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On the Stress State Analysis of Damage Materials with Voids and Elliptical Cracks

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Abstract We have considered mechanical modeling of damage materials including voids and 3-dimensional elliptical cracks based on a coupling damage model of void and crack.

We represented voids in material using the damage tensor based on concept of effective stress and modeled the elliptical cracks based on exact crack theory. According to this modeling, we determined characteristic quantity of stiffness in damage material and analyzed the stress state of damage material with voids and elliptical cracks.

Key words elliptical crack, void, damage

Introduction

Generally, some materials such as concrete and rock are the damage materials where exist various discontinuities such as micro-voids and cracks simultaneously.

The methods of continuum damage theory and micro mechanical damage theory were considered as the methods to evaluate the mechanical state of damage material.

The continuum damage mechanics is not related with physical background of damage and micro structural change inside of material, but represents the cracks, gaps and cavities as a concept of damage from phenomenological point of view and constructs the damage constitutive equation and evolution equation of materials by introducing damage factor in forms of scalar, vector or tensor and predicts the strain and fracture progress of material theoretically[3].

In reference [4] the void and crack damage parameter was represented as different damage parameters on damage material with voids, cracks, considered their relation with total damage parameter, and applied the relation to analyze mechanical state of fiber-reinforced solidification composite material with void and crack damage. On the other hand, unlike void damage, the micromechanical damage theory to solve damage problem on crack damage was proposed and many studies have been performed.

Micromechanical damage theory gives the comparatively clear mechanical background for the damage factor and the damage evolution law by including geometrical and mechanical characteristics of various micro-damage of material.

In order to estimate the effective elastic modulus of solid including microcracks by micromechanical damage theory, the equivalent effective medium method is used usually.

In this method the microcrack is considered to be in a equivalent elastic medium and assumption in which this method is effected is to recognize that external field around each microcrack is considered to be independent of correct position of microcrack.

There are several typical methods to estimate effective elastic characteristics of solid including microcrack. These are Taylor method or DCM(Dilute Concentration Method), SCM(Self-Consistent Method), the simplest method which neglects interaction between microdefects perfectly, DM(Differential Method), Mori-Tanaka method, GSCM(Generalized SCM), the Hashin-Shtrikman limit method which consider weak interaction between microdefects and the statistical micromechanics method which considers strong interaction between microdefects and so on[6]. Taylor method(or DCM) neglects interaction between microcrack, assuming that each microcrack is in elastic basis medium without damage and the force acting to microcrack equals to long distance stress field, and has sufficient accuracy under condition of infrequent microcrack.

SCM (Self-Consistent Method) was proposed to consider weak interaction between microcrack and represents damage in terms of a scalar variable, which is crack density and has comparative simple form and high accuracy.

In reference [7] they have determined compliance matrix of cracked material by quasi-micromechanical method (Mori-Tanaka method) on brittle solid with microcrack. In this method the strain of element is considered as the sum of matrix strain and variation of global average strain caused due to crack, which was represented by average displacement discontinuity of crack face.

But since mechanical states of void and crack are different from each other, the stress-strain state of damage material including voids and cracks can not be considered by only continuum damage theory and micromechanical damage theory exactly.

In order to solve this shortcoming, the coupling damage model of voids and cracks was proposed, where the void damage within material was represented by using damage tensor based on the concept of effective stress and the plane cracks by using crack theory.

On the basis of it the structural analysis method of plane problem was considered [1, 2].

In this paper, we determined the effective elastic stiffness characteristic quantities of damage material including voids and 3-dimensional elliptical crack on the basis of coupling damage model of voids and cracks and analyzed the stresses and displacements of damage material by using ANSYS on the basis of it.

1. The Coupling Damage Model of Damage Material including Voids and Elliptical Cracks

Let assume that size of an element is so enough big to involve the voids and cracks in it and is much smaller than structural size to represent the change of stress and strain in structure.

