Journal of **Kim Il Sung** University
(Natural Science)

Vol. 4 No. 1
Juche104(2015)

# On Fractional Differential Compatibility Equation of Viscoelastic Body and Its Application

*Kim Jun Yong*

**Abstract** This paper deals with the expression of compatibility equations of viscoelastic material in terms of fractional derivatives and its applicability and validity by analyzing a number of quasi-static and dynamic problems of viscoelastic body.

**Key word** viscoelastic body

## Introduction

Recently the compatibility equations of viscoelastic materials in the form of fractional derivatives are widely used in the field of viscoelasticity[1－12]. As these equations are formulated on the basis of the consideration of viscoelastic materials as the combination of finite number of elastic members and spring-pots and are based on the assumption that volume strain is elastic, these have some shortcomings in the analysis of multi-axial stress-strain state of viscoelastic bodies, the influence of viscosity on the volume strain, the hysteresis characteristic of the viscoelastic bodies and come other problems, and in their solving methods.

We will derive the fractional differential compatibility equations from these compatibility equations of integral form and investigate their applicability and validity through the analysis of a number of quasi-static and dynamic problems of viscoelastic plate.

## 1. Fractional Differential Compatibility Equations of Viscoelastic Materials

The compatibility equations of integral form of the viscoelastic materials are as follows.

$$S_{ij}(t) = 2G_0 e_{ij}(t) - \int_0^t \Gamma_s(t-\tau) e_{ij}(\tau) d\tau \tag{1}$$

$$\sigma(t) = K_0 \theta(t) - \int_0^t \Gamma_v(t-\tau) \theta(\tau) d\tau . \tag{2}$$

where $S_{ij}(t)$, $e_{ij}(t)$ are stress and strain deviators respectively, $\sigma(t)$ is average normal stress, $\theta(t)$ is relative volume strain, $G_0$ is instantaneous shear modules, $K_0$ is instantaneous volume strain coefficient, and the functions, $\Gamma_s(t)$ and $\Gamma_v(t)$ are shearing relaxation nucleus and volumetric relaxation nucleus.

Similarly to Rabotnov nucleus, shearing relaxation nucleus $\Gamma_s(t)$ is expressed as

$$\Gamma_s(t) = 2G_0 \sum_{k=0}^{\infty} \frac{\lambda^k t^{(k+1)\alpha - 1}}{\Gamma((k+1)\alpha)}, \quad \lambda < 0, \ 0 < \alpha < 1 \tag{3}$$

and volumetric relaxation nucleus $\Gamma_v(t)$ is expressed as

$$\Gamma_v(t) = K_0 \sum_{k=0}^{\infty} \frac{\lambda_v^k \, t^{(k+1)\alpha - 1}}{\Gamma((k+1)\alpha)}, \quad \lambda_v < 0, \quad 0 < \alpha < 1 \tag{4}$$

where $\Gamma((k+1)\alpha)$ is Gamma function, $\lambda$, $\lambda_v$, $\alpha$ are material parameters.

Substituting equation (3) into (1), employing the definition of fractional derivatives and simplifying yield

$$2G_0 D_t^{\alpha} e_{ij}(t) - 2G_0(1+\lambda) e_{ij}(t) = D_t^{\alpha} S_{ij}(t) - \lambda S_{ij}(t) \tag{5}$$

where $D_t^{\alpha}$ is Riemann-Liouville fractional differential operator[1].

Similarly to above derivation, we obtain following equation from (2) and (4)

$$K_0 D_t^{\alpha} \theta(t) - K_0(1+\lambda_v)\theta(t) = D_t^{\alpha}\sigma(t) - \lambda_v \sigma(t)\,. \tag{6}$$

Considering equation (6) and the relation between the deviator components and the tensor components of strain and stress, (5) leads to

$$D_t^{\alpha}\varepsilon_{ij}(t) - (1+\lambda)\varepsilon_{ij}(t) + (\lambda - \lambda_v)\theta(t)\delta_{ij}/3 = \\ = \frac{D_t^{\alpha}}{2G_0}\sigma_{ij}(t) - \frac{\lambda}{2G_0}\sigma_{ij}(t) + \left[\left(\frac{1}{3K_0} - \frac{1}{2G_0}\right) D_t^{\alpha}\sigma(t) + \left(\frac{-\lambda_v}{3K_0} + \frac{\lambda}{2G_0}\right)\sigma(t)\right]\delta_{ij}\,. \tag{7}$$

Equation (7) is the compatibility equation of viscoelastic materials expressed in terms of fractional derivatives.

