CSC236 Term Test 1 Version 2 Solution

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Question 1

- *Proof.* Define $P(n) : f(n) = 3^n$.

  I will use complete induction to prove that $\forall n \in \mathbb{N}, n > 2 \Rightarrow P(n)$.

  Base Case ($n = 0$):

  Let $n = 0$.

  Then, the definition of $f(n)$ tells us $f(n) = 1$.

  Then, we have

  $$\begin{align} f(n) &= 3^0 \tag{1} \\ &= 3^n \tag{2} \end{align}$$

  Thus, $P(n)$ follows.

  Base Case ($n = 1$):

  Let $n = 1$.

  Then, the definition of $f(n)$ tells us $f(n) = 3$.

  Then, we have

  $$\begin{align} f(n) &= 3^1 \tag{3} \\ &= 3^n \tag{4} \end{align}$$

Thus, $P(n)$ follows.

Base Case ($n = 2$):

Let $n = 2$.

Then, the definition of $f(n)$ tells us $f(n) = 9$.

Then, we have

$$
\begin{align}
f(n) &= 3^2 \tag{5}\\
&= 3^n \tag{6}
\end{align}
$$

Thus, $P(n)$ follows.

Case ($n > 2$):

Assume $n > 2$.

Then, since $0 \leq n - 1 < n$, $0 \leq n - 2 < n$, and $0 \leq n - 3 < n$, the complete induction tells us $P(n-1)$, $P(n-2)$, and $P(n-3)$, i.e. $f(n-1) = 3^{n-1}$, $f(n-2) = 3^{n-2}$, and $f(n-3) = 3^{n-3}$, respectively.

Then, using these facts, we can write

$$
\begin{align}
f(n) &= f(n-1) + 3f(n-2) + 9f(n-3) \tag{7}\\
&= 3^{n-1} + 3 \cdot 3^{n-2} + 3^2 \cdot 3^{n-3} \tag{8}\\
&= 3^{n-1} + 3^{n-1} + 3^{n-1} \tag{9}\\
&= 3^{n-1}(1 + 1 + 1) \tag{10}\\
&= 3^{n-1}3 \tag{11}\\
&= 3^n \tag{12}
\end{align}
$$

Thus, $P(n)$ follows. □

Correct Solution:

Define $P(n) : f(n) = 3^n$.

I will use complete induction to prove that $\forall n \in \mathbb{N},\ P(n)$.

Inductive Step:

Let $n \in \mathbb{N}$. Assume $H(n) : \bigwedge_{i=0}^{n-1} P(i)$. I will prove $P(n)$ follows. That is, $f(n) = 3^n$.

Base Case ($n = 0$):

Let $n = 0$.

Then, the definition of $f(n)$ tells us $f(n) = 1$.

Then, we have

$$\begin{aligned} f(n) &= 3^0 && (13) \\ &= 3^n && (14) \end{aligned}$$

Thus, $P(n)$ follows.

Base Case ($n = 1$):

Let $n = 1$.

Then, the definition of $f(n)$ tells us $f(n) = 3$.

Then, we have

$$\begin{aligned} f(n) &= 3^1 && (15) \\ &= 3^n && (16) \end{aligned}$$

Thus, $P(n)$ follows.

Base Case ($n = 2$):

Let $n = 2$.

Then, the definition of $f(n)$ tells us $f(n) = 9$.

Then, we have

$$\begin{aligned} f(n) &= 3^2 && (17) \\ &= 3^n && (18) \end{aligned}$$

Thus, $P(n)$ follows.

Case $(n > 2)$:

Assume $n > 2$.

Then, since $0 \leq n-1 < n$, $0 \leq n-2 < n$, and $0 \leq n-3 < n$, the complete induction tells us $P(n-1)$, $P(n-2)$, and $P(n-3)$, i.e. $f(n-1) = 3^{n-1}$, $f(n-2) = 3^{n-2}$, and $f(n-3) = 3^{n-3}$, respectively.

