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Some Properties of the Projective-Conformal Semi-Symmetric Connection in a Riemannian Manifold

Ho Tal Yun, Pak Pom I

In [1] a conformal semi-symmetric connection was defined as a connection that is conformally equivalent to Levi-Civita connection and discovered its properties, and based on it, proved that there exists a projective conformal semi-symmetric connection in a Riemannian manifold. In [5] it was discovered that the projective conformal symmetric connection is only the Levi-Civita connection itself. In [2] the projective conformal transformation and its invariant were considered in a statistical manifold. And in [6] the curvature copy problem between symmetric connections was studied in a Riemannian manifold. In [7] it was pointed out that there is no notion of minimum path length associated with the geodesics of any non-metric connection in a Riemannian manifold.

On the basis of the preceding study we considered properties of the projective-conformal semi-symmetric connection, namely the one form of non-metric connection, and proved that geodesic is minimum path length associated with geodesics of the connection in a Riemannian manifold.

In a Riemannian manifold the conformal semi-symmetric connection $\overline{\nabla}$ is a connection satisfying the equation

$$g_{ij} \to \overline{g}_{ij} = e^{-2\sigma} g_{ij} \tag{1}$$

for a 1-form π and a conformal transformation of metric

$$\overline{\nabla}_k \overline{g}_{ij} = 0,\ \ \overline{T}_{ij}{}^k = \varphi_j \delta_i^k - \varphi_i \delta_j{}^k \tag{2}$$

where g_{ij} is the Riemannian metric and $\sigma(x) \in C^{\infty}(M)$ and $\overline{T}_{ij}^k$ is the torsion tensor component of the connection $\overline{\nabla}$ and φ_i is a component of the 1-form π. The connection coefficient $\overline{\Gamma}_{ij}^k$ of the conformal semi-symmetric connection $\overline{\nabla}$ is

$$\overline{\Gamma}_{ij}^k = \left\{ {k \atop ij} \right\} - \left(\sigma_i \delta_j{}^k + \sigma_j \delta_i^k - g_{ij} \delta^k \right) + \varphi_j \delta_i^k - g_{ij} \varphi^k \tag{3}$$

where $\left\{ {k \atop ij} \right\}$ is the Christofell symbol and $\sigma_i = \partial_i \sigma$.

In a Riemannian manifold the connection coefficient $\overset{p}{\Gamma}{}_{ij}^k$ of the projective semi-symmetric connection $\overset{p}{\nabla}$ is represented by

$$\overset{p}{\Gamma}{}_{ij}^k = \left\{ {k \atop ij} \right\} + \frac{1}{2} \left(\psi_i \delta_j{}^k + \psi_j \delta_i{}^k + \varphi_j \delta_i^k - \varphi_i \delta_j{}^k \right) \tag{4}$$

where ψ_i is a component of the 1-form [1, 4].

In a Riemannian manifold the projective-conformal semi-symmetric connection ∇ is the connection satisfying $\overline{\nabla} = \overset{p}{\nabla}$, and that is a semi-symmetric connection projectively and conformally equivalent to Levi-Civita connection $\overset{\circ}{\nabla}$. Therefore its connection coefficient is

$$\Gamma_{ij}^{k} = \left\{ {k \atop ij} \right\} - \sigma_i \delta_j^k \tag{5}$$

In this case from (3) and (4), $\varphi_i = \sigma_i$, $\psi_i = -\sigma_i$.

Remark. If $\pi = 0$, then $\varphi_i = \sigma_i = 0$. Hence from (5) the projective-conformal symmetric connection is the Levi-Civita connection itself [5].

1. Properties of the Projective Conformal Semi-symmetric Connection

Definition 1 A connection ∇ that is projectively and conformally equivalent to the Levi-Civita connection in a Riemannian manifold is called a projective conformal semi-symmetric connection.

By (5) the connection coefficient Γ_{ij}^k of the projective conformal semi-symmetric connection ∇ is $\Gamma_{ij}^{k} = \left\{ {k \atop ij} \right\} - \sigma_i \delta_j^k$ or $\Gamma_{ij}^{k} = \left\{ {k \atop ij} \right\} - \varphi_i \delta_j^k$.

Hence the projective conformal semi-symmetric connection ∇ is a semi-symmetric non-metric connection satisfying the equation

$$\nabla_k g_{ij} = 2\sigma_k g_{ij} \quad , T_{ij}^k = \sigma_j \delta_i^k - \sigma_i \delta_j^k \tag{6}$$

for the conformal transformation (2) of the metric.

