# Problem Set 3 Solution

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## Question 1

1. Let $x \in \mathbb{R}$.

   **Base Case (n = 0):**

   Let $n = 0$.

   Then,

   $$a_0 = 0 \tag{1}$$

   Then it follows from above that the base case holds.

   **Inductive Case ($n > 0$):**

   Let $k \in \mathbb{N}$, and assume $a_n = x \prod_{i=0}^{n-1} a_i$.

   Then,

   $$\begin{align} x \prod_{i=0}^{n-1} a_i \cdot a_n &= x \prod_{i=0}^{n} a_i \tag{1} \\ &= a_{n+1} \tag{2} \end{align}$$

Then it follows from above that the recursive sequence of numbers is true for all natural numbers.

2. From the following table

| String Length | Number of Even (Digit Sum) | Number of Odd (Digit Sum) | Total |
|---|---|---|---|
| 1 | 2 | 1 | 3 |
| 2 | 5 | 4 | 9 |
| 3 | 14 | 13 | 27 |

we see that $E_n = \frac{3^n+1}{2}$ and $O_n = \frac{3^n-1}{2}$.

As well, we see that the number of new elements in $E_{n+1}$ is $3^n$.

Now, we will prove that $E_n$ and $O_n$ are true for all natural numbers using the induction hypothesis.

**Base Case (n = 1):**

Let $n = 1$.

Then, $E_n = \frac{4}{2} = 2$ and $O_n = \frac{2}{2} = 1$.

Since the result matches to data in table, the base case holds.

**Inductive Case:**

Let $n \in \mathbb{N}$. Assume $E_n = \frac{3^n+1}{2}$ and $O_n = \frac{3^n-1}{2}$.

Then,

$$\begin{aligned}
E_{n+1} &= \frac{3^n+1}{2} + 3^n && (1)\\
&= \frac{3^n+1}{2} + \frac{2 \cdot 3^n}{2} && (2)\\
&= \frac{3 \cdot 3^n + 1}{2} && (3)\\
&= \frac{3^{n+1}+1}{2} && (4)
\end{aligned}$$

Then, it follows from above that the inductive step for $E_n$ holds.

Similarly, for $O_n$,

$$\begin{aligned} O_{n+1} &= \frac{3^n - 1}{2} + 3^n && (5)\\ &= \frac{3^n - 1}{2} + \frac{2 \cdot 3^n}{2} && (6)\\ &= \frac{3 \cdot 3^n - 1}{2} && (7)\\ &= \frac{3^{n+1} - 1}{2} && (8) \end{aligned}$$

Then, it follows from above that the inductive step for $O_n$ holds.

Then, it follows from the definition of induction hypothesis that the value of $E_n$ and $O_n$ are true for all $n$.

# Question 2

a. Since first 1 repeats every $4i - 1$ times and the second 1 repeats every $4i$ times,

$$\begin{aligned} (0.\overline{0011})_2 &= \sum_{i=1}^{\frac{n}{4}} \left(\frac{1}{2}\right)^{4i} + \sum_{i=1}^{\frac{n}{4}} \left(\frac{1}{2}\right)^{4i-1} && (1)\\ &= \sum_{i=1}^{\frac{n}{4}} \left(\frac{1}{16}\right)^{i} + 2 \cdot \sum_{i=1}^{\frac{n}{4}} \left(\frac{1}{16}\right)^{i} && (2)\\ &= \frac{1}{16} \cdot \sum_{i=0}^{\frac{n}{4}-1} \left(\frac{1}{16}\right)^{i} + \sum_{i=0}^{\frac{n}{4}-1} \left(\frac{1}{16}\right)^{i} && (3)\\ &= \frac{3}{16} \cdot \sum_{i=0}^{\frac{n}{4}-1} \left(\frac{1}{16}\right)^{i} && (4) \end{aligned}$$

Then,

$$\frac{3}{16} \cdot \sum_{i=0}^{\frac{n}{4}-1} \left(\frac{1}{16}\right)^i = \frac{3}{16} \cdot \left(\frac{1 - \frac{1}{16}^{\frac{n}{4}}}{1 - (\frac{1}{16})}\right) \tag{5}$$

by using the formula $\forall n \in \mathbb{Z}^+ and r \in \mathbb{R},\ r \neq 1 \Rightarrow \sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$.