Then, using these facts, we can write

$$\begin{aligned}
f(n) &= f(n-1) + 3f(n-2) + 9f(n-3) && \text{[By definition, since } n > 2\text{]} && (19)\\
&= 3^{n-1} + 3 \cdot 3^{n-2} + 3^2 \cdot 3^{n-3} && && (20)\\
&= 3^{n-1} + 3^{n-1} + 3^{n-1} && && (21)\\
&= 3^{n-1}(1+1+1) && && (22)\\
&= 3^{n-1}3 && && (23)\\
&= 3^n && && (24)
\end{aligned}$$

Thus, $P(n)$ follows.

Notes:

1. Learned that $n > i$ in $\forall n \in \mathbb{N}, n > i \Rightarrow P(n)$ is used when $P(n)$ is true starting $i+1$.

   If $P(n)$ is true for all natural numbers, then $\forall n \in \mathbb{N}, P(n)$ is used.

2. Learned that 'Assume $n > 2$' in 'Let $n \in \mathbb{N}$. Assume $n > 2$. Assume $H(n) : \bigwedge_{i=0}^{n-1} P(i)$'. is used when $P(i)$ is true starting $n = 3$.

   If $P(i)$ is true for all natural numbers, then, 'Let $n \in \mathbb{N}$. Assume $H(n) : \bigwedge_{i=0}^{n-1} P(i)$' is used.

Question 2

- Given the statement to prove

  $P(x, y, z, w)$: There are no positive integers x, y, z, w such that $x^4 + 3y^4 + 9z^4 = 27w^4$.

Proof. I will prove $P(x, y, z, w)$ using proof by contradiction.

Assume $\exists x, y, z, w \in \mathbb{N}^+$, $x^4 + 3y^4 + 9z^4 = 27w^4$.

Then, the set $X = \{x \in \mathbb{N}^+ \mid \exists y, z, w \in \mathbb{N}^+, x^4 + 3y^4 + 9z^4 = 27w^4\}$ is not empty.

Then, by principle of well-ordering, there is smallest positive integer $x_0 \in X$, and postive integers $y_0, z_0, w_0 \in \mathbb{N}^+$ such that $x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4$.

Then,

$$\begin{aligned} x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 &\Rightarrow x_0^4 = 27w_0^4 - 3y_0^4 - 9z_0^4 \\ &\Rightarrow 3 \mid x_0^4 \Rightarrow 3 \mid x_0 \end{aligned} \qquad \text{[By hint]} \qquad (1)$$

$$\begin{aligned} \text{Let } \exists x_1 \in \mathbb{N}^+,\ x_0 = 3x_1 &\Rightarrow x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \\ &\Rightarrow 3y_0^4 = 27w_0^4 - 9z_0^4 - x_0^4 \\ &\Rightarrow 3y_0^4 = 27w_0^4 - 9z_0^4 - 3^4x_1^4 \\ &\Rightarrow y_0^4 = 9w_0^4 - 3z_0^4 - 3^3x_1^4 \\ &\Rightarrow 3 \mid y_0^4 \Rightarrow 3 \mid y_0 \end{aligned} \qquad \text{[By hint]} \qquad (2)$$

$$\begin{aligned} \text{Let } \exists y_1 \in \mathbb{N}^+,\ y_0 = 3y_1 &\Rightarrow x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \\ &\Rightarrow 9z_0^4 = 27w_0^4 - 3y_0^4 - x_0^4 \\ &\Rightarrow 9z_0^4 = 27w_0^4 - 3^5y_1^4 - 3^4x_1^4 \\ &\Rightarrow z_0^4 = 3w_0^4 - 3^3y_1^4 - 3^2x_1^4 \\ &\Rightarrow 3 \mid z_0^4 \Rightarrow 3 \mid z_0 \end{aligned} \qquad \text{[By hint]} \qquad (3)$$

$$\begin{aligned} \text{Let } \exists w_1 \in \mathbb{N}^+,\ w_0 = 3w_1 &\Rightarrow x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \\ &\Rightarrow 3^4x_1^4 + 3^5y_1^4 + 3^6z_1^4 = 3^7w_1^4 \\ &\Rightarrow x_1^4 + 3y_1^4 + 3^2z_1^4 = 3^3w_1^4 \\ &\Rightarrow x_1^4 + 3y_1^4 + 9z_1^4 = 27w_1^4 \\ &\Rightarrow x_1 \in X \end{aligned} \qquad (4)$$

Then, this leads to contradiction, because we know $x_1 < x_0$, $x_1 \in X$, but x_0 is the smallest number in X.

