

# CSC236 Assignment 1

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## Question 1

a. Yes. We can prove  $P(235)$  follows from  $P(234)$ .

*Proof.* Let  $b$  be the bipartite graph with 235 vertices where 117 vertices are in one partition and 118 vertices in the other partition (Note this is the configuration where maximum number of edges form).

The bipartite graph with 117 vertices on both sides of partition has  $\frac{234^2}{4}$  edges, and the assumption tells us this is the maximum number of edges the bipartite graph could form.

Since we know  $b$  has 117 more edges than the bipartite graph with 117 vertices on both sides, using these facts, we can conclude the upper bound number of edges for the bipartite graph with 235 vertices is

$$\frac{234^2}{4} + 117 = \frac{234^2}{4} + \frac{4 \cdot 117}{4} \tag{1}$$

$$= \frac{234^2 + 2 \cdot 234}{4} \tag{2}$$

$$\leq \frac{234^2 + 2 \cdot 234 + 1}{4} \tag{3}$$

$$= \frac{(234 + 1)^2}{4} \tag{4}$$

$$= \frac{(235)^2}{4} \tag{5}$$

□

### Attempt #2:

Assume  $P(234)$ . That is, every bipartite graph on 234 vertices has no more than  $\frac{234^2}{4}$  edges.

We need to prove  $P(235)$  follows. That is, every bipartite graph on 235 vertices has no more than  $\frac{235^2}{4}$  edges.

Let  $b$  be the bipartite graph with 235 vertices. Let  $b'$  be the bipartite graph with a vertex removed from the larger of two partitions in  $b$  along with its edges.

Since we know the maximum number of edges occur in  $b'$  when there are 117 vertices in both sides of the partitions, and since we know the edges of removed vertex forms edges with partition with bigger number of vertices, we can conclude the removed vertex forms at most 117 edges.

Since the assumption tells us  $b'$  has at most  $\frac{234^2}{4}$  edges, we can conclude the upper bound number of edges for the bipartite graph with 235 vertices is

$$\frac{234^2}{4} + 117 = \frac{234^2}{4} + \frac{4 \cdot 117}{4} \quad (6)$$

$$= \frac{234^2 + 2 \cdot 234}{4} \quad (7)$$

$$\leq \frac{234^2 + 2 \cdot 234 + 1}{4} \quad (8)$$

$$= \frac{(234 + 1)^2}{4} \quad (9)$$

$$= \frac{(235)^2}{4} \quad (10)$$

So  $P(235)$  follows.

### Notes:

- I have a stuck feeling as to how I can formulate this kind of proof.
- 5+ hours spent on this problem
- I feel I wronged the proof by using existential quantifier
- Noticed professor generalized his statement instead of using bipartite with  $x$  number of vertices in one partition and  $y$  vertices in other partition example

**proof:** Assume  $P(234)$  and let  $G$  be an arbitrary bipartite graph with 235 vertices. Remove a vertex, together with its edges from the larger of the two partitions to form a new bipartite graph,  $G'$  with 234 vertices. Notice that, since all edges must run from the removed vertex to the other (smaller) partition, there are no more than 117 edges.

By  $P(234)$   $G'$  has no more than  $234^2/4$  edges, so  $G$  has no more than:

$$\frac{234^2}{4} + 117 = \frac{234^2 + 4(117)}{4} = \frac{234^2 + 2(234)}{4} < \frac{234^2 + 2(234) + 1}{4} = \frac{235^2}{4}$$

So  $P(235)$  follows.

b. No.  $P(236)$  doesn't follow from  $P(235)$ .

*Proof.* Assume  $P(235)$ . That is, every bipartite graph with 235 vertices has no more than  $\frac{235^2}{4}$  edges.

We need to show  $P(236)$  doesn't follow. That is, there is a bipartite graph  $b$  with 236 vertices that has more than  $\frac{236^2}{4}$  edges.

Let  $b$  be bipartite graph with 118 vertices in  $V_1$  and 118 vertices in  $V_2$  (Notice that this forms the most number of edges).

The assumption tells us that every bipartite graph with 235 vertices has  $\frac{235^2}{4}$  edges.

Since we know that by removing a vertex with 118 edges from either one of the partition results in bipartite graph with 235 vertices, using above fact, we can write that  $b$  has

$$\frac{235^2}{4} + 118 = 13924.25 \quad (1)$$

edges.

