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A Constancy of Curvature of Statistical Manifolds with $\alpha-$Connection

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In this paper, we study a constancy of curvature of a statistical manifold, which is importantly studied in the information geometry recently.

In [2], an $\alpha-$connection was introduced into a statistical manifold and a condition for a statistical manifold to be $\alpha-$flat was studied, in [3], a condition for a statistical manifold with an $\alpha-$connection to be conjugate symmetric, and in [1], a condition for a statistical manifold with an $\alpha-$connection to be conjugate Ricci- symmetric

It has been found that an equiaffine structure of a statistical manifold is concerned with a constancy of curvature of a statistical manifold and there has been consideration of properties of a Hessian manifold of a constant Hessian curvature, which is a special statistical manifold (see [3, 4]).

The studies in [4] imply a necessity for considering a constancy of curvature of a statistical manifold with an $\alpha-$connection. Such a study is also a continuation of the previous study [1－3].

Hence, we study conditions that a statistical manifold with an $\alpha-$connection is of constant curvature.

For any $\alpha\in\boldsymbol{R}$, a $\alpha-$connection is defined in a statistical manifold $(M,\ g,\ \nabla)$ by

$$\nabla^{(\alpha)}=\frac{1+\alpha}{2}\nabla+\frac{1-\alpha}{2}\nabla^{*},$$

where ∇ and ∇^{*} are dual connections on M. Then $(M,\ g,\ \nabla^{(\alpha)})$ is also a statistical manifold.

Let's considerate the condition that a statistical manifold $(M,\ g,\ \nabla^{(\alpha)})$ is of constant curvature for any $\alpha\in\boldsymbol{R}$.

Theorem 1 A statistical manifold $(M,\ g,\ \nabla^{(\alpha)})$ is of constant curvature for any $\alpha\in\boldsymbol{R}$ if there exist $\alpha_1,\ \alpha_2\in\boldsymbol{R}$ $(|\alpha_1|\neq|\alpha_2|)$ such that statistical manifolds $(M,\ g,\ \nabla^{(\alpha_1)})$ and $(M,\ g,\ \nabla^{(\alpha_2)})$ are of constant curvature.

Proof Without generality, we assume $\alpha_1\neq 0$. Then since $\nabla^{(\alpha)}=\frac{\alpha_1+\alpha}{2\alpha_1}\nabla^{(\alpha_1)}+\frac{\alpha_1-\alpha}{2\alpha_1}\nabla^{(-\alpha_1)}$ holds for any $\alpha\in\boldsymbol{R}$, the following relation

$$R^{(\alpha)}(X,\ Y)Z=\frac{\alpha_1+\alpha}{2\alpha_1}R^{(\alpha_1)}(X,\ Y)Z+\frac{\alpha_1-\alpha}{2\alpha_1}R^{(-\alpha_1)}(X,\ Y)Z+$$
$$+(\alpha_1^2-\alpha^2)[K(Y,\ K(Z,\ X))-K(X,\ K(Y,\ Z))]$$

holds, where $K(X,Y)=\nabla_X Y-\nabla_X^g Y$ is the difference tensor field of a statistical manifold.

From the relations

$$R^{(\alpha_1)}(X,\ Y)Z=k_1\{g(Y,\ Z)X-g(X,\ Z)Y\},\tag{1}$$

$$R^{(\alpha_2)}(X,\ Y)Z=k_2\{g(Y,\ Z)X-g(X,\ Z)Y\}\tag{2}$$

the relation

$$R^{(\alpha)}(X,\ Y)Z=\frac{k_2\alpha_1^2-k_1\alpha_2^2+(k_1-k_2)\alpha^2}{\alpha_1^2-\alpha_2^2}\{g(Y,\ Z)X-g(X,\ Z)Y\}$$

holds, that is, a statistical manifold $(M,\ \nabla^{(\alpha)},\ g)$ is of constant curvature $\dfrac{k_2\alpha_1^2-k_1\alpha_2^2+(k_1-k_2)\alpha^2}{\alpha_1^2-\alpha_2^2}$.

