Journal of **Kim Il Sung** University (Natural Science) Vol. 3 No. 2 Juche103(2014)

The Existence and Uniqueness of Solutions for Infinite Horizon Forward-Backward Stochastic Differential Equations with Poission Jumps

Sin Myong Guk, Mun Chol

Abstract The properties of the infinite horizon forward-backward stochastic differential equation with jumps (JFBSDE) are investigated. First of all, we proved the existence and uniqueness of solutions of the infinite horizon backward stochastic differential equation with jumps(JBSDE). And under monotonicity conditions, the existence and uniqueness of solutions of the forward-backward stochastic differential equation with jumps are proved.

Key word infinite horizon forward-backward stochastic differential equation, Poission jump

Introduction

The great leader Comrade **Kim Il Sung** said as follows.

"**Today scientists and technicians are faced with a very important task. They must forge ahead more energetically with scientific research work in order to make a great contribution to raising the scientific and technological level of the country to a higher stage and developing the national economy at a rapid pace**."("**KIM IL SUNG** WORKS" Vol. 37 P. 359)

The forward stochastic differential equations and the backward stochastic differential equations are mainly considered before respectively.

However, in setting the stochastic controlled problem with the random coefficient and studying its solutions, it was mentioned in many works that combining and considering the two kinds of equations were reasonable and important ways.

With the study of stochastic control theory, the backward stochastic differential equation was established, and it had many adapted fields intensifying the studies.

It was firstly suggested and investigated by [7].

In their work, they suggested the stochastic differential equation $-dy(t)=g(t,\ y(t),\ z(t))dt-z(t)dw_t,\ \ y(T)=\xi$ and called it the backward stochastic differential equation; under the conditions that ξ was $\Im_T-$ measurable and square integrable random variable and that $g(t,\ y,\ z)$ satisfied with Lipschitz condition on $(y,\ z)$ and square integrable on ω, they proved that it had a unique solution $(y(t),\ z(t))$.

After, in [8], $Y_t=\xi+\int_t^\tau g(s,\ Y(s),\ Z(s))ds-\int_t^\tau Z(s)dw_s$ the existence and uniqueness, safety of solutions to backward stochastic differential equation with bounded random terminal times were established and investigated.

In [1] the existence, uniqueness of the solution to backward stochastic differential equation with Poisson jumps was proved.

$$Y(t)=\xi+\int_t^T g(s,\ U_s)ds-\int_t^T U_s(\alpha)\tilde{N}_p(ds,\ d\alpha),\ t\in[0,\ T]$$

In [2] $\begin{cases} dy_t=-g(t,\ y_t,\ z_t,\ u_t)dt+z_t dw_t+\int_{R_o^d} u_t(\alpha)\tilde{N}_p(dt,\ d\alpha),\ t\in[0,\ T] \\ y(T)=\xi \end{cases}$ the backward stochastic differential equation which had not continuous but jumps was firstly suggested and its unique solution $(y(t),\ z(t),\ u(t))$ existed under the some conditions.

And there were many studies on the problem combining backward stochastic differential equation with forward stochastic differential equation.

In [8], it was investigated the uniqueness and existence problem of solution to backward stochastic differential equation $y_t=\xi+\int_t^T g(s,\ x_s,\ y_s,\ z_s)ds-\int_t^T z(s)dw_s$ associated with the unique solution x_t of forward stochastic differential equation $x_t=x+\int_0^t b(s,\ x_s)ds+\int_0^t \sigma(s,\ x_s)dw_s,\ t\in[0,\ T]$ and applications in stochastic control and financial mathematical theory.

