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Property of Goodness-of-Fit Test of the Errors in Nonlinear Autoregressive Time Series Models with α−Mixing Errors

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Abstract We derived the asymptotic normality of test statistics T_n, $\hat{T}_n$ in nonlinear autoregressive time series models with stationary $\alpha-$mixing error terms; therefore we extended previous research results obtained in the model with i.i.d error terms to more general case.

Key word nonlinear autoregressive model

Introduction

The great leader Comrade **Kim Il Sung** said as follows.

"**We should actively develop the major areas of basic sciences such as mathematics, physics, chemistry and biology so as to raise the national standard of science and technology and find more effective solutions to the scientific and technical problems that arise in the different branches of the national economy.**"("**KIM IL SUNG** WORKS" Vol. 35 P. 313)

It is meaningful to consider the goodness-of-fit-test of the error distribution in time series models.

In autoregressive time series models, the goodness-of-fit test based on the residual empirical process has been extensively studied.

In [3, 4] researchers suggested the Bickel-Rosenblatt test statistic firstly based on the integrated squared error of the kernel type density estimator from the residuals.

In [5] derived the asymptotic normality of the test statistic under the null-hypothesis in linear autoregressive models with i.i.d error terms.

In [2] showed that the Bickel- Rosenblatt test statistic also converges asymptotically to a normal distribution under a fixed alternative.

In [1] derived asymptotic normality of the Bickel- Rosenblatt test statistic in nonlinear autoregressive time series models with i.i.d. errors.

We consider the asymptotic normality of the Bickel-Rosenblatt test statistic for goodness-of-fit test of error density in nonlinear autoregressive time series models with $\alpha-$mixing errors.

The test statistic is based on the integrated squared error of the nonparametric error density estimate and the error density under the null-hypothesis.

1. Model and Statistics

Let $\{X_i,\ i=0,\ \pm 1, \pm 2, \cdots\}$ be a strictly stationary process of real random variables obeying the model

$$X_i = r_\theta(X_{i-1}, \cdots, X_{i-p}) + \varepsilon_i \tag{1}$$

where $r_\theta,\ \theta \in \Theta$, is a family of known measurable functions for $\theta = (\theta_1, \cdots, \theta_q)$ from $R^p \to R$. Moreover, the errors $\{\varepsilon_i\}$ are assumed to be $\alpha-$mixing random variables with common density f and $X_{i-1}, \cdots, X_{i-p}$ are independent of $\{\varepsilon_i,\ i=1, 2, \cdots\}$.

The problem of interest is to test the hypothesis

$$H_0: f = f_0,\ \ H_1: f \neq f_0 \tag{2}$$

where f_0 is a prescribed density, based on the data $\{X_{1-p}, \cdots, X_0, X_1, \cdots, X_n\}$.

The proposed test is based on the integrated square deviation of a kernel type density estimator form the expectation of the kernel error density.

Especially, let $\hat\theta = (\hat\theta_1, \cdots, \hat\theta_q)^T$ be an estimator for θ, and define the residuals

$$\hat\varepsilon_i = X_i - r_{\hat\theta}(X_{i-1}, \cdots, X_{i-p}),\ i = 1,\ 2, \cdots. \tag{3}$$

Next, let K be a kernel density function and $h \equiv h_n$ be a sequence of positive numbers tending to zero, and we will define a kernel type estimator of the error density $f(t)$ to be

$$\hat f_n(t) = \frac{1}{n}\sum_{i=1}^n K_{h_n}(t - \hat\varepsilon_i),\quad t \in R$$

where $K_h(t) = (1/h)K(t/h)$.

We also consider the kernel error density based on the true errors $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$

$$f_n(t) = \frac{1}{n}\sum_{i=1}^n K_{h_n}(t - \varepsilon_i),\ \ t \in R.$$

We use the integrated squared deviation of $\hat f_n$ from $\mathrm{E} f_n(t) = \int K(x) f(t - h_n(x))dx = K_h * f(t)$, to test (2), i.e., we reject the null-hypothesis H_0 for the large values of the statistic

$$\hat T_n = \int [\hat f_n(t) - K_h * f_0(t)]^2 dt. \tag{4}$$

Note that $\hat T_n$ is an analogue of the Bickel-Rosenblatt statistic proposed in the case of the observable ε_i's

$$T_n = \int [f_n(t) - \mathrm{E}(f_0(t))]^2 dt \tag{5}$$

2. Main Assumptions

We first introduce the following assumption on the autoregressive function r_θ and on the estimator θ for $\hat\theta$.

Assumption 1 Let $U \subset \Theta \subset R^q$ be an open neighborhood of θ.

