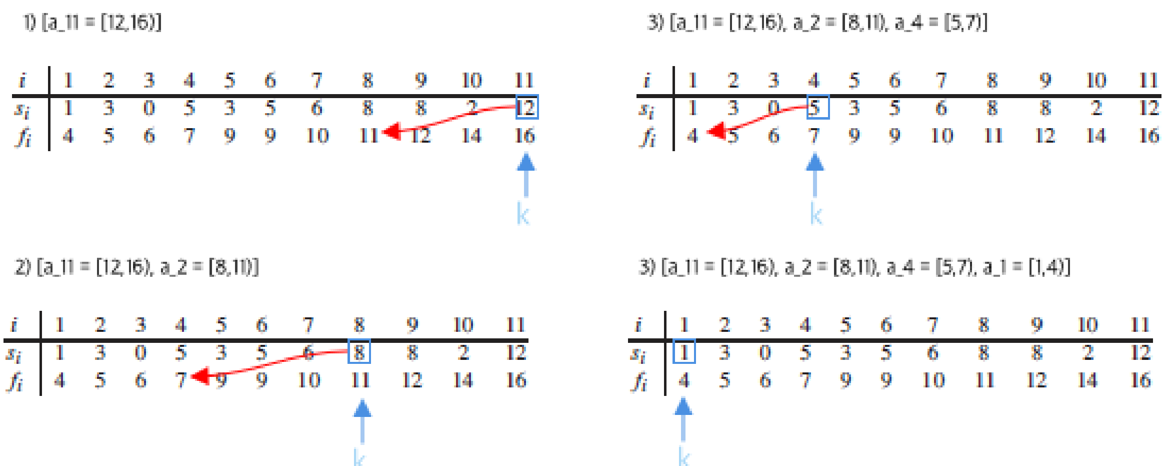
CSC373 Worksheet 2 Solution

July 27, 2020

1. 

1) [a_11 = [12,16)]

i	1	2	3	4	5	6	7	8	9	10	11
s_i	1	3	0	5	3	5	6	8	8	2	$\boxed{12}$
f_i	4	5	6	7	9	9	10	11	12	14	16

k

3) [a_11 = [12,16), a_2 = [8,11), a_4 = [5,7)]

i	1	2	3	4	5	6	7	8	9	10	11
s_i	1	3	0	$\boxed{5}$	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16

k

2) [a_11 = [12,16), a_2 = [8,11)]

i	1	2	3	4	5	6	7	8	9	10	11
s_i	1	3	0	5	3	5	6	$\boxed{8}$	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16

k

3) [a_11 = [12,16), a_2 = [8,11), a_4 = [5,7), a_1 = [1,4)]

i	1	2	3	4	5	6	7	8	9	10	11
s_i	$\boxed{1}$	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16

k

This approach is a greedy algorithm because algorithm

1) Has the greedy choice: selecting the last activity to start that is compatible with all previously selected activites
2) Has the greedy choice that is always part of optimal solution:

   Claim:

   Consider any nonempty subproblem S_k. Let a_m be an activity in S_k with the last activity to start that is compatible with all previously selected activities. Then a_m is included in some maximum-size subset of mutually compatible activities of S_k

   Proof. Let A_k be a maximum-size subset of mutually compatible activities in S_k, and let a_j be the activity in A_k with the last activity to start that is compatible with all previously selected activities.

   If $a_j = a_m$, we are done, since we have shown that a_m is the maximum-size subset of mutually compatible activities of S_k.

If $a_j \neq a_m$, let the set $A'_k = A_k = \{a_j\} \cup \{a_m\}$ be A_k but subtituting a_m for a_j. The activities in A'_k are disjoint, which follow because the activities in A_k are disjoint, a_j is the first activity in A_k to finish, and $s_j \leq s_m$.

Since $|A'_k| = |A_k|$, we conclude that A'_k is a maximum-size subset of mutually compatible activities of S_k, and it includes a_m. □

Notes:

- Greedy Algorithm
  - Always makes the choice that looks best at the moment
    * Locally optimal solution leads to globally optimal solution
- Activity-selection Problem (Greedy algorithm using dynamic programming)
  - Goal: Selecting maximum size set of mutually compatible activities

    Example:

i	1	2	3	4	5	6	7	8	9	10	11
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16

  - Suppose a set exists $S = \{a_1 = [s_1, f_1), a_2 = [s_2, f_2), ..., a_n = [s_n, f_n)\}$
    * a_i represents an i^{th} activity
    * s_i represents starting time
    * f_i represents finishing time
    * $0 \leq s_i < f_i < \infty$
    * $a_1, ..., a_n$ sorted in monotonically increasing order of finish time

      i.e.

      $f_1 \leq f_2 \leq f_3 \leq ... \leq f_{n-1} \leq f_n$

    * a_i and a_j are **compatible**, if intervals $[s_i, f_i)$ and $[s_j, f_j)$ don't overlap

      i.e

      $s_i \geq f_j$ and $s_j \geq f_i$

  - Steps
    1. Think about dynamic programming solution
       * Construct optimal solution using two subproblems

S_{ij}**:** activities that start after activity a_i finishes and before activity a_j starts

i.e.

