CSC236 Worksheet 5 Solution

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Question 1

a. *Proof.* For convenience, define $H(k) : R(3^k) = 3^k k$. Notice that when $n = 3^k$, $3^k k$ is the same as $n \log_3 n$. I will use simple induction to prove $\forall k \in \mathbb{N},\ H(k)$.

<u>**Base Case ($k = 0$):**</u>

Let $k = 0$.

Then,

$$\begin{aligned} 3^k \cdot k &= 3^0 \cdot 0 && (1) \\ &= 0 && (2) \\ &= R(n) && \text{[By def.]} \quad (3) \end{aligned}$$

Thus, $H(0)$ is verified.

<u>**Inductive Step:**</u>

Let $k \in \mathbb{N}$. Assume $H(k)$. That is $R(3^k) = 3^k k$.

I will show that $H(k+1)$ follows. That is, $R(3^{k+1}) = (k+1)3^{k+1}$.

The definition of $R(3^{k+1})$ tells us,

$$\begin{aligned} R(3^{k+1}) &= 3^{k+1} + 3R(\lceil 3^{k+1}/3 \rceil) && (4) \\ &= 3^{k+1} + 3R(3^k) && (5) \end{aligned}$$

Then, using this fact, we can conclude

$$R(3^{k+1}) = 3^{k+1} + 3 \cdot 3^k k \qquad \text{[By I.H]} \tag{6}$$
$$= 3^{k+1} + 3^{k+1}k \tag{7}$$
$$= 3^{k+1}(k+1) \tag{8}$$

$\square$

Correct Solution:

For convenience, define $H(k) : R(3^k) = 3^k k$. Notice that when $n = 3^k$, $3^k k$ is the same as $n \log_3 n$. I will use simple induction to prove $\forall k \in \mathbb{N},\ H(k)$.

Base Case ($k = 0$):

Let $k = 0$.

Then,

$$3^k \cdot k = 3^0 \cdot 0 \tag{9}$$
$$= 0 \tag{10}$$
$$= R(n) \qquad \text{[By def.]} \tag{11}$$

Thus, $H(0)$ is verified.

Inductive Step:

Let $k \in \mathbb{N}$. Assume $H(k)$. That is $R(3^k) = 3^k k$.

I will show that $H(k+1)$ follows. That is, $R(3^{k+1}) = (k+1)3^{k+1}$.

Since $k + 1 > 0$, $3^{k+1} > 1$.

So the definition of $R(3^{k+1})$ tells us,

$$R(3^{k+1}) = 3^{k+1} + 3R(\lceil 3^{k+1}/3 \rceil) \tag{12}$$
$$= 3^{k+1} + 3R(3^k) \tag{13}$$

Then, using this fact, we can conclude

$$
\begin{aligned}
R(3^{k+1}) &= 3^{k+1} + 3 \cdot 3^k k && \text{[By I.H]} && (14)\\
&= 3^{k+1} + 3^{k+1} k && && (15)\\
&= 3^{k+1}(k+1) && && (16)
\end{aligned}
$$

Notes:

- Noticed that professor used the phrase 'Notice that when $n = 3^k$, $3^k k$ is the same as $n \log_3 n$.' to express n interms of 3^k.
- I feel I should review this problem to make sure I understood.

b. *Proof.* Define $P(k)$ as

$$P(k) : \bigwedge_{m=1}^{m=k} R(m) \leq R(k)$$

I will prove $\forall n \in \mathbb{N}^+, P(n)$ using complete induction.

Base Case ($n = 1$):

Let $n = 1$.

Then, $\bigwedge_{m=1}^{m=1} R(m) = R(1)$.

So, $P(1)$ follows.

Base Case ($n = 2$):

Let $n = 2$.

I need to show $P(2)$ holds. That is, $R(1) \leq R(2)$ and $R(2) \leq R(2)$.

I will do so in parts.

Part 1 ($R(1) \leq R(2)$):

In this part, $R(1) = 0$ and $R(2) = 2 + R(\lceil 2/3 \rceil) = 2 + R(1) = 2$.

Since $0 \leq 2$, we can conclude $R(1) \leq R(2)$.

Part 2 ($R(2) \leq R(2)$):

In this part, $R(2) = R(2)$, so we can conclude $R(2) \leq R(2)$.

Case ($n > 2$):

Since $n > 2$, $1 \leq n - 1 < n$, so we know $P(n-1)$ holds.

Then, it is suffice to prove $R(n-1) \leq R(n)$.

Starting from $R(n-1)$, we have

$$
\begin{aligned}
R(n-1) &= (n-1) + R(\lceil (n-1)/3 \rceil) && \text{[By def.]} \quad (17)\\
&\leq n + R(\lceil (n-1)/3 \rceil) && (18)\\
&\leq n + R(\lceil n/3 \rceil) && \text{[By I.H, since } 1 \leq \lceil (n-1)/3 \rceil \leq \lceil n/3 \rceil < n\text{]} \quad (19)\\
&= R(n) && (20)
\end{aligned}
$$

□