Worksheet 8 Review

March 27, 2020

Question 1

a. $\forall n \in \mathbb{N}, (0 \leq 1) \wedge (n \leq 2^n) \Rightarrow (n+1) \leq 2^{n+1}$

Note:

- **Induction:** $\forall n \in \mathbb{N},\ P(0) \wedge P(n) \Rightarrow P(n+1)$

b. We will prove this statement by induction on n.

Base Case:

Let $n = 0$.

Then,

$$0 \leq 2^0 \quad (1)$$
$$0 \leq 1 \quad (2)$$

Since the above inequality is true, the base case holds.

Inductive Case:

Let $n \in \mathbb{N}$. Assume $P(n)$.

Then,

$$n \leq 2^n \tag{3}$$
$$n + 1 \leq 2^n + 1 \tag{4}$$
$$n + 1 \leq 2^n + 2^n \tag{5}$$
$$n + 1 \leq 2^{n+1} \tag{6}$$

by the fact $2^k + 2^k = 2^{k+1}$.

Then, it follows from proof by induction that the statement $n \leq 2^n$ is true for all n.

Correct Solution:

We will prove this statement by induction on n.

Base Case:

Let $n = 0$.

Then,

$$0 \leq 2^0 \tag{7}$$
$$0 \leq 1 \tag{8}$$

Since the above inequality is true, the base case holds.

Inductive Case:

Let $n \in \mathbb{N}$. Assume $P(n)$.

We want to show $n + 1 \leq 2^{n+1}$.

Then,

$$
\begin{aligned}
n &\leq 2^n & (9)\\
n+1 &\leq 2^n + 1 & (10)\\
n+1 &\leq 2^n + 2^n & (11)\\
n+1 &\leq 2^{n+1} & (12)
\end{aligned}
$$

by the fact $2^n + 2^n = 2^{n+1}$.

Then, it follows from proof by induction that the statement $n \leq 2^n$ is true for all n.

Notes:

- professor specifically states what we want to show in inductive case part of the proof. I thought it was obvious, and not necessary.
- When are the times 'we want to show x' in proof can be omitted?

Question 2

- We will prove the statement by induction on natural number n.

  Base Case:

  Let $n = 1$.

  Then,

$$
\begin{aligned}
\sum_{j=1}^{1} T_j &= 1\frac{\cdot(1+1)(1+2)}{6} & (1)\\
&= 1 & (2)
\end{aligned}
$$

  Since the data also shows value 1 at $n = 1$, the base case holds.

Inductive Case:

Let $n \in \mathbb{N}$. Assume $\sum_{j=0}^{n} T_j = \frac{n\cdot(n+1)(n+2)}{6}$.

We want to show $\sum_{j=0}^{n+1} T_j = \frac{(n+1)(n+2)(n+3)}{6}$.

It follows from the following table

n	1	2	3	4	5
$T_i = \frac{n\cdot(n+1)}{2}$	1	3	6	10	15
$\sum_{j=1}^{n} T_j$	1	4	10	20	35

that $n+1^{th}$ value of the summation is $\frac{(n+1)(n+2)}{2}$ more than the n^{th} sum.

Then,

$$\sum_{j=0}^{n+1} T_j = \frac{n\cdot(n+1)(n+2)}{6} + \frac{(n+1)(n+2)}{2} \tag{3}$$

$$= \frac{n\cdot(n+1)(n+2)}{6} + \frac{3(n+1)(n+2)}{6} \tag{4}$$

$$= \frac{(n+1)(n+2)(n+3)}{6} \tag{5}$$

Correct Solution:

We will prove the statement by induction on natural number n.

Base Case:

Let $n = 0$**.**

Then,

$$\sum_{j=0}^{1} T_j = \frac{0 \cdot (0+1)(0+2)}{6} \tag{1}$$
$$= 0 \tag{2}$$

Since

$$\sum_{j=0}^{0} T_j = T_0 \tag{3}$$

and,

$$T_0 = \frac{0 \cdot (0+1)}{2} \tag{4}$$
$$= 0 \tag{5}$$

, the base case holds.

Inductive Case:

Let $n \in \mathbb{N}$. Assume $\sum\limits_{j=0}^{n} T_j = \frac{n \cdot (n+1)(n+2)}{6}$.

We want to show $\sum\limits_{j=0}^{n+1} T_j = \frac{(n+1)(n+2)(n+3)}{6}$.

It follows from the following table

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that $n+1^{th}$ value of the summation is $\frac{(n+1)(n+2)}{2}$ more than the n^{th} sum.

Then,

$$\sum_{j=0}^{n+1} T_j = \frac{n \cdot (n+1)(n+2)}{6} + \frac{(n+1)(n+2)}{2} \tag{6}$$

$$= \frac{n \cdot (n+1)(n+2)}{6} + \frac{3(n+1)(n+2)}{6} \tag{7}$$

$$= \frac{(n+1)(n+2)(n+3)}{6} \tag{8}$$

Notes:

- I wasn't explicit about where the value 1 in data came from.

Question 3

a. Let $x \in \mathbb{R}$.