The physical meanings of parameters $\lambda$, $\lambda_v$ are $1/\lambda = (G_\infty - G_0)/G_0$, $1/\lambda_v = (K_\infty - K_0)/K_0$, $\alpha$ is the parameter which reflects the influence of hysteresis on stress-strain of viscoelatic material, $G_\infty$, $K_\infty$ are long shear stress and long strain modules respectively.

Equation (7) can be reduced in the case of small influence of viscosity on volume strain into

$$(D_t^{\alpha} - (1+\lambda))\varepsilon_{ij}(t) = (D_t^{\alpha} - \lambda)\sigma_{ij}(t)/(2G_0) + (1/(3K_0) - 1/(2G_0))(D_t^{\alpha} - \lambda)\sigma(t)\delta_{ij} \tag{8}$$

and in the case of incompressible materials into

$$(D_t^{\alpha} - (1+\lambda))\varepsilon_{ij}(t) = [(D_t^{\alpha} - \lambda)\sigma_{ij}(t) - (D_t^{\alpha} - \lambda)\sigma(t)\delta_{ij}]/(2G_0)\,. \tag{9}$$

## 2. Analysis of Some Viscoelastic Problems by Fractional Differential Compatibility Equation

### 2.1. Bending of isotropic viscoelastic plate

Let's consider the bending problem of thin plate with isotropic viscoelastic material.

The compatibility equation of viscoelastic material (8) gives the following bending equation of viscoelastic plate.

$$D_0(D_t^{\alpha} - (1+\lambda))\nabla^2\nabla^2 w = (D_t^{\alpha} - \lambda) g \tag{10}$$

where $D_0$ is instantaneous bending rigidity and $g$ is intensity of distributed load.

The rectangular plate of length $a$ and width $b$ is hinge support at all sides and subjected to the uniformly distributed load $g(t) = g_0$.

In this case the type of deflection is assumed as $w(x,\ y,\ t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \varphi_{mn}(t)\sin\gamma x \cdot \sin\delta y$.

where $\gamma = m\pi / a$, $\delta = n\pi / b$ and $m$, $n$ are integers.

The distributed load $g_0$ is also assumed as $g_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} g_{mn} \sin\gamma x \cdot \sin\delta y$.

Substituting this equation into (10) and simplifying, following equation is obtained

$$(D_t^{\alpha} - (1+\lambda))\varphi(t) = g\,t^{-\alpha} / \Gamma(1-\alpha) - \lambda g\,.$$

The solution of this equation is

$$\varphi(t) = C\frac{g}{D_0}\sum_{k=0}^{\infty}(1+\lambda)^k f_{(k+1)\alpha}(t) + \frac{g}{D_0}\sum_{k=0}^{\infty}(1+\lambda)^k (f_{k\alpha+1}(t) - \lambda f_{(k+1)\alpha+1}(t)) \tag{11}$$

where $D_t^{\alpha-1}\varphi(t)|_{t=0} = C$.

If there is no strain before consideration (zero initial condition), $C = 0$, so that the deflection is expressed only in terms of the second term of expression (11).

The first term is the deflection produced by the strain before consideration.

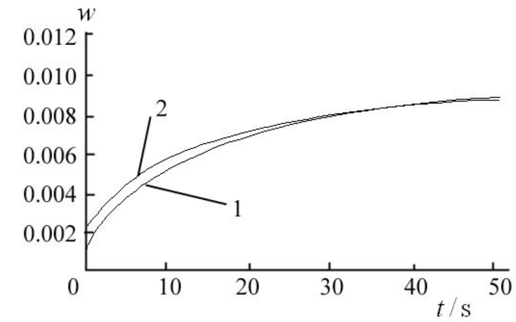


Fig. 1. The deflection behavior of viscoelastic plate according to time
1—the case that $c = 0$, 2—the case that $c \neq 0$

The deflection behavior of viscoelastic plate with respect to time is shown in Fig. 1.