We assume that, for all $\forall y \in R^q,\ \theta = (\theta_1, \cdots, \theta_q) \in U,\ j,\ k = 1, \cdots, q$

$$\left|\frac{\partial}{\partial\theta_j} r_\theta(y)\right| \le M_1(y), \quad \left|\frac{\partial^2}{\partial\theta_j\partial\theta_k} r_\theta(y)\right| \le M_2(y)$$

where $\mathrm{E}M_1^4(X_{i-1}, \cdots, X_{i-p}) < +\infty$, and $\mathrm{E}M_2^4(X_{i-1}, \cdots, X_{i-p}) < +\infty$ for $i \ge 1$.

For $1 \le i \le n$ and $1 \le j \le q$, set $Y_{ij} = \frac{\partial}{\partial\theta_j} r_\theta(X_{i-1}, \cdots, X_{i-p})$.

Assumption 2 We assume that there exists $\alpha < 1$ such that Y_{ij} satisfying

$$\sum_{i=1}^n Y_{ij} = O_p(n^\alpha), \; j = 1, 2, \cdots, q.$$

Assumption 3 The any estimator $\hat\theta = (\hat\theta_1, \cdots, \hat\theta_q)^T$ for θ based on $X_0, X_1, \cdots, X_n$ satisfies the law of iterated logarithm, i.e., there exists a constant C_1 $(0 < C_1 < \infty)$ such that

$$\limsup_{n\to\infty} \sqrt{\frac{n}{\log(\log n)}}\, |\theta - \hat\theta| \le C_1.$$

where

$$|\theta - \hat\theta| = \sqrt{\sum_{j=1}^q (\hat\theta_j - \theta_j)^2}. \tag{6}$$

We shall derive the asymptotic distribution of T_n under H_0.

In order to calculate the probability of type II error when $\hat T_n$ is used to test hypothesis (2), we shall consider the asymptotic distribution of $\hat T_n$ under in H_1 of hypothesis (2) in the sense of $d(f, f_0) = \int (f - f_0)^2(x)dx > 0$.

We conclude this part with some basic assumptions on the error density function f, the kernel density function K and bandwidth h_n.

Assumption 4 f is two order continuously differentiable with bounded first and second derivatives, and f^2 is integrable.

Assumption 5 K is a continuous bounded symmetric kernel with compact support.

Assumption 6 K''' exists and is bounded. K', $(K')^2$, K'', $(K'')^2$ are integrable.

Assumption 7 $nh_n^2 \to \infty$ and $h_n \to 0\ (n \to \infty)$.

Assumption 5 implies that $\int x^2 K(x)dx < \infty$ and $\int K^2(x)dx < \infty$.

3. Results

Let 's consider the asymptotic distribution of Bickel- Rosenblatt test statistic T_n.

First, we formulate a property of weak stationary random variables $\alpha-$mixing.

Lemma 1 Suppose that stationary sequence $\{X_t\}$ satisfies strong mixing condition.

If random variable ξ is measurable for $\Im_{-\infty}^t$ and random variables η is measurable for $\Im_{t+\tau}^\infty$ and there exists following properties $|\xi| < C_1$, $|\eta| < C_2$.

Then we obtain $|\mathrm{E}\xi\eta - \mathrm{E}\xi \cdot \mathrm{E}\eta| \le 4C_1C_2\alpha$.

Lemma 2 Under assumptions 1, 3, we have the following.

$$\sum_{i=1}^{n}(\hat{\varepsilon}_i-\varepsilon_i)^2=O_P(\log(\log n))$$

Lemma 3 Under assumptions 4－6, we have

$$\int\left[\mathrm{E}\left(K'\left(\frac{t-\varepsilon_i}{h_n}\right)\right)\right]^2 dt=O(h_n^2),\quad \int\left[\mathrm{E}\left(K'\left(\frac{t-\varepsilon_1}{h_n}\right)\right)\right]^2 dt=O(h_n),$$

$$\int\left[\mathrm{E}\left(K''\left(\frac{t-\varepsilon_i}{h_n}\right)\right)\right]^2 dt=O(h_n^2),\quad \int\left[\mathrm{E}\left(K''\left(\frac{t-\varepsilon_1}{h_n}\right)\right)\right]^2 dt=O(h_n).$$

Lemma 4 Suppose assumptions 1－7 hold.

