Non-archimedean SYZ mirror symmetry

Mirror symmetry? come from string theory.

mirror pairs

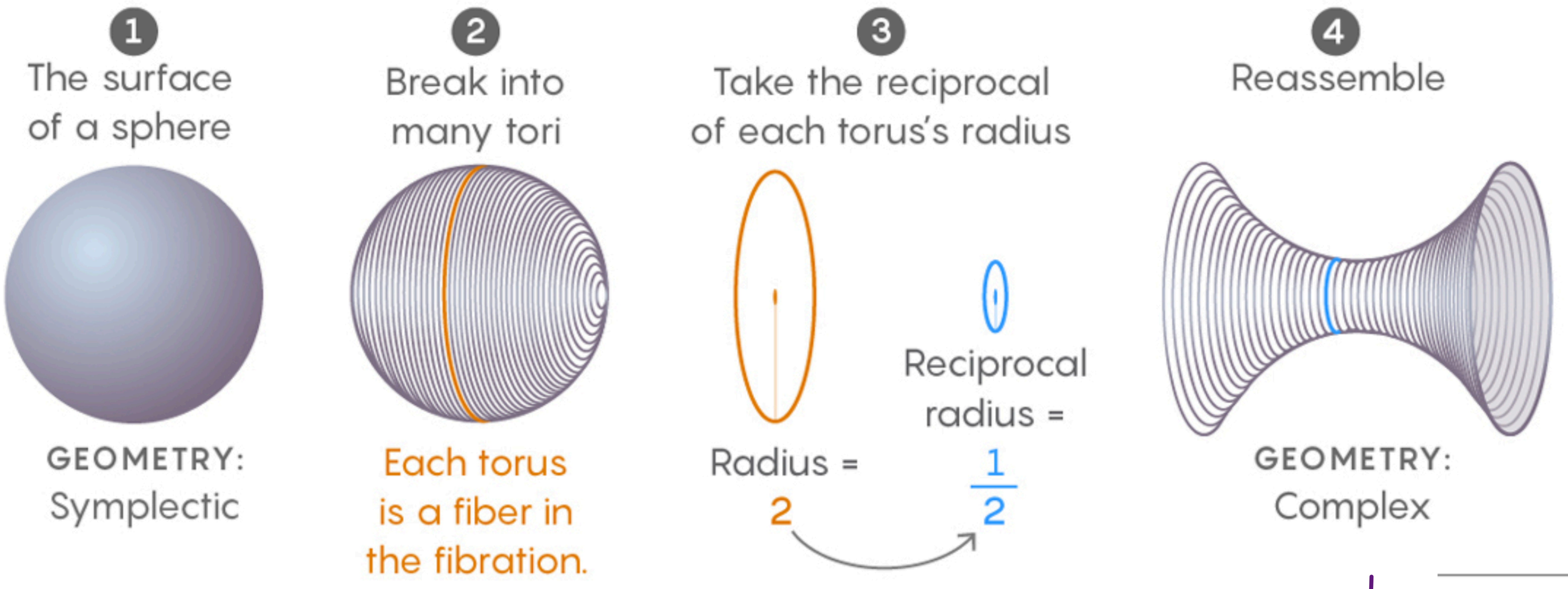
Roughly, it predicts that Calabi-Yau manifolds should exist in pairs $(X, X^\vee)$ s.t.

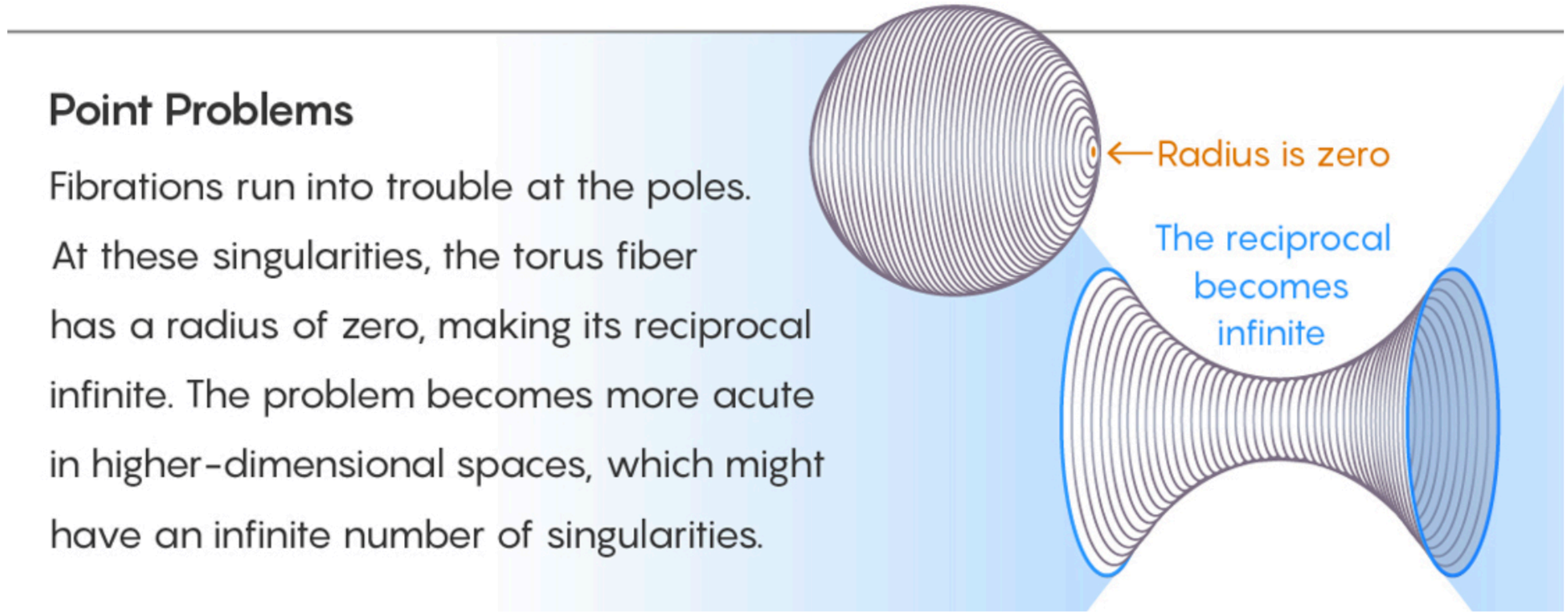
$$\boxed{\text{symplectic invariants of } X} \text{ ``}\Longleftrightarrow\text{''} \boxed{\text{complex invariants of } X^\vee}$$

1994 ICM **Kontsevich's homological mirror symmetry conjecture**

$$\boxed{\text{Fukaya category of } X} \cong \boxed{\text{Derived category of } X^\vee}$$

Question If we know X, can we obtain its mirror space $X^\vee$?





Conjecture 1 (Strominger-Yau-Zaslow [35]) *Mirror Calabi–Yau manifolds are equipped with special Lagrangian fibrations*

$$\begin{tikzcd} X \arrow[dr, "\mu"'] & & \check{X} \arrow[dl, "\check{\mu}"] \\ & B & \end{tikzcd}$$

μ $\check{\mu}$

B_0 open interval

B closed interval

such that μ and $\check{\mu}$ are dual torus fibrations over a dense open locus $B_0 \subset B$ of the base.