And in [4, 5]

$$x_t=x+\int_0^t b(s,\ x_s,\ y_s,\ z_s)ds+\int_0^t \sigma(s,\ x_s,\ y_s,\ z_s)dw_s,\ \ y_t=\xi+\int_t^T g(s,\ x_s,\ y_s,\ z_s)ds-\int_t^T z(s)dw_s$$

the FBSDE was established and they proved the unique existence of its solution and comparison theorem; they proved the existence of the equilibrium point of stochastic differential game problem with random coefficient and sufficient condition for the equilibrium point on based it. Here state equation and objective function are as follows.

$$X_t^u=x+\int_0^t\left(A_s X_s^u+\sum_{i=1}^n C_s^i u_s^i+\beta_s\right)ds+\int_0^t(\sigma_s X_s^u+\alpha_s)dw_s,\ t\le T$$

$$J^i(u)=\frac{1}{2}\mathrm{E}\left[\int_0^T (X_s^u.M_s^i X_s^u+u_s^i.N_s^i u_s^i)ds+X_T^u.Q^i X_T^u\right],\ i=\overline{1,\ n}$$

In [9] it was illustrated the problems on the optimal control of stochastic control problem with random coefficient.

While, in [5]

$$\begin{cases} dX(t)=b(t,\ X(t),\ Y(t),\ Z(t))dt+\sigma(t,\ X(t),\ Y(t),\ Z(t))dB_t \\ -dY(t)=f(t,\ X(t),\ Y(t),\ Z(t))dt-Z(t)dB_t \\ X(0)=x_0,\ (X(\cdot),\ Y(\cdot),\ Z(\cdot))\in M^2(0,\ \infty;\ R^n\times R^n\times R^{n\times d}) \end{cases}$$

it gave the unique existence of solution to the FBSDE on the infinite time interval and comparison theorem.

And, in [6] they proved the unique existence results of solution to the linear forward-backward stochastic differential equation with random coefficients, in [3] the unique existence results of solution to the forward-backward stochastic differential equation which had not jumps was illustrated.

The forward-backward stochastic differential equation which had continuous and discontinuous perturbation was investigated; however, it was the special case that the solution of the backward stochastic differential equation was independent to forward stochastic differential equation.

In [6] discussed the existence and uniqueness of solution for finite horizon BSDE and FBSDE with jumps.

In this paper, we set the infinite horizon forward-backward stochastic differential equation with jumps and proved the existence and uniqueness of its solution.

1. Setting of the Problem

At first, let make the probability space setting the problem and introduce some notations using in this paper.

Let $(\Omega,\ \Im,\ F,\ P)$ be complete stochastic basis.

R^d is d-dimensional Euclidean space and $R_0^d = R^d \setminus \{0\}$. $W=(W_t)_{t\ge 0}$ is d-dimensional standard Brownian motion or Winner process. $N_p(t,\ U)$ is Poisson counting measure of a Poisson point process and $\widetilde{N}_p(t,\ U)$ is the centered martingale measure of $N_p(t,\ U)$. That is, for the $U\in \boldsymbol{B}(R_0^d)$, $\widetilde{N}_p(t,\ U):=N_p(t,\ U)-\mathrm{E}N_p(t,\ U)=N_p(t,\ U)-n(U)t$.

$L^2(0,\ t;R^n)$ is the set of all R^n-valued random variable v_t such that $\mathrm{E}\int_0^t |v_t|^2 dt<\infty$.

We define $UF(d,\ n)$ as the set of $\boldsymbol{B}(R_0^d)$-measurable function which maps R_0^d to R^n such that $\|u\|_2^2:=\int_{R_0^d}|u(\alpha)|^2\ n(d\alpha)<\infty$.

