Worksheet 6 Review 2

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Question 1

a. $\forall x \in \mathbb{N},\ P(123) \wedge P(x) \Rightarrow x \leq 123$

Correct Solution:

$P(123) \wedge (\forall x \in \mathbb{N}, P(x) \Rightarrow x \leq 123)$

b. $IsCD(x, y, d):\ d \mid x \wedge d \mid y$, where $x, y, d \in \mathbb{Z}$

$IsGCD(x, y, d):\ \forall n \in \mathbb{N},\ IsCD(x, y, n) \Rightarrow \exists d \in \mathbb{N},\ IsCD(x, y, d) \wedge n \leq d$

Correct Solution:

$IsCD(x, y, d):\ d \mid x \wedge d \mid y$, where $x, y, d \in \mathbb{Z}$

$IsGCD(x, y, d) : (x = 0 \wedge y = 0 \Rightarrow d = 0) \wedge (x \neq 0 \wedge y \neq 0 \Rightarrow IsCD(x, y, d) \wedge (\forall d_1 \in \mathbb{Z},\ IsCD(x, y, d_1) \Rightarrow d_1 \leq d))$, where $x, y, d \in \mathbb{Z}$

Notes:

- Realized the definition of $IsGCD$ extends from previous question
- Noticed professor defines if...else conditions in a predicate logic the following way

  (case 1 $\Rightarrow$ statement 1) $\wedge$ (case 2 $\Rightarrow$ statement 2)

- Hm... I feel puzzled about $\wedge$ operator used in between cases (i.e. $(x = 0 \wedge y = 0 \Rightarrow d = 0) \wedge (x \neq 0 ...)$). At glimpse, I felt $\vee$ is more appropriate since if this case is not true, then we want other case should be true.

c. **Statement:** $IsCD(x, 0, x) \wedge (\forall d_1 \in \mathbb{Z}, IsCD(x, 0, d_1) \Rightarrow d_1 \leq x)$

Proof. Let $x \in \mathbb{Z}^+$

We need to prove x is a common divisor to both 0 and x, and we need to prove all common divisors d_1 of 0 and x is less than or equal to x.

First, we need to show there is $k_1 \in \mathbb{Z}$ such that $x = k_1 \cdot x$ and we need to show $k_2 \in \mathbb{Z}$ such that $0 = k_2 \cdot x$.

Let $k_1 = 1$ and $k_2 = 0$.

Then, we can calculate that

$$x = 1 \cdot x = k_1 \cdot x \tag{1}$$
$$0 = 0 \cdot x = k_2 \cdot x \tag{2}$$

Now, we need to show all integers d_1 that is a common divisor to both 0 and x is less than equal to x.

Let $d_1 \in \mathbb{Z}$ and assume $d_1 \mid x$ and $d_1 \mid 0$.

We need to show $d_1 \leq x$.

The hint tells us

$$\forall n \in \mathbb{Z}^+,\ \forall d \in \mathbb{Z},\ d \mid n \Rightarrow d \leq n \tag{3}$$

Because we know from assumption that $d_1 \mid x$, by using the hint, we can conclude

$$d_1 \leq x \tag{4}$$

□

<u>Pseudoproof:</u>

Let $x \in \mathbb{Z}^+$

We need to prove x is a common divisor to both 0 and x, and we need to prove all common divisors d_1 of 0 and x is less than or equal to x.

1. Show $IsCD(x, 0, x)$

   We need to show there is $k_1 \in \mathbb{Z}$ such that $x = k_1 \cdot x$ and we need to show $k_2 \in \mathbb{Z}$ such that $0 = k_2 \cdot x$.

   Let $k_1 = 1$ and $k_2 = 0$.

   - Show $x = k_1 \cdot x$ and $0 = k_2 \cdot 0$

     Then, we can calculate that

     $$x = 1 \cdot x = k_1 \cdot x \tag{5}$$
     $$0 = 0 \cdot x = k_2 \cdot x \tag{6}$$

2. Show $\forall d_1 \in \mathbb{Z}, IsCD(x, 0, d_1) \Rightarrow d_1 \leq x$

   Let $d_1 \in \mathbb{Z}$ and assume $d_1 \mid x$ and $d_1 \mid 0$.

