Midterm 2 Version 1 Solution

April 3, 2020

Question 1

a.

$$\begin{aligned}
100 \div 2 &= 50, \text{ Remainders } \mathbf{0} \\
50 \div 2 &= 25, \text{ Remainders } \mathbf{0} \\
25 \div 2 &= 12, \text{ Remainders } \mathbf{1} \\
12 \div 2 &= 6, \text{ Remainders } \mathbf{0} \\
6 \div 2 &= 3, \text{ Remainders } \mathbf{0} \\
3 \div 2 &= 1, \text{ Remainders } \mathbf{1} \\
1 \div 2 &= 0, \text{ Remainders } \mathbf{1}
\end{aligned}$$

Then, it follows from above that the binary representation of 100 is $(1100100)_2$.

b. The smallest number that can be expressed by an n-digit balanced ternary representation is

$$\sum_{i=0}^{n-1} d_i \cdot 3^i, \text{ where } d_i \in \{0, 1, 2\} \tag{1}$$

Correct Solution:

The smallest number that can be expressed by an n-digit balanced ternary representation is

$$-\left[\sum_{i=0}^{n-1} 3^i\right] \tag{1}$$

Notes:

- Realized professor is asking for an example of the smallest number.
- Ternary representation of a number

$$\sum_{i=0}^{n-1} d_i \cdot 3^i, \text{ where } d_i \in \{0, 1, 2\}$$

- Learned a negative number could be expressed in in ternary or binary representation of numbers.

c.

$f(n) \in \Omega(n)$	True	$g(n) \in \Omega(n)$	False	$f(n) \in \mathcal{O}(g(n))$	False
$f(n) \in \Theta(g(n))$	False	$g(n) \in \Theta(\log_3 n)$	True	$f(n) + g(n) \in \Theta(f(n))$	True

Notes:

- $\forall g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, and all numbers $a \in \mathbb{R}^{\geq 0}$, if $g \in \mathcal{O}(f)$, then $f + g \in \mathcal{O}(f)$
- $g \in \Theta(f) :\ g \in \mathcal{O}(f) \wedge g \in \Omega(f)$
  or
  $g \in \Theta(f) :\ \exists c_1, c_2, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n)$, where $f, g :\ \mathbb{N} \to \mathbb{R}^{\geq 0}$
- $g \in \Omega(f) :\ \exists c, n_o \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \geq cf(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

- $g \in \mathcal{O}(f) : \exists c, n_o \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq cf(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

d.

k	0	1	2
i_k	$3 = 3^1$	$9 = 3^2$	$81 = 3^4$

The value of i_k is

$$3^{2^k} \tag{1}$$

Notes:

- Realized we are only concerned with the lines **i = i * i** and **i = 3**

e. The number of iterations the function's loop will run is

$$\lceil \log_2 \log_3 n \rceil - 1 \tag{1}$$

Notes:

- The loop terminates when $3^{2^{(k+1)}} = i_{k+1} = i_k \cdot i_k \geq n$.
- $\forall x \in \mathbb{Z}, \forall y \in \mathbb{R}, \lfloor x + y \rfloor = x + \lfloor y \rfloor$
- Feel more confident there is no need to add an extra **+1**. Done by playing with examples (i.e is $\lceil \log \log_3(82) \rceil - 1$ true? Would the loop run only once?)

Question 2

- **Predicate Logic:** $\forall n \in \mathbb{N}, n \geq 3 \Rightarrow 5^n + 50 < 6^n$

  Proof. Let $n \in \mathbb{N}$.

  We will prove the statement by induction on n.

Base Case ($n = 3$):

Let $n = 3$.

We want to show $5^3 + 50 < 6^3$.

Starting from $5^3 + 50$, we can calculate

$$\begin{aligned} 5^3 + 50 &= 125 + 50 && (1)\\ &= 175 && (2)\\ &< 216 && (3)\\ &< 6^3 && (4) \end{aligned}$$

Inductive Case:

Let $n \in \mathbb{N}$. Assume $n \geq 3$ and $5^n + 50 < 6^n$.

We want to show $5^{n+1} + 50 < 6^{n+1}$.

