Existence Conditions of Solutions for the Integral Boundary Value Problems of Nonlinear Multi-Term Fractional Differential Equations with a Generalized Order

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Supreme Leader **Kim Jong Un** said as follows.

"This is the age of science and technology, and its level of scientific and technological development is a criterion for judging the overall strength and status of a country."

Many physical systems can be better described by integral boundary conditions that are encountered in various applications such as population dynamics, blood flow models, chemical engineering and cellular systems.

The existence and uniqueness of the solution for the integral boundary value problem was studied in [1] by Schauder's fixed-point theorem and Banach's fixed point theorem in ordered Banach spaces, and in [2], they studied the existence of the solution for the boundary value problem with fractional integral boundary conditions. In [3] they investigated the existence and uniqueness of the solution for the nonlinear fractional integro-differential equation with integral boundary conditions and in [4] the existence of the solution for the system of fractional differential equations with nonlocal integral boundary conditions was studied. Also, in [5], they discussed the existence and uniqueness of the solution for the fractional differential equation with integral boundary conditions, and the existence and uniqueness of the solution for single term fractional differential equations with various kinds of integral boundary conditions was dealt in [6-8].

In this paper we study the existence and uniqueness of the solution for the integral boundary value problem of multi term nonlinear fractional differential equations in which σ, the order of the derivative in the nonlinear term f, is extended to $0<\sigma<2$ and boundary conditions are more generalized than those in [1-8].

Our integral boundary value problem is as follows:

$$ {}^{c}D_{0+}^{q}u(t) = f(t,\ u(t),\ {}^{c}D_{0+}^{\sigma}u(t)), \quad t \in [0,\ 1] \qquad (1)$$

$$\alpha_1 u(0)+\beta_1 u(1)+\gamma_1 u'(0)+\delta_1 u'(1)=\int_0^1 g(s,\ u)ds, \qquad (2)$$

$$\alpha_2 u(0)+\beta_2 u(1)+\gamma_2 u'(0)+\delta_2 u'(1)=\int_0^1 h(s,\ u)ds \qquad (3)$$

where $1<q\le 2$, $0<\sigma<q\le 2$ and $\begin{vmatrix}\alpha_1+\beta_1 & \beta_1+\gamma_1+\delta_1\\ \alpha_2+\beta_2 & \beta_2+\gamma_2+\delta_2\end{vmatrix}=: p\ne 0$

Remark 1.It was supposed in [1] that $\alpha,\ \delta>0,\ \beta,\ \gamma\ge 0$ or $\alpha,\ \delta\ge 0,\ \beta,\ \gamma>0$. They excluded the case of $u(0)=\int_0^1 g(s,u)ds,\quad u(1)=\int_0^1 h(s,u)ds$.

But we discuss this case too. In fact, when

$\alpha_1=1,\ \ \gamma_1=\delta=\beta_1=0,\ \gamma_2=\delta_2=\alpha_2=0,\ \ \beta_2=1$, we can obviously get

$\begin{vmatrix}\alpha_1 & 0\\ \beta_2 & \beta_2\end{vmatrix}=1\ne 0$

Remark 2. The boundary condition (2) in this paper includes the boundary conditions in [1-9] as special cases. Without a loss of generality, we assume that $1<\sigma<q\le 2$.

Lemma 1.[10] If $\alpha>0$ and $f(x)\in L_p(a,\ b)\ (1\le p\le\infty)$, then the following equality $(D^{\alpha}_{a+}I^{\alpha}_{a+}f)(x)=f(x)$ hold almost everywhere on $[a,\ b]$.

Lemma 2. [10] If $\alpha>0$ and $f\in I^{\alpha}_{a+}(L_p),\ \ (1\le p\le\infty)$, then $(I^{\alpha}_{a+}D^{\alpha}_{a+})f(x)=f(x)$.