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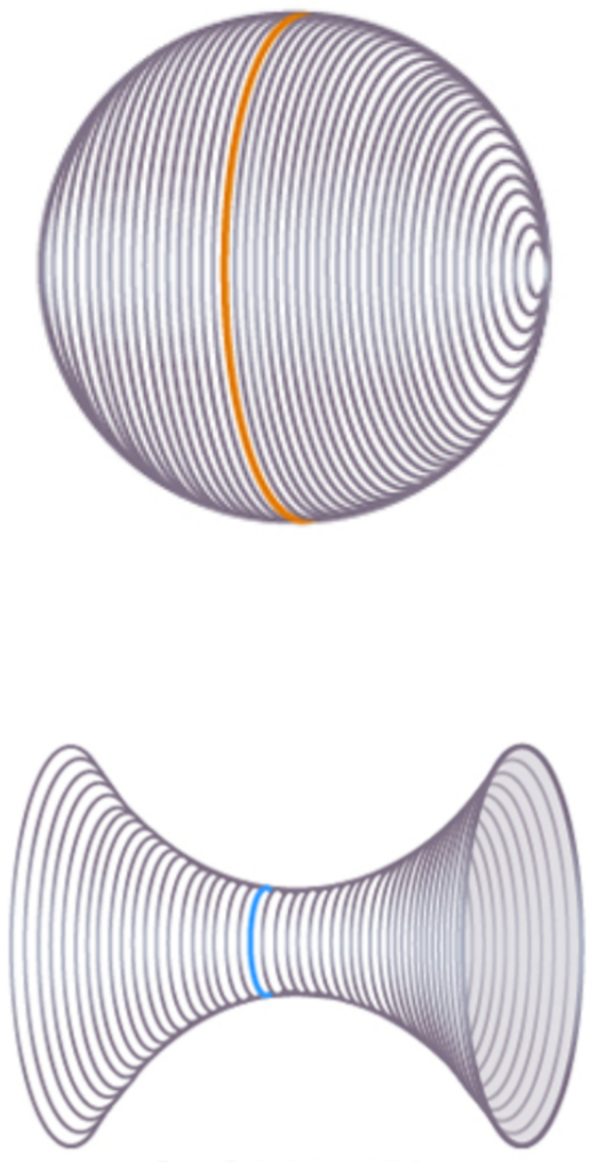