$S_{19} = \{a_4 = [5, 7), a_6 = [5, 9), a_7 = [6, 10)\}$

A_{ij}**:** maximum set of mutually compatible activities in S_{ij} (including a_k)

- $A_{ik} = A_{ij} \cap S_{ik}$
- $A_{kj} = A_{ij} \cap S_{kj}$
- $A_{ij} = A_{ik} \cup \{a_k\} \cup Akj$
- So, $|A_{ij}| = |A_{ik}| + |A_{kj}| + 1$

∗ Verify that optimal solution A_{ij} must include optimal solution to the two subproblems for S_{kj}

Let A'_{kj} be another mutually compatible activities in S_{kj} where $|A'_{kj}| > |A_{kj}|$.

Then we could use A'_{kj} in a solution to subproblem of S_{ij}

Then we have $|A_{ik}| + |A'_{kj}| + 1 > |A_{jk}| + |A_{kj}| + 1 = |A_{ij}|$ mutually compatible activites

This contradicts assumption that A_{ij} is an optimal solution

∗ Verify that optimal solution A_{ij} must include optimal solution to the two subproblems for S_{ik}

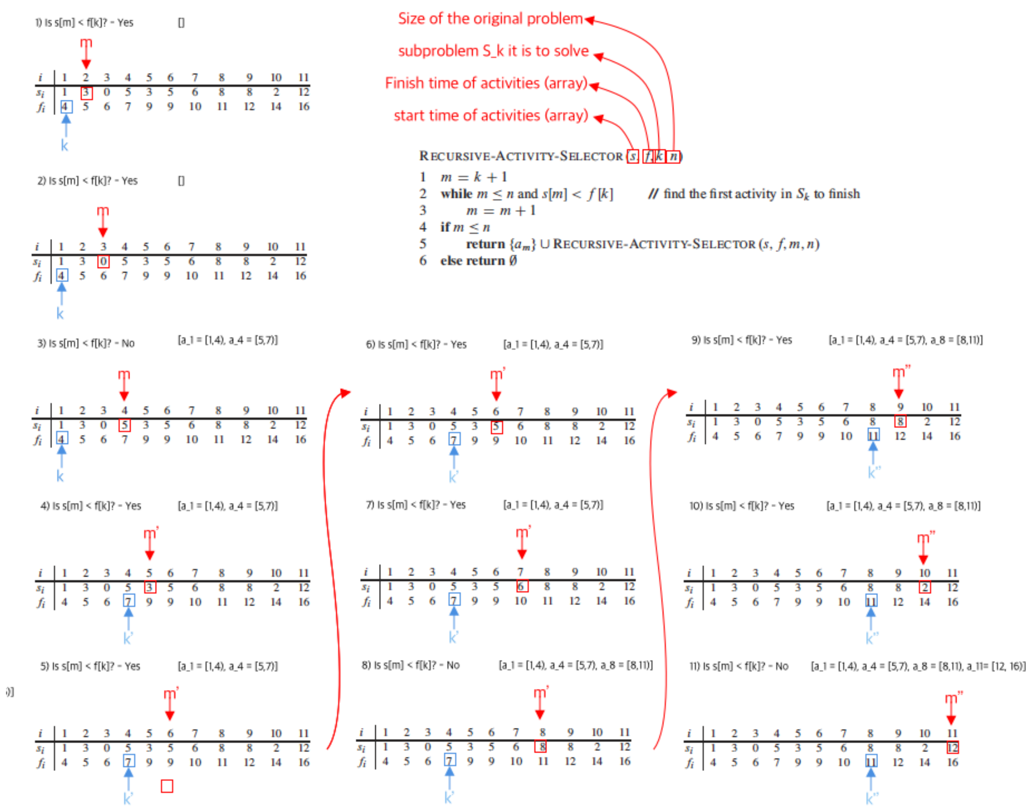
The same applies for activities in S_{ik}

2. Observe that only one choice - greedy choice, and that when we make the greedy choice, only one subproblem remains
   ∗ Steps
   1. Make a greedy choice
      - Choose an activity that makes the most resource possible (intuition)
      - Choose an acitivty that finishes the earliest (intuition)
   2. Solve a subproblem: Find activities that start after a_1 finishes
   3. Verify that making greedy choices always arrive at optimal solution

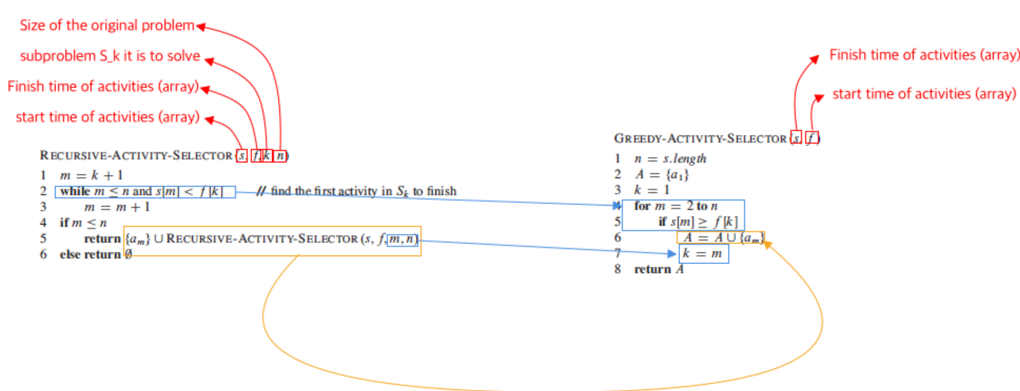
Theorem 16.1 (Page 418):

