Problem Set 3 Solution

March 24, 2020

Question 1

1. Let $x \in \mathbb{R}$.

   Base Case (n = 0):

   Let $n = 0$.

   Then,

$$a_0 = 0 \tag{1}$$

   Then it follows from above that the base case holds.

   Inductive Case ($n > 0$):

   Let $k \in \mathbb{N}$, and assume $a_n = x \prod_{i=0}^{n-1} a_i$.

   Then,

$$\begin{aligned} x \prod_{i=0}^{n-1} a_i \cdot a_n &= x \prod_{i=0}^{n} a_i && (1) \\ &= a_{n+1} && (2) \end{aligned}$$

Then it follows from above that the recursive sequence of numbers is true for all natural numbers.

2. From the following table

String Length	Number of Even (Digit Sum)	Number of Odd (Digit Sum)	Total
1	2	1	3
2	5	4	9
3	14	13	27

we see that $E_n = \frac{3^n+1}{2}$ and $O_n = \frac{3^n-1}{2}$.

As well, we see that the number of new elements in E_{n+1} is 3^n.

Now, we will prove that E_n and O_n are true for all natural numbers using the induction hypothesis.

Base Case (n = 1):

Let $n = 1$.

Then, $E_n = \frac{4}{2} = 2$ and $O_n = \frac{2}{2} = 1$.

Since the result matches to data in table, the base case holds.

Inductive Case:

Let $n \in \mathbb{N}$. Assume $E_n = \frac{3^n+1}{2}$ and $O_n = \frac{3^n-1}{2}$.

Then,

$$\begin{aligned} E_{n+1} &= \frac{3^n+1}{2} + 3^n && (1) \\ &= \frac{3^n+1}{2} + \frac{2 \cdot 3^n}{2} && (2) \\ &= \frac{3 \cdot 3^n + 1}{2} && (3) \\ &= \frac{3^{n+1}+1}{2} && (4) \end{aligned}$$

Then, it follows from above that the inductive step for E_n holds.

Similarly, for O_n,

$$\begin{aligned}
O_{n+1} &= \frac{3^n - 1}{2} + 3^n && (5)\\
&= \frac{3^n - 1}{2} + \frac{2 \cdot 3^n}{2} && (6)\\
&= \frac{3 \cdot 3^n - 1}{2} && (7)\\
&= \frac{3^{n+1} - 1}{2} && (8)
\end{aligned}$$

Then, it follows from above that the inductive step for O_n holds.

Then, it follows from the definition of induction hypothesis that the value of E_n and O_n are true for all n.

Question 2

a. Since first 1 repeats every $4i - 1$ times and the second 1 repeats every $4i$ times,

$$\begin{aligned}
(0.\overline{0011})_2 &= \sum_{i=1}^{\frac{n}{4}} \left(\frac{1}{2}\right)^{4i} + \sum_{i=1}^{\frac{n}{4}} \left(\frac{1}{2}\right)^{4i-1} && (1)\\
&= \sum_{i=1}^{\frac{n}{4}} \left(\frac{1}{16}\right)^{i} + 2 \cdot \sum_{i=1}^{\frac{n}{4}} \left(\frac{1}{16}\right)^{i} && (2)\\
&= \frac{1}{16} \cdot \sum_{i=0}^{\frac{n}{4}-1} \left(\frac{1}{16}\right)^{i} + \sum_{i=0}^{\frac{n}{4}-1} \left(\frac{1}{16}\right)^{i} && (3)\\
&= \frac{3}{16} \cdot \sum_{i=0}^{\frac{n}{4}-1} \left(\frac{1}{16}\right)^{i} && (4)
\end{aligned}$$

Then,

$$\frac{3}{16} \cdot \sum_{i=0}^{\frac{n}{4}-1} \left(\frac{1}{16}\right)^i = \frac{3}{16} \cdot \left(\frac{1 - \frac{1}{16}^{\frac{n}{4}}}{1 - (\frac{1}{16})}\right) \tag{5}$$

by using the formula $\forall n \in \mathbb{Z}^+ and r \in \mathbb{R},\ r \neq 1 \Rightarrow \sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$.

