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Problem of Minimal Volume Design of Truss Structures Considering Strength Constraints and Local Buckling Constraints

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Abstract We considered the problem of minimal volume design of truss structures with consideration of strength constraints for tensile members and local buckling constraints for compressive members. We derived an equalization criterion of energy density and optimization, proposed an algorithm for designing truss structures with minimal volume under different constraints for tension and compression, and verified validity of proposed method by using an example of the truss structure with 25 members.

Key words equalization criterion, energy density, local buckling constraint, minimum volume design

Introduction

The great leader Comrade **Kim Il Sung** said as follows.

"Research in the basic sciences should be intensified. Now that science and technology are advancing at a rapid pace and the role of modern technical devices is increasing rapidly in production and construction, there is an urgent need for the basic sciences to be developed still further."("**KIM IL SUNG** WORKS" Vol. 35 P. 312)

Many studies on the problem for optimal design of several structures, particularly to determine optimal cross-sections of members of truss have done using equalization criterions of energy density.

The equalization criterion of energy density and its algorithm of the problem for optimal design of the structure with maximum stiffness and eigen frequency considering strength and stiffness of truss structure was proposed in [1－3], the equalization criterion of energy density of the problem for determination of reasonable reinforcement length of steel column giving the specified critical load was considered in [4] , and besides the equalization criterions of energy density and algorithms for optimal solution of the problem for optimal design of several structures with different objective functions and constraints were studied in large number. Methods of optimal design considering strength constraints by using difference-quotient algorithm, genetic algorithm and evolution structure optimization algorithm were also considered in [5－10]. But the symmetric constraints for both tension and compression were being considered in the problem of minimal volume design of truss structures in preceding works. Optimal solution algorithm considering both strength constraints for tensile members and local buckling constraints for compressive members is not studied yet.

In this paper we considered the problem of minimal volume design of truss structures with consideration of both strength constraints for tensile members and local buckling constraints for compressive members by using equalization criterions of energy density.

1. Establishment of Problem

It is known in the process for finding optimization criteria of differential structures that the structure with minimal volume in case global stiffness of structure is given becomes such a structure in which its strain energy density in everywhere is equalized to average strain energy density of the whole structure. This can be obvious in view of mechanical state of the structure.

Although equalization of energy density grades up with equalization of stress distribution of structure, in general structures it is difficult to have strength limit directly corresponding to energy density because it is represented by shape strain energy. Particularly in the past it was difficult to consider strength condition in optimal design using the equalization criterion of energy density because stress limit of compressive member is related to Euler's stress in truss structure.

We can replace strength constraints for tensile members and local buckling constraints for compressive members in truss structure with energy constraints in following way.

Strength condition of truss structure can be represented as follows:

$$\sigma_i \le [\sigma_{01}]\ (i \in I_1)\ \text{(tension)},\quad \sigma_i \ge [\sigma_{02}]\ (i \in I_2)\ \text{(compression)} \tag{1}$$

where I_1, I_2 is sets of numbers subjected to tension and compression respectively, $[\sigma_{01}]$ is the limit stress of member subjected to tension and equals to the yielding stress of material, $[\sigma_{02}]$ is the limit stress of member subjected to compression and equals to the critical stress of Euler in case of considering buckling.

Since Euler's critical load for compressive member is $P_{\mathrm{c}i} = \pi^2 E J_i / l_i^2$, the critical stress is represented as

$$[\sigma_{02}] = \sigma_{\mathrm{c}i} = \pi^2 E J_i /(F_i\, l_i^2) \tag{2}$$

where E-elastic modulus, J_i, F_i, l_i- minimal inertia moment, sectional area and length of i th member, respectively.

Then, representing strain energy of i th truss member by stress $U_i = \sigma_i^2 / E \cdot V_i / 2$.