Starting from $5^{n+1} + 50$, we can calculate

$$\begin{aligned} 50^{n+1} + 50 &= 5^n \cdot 5 + 50 && (5)\\ &< 5^n \cdot 5 + 50 \cdot 5 && (6)\\ &< 5(5^n + 50) && (7) \end{aligned}$$

Then,

$$\begin{aligned} 50^{n+1} + 5 &< 5 \cdot 6^n && (8)\\ &< 6 \cdot 6^n && (9)\\ &< 6^{n+1} && (10) \end{aligned}$$

by using inductive hypothesis (i.e $5^n + 50 < 6^n$)

$\square$

Correct Solution:

Let $n \in \mathbb{N}$.

We will prove the statement by induction on n.

Base Case ($n = 3$):

Let $n = 3$.

We want to show $5^3 + 50 < 6^3$.

Starting from $5^3 + 50$, we can calculate

$$
\begin{align}
5^3 + 50 &= 125 + 50 \tag{1}\\
&= 175 \tag{2}\\
&< 216 \tag{3}\\
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\end{align}
$$

Inductive Case:

Let $n \in \mathbb{N}$. Assume $n \geq 3$ and $5^n + 50 < 6^n$.

We want to show $5^{n+1} + 50 < 6^{n+1}$.

Starting from $5^{n+1} + 50$, we can calculate

$$
\begin{align}
50^{n+1} + 50 &= 5^n \cdot 5 + 50 \tag{5}\\
&{\color{red}=} 5^n \cdot 5 + 50 \cdot 5 \tag{6}\\
&< 5(5^n + 50) \tag{7}
\end{align}
$$

Then,

$$
\begin{align}
50^{n+1} + 5 &< 5 \cdot 6^n \tag{8}\\
&< 6 \cdot 6^n \tag{9}\\
&= 6^{n+1} \tag{10}
\end{align}
$$

by using inductive hypothesis (i.e $5^n + 50 < 6^n$)

Notes:

– Noticed professor uses '=' sign if the expression's value remains unchanged from the one before

See equation 5 and 6 for example.

Question 3

- **Statement:** $\exists a \in \mathbb{R}^+,\ an + 1 \in \Theta(n^3)$

  Negation of Statement: $\forall a \in \mathbb{R}^+,\ \forall c_1, c_2, n_0 \in \mathbb{R}^+,\ \exists n \in \mathbb{N},\ (n \geq n_0) \wedge \big((an + 1 < c_1 n^3) \vee (an + 1 > c_2 n^3)\big)$

  Proof. Let $n = \left\lceil max(n_0, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{1}{c_1}}) \right\rceil + 1$.

  We will disprove the statement by showing $n \geq n_0$ and $an + 1 < c_1 n^3$

  Part 1 (Showing $n \geq n_0$):

  Using the fact that $\left\lceil max(n_0, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{1}{c_1}}) \right\rceil$ will result in a value greater than or equal to n_0, we can calculate

$$n_0 \leq \left\lceil max(n_0, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{1}{c_1}}) \right\rceil \tag{1}$$
$$\leq \left\lceil max(n_0, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{1}{c_1}}) \right\rceil + 1 \tag{2}$$

Then, because we know $n = \left\lceil max(n_0, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{1}{c_1}}) \right\rceil + 1$, we can conclude

$$n_0 \leq n \tag{3}$$

$\square$

Part 2 (Showing $an + 1 < c_1 n^3$):

We will prove $an + 1 < c_1 n^3$ by showing $an < \frac{c_1}{2} n^3$ and $1 < \frac{c_1}{2} n^3$, and then combining the two together.

For the first inequality, because we know $n = \left\lceil max(n_0, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{1}{c_1}}) \right\rceil + 1 > \sqrt{\frac{2a}{c_1}}$, we can conclude

$$\sqrt{\frac{2a}{c_1}} < n \tag{4}$$
$$\frac{2a}{c_1} < n^2 \tag{5}$$
$$a < \frac{c_1}{2} n^2 \tag{6}$$
$$an < \frac{c_1}{2} n^3 \tag{7}$$

For the second inequality, because we know $n = \left\lceil max(n_0, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{1}{c_1}}) \right\rceil + 1 > \sqrt[3]{\frac{1}{c_1}}$, we can conclude

$$\sqrt[3]{\frac{1}{c_1}} < n \tag{8}$$
$$\frac{1}{c_1} < n^3 \tag{9}$$
$$1 < n^3 \tag{10}$$

Then,

$$an + 1 < \frac{c_1}{2} \cdot n^3 + \frac{c_1}{2} \cdot n^3 \tag{11}$$
$$an + 1 < c_1 n^3 \tag{12}$$

Notes:

- I struggled on this question.
- Learned **+1** in $\left\lceil max(n_0, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{1}{c_1}})\right\rceil + 1 > \sqrt[3]{\frac{1}{c_1}}$ is to allow the use of inequality sign '<'.
- Learned that when c_1 is in inequality, with multiple terms like $an+1$ on the other side, and is asking to disprove it, I should first divide them up, find valid n for each term, and then recombine to create a valid n.

