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Differential Subordination and Superordination of a Subclass of Analytic Functions Defined by Sălăgean Operator

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The great leader Kim Jong Il said:

"Only when the basic knowledge of science, the foundation of science and technology, is sound can we build the tower of specialized science and technology high and rapidly develop science and technology in general."

In this paper we obtained some differential subordination and superordination for analytic functions defined by Sălăgean operator.

[1] obtained sufficient conditions for functions $f \in A$ to satisfy $\frac{zf'(z)}{f(z)} \prec q(z),\ \frac{z^2 f'(z)}{(f(z))^2} \prec q(z),\ z \in U$ and [2] obtained sufficient conditions to satisfy $\frac{H_l^m[\alpha_1+1]f(z)}{H_l^m[\alpha_1]f(z)} \prec q(z), \frac{z^{\alpha-1}H_l^m[\alpha_1+1]f(z)}{(H_l^m[\alpha_1]f(z))^\alpha} \prec q(z), z \in U$ by generalizing the result of [1] for the subclass of analytic functions defined by a linear operator $H_l^m[\alpha_1]: A \to A$. [3] considered the inclusion definition and differential subordination of the subclass $S_\alpha^n(\beta)$ of analytic functions defined by the Sălăgean operator $D^n: A \to A$.

This article generalized the results of [1] by obtaining the sufficient conditions to satisfy differential subordination

$$\frac{D^{n+1}f(z)}{D^n f(z)} \prec q(z),\quad \frac{z^2(D^n f(z))'}{(D^n f(z))^2} \prec q(z),\quad z \in U$$

and the sufficient conditions to satisfy superordination

$$q(z) \prec \frac{D^{n+1}f(z)}{D^n f(z)},\quad z \in U.$$

In the unit circle $U = \{z \in \boldsymbol{C} : |z| < 1\}$, the set of the analytic function $p(z) = a + p_1(z) + \cdots$ is labeled $H[a, 1]$ and the set of the analytic functions that are standardized by the conditions $f(0) = 0,\ f'(0) = 1$ are labeled A.

$$S^* := \{f \in A : \mathrm{Re}\frac{zf'(z)}{f(z)} > 0, z \in U\},\ K := \{f \in A : \mathrm{Re}(1 + \frac{zf''(z)}{f'(z)}) > 0, z \in U\}$$

These are called a star function and a convex function respectively.

As for the two functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ in the unit circle U, the convolution or Hadamard product of f and g is defined by $(f*g)(z):=z+\sum_{n=2}^{\infty}a_nb_nz^n$.

The following linear operator is called Sălăgean operator[4] $D^n: A\to A$ $(n\in \boldsymbol{N}\cup\{0\})$;

$$D^0f(z)=f(z),\ D^1f(z)=zf'(z)=Df(z),\cdots,\ D^nf(z)=D(D^{n-1}f(z))$$

The Dziok-Srivastava operator $H_l^m[\alpha_1]: A\to A$ is defined by

$$H_l^m[\alpha_1]f(z)=H_l^m(\alpha_1,\cdots,\ \alpha_l;\beta_1,\cdots,\ \beta_m)f(z)=z+\sum_{n=2}^{\infty}\frac{(\alpha_1)_{n-1}\cdots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}\cdots(\beta_m)_{n-1}}\frac{a_nz^n}{(n-1)!}$$

where

$$(a)_n:=\frac{\Gamma(a+n)}{\Gamma(a)}=\begin{cases}1, & n=0\\ a(a+1)\cdots(a+n-1), & n\in\boldsymbol{N}\end{cases},\ \ l,\ m\in\boldsymbol{N}\cup\{0\},\ \ l\le m+1.$$

Definition 1 When f and g are analytic functions in the unit circle $\overline{U}=\{z\in\boldsymbol{C}:|z|\le 1\}$, f is the subordinate to g (g is superordinate to f) and $f\prec g$ or $f(z)\prec g(z),\ z\in U$, if there exists a Schwarz function w analytic in U with $w(0)=0, |w(z)|<1, z\in U$ such that $f(z)=g(w(z)), z\in U$. If f is subordinate to g, g is called dominant of f. For all dominants $\tilde{g}$ of f, if $\tilde{g}\prec g$ then g is called the best dominant of f.And if g is superordinate to f, f is called the best subordinant if $\tilde{f}\prec f$ for all subordinants $\tilde{f}$ of g.

Definition 2 Let $\overline{U}=\{z\in\boldsymbol{C}:|z|\le 1\}$. Denote by Q the set of all functions f that are analytic and injective on $\overline{U}\setminus E(f)$, where $E(f):=\left\{\zeta\in\partial U:\lim_{z\to\zeta}f(z)=\infty\right\}$, and are such that $f'(\zeta)=0,\ \ \zeta\in\partial U\setminus E(f)$.