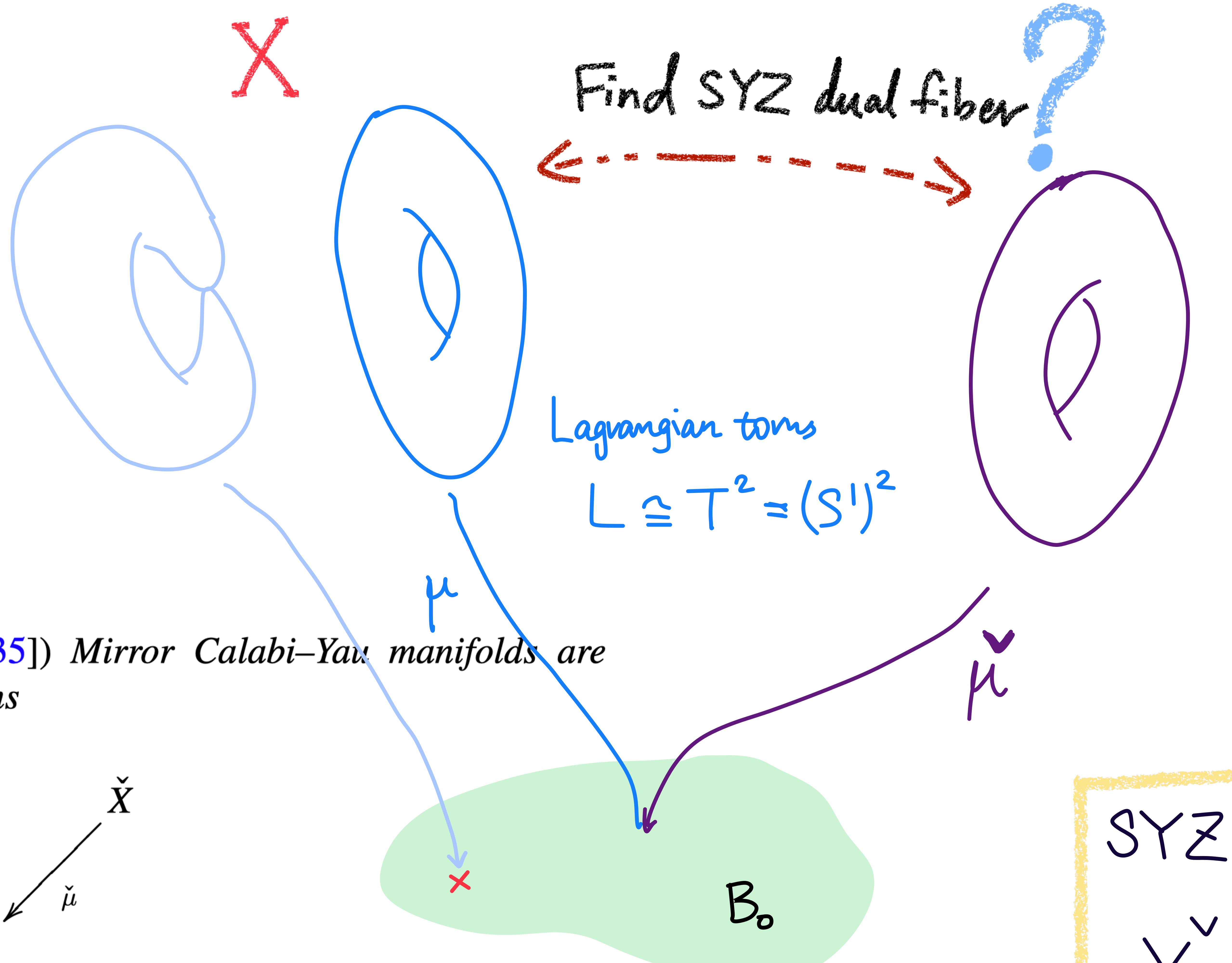

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Previous expectation

dual torus

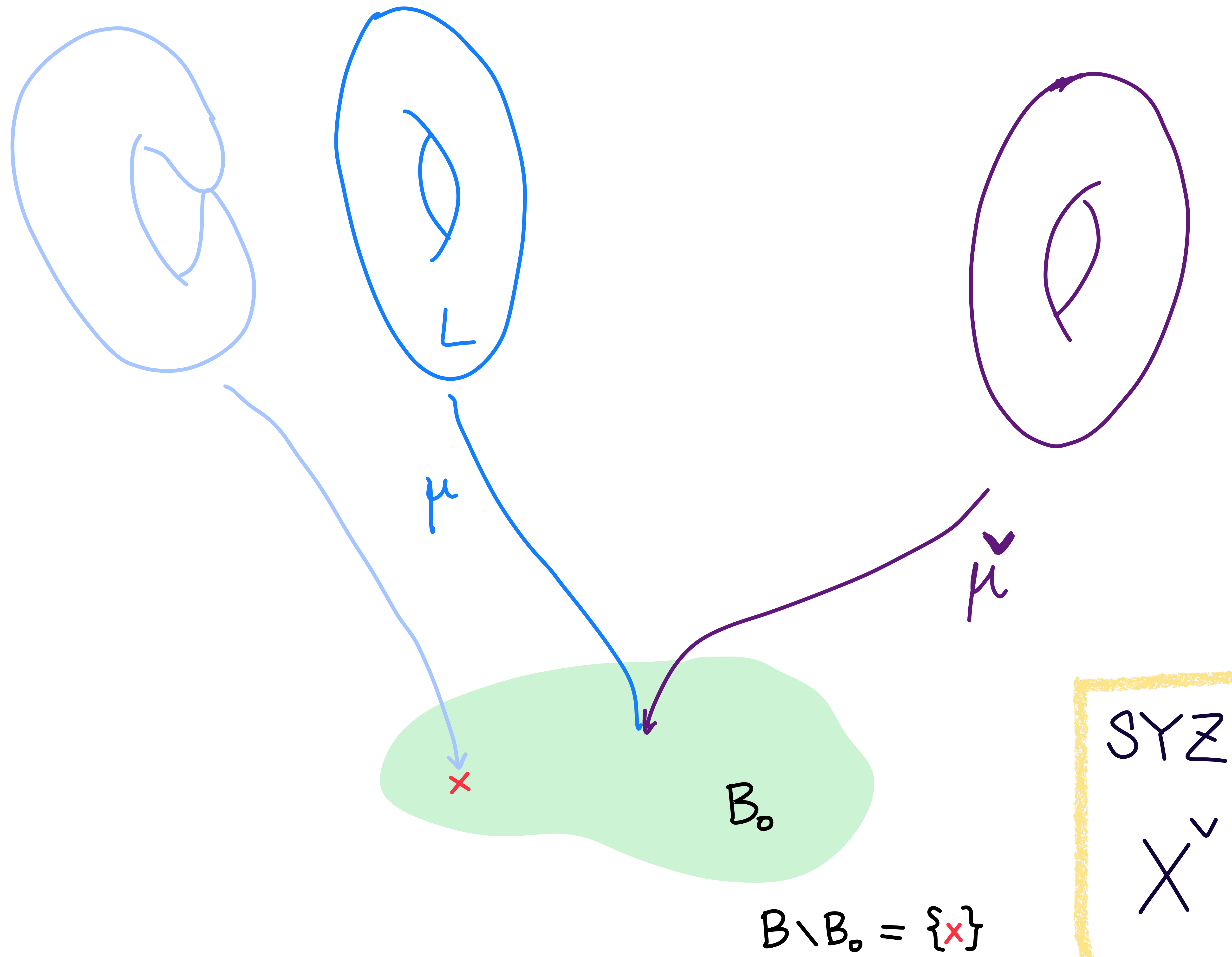
$$H^1(L; U(1)) \cong U(1)^2 = T^2$$

SYZ mirror construction

$$X^\vee \text{ “=” } \bigcup_L H^1(L; U(1))$$

Trouble Need to be modified by "quantum corrections"

namely, the counts of holomorphic disks :



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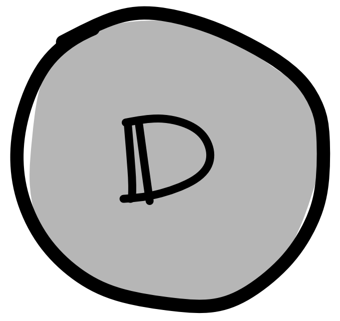
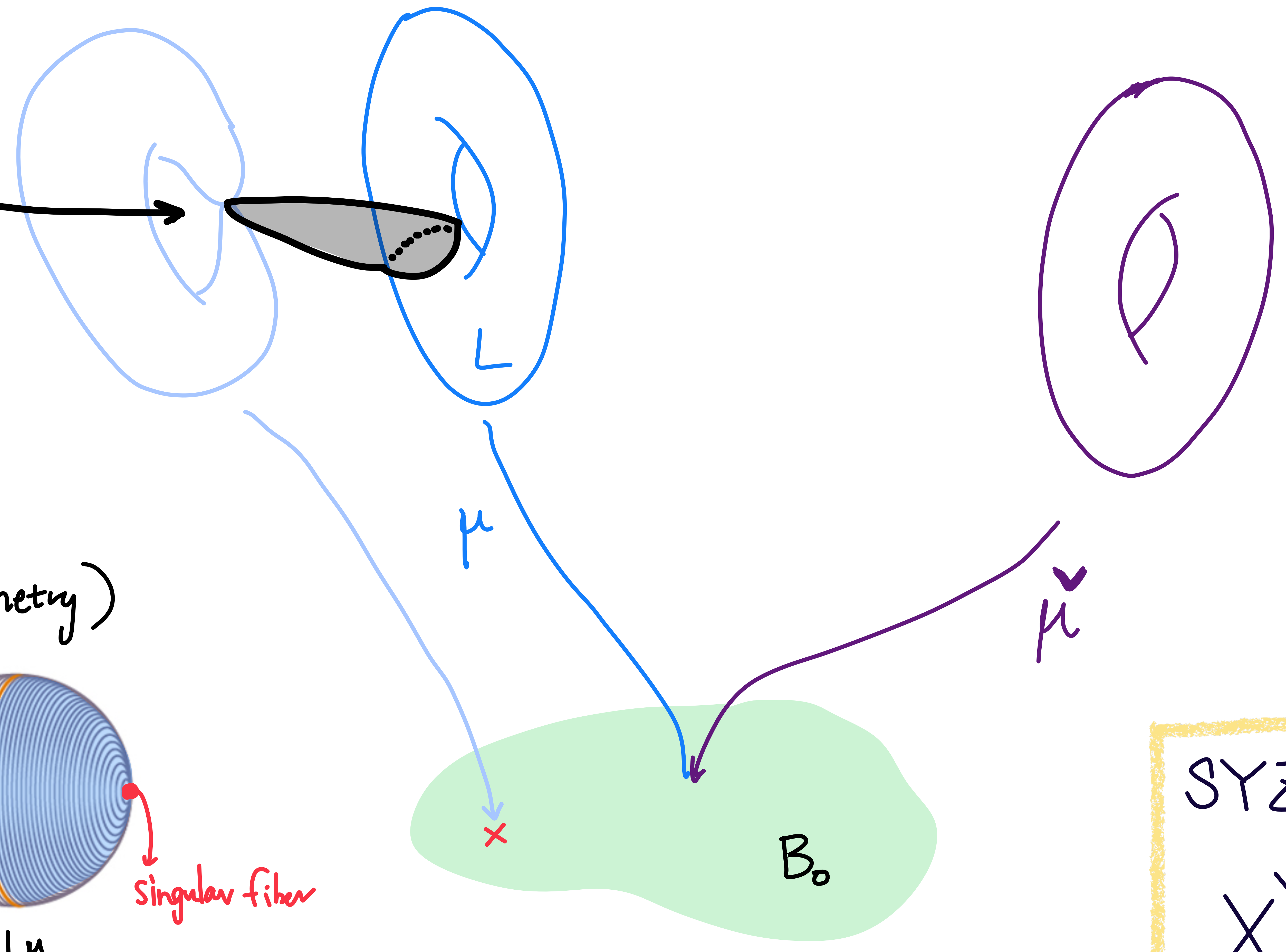
We also need to modify the dual side

unit disk in $\mathbb{C}$.

u: 

