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Numerical Solution of a Voltera Stochastic Integral Equation with Boundary Condition

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The great leader Kim Jong Il said:

"We must work actively to embrace advanced science and technology.

Embracing up-to-date science and technology is an important way to develop national science and technology quickly."

In [2, 4] we considered existence, uniqueness and some properties of the solution of Voltera stochastic integral equation, in [1] we considered existence and uniqueness of Voltera stochastic integral equation with boundary condition. In [3] we studied existence of solution of boundary value problem.

We construct approximation equation of the Voltera stochastic integral equation (SIE) with boundary condition;

$$\begin{cases} X_t = X_0 + \int_0^t f(t,\ r,\ X_r)dr + \int_0^t g(t,\ r)dW_r,\ \ t \in [0,\ 1] \\ X_0 = \mathrm{E}\psi(X_1) \end{cases}, \tag{1}$$

and we prove the existence and uniqueness of its solution and evaluate its convergence.

First of all, we will consider the Voltera SIE for any initial condition x.

1. Approximation Solution of Initial Value Problem and Its Property

If $\varphi_t(x)$ indicates the solution relevant to initial condition x of SIE

$$X_t = x + \int_0^t f(t,\ r,\ X_r)dr + \int_0^t g(t,\ r)dW_r, \quad t \in [0, 1]$$

then we can write

$$\varphi_t(x) = x + \int_0^t f(t, r, \varphi_r(x))dr + \int_0^t g(t, r)dW_r,\ t \in [0, 1]. \tag{2}$$

We assume that function f and g satisfy the following conditions.

H_1: $\exists K_1 > 0, \quad \forall t,\ s \in [0, 1], \quad \forall x,\ y \in R$

$$|f(t_1, s_1, x) - f(t_2, s_2, y)| \le K_1(|t_1 - t_2| + |s_1 - s_2| + |x - y|)$$

H_2: $M := \sup\limits_{s,\ t \in [0,\ 1]} |f(t,\ s,\ 0)| < \infty$.

H_3: $L := \sup\limits_{s,\ t \in [0,\ 1]} |g(t,\ s)| < \infty$

$H_4: \ \exists K_2>0, \quad \forall t_1, t_2, s_1, s_2 \in [0,1]$

$$|g(t_1,\ s_1)-g(t_2,\ s_2)| \le K_2(|t_1-t_2|+|s_1-s_2|)$$

For any partition $0=t_0<t_1<\cdots<t_m=1$ we define $n_t := \max\{n: t_n \le t\}$, $\delta := \{t_{i+1}-t_i\}$ and approximation scheme of (2) is constructed as follows;

$$\begin{aligned}\hat{\varphi}_{t_n}(x) &= x+\int_0^{t_n} f(t_n, t_{n_r}, \hat{\varphi}_{t_{n_r}}(x))dr+\int_0^{t_n} g(t_n, t_{n_r}))dW_r \\ &= x+\sum_{i=0}^{n-1} f(t_n,\ t_i,\ \hat{\varphi}_{t_i}(x))\Delta t_i+\sum_{i=0}^{n-1} g(t_n,\ t_i)\Delta W_{t_i}\end{aligned} \qquad t_n \in [0,\ 1],\ n=1,\cdots,m. \quad (3)$$

Then the following theory is established.

Theory 1 If the function f is a measurable function satisfying condition H_1, H_2 and g is the function satisfying condition H_3, H_4, then exist suitable $k>0$ under $E|x|<\infty$ satisfying $\sup\limits_{s\in[0,\ t]} E|\hat{\varphi}_{t_{n_s}}(x)-\varphi_s(x)|^2 \le K\delta, \quad t\in[0,1]$.

Proof First, evaluate $E|\hat{\varphi}_{t_{n_s}}(x)-\varphi_s(x)|^2$.

$$\mathrm{E}\,|\,\hat{\varphi}_{t_{n_s}}(x)-\varphi_s(x)\,|^2 \le$$

$$\le \mathrm{E}\left|\int_0^{t_{n_s}} f(t_{n_s},\ t_{n_r},\ \hat{\varphi}_{t_{n_r}}(x))dr+\int_0^{t_{n_s}} g(t_{n_r},\ t_{n_r}))dW_r-\int_0^s f(t,\ r,\ \varphi_r(x))dr-\int_0^s g(t,\ r)dW_r\right|^2 \le$$

$$\le 4\left\{\mathrm{E}\left|\int_0^{t_{n_s}}[f(t_{n_s},\ t_{n_r},\ \hat{\varphi}_{t_{n_r}}(x))-f(t,\ r,\ \varphi_r(x))]dr\right|^2+\mathrm{E}\left|\int_0^{t_{n_s}}[g(t_{n_s},\ t_{n_r})-g(t,\ r)]dW_r\right|^2+\right. \quad (4)$$

$$\left.+\mathrm{E}\left|\int_{t_{n_s}}^s f(t,\ r,\ \varphi_r(x))dr\right|^2+\mathrm{E}\left|\int_{t_{n_s}}^s g(t,\ r)dW_r\right|^2\right\}=4(I_1+I_2+I_3+I_4)$$

Then evaluate every term of right side of (4).

