Worksheet 16 Solution

March 29, 2020

Question 1

a. **Part 1.a - Finding minimum possible change for a loop in a single iteration**

The minimum possible change in a loop occurs when i increments by 1.

Part 1.b - Finding maximum possible change for a loop in a single iteration

The maximum possible change in a loop occurs when i increments by 6.

Part 2.a - Determine formula for an exact lower bound on the value

Since the loop starts at $i = 0$ and ends at $n - 1$, the loop has

$$n - 1 + 1 = n \tag{1}$$

iterations.

Since the smallest step increases by 1 per iteration, the total cost of the loop at minimum possible change is

$$(n) \cdot 1 = n \tag{2}$$

steps.

Part 2.a - Determine formula for an exact upper bound on the value

Since the loop starts at $i = 0$ and ends at $n - 1$, the loop has

$$n - 1 + 1 = n \tag{3}$$

iterations.

Part 2.b - Determine formula for an exact lower bound on the value

Since the largest step increases by 6 per iteration, the total cost of the loop at minimum possible change is

$$\left\lceil \frac{n}{6} \right\rceil \tag{4}$$

steps.

Part 3.a - Determine formula for an exact upper bound on the value Is it n?

Part 3.a - Determine formula for an exact upper bound on the value Is it $\lceil \frac{n}{6} \rceil$?

Part 4 - Determine Big Oh and Big Omega

The big Oh bound of running time is $\mathcal{O}(n)$, and the big theta of running time is $\Omega(n)$.

Since n in $\mathcal{O}(n)$ and $\Omega(n)$ are the same, $\Theta(n)$ is also true.

<u>**Correct Solution:**</u>

Part 1.a - Finding minimum possible change for a loop in a single iteration

The minimum possible change in a loop occurs when i increments by 1.

**Part 1.b - Finding maximum possible change for a loop in a

single iteration

The maximum possible change in a loop occurs when i increments by 6.

Part 2.a - Determine formula for an exact upper bound on the value

The upper bound of loop termination is when $k \geq n$

Part 2.b - Determine formula for an exact lower bound on the value

The lower bound of loop termination is when $6k \leq n$

Part 3.a - Use the formula to determine the exact number of loops that will occur for upper bound

Since the loop starts from 0 and ends at $n-1$, the loop has total of

$$n - 1 - 0 + 1 = n \tag{5}$$

iterations.

Since 1 step is taken for each iteration, the upper bound total cost of loop iteration is

$$n \cdot 1 = n \tag{6}$$

Since the statement on line 2 has cost of 1, the upper bound total cost of the algorithm is $n + 1$, or $\mathcal{O}(n)$.

Part 3.b - Use the formula to determine the exact number of loops that will occur for lower bound

Since the loop starts from 0 and ends at $n-1$, the loop has total of

$$n - 1 - 0 + 1 = n \tag{7}$$

iterations.

Since 6 steps are taken for each iteration, the lower bound total cost of loop iteration is

$$\left\lceil \frac{n}{6} \right\rceil \tag{8}$$

Since the statement on line 2 has cost of 1, the lower bound total cost of the algorithm is $\left\lceil \frac{n}{6} \right\rceil + 1$, or $\Omega(n)$

Part 4 - Determine Big Oh and Big Omega

The big Oh bound of running time is $\mathcal{O}(n)$, and the big theta of running time is $\Omega(n)$.

Since n in $\mathcal{O}(n)$ and $\Omega(n)$ are the same, $\Theta(n)$ is also true.

b. **Part 1.a - Finding minimum possible change for a loop in a single iteration**

The minimum possible change for a look in a single iteration is when i increases by a factor of 2

Part 1.b - Finding maximum possible change for a loop in a single iteration

The maximum possible change for a look in a single iteration is when i increases by a factor of 3

Part 2.a - Determine formula for an exact upper bound of the loop variable after k iterations

The exact upper bound of the loop variable after k iteration is $2^k \geq n$

Part 2.b - Determine formula for an exact lower bound of the loop variable after k iterations

The exact lower bound of the loop variable after k iteration is $3^k \geq n$

Part 3.a - Use the formula to determine the exact number of loops that will occur for upper bound

The upper bound of loop iteration is $\lceil \log n \rceil$, or $\mathcal{O}(\log n)$

Part 3.b - Use the formula to determine the exact number of loops that will occur for lower bound

The lower bound of loop iteration is $\lceil \log_3 n \rceil$, or $\Omega(\log n)$

Part 4 - Determine Big Oh and Big Omega

For the upper bound, we have $\mathcal{O}(\log n)$.

