2)

Part 2

Eigenvalues of M

$Mx = \lambda x$

$(M - \lambda I)x = 0$

$$M = \begin{bmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & -4 & 2 \\ -4 & 1-\lambda & -2 \\ 2 & -2 & -2-\lambda \end{bmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ -2 & -2-\lambda \end{vmatrix} - -4\begin{vmatrix} -4 & -2 \\ 2 & -2-\lambda \end{vmatrix} + 2\begin{vmatrix} -4 & 1-\lambda \\ 2 & -2 \end{vmatrix}$$

$$= (1-\lambda)(\lambda^2 + \lambda - 6) + 4(4\lambda + 12) + 2(2\lambda + 6)$$

$$= -\lambda^3 + 7\lambda - 6 + 16\lambda + 48 + 4\lambda + 12$$

$$= -\lambda^3 + 27\lambda + 54$$

$$-\lambda^3 + 27\lambda + 54 = 0$$

$$-(\lambda + 3)^2(\lambda - 6) = 0$$

$$\lambda = [-3, 6]$$

2) Part 2

Eigenvectors of Matrix M

Solve $M - \lambda I$ Eigen value of -3

$$\begin{bmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{bmatrix} - \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$R2 \leftarrow 2R3 + R2$

$$= \begin{bmatrix} 4 & -4 & 2 \\ 0 & 0 & 0 \\ 2 & -2 & 1 \end{bmatrix}$$

$R3 \leftarrow -\frac{1}{2}R1 + R3$

$$= \begin{bmatrix} 4 & -4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R1 \leftarrow \frac{1}{4}R1$

$$= \begin{bmatrix} 1 & -1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \quad x - y + \frac{1}{2}z = 0$$

$$x = y - \frac{1}{2}z$$

$$= \begin{bmatrix} y - \frac{1}{2}z \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}z \\ 0 \\ z \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}}$$

2) Part 2

Eigenvectors of Matrix M

solve $M - \lambda I$

Eigenvalue of 6

$$\begin{bmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{bmatrix} - 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{bmatrix}$$

$R2 \leftarrow 2R3 + R2$

$$= \begin{bmatrix} -5 & -4 & 2 \\ 0 & -9 & -18 \\ 2 & -2 & -8 \end{bmatrix}$$

$R_3 \leftarrow R_3 + \frac{2}{5}R1$

$$= \begin{bmatrix} -5 & -4 & 2 \\ 0 & -9 & -18 \\ 0 & -\frac{9}{5} & -\frac{18}{5} \end{bmatrix}$$

$R_3 \leftarrow -\frac{1}{5}R_2 + R_3$

$$= \begin{bmatrix} -5 & -4 & 2 \\ 0 & -9 & -18 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \leftarrow -\frac{1}{9}R_2$ $\quad$ $R_1 \leftarrow R_1 + 4R_2$ $\quad$ $R_1 \leftarrow -\frac{1}{5}R_1$

$$= \begin{bmatrix} -5 & -4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 10 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

[2] Part 2

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x - 2z = 0$

$y + 2z = 0$

$x = 2z$

$y = -2z$

$$v = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

Eigen vector for $-3, 6$

*

$$= \left[\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right.$$

Part3

$$A = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

$$f(A) = x_{11}^2 x_{22} x_{23} + x_{11} x_{12} x_{13} x_{31} - x_{33}^2 x_{32} x_{21}$$

$$\frac{\partial}{x_{11}} = \frac{\partial}{x_{11}} (x_{11}^2 x_{22} x_{23}) + \frac{\partial}{x_{11}} (x_{11} x_{12} x_{13} x_{31}) \cancel{- \frac{\partial}{x_{11}} (-x_{33}^2 x_{32} x_{21})}$$

