

# CSC373 Worksheet 5 Solution

August 9, 2020

1. *Proof.* Assume that a flow network  $G = (V, E)$  violates the assumption that the network contains a path  $s \rightsquigarrow v \rightsquigarrow t$  for all vertices  $v \in V$ . Let  $u$  be a vertex for which there is no path  $s \rightsquigarrow u \rightsquigarrow t$ .

I must show such that there is no flow at vertex  $u$ . That is, there exists a maximum flow  $f$  in  $G$  such that  $f(u, v) = f(v, u) = 0$  for all vertices  $v \in V$ .

Assume for the sake of contradiction that there is some vertex  $u$  with flow  $f$ . That is, there exists some vertices  $v \in V$  such that  $f(u, v) > 0$  or  $f(v, u) > 0$ .

I see that three cases follows, and I will prove each separately.

1. **Cases 1:**  $f(u, v) = 0$  and  $f(v, u) > 0$

Here, assume that  $f(u, v) = 0$  for all  $v \in V$  and  $f(v, u) > 0$  for some  $v \in V$ .

Then, we can write  $\sum_{v \in V} f(u, v) = 0$  and  $\sum_{v \in V} f(v, u) > 0$

But this violates the flow conservation property (i.e  $\sum_{v \in V} f(u, v) = \sum_{v \in V} f(v, u)$ )

Thus, by proof by contradiction,  $f(u, v) = 0$  and  $f(v, u) = 0$  for all  $v \in V$  and all  $u \in V$  with no path  $s \rightsquigarrow u \rightsquigarrow t$ .

2. **Cases 2:**  $f(u, v) > 0$  and  $f(v, u) = 0$

Here, assume that  $f(u, v) > 0$  for some  $v \in V$  and  $f(v, u) = 0$  for all  $v \in V$ .

Then, by similar work as case 1, the same result follows.

### 3. Cases 3: $f(u, v) > 0$ and $f(v, u) > 0$

Here, assume that  $f(u, v) > 0$  and  $f(v, u) > 0$  for some  $v \in V$ .

Since  $s \rightsquigarrow v \rightsquigarrow t$  and  $u$  is connected by some vertices  $v$ , we can write  $s \rightsquigarrow u \rightsquigarrow t$ .

Then, this violates the fact in header that the vertex  $u$  has no path  $s \rightsquigarrow u \rightsquigarrow t$ .

Thus, by proof by contradiction,  $f(u, v) = 0$  and  $f(v, u) = 0$  for all  $v \in V$  and all  $u \in V$  with no path  $s \rightsquigarrow u \rightsquigarrow t$ .

□

## Notes

### • Maximum Flow:

- Finds a flow of maximum value <sup>[1]</sup>

### Example



Here, the maximum flow is  $10 + 5 + 13 = 28$

### • Flow Network:

- $G = (V, E)$  is a directed graph in which each edge  $(u, v) \in E$  has a nonnegative capacity  $c(u, v) \geq 0$ .
- Two vertices must exist: **source**  $s$  and **sink**  $t$
- **path** from source  $s$  to vertex  $v$  to sink  $t$  is represented by  $s \rightsquigarrow v \rightsquigarrow t$



- **Capacity:**

- Is a non-negative function  $f : V \times V \rightarrow \mathbb{R}_{\geq 0}$
- Has **capacity constraint** where for all  $u, v \in V$   $0 \leq f(u, v) \leq c(u, v)$ 
  - \* Means flow cannot be above capacity constraint

- **Flow:**

- Is a real valued function  $f : V \times V \rightarrow \mathbb{R}$  in  $G$
- Satisfies **capacity constraint** (i.e for all  $u, v \in V$ ,  $0 \leq f(u, v) \leq c(u, v)$ )
- Satisfies **flow conservation**

For all  $u \in V - \{s, t\}$ , we require

$$\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v) \quad (1)$$

\* Means flow into vertex  $u$  is the same as flow going out of vertex  $u$ . <sup>[1]</sup>

\*  $\sum_{v \in V} f(u, v)$  means flow out of vertex  $u$

\*  $\sum_{v \in V} f(v, u)$  means flow into vertex  $u$

\*  $v \in V$  in  $\sum_{v \in V} f(u, v)$  means all vertices that are an edge away from vertex  $u$

**Example:**



## References

- 1) Princeton University, Network Flow 1, link
2. I need to formulate the problem of determining whether both of professor Adam's two children can go to the same school as maximum-flow problem.