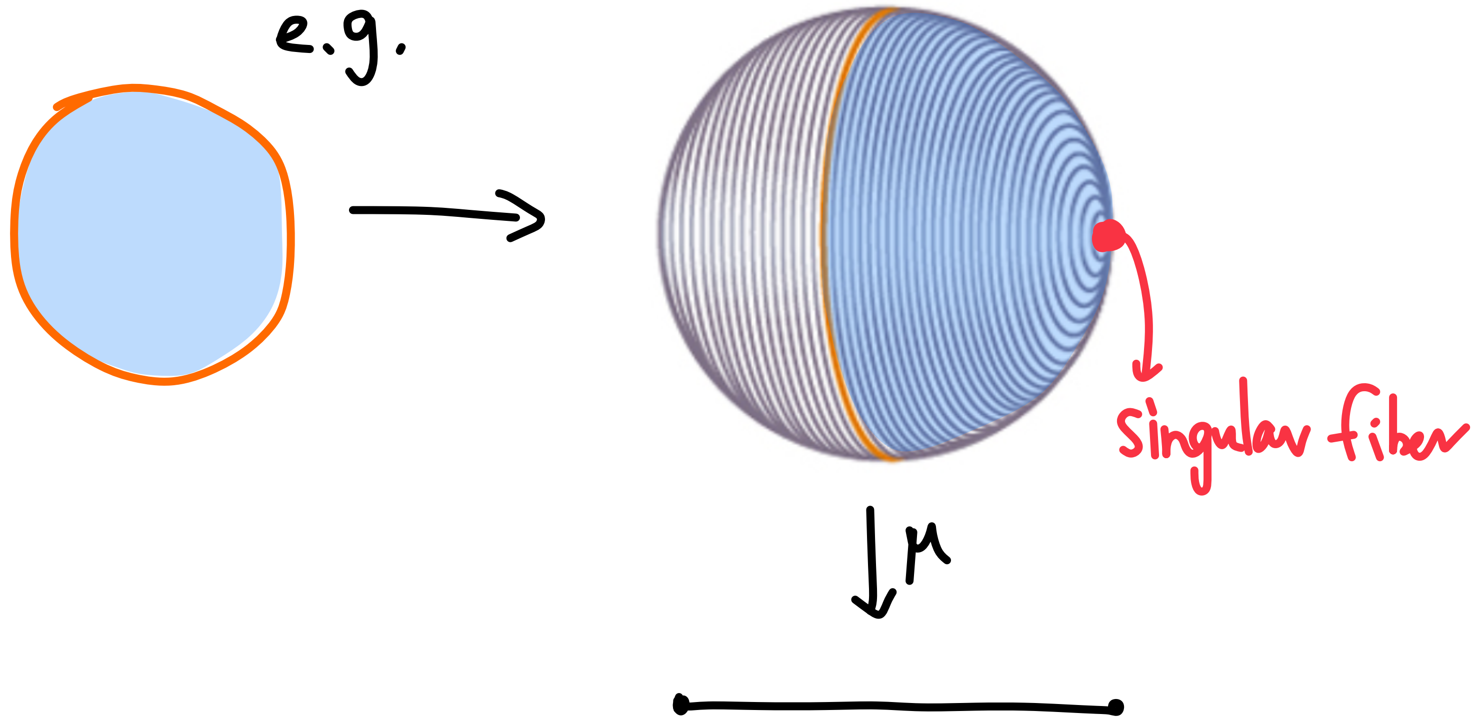


$u : (\mathbb{D}, \partial\mathbb{D}) \longrightarrow (X, L)$

J-holomorphic curves

(basic object in symplectic geometry)

e.g.



$B \setminus B_o = \{x\}$

SYZ mirror construction

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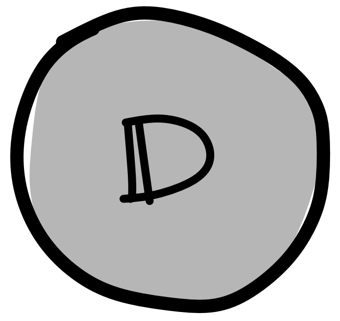
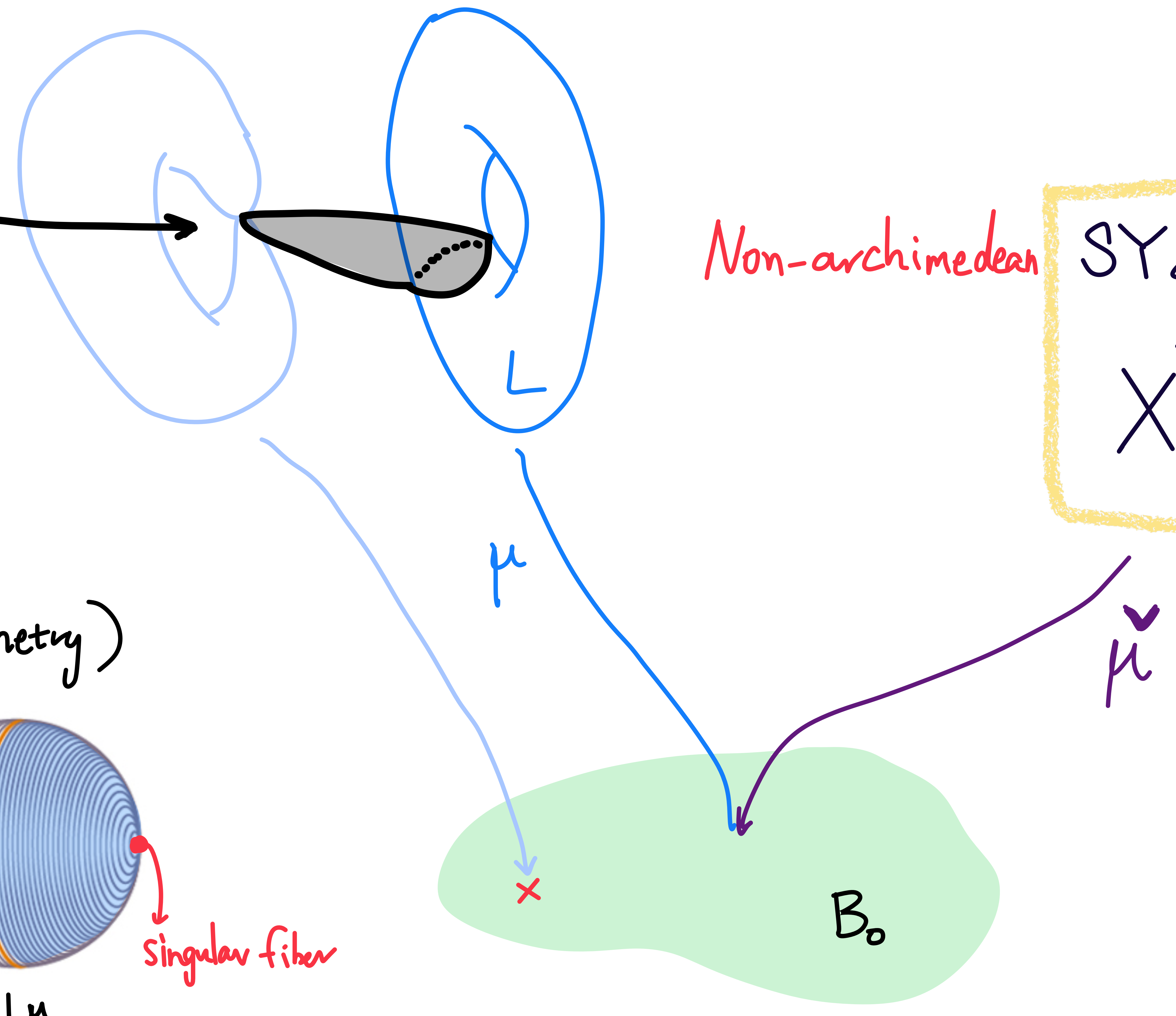
$u:$ 