$L^2(0,\ t;\ R^n\times R^n\times R^{n\times d}\times UF(d,\ n))$ is the set of all quadruple $(X_t,\ Y_t,\ Z_t,\ U_t)$ such that $X_t\in R^n$, $Y_t\in R^n$, $Z_t\in R^{n\times d}$, $U_t\in UF(d,\ n)$ are $\Im_t-$adapted processes,

$$\mathrm{E}\int_0^t |X_t(\omega)|^2 dt<\infty,\ \mathrm{E}|Y_t(\omega)|^2\ dt<\infty$$

$$\mathrm{E}\int_0^t \|Z_t(\omega)\|_1^2 dt<\infty,\ \ \mathrm{E}\int_0^t \|U_t(\omega)\|_2^2 dt<\infty .$$

$$\|Z\|_1^2:=\sum_{i=1}^n\sum_{j=1}^n (Z_{ij})^2$$

On the complete stochastic basis $(\Omega,\ \Im,\ F,\ P)$, consider the infinite horizon FBSDE with following jumps:

$$dX(t) = b(t, X_t, Y_t, Z_t, U_t)dt + \sigma(t, X_t, Y_t, Z_t, U_t)dW_t + \\ + \int_{R_0^d} c(X_t, Y_t, Z_t, U_t(\alpha))\widetilde{N}_p(dt, d\alpha), \; t \le T \tag{1}$$

$$X(0) = x_0$$

$$-dY(t) = f(t, X_t, Y_t, Z_t, U_t)dt - Z(t)dW_t - \int_{R_0^d} U_t(\alpha)\widetilde{N}_p(dt, d\alpha) \tag{2}$$

where $x_0 \in R^n$, b, σ, f are defined on $\Omega \times [0, \infty] \times R^n \times R^n \times R^{n\times d} \times UF(d, n)$ with values in R^n, $R^{n\times d}$, R^n, respectively, and c is defined on $\Omega \times [0, \infty] \times R^n \times R^n \times R^{n\times d} \times UF(d, n) \times R^d$ with values in R^n, W_t is d-dimensional Winner process. $N_p(t, U)$ is Poisson counting measure of a Poisson point process and $\widetilde{N}_p(t, U)$ is the centered martingale measure of $N_p(t, U)$.

We have to prove the existence and uniqueness of the quadruple (X_t, Y_t, Z_t, U_t) of $\Im_t$-adapted processes which satisfies with the FBSDEs(1), (2) under the some conditions.

2. The Existence and Uniqueness of Solution for the Infinite Horizon BSDE with Poisson Jumps

We are going to prove the existence and uniqueness of solution for infinite horizon backward stochastic differential equation and illustrate the existence and uniqueness of solution for forward-backward stochastic differential equation and its properties based on it in next section.

2.1. Preliminaries

Let consider the following backward stochastic differential equation on the complete stochastic basis $(\Omega, \Im, F, P)$

$$-dY_t = (G(t, Y_t, Z_t, U_t) + \varphi_t)dt - Z_t dW_t - \int_{R_0^d} U_t(\alpha)\widetilde{N}_p(dt, d\alpha) \tag{3}$$

where $G(t, Y, Z, U)$ is the random process with values in R^n defined on $[0, \infty) \times R^n \times R^{n\times d} \times UF(d, n)$, $\varphi_t \in L^2(0, \infty : R^n)$.

Definition A triple of processes (Y, Z, U) on $(\Omega, \Im, F, P)$ is called a solution of the backward stochastic differential equation (3) on $(\Omega, \Im, F, P)$, if

$$(Y, Z, U) \in L^2(0, \infty : R^n \times R^{n\times d} \times UF(d, n))$$

is satisfied.

For the some positive number $K > 0$, $L^{2,K}(0, \infty : R^n)$ denote the set of all the $\Im_t$ − adapted process Y_t such that $\mathrm{E}\int_0^\infty |Y_t(\omega)|^2 e^{-Kt} dt < \infty$.

