Journal of **Kim Il Sung** University (Natural Science) Vol. 3 No. 3 Juche103(2014)

Asymptotic Properties of Variance Estimator in Nonlinear Autoregressive Time Series Models with α−Mixing Errors Term

Kim Kyong Hui, Kim Ok Gyong

Abstract We considered the asymptotic properties of the variance estimator of errors in nonlinear autoregressive time series models with $\alpha-$mixing errors.

The estimator based on the residuals is shown to be consistent for the error variance. And the asymptotic distribution of the variance estimator is proved to be normal.

Key word nonlinear autoregressive model

The great leader Comrade **Kim Il Sung** said as follows.

"**We should actively develop the major areas of basic sciences such as mathematics, physics, chemistry and biology so as to raise the national standard of science and technology and find more effective solutions to the scientific and technical problems that arise in the different branches of the national economy**."("**KIM IL SUNG** WORKS" Vol. 35 P. 313)

Nonlinear autoregressive time series model is as follows.

$$X_i = r_\theta(X_{i-1}, \cdots, X_{i-p}) + \varepsilon_i \tag{1}$$

Let $\{X_i, i=0, \pm 1, \pm 2, \cdots\}$ be a stationary sequence of real variables satisfying above model, where r_θ, $\theta \in \Theta$ is a family of known functions from $R^p \to R$ on $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_q)^T \in \Theta \subset R^q$.

Moreover, the errors are assumed to be a $\alpha-$ mixing stationary sequence with mean being 0, common variance σ^2 and density f and $X_{i-1}, \cdots, X_{i-p}$ are independent with $\{\varepsilon_j, j=i, i+1, \cdots\}$.

The consideration for asymptotic properties of variance estimators of the error in time series models is very meaningful.

In researching result [2] it is derived asymptotic normality of the Bickel- Rosenblatt test statistic, based on the integrated squared error of the nonparametric error density estimators and the null error density, in nonlinear autoregressive time series models with i.i.d. errors.

In researching result [1] it is considered that the variance estimator, based on the residuals, is consistent with $\sqrt{n}$ rate for the error variance and asymptotic distribution of estimator is normal in case with i.i.d. errors.

We extend the model to more general case with $\alpha-$ mixing errors and derive asymptotic normal of variance estimator.

Assumption 1 Errors are $\alpha-$ mixing stationary sequences with mean being 0, common variance σ^2. For any $\varepsilon>0$, there exists $\delta>8/\varepsilon$ such $\alpha(\tau)=o(\tau^{-(1+\varepsilon)})$.

Suppose that $\mathrm{E}(\varepsilon_i)^{4+\delta}<\infty$ and that (ε_i^2) has a spectral density function that a bounded and continuance.

Assumption 2 When $r_\theta(\boldsymbol{y})=\boldsymbol{a}^T\boldsymbol{y}+o(\|\boldsymbol{y}\|)$ is defined as $\boldsymbol{A}=(\boldsymbol{a}, \boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_{p-1})\in R^q$, the spectral radius of $\boldsymbol{A}$ is $\rho(\mathbf{A})<1$. Where e_j is the j^{th} unit vector of R^q.

Assumption 3 Let $U\subset\Theta\subset R^q$ be any opened neighbored of $\boldsymbol{\theta}$, we assume that, for all $y\in R^p$, $\theta\in U$, $j, k=1, \cdots, q$

$$\left|\frac{\partial r_\theta(\boldsymbol{y})}{\partial\theta_j}\right|\le M_1(\boldsymbol{y}) \tag{2}$$

$$\left|\frac{\partial^2 r_\theta(\boldsymbol{y})}{\partial\theta_j\partial\theta_k}\right|\le M_2(\boldsymbol{y}) \tag{3}$$

where $\mathrm{E}M_1^2(X_{i-1}, \cdots, X_{i-p})<+\infty$, $\mathrm{E}M_2^2(X_{i-1}, \cdots, X_{i-p})<+\infty$.

For $1\le i\le n$, $1\le j\le q$, set

$$Y_{ij}=\frac{\partial}{\partial\theta_j}r_\theta(X_{i-1}, \cdots, X_{i-p}). \tag{4}$$

Then ε_i and $Y_{1j}, Y_{2j}, \cdots, Y_{ij}$ are independent each other.

Let $\hat\theta=(\hat\theta_1, \cdots, \hat\theta_q)$ be an estimator for $\boldsymbol{\theta}$ satisfying the iterated logarithm law. Therefore, there exists a constant $(0<C<\infty)$ such that

$$\lim\sup\sqrt{\frac{n}{\log(\log n)}}\,|\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}|\le C \tag{5}$$

where $|\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}|=\sqrt{\sum_{j=1}^q(\hat\theta_j-\theta_j)^2}$.

The above assumption on $\hat{\boldsymbol{\theta}}$ is satisfied for least square estimator under certain conditions.

