

# Worksheet 9 Review

March 28, 2020

## Question 1

- a. For every set of size 0 has 0 subsets of size 2.
- b. Let  $n = 0$ . Let  $S$  be an arbitrary set. Assume  $S$  has size 0.

Since  $S$  has size 0, empty subsets are the **only** subsets that can be included in  $S$ .

Then, because we know empty subsets have size 0, we can conclude there are 0 subsets of size 2.

It follows from above that the base case holds.

### Correct Solution:

**We want to show every set  $S$  of size 0 has 0 subsets of size 2.**

Since  $S$  has size 0, empty subsets are the **only** subsets that can be included in  $S$ .

Then, because we know an empty subset have size 0, we can conclude there are 0 subsets of size 2.

**Notes:**

- Professor specifically mentions **We want to show every set  $S$  of size  $0$  has  $0$  subsets of size  $2$**
- Professor doesn't include conclusion at the end of proof
- Under which cases conclusion to a proof are included.

c. Now we will prove inductive step.

Let  $k \in \mathbb{N}$ . Assume every set of size  $k$  has  $\frac{k(k-1)}{2}$  subsets of size  $2$ .

We want to show a set of size  $k+1$  has  $\frac{(k+1)k}{2}$  subsets of size  $2$ .

### Part 1: counting subsets of $S$ of size $2$ that contain $s_{k+1}$

It follows from the table below,

k	Sets	Subsets of Size 2 with $s_{k+1}$
0	$\{0, 1\}$	1
1	$\{0, 1, 2\}$	2
2	$\{0, 1, 2, 3\}$	3
2	$\{0, 1, 2, 3, 4\}$	4

that the number of subsets of size  $2$  that contain  $s_{k+1}$  is  $k+1$ .

### Part 2: counting subsets of $S$ of size $2$ that doesn't contain $s_{k+1}$

Because we know the subset of  $S$  that doesn't contain  $s_{k+1}$  is a set  $S$  of size  $k$ , we can conclude using induction hypothesis that there are

$$\frac{k(k-1)}{2} \tag{1}$$

subsets of size  $2$ .

### Part 3: Putting the counts together

Then,

$$\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2} \quad (2)$$

Then, it follows from proof by induction that the statement ' $\forall n \in \mathbb{N}$ , every set of size  $n$  has  $\frac{n(n-1)}{2}$  subsets of size 2' is true for all natural numbers  $n$ .

**Correct Solution:**

**Part 1: counting subsets of  $S$  of size 2 that contain  $s_{k+1}$**

It follows from the table below,

k	Sets	Subsets of Size 2 with $s_{k+1}$
2	$\{0, 1\}$	1
3	$\{0, 1, 2\}$	2
4	$\{0, 1, 2, 3\}$	3
5	$\{0, 1, 2, 3, 4\}$	4

that the number of subsets of size 2 that contain  $s_{k+1}$  is  $k$ .

**Part 2: counting subsets of  $S$  of size 2 that doesn't contain  $s_{k+1}$**

Because we know the subset of  $S$  that doesn't contain  $s_{k+1}$  is a set  $S$  of size  $k$ , we can conclude using induction hypothesis that there are

$$\frac{k(k-1)}{2} \quad (3)$$

subsets of size 2.

**Part 3: Putting the counts together**

Then,

$$\frac{k(k-1)}{2} + k = \frac{(k+1)k}{2} \quad (4)$$

Then, it follows from proof by induction that the statement ' $\forall k \in \mathbb{N}$ , every set of size  $k$  has  $\frac{k(k-1)}{2}$  subsets of size 2' is true for all natural numbers  $k$ .

**Notes:**

- I forgot that  $k$  represents number of elements in a set.

d. **Statement:** For every  $n \in \mathbb{N}$ , every finite set  $S$  of size  $n$ , has

$$\frac{n(n-1)(n-2)}{6} \quad (1)$$

subsets of size 3.

We will prove this statement by using induction on  $n$ .

**Base Case:**

Let  $n = 0$ .

Then, only the empty subsets can be included in  $S$ .

Because an empty subset has size 0, there are 0 subsets of size 3 in  $S$ .

Since

$$\frac{0 \cdot (0-1)(0-2)}{6} = 0 \quad (2)$$

the base case holds.

### Inductive Case:

Let  $k \in \mathbb{N}$ . Assume every finite set  $S$  of size  $k$  has exactly  $\frac{k(k-1)(k-2)}{6}$  subsets of size 3.

We want to show finite set  $S$  of size  $k + 1$  contains  $\frac{(k+1)k(k-1)}{6}$  subsets of size 3.

It follows from the table below

k	Sets	# of Subsets of Size 3	# of Subsets of Size 2
0	$\{\}$	0	0
1	$\{s_0\}$	0	0
2	$\{s_0, s_1\}$	0	1
3	$\{s_0, s_1, s_2\}$	1	3
4	$\{s_0, s_1, s_2, s_3\}$	4	6
5	$\{s_0, s_1, s_2, s_3, s_4\}$	10	10

we can deduce that given a set  $S$  size  $k + 1$ , the number of subsets of size 3 that contains  $s_{k+1}$  is the sum of # of subsets of size 3 that doesn't contain  $s_{k+1}$  and # of subsets of size 2 that doesn't contain  $s_{k+1}$ .

Then,

$$\frac{k(k-1)(k-2)}{6} + \frac{k(k-1)}{2} = \frac{k(k-1)(k-2)}{6} + \frac{3k(k-1)}{6} \quad (3)$$

$$= \frac{k(k-1)(k-2+3)}{6} \quad (4)$$

$$= \frac{k(k-1)(k+1)}{6} \quad (5)$$

## Question 2

## Question 3