Worksheet 20 Solution

Hyungmo Gu

April 16, 2020

Question 1

a. *Proof.* Let $V = \{1, 2, 3, 4, 5, 6\}$, $E = \{(1, 2), (1, 6), (2, 3), (3, 4), (4, 5), (5, 6)\})$.

We need to prove the graph $G = (V, E)$ is bipartite by proving the following properties:

1. There exists subsets $V_1, V_2 \subset V$ such that $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V.
2. Every edge in E has exactly one endpoint in V_1 and one in V_2.

We will prove the properties in parts.

<u>**Part 1 (Proving $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V):**</u>

Let $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

We need to prove $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V, i.e $V_1 \cup V_2 = V \wedge V_1 \cap V_2 = \emptyset$.

First, we need to show the subsets V_1 and V_2 are non-empty.

The header tells us both subsets V_1 and V_2 have more than 1 elements.

Then, using these facts, we can conclude $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$.

Finally, we need to show $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$.

The header tells us $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

Then, we can calculate

$$V_1 \cup V_2 = \{1, 2, 3, 4, 5, 6\} = V \quad (1)$$
$$V_1 \cap V_2 = \emptyset \quad (2)$$

Part 2 (Proving every edge in E has exactly one endpoint in V_1 and one in V_2):

Let $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

We need to show every edge in E has exactly one endpoint in V_1 and one in V_2.

The header tells us $V_1 = \{1, 3, 5\}$, $V_2 = \{2, 4, 6\}$, and $E = \{(1, 2), (1, 6), (2, 3),$ $(3, 4), (4, 5), (5, 6)\})$.

Using these facts, we can generate the following table.

Edge (1,2)	- 1 is in V_1 - 2 is in V_2	Edge (3,4)	- 3 is in V_1 - 4 is in V_2
Edge (1,6)	- 1 is in V_1 - 6 is in V_2	Edge (4,5)	- 4 is in V_2 - 6 is in V_1
Edge (2,3)	- 2 is in V_2 - 3 is in V_1	Edge (5,6)	- 5 is in V_1 - 6 is in V_2

Then, it follows from observation that every edge in E has one endpoint in V_1 and one in V_2.

$\square$

Pseudoproof:

Let $V = \{1, 2, 3, 4, 5, 6\}$, $E = \{(1, 2), (1, 6), (2, 3), (3, 4), (4, 5), (5, 6)\})$.

We need to prove the graph $G = (V, E)$ is bipartite by proving the following properties:

1. There exists subsets $V_1, V_2 \subset V$ such that $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V.
2. Every edge in E has exactly one endpoint in V_1 and one in V_2.

We will prove the properties in parts.

1. Show there exists subsets $V_1, V_2 \subset V$ such that $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V

Let $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

We need to prove $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V, i.e $V_1 \cup V_2 = V \wedge V_1 \cap V_2 = \emptyset$.

1. Show $V_1 \neq \emptyset, V_2 \neq \emptyset$
   First, we need to show the subsets V_1 and V_2 are non-empty.

   > The header tells us both subsets V_1 and V_2 have more than 1 elements.
   >
   > Then, using these facts, we can conclude $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$.

2. Show $V_1 \cup V_2 = V \wedge V_1 \cap V_2 = \emptyset$

   Second, we need to show $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$.

   > The header tells us $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.
   >
   > Then, we can calculate
   >
   > $$V_1 \cup V_2 = \{1, 2, 3, 4, 5, 6\} = V \tag{3}$$
   > $$V_1 \cap V_2 = \emptyset \tag{4}$$

Part 1:

Let $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

We need to prove $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V, i.e $V_1 \cup V_2 = V \wedge V_1 \cap V_2 = \emptyset$.

First, we need to show the subsets V_1 and V_2 are non-empty.

The header tells us both subsets V_1 and V_2 have more than 1 elements.

Then, using these facts, we can conclude $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$.

Finally, we need to show $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$.

The header tells us $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

Then, we can calculate

$$V_1 \cup V_2 = \{1, 2, 3, 4, 5, 6\} = V \tag{5}$$
$$V_1 \cap V_2 = \emptyset \tag{6}$$

2. Show every edge in E has exactly one endpoint in V_1 and one in V_2.

   Let $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

   We need to show every edge in E has exactly one endpoint in V_1 and one in V_2.

The header tells us $V_1 = \{1, 3, 5\}$, $V_2 = \{2, 4, 6\}$, and $E = \{(1, 2), (1, 6), (2, 3), (3, 4), (4, 5), (5, 6)\}$).

Using these facts, we can generate the following table.