According to the coupling damage model of voids and cracks, strain energy U for the elastic material including voids and cracks is represented as follows[1].

$$U = \phi_{pj} C^0_{ipkl} \varepsilon_{kl} \varepsilon_{ij} / 2 - \Delta V_c (\bar{\sigma}_{ij}) \tag{1}$$

where $\phi_{pj} = \delta_{pj} - D_{pj}$ is damage effect tensor, δ_{pj} -Kronecker delta, D_{pj} -damage tensor, C^0_{ipkl} -elastic coefficient tensor of material without damage, ΔV_c -complementary energy due to crack, which is related with effective stress tensor $\bar{\sigma}_{ij}$ ($\sigma_{ij} = \phi_{pj} \bar{\sigma}_{ip}$) reflecting the effect of void damage on effective crack damage element obtained by removing the voids in the element(Fig. 1).

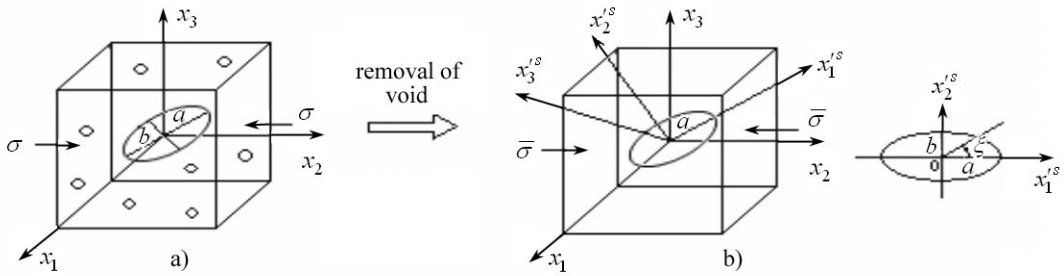

Fig. 1. Voids and elliptical crack damage model
a) damaged element, b) effective crack damage element

If standard coordinates system coincides with principal damage coordinates system, ϕ_{pj} is represented by principal value of damage tensor $\phi_i = 1 - D_i \quad (i = 1, 2, 3)$.

1.1. The coupling damage model of void and elliptical opened crack

The complementary energy due to crack for the 3-dimensional elliptical opened crack is expressed as follows [5].

$$\Delta V_c = \sum_{s=1}^{m} \frac{1-\nu^2}{EV} \int_A \left({K_{\mathrm{I}}^s}^2 + {K_{\mathrm{II}}^s}^2 + \frac{1}{1-\nu} {K_{\mathrm{III}}^s}^2 \right) dA \tag{2}$$

where s is the number of cracks, m -a number of cracks in the element, V -the volume of element, E, ν -elastic constants of material, K_{I}^s , K_{II}^s , K_{III}^s is I, II, III-type stress intensity coefficient of elliptical crack respectively and these are expressed by effective stress component $\bar{\sigma}^s_{33}, \bar{\sigma}^s_{31}, \bar{\sigma}^s_{32}$ on crack face in crack coordinates system as follows.

$$\begin{aligned}
K_{\mathrm{I}}^s &= \bar{\sigma}^s_{33} \frac{\sqrt{\pi b^s}}{E(k)} (1 - k^2 \cos^2 \xi)^{1/4}, \\
K_{\mathrm{II}}^s &= \left[\bar{\sigma}^s_{31} \frac{k' \cos\xi}{B(k,\ \nu)} + \bar{\sigma}^s_{32} \frac{\sin\xi}{C(k,\ \nu)} \right] \frac{\sqrt{\pi b^s}\, k^2}{(1 - k^2 \cos^2 \xi)^{1/4}}, \\
K_{\mathrm{III}}^s &= \left[\bar{\sigma}^s_{31} \frac{k' \cos\xi}{B(k,\ \nu)} - \bar{\sigma}^s_{32} \frac{\sin\xi}{C(k,\ \nu)} \right] \frac{\sqrt{\pi b^s}\, k^2}{(1 - k^2 \cos^2 \xi)^{1/4}}
\end{aligned} \tag{3}$$

where $k' = b^s / a^s$, a^s , b^s is long and short radius of s^{th} elliptical crack respectively, $k^2 = 1 - k'^2$, $B(k,\ \nu) = (k^2 - \nu) E(k) + \nu k'^2 K(k)$ and $C(k,\ \nu) = (k^2 + \nu k'^2) E(k) - \nu k'^2 K(k)$,