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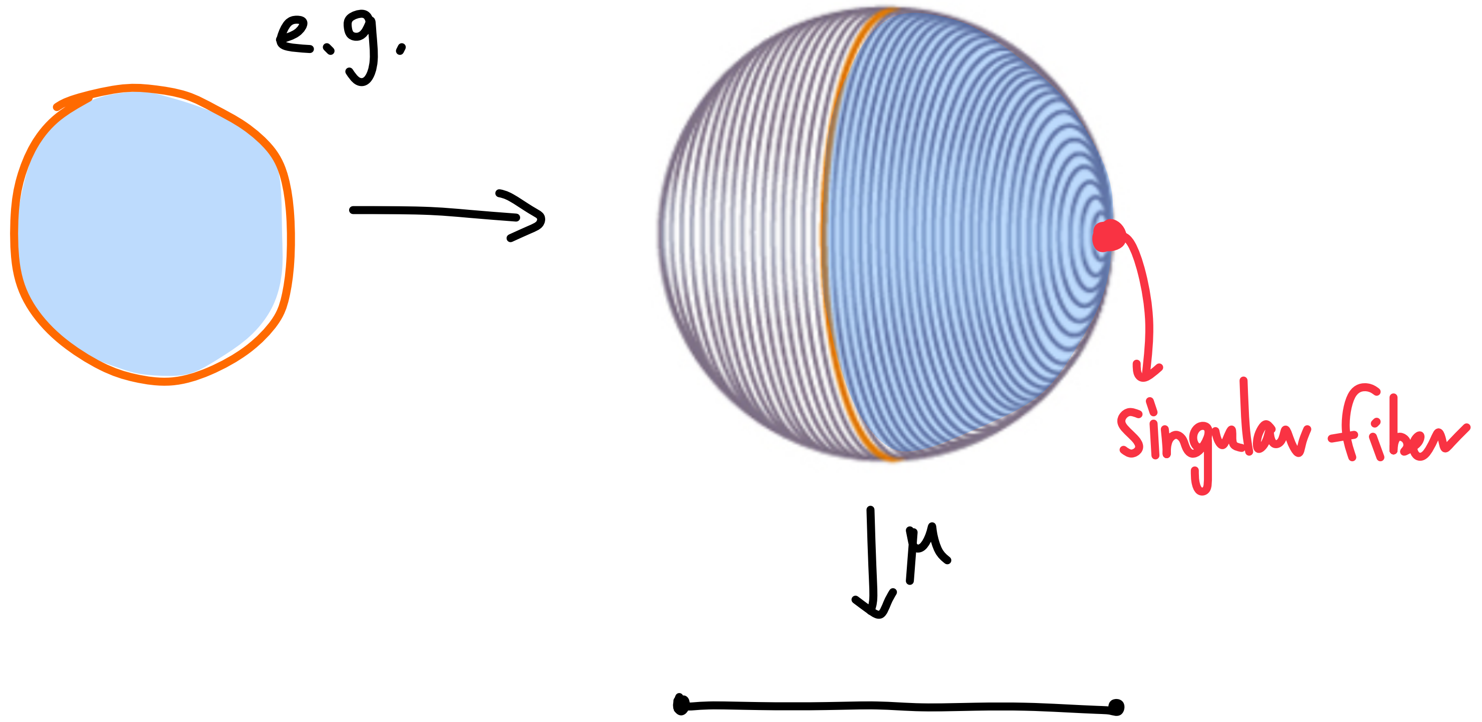


$B \setminus B_0 = \{x\}$

Non-archimedean SYZ mirror construction

$$X^{\vee} \text{ "="} \bigcup_{L} H^1(L; U(1))$$

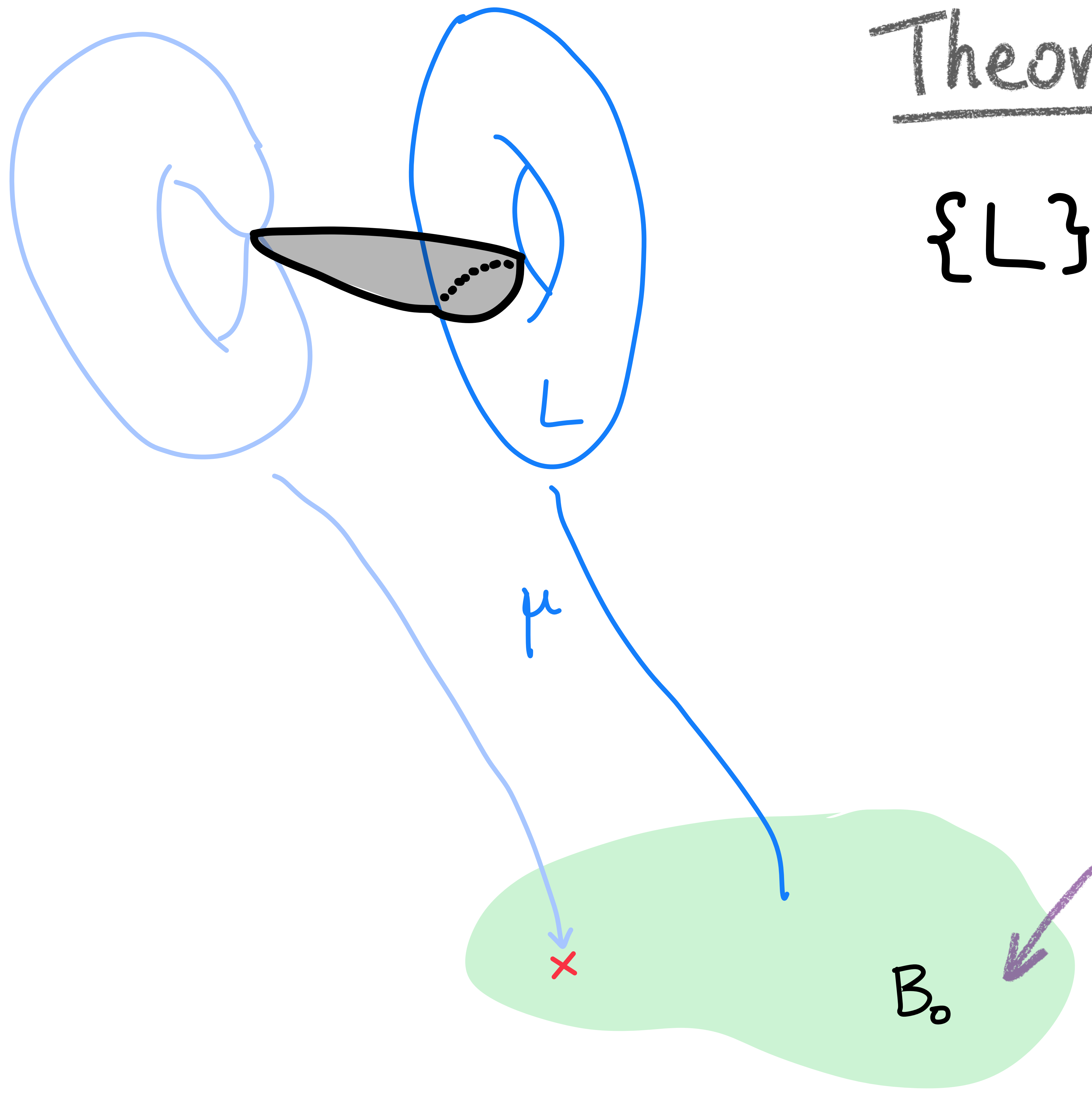
$U(1) \subseteq U_{\Lambda}$

multiplicative subgroup of the Novikov field Λ

Λ is a non-archimedean field

My thesis : realize the "quantum corrections"

(answer a conjecture by Denis Auroux)



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Theorem Suppose there is a good Lagrangian torus fibration $\{L\}$ on X, the set

include special Lagrangian fib.

$$\bigcup_L H^1(L; U_\Lambda) =: X^\vee$$

admits a rigid analytic space structure which is determined by the disk-counting

(Non-archimedean SYZ mirror construction)

* John Tate (1962) rigid analytic geometry

Archimedean : $|x+y| \le |x| + |y|$ $\mathbb{Q}, \mathbb{R}$

Non-archimedean : $|x+y| \le \max\{|x|, |y|\}$

* an analogue of a complex analytic space over a non-archimedean field ?