Edge (1,2)	- 1 is in V_1 - 2 is in V_2	Edge (3,4)	- 3 is in V_1 - 4 is in V_2
Edge (1,6)	- 1 is in V_1 - 6 is in V_2	Edge (4,5)	- 4 is in V_2 - 6 is in V_1
Edge (2,3)	- 2 is in V_2 - 3 is in V_1	Edge (5,6)	- 5 is in V_1 - 6 is in V_2

Then, it follows from observation that every edge in E has one endpoint in V_1 and one in V_2.

<u>Part 2:</u>

Let $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

We need to show every edge in E has exactly one endpoint in V_1 and one in V_2.

The header tells us $V_1 = \{1, 3, 5\}$, $V_2 = \{2, 4, 6\}$, and $E = \{(1, 2), (1, 6), (2, 3), (3, 4), (4, 5), (5, 6)\}$).

Using these facts, we can generate the following table.

Edge (1,2)	- 1 is in V_1 - 2 is in V_2	Edge (3,4)	- 3 is in V_1 - 4 is in V_2
Edge (1,6)	- 1 is in V_1 - 6 is in V_2	Edge (4,5)	- 4 is in V_2 - 6 is in V_1
Edge (2,3)	- 2 is in V_2 - 3 is in V_1	Edge (5,6)	- 5 is in V_1 - 6 is in V_2

Then, it follows from observation that every edge in E has one endpoint in V_1 and one in V_2.

b. Let $G = (V, E)$ be a complete bipartite graph.

Then, by property 3, we can conclude each vertex in V_1 is adjacent to all verticies in V_2.

Since there are n many edges for each vertex in V_1, and since there are m many verticies in V_1, we can calculate that the vertices in V_1 has

$$nm \tag{1}$$

edges.

Then, since there are no new edges for each vertex in V_2, we can conclude the graph has nm edges.

c. **Conjecture:** The length of every cycle in a bipartite graph is even (i.e. $\forall G = (V, E)$, $Bipartite(G) \Rightarrow \forall k \in \mathbb{N}, C = v_0, \ldots, v_k \wedge Cycle(C, G) \Rightarrow \exists d \in \mathbb{Z}, k = 2d$)

Pseudoproof:

Let $G = (V, E)$, and assume G is bipartite, with bipartition V_1, V_2. Let $C = v_0, ..., v_k$ form a cycle in G. Without loss of generality, assume $v_0 \in V_1$, $v_i \in V_1$ if $Even(i)$, and $v_i \in V_2$ if $Odd(i)$.

We will prove that $|C| = k$ is even by using induction on k.

1. Case 1 (Base case):

   Let $k = 3$.

   We need to show the sequence of verticles $C = v_1, v_2, v_3$ in G do not form a cycle. That is, there is a consecutive pair of vertices that's not adjacent.

   Assume $v_3 = v_0$.

   - Show v_2 is in V_1.

     First, we need to show v_2 is in V_1.

     The header tells us all even vertices in C are in V_1.

     Since 2 is even, we can conclude v_2 is in V_1.

   - Conclude v_2, v_3 are not adjacent using the properties of bipartite that no two vertices in V_1 are adjacent.

     Finally, we need to show v_2, v_3 are not adjacent.

The assumption tells us $v_3 = v_0$.

Then, since v_0, v_2 are in V_1 and the second property of bipartite graph tells us that no two verticies in V_1 are adjacent, we can conclude v_0, v_2 are not adjacent.

First, we need to show v_2 is in V_1.

The header tells us all even vertices in C are in V_1.

Since 2 is even, we can conclude v_2 is in V_1.

Finally, we need to show v_2, v_3.

The assumption tells us $v_3 = v_0$.

Then, since v_0, v_2 are in V_1 and the second property of bipartite graph tells us that no two verticies in V_1 are adjacent, we can conclude v_0, v_2 are not adjacent.

2. Case 2 (Inductive case):

   Let $k \in \mathbb{N}$. Assume $C = v_0, v_1, \ldots, v_k$ is a cycle in G, and $\exists d \in \mathbb{Z}, k = 2d$.

   We need to prove the cycle $C = v_1, \ldots, v_{k+1}$ that forms in G has even length.

   - .

Notes:

- Cycle with odd number of verticies - Not bipartite
- Cycle with even number of verticies - Bipartite
- 뚜퍼맨!! 영차! 영차! 형모 풀뚜있쪄!!
- 할뚜있다 형모야!!
- 형모 많이 틀렸죠
- 형모 틀리면 틀리면서 배우면 되느니라. 흠허허허허!!
- 형모 화이팅!!
- 파이팅 파이팅!!
- 형모 해낼 수 있죠!!!
- 형모야. 한걸음 더.
- 고마워요