Theorem 1. In a Riemannian manifold $(M,\ g)$ the projective conformal semi-symmetric connection ∇ has a curvature copy of the Levi-Civita connection

Proof Using (5), the curvature tensor $R_{ijk}{}^{l}$ of the projective conformal semi-symmetric connection ∇ is

$$R_{ijk}{}^{l} = K_{ijk}{}^{l} - \delta_k^l \left(\overset{\circ}{\nabla}_i \sigma_j - \overset{\circ}{\nabla}_j \sigma_i \right) \tag{7}$$

where $K_{ijk}{}^{l}$ is the curvature tensor of the Levi-Civita connection $\overset{\circ}{\nabla}$. By $\sigma_i = \partial_i \sigma$, $\overset{\circ}{\nabla}_i \sigma_j = \overset{\circ}{\nabla}_j \sigma_i$. Hence $R_{ijk}{}^{l} = K_{ijk}{}^{l}$.

Definition 2 When two different connections have the same connection components, we say that each connection is an equivalent of the other.

Theorem 2 In a Riemannian manifold $(M,\ g)$ the conformal semi-symmetric connection $\overline{\nabla}$ satisfying the equation

$$\overline{\nabla}_k \overline{g}_{ij} = 0 \ , \overline{T}_{ij}^k = \sigma_j \delta_i^k - \sigma_j \delta_i^k \tag{8}$$

for the conformal transformation (2) is an equivalent to the projective conformal semi-symmetric connection ∇ .

Proof The connection coefficient of the conformal semi-symmetric connection $\overline{\nabla}$ satisfying (8) is $\Gamma_{ij}^{k}=\left\{\begin{smallmatrix}\bar{k}\\ ij\end{smallmatrix}\right\}+\sigma_j\delta_i^k-g_{ij}\delta^k=\left\{\begin{smallmatrix}k\\ ij\end{smallmatrix}\right\}-\sigma_i\delta_j^k$.

In comparison with (5), $\overline{\nabla}$ is an equivalent to ∇.

Theorem 3 If the tangent vectors of the geodesics for the Levi-Civita connection $\overset{\circ}{\nabla}$ are $\overset{\circ}{X}$ and the tangent vectors of the geodesics for the projective conformal semi-symmetric connection ∇ are X, then

$$\| X \|=\| \overset{\circ}{X} \| \tag{9}$$

where $\|X\|^2=\bar{g}(X,\ X)$.

Proof The geodesics of the connection $\overset{\circ}{\nabla}$ are given in a local coordinate system by

$$\frac{d^2x^k}{ds^2}+\left\{\begin{matrix}k\\ ij\end{matrix}\right\}\frac{dx^i}{ds}\frac{dx^j}{ds}=0 \tag{10}$$

where s is an affine parameter. And the geodesics of the connection ∇ are given in a local coordinate system by

$$\frac{d^2x^k}{dt^2}+\Gamma_{ij}^k\frac{dx^i}{dt}\frac{dx^j}{dt}=0 \tag{11}$$

where t is an affine parameter. On one hand $\frac{d^2x^k}{dt}=\frac{dx^k}{ds}\frac{ds}{dt}$, $\frac{d^2x^k}{dt^2}=\frac{d^2x^k}{dx^2}\left(\frac{ds}{dt}\right)^2+\frac{dx^k}{ds}\frac{d^2s}{dt^2}$ substituting these expressions into (11)

$$\left(\frac{d^2x^k}{ds^2}+\left\{\begin{smallmatrix}k\\ ij\end{smallmatrix}\right\}\frac{dx^i}{ds}\frac{dx^j}{ds}\right)\left(\frac{ds}{dt}\right)^2+\frac{dx^k}{ds}\left(\frac{d^2s}{dt^2}-\frac{d\sigma}{dt}\frac{ds}{dt}\right)=0$$

where $\frac{d\sigma}{dt}=\sigma_i\frac{dx^i}{dt}$.

Using $\frac{dx^k}{ds}\neq 0$ and expression (10) $\frac{d^2s}{dt^2}-\frac{d\sigma}{dt}\frac{ds}{dt}=0$.

Hence $\ln\frac{ds}{dt}=\sigma+c$.

If $\sigma\big|_{t=0}$, $\left.\frac{ds}{dt}\right|_{t=0}=1$, then $c=0$. Consequently

$$\frac{ds}{dt}=e^{\sigma} \tag{12}$$

namely $s=\int_0^k e^{\sigma(x(t))}dt$

Hence by theorem 2 and expressions (1) and (2)

$$\| X \|^2=\bar{g}(X,\ X)=\bar{g}_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}=e^{-2\sigma}g_{ij}e^{\sigma}\frac{dx^i}{ds}e^{\sigma}\frac{dx^j}{ds}=g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}=\| \overset{\circ}{X} \|^2$$

2. The Variation of the Geodesics according to the Projective Conformal Semi-Symmetric Connection

Now we consider the variation of the geodesics according to the projective conformal semi-symmetric connection ∇. This becomes a variational problem of the geodesics according to the conformal semi-symmetric connection $\overline{\nabla}$ defined as expression (8) by theorem 2 in a Riemannian manifold.