$K(k)=\int_0^{\pi/2}(1-k^2\sin^2\xi)^{-1/2}d\xi$, $E(k)=\int_0^{\pi/2}(1-k^2\sin^2\xi)^{1/2}d\xi$ is the perfect elliptical integral of the 1, 2 type respectively.

If we substitute expression (3) in (2) and integrate by considering that $dA=k'adad\xi$, then we obtain as follows.

$$\Delta V_c=\sum_{s=1}^{m}\frac{4\pi(a^s)^3k'^2}{3V}\frac{1-\nu^2}{E}\left[\frac{(\overline{\sigma}_{33}^s)^2}{E(k)}+\frac{k^2(\overline{\sigma}_{31}^s)^2}{B(k,\ \nu)}+\frac{k^2(\overline{\sigma}_{32}^s)^2}{C(k,\ \nu)}\right] \tag{4}$$

Stress-strain relation is as follows.

$$\overline{\sigma}_{ij}^s=\lambda\varepsilon_{ll}^s\delta_{ij}+2Ge_{ij}^s \tag{5}$$

where λ is Rame constant, e_{ij} -strain deviator component and G-shear elastic coefficient.

On the other hand if the direction cosines between the coordinate axes of the standard coordinates system $(x_1,\ x_2,\ x_3)$ and the crack coordinates system $(x_1'^s,\ x_2'^s,\ x_3'^s)$ are $d_{ij}^s=\cos(x_i,\ x_j'^s)\ \ (i,\ j=1,\ 2,\ 3)$, we have

$$\varepsilon_{ij}^s=d_{ki}^s\,d_{lj}^s\,\varepsilon_{kl}\ . \tag{6}$$

Therefore substituting expression (4)－(6) in (1), we obtain

$$\begin{aligned}U=&\frac{1}{2}\phi_{pj}C_{ipkl}^0\varepsilon_{kl}\varepsilon_{ij}-\sum_{s=1}^{m}\frac{4\pi(a^s)^3k'^2}{3V}\frac{1-\nu^2}{E}\{[(\lambda+2G)^2d_{i3}^s\,d_{k3}^s\,\varepsilon_{ik}d_{j3}^s\,d_{l3}^s\,\varepsilon_{jl}+\\&+\lambda^2\,d_{i1}^s\,d_{k1}^s\,\varepsilon_{ik}d_{j1}^s\,d_{l1}^s\,\varepsilon_{jl}+\lambda^2\,d_{i2}^s\,d_{k2}^s\,\varepsilon_{ik}d_{j2}^s\,d_{l2}^s\,\varepsilon_{jl}+2\lambda(\lambda+2G)d_{i3}^s\,d_{k3}^s\,\varepsilon_{ik}d_{j1}^s\,d_{l1}^s\,\varepsilon_{jl}+\\&+2\lambda(\lambda+2G)d_{i3}^s\,d_{k3}^s\,\varepsilon_{ik}d_{j2}^s\,d_{l2}^s\,\varepsilon_{jl}+2\lambda^2d_{i1}^s\,d_{k1}^s\,\varepsilon_{ik}d_{j2}^s\,d_{l2}^s\,\varepsilon_{jl}]/E(k)+\\&+k^2G^2d_{i3}^s\,d_{k1}^s\,\varepsilon_{ik}d_{j3}^s\,d_{l1}^s\,\varepsilon_{jl}/B(k,\ \nu)+k^2G^2d_{i3}^s\,d_{k2}^s\,\varepsilon_{ik}d_{j3}^s\,d_{l2}^s\,\varepsilon_{jl}/C(k,\nu)\}.\end{aligned} \tag{7}$$

Then by using Green's theorem, stress-strain relation is expressed as follows.