Then,

$$\begin{aligned}
\frac{3}{16} \cdot \left(\frac{1 - \frac{1}{16}^{\frac{n}{4}}}{1 - (\frac{1}{16})}\right) &= (\frac{1 - \frac{1}{2}^n}{\frac{15}{16}}) && (6)\\
&= \frac{1}{5} \cdot \left(1 - \frac{1}{2}^n\right) && (7)\\
&= \frac{1}{5} \cdot \left(\frac{2^n - 1}{2^n}\right) && (8)
\end{aligned}$$

Then,

$$\begin{aligned}
0.2 - \frac{1}{5} \cdot \left(\frac{2^n - 1}{2^n}\right) &= \frac{1}{5} - \frac{1}{5} \cdot \left(\frac{2^n - 1}{2^n}\right) && (9)\\
&= \frac{2^n}{5 \cdot 2^n} - \frac{1}{5} \cdot \left(\frac{2^n - 1}{2^n}\right) && (10)\\
&= \frac{1}{5 \cdot 2^n} && (11)
\end{aligned}$$

Then, it follows from above that $\forall n \in \mathbb{Z}^+,\ 4 \mid n \Rightarrow \frac{1}{5 \cdot 2^n}$

b. Let $n \in \mathbb{Z}^+$, and $x \in \{x \mid x \in \mathbb{R}^+, 0 \leq x < 1\}$.

We will prove that the statement $\forall n \in \mathbb{Z}^+, \forall x \in S, \exists x_1 \in S,\ FB(n, x_1) \wedge 0 \leq x - x_1 < 1$ is true using induction hypothesis.

Let $n = 1$.

Case 1 ($0 \leq x < 0.5$, from $S = x \mid x \in \mathbb{R}, 0 \leq x < 1$):

Let $x_1 = 0$.

Then,

$$0 = (0.0)_2 \tag{1}$$
$$= \sum_{i=1}^{1} \frac{b_i}{2} \tag{2}$$

by the fact that $b_i = 0$.

Then, it follows from above that $FB(1, x_1)$ is true.

Now we will prove that $0 \leq x - x_1 < \frac{1}{2}$ is true.

Let $x_1 = 0$. Assume $0 \leq x < 0.5$.

Then,

$$0 \leq x < 0.5 \tag{3}$$
$$0 - x_1 \leq x - x_1 < \frac{1}{2} - x_1 \tag{4}$$
$$0 \leq x - x_1 < \frac{1}{2} \tag{5}$$

Then, it follows from above that $FB(n, x_1) \wedge 0 \leq x - x_1 < \frac{1}{2}$ hold for the base case with $0 \leq x < 0.5$.

Case 2 ($0.5 \leq x < 1$ from $S = \{x \mid x \in \mathbb{R}^{\geq 0}, 0 \leq x < 1\}$):

First, we will prove that $FB(1, x_1)$ is true.

Let $x_1 = 0.5$.

Then,

$$0.5 = \frac{1}{2} \tag{6}$$
$$= \sum_{i=1}^{1} \frac{b_i}{2} \tag{7}$$

where $b_i = 1$.

Then, it follows from the definition of finite fractional binary representation that x has fractional binary representation with 1 bits, and $FB(1, x_1)$ is true.

Now, we will prove that $0 \leq x - x_1 < 0.5$.

Let $x_1 = 0.5$. Assume $0.5 \leq x < 1$.

Then,

$$0.5 - x_1 \leq x - x_1 < 1 - x_1 \tag{8}$$
$$0 \leq x - x_1 < 0.5 \tag{9}$$

Then, it follows from above that $0 \leq x - x_1 < 0.5$ is true.

Then, since $0 \leq x - x_1 < 0.5$ is true and $FB(1, x_1)$ is true, $FB(1, x_1) \wedge 0 \leq x - x_1 < 0.5$ is true for the case $0.5 \leq x < 1$.

Then, by combining results from case 1 and case 2, we can conclude that the statement holds for the base case.

Now, let $n \in \mathbb{Z}^+$, and $x \in S$. Assume $\exists x_1 \in S$, $FB(n, x_1) \wedge 0 \leq x - x_1 \leq \frac{1}{2^n}$.

