CSC236 Worksheet 5 Solution

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May 7, 2020

Question 1

a. *Proof.* For convenience, define $H(k) : R(3^k) = 3^k k$. Notice that when $n = 3^k$, $3^k k$ is the same as $n \log_3 n$. I will use simple induction to prove $\forall k \in \mathbb{N},\ H(k)$.

Base Case ($k = 0$):

Let $k = 0$.

Then,

$$
\begin{aligned}
3^k \cdot k &= 3^0 \cdot 0 && (1)\\
&= 0 && (2)\\
&= R(n) && \text{[By def.]} \quad (3)
\end{aligned}
$$

Thus, $H(0)$ is verified.

Inductive Step:

Let $k \in \mathbb{N}$. Assume $H(k)$. That is $R(3^k) = 3^k k$.

I will show that $H(k+1)$ follows. That is, $R(3^{k+1}) = (k+1)3^{k+1}$.

The definition of $R(3^{k+1})$ tells us,

$$
\begin{aligned}
R(3^{k+1}) &= 3^{k+1} + 3R(\lceil 3^{k+1}/3 \rceil) && (4)\\
&= 3^{k+1} + 3R(3^k) && (5)
\end{aligned}
$$

Then, using this fact, we can conclude

$$
\begin{aligned}
R(3^{k+1}) &= 3^{k+1} + 3 \cdot 3^k k && \text{[By I.H]} && (6)\\
&= 3^{k+1} + 3^{k+1} k && && (7)\\
&= 3^{k+1}(k+1) && && (8)
\end{aligned}
$$

$\square$

Correct Solution:

For convenience, define $H(k) : R(3^k) = 3^k k$. Notice that when $n = 3^k$, $3^k k$ is the same as $n \log_3 n$. I will use simple induction to prove $\forall k \in \mathbb{N},\ H(k)$.

Base Case ($k = 0$):

Let $k = 0$.

Then,

$$
\begin{aligned}
3^k \cdot k &= 3^0 \cdot 0 && && (9)\\
&= 0 && && (10)\\
&= R(n) && \text{[By def.]} && (11)
\end{aligned}
$$

Thus, $H(0)$ is verified.

Inductive Step:

Let $k \in \mathbb{N}$. Assume $H(k)$. That is $R(3^k) = 3^k k$.

I will show that $H(k+1)$ follows. That is, $R(3^{k+1}) = (k+1)3^{k+1}$.

Since $k + 1 > 0$, $3^{k+1} > 1$.

So the definition of $R(3^{k+1})$ tells us,

$$
\begin{aligned}
R(3^{k+1}) &= 3^{k+1} + 3R(\lceil 3^{k+1}/3 \rceil) && (12)\\
&= 3^{k+1} + 3R(3^k) && (13)
\end{aligned}
$$

Then, using this fact, we can conclude

$$
\begin{aligned}
R(3^{k+1}) &= 3^{k+1} + 3 \cdot 3^k k && \text{[By I.H]} \qquad (14)\\
&= 3^{k+1} + 3^{k+1} k && (15)\\
&= 3^{k+1}(k+1) && (16)
\end{aligned}
$$