$$\sigma_{ij}=\frac{\partial U}{\partial\varepsilon_{ij}}=S_{ijkl}\varepsilon_{kl}\quad(i,\ j,\ k,\ l=1,\ 2,\ 3) \tag{8}$$

At this time elastic stiffness matrix of damage material is expressed as follows.

$$S_{ijkl}=S_{ijkl}'-\Delta S_{ijkl} \tag{9}$$

where $S_{ijkl}'=\phi_{pj}S_{ipkl}^0$ is elastic stiffness matrix reflecting the effect of void damage, ΔS_{ijkl} - elastic stiffness change matrix due to the cracks, therefore it is determined as follows.

$$\Delta S_{ijkl}=\sum_{s=1}^{m}[C_1L_{ij}^sL_{kl}^s+C_2M_{ij}^sM_{kl}^s+C_3N_{ij}^sN_{kl}^s] \tag{10}$$

$$\begin{cases}L_{ij}^s=\alpha_{ij}(\lambda d_{il}^s\,d_{jl}^s+2Gd_{i3}^sd_{j3}^s)\\M_{ij}^s=\dfrac{\alpha_{ij}G(d_{i3}^s\,d_{j1}^s+d_{i1}^sd_{j3}^s)}{2}\\N_{ij}^s=\dfrac{\alpha_{ij}G(d_{i3}^s\,d_{j2}^s+d_{i2}^sd_{j3}^s)}{2}\end{cases},\quad\alpha_{ij}=\begin{cases}1 & i=j\\2 & i\neq j\end{cases},\quad\begin{cases}C_1=\dfrac{8\pi a^3k'^2}{3V}\dfrac{1-\nu^2}{E}\dfrac{1}{E(k)}\\C_2=\dfrac{8\pi a^3k'^2}{3V}\dfrac{1-\nu^2}{E}\dfrac{k^2}{B(k,\ \nu)}\\C_3=\dfrac{8\pi a^3k'^2}{3V}\dfrac{1-\nu^2}{E}\dfrac{k^2}{C(k,\ \ \nu)}\end{cases}$$

1.2. The coupling damage model of voids and elliptical closed cracks

For stress σ'_{ij} in local (crack) coordinates system$(x_1'^s,\ x_2'^s,\ x_3'^s)$, in case of

$$|\sigma'_{31}|>\mu|\sigma'_{33}|,\quad |\sigma'_{32}|>\mu|\sigma'_{33}|,\quad \sigma'_{33}\le 0\ . \tag{11}$$

Friction sliding is happened between crack faces and the tangent stress components acting to crack faces at this time can be expressed as follows[2].

$$\sigma_{3j}^{c}=\sigma'_{3j}-\mu f|\sigma'_{33}|\quad (j=1,\ 2) \tag{12}$$

where μ is friction coefficient between crack faces and $f=\mathrm{sgn}(\sigma'_{ij})$.

At this time, the complementary energy due to elliptical closed crack is expressed as follows.

$$\Delta V_c=\sum_{s=1}^{m}\frac{1-\nu^2}{EV}\int_A\left({K_{\mathrm{II}}^{s}}^2+\frac{1}{1-\nu}{K_{\mathrm{III}}^{s}}^2\right)dA \tag{13}$$

where K_{II}^s, K_{III}^s are expressed by the effective stress component $\bar{\sigma}_{31}^{cs}$, $\bar{\sigma}_{32}^{cs}$ on crack face in crack coordinates system as follows.

$$K_{\mathrm{II}}^{s}=\left[\bar{\sigma}_{31}^{cs}\frac{k'\cos\xi}{B(k,\ \nu)}+\bar{\sigma}_{32}^{cs}\frac{\sin\xi}{C(k,\ \nu)}\right]\frac{\sqrt{\pi b^s}\,k^2}{(1-k^2\cos^2\xi)^{1/4}},$$
$$K_{\mathrm{III}}^{s}=\left[\bar{\sigma}_{31}^{cs}\frac{k'\cos\xi}{B(k,\ \nu)}-\bar{\sigma}_{32}^{cs}\frac{\sin\xi}{C(k,\ \nu)}\right]\frac{\sqrt{\pi b^s}\,k^2}{(1-k^2\cos^2\xi)^{1/4}} \tag{14}$$