Lemma 3. [9] Let $\alpha>0$. Assume that $(f_k)_{k=1}^{\infty}$ is a uniformly convergent sequence

of continuous functions on $[a,\ b]$. Then we may interchange the fractional integral operator and the limit process, i.e. $(I^{\alpha}_a \lim\limits_{k\to\infty} f_k)(x)=(\lim\limits_{k\to\infty} I^{\alpha}_a f_k)(x)$.

In particular, the sequence of functions $(I^{\alpha}_a f_k)_{k=1}^{\infty}$ is uniformly convergent.

Denote as follows:

$$W^{\sigma}[0,\ 1]:=\{u\,|\,u\in C^{n-1}[0,\ 1],\ \ {}^{c}D^{\sigma}_{0+}u\in C[0,\ 1],\ u(x)-\sum_{k=0}^{n-1}\frac{u^{(k)}(0)}{k!}x^k\in I^{\sigma}_{0+}(C[0,\ 1])\}, \qquad (4)$$

$$n-1<\sigma\le n$$

Define the norm in $W^{\sigma}[0,\ 1]$ by

$$\forall u\in W^{\sigma}[0,\ 1],\ \ \|u\|_{W^{\sigma}[0,\ 1]}:=\sum_{k=0}^{n-1}\left\|u^{(k)}\right\|_{C[0,\ 1]}+\left\|{}^{c}D^{\sigma}_{0+}u\right\|_{C[0,\ 1]} \qquad (5)$$

Theorem 1. $\left(W^{\sigma}[0,\ 1],\ \|\cdot\|_{W^{\sigma}[0,\ 1]}\right)$ is a Banach space.

Proof. If $\sigma\in N$, since $W^{\sigma}[0,\ 1]=C^{\sigma}[0,\ 1]$, we'll omit it.

Consider the case where $\sigma\notin N$.

First we prove when $n=1$, i.e. $0<\sigma<1$ in (4).

By (4), (5), we can see that

$$W^{\sigma}[0,\ 1]:=\{u|u\in C^{0}[0,\ 1]=C[0,\ 1],\ \ {}^{c}D_{0+}^{\sigma}u\in C[0,\ 1],\ u(x)-u(0)\in I_{0+}^{\sigma}(C[0,\ 1])\},$$

$$0<\sigma<1$$

$$\|u\|_{W^{\sigma}[0,\ 1]}=\|u\|_{C[0,\ 1]}+\left\|{}^{c}D_{0+}^{\sigma}u\right\|_{C[0,\ 1]}$$

Then we can prove the compactness of $(W^{\sigma}[0,\ 1],\ \|\cdot\|_{W^{\sigma}[0,\ 1]}),\ (0<\sigma<1)$.

In fact, suppose that $\{u_n\}$ is a Cauchy sequence in $W^{\sigma}[0,\ 1]\,(0<\sigma<1)$, then $\{u_n\}$ is a Cauchy sequence in $C[0,\ 1]$ and $\left\{{}^{c}D_{0+}^{\sigma}u_n\right\}$ is also a Cauchy sequence in $C[0,\ 1]$

From the compactness of $C[0,\ 1]$, we can get

$$\exists u_*\in C[0,\ 1];\ \ u_n\xrightarrow{C[0,\ 1]}u_*\ ,\ (n\to\infty),\exists v_*\in C[0,\ 1];\quad {}^{c}D_{0+}^{\sigma}u_n\xrightarrow{C[0,\ 1]}v_*\ ,\ (n\to\infty)$$

Then we can obtain $u_*\in W^{\sigma}[0,\ 1](0<\sigma<1)$. In fact,

since $\left\{{}^{c}D_{0+}^{\sigma}u_n\right\}$ uniformly converges to v_* in $C[0,\ 1]$, by using Lemma 3,

$\left\{I_{0+}^{\sigma}\,{}^{c}D_{0+}^{\sigma}u_n\right\}$ also uniformly converges to $I_{0+}^{\sigma}v_*$ in $C[0,\ 1]$.

