

# CSC236 Worksheet 2 Solution

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## Question 1

- **Statement:** Any full binary tree with at least 1 node has more leaves than internal nodes.

*Proof.* Let  $n$  be the total number of nodes in a full binary tree.

We will prove the statement by complete induction on  $n$ .

### Base Case ( $n = 1$ ):

Let  $n = 1$ .

We need to prove the full binary tree with 1 total number of nodes has more leaves than internal nodes.

There is only one full binary tree exists with 1 total number of node. That is, the full binary tree with 1 child node and 0 internal nodes.

Then, since we know a node is a leaf if it has no children, and since we know the child node has 0 children, we can write the child node is a leaf node.

Then, because we know the full binary tree has 1 leaf node, and 0 internal node, we can conclude it has more leaves than internal nodes.

### Base Case ( $n = 2$ ):

Let  $n = 2$ .

We need to prove the full binary tree with 2 total number of nodes has more leaves than internal nodes.

The definition of full binary tree tells us that a binary tree is a full binary tree if every internal node has two children.

Since we know the tree with 2 total number of nodes has 1 leaf and 1 internal node, using above fact, we can write the tree is not a full binary tree.

Then, by vacuous truth, we can conclude the tree has more leaves than internal nodes.

### **Base Case ( $n = 3$ ):**

Let  $n = 3$ .

We need to prove the full binary tree with 3 total number of nodes has more leaves than internal nodes.

The definition of full binary tree tells us that a binary tree is a full binary tree if every internal node has two children.

Because we know there are two types of binary trees possible, one is the tree with 2 internal nodes and 1 child and the other is 1 internal node and 2 children, using above fact, we can write the only possible full binary tree with 3 total nodes is 1 internal node and 2 children.

Now, the definition of leaf tells us leaf is a node that has no children.

Because we know by observation that the 2 child nodes don't have children, we can write the full binary tree has 2 leaves.

So, because we know the full binary tree has 1 internal node and 2 leaves, we can conclude the full binary tree has more leaves than internal node.

### **Inductive Step:**

Let  $k \geq 1$  be an arbitrary natural number. Assume that for all natural number  $i$  satisfying  $1 \leq i \leq k$ , any full binary trees with  $i$  total number of nodes has more leaves than internal nodes.

Let  $T$  be an arbitrary full binary tree with  $k + 1$  nodes. Let  $T'$  be the binary tree obtained by removing 2 leaves from the same parent node.

Let  $\ell$  be the number of leaves of  $T$ , and  $m$  be the number of internal nodes of  $T$ . Similarly, let  $\ell'$  be the number of leaves of  $T'$  and  $m'$  be the number of internal nodes of  $T'$ . We must prove  $\ell > m$ .

First, we need to show  $\ell' > m'$ .

The header tells us that  $T'$  is a full binary tree as a result of removing 2 leaves from the parent node of  $T$ .

Using this fact, we can calculate  $T'$  has

$$k + 1 - 2 = k - 1 \quad (1)$$

nodes.

Then, because we know  $1 \leq k - 1 \leq k$ , using induction hypothesis, we can write

$$\ell' > m' \quad (2)$$

Second, we need to show  $\ell = \ell' + 1$  and  $m = m' + 1$ .

The total number of both the leaf nodes and the internal nodes increase by 1 when 2 nodes are added to the same leaf node in a full binary tree.

Since 2 nodes added to the same leaf node of  $T'$  is  $T$ , we can write  $\ell = \ell' + 1$  and  $m = m' + 1$ .

Finally, putting together, because we know  $\ell' > m'$ ,  $\ell = \ell' + 1$  and  $m = m' + 1$ , we can conclude

$$\ell' + 1 > m' + 1 \quad (3)$$

$$\ell > m \quad (4)$$

□

### Notes:

– Complete Induction

\* **Statement:**  $\forall i \in \mathbb{N}, \forall n \in \mathbb{N}, n < i \Rightarrow A(n) \Rightarrow \forall i \in \mathbb{N}, A(i)$

\* **Statement Alt.:**  $\left( \forall n \in \mathbb{N}, \left[ \bigwedge_{k=0}^{k=n-1} P(k) \right] \Rightarrow P(n) \right) \Rightarrow \forall n \in \mathbb{N}, P(n)$

\* **Simple Example 1:**

**Statement:**  $\forall n \in \mathbb{N}, n \geq 0 \Rightarrow 10 \mid (n^5 - n)$

We will prove the statement by strong induction on  $n$ .