Based on the estimator $\hat{\boldsymbol{\theta}}$, we define the residuals

$$\hat\varepsilon_i=X_i-r_{\hat\theta}(X_{i-1}, \cdots, X_{i-p}),\ i=1, 2, \cdots, n \tag{6}$$

Then we define the estimator $\hat\sigma^2$ for σ^2 as follows;

$$\hat\sigma^2=\frac{1}{n}\sum_{i=1}^n\hat\varepsilon_i^2 \tag{7}$$

The asymptotic properties of $\hat\sigma^2$ are given as follows.

Theorem 1 Under the assumptions 1_3 and (5), we have

$$\hat\sigma^2=\frac{1}{n}\sum_{i=1}^n\varepsilon_i^2+O_p\left(\frac{\log(\log n)}{n}\right) \tag{8}$$

further, under the assumption 1 $\mathrm{E}\varepsilon_1^4 < \infty$, we have

$$\sqrt{n}(\hat{\sigma}^2 - \sigma^2) = O_p(1). \tag{9}$$

Proof By model (1) and formula (6), it follows for any $i = 1, 2, \cdots, n$

$$\hat{\varepsilon}_i = \varepsilon_i - \sum_{j=1}^{q}(\hat{\theta}_j - \theta_j)Y_{ij} + \frac{1}{2}\sum_{j=1}^{q}\sum_{k=1}^{q}(\hat{\theta}_j - \theta_j)(\hat{\theta}_k - \theta_k)Z_{ijk}. \tag{10}$$

Combining (10) with the definition of $\hat{\sigma}^2$ in (7), we obtain that

$$\begin{aligned}\hat{\sigma}^2 = &\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i^2 + \frac{1}{n}\sum_{i=1}^{n}\left[\sum_{j=1}^{q}(\hat{\theta}_j - \theta_j)Y_{ij}\right]^2 + \frac{1}{4n}\sum_{i=1}^{n}\left[\sum_{j=1}^{q}\sum_{k=1}^{q}(\hat{\theta}_j - \theta_j)(\hat{\theta}_k - \theta_k)Z_{ijk}\right]^2 - \\ &-\frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{q}(\hat{\theta}_j - \theta_j)Y_{ij}\varepsilon_i - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{q}(\hat{\theta}_j - \theta_j)(\hat{\theta}_k - \theta_k)Z_{ijk}\varepsilon_i + \\ &+\frac{1}{n}\sum_{i=1}^{n}\left[\sum_{j=1}^{q}(\hat{\theta}_j - \theta_j)Y_{ij}\right]\left[\sum_{l=1}^{q}\sum_{k=1}^{q}(\hat{\theta}_l - \theta_l)(\hat{\theta}_k - \theta_k)Z_{ilk}\right].\end{aligned} \tag{11}$$

Since $\hat{\boldsymbol{\theta}}$ is a strong consistent estimator of $\boldsymbol{\theta}$ and $U \subset \Theta \subset R^q$ is an opened neighborhood of $\boldsymbol{\theta}$,

$$\exists N > 0 : \forall n > N,\ \hat{\boldsymbol{\theta}} \in U,\ \boldsymbol{\theta}^* \in U \text{ (a. s.)}.$$

Thus, we can use assumption 1 to obtain $\forall n > N,\ 1 \le i \le n,\ 1 \le j,\ k \le q$,

$$\begin{aligned}&\mathrm{E}(Z_{ijk}^2) \le \mathrm{E}M_2^2(X_{i-1}, \cdots, X_{i-p}) < \infty,\\ &\mathrm{E}(|Y_{ij}Z_{ikl}|) \le \sqrt{\mathrm{E}M_1^2(X_{i-1}, \cdots, X_{i-p})\mathrm{E}M_2^2(X_{i-1}, \cdots, X_{i-p})} < \infty,\\ &\mathrm{E}(\varepsilon_i Z_{ijk}) \le \sqrt{\mathrm{E}(\varepsilon_i^2)\mathrm{E}(Z_{ijk}^2)} \le \sigma\sqrt{\mathrm{E}M_2^2(X_{i-1}, \cdots, X_{i-p})} < \infty.\end{aligned}$$

Using above equations and the strong stationary of $\{X_i\}$, and Markov's inequality, it follows that

$$\left.\begin{aligned}&\sum_{i=1}^{n}Z_{ijk}^2 = O_p(n)\\ &\sum_{i=1}^{n}\varepsilon_i Z_{ijk} = O_p(n)\\ &\sum_{i=1}^{n}Y_{ij}Z_{ikl} = O_p(n)\end{aligned}\right\}. \tag{12}$$

By (5) and (12), we have

$$\frac{1}{n}\sum_{i=1}^{n}\left[\sum_{j=1}^{q}\sum_{k=1}^{q}(\hat{\theta}_j - \theta_j)(\hat{\theta}_k - \theta_k)Z_{ijk}\right]^2 \le \frac{q}{n}\sum_{j=1}^{q}\sum_{k=1}^{q}(\hat{\theta}_j - \theta_j)^2(\hat{\theta}_k - \theta_k)^2\sum_{i=1}^{n}Z_{ijk}^2 = O_p\left(\frac{(\log(\log n))^2}{n^2}\right) \tag{13}$$