Thus, we can conclude the assumption is false.

$\square$

Correct Solution:

Given the statement to prove

$P(x, y, z, w)$: There are no positive integers x, y, z, w such that $x^4 + 3y^4 + 9z^4 = 27w^4$.

Proof. I will prove $P(x, y, z, w)$ using proof by contradiction.

Assume $\exists x, y, z, w \in \mathbb{N}^+$, $x^4 + 3y^4 + 9z^4 = 27w^4$.

Then, the set $X = \{x \in \mathbb{N}^+ \mid \exists y, z, w \in \mathbb{N}^+, x^4 + 3y^4 + 9z^4 = 27w^4\}$ is not empty.

Then, since X is subset of $\mathbb{N}$, by principle of well-ordering, there is smallest positive integer $x_0 \in X$. Furthermore, there are postive integers $y_0, z_0, w_0 \in \mathbb{N}^+$ such that $x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4$.

Then,

$$\begin{aligned} x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 &\Rightarrow x_0^4 = 27w_0^4 - 3y_0^4 - 9z_0^4 \\ &\Rightarrow 3 \mid x_0^4 \Rightarrow 3 \mid x_0 \end{aligned} \qquad \text{[By hint]} \qquad (1)$$

$$\begin{aligned} \text{Let } \exists x_1 \in \mathbb{N}^+,\ x_0 = 3x_1 &\Rightarrow x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \\ &\Rightarrow 3y_0^4 = 27w_0^4 - 9z_0^4 - x_0^4 \\ &\Rightarrow 3y_0^4 = 27w_0^4 - 9z_0^4 - 3^4x_1^4 \\ &\Rightarrow y_0^4 = 9w_0^4 - 3z_0^4 - 3^3x_1^4 \\ &\Rightarrow 3 \mid y_0^4 \Rightarrow 3 \mid y_0 \end{aligned} \qquad \text{[By hint]} \qquad (2)$$

$$\begin{aligned} \text{Let } \exists y_1 \in \mathbb{N}^+,\ y_0 = 3y_1 &\Rightarrow x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \\ &\Rightarrow 9z_0^4 = 27w_0^4 - 3y_0^4 - x_0^4 \\ &\Rightarrow 9z_0^4 = 27w_0^4 - 3^5y_1^4 - 3^4x_1^4 \\ &\Rightarrow z_0^4 = 3w_0^4 - 3^3y_1^4 - 3^2x_1^4 \\ &\Rightarrow 3 \mid z_0^4 \Rightarrow 3 \mid z_0 \end{aligned} \qquad \text{[By hint]} \qquad (3)$$

$$\begin{aligned} \text{Let } \exists w_1 \in \mathbb{N}^+,\ w_0 = 3w_1 &\Rightarrow x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \\ &\Rightarrow 3^4x_1^4 + 3^5y_1^4 + 3^6z_1^4 = 3^7w_1^4 \\ &\Rightarrow x_1^4 + 3y_1^4 + 3^2z_1^4 = 3^3w_1^4 \\ &\Rightarrow x_1^4 + 3y_1^4 + 9z_1^4 = 27w_1^4 \\ &\Rightarrow x_1 \in X \end{aligned} \qquad (4)$$

Then, this leads to contradiction, because we know $x_1 < x_0$, $x_1 \in X$, but x_0 is the smallest number in X.

Thus, we can conclude the assumption is false.

$\square$

Note:

- Noticed professor wrote 'Divide by 3' as a reasoning in calculation.

$$
\begin{aligned}
\text{Let } z_1 \in \mathbb{N}^+, 3z_1 = z_0 \quad &\Rightarrow \quad 81z_1^4 = 3w_0^4 - (9x_1^4 + 27y_1^4) \\
&\Rightarrow \quad 27z_1^4 = w_0^4 - (3x_1^4 + 9y_1^4) \Rightarrow 3x_1^4 + 9y_1^4 + 27z_1^4 = w_0^4 \qquad \text{\# divide by 3} \\
&\Rightarrow \quad 3 \mid w_0^4 \Rightarrow 3 \mid w_0 \qquad \text{\# since 3 divides LHS and allowed assumption}
\end{aligned}
$$

Question 3