Using the condition H_1, H_2, H_3, H_4 and Kosch-Bunyakovski inequality, properties of Ito integral, we get

$$I_1=\mathrm{E}\left|\int_0^{t_{n_s}}[f(t_{n_s},\ t_{n_r},\ \hat{\varphi}_{t_{n_r}}(x))-f(t,\ r,\ \varphi_r(x))]dr\right|^2 \le 8\delta^2 K_1^2+2K_1^2\int_0^{t_{n_s}}\mathrm{E}\,|\,\hat{\varphi}_{t_{n_r}}(x)-\varphi_r(x)\,|^2\,dr, \quad (5)$$

$$I_2=\mathrm{E}\left|\int_0^{t_{n_s}}[g(t_{n_s},\ t_{n_r})-g(t,\ r)]dW_r\right|^2 \le 4K_2^2\delta^2, \quad (6)$$

$$I_3=\mathrm{E}\left|\int_{t_{n_s}}^s f(t,\ r,\ \varphi_r(x))dr\right|^2 \le M^2\delta^2, \quad (7)$$

$$I_4 = \mathrm{E}\left|\int_{t_{n_s}}^{s} g(t,\ r)dW_r\right|^2 \le L^2\delta\,, \tag{8}$$

and substitute (5)－(8) in (4) and

$$\mathrm{E}\,|\,\hat{\varphi}_{t_{n_s}}(x) - \varphi_s(x)\,|^2 \le K\delta + K'\int_0^{t_{n_s}} \mathrm{E}\,|\,\hat{\varphi}_{t_{n_r}}(x) - \varphi_r(x)\,|^2\, dr \tag{9}$$

In this expression, if $Z(t) = \sup\limits_{s\in[0,t]} E\left|\hat{\varphi}_{t_{n_s}}(x) - \varphi_s(x)\right|^2$ then we get $Z(t) \le K\delta + K'\int_0^t Z(r)dr$ from (9). When we apply Gronwall inequality, get the result of theorem.

Lemma 1 If $f,\ g$ are measurable function satisfying the condition $\mathrm{H}_1,\ \mathrm{H}_2\ ,\ \mathrm{H}_3,\ \mathrm{H}_4$, the solution $\hat{\varphi}_{t_n}(x)$ of approximation scheme (3), for any $t_n \in [0, 1],\ x_1,\ x_2 \in R$, is satisfying

$$|\,x_1 - x_2\,|\,e^{-K_1} \le |\,\hat{\varphi}_{t_n}(x_1) - \hat{\varphi}_{t_n}(x_2)\,| \le |\,x_1 - x_2\,|\,e^{K_1} \tag{10}$$

Proof By the condition H_1,

$$|\,\hat{\varphi}_{t_n}(x_1) - \hat{\varphi}_{t_n}(x_2)\,| \le |\,x_1 - x_2\,| + K_1\int_0^{t_n} |\,\hat{\varphi}_{t_{n_r}}(x_1) - \hat{\varphi}_{t_{n_r}}(x_2)\,|dr$$

and let $Z(t) = \sup\limits_{t_n\in[0,\ t]} \mathrm{E}\,|\,\hat{\varphi}_{t_n}(x_1) - \hat{\varphi}_{t_n}(x_2)\,|$ and $Z(0) = |\,x_1 - x_2\,|$, then $Z(t) \le Z(0) + K_1\int_0^t Z(r)dr$.

When we apply the Grolwall-Berman inequality, it gets (10).

Lemma 2 If f is the measurable function satisfying the condition $\mathrm{H}_1',\ \mathrm{H}_2$ and g is the measurable function satisfying the condition $\mathrm{H}_3,\ \mathrm{H}_4$, then it is true that

1° $\sup\limits_{t_n<t} \mathrm{E}(|\,\hat{\varphi}_{t_n}(x)\,|) \le (|\,x\,| + M + L)e^{K_1},\ \ t \in [0,\ 1]$

2° $\sup\limits_{t_n<t} \mathrm{E}(|\,\hat{\varphi}_{t_n}(x)\,|^2) \le 3(|\,x\,|^2 + 2M^2 + L^2)e^{6K_1^2},\ \ t \in [0,\ 1]$

for the solution $\hat{\varphi}_{t_n}(x)$.