Since  $\frac{236^2}{4} = 13924.0$ , we can conclude  $P(236)$  doesn't follow.

□

### Notes:

- Noticed that professor tried to show the bipartite graph with 234 vertices with 117 vertices in  $V_1$  and 117 vertices in  $V_2$  is the one and the only bipartite graph with 234 vertices that form the most number of edges.

c. *Proof.* For convenience, define

$H(n') :$  Every bipartite graph with  $n'$  vertices has no more than  $\frac{n'^2 - 1}{4}$  edges  
when  $n'$  is odd, or  $\frac{n'^2}{4}$  edges when  $n'$  is even.

I must prove  $\forall n \in \mathbb{N}, H(n)$ .

Base Case ( $n = 0$ ):

The definition of edge tells us that to form, it requires a pair of two distinct vertices.

Since the graph has 0 vertices, we can conclude no edge can be formed.

### Inductive Step:

Let  $n \in \mathbb{N}$ . And assume  $H(n)$ .

We must prove that  $H(n+1)$  is true. There are two cases:  $n+1$  is odd, or  $n+1$  is even.

#### **Case 1 ( $n+1$ is even):**

Assume  $n+1$  is even. That is,  $\exists k \in \mathbb{Z}$  such that  $n+1 = 2k$ .

Let  $b$  be an arbitrary bipartite graph with  $n+1$  vertices. Let  $v$  be the vertex in larger of two partitions.

We must prove  $H(n+1)$  is true. That is, every bipartite graph with  $n+1$  vertices has no more than  $\frac{(n+1)^2}{4} = \frac{(2k)^2}{4}$  edges.

First, we need to show  $k$  is the most number of edges  $v$  could have in  $b$ .

The definition of bipartite graph tells us the edges of vertex  $v$  runs from  $v$  to each vertex in the smaller partition.

Since we know the smaller partition has at most  $\frac{n+1}{2} = k$ , vertices, we can write the vertex  $v$  has no more than  $k$  edges.

Second, we need to show the bipartite graph with  $n$  vertices has at most  $\frac{4k^2-4k}{4}$  edges.

The assumption tells us every bipartite graph with  $n'$  vertices has no more than  $\frac{n'^2-1}{4}$  edges when  $n'$  is odd, or  $\frac{n'^2}{4}$  edges when  $n'$  is even.

Since we know  $n = 2k - 1$  is odd, we can write the bipartite graph with  $n$  vertices has no more than

$$\frac{n^2 - 1}{4} = \frac{(2k - 1)^2 - 1}{4} \tag{2}$$

$$= \frac{4k^2 - 4k + 1 - 1}{4} \tag{3}$$

$$= \frac{4k^2 - 4k}{4} \tag{4}$$

edges.

Finally, by combining the two together, and using the fact  $k \geq 1$  (note this is required for  $n$  to stay positive), we can conclude the bipartite graph has no more than

$$\frac{n^2 - 1}{4} + k = \frac{4k^2 - 4k}{4} + k \quad (5)$$

$$= \frac{4k^2 - 4k}{4} + \frac{4k}{4} \quad (6)$$

$$= \frac{4k^2}{4} \quad (7)$$

$$= \frac{(2k)^2}{4} \quad (8)$$

$$= \frac{(2k)^2}{4} \quad (9)$$

$$= \frac{(n+1)^2}{4} \quad (10)$$

edges.

## Case 2 ( $n + 1$ is odd):

Assume  $n + 1$  is odd. That is,  $\exists k \in \mathbb{Z}$  such that  $n + 1 = 2k - 1$ .

Let  $b$  be an arbitrary bipartite graph with  $n + 1$  vertices. Let  $v$  be the vertex in larger of two partitions.

We must prove  $H(n + 1)$  is true. That is, every bipartite graph with  $n + 1$  vertices has no more than  $\frac{(n+1)^2-1}{4} = \frac{n(n+2)}{4} = \frac{2k(2k-2)}{4}$  edges.

First, we need to show  $k - 1$  is the most number of edges  $v$  could have in  $b$ .

The definition of bipartite graph tells us the edges of vertex  $v$  runs from  $v$  to each vertex in the smaller partition.

Since we know the smaller partition has at most  $\frac{n}{2} = \frac{2k-2}{2} = k - 1$ , vertices, we can write the vertex  $v$  has no more than  $k - 1$  edges.