If we integrate it, we obtain as follows.

$$\Delta V_c=\sum_{s=1}^{m}\frac{4\pi(a^s)^3k'^2}{3V}\frac{1-\nu^2}{E}\left[\frac{k^2(\bar{\sigma}_{31}^{cs})^2}{B(k,\ \nu)}+\frac{k^2(\bar{\sigma}_{32}^{cs})^2}{C(k,\ \nu)}\right] \tag{15}$$

On the other hand, from expression (12)

$$\bar{\sigma}_{3r}^{cs}=\bar{\sigma}_{3r}'^{s}+\mu f\bar{\sigma}_{33}'^{s}=\xi_{ij}^{s(r)}\bar{\sigma}_{ij}\quad (r=1,\ 2), \tag{16}$$

$$\xi_{ij}^{s(r)}=d_{3i}^{s}d_{rj}^{s}+\mu\, fd_{3i}^{s}d_{3j}^{s}\quad (r=1,\ 2). \tag{17}$$

Therefore if we substitute expression (15)—(17) in (1), we obtain as follows.

$$U=\frac{1}{2}\phi_{pj}C_{ipkl}^{0}\varepsilon_{kl}\varepsilon_{ij}-\sum_{s=1}^{m}\frac{4\pi(a^s)^3k'^2k^2}{3V}\frac{1-\nu^2}{E}\cdot$$
$$\cdot\{[\xi_{ij}^{s(1)}(\lambda\varepsilon_{ll}\delta_{ij}+2Ge_{ij})]^2/B(k,\ \nu)++[\xi_{ij}^{s(2)}(\lambda\varepsilon_{ll}\delta_{ij}+2Ge_{ij})]^2/C(k,\ \nu)\} \tag{18}$$

The elastic stiffness change matrix ΔS_{ijkl} due to closed crack is determined as follows.

$$\Delta S_{ijkl}=\sum_{s=1}^{m}[C_1M_{ij}^{s(1)}M_{kl}^{s(1)}+C_2M_{ij}^{s(2)}M_{kl}^{s(2)}] \tag{19}$$

$$M_{ij}^{s(r)}=\begin{cases}\lambda\xi_{nn}^{s(r)}\delta_{ij}+2G\xi_{ij}^{s(r)}, & i=j\\ G\xi_{ij}^{s(r)}, & i\ne j\end{cases}$$

$$C_1=\frac{8\pi a^3k'^2}{3V}\frac{1-\nu^2}{E}\frac{k^2}{B(k,\ \nu)},\quad C_2=\frac{8\pi a^3k'^2}{3V}\frac{1-\nu^2}{E}\frac{k^2}{C(k,\ \nu)}$$

2. The Numerical Simulations by ANSYS

We have done the finite element simulation by using the model SOLID64 of ANSYS 12.0 and anisotropy material model for the case that uniaxial distribution load with $P = 0.5\text{MPa}$ (tension for the opened crack, compression for the closed crack) is acted on the cube with $3\text{m} \times 3\text{m} \times 3\text{m}$ in total dimension and an elliptical crack is in the centre (Fig. 2). The differences from usual elastic analysis are that we select anisotropy elastic material as material model and give the values of elastic stiffness matrix $C_{ijkl} = C'_{ijkl} - \Delta C_{ijkl}$ of voids and cracks damage material corresponding to upper triangular matrix elements D_{11}, D_{12}, D_{13}, D_{14}, D_{15}, D_{16}, D_{22}, D_{23}, D_{24}, D_{25}, D_{26}, D_{33}, D_{34}, D_{35}, D_{36}, D_{44}, D_{45}, D_{46}, D_{55}, D_{56}, D_{66} of the 6×6 matrix in anisotropy elastic material setting dialog box. The material properties used in numerical simulation is $E = 58\text{GPa}$, $\nu = 0.23$, the principal damage coordinate system for void damage coincides with the standard system and the long radius is parallel to the plane $(x_2,\ x_3)$ of the standard system for elliptical crack with $k' = 0.8$ and the crack is orthogonal to the plane.