Examples

(1) the field $\mathbb{Q}_p$ of p-adic numbers (Number theory)

(2) the Novikov field

$$\Lambda = \mathbb{C}((T^{\mathbb{R}})) = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \;\middle|\; a_i \in \mathbb{C},\ \lambda_i \in \mathbb{R},\ \lambda_i \nearrow \infty \right\}$$

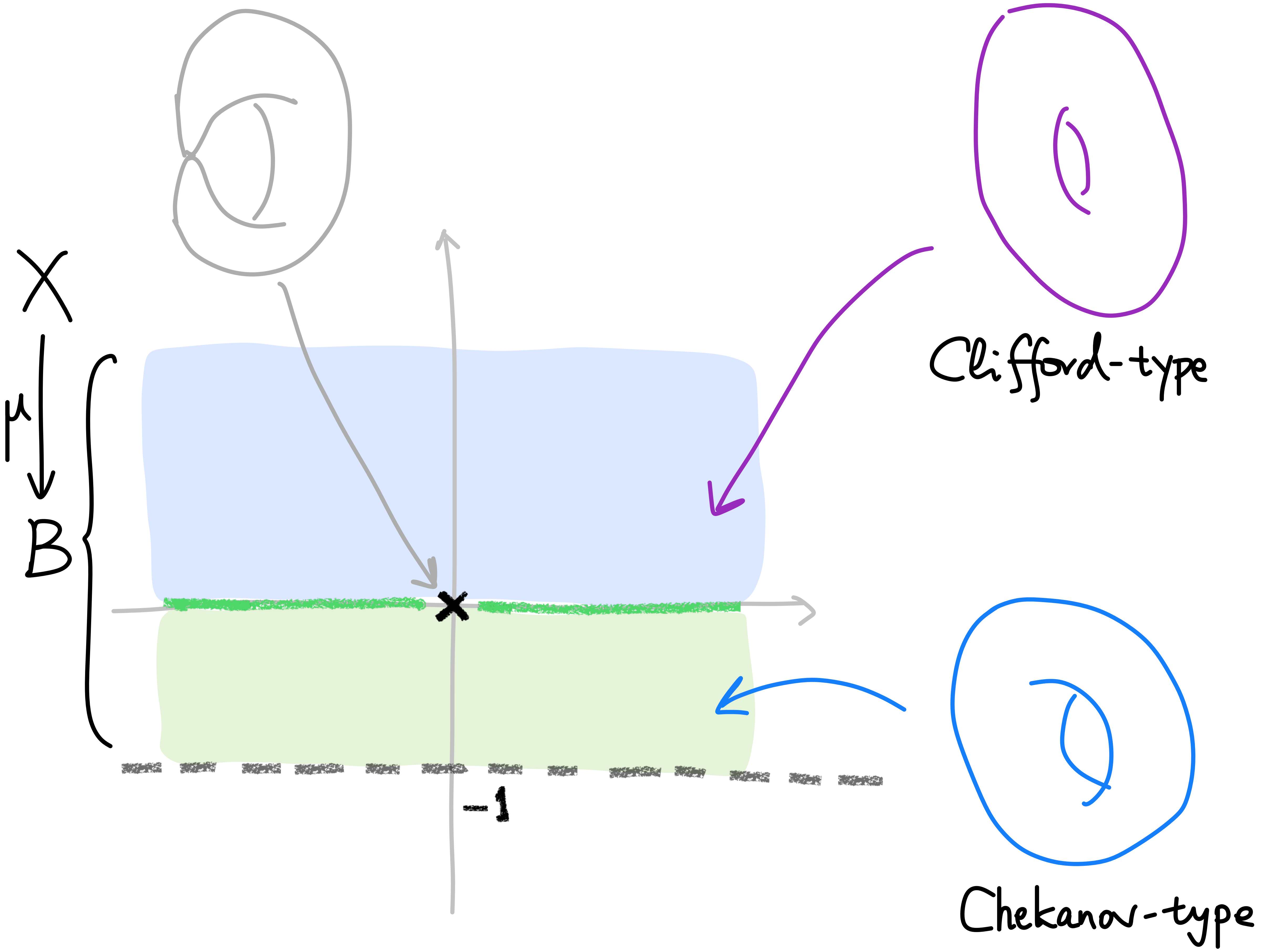
(Novikov field/ring is commonly used in symplectic geometry : Lagrangian Floer theory , Fukaya category , Symplectic cohomology)

rigid analytic space

This explains the title "Non-archimedean SYZ mirror symmetry"

Example & Application (Rough idea: disk-counting for different torus fibers are related)

Gross's Lagrangian fibration on $X = \mathbb{CP}^2$



- roughly equivalent to product tori $(S^1)^2 \subseteq \mathbb{C}^2$
- disk counting is easy to obtain

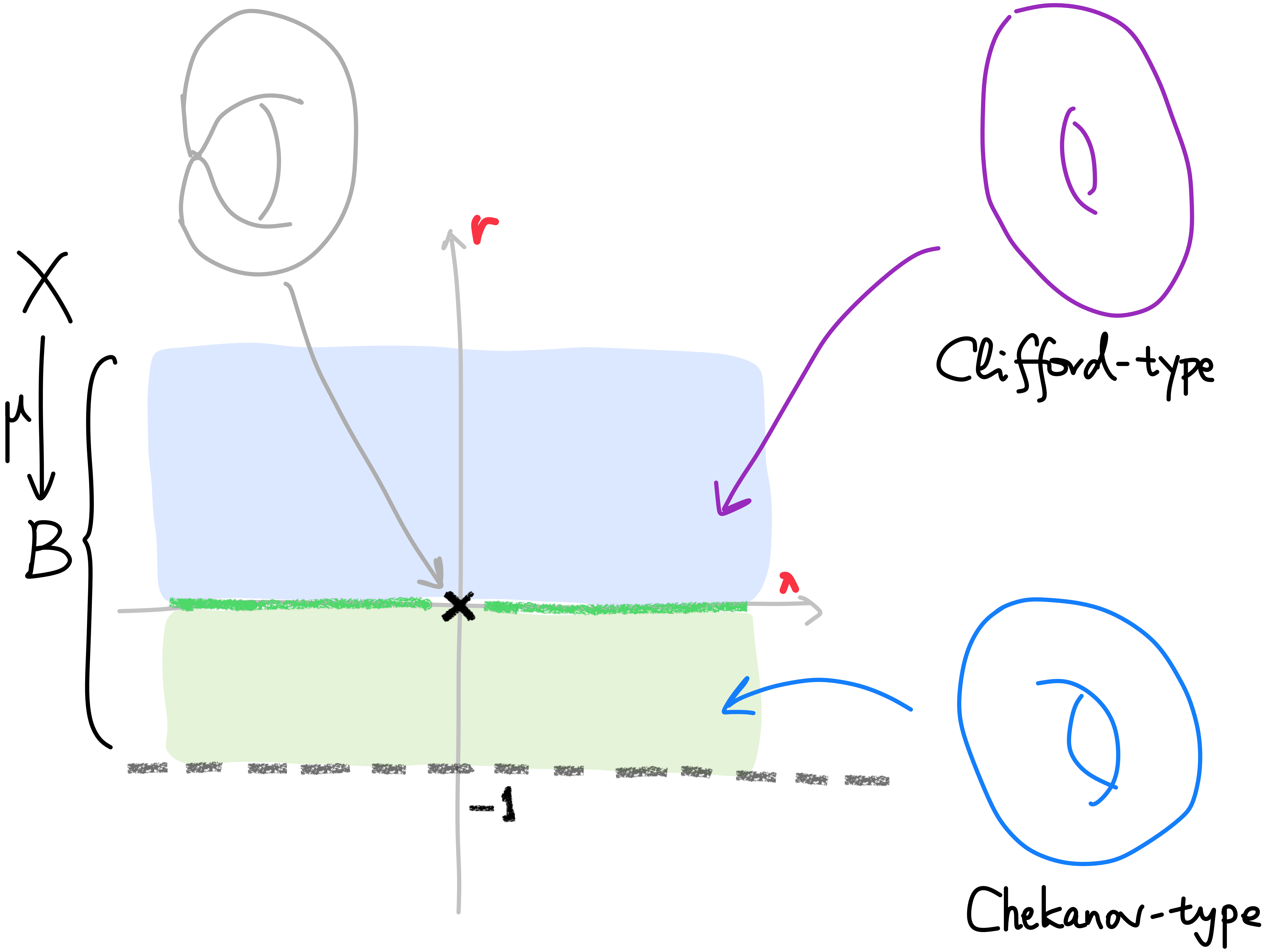
related by non-archimedean SYZ in my thesis