Consider any non-empty subproble S_k, and let a_m be an activity in S_k with the earliest finish time. Then a_m is included in some maximum-size subset of mutually compatible activities of S_k

3. Develop recursive greedy solution



4. Convert the recursive algorithm into iterative one



2. • Greedy Choice

  - Choose x_i that is greater than the current maximum as the upper bound of unit length closed interval
  - Choose x_i that is smaller than the current minimum as the lower bound of unit length closed interval

Example:

$\{0, 1, 2, 3, 4, 5\} \rightarrow [0, 5]$

$\{0, -1, 3, 5, 2\} \rightarrow [-1, 5]$

- Optimal Substructure

Let I be the following instance of the problem: Let n be the number of items, and let x_i be the i^{th} point in the set.

Let $A = [x_{\min}, x_{\max}]$ be the solution. The greedy algorithm works by assigning $x_{\min} = min(x_{\min}, x_n)$ and $x_{\max} = max(x_{\max}, x_n)$, and then continuing by solving the subproblem

$$I' = (n - 1, \{x_1, ..., x_{n-1}\}) \tag{1}$$

until $n = 0$.

We need to show that the strategy gives optimal solution.

Correct Solution:

1) Consider the left-most interval.
2) Set the left most point x in the set as its value (since we know it must contain the leftmost point)
3) For any point that is within the unit distance of the point x (i.e. $[x, x+1]$), remove the points since they are covered
4) Move to the next closest point not covered by the unit interval of x, and repeat until all points in the set are covered.
5) Since each step has a clearly optimal choice for where to put the leftmost interval, the final solution is optimal

Notes:

- I stopped because it's taking too much time.
- I struggled on this problem.

  - I had trouble understanding the meaning of unit interval
  - I felt there is missing knowledge regarding optimal substructure
  - I felt tunnel visioned to provide one interval that covers all

- I had difficulty arguing why the algorith is correct
  - i.e. How can i generate a claim?
- Unit length
  - $[1, 25, 2.25]$ includes all x_i such that $1.25 \leq x_i \leq 2.25$.
- Greedy-choice property and optimal substructure to problem are the two key ingredients
- Summary of Steps for Greedy Algorithm
  1. Determine the optimal structure of the problem
  2. Develop a recursive solution.
  3. Show that if we make the greedy choice, then only one subproblem remains
  4. Prove that it is always safe to make the greedy choice
  5. Develop a reursive algorithm that implements the greedy strategy
  6. Convert the recursive algorithm to an iterative algorithm
- Criteria for Greedy Algorithm
  1. Greedy-choice property
     - Exists if we can assemble a globally optimal solution by making a locally optimal (greedy) choices
  2. Optimal Substructure
     - Exists if an optimal solution to the problem contains within it optimal solutions to subproblems.
- Greedy vs Dynamic Programming
  - 0-1 Knapsack Problem

0-1 Knapsack Problem

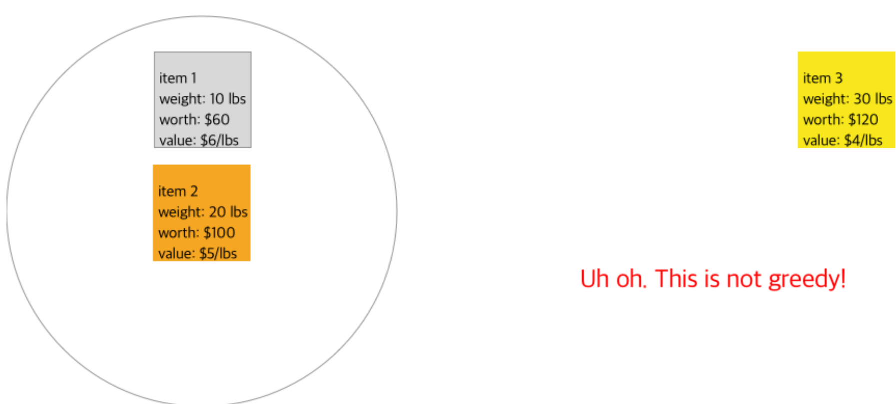
1)

Capacity: 50 lbs
Current: 50lbs

2)

Capacity: 50 lbs
current: 40 lbs

3)

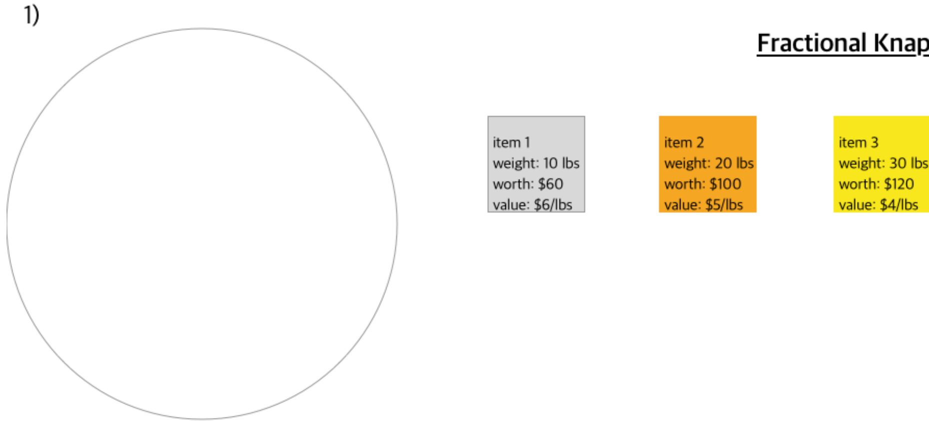


Capacity: 50 lbs
current: 20 lbs

– Fractional Knapsack Problem

Fractional Knapsack Problem

1)

