}$:

$$R = \{(1, 1), (1, 5), (2, 2), (2, 3), (2, 6), (3, 2), (3, 3), (3, 6), (4, 4), (5, 1), (5, 5), (6, 2), (6, 3), (6, 6)\}$$

Find the partition of A induced by R, i.e., find the equivalence classes of R.

Those elements related to 1 are 1 and 5 hence

$$[1] = \{1, 5\}$$

We pick an element which does not belong to [1], say 2. Those elements related to 2 are 2, 3, and 6, hence

$$[2] = \{2, 3, 6\}$$

The only element which does not belong to [1] or [2] is 4. The only element related to 4 is 4. Thus

$$[4] = \{4\}$$

Accordingly, the following is the partition of A induced by R:

$$[\{1, 5\}, \{2, 3, 6\}, \{4\}]$$