Then, under the following further assumptions on the bandwidth h_n:

$$n^{2(\alpha-1)}h_n^{-3/2}\log(\log n)\to\infty \tag{7}$$

we have

$$\int[\hat{f}_n(t)-f_n(t)]^2 dt=O_p\left(\frac{(\log(\log n))^2}{n^2h_n^4}+\frac{\log(\log n)}{n^{3-2\alpha}h_n^2}\right)=O_p\left(\frac{1}{n\sqrt{h_n}}\right). \tag{8}$$

Theorem 1 If the assumptions 4－7 hold, Bickel-Rosenblatt test statistics

$$T_n=\int[f_n(t)-K_h*f_0(t)]^2 dt$$

has the following properties:

① Under the null hypothesis $H_0: f=f_0$, as $n\to\infty$

$$n\sqrt{h_n}\left[T_n-\frac{1}{nh_n}\int K^2(x)dx\right]\to N\left(0,\ 2\int f_0^2(x)dx\int(K*K)^2(x)dx\right). \tag{9}$$

② Under the alternative $H_1: f\neq f_0$, as $n\to\infty$

$$\sqrt{n}[T_n-\int(K_h*(f-f_0))^2(x)dx]\to N(0,\ 4\mathrm{Var}[(f-f_0)(\varepsilon_1)]). \tag{10}$$

Proof Note that we will establish asymptotic normality under the null hypothesis $f=f_0$ and under fixed alternatives $f\neq f_0$ with different rates of convergence in both cases.

Let f denote the “true” density function of the random variables ε_i.

Recalling the definition of the statistic T_n and the density estimator f_n we obtain the following decomposition:

$$T_n=\int[f_n-K_h*f_0]^2 dx=\int[f_n-K_h*f]^2(x)dx+2\int[f_n-K_h*f](x)dxg_h(x)dx+\int g_h^2(x)dx=$$

$$=\frac{2}{n^2}\sum_{i<j}\int[K_h(x-\varepsilon_i)-e_h(x)][K_h(x-\varepsilon_i)-e_h(x)]dx+$$

$$+\frac{2}{n}\sum_{i=1}^{n}[(K_h*g_h)(\varepsilon_i)-E[(K_h*g_h)(\varepsilon_i)]]+\frac{1}{n^2}\sum_{i=1}^{n}\int[K_h(x-\varepsilon_i)-e_h(x)]^2 dx+g_h^2(x)dx$$

where the functions e_h, g_h are defined by $e_h=K_h*f$ and $g_h=K_h*(f-f_0)$, respectively.

A straightforward calculation shows $\frac{1}{n^2}\sum_{i=1}^{n}\int[K_h(x-\varepsilon_i)-e_h(x)]^2 dx=\frac{1}{nh}\int K^2(x)dx+O_P\left(\frac{1}{n}\right)$.

Consequently we obtain the stochastic expansion

$$T_n - \frac{1}{nh}\int K^2(x)dx - \int K_h * (f - f_0)]^2(x)dx = \frac{2}{n^2}\sum_{i<j} H_n(\varepsilon_i,\ \varepsilon_j) + \frac{2}{n}\sum_{i=1}^{n} Y_i + O_P\left(\frac{1}{n}\right)$$

where

$$H_n(\varepsilon_i,\ \varepsilon_j) = \int [K_h(x-\varepsilon_i) - e_h(x)][K_h(x-\varepsilon_j) - e_h(x)]dx,\ \ Y_i = (K_h * g_h)(\varepsilon_i) - \mathrm{E}[K_h * g_h(\varepsilon_i)]\,.$$

Define the first term in this decomposition as $U_n = \frac{2}{n^2}\sum_{i<j} H_n(\varepsilon_i,\ \varepsilon_j)$ and note that U_n does not depend on the density f_0 specified by the null hypothesis. Obviously, H_n is symmetric, $\lim_{n\to\infty}\mathrm{E}[H_n(\varepsilon_1,\ \varepsilon_2)\,|\,\varepsilon_1] = 0$, $\lim_{n\to\infty}\mathrm{E}[H_n^2(\varepsilon_1,\ \varepsilon_2)] < \infty$ for each $n \in N$.

In fact,

$$\mathrm{E}[H_n(\varepsilon_i,\ \varepsilon_j)\,|\,\varepsilon_i] = \int \mathrm{E}[(K_h(x-\varepsilon_i) - e_h(x))(K_h(x-\varepsilon_j) - e_h(x))\,|\,\varepsilon_i]dx$$

and random variable in the integrate symbol denotes η and consider $\sigma\{\varepsilon_s,\ s \le t\}$ as Lemma 1, we obtain that

$$\mathrm{E}\,|\,\mathrm{E}\{\eta\,|\,\Im_{-\infty}^0\} - \mathrm{E}\eta\,| = \mathrm{E}\{\xi_1(\mathrm{E}(\eta\,|\,\Im_{-\infty}^0) - \mathrm{E}\eta)\}.$$

where $\xi_1 = \mathrm{sgn}(\mathrm{E}(\eta\,|\,\Im_{-\infty}^0) - \mathrm{E}\eta)$ and it is measurable for $\Im_{-\infty}^0$, we have that

$$|\,\mathrm{E}\xi_1\eta - \mathrm{E}\xi_1\mathrm{E}\eta\,| \le 4\,|\,\mathrm{E}\xi_1\eta_1 - \mathrm{E}\xi_1\mathrm{E}\eta_1\,|$$

where $\eta_1 = \mathrm{sgn}(\mathrm{E}\{\xi_1\,|\,\Im_{-\infty}^0\} - \mathrm{E}\xi_1)$ and used $|\,\eta\,| \le 4$.