- more complicated
- disk counting is hard to obtain

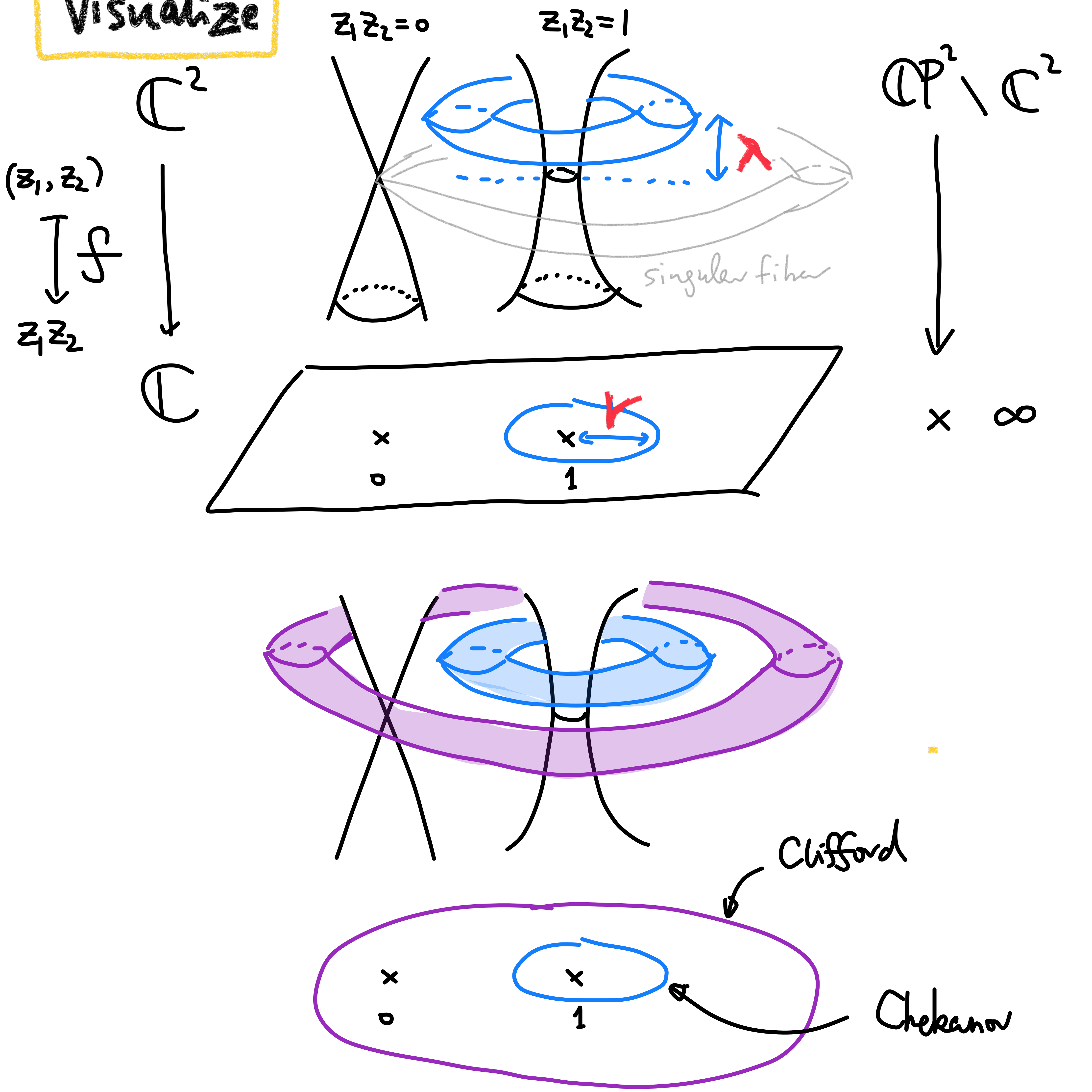
Example & Application

Explicitly $\mu: X \longrightarrow B$, $(z_1, z_2) \mapsto \Big(\overset{\lambda}{\tfrac{1}{2}|z_1|^2 - \tfrac{1}{2}|z_2|^2}, \overset{r}{|z_1 z_2 - 1| - 1}\Big)$