Proof 1° By using the similar method for the proof of Lemma 1, we get

$$\mathrm{E}\,|\,\hat{\varphi}_{t_n}(x)\,| \le |\,x\,| + \mathrm{E}\left|\int_0^{t_n} f(t_n,\ t_{n_r},\ \hat{\varphi}_{t_{n_r}}(x))dr\right| + \mathrm{E}\left(\left|\int_0^{t_n} g(t_n,\ t_{n_r})dW_r\right|\right) \le$$

$$\le |\,x\,| + \int_0^{t_n} \mathrm{E}\,|\,\hat{\varphi}_{t_{n_r}}(x)\,|dr + Mt + \left(\int_0^{t_n} (g(t,\ t_{n_r}))^2 dr\right)^{1/2} \le$$

$$\le (|\,x\,| + M + L) + K_1\int_0^{t_n} \mathrm{E}\,|\,\hat{\varphi}_{t_{n_r}}(x)\,|dr$$

from the property of the absolute value, the condition $\mathrm{H}_1,\ \mathrm{H}_2,\ \mathrm{H}_3,\ \mathrm{H}_4$ and the property of Ito integral. Therefore

$Z(t) \le (|x| + M + L) + K_1 \int_0^t Z(r)dr$, if $Z(t) = \sup_{t_n \le t} \mathrm{E} |\hat{\varphi}_{t_n}(x)|$. Applying the Gronwall inequality,

$$Z(t) \le (|x| + M + L)\left(1 + K_1 \int_0^t e^{K_1(t-r)} dr\right) = (|x| + M + L)e^{K_1}$$

2° By using the similar method for the proof of 1°, we get

$$\mathrm{E}|\hat{\varphi}_{t_n}(x)|^2 \le \mathrm{E}\left| x + \int_0^{t_n} f(t_n,\ t_{n_r},\ \hat{\varphi}_{t_{n_r}}(x))dr + \int_0^{t_n} g(t_n,\ t_{n_r})dW_r \right|^2 \le$$

$$\le 3\left\{ |x|^2 + \mathrm{E}\left| \int_0^{t_n} [f(t_n,\ t_{n_r},\ \hat{\varphi}_{t_{n_r}}(x)) - f(t_n,\ t_{n_r},0) + f(t_n,\ t_{n_r},\ 0)]dr \right|^2 + \mathrm{E}(\int_0^{t_n} [g(t_n,\ t_{n_r})]^2 dr \right\} \le$$

$$\le 3\left\{ |x|^2 + 2K_1^2 \int_0^{t_n} \mathrm{E}|\hat{\varphi}_{t_{n_r}}(x)|^2 dr + 2M^2 + L^2 \right\} \le 3(|x|^2 + 2M^2 + L^2) + 6K_1^2 \int_0^{t_n} \mathrm{E}|\hat{\varphi}_{t_{n_r}}(x)|^2 dr$$

and if $Z(t) = \sup_{t_n \le t} \mathrm{E}|\hat{\varphi}_{t_n}(x)|^2$, then $Z(t) \le 3(|x|^2 + 2M^2 + L^2) + 6{K_1}^2 \int_0^t Z(r)dr$.

Applying the Gronwall inequality, we get 2°.

2. The Approximation Equation with Boundary Condition

Here we construct the approximation scheme of boundary-value problem for ψ satisfying the condition H_5, and prove its convergence.

We can make the approximation scheme of (1) as follows;

$$\hat{\varphi}_{t_n}(x) = x + \int_0^{t_n} f(t_n,\ t_{n_r},\ \hat{\varphi}_{t_{n_r}}(x))dr + \int_0^{t_n} g(t_n,\ t_{n_r})dW_r,\ t_n \in [0,\ 1],\ n = 1, \cdots, m \qquad (11)$$

$$x = \mathrm{E}\psi(\hat{\varphi}_1(x)).$$

Theorem 2 Let the function f, g are the measurable function satisfying the condition H_1, H_2, H_3, H_4 and the function ψ is the continuous function satisfying

H_5: $0 < \exists \eta < e^{-K_1}$, $|\psi(x) - \psi(y)| \le \eta |x - y|$ for any $x,\ y \in R$.

Then the solution of the approximation equation (11) with boundary condition is the only one solution.

Proof Under the given conditions, the solution of the approximation equation

$$\hat{\varphi}_{t_n}(x) = x + \int_0^{t_n} f(t_n,\ t_{n_r},\ \hat{\varphi}_{t_{n_r}}(x))dr + \int_0^{t_n} g(t_n,\ t_{n_r})dW_r,\ t_n \in [0,\ 1],\ n = 1, \cdots, m$$

is the only one solution, so if x satisfying the boundary condition $x = \mathrm{E}\psi(\hat{\varphi}_1(x))$ exists only one, then that x is satisfying (11).