Then, we will prove that the statement $\forall n \in \mathbb{Z}^+, \forall x \in S, FB(n, x_1) \wedge 0 \leq x - x_1 \leq \frac{1}{2^n}$ is true for inductive case by separating $0 \leq x - x_1 \leq \frac{1}{2^n}$ into following cases.

Case 1 ($0 \leq x - x_1 < \frac{1}{2^{n+1}}$):

First, we will prove that $FB(n+1, x_2)$ is true.

Let $x_2 = x_1$.

Then,

$$
\begin{aligned}
x_2 &= \sum_{i=1}^{n} \frac{b_i}{2} && (10)\\
&= \sum_{i=1}^{n} \frac{b_i}{2} + \frac{b_{i+1}}{2} && (11)\\
&= \sum_{i=1}^{n+1} \frac{b_i}{2} && (12)
\end{aligned}
$$

by setting $b_{i+1} = 0$.

Then, it follows from above that $FB(n+1, x_2)$ is true.

Now, we will prove that $0 \leq x - x_2 < \frac{1}{2^{n+1}}$ is true.

Let $x_2 = x_1$. Assume $0 \leq x - x_1 < \frac{1}{2^{n+1}}$.

Then, it follows from assumption that $0 \leq x - x_2 < \frac{1}{2^{n+1}}$ is true.

Then, since $FB(n+1, x_2)$ is true and $0 \leq x - x_2 < \frac{1}{2^{n+1}}$ is true, $FB(n+1, x_2) \wedge 0 \leq x - x_2 < \frac{1}{2^{n+1}}$ is true for the case $0 \leq x - x_1 < \frac{1}{2^{n+1}}$.

Case 2 ($\frac{1}{2^{n+1}} \leq x - x_1 \leq \frac{1}{2^n}$):

First, we will prove that $FB(n+1, x_2)$ is true.

Let $x_2 = x_1 - \frac{1}{2^{n+1}}$.

Then,

$$x_2 = \sum_{i=1}^{n} \frac{b_i}{2^i} - \frac{1}{2^{n+1}} \tag{13}$$
$$= \sum_{i=1}^{n} \frac{b'_i}{2} \tag{14}$$

by the fact that $b'_i = b_i$, $b'_n = 0$ and $b'_{n+1} = 1$.

Then, it follows from the definition of finite fractional binary representation that $FBn+1, x_2$ is true.

Now, we will prove that $0 \leq x - x_2 \leq \frac{1}{2^{n+1}}$ is true.

Let $n \in \mathbb{Z}^+$, $x \in \mathbb{S}$, $x_2 = x_1 - \frac{1}{2^{n+1}}$. Assume $\frac{1}{2^{n+1}} \leq x - x_1 \leq \frac{1}{2^n}$.

Then,

$$\frac{1}{2^{n+1}} \leq x - x_1 \leq \frac{1}{2^n} \tag{15}$$
$$\frac{1}{2^{n+1}} - \frac{1}{2^{n+1}} \leq x - x_1 - \frac{1}{2^{n+1}} \leq \frac{1}{2^n} - \frac{1}{2^{n+1}} \tag{16}$$
$$0 \leq x - x_2 \leq \frac{1}{2^n} - \frac{1}{2^{n+1}} \tag{17}$$
$$0 \leq x - x_2 \leq \frac{2}{2^{n+1}} - \frac{1}{2^{n+1}} \tag{18}$$
$$0 \leq x - x_2 \leq \frac{1}{2^{n+1}} \tag{19}$$

Then it follows from above that $0 \leq x - x_2 \leq \frac{1}{2^{n+1}}$ is true.

Then, since $FB(n+1, x_2)$ is true, and $0 \leq x - x_2 \leq \frac{1}{2^{n+1}}$ is true, $FB(n+1, x_2) \wedge 0 \leq x - x_2 \leq \frac{1}{2^{n+1}}$ is true for the case $\frac{1}{2^{n+1}} \leq x - x_1 \leq \frac{1}{2^n}$.

Then, because the statement is true in both case 1 and case 2, the statement at inductive step holds.

Then, it follows from induction hypothesis that the statement is true for all natural numbers.