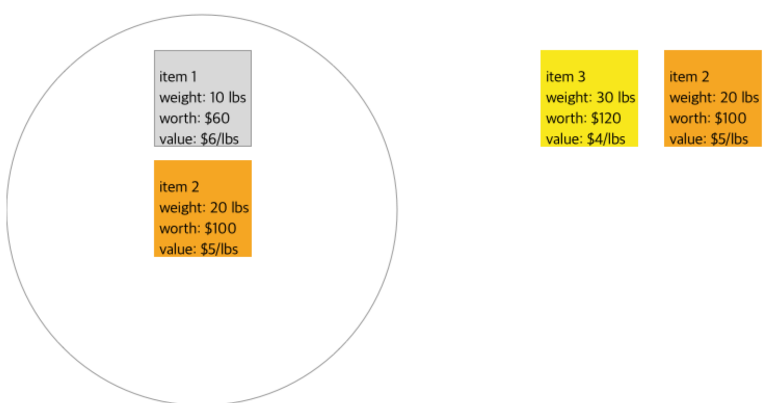


Capacity: 50 lbs
Current: 50lbs

2)

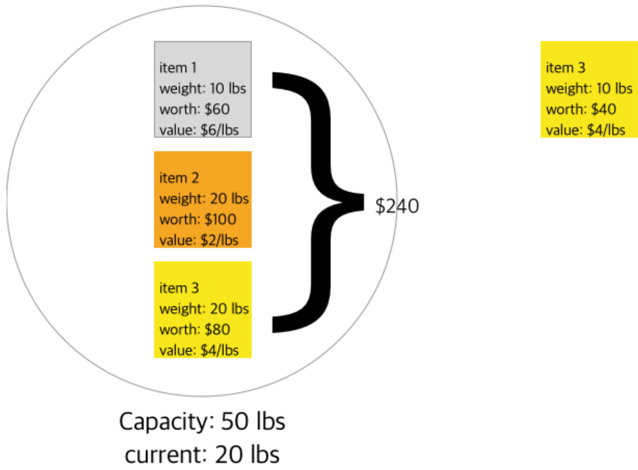
Capacity: 50 lbs
current: 40 lbs

3)



Capacity: 50 lbs
current: 20 lbs

4)



3. *Proof.* Let T be a binary tree corresponding to an optimal prefix code and suppose that T is not full. Let node n have a single child x. Let T' be the tree obtained by removing n and replacing it by x. Let m be leaf node which is descendent of x. Then we have:

My work:

$$\begin{aligned}
B(T') &\leq \sum_{c \in C \setminus \{m\}} c.freq \cdot d_T(c) + m.freq \cdot d_{T'}(m) && (1)\\
&= \sum_{c \in C \setminus \{m\}} c.freq \cdot d_T(c) + m.freq \cdot (d_T(m) - 1) && (2)\\
&< \sum_{c \in C \setminus \{m\}} c.freq \cdot d_T(c) + m.freq \cdot d_T(m) && (3)\\
&= \sum_{c \in C} c.freq \cdot d_T(c) && (4)\\
&= B(T) && (5)
\end{aligned}$$

which contradicts the fact that T was optimal. Therefore every binary tree corresponding to an optimal prefix code is full

$\square$

Notes:

- Optimal Substructure
  - A problem is said to have optimal substructure if an optimal solution can be constructed from optimal solutions of its subproblems.

- Huffman Codes
  - Is an algorithm that uses greedy algorithm for lossless (without loss of data) data compression
  - Has two types of codewords

	a	b	c	d	e	f
Frequency (in thousands)	45	13	12	16	9	5
Fixed-length codeword	000	001	010	011	100	101
Variable-length codeword	0	101	100	111	1101	1100

    * Fixed Length Code
      · has codeword with the same length
    * Variable Length
      · has codeword that may be of different lengths
  - Constructs optimal prefix codes
    * Means no codeword is a prefix of some other codewords