Second, we need to show the bipartite graph with  $n$  vertices has at most  $\frac{(2k-2)^2}{4}$  edges

The assumption tells us every bipartite graph with  $n'$  vertices has no more than  $\frac{n'^2-1}{4}$  edges when  $n'$  is odd, or  $\frac{n'^2}{4}$  edges when  $n'$  is even.

Since we know  $n = 2k - 2$  is even, we can write the bipartite graph with  $n$  vertices has no more than

$$\frac{n^2}{4} = \frac{(2k - 2)^2}{4} \quad (11)$$

edges.

Finally, by combining the two together, we can conclude the bipartite graph has no more than

$$\frac{n^2}{4} + k - 1 = \frac{(2k - 2)^2}{4} + k - 1 \quad (12)$$

$$= \frac{(2k - 2)^2}{4} + \frac{4k - 4}{4} \quad (13)$$

$$= \frac{4k^2 - 8k + 4}{4} + \frac{4k - 4}{4} \quad (14)$$

$$= \frac{2k(k - 2)}{4} \quad (15)$$

$$= \frac{(n + 2) \cdot n}{4} \quad (16)$$

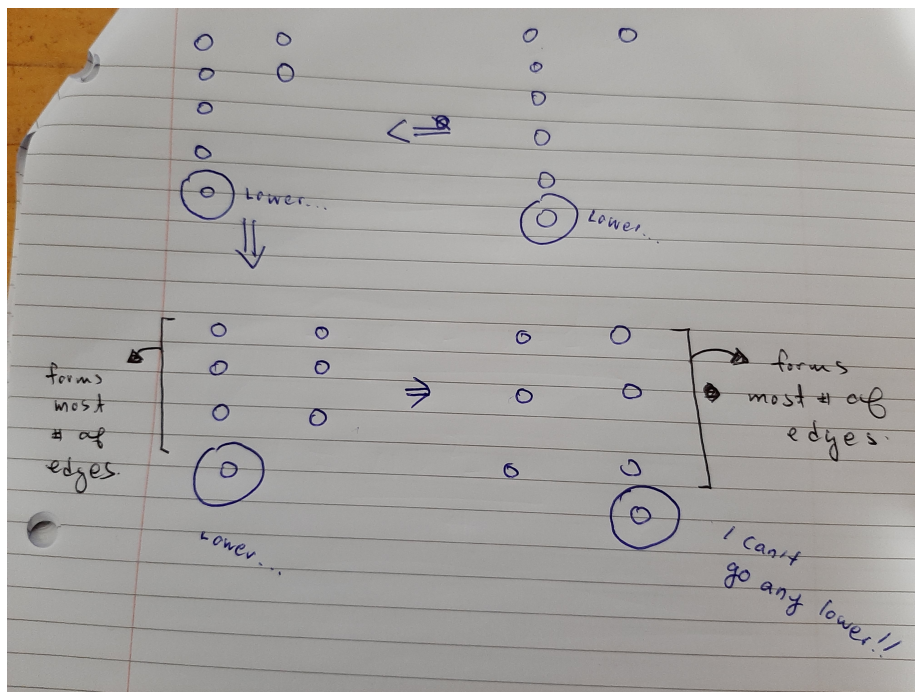
$$= \frac{(n + 1)^2 - 1}{4} \quad (17)$$

edges.

□

### Notes:

- Learned the line 'Remove a vertex, together with its edges from the larger of the two partitions to form a new bipartite graph ... Notice that, since all edges must run from the removed vertex to other smaller partition, there are no more than...' means the following



- This was tricky. And the trick is sneaky :).

## Question 2

a. Yes.  $P(29)$  follows from  $P(3)$ .

*Proof.* Assume  $P(3)$ . That is  $\exists k \in \mathbb{Z}, f(3) = 4k$ .

I must prove  $P(29)$  follows. That is,  $\exists d \in \mathbb{Z}, f(29) = 4d$ .

Let  $d = 4k^2 + k$ .

The definition of  $f(n)$  tells us

$$f(29) = [f(\lfloor \log_3 29 \rfloor)]^2 + f(\lfloor \log_3 29 \rfloor) \quad (1)$$

Using above fact, we can calculate

$$f(29) = [f(3)]^2 + f(3) \quad (2)$$

Then, since we know from the header that  $f(3) = 4k$ , we can write

$$f(29) = [4k]^2 + 4k \quad (3)$$

$$= 4 \cdot (4k^2 + k) \quad (4)$$

Then, since we know from the header that  $d = 4k^2 + k$ , we can conclude

$$f(29) = 4d \quad (5)$$

□

b. Yes.  $P(29)$  follows from  $P(4)$ .