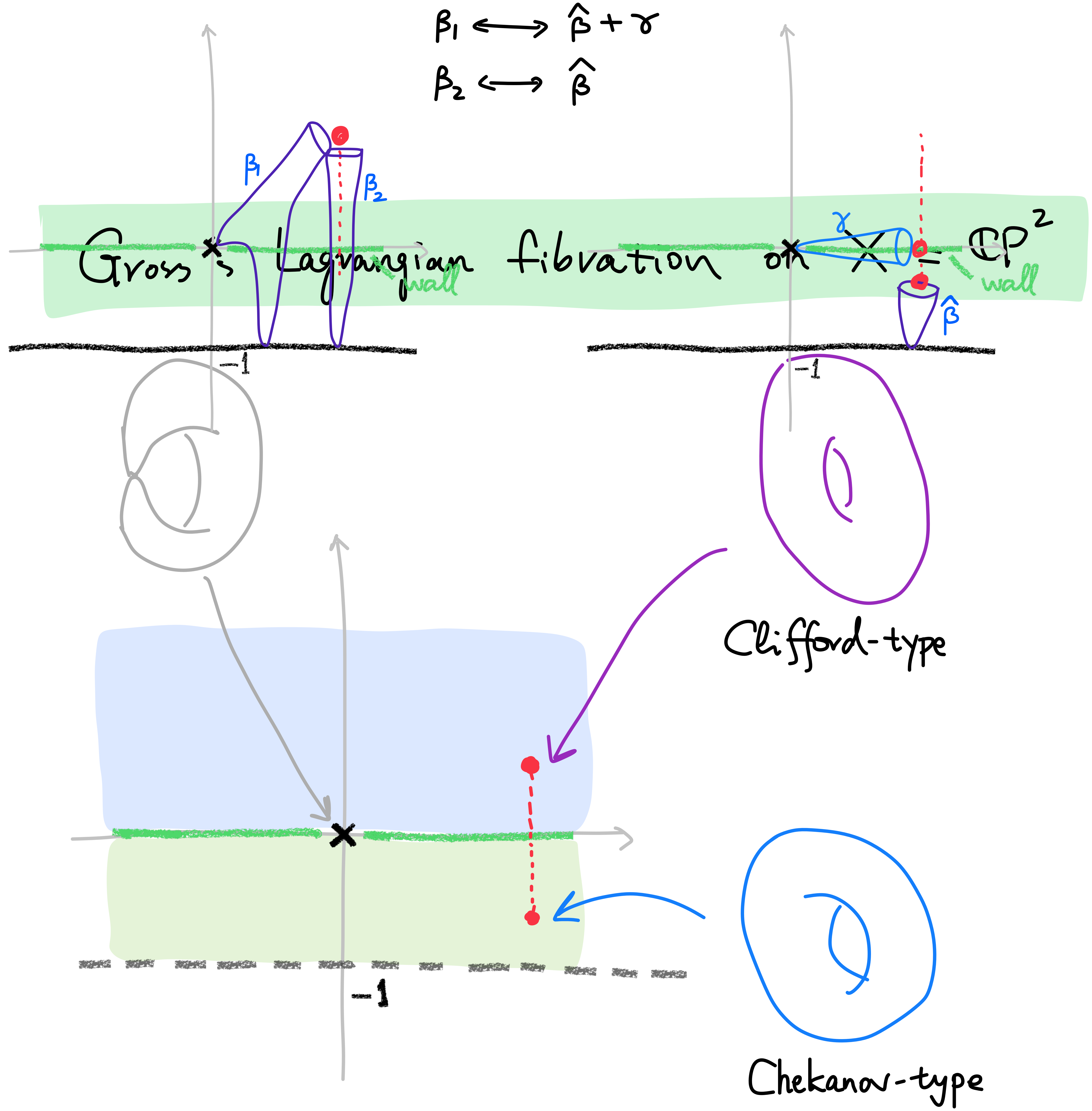
Gross's Lagrangian fibration on $X = \mathbb{CP}^2$



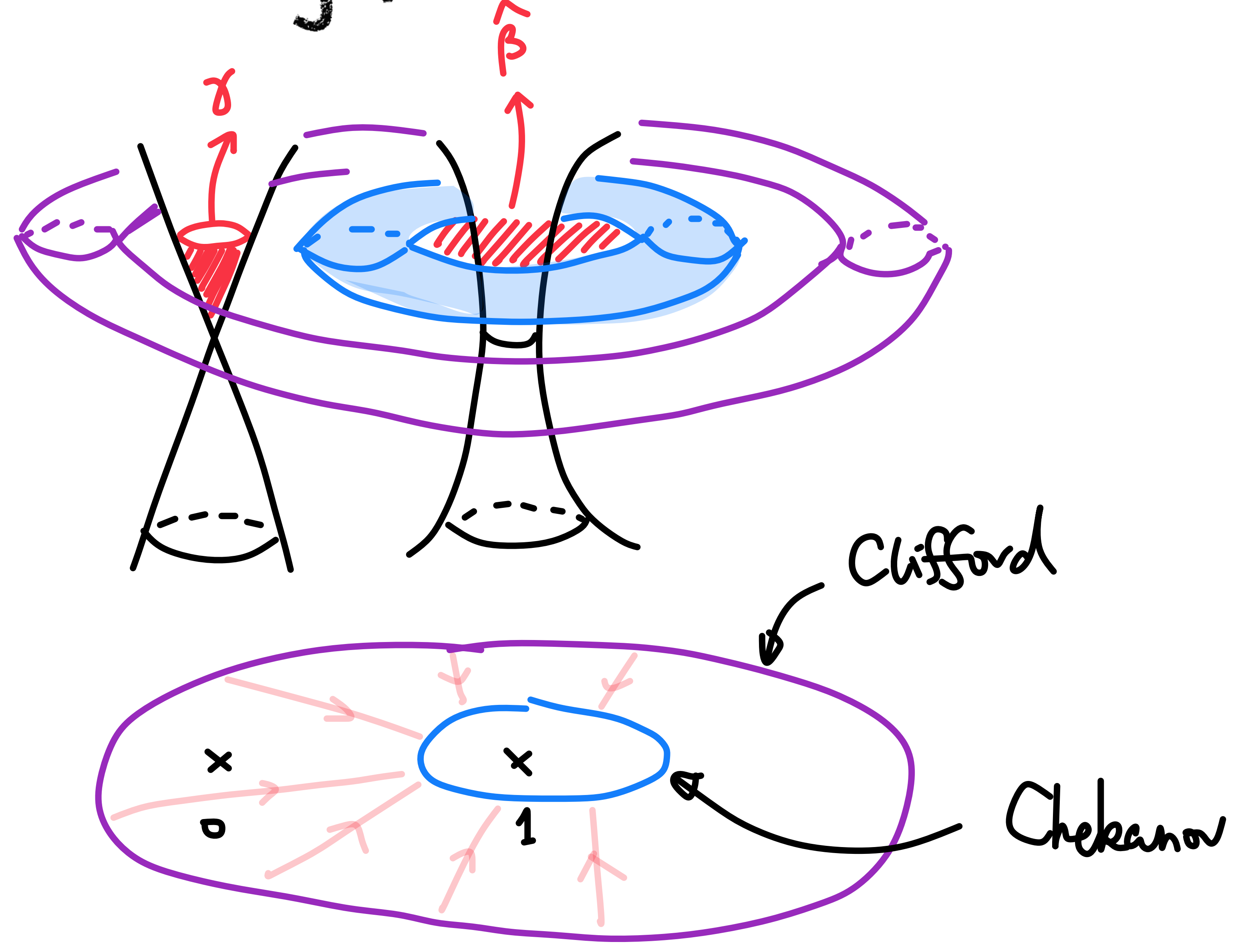
Visualize



Gross's Lagrangian fibration on $\mathbb{CP}^2$



wall crossing phenomenon



Disk counting for $\beta_1, \beta_2, \hat{\beta}$ and γ are related.

(There are other disks which are hard to draw)

<u>Recall</u> Disk counting for Clifford tori is easy

$\Downarrow$

obtain disk counting for Chekanov tori

The idea can be applied in higher dimensions:

$\boxed{\text{Corollary}}$

For a Chekanov-type Lagrangian torus L in $\mathbb{C}P^n$, we can compute all the non-trivial open Gromov-Witten invariants:

(disk counting)

$\dfrac{n!}{k_1! \cdots k_n!}$ with $k_1 + \cdots + k_n = n$

1 (exceptional), and all the multinomial coefficients in $(X_1 + \cdots + X_n)^n$

In particular, there are $1 + \binom{2n-1}{n-1}$ classes in $\pi_2(\mathbb{C}P^n, L)$ that contain holomorphic disks.

$\boxed{n=2}$ $(X_1+X_2)^2 = X_1^2 + 2X_1X_2 + X_2^2$

1, 2, 1

Rmk

- The similar results hold for $\mathbb{C}P^r \times \mathbb{C}P^{n-r}$
- When $n=2$, $r=1$, we retrieve the early results of Auroux, Chekanov-Schlenk
- In the case of $\mathbb{C}P^n$, it agrees with the Pascaleff-Tonkonog's work