Question 3

a. **Definition of Big-Oh:** $g \in \mathcal{O}(f) : \exists c, n_0 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \leq cf(n), where f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$

Since $a \leq b \wedge c \leq d \Rightarrow a + b \leq c + d$, we will prove that $n^4 + 165n^3 \leq c(n^4 - n^2)$ by validating the following inequalities, then combining it together

$$n^4 \leq \frac{c}{3}n^4 \tag{1}$$
$$165n^3 \leq \frac{c}{3}n^4 \tag{2}$$
$$cn^2 \leq \frac{c}{3}n^4 \tag{3}$$

Validating $(n^4 \leq \frac{c}{3}n^4)$:

Let $n \in \mathbb{N}$, $n_0 = 2$, and $c = 249$. Assume $n \geq n_0$.

Then,

$$n^4 \leq \frac{249}{3}n^4 \tag{4}$$
$$n^4 \leq 83n^4 \tag{5}$$

Then, it follows from above that $n^4 \leq \frac{c}{3}n^4$ holds.

Validating $(165n^3 \leq \frac{c}{3}n^4)$:
Let $n \in \mathbb{N}$, $n_0 = 2$, and $c = 249$. Assume $n \geq n_0$.

Then,

$$165n^3 \leq \frac{c}{3}n^4 \tag{6}$$
$$165n^3 \leq \frac{249}{3}n^4 \tag{7}$$
$$165n^3 \leq 83n^4 \tag{8}$$

Then,

$$165n^3 \leq 83n^4 \tag{9}$$
$$165(2)^3 \leq 83(2)^4 \tag{10}$$

because of the fact $n_0 = 2$ and the assumption $n \geq n_0$

Then,

$$165(2)^3 \leq 83(2)^4 \tag{11}$$
$$165 \cdot 8 \leq 83 \cdot 16 \tag{12}$$
$$1320 \leq 1328 \tag{13}$$

Then, it follows from above that $165n^3 \leq \frac{c}{3}n^4$ holds.

Validating $(cn^2 \leq \frac{c}{3}n^4)$:

Let $n \in \mathbb{N}$, $n_0 = 2$, and $c = 249$. Assume $n \geq n_0$.

Then,

$$\begin{align}
cn^2 &\leq \frac{c}{3}n^4 \tag{14}\\
249n^2 &\leq \frac{249}{3}n^4 \tag{15}\\
249n^2 &\leq 83n^4 \tag{16}\\
& \tag{17}
\end{align}$$

Then,

$$\begin{align}
249n^2 &\leq 83n^4 \tag{18}\\
249(2)^2 &\leq 83(2)^4 \tag{19}\\
& \tag{20}
\end{align}$$

because of the fact $n_0 = 2$ and the assumption $n \geq n_0$

Then,

$$\begin{align}
249n^2 &\leq 83n^4 \tag{21}\\
249 \cdot 4 &\leq 83 \cdot 16 \tag{22}\\
996 &\leq 1328 \tag{23}
\end{align}$$

Then, it follows from above that $cn^2 \leq \frac{c}{3}n^4$ holds.

Then, because we know $n^4 \leq \frac{c}{3}n^4$, $165n^3 \leq \frac{c}{3}n^4$, and $cn^2 \leq \frac{c}{3}n^4$ are true, we can conclude that $n^4 + 165n^3 + cn^2 \leq cn^4$ is true.

Then,

$$\begin{align}
n^4 + 165n^3 + cn^2 &\leq cn^4 \tag{24}\\
n^4 + 165n^3 &\leq c(n^4 - n^2) \tag{25}
\end{align}$$

Then, it follows from the definition of Big-Oh that the statement $n^4 + 165n^3 \in \mathcal{O}(n^4 - n^2)$ is true.

b. **Negation:** $\forall f : \mathbb{N} \to \mathbb{R}^+, \exists g : \mathbb{N} \to \mathbb{R}^{\geq 0}, (\forall c, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N}, n \geq n_0 \wedge (g(n) > cf(n))) \wedge (\forall d, m_0 \in \mathbb{R}^+, \exists m \in \mathbb{N}, m \geq m_0 \wedge (g(n) < cf(n)))$

Let

$$g(n) = \begin{cases} 0 & \text{if n even} \\ nf(n) & \text{otherwise} \end{cases}$$

We will prove the statement by separating into cases, and combine them together in the end.