Therefore total sum of strain energies for members under tension is as follows.

$$\sum_{i \in I_1} U_i = \frac{1}{2} \boldsymbol{u}^T \cdot \sum_{i \in I_1} \boldsymbol{K}_i \boldsymbol{u} = \frac{1}{2E} \cdot \sum_{i \in I_1} \sigma_i^2 V_i$$

Considering the strength condition for members under tension, strength constraint can be represented using limit value of energy as follows.

$$\frac{1}{2} \boldsymbol{u}^T \cdot \sum_{i \in I_1} \boldsymbol{K}_i \boldsymbol{u} \le U_{01},\quad i \in I_1 \tag{3}$$

where $U_{01} = \sum_{i \in I_1} U_{01i} = \frac{1}{2} \sum_{i \in I_1} \frac{[\sigma_{01}]^2}{E} V_i$ and stiffness matrix of member $\boldsymbol{K}_i$ is the augmented stiffness matrix of member element which has the same size as that of total structure.

Similarly, for compressive member we can write as follows.

$$\frac{1}{2} \boldsymbol{u}^T \cdot \sum_{i \in I_2} \boldsymbol{K}_i \boldsymbol{u} \le U_{02},\quad i \in I_2 \tag{4}$$

where $U_{02}=\sum_{i\in I_2}\frac{\sigma_{c\,i}^2}{2E}V_i=\frac{1}{2}\sum_{i\in I_2}\frac{\pi^4 E J_i^2}{F_i l_i^3}=\frac{1}{2}\sum_{i\in I_2}\pi^4 E\left(\frac{r_i}{l_i}\right)^4 V_i$. F_i :sectional area of i^{th} member, r_i : minimal inertia radius of cross-section($r_i=\sqrt{J_i/F_i}$).

Thus, we can formulate the problem of minimal volume design of truss structures with consideration of strength constraints for member under tension and local buckling constraints for member under compression as follows:

$$\left.\begin{aligned}&\sum_{i\in I_1\cup I_2}V_i\Rightarrow\min\\&\frac{1}{2}\boldsymbol{u}^T\sum_{i\in I_1}\boldsymbol{K}_i\boldsymbol{u}\le U_{01}\ \ (i\in I_1)\\&\frac{1}{2}\boldsymbol{u}^T\sum_{i\in I_2}\boldsymbol{K}_i\boldsymbol{u}\le U_{02}\ \ (i\in I_2)\\&\left(\sum_{i\in I}\boldsymbol{K}_i+\sum_{i\in I_2}\boldsymbol{K}_i\right)\boldsymbol{u}=\boldsymbol{P}\end{aligned}\right\}\tag{5}$$

Now, let’s derive the equalization criterion of energy density for the problem formulated by equation (5).

Constructing Lagrangian function as follows:

$$L(\boldsymbol{u},\ F_i)=\sum_{i\in I_1\cup I_2}V_i+\lambda_1\left(\frac{1}{2}\boldsymbol{u}^T\sum_{i\in I_1}\boldsymbol{K}_i\boldsymbol{u}-U_{01}\right)+\lambda_2\left(\frac{1}{2}\boldsymbol{u}^T\sum_{i\in I_2}\boldsymbol{K}_i\boldsymbol{u}-U_{02}\right)+\boldsymbol{\mu}^T\left[\left(\sum_{i\in I_1}\boldsymbol{K}_i+\sum_{i\in I_2}\boldsymbol{K}_i\right)\boldsymbol{u}-\boldsymbol{P}\right]\tag{6}$$

and using the condition in optimal point $\partial L/\partial\boldsymbol{u}=0$, we can obtain equations:

$$\frac{\partial L}{\partial\boldsymbol{u}}=\lambda_1\boldsymbol{u}^T\sum_{i\in I_1}\boldsymbol{K}_i+\lambda_2\boldsymbol{u}^T\sum_{i\in I_2}\boldsymbol{K}_i+\boldsymbol{\mu}^T\sum_{i\in I_1}\boldsymbol{K}_i+\boldsymbol{\mu}^T\sum_{i\in I_2}\boldsymbol{K}_i=0$$

and following expression results from above:

$$(\lambda_1\boldsymbol{u}^T+\boldsymbol{\mu}^T)\sum_{i\in I_1}\boldsymbol{K}_i+(\lambda_2\boldsymbol{u}^T+\boldsymbol{\mu}^T)\sum_{i\in I_2}\boldsymbol{K}_i=0$$