The problem statement tells us the following:

1. There is 1 supersource (location of home)
2. There is 1 sink (location of school)
3. There are two sources ( $s_1$  as child 1,  $s_2$  as child 2)
4. Edge  $(u, v)$  has capacity of 0 or more (0 representing unavailable sidewalk, 1 for sidewalk with capacity of 1, 2 for street with capacity of 2 and so on)
5. Each vertex represents corner of intersection, and two children can have their paths crossing here.
6. Has flow of 2, 1 or 0 (1 is where one of the two children walking on the road. 0 is none.)

Here we are to find whether children must go on to a vertex and out to the same edge with the flow of 2, or determine whether there is only edge to school with capacity of 1 or less.

If none, then both children can safely go to school.



### Notes:

- **Cross at a Corner**

- Means to walk across the street at a corner of the intersection.



- **Multiple Sources and Sinks**

- Has edges  $(s, s_i)$  where  $i = 1 \dots n$  and  $(t_j, t)$  where  $j = 1 \dots n$  with capacity of  $\infty$

### Example:

Lucky Puck Company having a set of  $m$  factories  $\{s_1, s_2, \dots, s_m\}$ , and a set of  $n$  warehouses and  $n$  warehouses  $\{t_1, t_2, \dots, t_n\}$



3. I need to show how to transform a flow network  $G = (V, E)$  with vertex capacities into an equivalent flow network  $G' = (V', E')$  without vertex capacities.

For each vertex capacities, change as follows.



After transformation, there will be  $m$  more edges and vertices, where  $m$  represents the number of vertex capacities in  $G$ .

Notes:

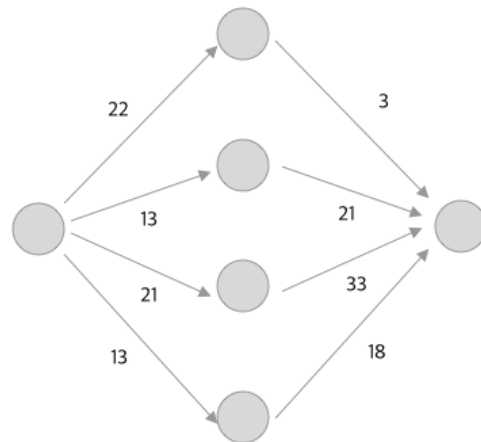
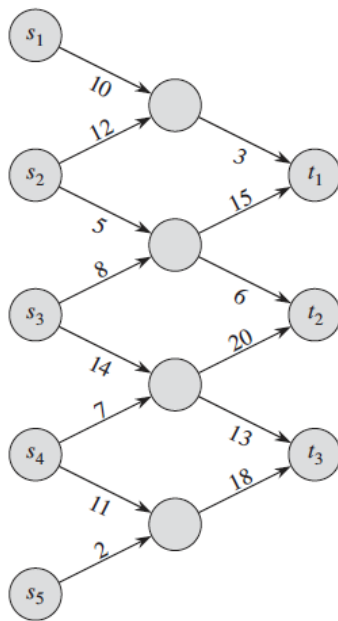
- **Vertex Capacities**

- Each vertex  $v$  has limit  $l(v)$  on how much flow can pass through  $v$

4. I need to show how to convert the problem of finding a flow  $f$  that obeys the constraints into the problem of finding a maximum flow in a single source, single-sink flow network

The steps are as follows:

- Combine all sources  $s_i$  into a single source  $s$
- Combine all sinks  $t_j$  into a single sink  $t$
- Connect source  $s$  to each adjacent vertex  $v$  with edge weight  $\sum_i f(s_i, v) = p_i$ 
  - The total edge weight from  $s$  should be  $\sum_i p_i$
- Connect each adjacent vertex  $v$  of  $t$  to  $t$  with edge weight  $\sum_j f(v, t_j) = q_j$ 
  - The total edge weight to  $t$  should be  $\sum_j q_j$
- Find a simple path from  $s$  to  $t$  with the maximum amount of total flow