For the lower bound, we have $\Omega(\log n)$

Since Big Oh and Big Omega have the same value, $\Theta(\log n)$ is also true.

Question 2

a. Since **helper1** has cost of n steps, and **helper2** has cost of n^2 steps, the algorithm has total runtime of $n^2 + n$ steps, or $\Theta(n^2)$

Attempt #2:

Since **helper1** has cost of n steps, and **helper2** has cost of n^2 steps, the algorithm has total **cost** of $n^2 + n$ steps, or $\Theta(n^2)$

Notes:

- Noticed professor uses **runtime** for $\Theta(n^2)$ or $\Theta(n)$ and **cost** for the exact cost of helper functions (i.e. $n^2 + n$)

b. Assume **helper1** has running time of $\Theta(n)$ steps and **helper2** has running time of $\Theta(n^2)$.

Because the outer loop 1 runs from $i = 0$ to $\lceil \frac{n}{2} \rceil - 1$, the outer loop 1 has

$$\left\lceil \frac{n}{2} \right\rceil - 1 + 1 = \left\lceil \frac{n}{2} \right\rceil \tag{1}$$

iterations.

Since the outer loop 1 takes n steps per iteration, the outer loop 1 has total cost of $\lceil \frac{n}{2} \rceil \cdot n$ steps.

Because the outer loop 2 runs from $j = 0$ to $j = 9$, it has

$$(9 - 0 + 1) = 10 \tag{2}$$

iterations.

Since the outer loop 2 takes n^2 steps per iteration, it has total cost of $10n^2$ steps.

Since $i = 0$ and $j = 0$ each have cost of 1, the total cost of the algorithm is $\lceil \frac{n}{2} \rceil \cdot n + 10n^2 + 2$ steps or $\Theta(n^2)$.

Notes:

- Noticed professor uses the phrase **each iteration requires n steps for the call to helper 1** to reference helper functions in loop.
- Noticed professor did not consider $i = 0$ and $j = 0$ into total costs. Should $i = 0$ and $j = 0$ be counted towards costs? If not, how come the cost of $len(lst)$ and **return** statement are considered in Question 1.a of worksheet 15? Are there rules such as what to include and what to omit when considering the statements with constant time?

c. Assume **helper1** function has runtime of $\Theta(n)$, and **helper2** function has runtime of $\Theta(n^2)$.

Since loop 1 runs from $i = 0$ to $n - 1$, the loop has

$$n - 1 - 0 + 1 = n \tag{1}$$

iterations.

Then, since each iteration of loop 1 requires n steps for the call to **helper1**, the loop has total cost of

$$n \cdot n = n^2 \tag{2}$$

steps.

Because we know the loop 2 runs from $j = 0$ to $j = 9$, we can conclude the loop has

$$9 - 0 + 1 = 10 \tag{3}$$

iterations.

Since each iteration of loop 2 requires n^2 steps for the call to **helper2**, the loop has total cost of

$$10 \cdot n^2 = 10n^2 \tag{4}$$

steps.

Since $i = 0$ and $j = 0$ each have cost of 1, the total cost of algorithm is

$$n^2 + 10n^2 + 2 = 11n^2 + 2 \tag{5}$$

steps, or $\Theta(n^2)$.

Correct Solution:

Let $n \in \mathbb{N}$. Assume **helper1** function has runtime of $\Theta(n)$, and **helper2** function has runtime of $\Theta(n^2)$.