Then,

$$\begin{aligned}
\frac{3}{16} \cdot \left(\frac{1 - \frac{1}{16}^{\frac{n}{4}}}{1 - (\frac{1}{16})}\right) &= (\frac{1 - \frac{1}{2}^n}{\frac{15}{16}}) && (6)\\
&= \frac{1}{5} \cdot \left(1 - \frac{1}{2}^n\right) && (7)\\
&= \frac{1}{5} \cdot \left(\frac{2^n - 1}{2^n}\right) && (8)
\end{aligned}$$

Then,

$$\begin{aligned}
0.2 - \frac{1}{5} \cdot \left(\frac{2^n - 1}{2^n}\right) &= \frac{1}{5} - \frac{1}{5} \cdot \left(\frac{2^n - 1}{2^n}\right) && (9)\\
&= \frac{2^n}{5 \cdot 2^n} - \frac{1}{5} \cdot \left(\frac{2^n - 1}{2^n}\right) && (10)\\
&= \frac{1}{5 \cdot 2^n} && (11)
\end{aligned}$$

Then, it follows from above that $\forall n \in \mathbb{Z}^+,\ 4 \mid n \Rightarrow \frac{1}{5 \cdot 2^n}$

b. Let $n \in \mathbb{Z}^+$, and $x \in \{x \mid x \in \mathbb{R}^+, 0 \leq x < 1\}$.

We will prove that the statement $\forall n \in \mathbb{Z}^+, \forall x \in S, \exists x_1 \in S,\ FB(n, x_1) \wedge 0 \leq x - x_1 < 1$ is true using induction hypothesis.

Let $n = 1$.

**Case 1** ($0 \leq x < 0.5$, from $S = x \mid x \in \mathbb{R}, 0 \leq x < 1$):

Let $x_1 = 0$.

Then,

$$\begin{align} 0 &= (0.0)_2 \tag{1} \\ &= \sum_{i=1}^{1} \frac{b_i}{2} \tag{2} \end{align}$$

by the fact that $b_i = 0$.

Then, it follows from above that $FB(1, x_1)$ is true.

Now we will prove that $0 \leq x - x_1 < \frac{1}{2}$ is true.

Let $x_1 = 0$. Assume $0 \leq x < 0.5$.

Then,

$$\begin{align} 0 \leq x < 0.5 \tag{3} \\ 0 - x_1 \leq x - x_1 < \frac{1}{2} - x_1 \tag{4} \\ 0 \leq x - x_1 < \frac{1}{2} \tag{5} \end{align}$$

Then, it follows from above that $FB(n, x_1) \wedge 0 \leq x - x_1 < \frac{1}{2}$ hold for the base case with $0 \leq x < 0.5$.

**Case 2** ($0.5 \leq x < 1$ from $S = \{x \mid x \in \mathbb{R}^{\geq 0}, 0 \leq x < 1\}$):

First, we will prove that $FB(1, x_1)$ is true.

Let $x_1 = 0.5$.

Then,

$$\begin{align} 0.5 &= \frac{1}{2} \tag{6} \\ &= \sum_{i=1}^{1} \frac{b_i}{2} \tag{7} \end{align}$$

where $b_i = 1$.

Then, it follows from the definition of finite fractional binary representation that $x$ has fractional binary representation with 1 bits, and $FB(1, x_1)$ is true.

Now, we will prove that $0 \leq x - x_1 < 0.5$.

Let $x_1 = 0.5$. Assume $0.5 \leq x < 1$.

Then,

$$\begin{align} 0.5 - x_1 \leq x - x_1 < 1 - x_1 \tag{8} \\ 0 \leq x - x_1 < 0.5 \tag{9} \end{align}$$

Then, it follows from above that $0 \leq x - x_1 < 0.5$ is true.

Then, since $0 \leq x - x_1 < 0.5$ is true and $FB(1, x_1)$ is true, $FB(1, x_1) \wedge 0 \leq x - x_1 < 0.5$ is true for the case $0.5 \leq x < 1$.

