and now we need to find a negative θ such that $M(\theta) < e^{\theta a}$. In particular, we need to focus on θ for which the moment generating function is finite. For this purpose let $\mathcal{D}(M) \triangleq \{\theta : M(\theta) < \infty\}$. Namely $\mathcal{D}(M)$ is the set of values θ for which the moment generating function is finite. Thus we call $\mathcal{D}$ the domain of M.

3 Moment generating function. Examples and properties

Let us consider some examples of computing the moment generating functions.

- **Exponential distribution.** Consider an exponentially distributed random variable X with parameter λ. Then

$$\begin{aligned} M(\theta) &= \int_0^\infty e^{\theta x} \lambda e^{-\lambda x} dx \\ &= \lambda \int_0^\infty e^{-(\lambda-\theta)x} dx. \end{aligned}$$

When $\theta < \lambda$ this integral is equal to $\frac{-1}{\lambda-\theta} e^{-(\lambda-\theta)x}\Big|_0^\infty = 1/(\lambda-\theta)$. But when $\theta \geq \lambda$, the integral is infinite. Thus the exp. moment generating function is finite iff $\theta < \lambda$ and is $M(\theta) = \lambda/(\lambda-\theta)$. In this case the domain of the moment generating function is $\mathcal{D}(M) = (-\infty, \lambda)$.

Standard Normal distribution. When X has standard Normal distribution, we obtain

$$\begin{aligned} M(\theta) = \mathbb{E}[e^{\theta X}] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{\theta x} e^{-\frac{x^2}{2}} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{x^2-2\theta x+\theta^2-\theta^2}{2}} dx \\ &= e^{\frac{\theta^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{(x-\theta)^2}{2}} dx \end{aligned}$$

Introducing change of variables $y = x - \theta$ we obtain that the integral is equal to $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{y^2}{2}} dy = 1$ (integral of the density of the standard Normal distribution). Therefore $M(\theta) = e^{\frac{\theta^2}{2}}$. We see that it is always finite and $\mathcal{D}(M) = \mathbb{R}$.

In a retrospect it is not surprising that in this case $M(\theta)$ is finite for all θ. The density of the standard Normal distribution "decays like" $\approx e^{-x^2}$ and