CSC236 Worksheet 3

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Question 1

Rough Work:

Predicate Logic: $\forall A \subseteq \mathbb{N},\ A \neq \emptyset \Rightarrow (\exists a \in A,\ \forall x \in A,\ a \leq x)$

Given the statement to prove

$$P(x, y, z) : \text{There are no positive integers } x, y, z \text{ such that } x^3 + 3y^3 = 9z^3$$

I will prove $P(x, y, z)$ using proof by contradiction.

Assume $\exists x, y, z \in \mathbb{N}^+,\ x^3 + 3y^3 = 9z^3$.

1. State that there is smallest element $x_0 \in X$ with $y_0, z_0 \in \mathbb{N}^+$ satisfying $x^3 + 3y^3 = 9z^3$, using well-ordering principle

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   The header tells us there are elements $x, y, z \in \mathbb{N}^+$, satisfying $x^3 + 3y^3 = 9z^3$.

   Then, we can write the set $X = \{x \mid x \in \mathbb{N}^+,\ \exists y, z \in \mathbb{N}^+,\ x^3 + 3y^3 = 9z^3\}$ is not empty.

   Then, using principle of well-ordering, we can write that there is smallest positive natural number $x_0 \in X$ along with $y_0, z_0 \in \mathbb{N}^+$ satisfying $x^3 + 3y^3 = 9z^3$.

2. Show that $x_1^3 = 9z_1^3 - 3y_1^3$ is satisfied, given $x_0 > x_1$

   Second, we need to show that $x_1^3 = 9z_1^3 - 3y_1^3$ is satisfied, given $x_0 > x_1$.

   We will do so in parts.

   - First, show that $x_0 = 3 \cdot x_1$, using $x_0^3 + 3y_0^3 = 9z_0^3$ and the fact if a prime number p divides a perfect cube n^3, then p also divides n.

     Part 1 (Showing $x_0 = 3 \cdot x_1$):

     The assumption tells us

     $$x_0^3 + 3y_0^3 = 9z_0^3 \quad (1)$$
     $$x_0^3 = 9z_0^3 - 3y_0^3 \quad (2)$$

     Since $3 \mid 9z_0^3 - 3y_0^3$, we can write that $3 \mid x_0^3$.

     Then, since 3 is a prime number, by using the hint provided in question 3 of assignment 1, we can write there is $x_1 \in \mathbb{Z}$, $x_0 = 3 \cdot x_1$.

     Then, because we know $x_0, 3 \in \mathbb{N}^+$, we can conclude $x_1 \in \mathbb{N}^+$.

   - Second, show that $y_0 = 3 \cdot y_1$, using $x_0^3 = 3^3 x_1^3 = 9z_0^3 - 3y_0^3$

Part 2 (Showing $y_0 = 3 \cdot y_1$):

The assumption tells us

$$x_0^3 + 3y_0^3 = 9z_0^3 \tag{3}$$
$$3y_0^3 = 9z_0^3 - x_0^3 \tag{4}$$

Then, using the fact from part 1 that $x_0 = 3 \cdot x_1$, we can calculate

$$3y_0^3 = 9z_0^3 - 3^3x_1^3 \tag{5}$$
$$y_0^3 = 3z_0^3 - 3^2x_1^3 \tag{6}$$

Since $3 \mid 3z_0^3 - 3^2x_1^3$, we can write that $3 \mid y_0^3$.

Then, since 3 is a prime number, by using the hint provided in question 3 of assignment 1, we can write there is $y_1 \in \mathbb{Z}$, $y_0 = 3 \cdot y_1$.

Then, because we know $y_0, 3 \in \mathbb{N}^+$, we can conclude $y_1 \in \mathbb{N}^+$.

- Third, show that $z_0 = 3 \cdot z_1$, using $x_0^3 = 3^3x_1^3 = 9z_0^3 - 3y_0^3 = 9z_0^3 - 3^4y_1^3$
- Finally, show $x_1^3 = 9z_1^3 - 3y_1^3$

Second, we need to show that $x_1^3 = 9z_1^3 - 3y_1^3$ is satisfied, given $x_0 > x_1$.

We will do so in parts.

3. Conclude that this contracts the original claim that x_0 is the smallest non-negative value satisfying $x^3 + 3y^3 = 9z^3$.

Notes:

- **Proof By Contradiction:** $\neg P \Rightarrow \neg Q \land Q$ (Assuming we are proving $P \Rightarrow Q$)
- **Principle of Well-Ordering:** Any nonempty subset A of $\mathbb{N}$ contains a minimum element; i.e. for any $A \subseteq \mathbb{N}$ such that $A \neq \emptyset$, there is some $a \in A$ such that for all $a' \in A$, $a \leq a'$.

- examples of well-ordered sets
    1. $\mathbb{N} \cup \{0\}$
    2. $\mathbb{N} \cup \{1, 2\}$
    3. $\{n \in \mathbb{N} : n > 5\}$
- examples of non-well-ordered sets
    1. $\mathbb{R}$ and the open interval $(0, 2)$
    2. $\mathbb{Z}$

Question 2

Question 3