$L^{2,K}(0, \infty : R^{n\times d})$, $L^{2,K}(0, \infty : UF(d, n))$ denote the set of all the $\Im_t$ − adapted process Z_t, U_t

such that $\mathrm{E}\int_0^\infty \| Z_t(\omega)\|_1^2 e^{-Kt}dt<\infty$, $\mathrm{E}\int_0^\infty \| U_t(\omega)\|_2^2 \cdot e^{-Kt}<\infty$ and $L^{2,K}(0,\infty:R^n\times R^{n\times d}\times UF(d,n))$ denote the set of triples of the $\Im_t$ – adapted process (Y_t, Z_t, U_t) such that

$$\mathrm{E}\int_0^\infty | Y_t(\omega)|^2+\| Z_t(\omega)\|_1^2+\| U_t(\omega)\|_2^2)\cdot e^{-Kt}dt<\infty .$$

Spaces mentioned above are all Hilbert space.

Assumption 1 ① $\forall (Y, Z, U)\in R^n\times R^{n\times d}\times UF(d, n)$;

$G(\cdot, Y, Z, U)$ is the $\Im_t$-adapted process defined on $[0,\infty)$.

$G(t,0,0,0)\equiv 0,\ \forall t\in R^+,\ G(t, Y, Z, U):\Omega\times[0,\infty)\times R^n\times R^{n\times d}\times UF(d, n)\to R^n$

② $\mathrm{E}\int_0^\infty | G(t, Y_t, Z_t, U_t)|^2 dt<\infty$

③ $G(t, Y, Z, U)$ satisfies with Lipschitz condition on (Y, Z, U). That is,

$$\exists c>0:| G(t, Y_1, Z_1, U_1)-G(t, Y_2, Z_2, U_2)|\le c(| Y_1-Y_2|+\| Z_1-Z_2\|_1+c_2\| U_1-U_2\|_2),$$
$$\forall Y_1, Y_2\in R^n,\ Z_1, Z_2\in R^{n\times d},\ U_1, U_2\in UF(d, n),\ t\in R^+.$$

④ The function G satisfies with weak monotonicity.

$$\exists\rho>0:<G(t, Y_1, Z, U)-G(t, Y_2, Z, U),\ Y_1-Y_2>\le-\rho| Y_1-Y_2|^2$$
$$\forall Y_1, Y_2\in R^n,\ Z\in R^{n\times d},\ U\in UF(d, n),\ t\in R^+$$

Lemma 1 Let (Y^1, Z^1, U^1), $(Y^2, Z^2, U^2)\in L^{2,K}(0,\infty:R^n\times R^{n\times d}\times UF(d, n))$ be solutions of equation (3) correspondent to $\varphi=\varphi^1,\ \varphi=\varphi^2$ $(\varphi^1,\ \varphi^2\in L^{2,K}(0,\infty:R^n))$ and the function G be a mapping $\Omega\times[0,\infty)\times R^n\times R^{n\times d}\times UF(d, n)$ to R^n and satisfied with assumption 1.

Then we have

$$\mathrm{E}\int_0^\infty\left[(-K+2\rho-4C^2-\delta)| Y_t^1-Y_t^2|^2+\frac{1}{2}\| Z_t^1-Z_t^2\|_1^2+\frac{1}{2}\| U_t^1-U_t^2\|_2^2\right]e^{-Kt}dt\le\frac{1}{\delta}\mathrm{E}\int_0^\infty|\varphi_t^1-\varphi_t^2|^2 e^{-Kt}dt .$$

where $\delta>0$ is defined in $-K+2\rho-4C^2-\delta>0$.

2.2. The existence and uniqueness of solution

Here we consider the existence and uniqueness theorem of solution for the backward stochastic differential equation (1) using the previous proved relation.

Let consider the following sequence to prove the existence and uniqueness.

$$\varphi_t^m\equiv I_{[0,m]}(t)\varphi_t,\ t\in[0,\infty),\ m=1, 2, 3, \cdots$$

Then $\varphi_t^m\in L^{2,K}(0,\infty:R^n)$ and $\varphi_t^m\to\varphi_t\ (m\to\infty)$.