Results

Theorem 1 Let q be convex in the unit circle U and let $\mathrm{Re}\{\beta/\delta+2\gamma q(z)/\delta+1+zq''(z)/q'(z)\}>0,\ \ z\in U\ \ \alpha,\ \beta,\ \gamma,\ \delta\in\boldsymbol{C},\ \ \delta\ne 0$ If $f\in A$ satisfies the following subordination

$$\alpha+\beta\frac{D^{n+1}f(z)}{D^nf(z)}+\gamma\left(\frac{D^{n+1}f(z)}{D^nf(z)}\right)^2+\delta\left(\frac{D^{n+2}f(z)}{D^nf(z)}-\left(\frac{D^{n+1}f(z)}{D^nf(z)}\right)^2\right)\prec\alpha+\beta q(z)+\gamma(q(z))^2+\delta zq'(z)$$

then $\dfrac{D^{n+1}f(z)}{D^nf(z)}\prec q(z),\ z\in U$ and q is the best dominant.

Proof By setting $\theta(w)=\alpha+\beta w+\gamma w^2,\ \ \phi(w)=\delta$, it can be easily observed that θ and ϕ are analytic in $\boldsymbol{C}$ and $\phi(w)\ne 0,\ \ w\in U$.

Also, letting $Q(z)=zq'(z)\phi(q(z))=\delta zq'(z),\ \ h(z)=\theta(q(z))+Q(z)=\alpha+\beta q(z)+\gamma(q(z))^2+\delta zq'(z)$, then we obtain the following result from the conditions of this theorem;

$$\mathrm{Re}\{zh'(z)/Q(z)\}=\mathrm{Re}\{\beta/\delta+2\gamma q(z)/\delta+1+zq''(z)/q'(z)\}>0,\ z\in U.$$

Let the function p be defined by $p(z):=\dfrac{D^{n+1}f(z)}{D^n f(z)}$, so that, by a straight computation, we have $zp'(z)=\dfrac{D^{n+2}f(z)}{D^n f(z)}-\left(\dfrac{D^{n+1}f(z)}{D^n f(z)}\right)^2$.

From the condition of this theorem,

$$\alpha+\beta\frac{D^{n+1}f(z)}{D^n f(z)}+\gamma\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right)^2+\delta\left(\frac{D^{n+2}f(z)}{D^n f(z)}-\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right)^2\right)\prec\alpha+\beta q(z)+\gamma(q(z))^2+\delta zq'(z)$$

by using the theorem 3.4h of [6], $\dfrac{D^{n+1}f(z)}{D^n f(z)}\prec q(z),\ z\in U$ and q is the best dominant.

Remark 1 By setting $n=0$, $\alpha=0$, $\beta=1-\alpha$, $\gamma=\delta=\alpha$, we obtain the theorem 3 of [1].

Theorem 2 Let q be analytic and univalent in the unit circle U with $q(0)=1$ and $zq'(z)/q(z)$ be star function in U.

If $f\in A$ satisfies the following subordination

$$\alpha+\gamma\left(1+\frac{(D^{n+1}f(z))'}{(D^n f(z))'}-2\frac{D^{n+1}f(z)}{D^n f(z)}\right)\prec\alpha+\gamma\frac{zq'(z)}{q(z)},\ z\in U,\ \alpha,\ \gamma\in\mathbf{C}$$

then $\dfrac{z^2(D^n f(z))'}{(D^n f(z))^2}\prec q(z),\ z\in U$ and q is the best dominant.

Proof By setting $\theta(w)=\alpha,\ \phi=\gamma/w$, it can be easily observed that θ is analytic in $\boldsymbol{C}$ and ϕ is analytic in $\boldsymbol{C}\setminus\{0\}$ and $\phi(w)\neq 0,\ w\in\boldsymbol{C}\setminus\{0\}$.

Also, letting $Q(z)=zq'(z)\phi(q(z))=\gamma zq'(z)/q(z)$, $h(z)=\theta(q(z))+Q(z)=\alpha+\gamma zq'(z)/q(z)$, then we obtain the following result from the conditions of this theorem;

$$\operatorname{Re}\frac{zh'(z)}{Q(z)}=\operatorname{Re}\left\{1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\}>0,\ \ z\in U.$$

Let the function p be defined by $p(z):=\dfrac{z^2(D^n f(z))'}{(D^n f(z))^2}$, so that, by a straight computation, we have $\dfrac{zp'(z)}{p(z)}=1+\dfrac{(D^{n+1}f(z))'}{(D^n f(z))'}-2\dfrac{D^{n+1}f(z)}{D^n f(z)}$.