Let $C:[\alpha, \beta]\to M$ be a differentiable path and $J=(-1, 1)$ in a Riemannian manifold. Let the differentiable mapping $V:[\alpha, \beta]\times J\to M$ be a variety satisfying $V(t, 0)=C(t)$ for $\forall t\in[\alpha, \beta], \ \forall\varepsilon\in J$. Let V be the proper variety satisfying $V(\alpha, \varepsilon)=C(\alpha), V(\beta, \varepsilon)=C(\beta)$.

The projective conformal semi-symmetric connection ∇ is the connection satisfying the equation

$$\nabla_Z g(X, Y)=2\pi(Z)g(X, Y), \ T(X, Y)=\pi(Y)X-\pi(X)Y \tag{13}$$

for $\forall X, Y, Z\in T(M)$, where $\pi(Z)$ is the true form. The conformal semi-symmetric connection $\overline{\nabla}$ equating with this connection is the connection satisfying the equation

$$\overline{\nabla}_Z\overline{g}(X, Y)=0, \ \overline{T}(X, Y)=\pi(Y)X-\pi(X)Y \tag{14}$$

where $\overline{g}(X, Y)=e^{-2\sigma}g(X, Y)$, and $d\sigma=\pi$.

In a Riemannian manifold the length of a variety of the path C is

$$L(\varepsilon)=\int_\alpha^\beta \| X(t, \varepsilon)\| dt \tag{15}$$

where $\| X(t, \varepsilon)\|=g(X, X)^{1/2}$ and $X(t, \varepsilon)$ is a tangent vector of variety V.

By theorem 3, if C is a geodesic according to $\overline{\nabla}$, then $\| X(t, \varepsilon)\|_{\varepsilon=0}=1$ and by expression (8)

$$Z<X, Y>=<\overline{\nabla}_Z X+<X, \overline{\nabla}_Z Y>$$

where $<X, Y>=\overline{g}(X, Y)$.

Theorem 4. Let $C:[\alpha, \beta]\to M$ be a geodesic according to the projective conformal semi-symmetric connection, $V:[\alpha, \beta]\times J\to M$ be the proper variety and $X, Y\in TV$ be

$$X=V_*D_2, \quad Y=V_*D_2 \ \left(D_1=\frac{\partial}{\partial t}, \ D_2=\frac{\partial}{\partial\varepsilon}\right).$$

If Y satisfies the conditions $<Y, X>\big|_{\varepsilon=0}=0$, $\nabla_{D_1}Y|_{\varepsilon=0}=0$, $\nabla_{D_2}Y|_{\varepsilon=0}=0$, $\nabla_{D_1}Y|_\alpha=Y_\beta=0$, then

$$L'(0)=0 \tag{16}$$

and

$$L''(0)=\int_\alpha^\beta \left\|d\sigma(X)Y+d\sigma(Y)X\right\|^2\Big|_{t=0} dt\geq 0 \tag{17}$$

Proof First we will consider $L'(0)$. By expression (15),

$$L'(0)=\int_\alpha^\beta D_2\|X\|\Big|_{\varepsilon=0}dt=\int_\alpha^\beta D_2<X,\ X>^{\frac{1}{2}}\Big|_{\varepsilon=0}dt=\int_\alpha^\beta \frac{<\overline{\nabla}_{D_2}X,\ X>}{\|X\|}\Big|_{\varepsilon=0}dt\,.$$

From $[D_1\ ,\ D_2]=0$ and expression (14)

$$T(X,\ Y)=\nabla_{D_1}Y-\nabla_{D_2}X=\pi(Y)X-\pi(X)Y=d\sigma(Y)X-d\sigma(X)Y$$

Hence from $\overline{\nabla}_{D_2}X=\overline{\nabla}_{D_1}Y-d\sigma(Y)X+d\sigma(X)Y$,

$$L'(0)=\int_\alpha^\beta \frac{<\overline{\nabla}_{D_1}Y,\ X>-d\sigma(Y)<X,\ X>+d\sigma(X)<Y,\ X>}{\|X\|}\Big|_{\varepsilon=0}dt\,. \tag{18}$$

By the assumption of the theorem, $\|X\|\big|_{\varepsilon=0}=1$, $\overline{\nabla}_{D_1}Y|_{\varepsilon=0}=0$ and $<Y,\ X>|_{\varepsilon=0}=0$. Thus

$$L'(0)=-\int_\alpha^\beta d\sigma(Y)|_{\varepsilon=0}\,dt=-\int_\alpha^\beta \frac{d\sigma(Y)}{dt}\Big|_{\varepsilon=0}dt=-\sigma(X)\Big|_\alpha^\beta=\sigma(Y|_\alpha)-\sigma(Y|_\beta)=0\,.$$

Hence (16) is proved.