But since $I_{0+}^{\sigma}\,{}^{c}D_{0+}^{\sigma}u_n=I_{0+}^{\sigma}D_{0+}^{\sigma}[u_n(x)-u_n(0)]$ and $u_n\in W^{\sigma}[0,\ 1]$, we have

$$u_n(x)-u(0)\in I_{0+}^{\sigma}(C[0,\ 1]).$$

By using Lemma 2, we can see that $I_{0+}^{\sigma}\,{}^{c}D_{0+}^{\sigma}u_n=u_n(x)-u_n(0)$ and $u_n(x)-u_n(0)$ uniformly converges to $I_{0+}^{\sigma}v_*$. Thus we can get $u_*(x)-u_*(0)=I_{0+}^{\sigma}v_*$

Since $v_*\in C[0,\ 1]$, we obtain $u_*(x)-u_*(0)\in I_{0+}^{\sigma}(C[0,\ 1])$.

On the other hand, since $u_*(x)-u_*(0)=I_{0+}^{\sigma}v_*$, $D_{0+}^{\sigma}[u_*(x)-u_*(0)]$ exists and from Lemma 1, we have ${}^{c}D_{0+}^{\sigma}u_*(x)=D_{0+}^{\sigma}[u_*(x)-u_*(0)]=D_{0+}^{\sigma}I_{0+}^{\sigma}v_*=v_*$. That is, ${}^{c}D_{0+}^{\sigma}u_*\in C[0,\ 1]$.

Therefore we can prove $\|u_n-u_*\|_{W^{\sigma}[0,\ 1]}\to 0\ \ (n\to\infty)$

From this, we can conclude that $(W^{\sigma}[0,\ 1],\ \|\cdot\|_{W^{\sigma}[0,\ 1]}),\ (0<\sigma<1)$ is a Banach space.

Next, we prove when $n=2$, i.e. $1<\sigma\le 2$ in (4).

By (4), (5) we can see that

$$W^{\sigma}[0,\ 1]:=\{u|u\in C^{1}[0,\ 1],\ \ {}^{c}D_{0+}^{\sigma}u\in C[0,\ 1],\ u(x)-u(0)-u'(0)x\in I_{0+}^{\sigma}(C[0,\ 1]),1<\delta<2\},$$

$$\|u\|_{W^{\sigma}[0,\ 1]}=\|u\|_{C[0,\ 1]}+\|u'\|_{C[0,\ 1]}+\left\|{}^{c}D_{0+}^{\sigma}u\right\|_{C[0,\ 1]}.$$

Then $(W^{\sigma}[0,\ 1],\ \|\cdot\|_{W^{\sigma}[0,\ 1]})\,(1<\sigma<2)$ is a compact space.

In fact, suppose that $\{u_n\}$ is a fundamental sequence in $W^{\sigma}[0,\ 1]\,(1<\sigma<2)$.

Then we have

$$\exists u_*\in C[0,\ 1];\ u_n\Rightarrow u_*\ ,\ \exists v_*\in C[0,\ 1];\ u_n'\Rightarrow v_*,\ \exists w_*\in C[0,\ 1];\ {}^{c}D_{0+}^{\sigma}u_n\Rightarrow w_*.$$

Then we can prove that $u_* \in W^{\sigma}[0,\ 1](1<\sigma<2)$. The reason for that is as follows.

Since u_n' uniformly converges to v_*, $\int_0^x u_n'(t)dt$ uniformly converges to $\int_0^x v_*(t)dt$.

So we can get $u_*(x)-u_*(0)=\int_0^x v_*(t)dt$.

Since $v_* \in C[0,\ 1]$, we have $u_* \in C^1[0,\ 1],\ u_*'(x)=v_*(x)$.

On the other hand, since $\left\{{}^cD_{0+}^{\sigma}u_n\right\}$ uniformly converges to w_*, $\left\{I_{0+}^{\sigma}\,{}^cD_{0+}^{\sigma}u_n\right\}$ also uniformly converges to $I_{0+}^{\sigma}w_*$.