In the case where a load acted before consideration and vanished after consideration, the deflection is equal to the first term of expression (11). This shows that the deflection of a moment is composed of the sum of the deflections by the load that has been acting throughout since the moment of consideration and by the load that acted before consideration.

### 2.2. Analysis of reinforced viscoelastic cylinder with varying internal diameter

A viscoelastic cylinder under the internal pressure $P_0$ is reinforced outside by thin elastic cylinder with thickness $h$ and its internal diameter $2a(t)$ is varying with the time and it rotates at angular velocity $\omega$ around its axis.

Suppose that it is axis symmetric plane strain problem and introduce cylindrical coordinate system $\{0\ ;\ r,\ \theta,\ z\}$ where the symmetric axis is $z$-axis.

Basic equations can be established as follows.

$$\varepsilon_r = \frac{\partial u}{\partial r},\ \ \varepsilon_\theta = \frac{u}{r},\ \ \varepsilon_r + \varepsilon_\theta = 0 \tag{12}$$

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + \rho\omega^2 r = 0 \tag{13}$$

The following equation is established by the compatibility equation (9) of incompressible viscoelastic material

$$(D_t^{\alpha} - (1+\lambda))\varepsilon_r = [(D_t^{\alpha} - \lambda)\sigma_r + (D_t^{\alpha} - \lambda)\sigma]/(2G_0)\,, \tag{14}$$

$$(D_t^\alpha-(1+\lambda))\varepsilon_\theta=[(D_t^\alpha-\lambda)\sigma_\theta+(D_t^\alpha-\lambda)\sigma]/(2G_0). \tag{15}$$

where $\rho$ is the density of viscoelastic material, $G_0$ is instantaneous shear modules, $\lambda$, $\alpha$ are characteristic parameters of viscoelastic material and $\sigma$ is average normal stress.

Boundary conditions are $\sigma_r(a(t))=-P_0$, $\sigma_r(b,\ t)=-B\,\varepsilon_0(b,\ t)$. Where $B=\dfrac{E_\vartheta h}{(1-\nu_\vartheta^2)b}$, $b$ is external diameter and $E_\vartheta$, $\nu_\vartheta$ are elastic modulus and Poisson ratio of reinforcing layer respectively.

Suppose that the reinforcing layer is a thin shell and its inertia is ignored.

Equation (12) leads to

$$u=\varphi(t)/r,\ \ \varepsilon_\theta=\varphi(t)/r^2,\ \ \varepsilon_r=-\varphi(t)/r^2. \tag{16}$$

Equation (13)—(16) leads to

$$D_t^\alpha\varphi(t)+A_3(t)D_t^{\alpha-1}\varphi(t)+A_4(t)\varphi(t)=F(t). \tag{17}$$

where $A_3(t)=\eta_1(t)/\eta(t),\ A_4(t)=\eta_2(t)/\eta(t),\ F(t)=F_1(t)/\eta(t)$, $\eta(t)=1/b^2-1/a^2(t)-B/(2G_0b^2)$,

$$\eta_1(t)=\alpha D_t^1(1/b^2-1/a^2(t))=-\alpha\ D_t^1/a^2(t),\ \ F_1(t)=(D_t^\alpha-\lambda)(-\sigma(a)+\rho\omega^2(a^2(t)-b^2)/2)/(2G_0),$$

$$\eta_2(t)=-(1+\lambda)(1/b^2-1/a^2(t))+B\cdot\lambda/(2G_0b^2)=1/b^2-1/a^2(t)-\lambda\eta(t).$$

The solution of fractional differential equation (17) is

$$\varphi(t)=\int_0^t G_1(t-\tau)F(\tau)d\tau. \tag{18}$$

where $G_1(t-\tau)=\dfrac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}+\sum\limits_{k=0}^{\infty}(-1)^{k+1}I_{\tau+}^\alpha[A_3(t)I_{\tau+}^{\alpha-1}+A_4(t)I_{\tau+}^\alpha]^k\cdot\left[A_3(t)+A_4(t)\dfrac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\right].$

**2.3. Analysis of vibration of viscoelastic plate by inplane force varying periodically**

The bending vibration equation of viscoelastic plate under the inplane force $N_x$ is

$$D_0(D_t^\alpha-(1+\lambda))\nabla^2w+(D_t^\alpha-\lambda)\left(\rho h\frac{\partial^2w}{\partial t^2}-N_x\frac{\partial^2w}{\partial x^2}\right)=0. \tag{19}$$

Assume that the type of its solution is $w(x,\ y,\ t)=\sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty}\overline{\varphi}_{mn}(t)\sin\dfrac{m\pi x}{a}\sin\dfrac{n\pi y}{b}$.