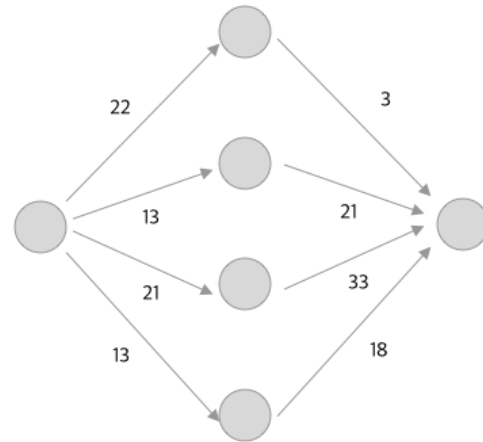
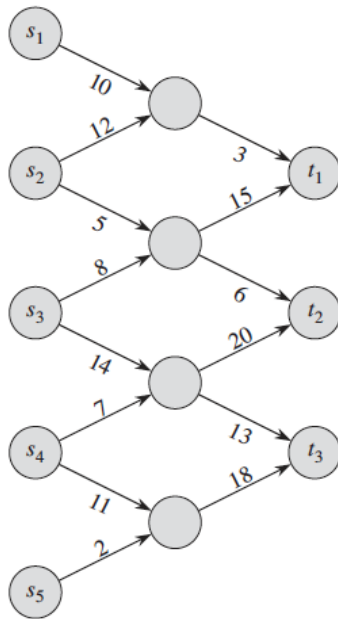


### Rough Works:

I need to show how to convert the problem of finding a flow  $f$  that obeys the constraints into the problem of finding a maximum flow in a single source, single-sink flow network

The steps are as follows:

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  - The total edge weight from  $s$  should be  $\sum_i p_i$
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  - The total edge weight to  $t$  should be  $\sum_j q_j$
- Find a simple path from  $s$  to  $t$  with the maximum amount of total flow



### Notes:

- **Ford-Fulkerson Method**

- Is a greedy algorithm that solves the maximum-flow problem
  - \* Determines maximum flow from start vertex to sink vertex in a graph
- Called method (not algorithm) because several different implementations with different running time is used

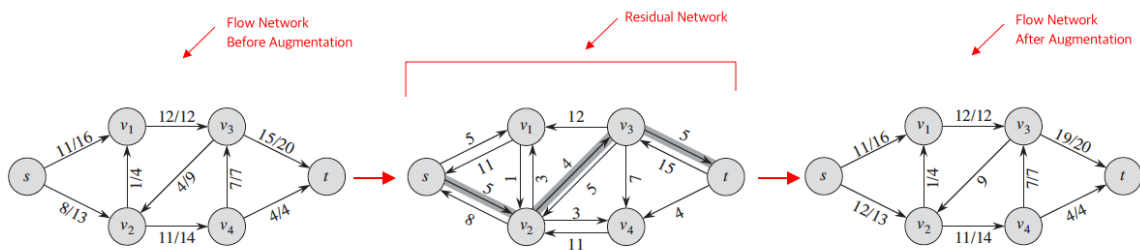
#### **FORD-FULKERSON-METHOD** ( $G, s, t$ )

- 1 initialize flow  $f$  to 0
- 2 **while** there exists an augmenting path  $p$  in the residual network  $G_f$
- 3     augment flow  $f$  along  $p$
- 4 **return**  $f$



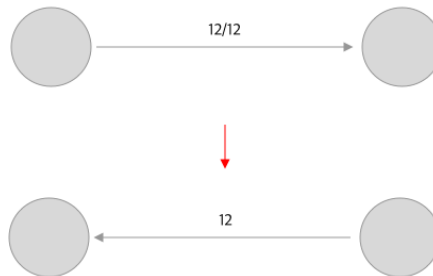
## • Residual Network

- Indicates how much more flow is allowed in each edge in the network graph <sup>[1]</sup>
- Consists of edges with capacities that represents how we can change the flow on edges of  $G$ .
- Provides roadmap for adding flow to the original flow network