But we can easily see that

$$I_{0+}^{\sigma}\,{}^cD_{0+}^{\sigma}u_n = I_{0+}^{\sigma}D_{0+}^{\sigma}[u_n(x)-u_n(0)-u_n'(0)]$$

$$u_n \in W^{\sigma}[0,\ 1](1<\sigma<2).$$

And since $u_n(x)-u_n(0)-u_n'(0)x \in I_{0+}^{\sigma}(C[0,\ 1])$, $I_{0+}^{\sigma}\,{}^cD_{0+}^{\sigma}u_n = u_n(x)-u_n(0)-u_n'(0)x$ uniformly converges to $u_*(x)-u_*(0)-v_*(0)x = u_*(x)-u_*(0)-u_*'(0)x$.

Therefore we have $u_*(x)-u_*(0)-u_*'(0)x = I_{0+}^{\sigma}w_*$.

On the other hand, we can see that ${}^cD_{0+}^{\sigma}u_*(x) = D_{0+}^{\sigma}[u_*(x)-u_*(0)-u_*'(0)x] = D_{0+}^{\sigma}I_{0+}^{\sigma}w_* = w_*$.

Thus we conclude that

${}^cD_{0+}^{\sigma}u_*(x)\in C[0,\ 1]$, $u_* \in W^{\sigma}[0,\ 1](1<\sigma<2)$, $\|u_n-u_*\|_{W^{\sigma}[0,\ 1]} \to 0\ \ (n\to\infty)$.

That is, $\left(W^{\sigma}[0,\ 1],\ \|\cdot\|_{W^{\sigma}}\right)(1<\sigma<2)$ is a Banach space.

By using induction, we can prove that for any $n\in N$, when $n-1<\sigma<n$, $(W^{\sigma}[0,\ 1],\ \|\cdot\|_{W^{\sigma}})$ is a Banach space.

Lemma 4. Suppose that $1<q\le 2$ and $y, \varphi,\ \psi \in C[0,\ 1]$.

Then the solution $u\in W^{q}[0,\ 1](1<q\le 2)$ of the integral boundary value problem

$${}^cD_{0+}^{q}u(t)=y(t),\quad t\in[0,\ 1] \tag{6}$$

$$\alpha_1 u(0)+\beta_1 u(1)+\gamma_1 u'(0)+\delta_1 u'(1)=\int_0^1\varphi(s)ds, \tag{7}$$

$$\alpha_2 u(0)+\beta_2 u(1)+\gamma_2 u'(0)+\delta_2 u'(1)=\int_0^1\psi(s)ds \tag{8}$$

is given by

$$u(t)=I_0^q y(t)+\frac{(\beta\beta_2-\delta\beta_1+t\gamma\beta_1-t\alpha\beta_2)}{p}I_0^q y(t)\,|_{t=1}+\frac{(\beta\delta_2-\delta\delta_1+t\gamma\delta_1-t\alpha\delta_2)}{p}I_0^{q-1}y(t)\,|_{t=1}+A(t),$$

where

$$A(t):=\frac{(\delta-t\gamma)}{p}\int_0^1\varphi(s)ds-\frac{(\beta-t\alpha)}{p}\int_0^1\psi(s)ds,$$

$$\alpha=\alpha_1+\beta_1,\quad \beta=\beta_1+\gamma_1+\delta_1,\quad \gamma=\alpha_2+\beta_2,\quad \delta=\beta_2+\gamma_2+\delta_2.$$

Theorem 2. Suppose that $f\in C([0,\ 1]\times R^2)$.