Then from equation (19)

$$D_t^{2+\alpha}\varphi-\lambda D_t^2\varphi+A_1(t)D_t^\alpha\varphi+A_2(t)D_t^{\alpha-1}\varphi+A_3(t)\varphi=0. \tag{20}$$

where $\varepsilon_a=m^2\pi^2/(a^2\rho h),\ \omega_0=D_0/\rho h\cdot(m^2/a^2+n^2/b^2)^2\cdot\pi^4,\ \varphi=\overline{\varphi}_{mn}^{(t)}/h$,

$$A_1(t)=-\varepsilon_aN_x+\omega_0,\ \ A_2(t)=-\varepsilon_a\alpha\,D_t^1N_x,\ \ A_3(t)=-(\omega_0+\lambda A_1(t)).$$

If the inplane force is expressed as the type of $N_x=F_0+F_1\cos\omega t$,

$$A_1(t)=\left[\frac{F_*-F_0}{F_1}-\cos\omega t\right]\varepsilon_aF_1.$$

where $F_*=\dfrac{\omega_0}{\varepsilon_a}=D_0\left(\dfrac{m^2}{a^2}+\dfrac{n^2}{b^2}\right)^2\pi^2\Bigg/\dfrac{m^2}{a^2}$ is coincide with the static critical load of correspondent elastic plate(elastic plate of which elastic modulus is equal to instantaneous elastic modulus of viscoelastic plate).

In case that $A_2(t)=\alpha\omega\varepsilon_a F_1 \sin\omega t$, $A_3(t)=-(\omega_0+\lambda A_1(t))$, the solution of equation (20) is

$$\varphi(t)=\sum_{i=1}^{3} C_i\varphi_i(t). \tag{21}$$

where

$$\varphi_i(t)=f_{2+\alpha-i+1}^{(t)}+\sum_{\ell=0}^{\infty}(-1)^{\ell+1}I^{2+\alpha}(-\lambda I^{\alpha}+A_1(t)I^2+A_2(t)I^3+A_3(t)I^{2+\alpha})^{\ell}\cdot$$

$$\cdot(-\lambda f_{\alpha-i+1}(t)+A_1(t)f_{2-i+1}(t)+A_2(t)f_{3-i+1}(t)+A_3(t)f_{2+\alpha-i+1}(t))$$

$$C_i=D_t^{2+\alpha-i}\varphi(t)|_{t=0},\ i=1,\,2,\,3.$$

The vibration behavior of the viscoelastic plate expressed as expression (21) is shown in Fig. 3.

