ECS 20: Discrete Mathematics for Computer Science

Winter 2021

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Outline

- Recap of Asymptotic Notation (e.g. Big O)
- Graphs and Trees

Big O Recap: Frequently used functions

a. Polynomials:
$p(n) = \sum_{i=0}^{d} a_i n^i$ is called a polynomial in n of degree d where $a_d \neq 0$.
  - $p(n) = O(n^d)$

b. Exponentiation:
a^n is the basic form of an exponential, where a is called base.
  - $(a^n)^m = a^{mn} = (a^m)^n$

c. Logarithms:
The inverse function to exponentiation, namely:
$\log_a n = x$ exactly if $a^x = n$ and $n > 0$ and $a > 0$ and $a \neq 1$.

Usually, base $a = 2, e, 10$ (e is the constant base ≈ 2.718 for natural logarithm)

For all real $a > 0$, $b > 0$, $c > 0$, and n, we have:

  i. $a = b^{\log_b a}$
  ii. $\log_c(ab) = \log_c a + \log_c b$
  iii. $\log_b a^n = n \log_b a$

Big O Recap: Growth relation between functions

Frequently used functions	Order of Growth	Example
constant	$O(1)$	$f(n) = 10$
logarithm	$O(\log n)$	$f(n) = 4\log(n+2)$
linear	$O(n)$	$f(n) = 2n + 1$
linearithmic	$O(n \log n)$	$f(n) = 3n \log 4n + 1$
quadratic	$O(n^2)$	$f(n) = 2n^2 + 3n + 1$
cubic	$O(n^3)$	$f(n) = n^3 + 1$
exponential	$O(a^n), a \geq 2$	$f(n) = 3^n$

Big O Recap: Problem 2 in HW8

Show that $(x^3 + 2x)/(2x + 1)$ is $O(x^2)$.

Graphs and Trees: Outline (1)

1. Notion of graphs
   - Vertex (vertices) and edge(s)
   - Incident, adjacent and degree
   - Handshaking Theorem
   - Simple graphs and multigraphs
   - Graph isomorphism
2. Graph representation
   - Adjacency list
   - Adjacency matrix
   - Incidence matrix
3. Special types of graphs
   - Complete graph K_n
   - Cycle, Wheel and n-cube Q_n
   - Bipartite graph (subgraphs, union)
   - Complete bipartite graph $K_{m,n}$

Graphs and Trees: Outline (2)

4. Connectivity
   - Path and cycle
   - Connected graphs
   - Euler path (cycle)
   - Hamiltonian path (cycle)
5. Planar graph
   - Definition
   - Euler's formula
   - k-colorable
   - Elementary subdivision, homeomorphism and Kuratowaski's Theorem
6. Trees
   - Definition
   - Rooted tree
   - m-ary tree, e.g. binary tree
   - Binary search tree

Graphs and Trees: Terminology (1)

1. **Path**: an alternating sequence of vertices and edges.
2. **Simple Path**: a path in which all vertices are distinct.
3. **Trail**: a path in which all edges are distinct.
4. **Distance**: the length of the shortest path between u and v.
5. **Diameter**: the maximum distance between any pairs of vertices.

Graphs and Trees: Terminology (2)

Problem 8.34. Consider the graph G in Fig. 8-58. Find:

(a) degree of each vertex, and verify Handshaking Theorem;

(b) all simple paths from A to L;

(c) all **trails** (distinct edges) from B to C;

(d) $d(A, C)$, **distance** from A to C;

(e) diam(G), the **diameter** of G.

Graphs and Trees: Graph isomorphism (1)

To show that two simple graphs are not isomorphic, we can show that they do not share an invariant property, such as the same number of vertices, edges, degrees, etc.

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If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex $\{w\}$ together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called **elementary subdivision**.

Two graphs G_1 and G_2 are called **homeomophic** if they can be obtained from the same or isomorphic graph by a sequence of elementary subdivisions.

Graphs and Trees: Graph isomorphism (2)

Problem 8.36. Consider the graph G in Fig. 8-58. Find the subgraph $H = \{V', E'\}$ of G where V' equals:

(a) $\{B, C, D, J, K\}$

(b) $\{A, C, J, L, M\}$

(c) $\{B, D, J, M\}$

(d) $\{C, K, L, M\}$

Which of them are isomorphic and which homeomorphic?

Graphs and Trees: Euler cycle and Hamiltonian cycle (1)

	Euler path (cycle)	Hamiltonian path (cycle)
Definition	a path (cycle) containing every **edge** of G exactly once	a path (cycle) containing every **vertex** of G exactly once
Theorem	Euler cycle: *iff* each vertex has **even** degree Euler path but not cycle: *iff* G has exactly two vertices of **odd** degrees	Hamiltonian cycle: $\#$ vertices $= n \geq 3$ and $deg(v) \geq n/2$

Graphs and Trees: Euler cycle and Hamiltonian cycle (2)

Problem 8.44. & 8.45. Consider the graphs $K_5, K_{3,3} and K_{2,3}$ in Fig. 8-59. Find an Euler (traversable) path or an Euler circuit (cycle) of each graph, if it exists. If not, why? How about Hamiltonian path or Hamiltonian circuit?

Thank you!

Wish you every success in your future endeavors!