Then $u\in W^q[0,\ 1](1<q\le 2)$ is a solution of the boundary value problem (1)-(3) if and only if $u\in W^q[0,\ 1](1<q\le 2)$ is a solution of the fractional integro-differential equation:

$$\begin{aligned}u(t)&=I_0^q f(t,\ u(t),\ {}^cD_{0+}^{\sigma}u(t))+\frac{(\beta\beta_2-\delta\beta_1+t\gamma\beta_1-t\alpha\beta_2)}{p}I_0^q f(t,\ u(t),\ {}^cD_{0+}^{\sigma}u(t))|_{t=1}+\\&+\frac{(\beta\delta_2-\delta\delta_1+t\gamma\delta_1-t\alpha\delta_2)}{p}I_0^{q-1}f(t,\ u(t),\ {}^cD_{0+}^{\sigma}u(t))|_{t=1}+\\&+\frac{(\delta-t\gamma)}{p}\int_0^1 g(s,u)ds-\frac{(\beta-t\alpha)}{p}\int_0^1 h(s,u)ds.\end{aligned}\tag{9}$$

Let study the existence and uniqueness of the solution by using the Banach contraction principle.

Suppose that the following hypotheses hold.

(H1)

$$\exists k_0>0:\forall t,\quad \forall(u,\ v),\ (\bar{u},\ \bar{v})\in R^2,\ |f(t,\ u,\ v)-f(t,\ \bar{u},\ \bar{v})|\le k_0(|u-\bar{u}|+|v-\bar{v}|)$$

(H2) $\exists k_1>0:\ \forall t,\quad \forall u,\ \bar{u}\in R,\ |g(t,\ u)-g(t,\ \bar{u})|\le k_1|u-\bar{u}|$

(H3) $\exists k_2>0:\forall t,\quad \forall u,\ \bar{u}\in R,\ |h(t,\ u)-h(t,\ \bar{u})|\le k_2|u-\bar{u}|$

Denote as follows.

$$d_1(t):=\frac{(\beta\beta_2-\delta\beta_1+t\gamma\beta_1-t\alpha\beta_2)}{p},\quad d_2(t):=\frac{(\beta\delta_2-\delta\delta_1+t\gamma\delta_1-t\alpha\delta_2)}{p},$$

$$d_3(t):=\frac{(\delta-t\gamma)}{p},\quad d_4(t):=-\frac{(\beta-t\alpha)}{p}.$$

Then we can rewrite (9) as

$$\begin{aligned}T(u)(t)&=I_0^q f(t,\ u(t),\ {}^cD_{0+}^{\sigma}u(t))+d_1(t)I_0^q f(t,\ u(t),\ {}^cD_{0+}^{\sigma}u(t))|_{t=1}+\\&+d_2(t)I_0^{q-1}f(t,\ u(t),\ {}^cD_{0+}^{\sigma}u(t))|_{t=1}+d_3(t)\int_0^1 g(s,\ u)ds+d_4(t)\int_0^1 h(s,\ u)ds.\end{aligned}\tag{10}$$

For $u,\ \bar{u}\in W^{\sigma}[0,\ 1](1<\sigma<q\le 2)$, put

$$\hat{d}_1:=\max_i\max_t|d_i(t)|,\ \hat{d}_2:=\max_i\max_t|d_i'(t)|,$$

$$\hat{d}_3:=\max_i\max_t|{}^cD_{0+}^{\sigma}d_i(t)|,\quad i=1,\ 2,\ 3,\ 4$$

$\omega_0 := \max\{(\frac{1+\hat{d}_1}{\Gamma(q+1)}+\frac{\hat{d}_1}{\Gamma(q)})k_0,\ \ (\hat{d}_1\cdot k_1+\hat{d}_1\cdot k_2)\}$,

$\omega_1 := \max\{(\frac{\hat{d}_2}{\Gamma(q+1)}+\frac{1+\hat{d}_2}{\Gamma(q)})k_0,\ (\hat{d}_2\cdot k_1+\hat{d}_2\cdot k_2)\}$,

$\omega_2 := \max\{(\frac{1}{\Gamma(q-\sigma+1)}+\frac{\hat{d}_3}{\Gamma(q+1)}+\frac{\hat{d}_3}{\Gamma(q)})k_0,\ (\hat{d}_3\cdot k_1+\hat{d}_3\cdot k_2)\}$

Theorem 3. If $\omega_0+\omega_1+\omega_2<1$, then the solution of problem (1)-(3) exists uniquely in $W^{\sigma}[0,\ 1](1<\sigma<q\le 2)$.