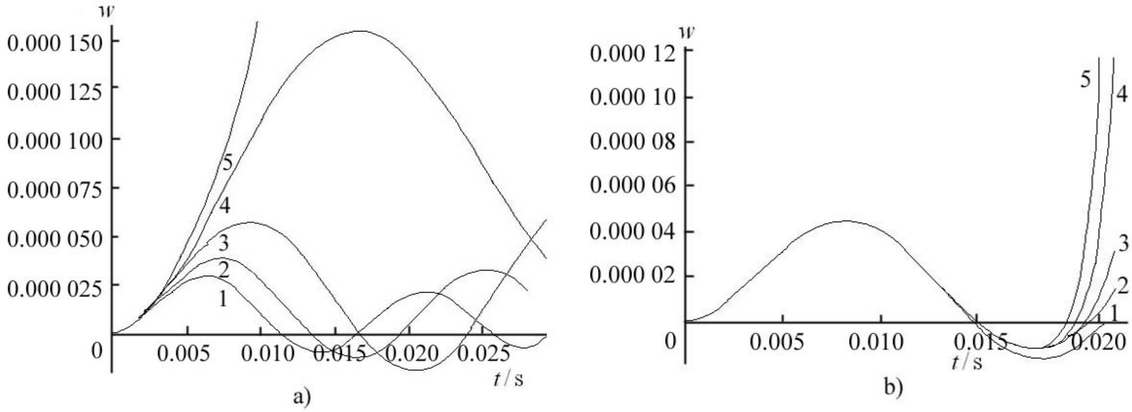


Fig. 2. The vibration behavior of the viscoelastic plate

where curve 1 represents the case $F_0+F_1<<F_*$, 2－the case $F_0+F_1<(1+\lambda)/\lambda F_*$, 3－the case $F_0+F_1<(1+\lambda)/\lambda F_*$ and the difference is slight, 4－the case $(1+\lambda)/\lambda F_*<F_0+F_1<F_*$ and 5 － $F_*<F_0+F_1$. Through the above 5 curves the behaviors of vibration are compared(Fig. 2. a).

The cases 1 and 2 represent damped vibrations, the cases 3 and 4 represent the increasing vibrations and the case 5 represent abrupt increase of displacement.

Additionally, it is obvious that the more the frequency of external force approaches the natural frequency, the more unstable vibration becomes (Fig. 2. b). Here $\omega_1$, $\omega_2$ represent the case $\omega<<\omega_0$, $\omega_3$ and $\omega_4$ the case $\omega<\omega_0$ and at the same time $\omega_1$ is close to $\omega_0$, and $\omega_5$ the case $\omega_0<\omega_5$ and at the same time $\omega_5$ is close to $\omega_0$.

Equation (21) not only shows well the general behavior of parametric vibration of viscoelastic plate, but gives an accurate estimation of its displacement.

### 2.4. Analysis of dynamic displacement of viscoelastic cylinder by impact internal pressure

Let's consider the case in which the outside of viscoelastic cylinder is reinforced by thin elastic cylinder with thickness *h*, the whole cylinder rotates at uniform angular velocity of $\omega$ around its axis and internal pressure $P_0$ acts suddenly inside the cylinder. In this case we will determine the dynamic displacement of the viscoelastic cylinder.

Suppose that the material is incompressible and its density is $\rho$.

Suppose that it is axisymmetric plane strain problem and introduce cylindrical coordinates $\{0\,;\ r,\ \theta,\ z\}$ of which symmetric axis is $z$-axis.

The basic equations are

$$\frac{\partial \sigma_r}{\partial r}+\frac{\sigma_r-\sigma_\theta}{r}+\rho\omega^2 r=\rho\frac{\partial^2 u}{\partial t},\quad \varepsilon_r=\frac{\partial u}{\partial r},\quad \varepsilon_\theta=\frac{u}{r}.$$

The compatibility equations are given from equation (9)

$$(D_t^\alpha-(1+\lambda))\varepsilon_r=\frac{1}{2G_0}[(D_t^\alpha-\lambda)\sigma_r-(D_t^\alpha-\lambda)\sigma],$$

$$(D_t^\alpha-(1+\lambda))\varepsilon_\theta=\frac{1}{2G_0}[(D_t^\alpha-\lambda)\sigma_\theta-(D_t^\alpha-\lambda)\sigma].$$

Solve above equations simultaneously.

From the assumption of incompressible material $u=\varphi(t)/r,\ \varepsilon_r=-\varphi(t)/r^2,\ \varepsilon_\theta=\varphi(t)/r^2$.

Boundary conditions are $\sigma_r=-P_0$ (in $r=a$, internal boundary), $\sigma_r=-g(t)$ (in $r=b$, boundary with reinforced layer).

As the reinforced layer is an elastic shell $\sigma_r\,|_{r=b}=-g(t)=-\dfrac{E_{\text{rein}}h}{(1-\nu_{\text{rein}}^2)b}\cdot\dfrac{\varphi(t)}{b^2}-\dfrac{\rho_{\text{rein}}h}{b}D_t^2\varphi(t)$,

where $E_{\text{rein}}$, $\nu_{\text{rein}}$ are respectively elastic modulus and lateral strain rate of the reinforced layer and $\rho_{\text{rein}}$ is its density.