We can find that above equation holds when undetermined multipliers to $\sum_{i\in I_1}\boldsymbol{K}_i$ and $\sum_{i\in I_2}\boldsymbol{K}_i$ are different to each other at least as follows.

$$\lambda_1\boldsymbol{u}^T=-\boldsymbol{\mu}^T\ \ (i\in I_1),\ \ \lambda_2\boldsymbol{u}^T=-\boldsymbol{\mu}^T\ \ (i\in I_2)\tag{7}$$

Then, following expression results from the condition in optimal point $\partial L/\partial F_i=0$:

$$\begin{aligned}\frac{\partial L}{\partial F_i}&=l_i+\lambda_1\left[\frac{1}{2}\mathbf{u}^T\frac{\partial\mathbf{K}_i^{(I_1)}}{\partial F_i}+\mathbf{u}^T\sum_{i\in I_1}\mathbf{K}_i\frac{\partial\mathbf{u}}{\partial F_i}-\frac{\partial U_{01}}{\partial F_i}\right]+\lambda_2\left[\frac{1}{2}\mathbf{u}^T\frac{\partial\mathbf{K}_i^{(I_2)}}{\partial F_i}+\mathbf{u}^T\sum_{i\in I_2}\mathbf{K}_i\frac{\partial\mathbf{u}}{\partial F_i}-\frac{\partial U_{02}}{\partial F_i}\right]+\\&+\boldsymbol{\mu}^T\frac{\partial\mathbf{K}_i^{(I_1)}}{\partial F_i}\mathbf{u}+\boldsymbol{\mu}^T\sum_{i\in I_1}\mathbf{K}_i\frac{\partial\mathbf{u}}{\partial F_i}+\boldsymbol{\mu}^T\frac{\partial\mathbf{K}_i^{(I_2)}}{\partial F_i}\mathbf{u}+\boldsymbol{\mu}^T\sum_{i\in I_2}\mathbf{K}_i\frac{\partial\mathbf{u}}{\partial F_i}=0\ \ (i\in I_1\cup I_2)\end{aligned}\tag{8}$$

Substituting $\boldsymbol{\mu}^T$ from (7) in (8) and rearranging,

$$l_i - \frac{1}{2}\lambda_1 \boldsymbol{u}^T \frac{\partial \boldsymbol{K}_i^{(I_1)}}{\partial F_i}\boldsymbol{u} - \frac{1}{2}\lambda_2 \boldsymbol{u}^T \frac{\partial \boldsymbol{K}_i^{(I_2)}}{\partial F_i} - \lambda_1 \frac{\partial U_{01}}{\partial F_i} - \lambda_2 \frac{\partial U_{02}}{\partial F_i} = 0 \quad (i \in I_1,\ I_2). \tag{9}$$

From (9), when i^{th} member is under tension, $l_i - \frac{1}{2}\lambda_1 \boldsymbol{u}^T \frac{\partial \boldsymbol{K}_i^{(I_1)}}{\partial F_i}\boldsymbol{u} - \lambda_1 \frac{\partial U_{01}}{\partial F_i} = 0 \quad (i \in I_1)$ and when i^{th} member is under compression, $l_i - \frac{1}{2}\lambda_2 \boldsymbol{u}^T \frac{\partial \boldsymbol{K}_i^{(I_2)}}{\partial F_i}\boldsymbol{u} - \lambda_2 \frac{\partial U_{02}}{\partial F_i} = 0 \quad (i \in I_2)$.

