Multiparameter Models (two dimensions)

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Nuisance Parameter I

- In most statistical models, the parameter is divided into two parts $\theta = (\theta_1, \theta_2)$: the part of primary interest or the parameter of interests θ_1, and the part of secondary interest or the nuisance parameter θ_2.

- When this is the case, how to eliminate the nuisance parameter in the statistical inference problem is one of major statistical problem

Nuisance Parameter II

- For this, The Bayesian framework provides a natural solution to this problem.

- The posterior of $\theta = (\theta_1, \theta_2)$ is given by

$$\pi(\theta_1, \theta_2 | D_{1:n}) \propto L(\theta_1, \theta_2)\pi(\theta_1, \theta_2),$$

where $\pi(\theta_1, \theta_2) = \pi(\theta_1)\pi(\theta_2|\theta_1)$ is the prior of θ, and $D = (y_1, \ldots, y_n)$.

Nuisance Parameter III

- Since the uncertainty for θ_1 is represented by the marginal posterior distribution of θ_1, the Bayesian inference for θ_1 is completely determined by

$$\pi(\theta_1|D_{1:n}) = \int \pi(\theta_1, \theta_2|D_{1:n})d\theta_2.$$

Nuisance Parameter IV

- Note

$$
\begin{aligned}
\pi(\theta_1|D_{1:n}) &= \pi(\theta_1, \theta_2|D_{1:n})d\theta_2 \\
&= C \int L(\theta_1, \theta_2)\pi(\theta_2|\theta_1)\pi(\theta_1)d\theta_2 \\
&= C \int L(\theta_1, \theta_2)\pi(\theta_2|\theta_1)d\theta_2\pi(\theta_1).
\end{aligned}
$$

- Thus, some Bayesians recommend use of the integrated likelihood such as $L(\theta_1) = C \int L(\theta_1, \theta_2)\pi(\theta_2|\theta_1)d\theta_2$.

Normal Model with Non-informative Prior I

Prior: $\pi(\mu, \tau^2) \propto (1/\tau^2) d\mu d\tau^2$.

Model: $y_1, \ldots, y_n \sim i.i.d. \mathcal{N}(\mu, \tau^{-2})$.

Normal Model with Non-informative Prior II

Posterior :

$$
\begin{aligned}
\pi(\mu, \tau^2 | y_1, \ldots, y_n) &\propto \tau^{-2} \prod_{i=1}^{n} \left[\tau \exp\left(-\frac{1}{2}\tau^2 (y_i - \mu)^2 \right) \right] \\
&\propto (\tau^2)^{n/2-1} \exp\Big(-\frac{1}{2}\tau^2 \Big(\sum_{i=1}^{n} (y_i - \bar{y})^2 + n(\bar{y} - \mu)^2 \Big) \Big) \\
&\propto (\tau^2)^{n/2-1/2-1} \exp\left(-\frac{1}{2}\tau^2 (n-1) s^2 \right) \\
&\quad \tau \exp\left(-\frac{1}{2}\tau^2 n(\mu - \bar{y})^2 \right).
\end{aligned}
$$

Normal Model with Non-informative Prior III

- Conditional posterior

$$
\begin{aligned}
\tau^2 | y_1, \ldots, y_n &\sim \mathcal{G}amma\left((n-1)/2, (n-1)s^2/2\right), \\
\mu | \tau^2, y_1, \ldots, y_n &\sim N\left(\bar{y}, n^{-1}\tau^{-2}\right).
\end{aligned}
$$

- Marginal posterior of τ^2

$$
\begin{aligned}
\pi(\tau^2 | y_1, \ldots, y_n) &\propto \int (\tau^2)^{n/2-1} \exp\left(-\frac{1}{2}\tau^2 \left\{(n-1)s^2 + n(\bar{y}-\mu)^2\right\}\right) d\mu \\
&\propto (\tau^2)^{n/2-1} \exp\left(-\frac{1}{2}\tau^2 (n-1)s^2\right) \sqrt{\frac{2\pi}{n\tau^2}}.
\end{aligned}
$$

Normal Model with Non-informative Prior IV

* $\mathcal{G}amma((n-1)/2, (n-1)s^2/2)$.