Then, by combining results from case 1 and case 2, we can conclude that the statement holds for the base case.

Now, let $n \in \mathbb{Z}^+$, and $x \in S$. Assume $\exists x_1 \in S$, $FB(n, x_1) \wedge 0 \leq x - x_1 \leq \frac{1}{2^n}$.

Then, we will prove that the statement $\forall n \in \mathbb{Z}^+, \forall x \in S, FB(n, x_1) \wedge 0 \leq x - x_1 \leq \frac{1}{2^n}$ is true for inductive case by separating $0 \leq x - x_1 \leq \frac{1}{2^n}$ into following cases.

**Case 1** ($0 \leq x - x_1 < \frac{1}{2^{n+1}}$):

First, we will prove that $FB(n+1, x_2)$ is true.

Let $x_2 = x_1$.

Then,

$$\begin{aligned} x_2 &= \sum_{i=1}^{n} \frac{b_i}{2} && (10) \\ &= \sum_{i=1}^{n} \frac{b_i}{2} + \frac{b_{i+1}}{2} && (11) \\ &= \sum_{i=1}^{n+1} \frac{b_i}{2} && (12) \end{aligned}$$

by setting $b_{i+1} = 0$.

Then, it follows from above that $FB(n+1, x_2)$ is true.

Now, we will prove that $0 \leq x - x_2 < \frac{1}{2^{n+1}}$ is true.

Let $x_2 = x_1$. Assume $0 \leq x - x_1 < \frac{1}{2^{n+1}}$.

Then, it follows from assumption that $0 \leq x - x_2 < \frac{1}{2^{n+1}}$ is true.

Then, since $FB(n+1, x_2)$ is true and $0 \leq x - x_2 < \frac{1}{2^{n+1}}$ is true, $FB(n+1, x_2) \wedge 0 \leq x - x_2 < \frac{1}{2^{n+1}}$ is true for the case $0 \leq x - x_1 < \frac{1}{2^{n+1}}$.

**Case 2** ($\frac{1}{2^{n+1}} \leq x - x_1 \leq \frac{1}{2^n}$):

First, we will prove that $FB(n+1, x_2)$ is true.

Let $x_2 = x_1 - \frac{1}{2^{n+1}}$.

Then,

$$x_2 = \sum_{i=1}^{n} \frac{b_i}{2^i} - \frac{1}{2^{n+1}} \tag{13}$$
$$= \sum_{i=1}^{n} \frac{b'_i}{2} \tag{14}$$

by the fact that $b'_i = b_i$, $b'_n = 0$ and $b'_{n+1} = 1$.

Then, it follows from the definition of finite fractional binary representation that $FBn+1, x_2$ is true.

Now, we will prove that $0 \leq x - x_2 \leq \frac{1}{2^{n+1}}$ is true.

Let $n \in \mathbb{Z}^+$, $x \in \mathbb{S}$, $x_2 = x_1 - \frac{1}{2^{n+1}}$. Assume $\frac{1}{2^{n+1}} \leq x - x_1 \leq \frac{1}{2^n}$.

Then,

$$\frac{1}{2^{n+1}} \leq x - x_1 \leq \frac{1}{2^n} \tag{15}$$
$$\frac{1}{2^{n+1}} - \frac{1}{2^{n+1}} \leq x - x_1 - \frac{1}{2^{n+1}} \leq \frac{1}{2^n} - \frac{1}{2^{n+1}} \tag{16}$$
$$0 \leq x - x_2 \leq \frac{1}{2^n} - \frac{1}{2^{n+1}} \tag{17}$$
$$0 \leq x - x_2 \leq \frac{2}{2^{n+1}} - \frac{1}{2^{n+1}} \tag{18}$$
$$0 \leq x - x_2 \leq \frac{1}{2^{n+1}} \tag{19}$$

Then it follows from above that $0 \leq x - x_2 \leq \frac{1}{2^{n+1}}$ is true.

Then, since $FB(n+1, x_2)$ is true, and $0 \leq x - x_2 \leq \frac{1}{2^{n+1}}$ is true, $FB(n+1, x_2) \wedge 0 \leq x - x_2 \leq \frac{1}{2^{n+1}}$ is true for the case $\frac{1}{2^{n+1}} \leq x - x_1 \leq \frac{1}{2^n}$.