Multiplying F_i to both sides of two expressions and considering structure of stiffness matrix

$$V_i - \lambda_1 U_i^{(I_1)} - \lambda_1 U_{01i} = 0 \quad (i \in I_1),\ \ V_i - \lambda_2 U_i^{(I_2)} - \lambda_2 U_{02i} = 0 \quad (i \in I_2) \tag{10}$$

where $U_{01i} = \frac{[\sigma_{01}]^2}{2E} V_i$, $U_{02i} = \frac{1}{2}\pi^4 \alpha \frac{EJ_i^2}{F_i\, l_i^3}$, α is a constant determined by approximate relation between J_i and F_i, namely the shape of cross-section of member (2 for circular and rectangular section)

Summing (10) for all members under tension and compression, respectively,

$$\sum_{i \in I_1} U_i - \lambda_1 [(U_i^{(I_1)} + U_{01i})] = 0 \ \ (i \in I_1),\quad \sum_{i \in I_2} U_i - \lambda_2 [(U_i^{(I_2)} + U_{02i})] = 0 \ \ (i \in I_2).$$

And eliminating unknown multipliers λ_1, λ_2 in expression and (10), we obtain an equalization criterion of energy density as follows:

$$\frac{U_i^{(I_1)} + U_{01i}}{V_i} = \frac{\sum_{i \in I_1} [U_i^{(I_1)} + U_{01i}]}{\sum_{i \in I_1} V_i} \ \ (i \in I_1),\quad \frac{U_i^{(I_2)} + U_{02i}}{V_i} = \frac{\sum_{i \in I_1} [U_i^{(I_2)} + U_{02i}]}{\sum_{i \in I_2} V_i} \ \ (i \in I_2)$$

From condition in the optimal point: $\left(\frac{1}{2}\boldsymbol{u}^T \cdot \sum_{i \in I_1} \boldsymbol{K}_i \boldsymbol{u} - U_{01}\right) = \left(\frac{1}{2}\boldsymbol{u}^T \sum_{i \in I_2} \boldsymbol{K}_i \boldsymbol{u} - U_{02}\right) = 0$, we obtain the criterion as follows:

$$\frac{U_i^{(I_1)} + U_{01i}}{V_i} = \frac{2U_{01}}{\sum_{i \in I_1} V_i} \ \ (i \in I_1),\quad \frac{U_i^{(I_2)} + U_{02i}}{V_i} = \frac{2U_{02}}{\sum_{i \in I_2} V_i} \ \ (i \in I_2). \tag{11}$$

The structure in which sum of its strain energy densities under real stress and limit stress for each tensile and compressive member respectively is equalized to limit value of total strain energy density corresponding to the whole structure becomes optimal structure.

2. An Algorithm for Finding Optimal Structure and Example

From criteria (11), we can obtain

$$F_i = \frac{(U_i^{(I_1)} + U_{01i}) \sum_{i \in I_1} V_i}{2U_{01}\, l_i} \ \ (i \in I_1),\quad F_i = \frac{(U_i^{(I_1)} + U_{02i}) \sum_{i \in I_2} V_i}{2U_{02}\, l_i} \ \ (i \in I_2) \tag{12}$$

and find optimal solution by the following iteration scheme:

$$F_i^{(k+1)} = \frac{(U_i^{(I_1)} + U_{01i}) \sum_{i \in I_1} V_i^{(k)}}{2U_{01}\, l_i} \quad (i \in I_1), \quad F_i^{(k+1)} = \frac{(U_i^{(I_2)} + U_{02i}) \sum_{i \in I_2} V_i^{(k)}}{2U_{02}\, l_i} \quad (i \in I_2) \quad (13)$$

The algorithm is as follows:

① For $k = 0$, Solve equations $\boldsymbol{Ku} = \boldsymbol{P}$, find $\boldsymbol{u}^{(k)}$ and assort tensile and compressive member, then compute $U_{01i}^{(k)}$, $U_{02i}^{(k)}$.

② Compute $U_i^{(k)(I_1)} = \frac{1}{2}\boldsymbol{u}^{(k)T} \sum_{i \in I_1} \boldsymbol{K}_i^{(k)} \boldsymbol{u}^{(k)}$, $U_i^{(k)(I_2)} = \frac{1}{2}\boldsymbol{u}^{(k)T} \sum_{i \in I_2} \boldsymbol{K}_i^{(k)} \boldsymbol{u}^{(k)}$.