As the proof of lemma 1, we have

$$\begin{aligned}|\,\mathrm{E}\xi_1\eta_1 - \mathrm{E}\xi_1 E\eta_1\,| = |\,& P(AB) + P(\overline{A}\overline{B}) - P(\overline{A}B) - P(A\overline{B}) - \\ & - P(A)P(B) - P(\overline{A})P(\overline{B}) + P(\overline{A})P(B) + P(A)P(\overline{B})\,| \le 4\alpha\end{aligned}$$

therefore, we have $\mathrm{E}\,|\,\mathrm{E}\{\eta\,|\,\Im_{-\infty}^0\} - \mathrm{E}\eta\,| \le 16\alpha(\tau)$ where $\tau = |\,i-j\,|$.

Thus since $\{\varepsilon_t\}$ is a strictly mixing sequence with coefficient $\alpha(\tau)$ such that left-hand side of above equation converges zero as $\tau \to \infty$.

And Let's see $\xi = K_h(x-\varepsilon_i)$, $\eta = K_h(x-\varepsilon_j)$. By applying the result of lemma 1 again, we obtain $|\,\mathrm{E}\eta\,| \le 4\alpha(\tau)$.

Therefore, we have that $\lim_{n\to\infty}\mathrm{E}[H_n(\varepsilon_i,\ \varepsilon_j)\,|\,\varepsilon_i] = 0$.

As the same way, we can proof the other moment limit results.

Applying the central limit theorem for degenerate *U*-statistics, we obtain the main result of theorem 1.□

Theorem 2 Suppose that assumptions 1－7 are satisfied.

And suppose that the bandwidth h_n satisfies the following:

$$n^{2(\alpha-1)}h_n^{-2}\log(\log n) \to 0 \qquad (11)$$

$$n^{-1}h_n^{-4}(\log(\log n))^2 \to 0 \qquad (12)$$

Then, the test statistics $\hat{T}_n$ has the following properties:

① Under the null hypothesis $H_0: f = f_0$, as $n \to \infty$

$$n\sqrt{h_n}\left[\hat{T}_n - \frac{1}{nh_n}\int K^2(x)dx\right] \to N\left(0,\ \ 2\int f_0^2(x)dx\int (K*K)^2(x)dx\right).$$

② Under the alternative $H_1 : f \neq f_0$, as $n \to \infty$

$$n\sqrt{h_n}\left[\hat{T}_n - \int K_h * (f - f_0))^2(x)dx\right] \to N(0,\ 4\operatorname{var}[(f - f_0)(\varepsilon_1)]).$$

Proof By (9) it is sufficient to show that $n\sqrt{h_n}(\hat{T}_n - T_n) = o_p(1)$.

Using the definition of $\hat{T}_n$ and T_n, we can obtain that

$$|\hat{T}_n - T_n| \le \int (\hat{f}_n(t) - f_n(t))^2 dt + 2\left[\int (\hat{f}_n(t) - f_n(t))^2 dt\right]^{1/2} \sqrt{T_n}\ . \tag{13}$$

Therefore, from lemma 3 and the fact $T_n = O_P\left(\frac{1}{nh_n}\right)$ it follows that

$$|\hat{T}_n - T_n| = o_p\left(\frac{1}{n\sqrt{h_n}}\right) + O_p\left(\frac{1}{n\sqrt{h_n}}\sqrt{\frac{(\log(\log n))^2}{nh_n^4} + \frac{\log(\log n)}{n^{2-2\alpha}h_n^2}}\right) = o_p\left(\frac{1}{n\sqrt{h_n}}\right)$$

where (11) and (12) are also used. Hence, we completed the proof of the first part of theorem 2.

Based on (10), to prove $\sqrt{n}(\hat{T}_n - T_n) = o_P(1)$.

By (10), we can obtain that

$$T_n = O_P(1) \tag{14}$$

Therefore, by (8), (13) and (14), it follows that

$$|\hat{T}_n - T_n| = o_P\left(\frac{1}{\sqrt{n}\sqrt{nh_n}}\right) + O_P\left(\frac{1}{\sqrt{n}}\sqrt{\frac{(\log(\log n))^2}{nh_n^4} + \frac{\log(\log n)}{n^{2-2\alpha}h_n^2}}\right) = o_P\left(\frac{1}{\sqrt{n}}\right)$$

where we also used (11), (12) and the fact $nh_n \to \infty$.

Hence, we completed the proof of theorem 2.□

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