- Interestingly, we have

$$\frac{(n-1)s^2}{\sigma^2} | y_1, \ldots, y_n \sim \mathcal{G}amma((n-1)/2, 1/2) \equiv \chi^2(n-1).$$

Normal Model with Non-informative Prior V

- Marginal posterior of μ

$$
\begin{aligned}
\pi(\mu|y_1,\ldots,y_n) &\propto \int (\tau^2)^{n/2-1} \exp\left(-\frac{1}{2}\tau^2\left\{(n-1)s^2+n(\bar{y}-\mu)^2\right\}\right) d\tau^2 \\
&= \Gamma(n/2)\left[\frac{1}{2}\left\{(n-1)s^2+n(\bar{y}-\mu)^2\right\}\right]^{-n/2} \\
&\propto \left[1+\frac{n(\bar{y}-\mu)^2}{(n-1)s^2}\right]^{-n/2}.
\end{aligned}
$$

* $t_{n-1}(\bar{y}, s^2/n)$, it means the distribution of $\frac{t-\bar{y}}{s/\sqrt{n}}$ where $t \sim t(n-1)$.

Normal Model with Non-informative Prior VI

- Thus, the Bayesian credible set and frequentist confidence interval exactly match.

Normal Model with Non-informative Prior VII

- Predictive distribution: Let $\tilde{y}$ is a new observation. Then,

$$\begin{aligned} p(\tilde{y}|y_1,\ldots,y_n) &\propto \int p(\tilde{y}|\mu,\tau^2,y_1,\ldots,y_n)\pi(\mu,\tau^2|y_1,\ldots,y_n)d\mu d\tau^2 \\ &= \int p(\tilde{y}|\mu,\tau^2)\pi(\mu,\tau^2|y_1,\ldots,y_n)d\mu d\tau^2. \end{aligned}$$

* $t_{n-1}(\bar{y},(1+1/n)s^2)$.

Inverse-Gamma and Inverse-χ^2 I

- Note Inverse-Gamma distribution is that

$$X \sim \mathcal{I}nv - Gamma(\alpha, \beta),$$

where $X = 1/Y, Y \sim \mathcal{G}amma(\alpha, \beta)$.

1. Density of X is

$$\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x).$$

2. Density of Y is

$$\frac{\beta^\alpha}{\Gamma(\alpha)} y^{-\alpha+1} \exp(-\beta/y) \left|\frac{dx}{dy}\right| = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{-\alpha-1} \exp(-\beta/y).$$

Inverse-Gamma and Inverse-χ^2 II

- Inverse-χ^2 distribution
  - $\chi^2(\nu) \equiv \mathcal{G}amma(\nu/2, 1/2)$.
  - $Y \sim \mathcal{I}nverse\text{-}\chi^2(\nu)$ if $1/Y \sim \chi^2(\nu)$.
  - Thus
    1. $\mathcal{I}nverse\text{-}\chi^2(\nu) \equiv \mathcal{I}nverse\text{-}Gamma(\nu/2, 1/2)$.
    2. $\mathcal{I}nverse\text{-}\chi^2(\nu, s^2) \equiv \mathcal{I}nverse\text{-}Gamma(\nu/2, \nu s^2/2)$.

Normal Model with Conjugate Prior or Semi-conjugate Prior I

- Conjugate prior

$$\begin{aligned} \mu|\tau^2 &\sim \mathcal{N}\left(\mu_0, \sigma^2/\kappa_0 = (\tau^2\kappa_0)^{-1}\right), \\ \tau^2 &\sim \mathcal{G}amma(\alpha, \beta). \end{aligned}$$