Example 1 Let $(M,\ g)$ be a family of normal distributions:

$$M=\left\{p(x,\ \theta)\ \middle|\ p(x,\ \theta)=\frac{1}{\sqrt{2\pi(\theta^2)^2}}\exp\left\{-\frac{1}{2(\theta^2)^2}(x-\theta^1)^2\right\}\right\},$$

$$g:=2(\theta^2)^{-2}\sum d\theta^i d\theta^i\quad x\in P\ ,\ \ \theta^1\in P\ ,\ \ \theta^2>0$$

We define an $\alpha-$connection by the following:

$$\nabla^{(\alpha)}_{\frac{\partial}{\partial\theta^1}}\frac{\partial}{\partial\theta^1}=(-1+2\alpha)(\theta^2)^{-1}\frac{\partial}{\partial\theta^2}\quad,\quad \nabla^{(\alpha)}_{\frac{\partial}{\partial\theta^2}}\frac{\partial}{\partial\theta^2}=(1+\alpha)(\theta^2)^{-1}\frac{\partial}{\partial\theta^2}$$

$$\nabla^{(\alpha)}_{\frac{\partial}{\partial\theta^1}}\frac{\partial}{\partial\theta^2}=\nabla^{(\alpha)}_{\frac{\partial}{\partial\theta^2}}\frac{\partial}{\partial\theta^1}=0\ .$$

Then the statistical manifold $(M,\ g,\ \nabla^{(0)})$ is of constant curvature $\left(-\dfrac{1}{2}\right)$,and the statistical manifold $(M,\ g,\ \nabla^{(1)})$ is of constant curvature 0. Hence for any $\alpha\in\boldsymbol{R}$, the statistical manifold $(M,\ g,\ \nabla^{(\alpha)})$ is of constant curvature $((\alpha^2-1)/2)-$.

Example 2 Let $(M,\ g)$ be a family of random walk distributions [1]:

$$M=\left\{p(x;\theta^1,\ \theta^2)\ \middle|\ p(x;\theta^1,\ \theta^2)=\sqrt{\frac{\theta^2}{2\pi x}}\exp\left\{-\frac{\theta^2 x}{2}+\frac{\theta^2}{\theta^1}-\frac{\theta^2}{2(\theta^1)^2 x}\right\},\ \ x,\ \mu,\ \lambda>0\right\},$$

$$g=\frac{\theta^2}{(\theta^1)^3}(d\theta^1)^2+\frac{1}{2(\theta^2)^2}(d\theta^2)^2.$$

We define an $\alpha-$connection by the following:

$$\nabla^{(\alpha)}_{\frac{\partial}{\partial\theta^1}}\frac{\partial}{\partial\theta^1}=\frac{-3(1+\alpha)}{2}(\theta^1)^{-1}\frac{\partial}{\partial\theta^1}\ +(-1+\alpha)(\theta^1)^{-3}(\theta^2)^2\frac{\partial}{\partial\theta^2}$$

$$\nabla^{(\alpha)}_{\frac{\partial}{\partial\theta^1}}\frac{\partial}{\partial\theta^2}=\nabla^{(\alpha)}_{\frac{\partial}{\partial\theta^2}}\frac{\partial}{\partial\theta^1}=-\frac{-1+\alpha}{2}(\theta^2)^{-1}\frac{\partial}{\partial\theta^1},\ \ \nabla^{(\alpha)}_{\frac{\partial}{\partial\theta^2}}\frac{\partial}{\partial\theta^2}=(-1+\alpha)(\theta^2)^{-1}\frac{\partial}{\partial\theta^2}\quad .$$

Then the statistical manifold $(M,\ g,\nabla^{(0)})$ is of constant curvature $\left(-\frac{1}{2}\right)$,and the statistical manifold $(M, g,\ \nabla^{(1)})$ constant curvature 0. Hence for any $\alpha\in\boldsymbol{R}$, the statistical manifold $(M, g,\ \nabla^{(\alpha)})$ is of constant curvature $\frac{\alpha^2-1}{2}$.

Theorem 1 implies the following facts.