   We need to show $d_1 \leq x$.

   1. Use fact '$\forall n \in \mathbb{Z}^+, \forall d \in \mathbb{Z},\ d \mid n \Rightarrow d \leq n$' to show $d_1 \leq x$.

   The hint tells us

   $$\forall n \in \mathbb{Z}^+, \forall d \in \mathbb{Z},\ d \mid n \Rightarrow d \leq n \tag{7}$$

   Because we know from assumption that $d_1 \mid x$, by using the hint, we can conclude

   $$d_1 \leq x \tag{8}$$

d. $\forall a, b \in \mathbb{Z},\ (a \neq 0) \vee (b \neq 0) \Rightarrow \exists p, q \in \mathbb{Z},\ pa + qb = gcd(a, b)$

Question 2

a. *Proof.* Assume $Even(n)$. That is $\exists k \in \mathbb{Z},\ n = 2k$.

   We need to show there is an integer k_1 such that $n^2 - 3n = 2k_1$.

   Let $k_1 = (2k^2 - 3k)$.

The assumption tells us $n = 2k$.

Then, by using this fact, we can write

$$\begin{aligned} n^2 - 3n &= (2k)^2 - 3(2k) && (1) \\ &= 4k^2 - 6k && (2) \\ &= 2(2k^2 - 3k) && (3) \\ &= 2k_1 && (4) \end{aligned}$$

□

Pseudoproof:

Assume $Even(n)$. That is $\exists k \in \mathbb{Z},\ n = 2k$.

We need to show there is an integer k_1 such that $n^2 - 3n = 2k_1$.

Let $k_1 = (2k^2 - 3k)$.

- Show $n^2 - 3n = 2k_1$ by using assumption.

> The assumption tells us $n = 2k$.
>
> Then, by using this fact, we can write
>
> $$\begin{aligned} n^2 - 3n &= (2k)^2 - 3(2k) && (5) \\ &= 4k^2 - 6k && (6) \\ &= 2(2k^2 - 3k) && (7) \\ &= 2k_1 && (8) \end{aligned}$$

b. *Proof.* In this case, assume $Odd(n)$. That is $\exists k \in \mathbb{Z},\ n = 2k - 1$.

We need to show there is an integer k_1 such that $n^2 - 3n = 2k_1$.

Let $k_1 = (2k^2 - 5k + 2)$.

The assumption tells us $n = 2k - 1$.

Then, by using this fact, we can write

$$
\begin{aligned}
n^2 - 3n &= (2k-1)^2 - 3(2k-1) && (9)\\
&= 4k^2 - 4k + 1 - 6k + 3 && (10)\\
&= 4k^2 - 10k + 4 && (11)\\
&= 2(2k^2 - 5k + 2) && (12)\\
&= 2k_1 && (13)
\end{aligned}
$$

□

Pseudoproof:

Assume $Odd(n)$. That is $\exists k \in \mathbb{Z},\ n = 2k - 1$.

We need to show there is an integer k_1 such that $n^2 - 3n = 2k_1$.

Let $k_1 = (2k^2 - 5k + 2)$.

- Show $n^2 - 3n = 2k_1$ by using assumption.

The assumption tells us $n = 2k - 1$.

Then, by using this fact, we can write

$$
\begin{aligned}
n^2 - 3n &= (2k-1)^2 - 3(2k-1) && (14)\\
&= 4k^2 - 4k + 1 - 6k + 3 && (15)\\
&= 4k^2 - 10k + 4 && (16)\\
&= 2(2k^2 - 5k + 2) && (17)\\
&= 2k_1 && (18)
\end{aligned}
$$

Notes:

- Noticed professor uses predicate logic when expanding definition in assumption.

Assume that n is odd, i.e. $\exists k \in \mathbb{Z},\ n = 2k - 1$.

Question 3