CSC373 Worksheet 6 Solution

August 12, 2020

1. **Rough Works:**

1. Multiply objective function by - 1

Maximize

$$-2x_1 - 7x_2 - x_3$$

Subject to

$$\begin{aligned} x_1 - x_3 &= 7 \\ 3x_1 + x_2 &\geq 7 \\ x_2 &\geq 0 \\ x_3 &\leq 0 \end{aligned}$$

2. Replace non-nonnegative constraints x_1

Maximize

$$-2x_1' + 2x_1'' - 7x_2 - x_3$$

Subject to

$$
\begin{aligned}
x_1' - x_1'' - x_3 &= 7 \\
3x_1' - 3x_1'' + x_2 &\geq 7 \\
x_1', x_1'', x_2 &\geq 0 \\
x_3 &\leq 0
\end{aligned}
$$

3. Replace non-nonnegative constraints x_3

Maximize

$$-2x_1' + 2x_1'' - 7x_2 - x_3' + x_3''$$

Subject to

$$
\begin{aligned}
x_1' - x_1'' - x_3' + x_3'' &= 7 \\
3x_1' - 3x_1'' + x_2 &\geq 7 \\
x_1', x_1'', x_2, x_3', x_3'' &\geq 0
\end{aligned}
$$

4. Replace equality constraints with $\geq$ and $\leq$

Maximize

$$-2x_1' + 2x_1'' - 7x_2 - x_3' + x_3''$$

Subject to

$$
\begin{aligned}
x_1' - x_1'' - x_3' + x_3'' &\leq 7 \\
x_1' - x_1'' - x_3' + x_3'' &\geq 7 \\
3x_1' - 3x_1'' + x_2 &\geq 7 \\
x_1', x_1'', x_2, x_3', x_3'' &\geq 0
\end{aligned}
$$

Notes:

- **Linear Programming**

  - Is a method to achieve the best outcome (such as maximum profit or lowest cost) in a mathematical model whose requirements are represented by linear relationships. [1]
  - Is named to make it sound cool for government funding
    * Like dynamic programming
  - Applications
    * Microeconomics (maximize profits, minimize costs)
    * Company management

- **Standard Form**
  - Is a form of linear programming
  - Are about <u>maximizing</u>, not minimizing [2]
  - All have a positivity constraint for each variable [2]
  - All other constraints are all of the form "linear combination of variables ≤ constant". [2]

3. Are about maximizing and not minimizing

$$\boxed{\text{Maximize}}\ c_1x_1 + c_2x_2 + \cdots c_nx_n$$

subject to

2. constraints of the form Σ a_ij x_j ≤ b_i

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m \\ \boxed{x_1, x_2, \ldots, x_n \ge 0}& \end{aligned}$$

1. non-negativity constraints for each variable

- **Converting Linear Programming to Standard Form**
  1) The objective function might be a minimization rather than a maximization
     - Negate coefficients of the objective function

multiply by -1

$$\begin{array}{ll} \boxed{\text{minimize} \quad -2x_1 + 3x_2} & \\ \text{subject to} & \\ & x_1 + x_2 = 7 \\ & x_1 - 2x_2 \le 4 \\ & x_1 \ge 0 \end{array} \qquad \begin{array}{ll} \boxed{\text{maximize} \quad 2x_1 - 3x_2} & \\ \text{subject to} & \\ & x_1 + x_2 = 7 \\ & x_1 - 2x_2 \le 4 \\ & x_1 \ge 0 \end{array}$$

2) There might be variables without nonnegativity constraints
   - Replace each non-nonnegative variable x_i with x_i' and x_i''
   - Modify linear program

Replace x_i with x_i' and x_i''

$$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2 \\ \text{subject to} & \\ & x_1 + x_2 = 7 \\ & x_1 - 2x_2 \le 4 \\ & x_1 \ge 0 \end{array} \qquad \begin{array}{ll} \text{maximize} & 2x_1 - 3x_2' + 3x_2'' \\ \text{subject to} & \\ & x_1 + x_2' - x_2'' = 7 \\ & x_1 - 2x_2' + 2x_2'' \le 4 \\ & x_1, x_2', x_2'' \ge 0 \end{array}$$

x_2 is not nonnegative :(They are now nonnegative :)! Yayy!!

3) There might be **equality constraints**, which have an equal sign rather than a less-than-or-equal-to sign
   - Replace equality constraint $f(x_1, x_2, ..., x_n) = b$ with $f(x_1, x_2, ..., x_n) \le b$ and $f(x_1, x_2, ..., x_n) \ge b$

Multiply incorrect constraints by -1

$$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2' + 3x_2'' \\ \text{subject to} & \\ & x_1 + x_2' - x_2'' \le 7 \\ & x_1 + x_2' - x_2'' \ge 7 \\ & x_1 - 2x_2' + 2x_2'' \le 4 \\ & x_1, x_2', x_2'' \ge 0\,. \end{array} \qquad \begin{array}{ll} \text{maximize} & 2x_1 - 3x_2 + 3x_3 \\ \text{subject to} & \\ & x_1 + x_2 - x_3 \le 7 \\ & -x_1 - x_2 + x_3 \le -7 \\ & x_1 - 2x_2 + 2x_3 \le 4 \\ & x_1, x_2, x_3 \ge 0 \end{array}$$

4) There might be **inequality constraints**, but instead of having a less-than-or-equal-to-sign
   - Multiply incorrect inequality constraints by -1

Replace = with ≤ and ≥

$$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2' + 3x_2'' \\ \text{subject to} & \\ & x_1 + x_2' - x_2'' = 7 \\ & x_1 - 2x_2' + 2x_2'' \le 4 \\ & x_1, x_2', x_2'' \ge 0 \end{array} \qquad \begin{array}{ll} \text{maximize} & 2x_1 - 3x_2' + 3x_2'' \\ \text{subject to} & \\ & x_1 + x_2' - x_2'' \le 7 \\ & x_1 + x_2' - x_2'' \ge 7 \\ & x_1 - 2x_2' + 2x_2'' \le 4 \\ & x_1, x_2', x_2'' \ge 0\,. \end{array}$$

References:

1) Wikipedia, Linear Programming, link
2) Instituto de Mathematicas, Standard form for Linear Programs, link