Proof. In order to use Banach's fixed point theorem, prove the following.

1) If $u(t)\in W^{\sigma}[0,\ 1](1<\sigma<q\le 2)$, $(Tu)(t)\in W^{\sigma}[0,\ 1](1<\sigma<q\le 2)$

2) For $u,\ \bar{u}\in W^{\sigma}[0,\ 1](1<\sigma<q\le 2)$, the following inequality holds:

$$\| T(u)-T(\bar{u})\|_{W_2}\le(\omega_0+\omega_1+\omega_2)\| u-\bar{u}\|_{W^{\sigma}[0,\ 1]},\ (1<\sigma<q\le 2)$$

Prove the first result. In (10) ,2nd, 3rd, 4th terms are linear , so $(Tu)(t)$ can be rewritten as

$$(Tu)(t)=I_0^q f(t,\ u(t),\ {}^cD_{0+}^{\sigma}u(t))+(At+B) \qquad (11)$$

It is sufficient to show that $(Tu)(t)\in W^{\sigma}[0,\ 1](1<\sigma<q\le 2)$ to prove that

。 $(Tu)(t)\in C^1[0,\ 1]$,

。 ${}^cD_{0+}^{\sigma}(Tu)(t)\in C[0,\ 1]$,

。 $(Tu)(t)-(Tu)(0)-(Tu)'(0)t\in I^{\sigma}(C[0,\ 1])$.

Since (11) and $q-1>0$, we can get $(Tu)'(t)=\left(I_0^{q-1}f(t,\ u(t),\ {}^cD_{0+}^{\sigma}u(t))+A\right)\in C[0,\ 1]$

So we have $(Tu)(t)\in C^1[0,\ 1]$.

On the other hand by using the definition and properties of Caputo fractional derivative and $1<\sigma<q<2$, we obtain

$$ {}^cD_{0+}^{\sigma}(Tu)(t)={}^cD_{0+}^{\sigma}I_0^q f(t,\ u(t),\ {}^cD_{0+}^{\sigma}u(t))+{}^cD_{0+}^{\sigma}(At+B)=$$
$$=D_{0+}^{\sigma}I_0^q f(t,\ u(t),\ {}^cD_{0+}^{\sigma}u(t))=I_0^{q-\sigma}f(t,\ u(t),\ {}^cD_{0+}^{\sigma}u(t))\in C[0,\ 1]$$

Next prove that $(Tu)(t)-(Tu)(0)-(Tu)'(0)\,t\in I^{\sigma}(C[0,\ 1])$.

Since $(Tu)(t)-(Tu)(0)-(Tu)'(0)t=I_0^q f(t,\ u(t),\ {}^cD_{0+}^{\sigma}u(t))$,

we should prove the existence of $\varphi(t)\in C[0,\ 1]$ that satisfies

$$I_0^q f(t,\ u(t),\ {}^cD_{0+}^{\sigma}u(t))=I_0^{\sigma}\varphi(t). \qquad (12)$$

Since $1<\sigma<q<2$, by using the semi-group property of fractional integral we have

$$I_0^q f(t,\ u(t),\ {}^cD_{0+}^{\sigma}u(t))=I_0^{\sigma}I_0^{q-\sigma}f(t,\ u(t),\ {}^cD_{0+}^{\sigma}u(t)).$$

$q-\sigma>0$ shows that $I_0^{q-\sigma}f(t,\ u(t),\ {}^cD_{0+}^{\sigma}u(t))\in C[0,\ 1]$, so we can prove the existence of $\varphi(t)\in C[0,\ 1]$ that satisfies (12) by putting $\varphi(t):=I_0^{q-\sigma} f(t,\ u(t),\ {}^cD_{0+}^{\sigma}u(t))$.