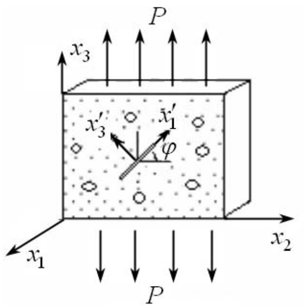

Fig. 2. Calculation model

In Fig. 3－6, we illustrated the variation feature of stress according to the void damage and crack parameters with the void damage measure $D = (D_1^2 + D_2^2 + D_3^2)^{1/2}$.

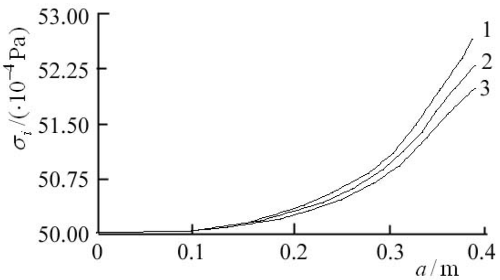

Fig. 3. Stress according to the length of opened crack ($\varphi = 45^\circ$)

1－ $D = 0$, 2－ $D = 0.1$, 3－ $D = 0.2$

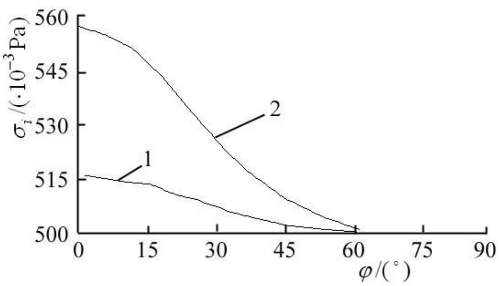

Fig. 4. Stress according to the direction of opened crack ($D = 0.1$)

1－ $a = 0.2$, 2－ $a = 0.3$

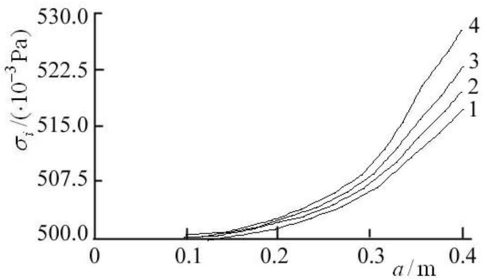

Fig. 5. Stress according to the length of closed crack ($\varphi = 45^\circ$)

1－ $D = 0$, 2－ $D = 0.1$, 3－ $D = 0.2$, 4－ $D = 0.3$

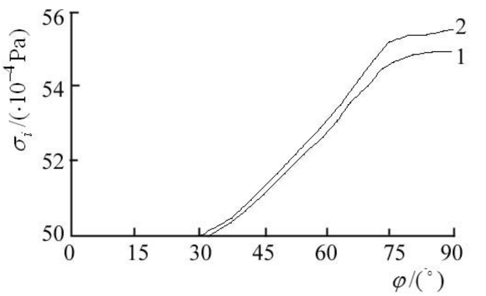

Fig. 6. Stress according to the direction of closed crack ($a = 0.2\text{m}$)

1－ $D = 0$, 2－ $D = 0.1$

As shown in figures, the maximum stress intensity σ_i increases with increasing of crack length and changes with the variation of crack direction φ within $0° \sim 90°$, while the stress intensity σ_i has maximal value on the edge of the cracks.

Conclusion

We determined the 3D elastic stiffness characteristic quantities of damage material containing the voids and elliptical crack damages by means of coupling damage model for the damage material with the voids and crack damages, and analyzed the stress state of the damaged structure using ANSYS. Stress increases with increasing of the voids and crack damages, and the direction of crack gives a significant influence on stress state.

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