Then, because the statement is true in both case 1 and case 2, the statement at inductive step holds.

Then, it follows from induction hypothesis that the statement is true for all natural numbers.

# Question 3

a. **Definition of Big-Oh:** $g \in \mathcal{O}(f) : \exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq cf(n), where f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

Since $a \leq b \wedge c \leq d \Rightarrow a+b \leq c+d$, we will prove that $n^4 + 165n^3 \leq c(n^4 - n^2)$ by validating the following inequalities, then combining it together

$$n^4 \leq \frac{c}{3}n^4 \tag{1}$$
$$165n^3 \leq \frac{c}{3}n^4 \tag{2}$$
$$cn^2 \leq \frac{c}{3}n^4 \tag{3}$$

**Validating** $(n^4 \leq \frac{c}{3}n^4)$:

Let $n \in \mathbb{N}$, $n_0 = 2$, and $c = 249$. Assume $n \geq n_0$.

Then,

$$n^4 \leq \frac{249}{3}n^4 \tag{4}$$
$$n^4 \leq 83n^4 \tag{5}$$

Then, it follows from above that $n^4 \leq \frac{c}{3}n^4$ holds.

**Validating** $(165n^3 \leq \frac{c}{3}n^4)$:
Let $n \in \mathbb{N}$, $n_0 = 2$, and $c = 249$. Assume $n \geq n_0$.

Then,

$$165n^3 \leq \frac{c}{3}n^4 \tag{6}$$
$$165n^3 \leq \frac{249}{3}n^4 \tag{7}$$
$$165n^3 \leq 83n^4 \tag{8}$$

Then,

$$165n^3 \leq 83n^4 \tag{9}$$
$$165(2)^3 \leq 83(2)^4 \tag{10}$$

because of the fact $n_0 = 2$ and the assumption $n \geq n_0$

Then,

$$165(2)^3 \leq 83(2)^4 \tag{11}$$
$$165 \cdot 8 \leq 83 \cdot 16 \tag{12}$$
$$1320 \leq 1328 \tag{13}$$

Then, it follows from above that $165n^3 \leq \frac{c}{3}n^4$ holds.

**Validating** $(cn^2 \leq \frac{c}{3}n^4)$:

Let $n \in \mathbb{N}$, $n_0 = 2$, and $c = 249$. Assume $n \geq n_0$.

Then,

$$cn^2 \le \frac{c}{3}n^4 \tag{14}$$
$$249n^2 \le \frac{249}{3}n^4 \tag{15}$$
$$249n^2 \le 83n^4 \tag{16}$$
$$\tag{17}$$

Then,

$$249n^2 \le 83n^4 \tag{18}$$
$$249(2)^2 \le 83(2)^4 \tag{19}$$
$$\tag{20}$$

because of the fact $n_0 = 2$ and the assumption $n \ge n_0$

Then,

$$249n^2 \le 83n^4 \tag{21}$$
$$249 \cdot 4 \le 83 \cdot 16 \tag{22}$$
$$996 \le 1328 \tag{23}$$

Then, it follows from above that $cn^2 \le \frac{c}{3}n^4$ holds.

Then, because we know $n^4 \le \frac{c}{3}n^4$, $165n^3 \le \frac{c}{3}n^4$, and $cn^2 \le \frac{c}{3}n^4$ are true, we can conclude that $n^4 + 165n^3 + cn^2 \le cn^4$ is true.

Then,

$$n^4 + 165n^3 + cn^2 \le cn^4 \tag{24}$$
$$n^4 + 165n^3 \le c(n^4 - n^2) \tag{25}$$

Then, it follows from the definition of Big-Oh that the statement $n^4 + 165n^3 \in \mathcal{O}(n^4 - n^2)$ is true.

b. **Negation:** $\forall f : \mathbb{N} \to \mathbb{R}^{+}, \exists g : \mathbb{N} \to \mathbb{R}^{\geq 0}, (\forall c, n_0 \in \mathbb{R}^{+}, \exists n \in \mathbb{N}, n \geq n_0 \wedge (g(n) > cf(n))) \wedge (\forall d, m_0 \in \mathbb{R}^{+}, \exists m \in \mathbb{N}, m \geq m_0 \wedge (g(n) < cf(n)))$

Let

$$g(n) = \begin{cases} 0 & \text{if n even} \\ nf(n) & \text{otherwise} \end{cases}$$

We will prove the statement by separating into cases, and combine them together in the end.

