

# Problem Set 2 Solution

March 17, 2020

## Question 1

a.

- b. **Predicate Logic:**  $\forall k, n \in \mathbb{Z}^+, \forall p \in \mathbb{N}, Prime(p) \wedge p^k < n < p^k + p \Rightarrow gcd(p^k, n) = 1$

Let  $k, n \in \mathbb{Z}^+$ , and  $p \in \mathbb{N}$ . Assume  $Prime(p)$ , and  $p^k < n < p^k + p$ .

Then,  $p^k$  can either be divided by 1 or  $p$  by fact 3.

Since,  $p^k < n < p^k + p$ ,  $n$  cannot be written in multiples of  $p$ .

Then, it follows from the definition of divisibility that  $p \nmid n$ .

Since  $p \nmid n$ , but  $1 \mid p^k$  and  $1 \mid n$ ,  $gcd(p^k, n) = 1$ .

- c. **Predicate Logic:**  $\forall m \in \mathbb{Z}, \forall n_0 \in \mathbb{N}, \exists n \in \mathbb{N} n > n_0 \wedge gcd(n, n+m) = 1$

Since there are infinitely many primes by fact 4, let  $Prime(n)$  and  $n > m$ .

Since  $Prime(n)$ , by fact 3,  $n$  can either be divided by 1 or  $n$ .

Since  $n \mid n$ , but  $n \nmid m$ ,  $n \nmid (n+m)$ , and  $n$  can't be chosen as the greatest common divisor of  $n$  and  $n+m$ .

Since  $\gcd(n, n + m) \neq n$  but  $1 \mid n$  and  $1 \mid (n + m)$ ,  $\gcd(n, n + m) = 1$ .

Then, it follows from above that the statement  $\forall m \in \mathbb{Z}, \forall n_0 \in \mathbb{N}, \exists n \in \mathbb{N}$   
 $n > n_0 \wedge \gcd(n, n + m) = 1$  is true.

- d. **Definition of Primary Gap:** Let  $a \in \mathbb{N}$ . We say that  $a$  is a prime gap when there exists a prime  $p$  such that  $p + a$  is also prime, and none of the numbers between  $p$  and  $p + a$  (exclusive) are prime.

**Case 1** ( $a > 2$ ):

Let  $a, p \in \mathbb{Z}^+$ . Assume  $PrimaryGap(a)$ ,  $Primary(p)$ , and  $a > 2$ .

Then,  $2 \nmid p$  and  $2 \nmid p + a$ .

Then,

$$2 \mid (p + a) - a \tag{1}$$

$$2 \mid a \tag{2}$$

by fact 1.

Then it follows from above that in case  $a > 2$ , primary gap is divisible by 2.

**Case 2** ( $a \leq 2$ ):

Let  $a, p \in \mathbb{Z}^+$ . Assume  $PrimaryGap(a)$ ,  $Primary(p)$ , and  $a \leq 2$ .

Then, only two primary numbers in  $\mathbb{Z}^+$  exist - 1 and 2.

Then,

$$a = 2 - 1 \tag{1}$$

$$a = 1 \tag{2}$$

Then, it follows from above that in case  $a \leq 2$ , the value of primary gap is 1.

## Question 2

a. Let  $n \in \mathbb{N}$ , and  $x \in \mathbb{R}$ .

Because we know  $\forall x \in \mathbb{R}, 0 \leq x - \lfloor x \rfloor < 1$  from fact 1, we can conclude  $\lfloor x \rfloor \leq x < 1 + \lfloor x \rfloor$ .

Then,

$$\lfloor nx \rfloor - n\lfloor x \rfloor \leq nx - n\lfloor x \rfloor \quad (1)$$

$$\leq n(x - \lfloor x \rfloor) \quad (2)$$

by using the above.

Then,

$$\lfloor nx \rfloor - n\lfloor x \rfloor \leq n(x - \lfloor x \rfloor) \quad (3)$$

$$< n \quad (4)$$

$$< k \quad (5)$$

by using fact 1 and choosing  $k = n$ .

Then, it follows that the statement the statement  $\forall n \in \mathbb{N}, \exists k \in \mathbb{N}, x \in \mathbb{R}, \lfloor nx \rfloor - n\lfloor x \rfloor \leq k$  is true.

b. **Negation of statement:**  $\forall k \in \mathbb{N}, \exists m \in \mathbb{N}, \exists x \in \mathbb{R}, \lfloor nx \rfloor - n\lfloor x \rfloor > k$

Let  $x = 0.5$  and  $n = 2(k + 1)$ .

Then,

$$\lfloor nx \rfloor - n\lfloor x \rfloor = \lfloor \frac{2(k+1)}{2} \rfloor - n\lfloor 0.5 \rfloor \quad (1)$$

$$= k + 1 - 0 \quad (2)$$

$$= k + 1 \quad (3)$$

$$> k \quad (4)$$

Then it follows that the statement  $\exists k \in \mathbb{N}, \forall n \in \mathbb{N}, \forall x \in \mathbb{R}, \lfloor nx \rfloor - n\lfloor x \rfloor \leq k$  is false.

### Question 3

a. **Predicate Logic:**  $\forall f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = f(-x) \wedge -f(-x) = f(x) \leftrightarrow f = 0$

#### Part 1: Proving in $\Rightarrow$ direction

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Assume  $f(x) = f(-x) \wedge -f(-x) = f(x)$ .

Then,

$$f(-x) - f(-x) = 2f(x) \tag{1}$$

$$0 = 2f(x) \tag{2}$$

by adding  $f(x) = f(-x)$  and  $-f(-x) = f(x)$  together.

Then,

$$0 = f(x) \tag{3}$$

Then it follows that the statement  $\forall f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = f(-x) \wedge -f(-x) = f(x) \Rightarrow f = 0$  is true.

#### Part 2: Proving in $\Leftarrow$ direction

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Assume  $f(x) = 0$ .

Then,

$$-f(-x) = -(-0) \tag{1}$$

$$= 0 \tag{2}$$

$$= f(x) \tag{3}$$

It follows from above that  $f(x) = 0$  is an odd function.

Also,

$$f(-x) = (-0) = 0 = f(x) \quad (4)$$

It follows from above that  $f(x) = 0$  is an odd function.

Because we know  $f(x) = 0$  is both even and odd, we can conclude that the statement  $\forall f : \mathbb{R} \rightarrow \mathbb{R}, f = 0 \Rightarrow f(x) = f(-x) \wedge -f(-x) = f(x)$  is true.