Since loop 1 runs from $i = 0$ to $n - 1$ where i represents the variable for loop 1, the loop has

$$n - 1 - 0 + 1 = n \tag{1}$$

iterations.

Then, since each iteration of loop 1 requires i steps for the call to **helper1**, the loop has total cost of

$$\sum_{i=0}^{n-1} i = \frac{n(n-1)}{2} \tag{2}$$

steps.

Because we know the loop 2 runs from $j = 0$ to $j = 9$ where j represents the variable for loop 2, we can conclude the loop has

$$9 - 0 + 1 = 10 \tag{3}$$

iterations.

Since each iteration of loop 2 requires j^2 steps for the call to **helper2**, the loop has total cost of

$$\sum_{j=0}^{9} j^2 = \frac{9 \cdot (9-1)(2(9)-1)}{6} \quad (4)$$
$$= \frac{9 \cdot 8 \cdot 17}{6} \quad (5)$$
$$= 204 \quad (6)$$

steps.

Since **the statements** $i = 0$ and $j = 0$ each have cost of 1, the total cost of algorithm is

$$\frac{n(n-1)}{2} + 204 + 2 = \frac{n(n-1)}{2} + 206 \quad (7)$$

steps, or $\Theta(n^2)$.

Notes:

- Missed that the helper functions depend on loop.
- Noticed that in solutions, the variables i, j, n are assumed to be in $\mathbb{N}$. But I feel worried applying the same assumption would get me into troubles. Would marks be deducted for not mentioning about the variables n, i and j? If not, when are the times the mentioning of variables can be omitted?

Question 3

a. **Predicate Logic:** $\forall x \in \mathbb{Z}^+$, (3 loops occur) $\Rightarrow \exists x_{final}, m \in \mathbb{Z}^+$, $x - x_{final} \geq 2^m$

Let $x \in \mathbb{Z}^+$. Assume 3 loop iterations occur.

We will prove the statement by dividing into cases. First case is where $x \bmod 2 == 0$ in all three loops. Second case is where $x \bmod 2 == 0$ runs once, then $x = 2 * x - 2$, and then $x \bmod 2 == 0$. The last case is where $x = 2 * x - 2$ is run, and the rest with $x \bmod 2 == 0$.

Case 1 ($\exists k \in \mathbb{Z},\ x = 2^k$):

Let $m = 2$. Assume there is some $k \in \mathbb{Z}$, $x = 2^k$.

We will show $x - x_{final} \geq 2^m$ by calculating the value of x_{final} and subtracting it from x.

It follows from the the statement $x = x//2$ being executed three times that the value of x_{final} is

$$x_{final} = x^{k-3} \tag{1}$$

Then, because we know the loop terminates when $x \leq 1$, we can conclude that

$$\begin{align} x^{k-3} &\leq 1 \tag{2}\\ \log x^{k-3} &\leq \log 1 \tag{3}\\ k - 3 &\leq 0 \tag{4}\\ k &\leq 3 \tag{5} \end{align}$$

Then, because we know $k < 3$ results in loop count less than 3, we can conclude that

$$k = 3 \tag{6}$$

Then,

$$\begin{aligned}
x_{final} &= 2^{3-3} && (7)\\
&= 2^0 && (8)\\
&= 1 && (9)
\end{aligned}$$

Then,

$$\begin{aligned}
x - x_{final} &= 2^3 - 1 && (10)\\
&= 8 - 1 && (11)\\
&= 7 && (12)\\
&\geq 4 && (13)\\
&\geq 2^2 && (14)\\
&\geq 2^m && (15)
\end{aligned}$$

Case 2 ($\exists k \in \mathbb{Z},\ x = 2 \cdot Odd(k)$):

Let $m = 1$. Assume $\exists k \in \mathbb{Z},\ x = 2(2k+1)$.

We will show $x - x_{final} \geq 2^m$ by calculating the value of x_{final} and subtracting it from x.