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{q}(\hat{\theta}_j - \theta_j)(\hat{\theta}_k - \theta_k)Z_{ijk}\varepsilon_i = \frac{1}{n}\sum_{j=1}^{q}\sum_{k=1}^{q}(\hat{\theta}_j - \theta_j)(\hat{\theta}_k - \theta_k)\sum_{i=1}^{n}Z_{ijk}\varepsilon_i = O_p\left(\frac{\log(\log n)}{n}\right) \tag{14}$$

$$\frac{1}{n}\sum_{i=1}^{n}\left[\sum_{j=1}^{q}(\hat{\theta}_j-\theta_j)Y_{ij}\right]\left[\sum_{l=1}^{q}\sum_{k=1}^{q}(\hat{\theta}_l-\theta_l)(\hat{\theta}_k-\theta_k)Z_{ilk}\right]=$$
$$=\frac{1}{n}\sum_{j=1}^{q}\sum_{l=1}^{q}\sum_{k=1}^{q}(\hat{\theta}_j-\theta_j)(\hat{\theta}_l-\theta_l)(\hat{\theta}_k-\theta_k)\sum_{i=1}^{n}Y_{ij}Z_{ilk}=O_p\left(\frac{(\log(\log n))^{3/2}}{n^{3/2}}\right). \tag{15}$$

And by assumption 2 and (5), we have

$$\frac{1}{n}\sum_{i=1}^{n}\left[\sum_{j=1}^{q}(\hat{\theta}_j-\theta_j)Y_{ij}\right]^2=O_p\left(\frac{\log(\log n)}{n}\right). \tag{16}$$

In fact, by assumption 2 and the definition of Y_{ij} in (4), we have

$$\mathrm{E}(Y_{ij}^2)\le \mathrm{E}M_1^2(X_{i-1},\cdots,X_{i-p})<\infty .$$

Thus, using the strong stationary of $\{X_i\}$ and Markov's inequality, it follows that

$$\sum_{i=1}^{n}Y_{ij}^2=O_p(n). \tag{17}$$

By Cauchy-Schwarz inequality, we have that

$$\left[\sum_{j=1}^{q}(\hat{\theta}_j-\theta_j)Y_{ij}\right]^2\le\sum_{j=1}^{q}(\hat{\theta}_j-\theta_j)^2\sum_{j=1}^{q}Y_{ij}^2 .$$

Combining (5) with (7), we obtain that (16) holds.

Now, we should estimate the end term in (11).

Under assumption 2 and (5), we have that

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{q}(\hat{\theta}_j-\theta_j)Y_{ij}\varepsilon_i=O_p\left(\frac{\sqrt{\log(\log n)}}{n}\right). \tag{18}$$

In fact, using the fact that Y_{ij} $(j=1,2,\cdots,q)$ and $(\varepsilon_i,\ \varepsilon_{i+1},\ \cdots)$ are independent and the strong stationary of $\{X_i\}$, we obtain that

$$\mathrm{E}\left(\left|\sum_{i=1}^{n}\varepsilon_iY_{ij}\right|^2\right)=\sum_{i=1}^{n}\mathrm{E}(\varepsilon_i^2Y_{ij}^2)=O(n) .$$

By this formula Chebyshev's inequality, it is easy to see that $\sum_{i=1}^{n}\varepsilon_iY_{ij}=O_p(n^{1/2})$.

Combining this with (5), it follows that

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{q}(\hat{\theta}_j-\theta_j)Y_{ij}\varepsilon_i=\frac{1}{n}\sum_{j=1}^{q}(\hat{\theta}_j-\theta_j)\sum_{i=1}^{n}Y_{ij}\varepsilon_i=O_p\left(\frac{\sqrt{\log(\log n)}}{n}\right).$$

Therefore, we obtain (18). Thus, by (11), (13)−(16), (18), we can conclude that (8) holds.

If errors (ε_i) satisfy assumption 1 as $\alpha-$mixing stationary sequences, according to the central limit theorem for (ε_i^2) we have that $\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i^2=\sigma^2+O_p(n^{-1/2})$.

Combining this with (8), we obtain (9).□

Theorem 2 Under all assumptions of theorem 1, we have that in distribution

$$\sqrt{n}(\hat{\sigma}^2 - \sigma^2) \to N(0,\ \mathrm{Var}(\varepsilon_1^2)) . \tag{19}$$

Proof Using (8), it follows that

$$\sqrt{n}(\hat{\sigma}^2 - \sigma^2) = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i^2 - \sigma^2\right) + O_p\left(\frac{\log(\log n)}{\sqrt{n}}\right) = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i^2 - \sigma^2\right) + O_p(1) . \tag{20}$$

Using assumption 1 for $\alpha-$mixing sequences (ε_i^2), central limit theorem is established, so we have that

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i^2 - \sigma^2\right) \to N(0,\ \mathrm{Var}(\varepsilon_1^2)) . \tag{21}$$

Combining (20) with (21), we have shown that (19) holds.□

References

[1] Fu Xia Cheng; Journal of Statistical Planning and inference, 141, 1588, 2011.

[2] F. Cheng et al.; Statistics and Probability Letters, 78, 50, 2008.