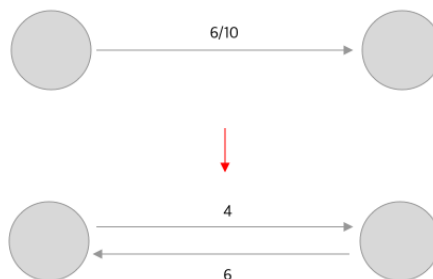
## Steps

- 1)  $Flow = Capacity$ : Opposite arrow



- 2)  $Flow < Capacity$ :

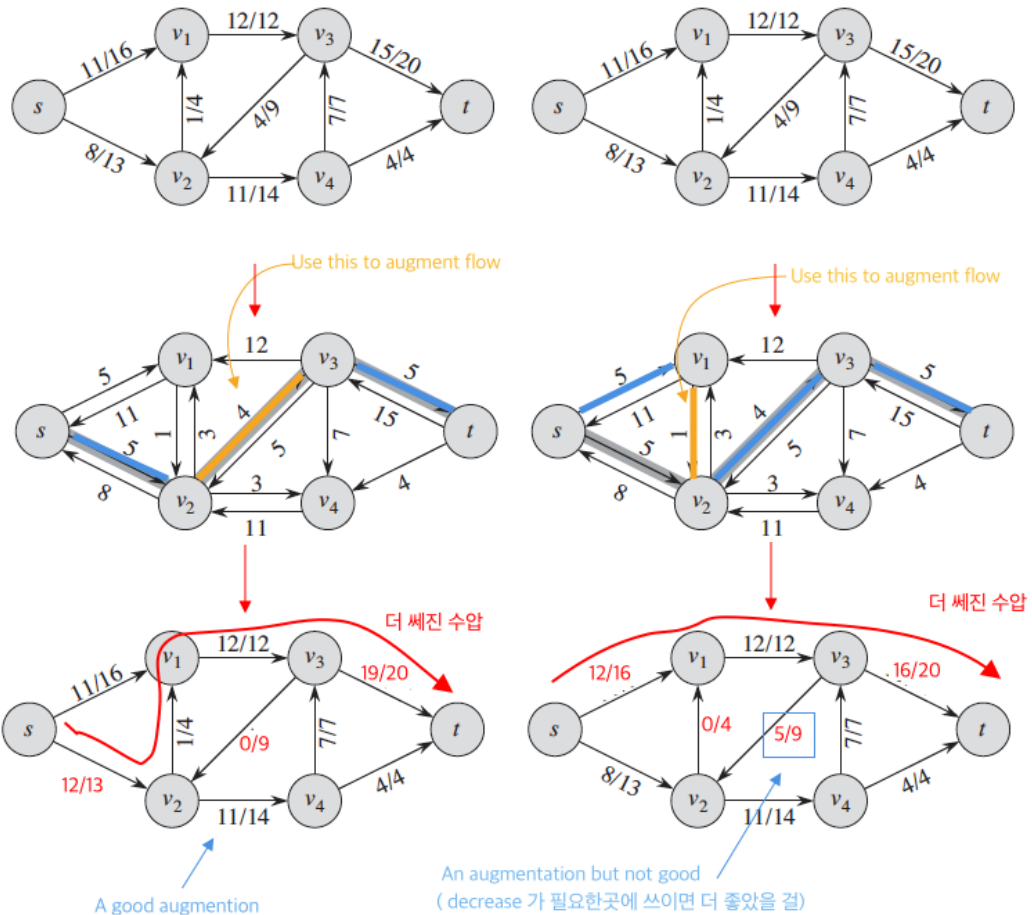
- $Flow$ : Opposite Arrow
- $Capacity - Flow$ : Current Arrow



## • Augmenting Path

- Is a path from source  $S$  to sink  $T$  where you can increase the amount of flow

- Is a path that doesn't contain cycle (simple path) <sup>[2]</sup>



- Edge  $(u, v)$  on simple path on an augmenting path can be increased by upto  $c_f(u, v)$  without violating the capacity constraint

- Augmentation

- 한국어로 '거꾸로 가는 수압 decrease 해서 앞으로 가는 수압 더 세게 만들기'
- Is symbolized by  $f \uparrow f'$ 
  - \*  $f$  is a flow in  $G$
  - \*  $f'$  is a flow in the residual network  $G_f$

## References

- 1) Hacker Earth, Maximum Flow, [link](#)
- 2) Stack Overflow, What Exactly Is Augmentation Path, [link](#)