Because we know $x > 1$ and $2 \mid x$ in first iteration, we can conclude that the new value of x, or x_2 is

$$\begin{aligned}
x_2 &= \left\lfloor \frac{2(2k+1)}{2} \right\rfloor && (16)\\
&= (2k+1) && (17)
\end{aligned}$$

In second iteration, because we know $x_2 > 1$ and $2 \nmid x_2$, we can conclude the statement $x = 2 * x - 2$ will run.

Then, the new value of x or x_3 is

$$\begin{aligned} x_3 &= 2 \cdot (2k+1) - 2 && (18) \\ &= 2 \cdot (2k+1-1) && (19) \\ &= 4k && (20) \end{aligned}$$

In final iteration, because we know $x_3 > 1$, and $2 \mid x_3$, the last value of x in last iteration, or x_{final} is

$$\begin{aligned} x_{final} &= \left\lfloor \frac{2 \cdot (2k+1) - 1}{2} \right\rfloor && (21) \\ &= 2k && (22) \end{aligned}$$

Then,

$$\begin{aligned} x - x_{final} &= 2(2k+1) - 2k && (23) \\ &= 2\left[(2k+1) - k\right] && (24) \\ &= 2(k+1) && (25) \end{aligned}$$

Then, because we know the termination occurs when $x \leq 1$, we can conclude that

$$\begin{aligned} 2(k+1) &\leq 1 && (26) \\ k &\leq 0 && (27) \end{aligned}$$

Then, because we know $x \in \mathbb{Z}^+$ and $k < 0$ results in $x < 0$, we can conclude that $k = 0$.

Then,

$$
\begin{aligned}
x - x_{final} &= 2(k+1) && (28)\\
&= 2(0+1) && (29)\\
&= 2 && (30)\\
&= 2^1 && (31)\\
&= 2^m && (32)\\
&\geq 2^m && (33)
\end{aligned}
$$

Case 3 ($\exists k \in \mathbb{Z},\ x = Odd(k)$):

Let $m = 1$. Assume $\exists k \in \mathbb{Z}$, $x = 2k + 1$.

We will show $x - x_{final} \geq 2^m$ by calculating the value of x_{final} and subtracting it from x.

In first iteration, because we know $x > 1$ and $2 \nmid x$, we can conclude that the line $x = 2 * x - 2$ will run, and the new value of x or x_2 is

$$
\begin{aligned}
x_2 &= 2 \cdot (2k+1) - 2 && (34)\\
&= 4k && (35)
\end{aligned}
$$

For the second iteration, because we know $x_2 > 1$ and $2 \mid x_3$, we can conclude the new value of x or x_3 is

$$
\begin{aligned}
x_3 &= \left\lfloor \frac{x_2}{2} \right\rfloor && (36)\\
&= \left\lfloor \frac{4k}{2} \right\rfloor && (37)\\
&= 2k && (38)
\end{aligned}
$$

Now in final iteration, because $x_3 > 1$ amd $2 \mid x_3$, we can conclude the final value of x_3 is

$$
\begin{aligned}
x_3 &= \left\lfloor \frac{x_3}{2} \right\rfloor && (39) \\
&= k && (40)
\end{aligned}
$$

Then, since termination occurs when $x_{final} \leq 1$, we can conclude

$$k = x_{final} \leq 1 \tag{41}$$

Then, because we know $k = 0$ results in $x = 1$, and since 3 loops cannot occur with $x = 1$, we can conclude

$$k = 1 \tag{42}$$

Then,

$$
\begin{aligned}
x - x_{final} &= 2k + 1 - k && (43) \\
&= k + 1 && (44) \\
&= 2 && (45) \\
&= 2^1 && (46) \\
&= 2^m && (47) \\
&\geq 2^m && (48)
\end{aligned}
$$

Notes:

- Oh my... I read the question wrong. I need to generalize this for all 3 iterations before and right before termination
- **By a factor** means $\frac{1}{2}$, and not $\left(\frac{1}{2}\right)^m$.
- Must always ask clarification question to professor. Don't dive when not so sure. It's not healthy. The future moe will appreciate it.

b.