1. Base Case ( $n = 0$ )

Let  $n = 0$ .

We need to prove  $10 \mid (n^5 - n)$  is true when  $n = 0$ . That is, there exists  $k \in \mathbb{Z}$  such that  $(n^5 - n) = 10k$ .

Let  $k = 0$ .

Starting from the left hand side, using the fact  $n = 0$ , we can write

$$(n^5 - n) = 0 \tag{5}$$

Then, because we know  $10k = 0$ , we can conclude

$$(n^5 - n) = 10k \tag{6}$$

2. Base Case ( $n = 1$ )

Let  $n = 1$ .

We need to prove  $10 \mid (n^5 - n)$  is true when  $n = 1$ . That is, there exists  $k \in \mathbb{Z}$  such that  $(n^5 - n) = 10k$ .

Let  $k = 0$ .

Starting from the left hand side, using the fact  $n = 1$ , we can write

$$(n^5 - n) = 1 - 1 \tag{7}$$

$$= 0 \tag{8}$$

Then, because we know  $10k = 0$ , we can conclude

$$(n^5 - n) = 10k \tag{9}$$

### 3. Inductive Step

Assume  $k \geq 1$ . Assume that for all natural number  $i$  satisfying  $0 \leq i \leq k$ ,  $10 \mid (i^5 - i)$ . That is,  $\exists d \in \mathbb{Z}$ ,  $(i^5 - i) = 10d$ .

We need to prove  $\exists \tilde{d} \in \mathbb{Z}$  such that  $((k+1)^5 - (k+1)) = 10\tilde{d}$ .

Let  $\tilde{d} = c + (k-1)^4 + 4 \cdot (k-1)^3 + 8 \cdot (k-1)^2 + 8 \cdot (k-1) + 3$ .

Starting from  $((k+1)^5 - (k+1))$ , using binominal theorem, we can write,

$$(k+1)^5 - (k+1) = \left[(k-1) + 2\right]^5 - \left[(k-1) + 2\right] \quad (10)$$

$$= \sum_{b=0}^5 \binom{5}{b} (k-1)^{5-b} \cdot 2^b \quad (11)$$

$$= (k-1)^5 + 10 \cdot (k-1)^4 + 40 \cdot (k-1)^3 + 80 \cdot (k-1)^2 + 80 \cdot (k-1) + 32 - \left[(k-1) + 2\right] \quad (12)$$

$$= \left[(k-1)^5 - (k-1)\right] + 10 \cdot (k-1)^4 + 40 \cdot (k-1)^3 + 80 \cdot (k-1)^2 + 80 \cdot (k-1) + 30 \quad (13)$$

(The reason why  $k-1$  is chosen instead of  $k-2$  and  $k-3$  is because of the last term  $2^5 = 32$ , i.e  $32 - 2 = 30$ )

Then, because we know  $0 \leq k-1 \leq k$  and  $10 \mid (k-1)^5 - (k-1)$  from the header, we can write  $\exists c \in \mathbb{Z}$  such that  $(k-1)^5 - (k-1) = 10c$ , and

$$(k+1)^5 - (k+1) = 10c + 10 \cdot (k-1)^4 + 40 \cdot (k-1)^3 + 80 \cdot (k-1)^2 + 80 \cdot (k-1) + 30 \quad (14)$$

$$(k+1)^5 - (k+1) = 10 \cdot \left[ c + (k-1)^4 + 4 \cdot (k-1)^3 + 8 \cdot (k-1)^2 + 8 \cdot (k-1) + 3 \right] \quad (15)$$

$$(16)$$

Then, because we know  $\tilde{d} = c + (k-1)^4 + 4 \cdot (k-1)^3 + 8 \cdot (k-1)^2 + 8 \cdot (k-1) + 3$  from the header, we can conclude

$$(k+1)^5 - (k+1) = 10\tilde{d} \quad (17)$$

**Question 2**

**Question 3**