Indeed, $\varphi_t\in L^{2,K}(0,\infty:R^n)$, hence $\mathrm{E}\int_0^\infty|\varphi_t^m|^2 e^{-Kt}dt=\mathrm{E}\int_0^m|\varphi_t|^2 e^{-Kt}dt\le\mathrm{E}\int_0^\infty|\varphi_t|^2 e^{-Kt}dt<+\infty$.

Lemma 2 If the solution to equation (3) correspondent to $\varphi=\varphi_t^m$ is (Y_t^m, Z_t^m, U_t^m), then it is a Cauchy sequence in $L^{2,K}(0, \infty : R^n \times R^{n\times d} \times UF(d, n))$.

Theorem 1 Under the assumption 1, for any $\varphi_t \in L^{2,K}(0, \infty : R^n)$, equation (3) has a unique solution (Y_t, Z_t, U_t) in $L^{2,K}(0, \infty : R^n \times R^{n\times d} \times UF(d, n))$.

3. The Existence and Uniqueness of Solution for Infinite Horizon FBSDE with Poisson Jumps

Let consider the infinite horizon FBSDE with Poisson jumps (1), (2) defined on the complete stochastic basis $(\Omega, \Im, F, P)$.

Assumption 2 ① $\exists \mu > 0 : < A(t, V) - A(t, V'), V - V' > \le -\mu |V - V'|^2$

$\forall V, V' \in R^n \times R^n \times R^{n\times d} \times UF(d, n)$, $\forall t \in R^+$

② $\forall v \in R^n \times R^n \times R^{n\times d} \times UF(d, n)$, $A(\cdot, v)$: $\Im_t$-adapted processes defined on $[0, \infty)$

③ $A(t, v)$ satisfies with Lipschitz condition on v That is,

$\exists l > 0 : | A(t, v) - A(t, v')| \le l | v - v'|$, $\forall v, v' \in R^n \times R^n \times R^{n\times d} \times UF(d, n)$, $\forall t \in R^+$.

④ For some suitable number $K' > 0$,

$|b(t, v)|^2 \le K'(1 + |v|^2)$, $|\sigma(t, v)|^2 \le K'(1 + |v|^2)$, $|c(t, v)|^2 \le K'(1 + |v|^2)$.

First, let consider the following equation parameterized by $\theta \in [0, 1]$.

$$\begin{aligned} dX^\theta(t) &= [\theta(t, V^\theta(t)) - \mu(1-\theta)Y^\theta(t) + \Phi(t)]dt + [\theta\sigma(t, V^\theta(t)) - \mu(1-\theta)Z^\theta(t) + \Psi(t)]dW_t + \\ &+ \int_{R_0^d} [\theta C(V^\theta(t), \alpha) - \mu(1-\theta)U^\theta(\alpha) + g(t)]\widetilde{N}_p(dt, d\alpha) \\ -dY^\theta(t) &= [\theta f(t, V^\theta(t)) + \mu(1-\theta)X^\theta(t) + \gamma(t)]dt - Z^\theta(t)dW_t - \int_{R_0^d} U^\theta(\alpha)\widetilde{N}_p(dt, d\alpha) \\ X^\theta(0) &= x_0 \end{aligned} \tag{4}$$

When $\theta = 1$, $\Phi = \Psi = \gamma = g = 0$, equation (1) is equivalent to (2).

We consider the case $\theta = 0$ in (4), that is,

$$\begin{aligned} dX^0(t) &= [-\mu Y^0(t) + \Phi(t)]dt + [-\mu Z^0(t) + \Psi(t)]dW_t + \int_{R_0^d} [-\mu U_t^0(\alpha) + g(t)]\widetilde{N}_p(dt, d\alpha) \\ -dY^0(t) &= [\mu X^0(t) + \gamma(t)]dt - Z^0(t)dW_t - \int_{R_0^d} U_t^0(\alpha)\widetilde{N}_p(dt, d\alpha) \\ X^0(0) &= x_0 \end{aligned} \tag{5}$$

Lemma 3 $\forall x_0 \in R^n$, $\Phi, \gamma, \Psi, g \in L^2(0, \infty)$, the stochastic differential equation (5) has a unique solution $(X^0, Y^0, Z^0, U^0) \in L^2(0, \infty : R^n \times R^n \times R^{n\times d} \times UF(d, n))$.

