Worksheet 6 Review

March 24, 2020

Question 1

a. $\forall n \in \mathbb{N},\ P(123) \wedge \neg(n > 123 \Rightarrow P(n))$

Correct Solution:

$P(123) \wedge (\forall n \in \mathbb{N}, P(n) \Rightarrow n \leq 123)$

b. $IsCD(x, y) :\ \exists x, y, d \in \mathbb{Z},\ d \mid x \wedge d \mid y$

$IsGCD(x, y) :\ \exists x, y, d \in \mathbb{Z}, d \mid x \wedge d \mid y \wedge (\forall n \in \mathbb{N}, n > d \Rightarrow n \nmid x \vee n \nmid y)$

Correct Solution:

$IsGCD(x, y, d) : \exists x, y, d \in \mathbb{Z}, (x = 0 \wedge y = 0 \Rightarrow d = 0) \wedge (x \neq 0 \vee y \neq 0 \Rightarrow IsCD(x, y, d) \Rightarrow \forall d' \in \mathbb{Z}, IsCD(x, y, d') \Rightarrow d' \leq d)$

c. Let $a = x$, $b = 0$, $d = x$ and $d' \in \mathbb{Z}$. Assume $IsCD(x, y, d')$.

Because we know $x \mid x$ and $x \mid 0$, we can conclude that d is a common divisor to a and b.

Since $d' \mid a$ and $d' \mid b$, and since $\forall n \in \mathbb{Z}^+, \forall d \in \mathbb{Z},\ d \mid n \Rightarrow d \leq n$, we can conclude that

$$d' \leq a \tag{1}$$

Then,

$$d' \leq d \tag{2}$$

by the fact that $d = a$.

Then it follows from above that the statement $\forall x \in \mathbb{Z}^+,\ IsGCD(x, 0, x)$ is true.

d. **Attempt 1:**

$a, b \in \mathbb{Z},\ a \neq 0 \vee b \neq 0 \Rightarrow (\exists d \in \mathbb{Z},\ d = GCD(a, b) \Rightarrow \forall d' \in \mathbb{Z}^+, \exists p, q \in \mathbb{Z})$

Attempt 2:

$a, b \in \mathbb{Z},\ \exists d \in \mathbb{Z},\ (a \neq 0 \vee b \neq 0) \wedge d = GCD(a, b) \Rightarrow \exists p, q \in \mathbb{Z},\ d = ap + bq \wedge d > 0 \wedge (\forall d' \in \mathbb{Z}^+, d' = ap + bq \Rightarrow d' \geq d))$

Question 2

a. Let $n \in \mathbb{Z}$. Assume that $\exists k \in \mathbb{Z}, n = 2k$.

Then,

$$\begin{align} n^2 - 3n &= 4n^2 - 6n \tag{1} \\ &= 2(n^2 - 3n) \tag{2} \\ &= 2m \tag{3} \end{align}$$

where $m = n^2 - 3n \in \mathbb{Z}$.

Then, by definition of even number, $n^2 - 3n$ is even.

b. Let $n \in \mathbb{Z}$. Assume $\exists k \in \mathbb{Z},\ n = 2k - 1$.

Then,

$$\begin{align} n^2 - 3n &= (2k-1)^2 - 3 \cdot (2k - 1) \tag{1} \\ &= 4k^2 - 4k + 1 - 6k + 3 \tag{2} \\ &= 4k^2 - 10k + 4 \tag{3} \\ &= 2(2k^2 - 5k + 2) \tag{4} \\ &= 2m \tag{5} \end{align}$$

where $m = 2k^2 - 5k + 2 \in \mathbb{Z}$.

Then, it follows from the definition of even number that $n^2 - 3n$ is even.

Question 3

a. $\forall a, b \in \mathbb{N},\ Prime(b) \Rightarrow 1 \geq gcd(a,b) \vee gcd(a,b) \geq b$

b. Let $a, b \in \mathbb{N}$. Assume $Prime(b)$.

We will prove the statement by considering two cases, when $b \mid a$, and when $b \nmid a$.

Case 1 ($b \mid a$):

Assume $b \mid a$.

Since b is a prime number, there are to possible divisors b and 1.

Since $b \mid a$ and $b \mid b$, $b = gcd(a,b)$.

Since $b \mid gcd(a,b)$, by the fact $\forall n \in \mathbb{Z}^+, d \in \mathbb{Z},\ d \mid n \Rightarrow d \leq n$, we can conclude that

$$b \leq gcd(a,b) \tag{1}$$

Case 2 ($b \nmid a$):

Assume $b \nmid a$.

Since b is a prime number, b has two divisors b and 1.

Since $b \nmid a$ and $1 \mid a$, $1 = gcd(a,b)$.

Since $gcd(a,b) \mid 1$, by the fact $\forall n \in \mathbb{Z}^+, d \in \mathbb{Z},\ d \mid n \Rightarrow d \leq n$, we can conclude that

$$1 \geq gcd(a,b) \tag{1}$$