  See figure 1 for example

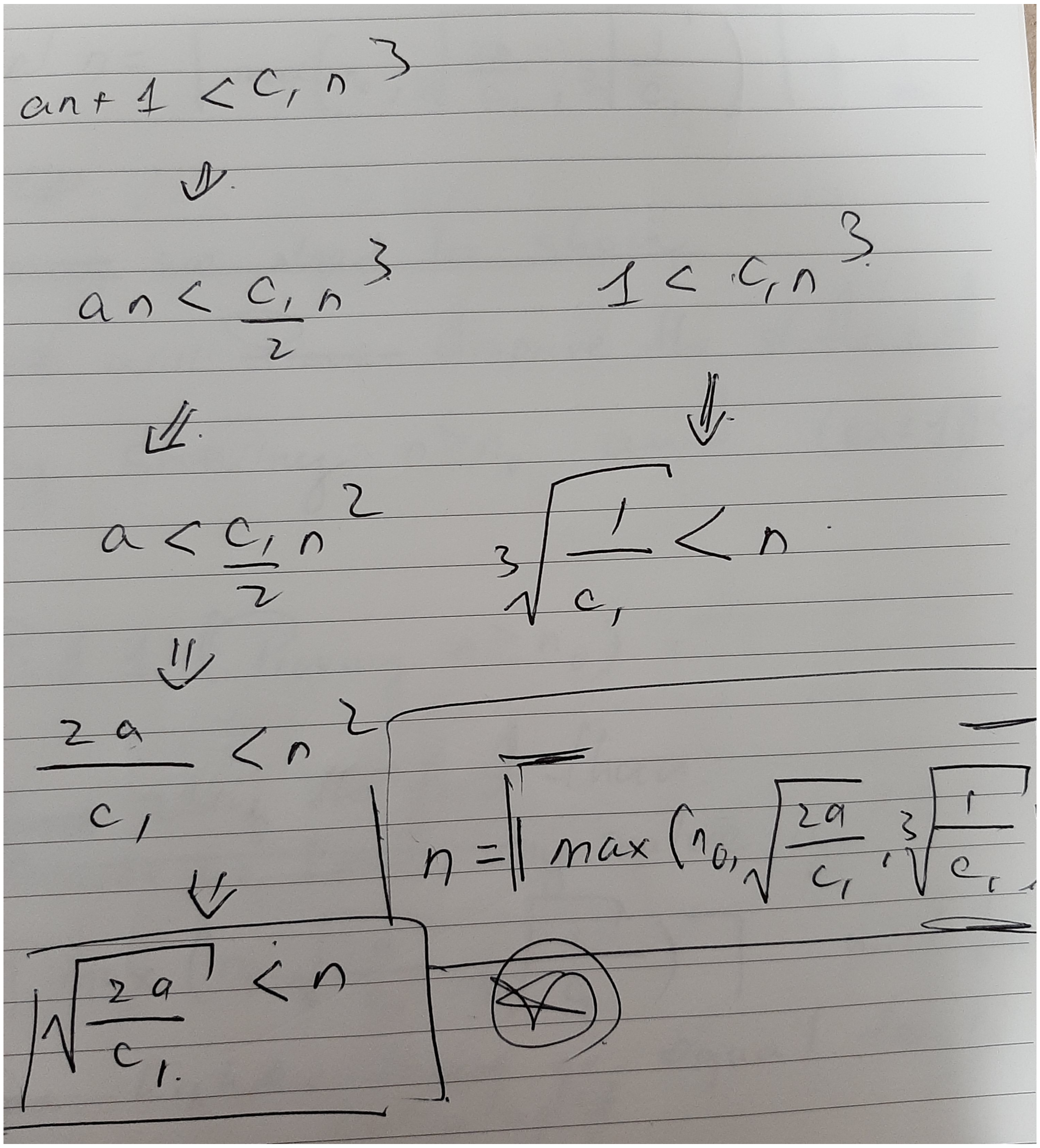

Figure 1: A sample work for question 3

Question 4