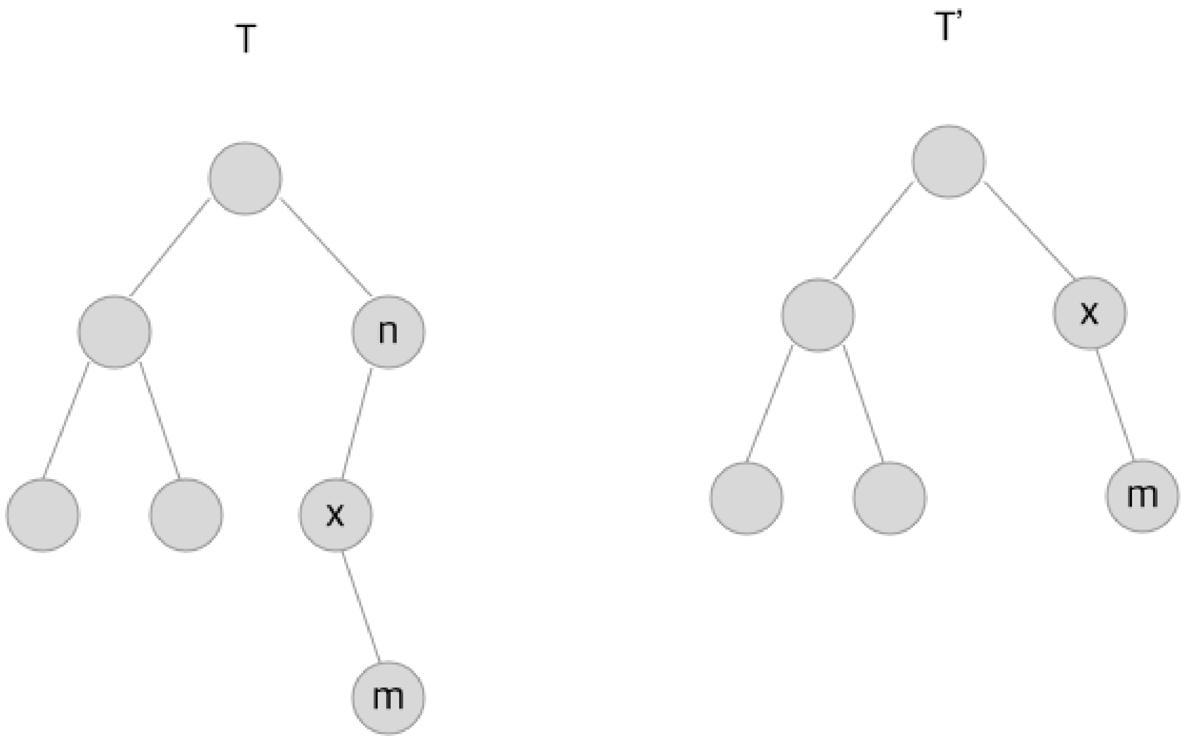
      e.g.
      The following is not prefix codes

      a - 110
      b - 1101

      e.g.
      The following is prefix codes

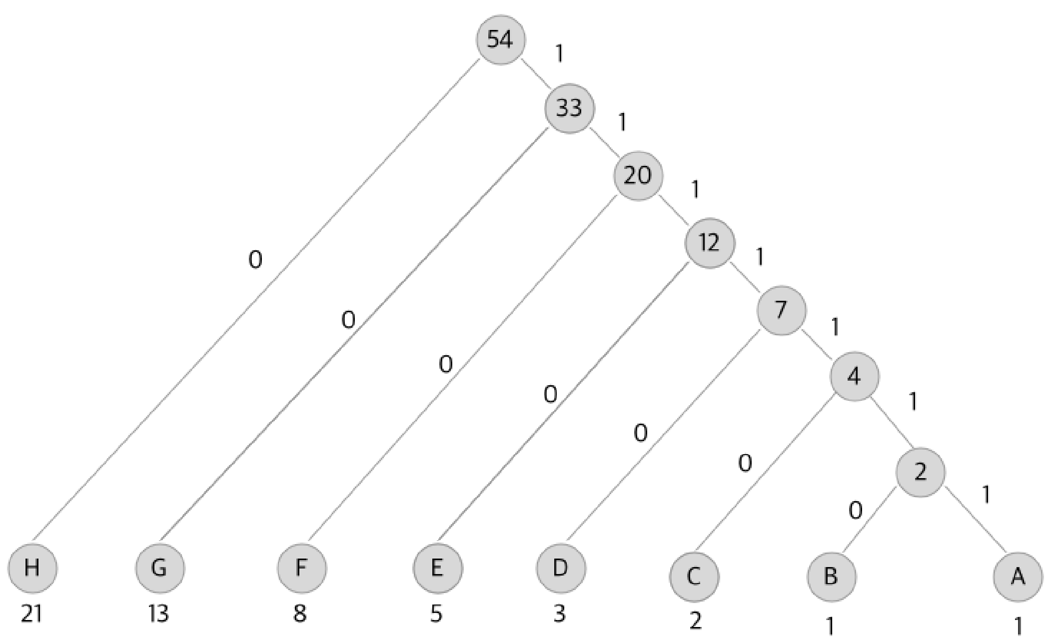
      a - 110
      b - 111

- Realized that I should learn with the solution. Otherwise, it will take too much time.
- Learned that the author used another but very similar tree T' to show the cost of bits in T is not minimum, which is the condition of prefix codes.
- Learned that the solution feels very similar to the proof of optimal substructure on page 416.
- Learned that the tree T and T' looks as follows:

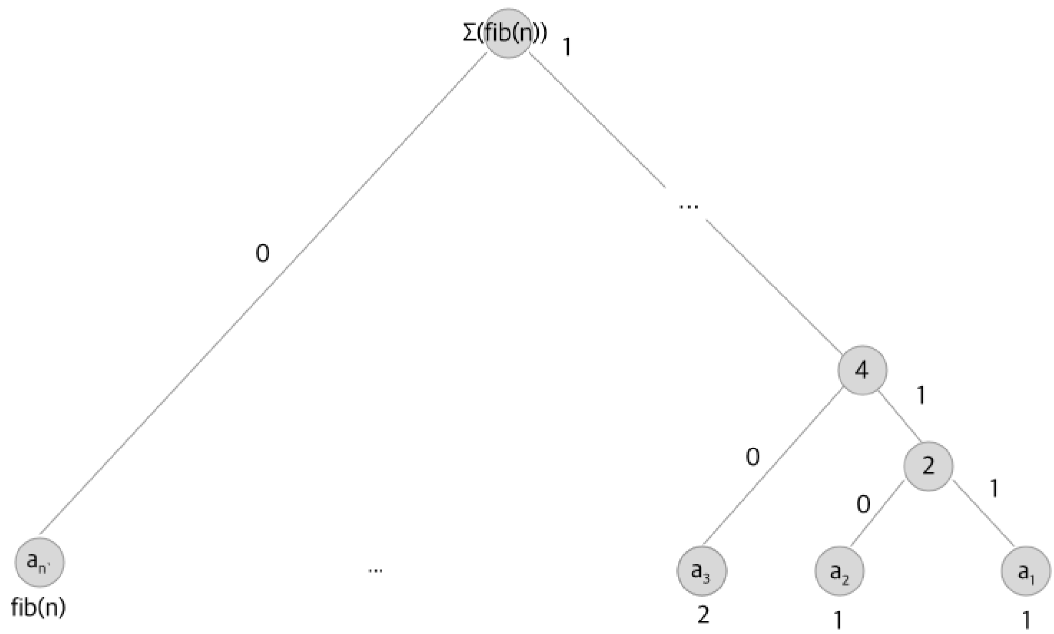


4. **Solution:**

- Finding optimal Huffman code



- Generalizing answer to find the optimal code when the frequencies are first n fibonacci numbers



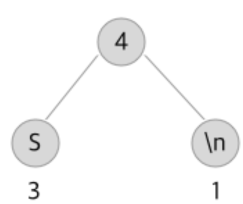
<u>**Notes**</u>

- Constructing Huffman Code

  <u>**Example:**</u>

char	A	E	I	S	T	P	\ n
Freq	10	15	12	3	4	13	1

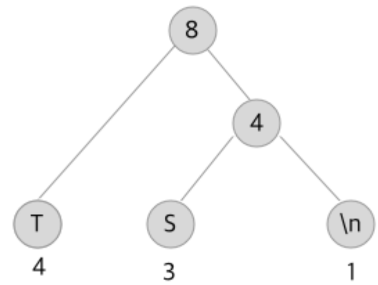
1. Take the 2 chars with the lowest frequency



2. Make a 2 leaf node tree from them

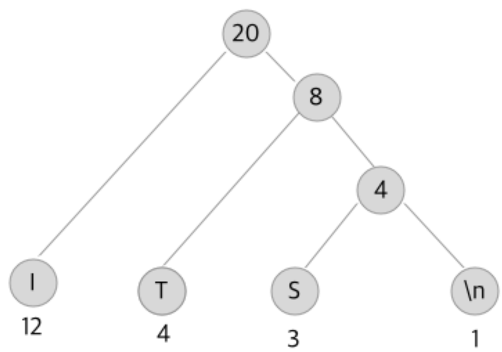
2)

char	A	E	I	S	T	P	\ n
Freq	10	15	12	~~3~~	~~4~~	13	~~1~~



3)

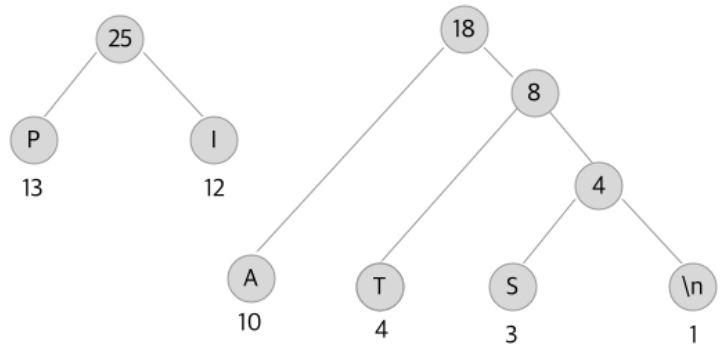
char	A	E	I	S	T	P	\ n
Freq	10	15	~~12~~	~~3~~	~~4~~	13	~~1~~



3. If the node has summed value that is higher than any other values in the table, then repeat 1 and 2 in another tree

4)

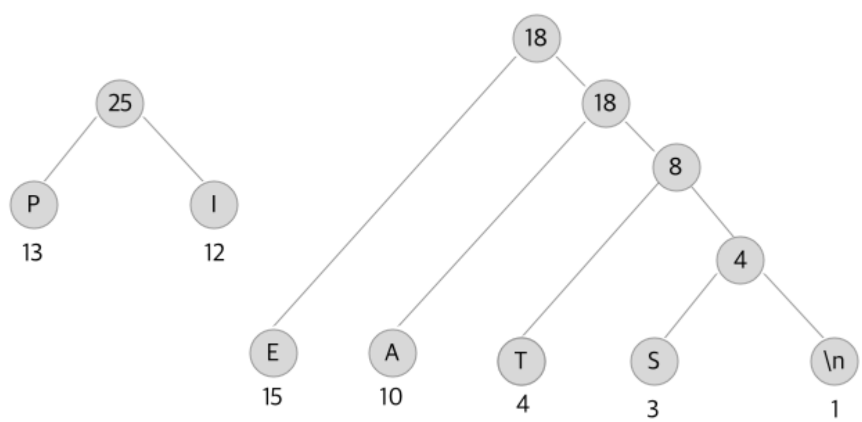
char	A	E	I	S	T	P	\n
Freq	~~10~~	15	~~12~~	~~3~~	~~4~~	~~13~~	~~1~~