- Model

$$y_1, \ldots, y_n \sim i.i.d. \mathcal{N}(\mu, \tau^{-2}).$$

Normal Model with Conjugate Prior or Semi-conjugate Prior II

- Posterior

$$
\begin{aligned}
& \pi(\mu, \tau^2 | y_1, \ldots, y_n) \\
\propto \quad & (\tau^2)^{\alpha-1} \exp(-\beta\tau^2)(\kappa_0\tau^2)^{1/2} \exp\left(-\frac{1}{2}\kappa_0\tau^2(\mu-\mu_0)^2\right) \times (\tau^2)^{n/2} \exp\left(-\frac{1}{2}\tau^2 \sum_{i=1}^{n}(y_i-\mu)^2\right) \\
\propto \quad & (\tau^2)^{\alpha+(n+1)/2-1} \exp\left(-\tau^2\left\{\beta + 0.5\kappa_0(\mu-\mu_0)^2 + 0.5(n-1)s^2 + 0.5n(\bar{y}-\mu)^2\right\}\right) \\
\propto \quad & (\tau^2)^{\alpha+(n+1)/2-1} \exp\left(-\tau^2\left\{\beta + 0.5(n-1)s^2\right\}\right) \\
& \times \exp\left(-\frac{1}{2}\tau^2\left\{(\kappa_0+n)\left(\mu - \frac{\kappa_0\mu_0+n\bar{y}}{\kappa_0+n}\right)^2 + \kappa_0\mu_0^2 + n\bar{y}^2 - \frac{(\kappa_0\mu_0+n\bar{y})^2}{\kappa_0+n}\right\}\right).
\end{aligned}
$$

Normal Model with Conjugate Prior or Semi-conjugate Prior III

$$
\begin{aligned}
\mu|\tau^2, y_1, \ldots, y_n &\sim N\left(\frac{\kappa_0\mu_0 + n\bar{y}}{\kappa_0 + n}, \frac{\tau^{-2}}{\kappa_0 + n}\right), \\
\tau^2|y_1, \ldots, y_n &\sim \mathcal{G}amma\Big(\alpha + n/2,\ \beta + \frac{1}{2}(n-1)s^2 \\
&\qquad + \frac{1}{2}(\kappa_0\mu_0^2 + n\bar{y}^2) - \frac{1}{2}\frac{(\kappa_0\mu_0 + n\bar{y})^2}{\kappa_0 + n}\Big).
\end{aligned}
$$

Normal Model with Conjugate Prior or Semi-conjugate Prior IV

- Semi-conjugate prior

$$\begin{aligned} \mu|\tau^2 &\sim N\left(\mu_0, \sigma_0^2 = \tau_0^{-2}\right), \\ \tau^2 &\sim Gamma(\alpha, \beta). \end{aligned}$$

- Model

$$y_1, \ldots, y_n \sim i.i.d. N(\mu, \tau^{-2}).$$

Normal Model with Conjugate Prior or Semi-conjugate Prior V

- Posterior

$$
\begin{aligned}
& \pi(\mu, \tau^2 | y_1, \ldots, y_n) \\
\propto \quad & (\tau^2)^{\alpha-1} \exp(-\beta\tau^2)(\tau_0^2)^{1/2} \exp\left(-\frac{1}{2}\tau_0^2(\mu-\mu_0)^2\right) \times (\tau^2)^{n/2} \exp\left(-\frac{1}{2}\tau^2 \sum_{i=1}^{n}(y_i-\mu)^2\right) \\
\propto \quad & (\tau^2)^{\alpha+n/2-1} \exp\left(-\beta\tau^2\right) \exp\left(-\frac{1}{2}\tau_0^2(\mu-\mu_0)^2\right) \exp\left(-\frac{1}{2}\tau^2 n(\mu-\bar{y})^2 - \frac{1}{2}\tau^2(n-1)s^2\right).
\end{aligned}
$$

Remark : This prior class is not conjugate, but the conditional posterior distributions of μ and τ^2 given everything else are easily identified. (the distribution of $\mu|\tau^2, y$ depends on τ.)

Normal Model with Conjugate Prior or Semi-conjugate Prior VI

$$\begin{aligned}
\mu|\tau^2, y_1, \ldots, y_n &\sim N\left(\frac{\tau_0^2\mu_0 + n\tau^2\bar{y}}{\tau_0^2 + n\tau^2}, \frac{1}{\tau_0^2 + n\tau^2}\right), \\
\tau^2|\mu, y_1, \ldots, y_n &\sim \mathcal{G}amma\left(\alpha + \frac{n}{2}, \beta + \frac{1}{2}(n-1)s^2 + \frac{1}{2}n(\mu - \bar{y})^2\right).
\end{aligned}$$