Therefore we can get $(Tu)(t)-(Tu)(0)-(Tu)'(0)\,t\in I^{\sigma}(C[0,\ 1])$, and $(Tu)(t)\in W^{\sigma}[0,\ 1](1<\sigma<q\le 2)$.

Next prove that for $u,\ \bar{u}\in W^{\sigma}[0,\ 1](1<\sigma<q\le 2)$, the following inequality holds:

$$\| T(u)-T(\bar{u})\|_{W^{\sigma}[0,\ 1]}\le(\omega_0+\omega_1+\omega_2)\| u-\bar{u}\|_{W^{\sigma}[0,\ 1]},\ (1<\sigma<q\le 2).$$

For any $u,\ \bar{u}\in W^{\sigma}[0,\ 1](1<\sigma<q\le 2)$, we can see that

$$| T(u)(t)-T(\bar{u})(t)|\le k_0(\| u-\bar{u}\|_{\infty}+\|{}^cD_{0+}^{\sigma}u-{}^cD_{0+}^{\sigma}\bar{u}\|_{\infty})\cdot$$
$$\cdot(I_0^q1+\hat{d}_1\cdot I_0^q1|_{t=1}+\hat{d}_1\cdot I_0^{q-1}1|_{t=1})+(\hat{d}_1\cdot k_1+\hat{d}_1\cdot k_2)\| u-\bar{u}\|_{\infty}.$$

Since $(I_0^q1+\hat{d}_1\cdot I_0^q1|_{t=1}+\hat{d}_1\cdot I_0^{q-1}1|_{t=1})\le\dfrac{1+\hat{d}_1}{\Gamma(q+1)}+\dfrac{\hat{d}_1}{\Gamma(q)}$ we have

$$| T(u)(t)-T(\bar{u})(t)|\le(\frac{1+\hat{d}_1}{\Gamma(q+1)}+\frac{\hat{d}_1}{\Gamma(q)})k_0\| u-\bar{u}\|_{W^{\sigma}[0,\ 1]}+(\hat{d}_1\cdot k_1+\hat{d}_1\cdot k_2)\| u-\bar{u}\|_{W^{\sigma}[0,\ 1]}$$
$$\le\omega_0\| u-\bar{u}\|_{W^{\sigma}[0,\ 1]},\ (1<\sigma<q\le 2).$$

Therefore, $\| T(u)-T(\bar{u})\|_{\infty}\le\omega_0\| u-\bar{u}\|_{W^{\sigma}[0,\ 1]},\ (1<\sigma<q\le 2)$.

On the other hand, since

$$(T'u)(t)=I_0^{q-1}f(t,\ u(t),\ {}^cD_{0+}^{\sigma}u(t))++d_1'(t)\,I_0^q f(t,\ u(t),\ {}^cD_{0+}^{\sigma}u(t))|_{t=1}+$$
$$+d_2'(t)\,I_0^{q-1}f(t,\ u(t),\ {}^cD_{0+}^{\sigma}u(t))|_{t=1}+d_3'(t)\int_0^1 g(s,\ u)ds+d_4'(t)\int_0^1 h(s,\ u)ds$$

We can see that

$$|(T'u)(t)-(T'\bar{u})(t)|\le k_0(\| u-\bar{u}\|_{\infty}+\|{}^cD_{0+}^{\sigma}u-{}^cD_{0+}^{\sigma}\bar{u}\|_{\infty})(I_0^{q-1}1+$$
$$+\hat{d}_2\cdot I_0^q1|_{t=1}+\hat{d}_2\cdot I_0^{q-1}1|_{t=1})+(\hat{d}_2\cdot k_1+\hat{d}_2\cdot k_2)\| u-\bar{u}\|_{\infty}$$

that is, $\| T'(u)-T'(\bar{u})\|_{\infty}\le\omega_1\| u-\bar{u}\|_{W^{\sigma}[0,\ 1]},\ (1<\sigma<q\le 2)$.