4. Attach an additional node to the subtree with the smallest value

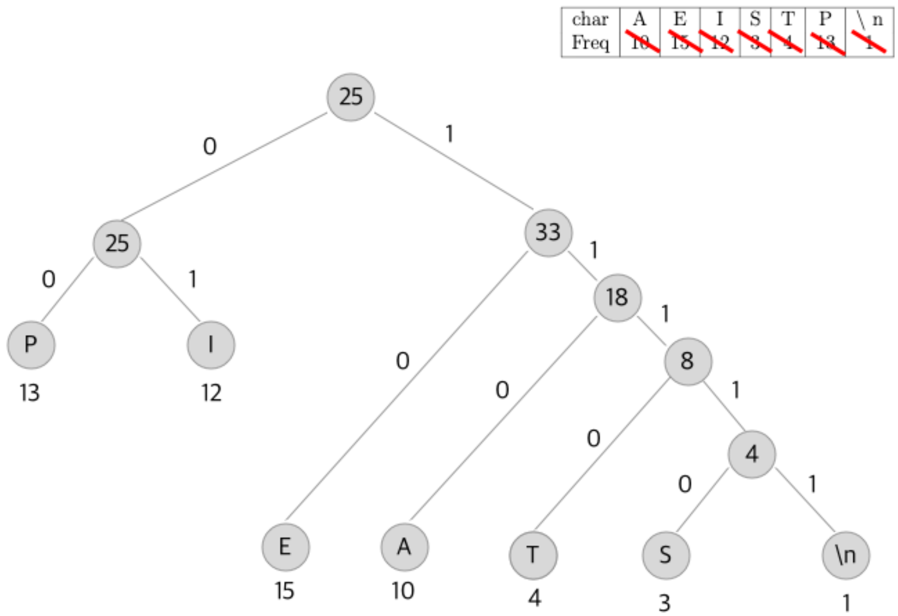
5)

char	A	E	I	S	T	P	\n
Freq	~~10~~	15	~~12~~	~~3~~	~~4~~	~~13~~	~~1~~



5. Repeat step 4 above until done

5)



5. Instead of grouping together the two with lowest frequency into pairs that have the smallest total frequency, we will group together the three with lowest frequency in order to have a final result that is a ternary tree. The analysis of optimality is almost identical to the binary case. We are placing the symbols of lowest frequency lower down in the final tree and so they will have longer codewords than the more frequently occurring symbols.

My Work (with a lot of help from textbook):

- Proof of greedy-choice property for Huffman's Algorithm (Ternary)

Lemma (Modification of lemma on page 433):

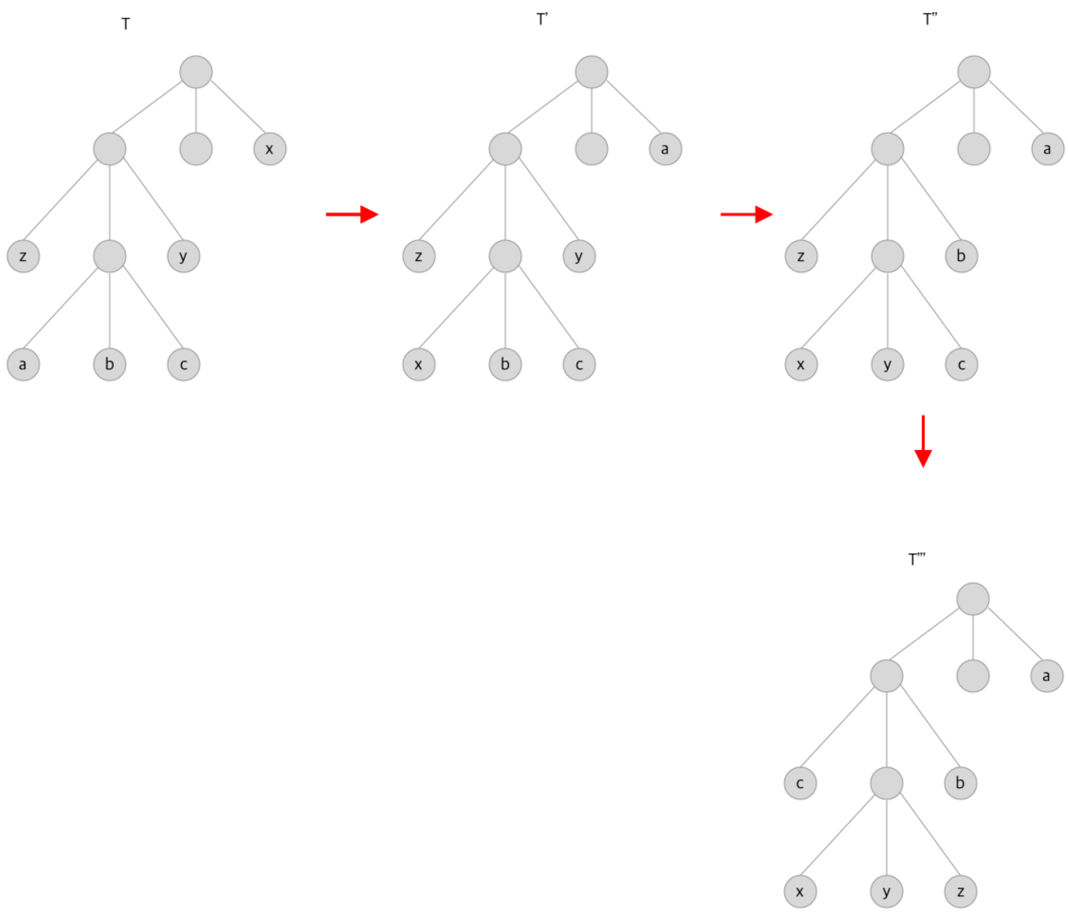
Let C be an alphabet in which each character $c \in C$ has frequency cfreq. Let x, y and z be two characters in C having the lowest frequencies. Then there exists an optimal prefix code for C in which the codewords for x, y and z have the smae length and differ only in the last bit.