Theorem 2 Under the assumption 2, if there exists $\exists \theta_0 \in [0, 1)$, $\forall x_0 \in R^n$, $\Phi, \gamma, \Psi, g \in L^2(0, \infty)$ such that equation (4) has a unique solution in $L^2(0, \infty : R^n \times R^n \times R^{n\times d} \times UF(d, n))$, then for any $\forall x_0 \in R^n$, $\Phi, \gamma, \Psi, g \in L^2(0, \infty)$, there exists $\exists \delta_0 > 0 : \forall \delta \in [0, \delta_0]$ such that equation (4) has a unique solution $(X^{\theta_0+\delta}(\cdot), Y^{\theta_0+\delta}(\cdot), Z^{\theta_0+\delta}(\cdot), U^{\theta_0+\delta}(\cdot))$ in $L^2(0, \infty : R^n \times R^n \times R^{n\times d} \times UF(d, n))$.

Proof By the assumption, for any quadruple

$$v=(x,\ y,\ z,\ u)\in L^2(0,\infty:R^n\times R^n\times R^{n\times d}\times UF(d,\ n)),$$

the following unique solution for FBSDE;

$$\begin{aligned}
dX(t)&=[\theta_0 b(t,\ V(t))-\mu(1-\theta_0)Y(t)+\delta(b(t,\ v(t))+\mu y(t))+\Phi(t)]dt+\\
&\quad+[\theta_0\sigma(t,\ V(t))-\mu(1-\theta_0)Z(t)+\delta(\sigma(t,\ v(t))+\mu z(t))+\Psi(t)]dW_t+\\
&\quad+\int_{R_0^d}[\theta_0 c(V(t),\ \alpha)-\mu(1-\theta_0)U_t(\alpha)+\delta(c(v(t),\ \alpha)+\mu u(t))+g(t)]\widetilde{N}_p(dt,\ d\alpha)\\
-dY(t)&=[\theta_0 f(t,\ V(t))+\mu(1-\theta_0)X(t)+\delta(f(t,\ v(t))-\mu x(t))+\gamma(t)]dt-\\
&\quad-Z(t)dW_t-\int_{R_0^d}U_t(\alpha)\widetilde{N}_p(dt,\ d\alpha)\\
X^\theta(0)&=x_0
\end{aligned}\tag{6}$$

From equation (6), we consider the operator

$I_{\theta_0+\delta}:v\in L^2(0,\ \infty:R^n\times R^n\times R^{n\times d}\times UF(d,\ n))\mapsto V\in L^2(0,\ \infty:R^n\times R^n\times R^{n\times d}\times UF(d,\ n))$.

Let

$$v'=(x',\ y',\ z',\ u'),\ \hat{v}=(x-x',\ y-y',\ z-z',\ u-u')=(\hat{x},\ \hat{y},\ \hat{z},\ \hat{u}),$$

$$V'=(X',\ Y',\ Z',\ U'),\ \hat{V}=(X-X',\ Y-Y',\ Z-Z',\ U-U')=(\hat{X},\ \hat{Y},\ \hat{Z},\ \hat{U})$$

and choose the sequence $\{T_i\}$ $(0\le T_1<T_2<\cdots<T_i<\cdots,\ T_i\to\infty)$ such that

$$\lim_{i\to\infty}\mathrm{E}<\hat{X}(T_i),\ \hat{Y}(T_i)>=0.$$

To apply the stochastic integral deformation formula to $<\hat{X},\ \hat{Y}>$ in $[0,\ T_i]$ and consider the conditional mathematical expectation with respect to $\Im_t$, we can obtain the following relation;