By above equations

$$D_t^{2+\alpha}\varphi(t)-\lambda D_t^2\varphi(t)+\mu_1 D_t^\alpha\varphi(t)+\mu_2\varphi(t)=F(t). \tag{22}$$

where $\rho_0=\rho\,\ell n\dfrac{a}{b}-\dfrac{\rho_{\text{rein}}h}{b}$, $\delta_0=-\left(2G_0\left(\dfrac{1}{a^2}-\dfrac{1}{b^2}\right)+\dfrac{E_{\text{rein}}h}{(1-\nu_{\text{rein}}^2)b^3}\right)$, $\delta_1=2G_0\left(\dfrac{1}{a^2}-\dfrac{1}{b^2}\right)-\lambda\delta_0$,

$$\mu_1=\frac{\delta_0}{\rho_0},\quad \mu_2=\frac{\delta_1}{\rho_0},\quad F(t)=\frac{1}{\rho_0}(D_t^\alpha-\lambda)\left(P_0-\frac{1}{2}\rho\omega^2(b^2-a^2)\right).$$

The solution of (22) is $\varphi(t)=\sum_{i=1}^{3}C_i\varphi_i(t)+\varphi_3(t)*F(t)$. Where

$$\varphi_i(t)=\sum_{k=0}^{\infty}(-1)^k\sum_{|\beta|=k}\frac{k!}{\beta_1!\beta_2!\beta_3!}(-\lambda)^{\beta_1}\mu_1^{\beta_2}\mu_2^{\beta_3}f_{\alpha_*}(t)$$

$$\alpha_*=\alpha\beta_1+2\beta_2+(2+\alpha)\beta_3+2+\alpha-i+1\quad(i=1,\ 2,\ 3),$$

$$C_i=D_t^{2+\alpha-i}\varphi(t)\,|_{t=0},$$

$$\varphi_3(t)*F(t)=\sum_{k=0}^{\infty}(-1)^k\sum_{|\beta|=k}\frac{k!}{\beta_1!\beta_2!\beta_3!}(-\lambda)^{\beta_1}\mu_1^{\beta_2}\mu_2^{\beta_3}(f_{\alpha_{**}-\alpha}(t)-\lambda f_{\alpha_{**}}))\cdot\frac{1}{\rho_0}\left(P_0+\frac{\rho\omega^2(b^2-a^2)}{2}\right),$$

$$\alpha_{**}=\alpha\beta_1+2\beta_2+(2+\alpha)\beta_3+\alpha+1.$$

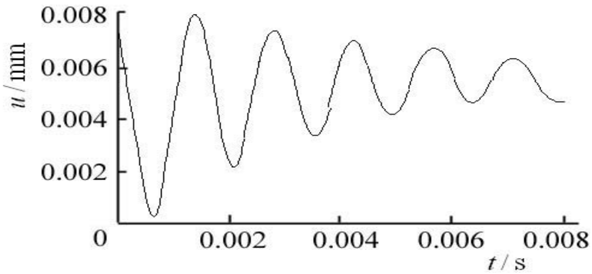


Fig. 3. The calculation results when there is no strain before consideration

When there is no strain before consideration, the result of calculation is shown in Fig. 3.

This shows that the analysis of viscoelastic bodies by means of fractional derivatives gives enough explanation to its mechanical behaviors.

In this case the equilibrium positions of vibration follow the curve similar to creep curve.

**Conclusion**

The fractional differential compatibility equation of viscoelastic materials derived in this paper gives good results in the analysis of quasi static and dynamic problems of viscoelastic bodies.

**References**

[1] 김명하 등; 분수계미적분과 그 응용, **김일성**종합대학출판사, 23～78, 주체95(2006).

[2] T. A. Сургуладзе и др.; Математика, механика, 4, 65, 2002.

[3] T. A. Сургуладзе и др.; Математика, механика, 5, 61, 2002.

[4] R. L. Bagley et al.; AIAA Journal, 21, 741, 1983.

[5] S. Sakakibara; Nonlinear Analysis, 47, 5449, 2001.

[6] R. C. Koeller; Journal of Applied Mechanics, 51, 299, 1984.

[7] R. C. Koeller; ACTA Mech., 58, 251, 1986.

[8] Y. A. Rossikhin et al.; Int. J. of Engineering Science, 39, 149, 2001.

[9] R. L. Bagley et al.; AIAA Journal, 23, 918, 1985.

[10] B. S. Liebst et al.; Measurement, Control, 118, 572, 1996.

[11] N. Heymans et al.; Rheol. Acta., 33, 210, 1994.

[12] 那順野 等; 日本機械學瀚会論文集, C, 69, 1493, 2003.