CSC236 Worksheet 5 Solution

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Question 1

a. *Proof.* For convenience, define $H(k) : R(3^k) = 3^k k$. Notice that when $n = 3^k$, $3^k k$ is the same as $n \log_3 n$. I will use simple induction to prove $\forall k \in \mathbb{N},\ H(k)$.

Base Case ($k = 0$):

Let $k = 0$.

Then,

$$\begin{aligned}
3^k \cdot k &= 3^0 \cdot 0 && (1)\\
&= 0 && (2)\\
&= R(n) \quad \text{[By def.]} && (3)
\end{aligned}$$

Thus, $H(0)$ is verified.

Inductive Step:

Let $k \in \mathbb{N}$. Assume $H(k)$. That is $R(3^k) = 3^k k$.

I will show that $H(k+1)$ follows. That is, $R(3^{k+1}) = (k+1)3^{k+1}$.

The definition of $R(3^{k+1})$ tells us,

$$\begin{aligned}
R(3^{k+1}) &= 3^{k+1} + 3R(\lceil 3^{k+1}/3 \rceil) && (4)\\
&= 3^{k+1} + 3R(3^k) && (5)
\end{aligned}$$

Then, using this fact, we can conclude

$$R(3^{k+1}) = 3^{k+1} + 3 \cdot 3^k k \qquad \text{[By I.H]} \tag{6}$$
$$= 3^{k+1} + 3^{k+1} k \tag{7}$$
$$= 3^{k+1}(k+1) \tag{8}$$

□

Correct Solution:

For convenience, define $H(k) : R(3^k) = 3^k k$. Notice that when $n = 3^k$, $3^k k$ is the same as $n \log_3 n$. I will use simple induction to prove $\forall k \in \mathbb{N},\ H(k)$.

Base Case ($k = 0$):

Let $k = 0$.

Then,

$$3^k \cdot k = 3^0 \cdot 0 \tag{9}$$
$$= 0 \tag{10}$$
$$= R(n) \qquad \text{[By def.]} \tag{11}$$

Thus, $H(0)$ is verified.

Inductive Step:

Let $k \in \mathbb{N}$. Assume $H(k)$. That is $R(3^k) = 3^k k$.

I will show that $H(k+1)$ follows. That is, $R(3^{k+1}) = (k+1)3^{k+1}$.

Since $k + 1 > 0$, $3^{k+1} > 1$.

So the definition of $R(3^{k+1})$ tells us,

$$R(3^{k+1}) = 3^{k+1} + 3R(\lceil 3^{k+1}/3 \rceil) \tag{12}$$
$$= 3^{k+1} + 3R(3^k) \tag{13}$$

Then, using this fact, we can conclude

$$
\begin{aligned}
R(3^{k+1}) &= 3^{k+1} + 3 \cdot 3^k k && \text{[By I.H]} && (14)\\
&= 3^{k+1} + 3^{k+1} k && && (15)\\
&= 3^{k+1}(k+1) && && (16)
\end{aligned}
$$

Notes:

- Noticed that professor used the phrase 'Notice that when $n = 3^k$, $3^k k$ is the same as $n \log_3 n$.' to express n interms of 3^k.
- I feel I should review this problem to make sure I understood.

b. **Rough Works:**

Define $P(k)$ as

$$P(k) : \bigwedge_{m=1}^{m=k} R(m) \leq R(k)$$

I will prove $\forall n \in \mathbb{N}^+, P(n)$ using complete induction. I will assume without proof, that if n is a natural number greater than 1, then $n > \lceil n/3 \rceil \geq 1$.

1. Inductive Step

   Let $n \in \mathbb{N}$. Assume $n \geq 1$. Assume $H(n) : \bigwedge_{i=1}^{n-1} P(i)$. I will prove that $P(n)$ follows.

2. Base Case ($n = 1$)

   Let $n = 1$.

   Then, $\bigwedge_{m=1}^{m=1} R(m) = R(1)$.

   So, $P(1)$ follows.

3. Base Case ($n = 2$)

Let $n = 2$.

I need to show $P(2)$ holds. That is, $R(1) \leq R(2)$ and $R(2) \leq R(2)$.

I will do so in parts.

Part 1 ($R(1) \leq R(2)$):

In this part, $R(1) = 0$ and $R(2) = 2 + R(\lceil 2/3 \rceil) = 2 + R(1) = 2$.

Since $0 \leq 2$, we can conclude $R(1) \leq R(2)$.

Part 2 ($R(2) \leq R(2)$):

In this part, since $R(2) = R(2)$, we can conclude $R(2) \leq R(2)$.

So, since $R(1) \leq R(2)$ and $R(2) \leq R(2)$ are true, we can conclude $P(2)$ holds.

4. Case ($n > 2$)