We can also see that

$$|{}^cD_{0+}^{\sigma}(Tu)(t)-{}^cD_{0+}^{\sigma}(T\bar{u})(t)|\le k_0(\| u-\bar{u}\|_{\infty}+\|{}^cD_{0+}^{\sigma}u-{}^cD_{0+}^{\sigma}\bar{u}\|_{\infty})\cdot$$
$$\cdot(I_0^{q-\sigma}1+\hat{d}_3\cdot I_0^q1|_{t=1}+\hat{d}_3\cdot I_0^{q-1}1|_{t=1})+(\hat{d}_3\cdot k_1+\hat{d}_3\cdot k_2)\| u-\bar{u}\|_{\infty}$$

Therefore we have $|{}^cD_{0+}^{\sigma}(Tu)(t)-{}^cD_{0+}^{\sigma}(T\bar{u})(t)|\le\omega_2\| u-\bar{u}\|_{W^{\sigma}[0,\ 1]}$ which implies that

$$\|{}^cD_{0+}^{\sigma}T(u)(t)-{}^cD_{0+}^{\sigma}T(\bar{u})(t)\|_{\infty}\le\omega_2\| u-\bar{u}\|_{W^{\sigma}[0,\ 1]},\ (1<\sigma<q\le 2)$$

that is, $\| T(u)-T(\bar{u})\|_{W^{\sigma}[0,\ 1]}\le(\omega_0+\omega_1+\omega_2)\| u-\bar{u}\|_{W^{\sigma}[0,\ 1]},\ (1<\sigma<q\le 2)$.

From the assumption that $\omega_0+\omega_1+\omega_2<1$, using the Banach fixed point theorem, the solution of problem (1)-(3) exists uniquely in $W^{\sigma}[0,\ 1](1<\sigma<q\le 2)$.

References

[1] R. A. Khan, M. U. Rehman, J. Henderson, Existence and uniqueness of solutions for nonlinear fractional differential equations with integral boundary conditions, Fractional Differential Calculus, 1(1) (2011), 29–43

[2] A. Guezane-Lakoud, R. Khaldi, Solvability of a fractional boundary value problem with fractional integral condition, *Nonlinear Anal.* **75** (2012), 2692–2700

[3] B. Ahmad, J. J. Nieto, Existence results for nonlinear boundary value problems of fractional integrodifferential equations with integral boundary conditions, Boundary Value Problems, Volume 2009, Article ID 708576, 11 pages

[4] C. Zhu, X. Zhang, Z. Wu, Solvability for a coupled system of fractional differential equations with integral boundary conditions, Taiwanese Journal of Mathematics, 17(6) (2013) 2039–2054

[5] X. Liu, M. Jia, B. Wu, Existence and uniqueness of solution for fractional differential equations with integral boundary conditions, Electronic Journal of Qualitative Theory of Differential Equations 69(2009), 1–10

[6] B. Ahmad, J. J. Nieto, Existence results for nonlinear boundary value problems of fractional integrodifferential equations with integral boundary conditions, Boundary Value Problems,Volume 2009, Article ID 708576, 11 pages

[7] M. Benchohra, F. Ouaar, Existence results for nonlinear fractional differential equations with integral boundary conditions, Bulletin of Mathematical Analysis and Applications 2(4) (2010), 7–15

[8] B. Ahmad, J. J. Nieto, A. Alsaedi, Existence and uniqueness of solutions for nonlinear fractional differential equations with non-separated type integral boundary conditions, Acta Mathematica Scientia 31B(6)(2011) 2 122–2 130

[9] K. Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag, Berlin, 2010

[10] A. A. Kilbas , H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, 2006

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