*Proof.* Assume  $P(4)$ . That is  $\exists k \in \mathbb{Z}, f(4) = 4k$ .

I must prove  $P(29)$  follows. That is,  $\exists d \in \mathbb{Z}, f(29) = 4d$ .

Let  $d = k \cdot (4k + 1)$ .

The definition of  $f(n)$  tells us

$$f(29) = [f(\lfloor \log_3 29 \rfloor)]^2 + f(\lfloor \log_3 29 \rfloor) \quad (1)$$

Using above fact, we can calculate

$$f(29) = [f(3)]^2 + f(3) \quad (2)$$

Then, since we know from the definition of  $f(n)$  that  $f(4) = f(3) = f(1)^2 + f(1)$ , we can write

$$f(29) = [f(1)^2 + f(1)]^2 + [f(1)^2 + f(1)] \quad (3)$$

$$= [f(4)]^2 + f(4) \quad (4)$$



Then, since we know from the header that  $f(4) = 4k$ , we can write

$$f(29) = [4k]^2 + 4k \quad (5)$$

$$= 4k \cdot (4k + 1) \quad (6)$$

Then, since we know from the header that  $d = k \cdot (4k + 1)$ , we can conclude

$$f(29) = 4d \quad (7)$$

□

c. **Rough Work:**

Define

$$P(n) : f(n) \text{ is a multiple of } 4$$

I must prove by complete induction that  $\forall n \in \mathbb{N} - \{0\}, P(n)$ .

1. Inductive step

Let  $n \in \mathbb{N} - \{0\}$ . Assume  $H(n) : \bigwedge_{i=1}^{i=n-1} P(i)$ .

I will show that  $P(n)$  follows, that  $f(n)$  is a multiple of 4. That is,  $\exists d \in \mathbb{Z}, f(n) = 4d$ .

- Base Case ( $n = 1$ )

Let  $n = 1$ . Let  $d = 3$ .

**Base Case ( $n = 1$ ):**

Let  $n = 1$ . Let  $d = 3$ .

Then,

$$f(1) = [f(\lfloor \log_3 1 \rfloor)]^2 + f(\lfloor \log_3 1 \rfloor) \quad (1)$$

$$= f(0)^2 + f(0) \quad (2)$$

$$= (3)^2 + 3 \quad (3)$$

$$= 12 \quad (4)$$

Then, since  $d = 3$ ,

$$f(1) = 4 \cdot (3) \quad (5)$$

$$f(1) = 4d \quad (6)$$

- Base Case ( $n = 2$ )

**Base Case ( $n = 2$ ):**

Let  $n = 2$ . Let  $d = 3$ .

Then,

$$f(1) = [f(\lfloor \log_3 1 \rfloor)]^2 + f(\lfloor \log_3 1 \rfloor) \quad (7)$$

$$= f(0)^2 + f(0) \quad (8)$$

$$= (3)^2 + 3 \quad (9)$$

$$= 12 \quad (10)$$

Then, since  $d = 3$ ,

$$f(1) = 4 \cdot (3) \quad (11)$$

$$f(1) = 4d \quad (12)$$

- Case  $n \geq 3$

Let  $n \geq 3$ , and  $d = 4k^2 + k$

- State  $f(n) = [f(\lfloor \log_3 n \rfloor)]^2 + f(\lfloor \log_3 n \rfloor)$  using the definition of  $f(n)$ .
- Conclude  $f(n)$  is divisible by 4 by using the fact  $1 \leq \lfloor \log_3 n \rfloor < n$  and strong induction.

**Case ( $n \geq 3$ ):**

Let  $n \geq 3$ , and  $d = 4k^2 + k$

Then, since  $n > 0$ ,

$$f(n) = f(\lfloor \log_3 n \rfloor)^2 + f(\lfloor \log_3 n \rfloor) \quad (13)$$

Then, since  $1 \leq \lfloor \log_3 n \rfloor < n$ , and  $P(\lfloor \log_3 n \rfloor)$ ,  $\exists k \in \mathbb{Z}$ ,

$$f(n) = [4k]^2 + 4k \quad (14)$$

$$= 4(4k^2 + k) \quad (15)$$

Then, since  $d = 4k^2 + k$ ,

$$f(n) = 4d \quad (16)$$

### Question 3

### Question 4