**Case 1** ($g \notin \mathcal{O}(f) : \forall c, n_0 \in \mathbb{R}^{+}, \exists n \in \mathbb{N}, n \geq n_0 \wedge g(n) > cf(n)$):

Let $c, n_0 \in \mathbb{R}^{+}$, $n = 2 \cdot \lceil max(c, n_0) \rceil$.

Because there are two parts in $max(c, n_0)$, we will prove by separating into cases $n_0 > c$, and $c > n_0$ and combine the result together in the end.

Consider the case $n_0 > c$.

Since $n_0 > c$ and $n = 2\lceil n_0 \rceil$, we can conclude that $n \geq n_0$.

Then,

$$\begin{aligned} g(n) &= nf(n) && (1) \\ &= 2\lceil n_0 \rceil f(n) && (2) \\ &> 2cf(n) && (3) \\ &> cf(n) && (4) \end{aligned}$$

Then, since $g(n) > cf(n)$ is true and $n \geq n_0$ is true, $n \geq n_0 \wedge g(n) > cf(n)$ is true.

Now, consider the case $c > n_0$.

Since $c > n_0$ and $n = 2\lceil c \rceil$, we can conclude that $n \geq n_0$.

Then,

$$\begin{aligned} g(n) &= nf(n) && (5) \\ &= 2\lceil c \rceil f(n) && (6) \\ &> 2cf(n) && (7) \\ &> cf(n) && (8) \end{aligned}$$

Then, since $g(n) > cf(n)$ is true and $n \geq n_0$ is true given $n_0 > c$, and $c > n_0$, $n \geq n_0 \wedge g(n) > cf(n)$ is true.

Then, it follows from above that the statement $g \notin \mathcal{O}(f)$ is true.

**Case 2** ($g \notin \Omega(f) : \forall c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \wedge g(n) < cf(n)$):

Let $c, n_0 \in \mathbb{R}^+$, $n = \lceil n_0 \rceil + 1$.

Then, we can conclude that $n \geq n_0$.

Then,

$$0 = g(n) < cf(n) \qquad (1)$$

by the fact that no values in codomain of $f$ is 0.

Then, $n \geq n_0 \wedge g(n) < cf(n)$ is true by the fact that $(n \geq n_0)$ is true and $(g(n) < cf(n))$ is true.

Then, it follows from above that the statement $g \notin \Omega(f)$ is true.

Since $g \notin \Omega(f)$ is true and $g \notin \mathcal{O}(f)$ is true given $g$ and $n$, it follows from the negation of statement that the statement $\exists f : \mathbb{N} \to \mathbb{R}^+, (\forall g : \mathbb{N} \to \mathbb{R}^{\geq 0}, g \in \mathcal{O}(f) \vee g \in \Omega(f))$ is false.

# Question 4

a. Let $n_0 = c^{-\frac{1}{b-a}}$. Assume $n \geq n_0$.

Then,

$$
\begin{align}
n &\geq n_0 \tag{1}\\
n &\geq c^{-\frac{1}{b-a}} \tag{2}\\
[n]^{-(b-a)} &\leq \left[c^{-\frac{1}{b-a}}\right]^{-(b-a)} \tag{3}\\
n^{a-b} &\leq c \tag{4}
\end{align}
$$

Then,

$$
\begin{align}
cf(n) &= cn^b \tag{5}\\
&\geq n^{a-b}n^b \tag{6}\\
&\geq n^{a-b+b} \tag{7}\\
&\geq n^a \tag{8}
\end{align}
$$

Then, it follows from the definition of little-oh that the statement $\forall a, b \in \mathbb{R}^+, a < b \Rightarrow n^a \in o(n^b)$ is true.