$$= 2 x_{11} x_{22} x_{23} + 1$$

$$\frac{\partial f}{\partial x_{11} x_{12} x_{13} x_{31}} = \frac{\partial}{\partial x_{11} x_{12} x_{13} x_{31}} \cancel{x_{11}^2 x_{22} x_{23}} + \frac{\partial}{\partial x_{11} x_{12} x_{13} x_{31}} x_{11} x_{12} x_{13} x_{31} - \frac{\partial}{\partial x_{11} x_{12} x_{13} x_{31}} \cancel{x_{33}^2 x_{32} x_{21}}$$

$$= 1$$

$$\frac{\partial f}{\partial x_{22} x_{23}} = x_{11}^2 + 0 + 0$$

$$= x_{11}^2$$

$$\frac{\partial f}{\partial x_{33}} = \frac{\partial f}{\partial x_{33}} - x_{33}^2 x_{32} x_{21} + \frac{\partial f}{\partial x_{33}} \cancel{x_{11}^2 x_{22} x_{23}} + \frac{\partial}{\partial x_{33}} \cancel{(x_{11} x_{12} x_{13} x_{31})}$$

$$= -2 x_{33} x_{32} x_{21}$$

$$\frac{\partial f}{\partial x_{32} x_{21}} = \frac{\partial f}{\partial x_{32} x_{21}} - x_{33}^2 x_{32} x_{21} + \frac{\partial}{\partial x_{32} x_{21}} \cancel{x_{11}^2 x_{22} x_{23}} + \frac{\partial}{\partial x_{32} x_{21}} \cancel{(x_{11} x_{12} x_{13} x_{31})}$$

$$= -x_{33}^2$$

Part 3

$$\boxed{\nabla f = \left(2x_{11}x_{22}x_{23}+1,\ 1,\ x_{2}^{11},\ -2x_{33}x_{32}x_{21},\ -x_{33}^{1} \right)}$$

Hessian

$$g(x,y,z) = x^3y + yz\sin(x) + xy^2z^5$$

First derivative

$$\frac{\partial}{x} = 3x^2y + yz\cos(x) + y^2z^5$$

$$= y\left(3x^2 + z\cos(x) + yz^5\right)$$

$$\frac{\partial}{y} = x^3 + z\sin(x) + 2yxz^5$$

$$\frac{\partial}{z} = 0 + y\sin(x) + 5z^4xy^2$$

$$= y\sin(x) + 5z^4xy^2$$

Second derivative

$$\frac{\partial}{xx} = 3x^2y + 2y\cos(x) + y^3z^5$$

$$= \boxed{6xy - 2y\sin(x)}$$

$$\frac{\partial}{xy} = \boxed{3x^2 + 2\cos(x) + 3y^2z^5}$$

$$\frac{\partial}{xz} = \boxed{5z^4y^3}$$

$$\frac{\partial}{yx} = x^3 + 2\sin(x) + 2yxz^5$$

$$= \boxed{3x^2 + 2\cos x + 2yz^5}$$

$$\frac{\partial}{yy} = x^3 + 2\sin(x) + 2yxz^5$$

$$= \boxed{2xz^5}$$

$$\frac{\partial}{yz} = x^3 + 2\sin(x) + 2yxz^5$$

$$= \boxed{5z^4 2yx}$$

$$\frac{\partial}{zx} = y\sin(x) + 5z^4xy^2$$

$$= \boxed{2y\cos(x) + 5z^4y^2}$$

$$\frac{\partial}{zy} = \boxed{\sin(x) + 10z^4xy}$$

$$\frac{\partial}{zz} = \boxed{20z^3xy^2}$$

$$H_g = \boxed{\begin{array}{ccc} x & y & z \\ \multicolumn{3}{c}{\begin{bmatrix} 6xy - 2y\cos x & 3x^2 + 2\cos(x) + 3y^2z^5 & 5z^4y^3 \\ 3x^2 + 2\cos x + 2yz^5 & 6z^4 2yx & 5z^4 2yx \\ 2y\cos(x) + 5z^4y^2 & \sin(x) + 10z^4xy & 20z^3xy^2 \end{bmatrix}} \end{array}}$$