Case 1 ($g \notin \mathcal{O}(f) : \forall c, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N}, n \geq n_0 \wedge g(n) > cf(n)$):

Let $c, n_0 \in \mathbb{R}^+$, $n = 2 \cdot \lceil max(c, n_0) \rceil$.

Because there are two parts in $max(c, n_0)$, we will prove by separating into cases $n_0 > c$, and $c > n_0$ and combine the result together in the end.

Consider the case $n_0 > c$.

Since $n_0 > c$ and $n = 2\lceil n_0 \rceil$, we can conclude that $n \geq n_0$.

Then,

$$\begin{aligned} g(n) &= nf(n) && (1) \\ &= 2\lceil n_0 \rceil f(n) && (2) \\ &> 2cf(n) && (3) \\ &> cf(n) && (4) \end{aligned}$$

Then, since $g(n) > cf(n)$ is true and $n \geq n_0$ is true, $n \geq n_0 \wedge g(n) > cf(n)$ is true.

Now, consider the case $c > n_0$.

Since $c > n_0$ and $n = 2\lceil c \rceil$, we can conclude that $n \geq n_0$.

Then,

$$
\begin{aligned}
g(n) &= nf(n) && (5)\\
&= 2\lceil c \rceil f(n) && (6)\\
&> 2cf(n) && (7)\\
&> cf(n) && (8)
\end{aligned}
$$

Then, since $g(n) > cf(n)$ is true and $n \geq n_0$ is true given $n_0 > c$, and $c > n_0$, $n \geq n_0 \wedge g(n) > cf(n)$ is true.

Then, it follows from above that the statement $g \notin \mathcal{O}(f)$ is true.

Case 2 ($g \notin \Omega(f) :\ \forall c, n_0 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \wedge g(n) < cf(n)$):

Let $c, n_0 \in \mathbb{R}^+$, $n = \lceil n_0 \rceil + 1$.

Then, we can conclude that $n \geq n_0$.

Then,

$$0 = g(n) < cf(n) \qquad (1)$$

by the fact that no values in codomain of f is 0.

Then, $n \geq n_0 \wedge g(n) < cf(n)$ is true by the fact that $(n \geq n_0)$ is true and $(g(n) < cf(n))$ is true.

Then, it follows from above that the statement $g \notin \Omega(f)$ is true.

Since $g \notin \Omega(f)$ is true and $g \notin \mathcal{O}(f)$ is true given g and n, it follows from the negation of statement that the statement $\exists f : \mathbb{N} \rightarrow \mathbb{R}^+$, $(\forall g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0},\ g \in \mathcal{O}(f) \vee g \in \Omega(f))$ is false.

Question 4

a. Let $n_0 = c^{-\frac{1}{b-a}}$. Assume $n \geq n_0$.

Then,

$$\begin{align} n &\geq n_0 \tag{1}\\ n &\geq c^{-\frac{1}{b-a}} \tag{2}\\ [n]^{-(b-a)} &\leq \left[c^{-\frac{1}{b-a}}\right]^{-(b-a)} \tag{3}\\ n^{a-b} &\leq c \tag{4}\end{align}$$

Then,

$$\begin{align} cf(n) &= cn^b \tag{5}\\ &\geq n^{a-b}n^b \tag{6}\\ &\geq n^{a-b+b} \tag{7}\\ &\geq n^a \tag{8}\end{align}$$

Then, it follows from the definition of little-oh that the statement $\forall a, b \in \mathbb{R}^+, a < b \Rightarrow n^a \in o(n^b)$ is true.

b. **Predicate logic in expanded form:** $\forall f, g : \mathbb{N} \to \mathbb{R}^+,\ (\forall c \in \mathbb{R}^+,\ \exists n_0 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \leq cf(n)) \Rightarrow (\forall d, m_0 \in \mathbb{R}^+, \exists m \in \mathbb{N},\ m \geq m_0 \wedge f(m) > dg(m))$

Rough idea

$$\frac{1}{2}g(n) < g(n) \leq cf(n) \tag{1}$$