Corollary 1 If there exist $\alpha_1,\ \alpha_2\in\boldsymbol{R}$ $(|\alpha_1|\neq|\alpha_2|)$ such that the statistical manifold $(M,\ g,\ \nabla^{(\alpha_1)})$ is of constant curvature k_1 and the statistical manifold $(M,\ g,\ \ \nabla^{(\alpha_2)})$ is of constant curvature k_2, and $k_1=k_2=k$ holds, then for $\alpha\in\boldsymbol{R}$, the statistical manifold $(M,\ g,\ \nabla^{(\alpha)})$ is of constant curvature k.

Corollary 2 If there exist $\alpha_1,\ \alpha_2\in\boldsymbol{R}$ $(|\alpha_1|\neq|\alpha_2|)$ such that the statistical manifold $(M,\ g,\ \nabla^{(\alpha_1)})$ is of constant curvature k_1 and the statistical manifold $(M,\ g,\ \ \nabla^{(\alpha_2)})$ constant curvature k_2 ,and $k_1\neq k_2$ holds, then for $\alpha\in\boldsymbol{R}$ satisfying that $\alpha^2=\frac{k_2\alpha_1^2-k_1\alpha_2^2}{k_2-k_1}$, the statistical manifold $(M,\ g,\ \ \nabla^{(\alpha)})$ is flat.

Example 3 $k_1=-1/2,\ k_2=0$, $\alpha_1=0$ and $\alpha_2=1$ hold in example 1 and example 2. Hence for $\alpha\in\boldsymbol{R}$ satisfying that $\alpha^2=1$, the statistical manifold $(M,\ g,\ \nabla^{(\alpha)})$ is flat.

Theorem 2 If the Hessian manifold $(M,\ g,\ \nabla)$ is of constant Hessian curvature, the statistical manifold $(M,\ g,\ \nabla^{(\alpha)})$ is of constant curvature for any $\alpha\in\boldsymbol{R}$.

Proof If the Hessian manifold $(M,\ g,\ \nabla)$ is of constant Hessian curvature,

$$(\nabla K)(Y,\ Z;\ X)=-\frac{c}{2}\{g(X,\ Y)Z+g(X,\ Z)Y\},\ c\in\boldsymbol{R}$$

holds. On the other hand, the curvature tensor R^{∇^g} of Levi-Civita connection ∇^g is written by

$$\begin{gathered}R^{\nabla^g}(X,\ Y)Z=R^{\nabla}(X,\ Y)Z-(\nabla K)(Y,Z\ ;X)+(\nabla K)(Z\ ,\ \ X\,;Y)+\\ +K(X,\ K(Y,\ Z))-K(Y,\ K(Z,\ X))\end{gathered},$$

where R is the curvature tensor of ∇ and K is difference tensor $K(X,\ Y):=\nabla_X Y-\nabla_X^g Y$. Then

$$\begin{aligned}(\nabla K)(Y,\ Z;X)-(\nabla K)(Z,\ X\,;Y)=2\{K(X,\ K(Y,Z))-K(Y,\ K(X,\ Z))\}+\\ +\frac{1}{2}\{R^{\nabla}(X,\ Y)Z-R^{\nabla^*}(X,\ Y)Z\}\end{aligned}$$

implies $R^{\nabla^g}(X,\ Y)Z=-\frac{c}{4}\{g(Y,\ Z)X-g(X,\ Z)Y\}$, where R^* is curvature tensor of dual connection ∇^*, that is, the statistical manifold $(M,\ \nabla^g,\ g)$ is of constant curvature. On the other hand, the statistical manifold $(M,\ \nabla,\ g)$ is flat, that is, constant curvature 0. Therefore we finish the proof of theorem from theorem 1.

3. Conclusion

We found a condition that the statistical manifold $(M, \nabla^{(\alpha)}, g)$ is of constant curvature for any $\alpha \in \boldsymbol{R}$. We also showed that if the Hessian manifold (M, ∇, g) is of constant Hessian curvature, the statistical manifold $(M, \nabla^{(\alpha)}, g)$ is of constant curvature for any $\alpha \in \boldsymbol{R}$,

References

[1] 민철림 등; 조선민주주의인민공화국 과학원통보, 2, 11, 주체98(2009).

[2] J. Zhang; AISM, 59, 161, 2007.

[3] H. Matsuzoe et al.; Diff. Geom. Appl., 24, 567, 2006.

[4] H. Furuhata; Diff. Geom. Appl., 27, 420, 2009.