Next we will consider $L''(0)$. Using expression (18)

$$L''(0)=\int_\alpha^\beta D_2\left(\frac{<\overline{\nabla}_{D_1}Y,\ X>-\pi(Y)<X,\ X>+\pi(X)<Y,\ X>}{\|X\|}\right)\Bigg|_{\varepsilon=0}dt=$$

$$=\int_\alpha^\beta\left[\frac{D_2(<\overline{\nabla}_{D_1}Y,\ X>-\pi(Y)<X,\ X>+\pi(X)<Y,\ X>)}{\|X\|}-\right. \tag{19}$$

$$\left.-\frac{(<\overline{\nabla}_{D_1}Y,\ X>-\pi(Y)<X,\ X>+\pi(X)<Y,\ X>)^2}{\|X\|^3}\right]\Bigg|_{\varepsilon=0}dt$$

From theorem 2,

$$D_2<\overline{\nabla}_{D_1}Y,\ X>=<\overline{\nabla}_{D_2}\overline{\nabla}_{D_1}Y,\ X>+<\overline{\nabla}_{D_1}Y,\ \overline{\nabla}_{D_2}X>,\quad \overline{R}(X,\ Y)Y=\overline{\nabla}_{D_1}\overline{\nabla}_{D_2}Y-\overline{\nabla}_{D_2}\overline{\nabla}_{D_1}Y$$

then $<\overline{\nabla}_{D_1}\overline{\nabla}_{D_2}Y,\ X>=D_1<\overline{\nabla}_{D_1}\overline{\nabla}_{D_2}Y,\ X>-<\overline{R}(Y,\ X)X,\ Y>$ and from $\overline{T}(X,\ Y)=\overline{\nabla}_{D_1}Y-\overline{\nabla}_{D_2}Y$

$$<\overline{\nabla}_{D_1}Y,\ \overline{\nabla}_{D_2}Y>=D_1<\overline{\nabla}_{D_1}Y+T(Y,\ X),\ Y>-<\overline{\nabla}^2_{D_1}Y+\overline{\nabla}_{D_1}\overline{T}(Y,\ X),\ Y>$$

then

$$D_2<\overline{\nabla}_{D_1}Y,\ X>=D_1(<\overline{\nabla}_{D_2}Y,\ X>+<\overline{\nabla}_{D_2}Y+\overline{T}(Y,\ X),Y>)-<\overline{\nabla}^2_{D_1}Y+\overline{\nabla}_{D_1}\overline{T}(Y,\ X)+\overline{R}(Y,\ X)X,\ Y>.$$

Using the assumption of the theorem,

$$\int_\alpha^\beta \frac{D_2<\overline{\nabla}_{D_1}Y,\ X>}{\|X\|}\Big|_{\varepsilon=0}dt=0 \tag{20}$$

And

$$D_2(\pi(Y)<X,\ X>)=\pi(\overline{\nabla}_{D_2}Y)<X,\ X>+2\pi(Y)(<\overline{\nabla}_{D_1}Y,\ X>-\pi(Y)<X,\ X>+\pi(X)<Y,\ X>)$$

Using the assumption of the theorem

$$D_2(\pi(Y)<X,\ X>)|_{\varepsilon=0}=-2\pi^2(Y)<X,\ X> \tag{21}$$

And

$$D_2(\pi(X) < Y,\ X >) = D_2(\pi(X)) < Y,\ X > + \pi(X) D_2 < Y,\ X >=$$
$$= \pi(\overline{\nabla}_{D_2} Y) < Y,\ X > + \pi(X)(< \overline{\nabla}_{D_2} Y\ ,\ X > + < Y,\ \overline{\nabla}_{D_2} Y >) =$$
$$= \pi(\overline{\nabla}_{D_2} Y) < Y,\ X > + \pi(X)(< \overline{\nabla}_{D_2} Y,\ X > + < \overline{\nabla}_{D_1} Y,\ Y > + \pi(X) < Y,\ Y > - \pi(X) < X,\ Y >)$$

Based on the assumption of the theorem

$$D_2(\pi(X) < Y,\ X >)|_{\varepsilon=0} = \pi^2(X) < Y,\ Y > \tag{22}$$

And by the assumption of the theorem

$$(< \overline{\nabla}_{D_2} Y,\ X > - \pi(Y) < X,\ X > + \pi(X) < Y,\ X >)^2 |_{\varepsilon=0} = \pi^2(Y) < X,\ X >|_{\varepsilon=0} \tag{23}$$

Substituting expressions (20), (21), (22) and (23) for (19), we obtain expression (17).

Theorem 4 shows that the geodesics according to the projective conformal semi-symmetric connection ∇ is a path of minimum length, if the conjugate point is not in its inner part.

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