③ Compute $F_i^{(k+1)}$ $(i \in I_1 \cup I_2)$ using (13).

If the following condition is satisfied: $\left| \sum_{i \in I_1 \cup I_2} V_i^{(k+1)} - \sum_{i \in I_1 \cup I_2} V_i^{(k)} \right| \le \varepsilon$, take $F_i^{(k+1)}$ as optimal solution. Otherwise, iterate from ① for $k = k + 1$.

For numerical example, we performed optimal design of 3D tower with 25 members widely used in optimal design of structures as typical example. (Fig, Table 1)

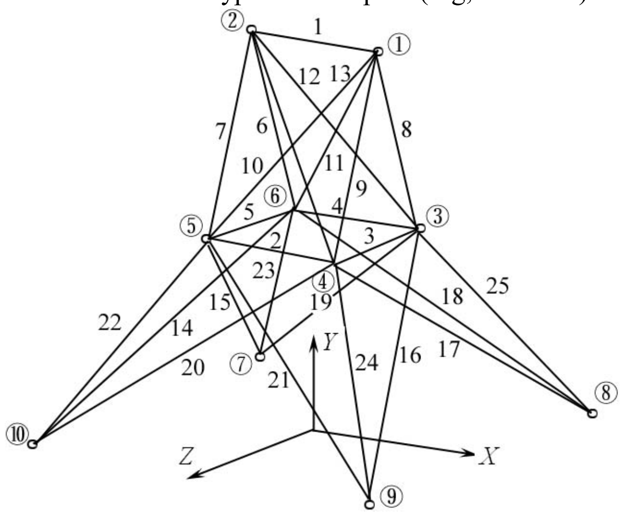

Fig. Truss with 25 members

Sections of each member is circular, material constants is $E = 68.9\text{GPa}$, $[\sigma_{01}] = 275.8\text{MPa}$ and sectional area of each member is $5.0 \times 10^{-4}\ \text{m}^2$ for initial structure.

Load condition in truss is shown as table 2.

Table 1. Nodal Coordinates

Nodal number	X/m	Y/m	Z/m	Nodal number	X/m	Y/m	Z/m
1	0.952 5	5.08	0	6	−0.952 5	2.54	−0.9525
2	−0.952 5	5.08	0	7	−2.540 0	0	−2.540 0
3	0.952 5	2.54	−0.952 5	8	2.540 0	0	−2.540 0
4	0.952 5	2.54	0.952 5	9	2.540 0	0	2.540 0
5	−0.952 5	2.54	0.952 5	10	-2.540 0	0	2.540 0

Table 2. Load condition(N)

Nodal number	Action direction of load		
	X/m	*Y*/m	*Z*/m
1	44 482	−22 241	4 448.2
2	44 482	−22 241	0
4	0	0	−2 224.1
5	0	0	−2 224.1

By selecting 6 design variables and limiting lower bounds of cross-sectional areas of members to $0.645\times10^{-4}\ m^2$, optimal solution is attained by iterations of about 3 times and then cross-sectional area of each member and total volume of structure is shown in table 3.

Table 3. Cross-sectional areas and total volume

Design variable	Number of member	Initial /($\times10^{-4}m^2$)	Under symmetric constrains for tension and compression /($\times10^{-4}m^2$)	Under buckling Constraints/($\times10^{-4}m^2$)
1	1	5.000	0.645 0	0.645 0
2	2-5	5.000	2.423 0	1.580 0
3	6-9	5.000	3.054 0	3.006 0
4	10-13	5.000	0.645 0	2.797 0
5	14-21	5.000	1.221 0	2.822 0
6	22-25	5.000	2.450 0	3.393 0
	Total volume /m^3	0.042	0.021 8	0.023 7

From table 3 it is shown that, comparing the results of optimal solutions under the symmetric constraints for both tension and compression and under constraints for tensile strength and buckling in compression, total volume of structure in the latter increases a little about 8%.

This shows that it is important to account for buckling constraint for compressive members of the truss structures in practice.

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