$$\begin{aligned}
&\mathrm{E}<\hat{X}(T_i),\ \hat{Y}(T_i)>-\mathrm{E}<\hat{X}(0),\ \hat{Y}(0)>=\theta_0\mathrm{E}\int_0^{T_i}<(A(t,\ V)-A(t,\ V')),\ \hat{V}>dt-\\
&\quad-\mu(1-\theta_0)\mathrm{E}\int_0^{T_i}(<\hat{X},\ \hat{X}>+<\hat{Y},\ \hat{Y}>+<\hat{Z},\ \hat{Z}>_1+<\hat{U},\ \hat{U}>_2)dt+\\
&\quad+\delta\mathrm{E}\int_0^{T_i}(\mu<\hat{X},\ \hat{x}>+\mu<\hat{Y},\ \hat{y}>+\mu<\hat{Z},\ \hat{z}>_1+\mu<\hat{U},\ \hat{u}>_2+\\
&\quad+<\hat{X},\ -\bar{f}>+<\hat{Y},\ \bar{b}>+<\hat{Z},\ \bar{\sigma}>_1+<\hat{U},\ \bar{c}>_2)dt\le\\
&\quad\le\mathrm{E}\int_0^{T_i}\Big[-\mu|\hat{V}|^2+\frac{1}{2}\delta\mu(|\hat{X}||\hat{x}|+|\hat{Y}||\hat{y}|+|\hat{Z}||\hat{z}|+|\hat{U}||\hat{u}|)+\\
&\quad+\frac{4}{2}\delta l(|\hat{X}|+|\hat{Y}|+|\hat{Z}|+|\hat{U}|)|\hat{V}|\Big]dt
\end{aligned}$$

where $\bar{f}=f(t,\ v)-f(t,\ v')$, $\bar{b}=b(t,\ v)-b(t,\ v')$, $\bar{\sigma}=\sigma(t,\ v)-\sigma(t,\ v')$, $\bar{c}=c(t,\ v)-c(t,\ v')$.

To consider the limit of the above inequality with respect to $i\to\infty$,

$$\mu\mathrm{E}\int_0^\infty|\hat{V}|^2\,dt\le\delta\mathrm{E}\int_0^\infty\frac{\mu+4l}{2}(|\hat{V}|^2+|\hat{v}|^2)dt,\ \delta_0=\frac{\mu}{2(\mu+4l)},\ \delta\in[0,\ \delta_0],\ \mathrm{E}\int_0^\infty|\hat{V}|^2\,dt\le\frac{1}{3}\mathrm{E}\int_0^\infty|\hat{v}|^2\,dt$$

therefore mapping $I_{\theta_0+\delta}(\cdot)$ is contraction.

Therefore it has a unique fixed point and we obtain the existence of the unique solution of (4) when we set $\theta = \theta_0 + \delta$ to substitute it to equation (6).

Theorem 3 Under the assumption 2, for any $x_0 \in R^n$, equation (4) has a unique solution in

$$L^2(0,\ \infty : R^n \times R^n \times R^{n\times d} \times UF(d,\ n)).$$

References

[1] **김일성**종합대학학보(자연과학), 46, 5, 3, 주체89(2000).

[2] **김일성**종합대학학보(자연과학), 46, 12, 2, 주체89(2000).

[3] F. Delarue; Stochastic Processes and Their Applications, 116, 1172, 2006.

[4] S. Hamadene; Stochastic Process Appl., 17, 1, 1998.

[5] S. Peng et al.; Stochastic Process Appl., 85, 75, 2000.

[6] J. Yong; Probab. Theory Relat. Fields, 135, 53, 2006.

[7] E. Paradoux et al.; System and Control Letters, 14, 55, 1990.

[8] S. Peng; Appl. Math. Opto., 27, 125, 1993.

[9] S. Tang; SIAM J. Control Optim., 42, 53, 2003.