Proof. Let a, b and c be three characters that are sibling leaves of maximum depth in T.

Without loss of generality, we assume that $a.freq \leq b.freq \leq c.freq$, $x.freq \leq y.freq \leq z.freq$ Since $x.freq$, $y.freq$ and $z.freq$ are the three lowest leaf frequencies, in order, and $a.freq$ and $b.freq$ are three arbitrary frequencies, in order, we have $x.freq \leq a.freq$, $y.freq \leq b.freq$, and $z.freq \leq c.freq$.

In the remainder of the proof, it is possible that we could have $x.freq = a.freq$, $y.freq = b.freq$, and $z.freq = c.freq$. However, if we had $x.freq = b.freq$ and $x.freq = c.freq$, then we would also have $a.freq = b.freq = c.freqx.freq = y.freq = z.freq$, and the lemma is trivially true. Thus, we will assume $x.freq \neq b.freq$ and $x.freq \neq c.freq$, which means that $x \neq b$ and $x \neq c$.

As the following image shows, we exchange the positions in T of a and x to produce a tree T', and then we exchange the positions in T' of b and y to produce a tree T'', and then we exchange the position in T'' of c and z to produce a tree T''' in which x, y and z are sibling leaves of maximum depth. By equation (16.4), the difference in cost between T and T' is



$$\begin{aligned} B(T) - B(T') &= \sum_{c \in C} c.freq \cdot d_T(c) - \sum_{c \in C} c.freq \cdot d_{T'}(c) && (6)\\ &= x.freq \cdot d_T(x) + a.freq \cdot d_T(a) - x.freq \cdot d_{T'}(x) - a.freq \cdot d_{T'}(a) && (7)\\ &= x.freq \cdot d_T(x) + a.freq \cdot d_T(a)_x.freq \cdot d_T(a) - a.freq \cdot d_T(x) && (8)\\ &= (a.freq - x.freq)(d_T(a) - d_T(x)) && (9)\\ &\geq 0 && (10) \end{aligned}$$

because both $a.freq - x.freq$ and $d_T(a) - d_T(x)$ are nonnegative. More specifically, $a.freq - x.freq$ is non-negative because x is a minimum-frequency leaf, and $d_T(a) - d_T(x)$ is non-negative because a is leaf of maximum depth in T. Similarly, exchanging y and b does not increase the cost, and so $B(T') - B(T'')$ is non-negative. Therefore $B(T'') \leq B(T)$, and since T is optimal we have $B(T) \leq B(T'')$, which implies $B(T) = B(T'')$. Thus T'' is an optimal tree in which x and y appear as sibling leaves of maximum depth, from which the lemma follows.

□

- Proof of optimal structure property for Huffman's Algorithm (Ternary)

Lemma (Modification of lemma on page 435):

Let C be a given alphabet with frequency $c.freq$ defined for each character $c \in C$. Let x, y and z be three characters in C with minimum frequency. Let C' be the alphabet C with characters x, y and z removed and a new character w added, so that $C' = C - \{x, y, z\} \cup \{w\}$. Define $freq$ for C' as for C, except taht $w.freq = x.freq + y.freq + z.freq$. Let T' be any tree representing an optimal prefix code for the alphabet C'. Then the tree T, obtained from T' by replacing the leaf node for w with an internal node having x, y and z as children, represents an optimal prefix code for the alphabet C.

Proof. We first show how to express the cost $B(T)$ of tree T in terms of the cost $B(T')$ of the Tree T' by considering the componentin equation 16.4. For each character $c \in C - \{x, y, z\}$, we have that $d_T(c) = d_{T'}(c)$, and hence $c.freq \cdot d_T(c) = c.freq \cdot d_{T'}(c)$. Since $d_T(w) = d_T(x) = d_T(y) = d_{T'}(z) + 1$, we have $w.freq = x.freq + y.freq + z.freq$. □

Notes:

- Lemma means “subsidiary or intermediate theorem in an argument of proof”
- Proof of greedy-choice property for Huffman’s Algorithm (Binary)

  Lemma 16.2:

  Let C be an alphabet in which each chapter $c \in C$ has frequency $c.freq$. Let x and y be two characters in C having the lowest frequencies. Then there exists an optimal prefix code for C in which the codewords for x and y have the same length and differ only in the last bit.

  Proof. $\square$

- Proof of optimal structure property for Huffman’s Algorithm (Binary)