From the condition of this theorem, $\alpha+\gamma\left[1+\dfrac{(D^{n+1}f(z))'}{(D^n f(z))'}-2\dfrac{D^{n+1}f(z)}{D^n f(z)}\right]\prec\alpha+\gamma\dfrac{zq'(z)}{q(z)},\ z\in U$, by using the theorem 3.4h of [6], $\dfrac{z^2(D^n f(z))'}{(D^n f(z))^2}\prec q(z),\ z\in U$ and q is the best dominant.

Remark 2 By setting $n=0$, $\alpha=0$, $\gamma=1$, we obtain the theorem 4 of [1].

Theorem 3 Let q be convex in the unit circle U with $q(z)\neq 0,\ z\in U$ and for $\beta,\gamma,\delta\in\boldsymbol{C}$, $\delta\neq 0$, $\operatorname{Re}\{\beta/\delta+2\gamma q(z)/\delta\}>0,\ z\in U$.

Let $f \in A$, $\dfrac{D^{n+1}f(z)}{D^n f(z)} \in H[q(0),\ 1] \cap Q$,

$\alpha+\beta\dfrac{D^{n+1}f(z)}{D^n f(z)}+\gamma\left(\dfrac{D^{n+1}f(z)}{D^n f(z)}\right)^2+\delta\left(\dfrac{D^{n+2}f(z)}{D^n f(z)}-\left(\dfrac{D^{n+1}f(z)}{D^n f(z)}\right)^2\right)$ be univalent in U.

If $f \in A$ satisfies the following subordination

$$\alpha+\beta q(z)+\gamma(q(z))^2+\delta zq'(z) \prec \alpha+\beta\frac{D^{n+1}f(z)}{D^n f(z)}+\gamma\left(\frac{D^{n+2}f(z)}{D^n f(z)}\right)^2+\delta\left(\frac{D^{n+2}f(z)}{D^n f(z)}-\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right)^2\right)$$

then $q(z) \prec \dfrac{D^{n+1}f(z)}{D^n f(z)}$, $z \in U$ and q is the best subordinant.

Proof By setting $\theta(w)=\alpha+\beta w+\gamma w^2$, $\phi(w)=\delta$, it can be easily observed that θ and ϕ are analytic in $\mathbf{C}$ and $\phi(w)\neq 0$, $w \in U$.

From the condition of this theorem, $\operatorname{Re}\dfrac{\theta'(q(z))}{\phi(q(z))}=\operatorname{Re}\left\{\dfrac{\beta}{\delta}+\dfrac{2\gamma}{\delta}q(z)\right\}>0$, $z \in U$.

By using the theorem of [7], $q(z) \prec \dfrac{D^{n+1}f(z)}{D^n f(z)}$, $z \in U$ and q is the best subordinant.

Theorem 4 Let q_1 and q_2 be convex in the unit circle U with $q_1(z)\neq 0,\ q_2(z)\neq 0$ and for $\alpha,\ \beta,\ \gamma,\ \delta \in \boldsymbol{C},\ \delta\neq 0$, $\operatorname{Re}\left\{\dfrac{\beta}{\delta}+\dfrac{2\gamma}{\delta}q_1(z)+1+\dfrac{zq_1''(z)}{q_1'9z)}\right\}>0,\ z\in U$, $\operatorname{Re}\left\{\dfrac{\beta}{\delta}+\dfrac{2\gamma}{\delta}q_2(z)\right\}>0,\ z \in U$.

Let $f \in A$, $\dfrac{D^{n+1}f(z)}{D^n f(z)} \in H[q(0),\ 1] \cap Q$ ($\alpha \in \mathbf{C}$),

$\alpha+\beta\dfrac{D^{n+1}f(z)}{D^n f(z)}+\gamma\left(\dfrac{D^{n+1}f(z)}{D^n f(z)}\right)^2+\delta\left(\dfrac{D^{n+2}f(z)}{D^n f(z)}-\left(\dfrac{D^{n+1}f(z)}{D^n f(z)}\right)^2\right)$ be univalent in U.

If $f \in A$ satisfies the following subordination

$$\begin{aligned}&\alpha+\beta q_2(z)+\gamma(q_2(z))^2+\delta zq_2'(z) \prec\\ &\prec \alpha+\beta\frac{D^{n+1}f(z)}{D^n f(z)}+\gamma\left(\frac{D^{n+2}f(z)}{D^n f(z)}\right)^2+\delta\left(\frac{D^{n+2}f(z)}{D^n f(z)}-\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right)^2\right)\prec\\ &\prec \alpha+\beta q_1(z)+\gamma(q_1(z))^2+\delta zq_1'(z),\quad z\in U\end{aligned}$$

then $q_2(z) \prec \dfrac{D^{n+1}f(z)}{D^n f(z)} \prec q_1(z)$, $z \in U$ and q_1 is the best dominant and q_2 is the best subordinant.

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