When we define $x_n = \mathrm{E}\psi(\hat{\varphi}_1(x_{n-1}))$, $n = 1, 2, \cdots$, we get

$$\begin{aligned} |x_n - x_{n-1}| &= |\mathrm{E}(\psi(\hat{\varphi}_1(x_{n-1})) - \psi(\hat{\varphi}_1(x_{n-2})))| \le \eta |\mathrm{E}(\hat{\varphi}_1(x_{n-1}) - \hat{\varphi}_1(x_{n-2}))| \le \\ &\le \eta e^{K_1} \mathrm{E}|x_{n-1} - x_{n-2}| \le \alpha \mathrm{E}|x_{n-1} - x_{n-2}| \end{aligned}.$$

Therefore

$$|x_n - x_{n-1}| \le \alpha \mathrm{E}|x_{n-1} - x_{n-2}| \le \alpha^n \mathrm{E}|x_1 - x_0|.$$

However $0 \le \alpha = \eta e^{K_1} \le 1$ by the condition H_5 and $\lim_{n\to\infty} \mathrm{E}|x_n - x_{n-1}| = 0$ for $n \to \infty$

Therefore $\{x_n\}$ is fundamental sequence. By the condition H_5

$$|x_n| = |\mathrm{E}\psi(\hat{\varphi}_1(x_{n-1}))| = |\mathrm{E}(\psi(\hat{\varphi}_1(x_{n-1})) - \psi(0)) + \psi(0) \le \eta|\hat{\varphi}_1(x_{n-1})| + |\psi(0)|$$

and if $\mathrm{E}|x_0| < \infty$ then $\mathrm{E}|\hat{\varphi}_t(x_0)| < \infty$ (almost). And ψ is the continuous function, so $\mathrm{E}|x_n| < \infty$ (almost). Therefore limitation of $\{x_n\}$ exists. If the limitation expresses $\hat{X}_0$, then it establishes $\hat{X}_0 = \mathrm{E}\psi(\hat{\varphi}_1(\hat{X}_0))$.

Let we consider the convergence the solution of approximation scheme (11) to the solution of boundary-value problem (1).

Lemma 3 Let the function f and g are the measurable functions satisfying the condition H_1, H_2, H_3, H_4 and ψ is the continuous function satisfying the condition H_5. Then $\mathrm{E}|\hat{X}_0 - X_0| \le C\delta^{1/2}$ existing the suitable constant $C > 0$.

Proof By the condition H_5 and theorem 1, lemma 3,

$$\begin{aligned}\mathrm{E}|\hat{X}_0 - X_0| &= \mathrm{E}|\mathrm{E}(\psi(\hat{\varphi}_1(\hat{X}_0)) - \psi(\varphi_1(X_0)))| \le \eta \mathrm{E}|\mathrm{E}(\hat{\varphi}_1(\hat{X}_0) - \varphi_1(X_0))| \le \\ &\le \eta \mathrm{E}|\hat{\varphi}_1(\hat{X}_0) - \varphi_1(\hat{X}_0)| + \eta \mathrm{E}|\varphi_1(\hat{X}_0) - \varphi_1(X_0)| \le \\ &\le \eta\sqrt{K}\delta^{1/2} + \eta e^{K_1}\mathrm{E}|\hat{X}_0 - X_0| = \eta\sqrt{K_1}\delta^{1/2} + \alpha\mathrm{E}|\hat{X}_0 - X_0|\end{aligned}$$

and $\mathrm{E}|\hat{X}_0 - X_0| \le \dfrac{\eta K_1}{1-\alpha}\delta^{1/2} = C\delta^{1/2}$ from $0 < \alpha = \eta e^{K_1} < 1$.

Theorem 3 Under the condition of lemma 3, it exists suitable constant $K' > 0$ that

$$\sup_{t\in[0,\,1]} \mathrm{E}|\hat{\varphi}_{t_{n_t}}(\hat{X}_0) - \varphi_t(X_0)| \le K'\delta^{1/2}.$$

Proof Using theorem 1 and lemma

$$\begin{aligned}&\sup_{t\in[0,\,1]} \mathrm{E}|\hat{\varphi}_{t_{n_t}}(\hat{X}_0) - \varphi_t(X_0)| \le \sup_{t\in[0,\,1]} \mathrm{E}|\hat{\varphi}_{t_{n_t}}(\hat{X}_0) - \varphi_t(\hat{X}_0)| + \sup_{t\in[0,\,1]} \mathrm{E}|\varphi_t(\hat{X}_0) - \varphi_t(X_0)| \le \\ &\le K\delta^{1/2} + e^{K_1}\mathrm{E}|\hat{X}_0 - X_0| \le K\delta^{1/2} + Ce^{K_1}\delta^{1/2} = (K + Ce^{K_1})\delta^{1/2} = K'\delta^{1/2}. \square\end{aligned}$$

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