CSC236 Worksheet 2 Review

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Question 3

- *Proof.* For convenience, define $P(n) : f(n) \leq 3^n$. I will use complete induction to prove that $\forall n \in \mathbb{N}$, $P(n)$.

  Inductive Step:

  Let $n \in \mathbb{N}$. Assume $H(n) : \bigwedge_{i=0}^{n-1} P(i)$. I will show $P(n)$ follows. That is $f(n) \leq 3^n$.

  Base Case ($n = 0$):

  Let $n = 0$.

  Then,

$$\begin{aligned} f(n) &= 1 && \text{[By def.]} && (1) \\ &= 3^0 && && (2) \\ &\leq 3^0 && && (3) \\ &= 3^n && && (4) \end{aligned}$$

  Thus, $P(n)$ follows.

  Base Case ($n = 1$):

  Let $n = 1$.

  Then,

$$
\begin{aligned}
f(n) &= 3 && \text{[By def.]} && (5)\\
&= 3^1 && && (6)\\
&\leq 3^1 && && (7)\\
&= 3^n && && (8)
\end{aligned}
$$

Thus, $P(n)$ follows.

Case $(n > 1)$:

Let $n \in \mathbb{N} \setminus \{0\}$.

Then, we have

$$
\begin{aligned}
f(n) &= 2(f(n-2) + f(n-1)) + 1 && \text{[By def., since } 1 < n\text{]} && (9)\\
&\leq 2(3^{n-2} + 3^{n-1}) + 1 && \text{[By I.H, since } 1 \leq n-2 < n-1 < n\text{]} && (10)\\
&= 2 \cdot 3^{n-2}(1+3) + 1 && && (11)\\
&= 8 \cdot 3^{n-2} + 1 && && (12)\\
&\leq 8 \cdot 3^{n-2} + 3^{n-2} && \text{[Since } 1 < n \text{ and } 0 \leq 3^{n-2}\text{]} && (13)\\
&= 9 \cdot 3^{n-2} && && (14)\\
&= 3^n && && (15)
\end{aligned}
$$

Thus, $P(n)$ follows. □

Correct Solution:

For convenience, define $P(n) : f(n) \leq 3^n$. I will use complete induction to prove that $\forall n \in \mathbb{N},\ P(n)$.

Inductive Step:

Let $n \in \mathbb{N}$. Assume $H(n) : \bigwedge_{i=0}^{n-1} P(i)$. I will show $P(n)$ follows. That is $f(n) \leq 3^n$.

Base Case $(n = 0)$:

Let $n = 0$.

Then,

$$
\begin{aligned}
f(n) &= 1 && \text{[By def.]} && (16)\\
&= 3^0 && && (17)\\
&\leq 3^0 && && (18)\\
&= 3^n && && (19)
\end{aligned}
$$

Thus, $P(n)$ follows in this case.

Base Case ($n = 1$):

Let $n = 1$.

Then,

$$
\begin{aligned}
f(n) &= 3 && \text{[By def.]} && (20)\\
&= 3^1 && && (21)\\
&\leq 3^1 && && (22)\\
&= 3^n && && (23)
\end{aligned}
$$

Thus, $P(n)$ follows in this case.

Case ($n > 1$):

Let $n \in \mathbb{N} \setminus \{0\}$.

Then, we have

$$
\begin{aligned}
f(n) &= 2(f(n-2) + f(n-1)) + 1 && \text{[By def., since } 1 < n\text{]} && (24)\\
&\leq 2(3^{n-2} + 3^{n-1}) + 1 && \text{[By I.H, since } 1 \leq n-2 < n-1 < n\text{]} && (25)\\
&= 2 \cdot 3^{n-2}(1+3) + 1 && && (26)\\
&= 8 \cdot 3^{n-2} + 1 && && (27)\\
&\leq 8 \cdot 3^{n-2} + 3^{n-2} && \text{[Since } 1 < n \text{ and } 1 \leq 3^{n-2}\text{]} && (28)\\
&= 9 \cdot 3^{n-2} && && (29)\\
&= 3^n && && (30)
\end{aligned}
$$

Thus, $P(n)$ follows from $H(n)$ in this case.

Notes:

- Learned $n \in \mathbb{N} \setminus \{0, \cdots, k\}$ is used to express $n > k$, where$n \in \mathbb{N}$.
- Noticed professor wrote '$\cdots$ in this case.' at the end of each case.

Question 2

- **Rough Work:**

Define $P(n)$: Postage of exactly n cents can be made using only 3-cent and 4-cent stamps

I will use complete induction to prove that $\forall n \in \mathbb{N},\ n \geq 6 \Rightarrow P(n)$.

1. Inductive Step

   Let $n \in \mathbb{N}$. Assume $H(n) : \bigwedge_{i=0}^{n-1} P(i)$. I will show $P(n)$ follows.

2. Base Case ($n = 6$)

   Let $n = 6$.

   Since $n = 6$ can be made using 2 3-cent stamps, $P(n)$ follows in this step.

3. Base Case ($n = 7$)

   Let $n = 7$.

   Since $n = 7$ can be made using 1 3-cent stamp and 1 4-cent stamp, $P(n)$ follows in this step.

4. Base Case ($n = 8$)

   Let $n = 8$.

   Since $n = 8$ can be made using 2 4-cent stamps, $P(n)$ follows in this step.

5. Base Case ($n = 9$)

Let $n = 9$.

Since $n = 9$ can be made using 3 3-cent stamps, $P(n)$ follows in this step.

6. Case $(n < 9)$
   I need to show $\exists d, e \in \mathbb{N}$, $n = d \cdot 3 + e \cdot 4$.

   Since $n > 9$, $6 \leq n - 4 < n$, so $P(n-4)$ is true. That is, postage of $n - 4$ cents can be made using postage of 4-cents and 3-cents. In other words, $\exists d', e' \in \mathbb{N}$, $n - 4 = d' \cdot 3 + e' \cdot 4$.

   Thus, we have

$$n - 4 + 4 = d' \cdot 3 + e' \cdot 4 + 4 \tag{31}$$
$$n = d' \cdot 3 + (e' + 1) \cdot 4 \tag{32}$$

   So, by choosing $d = d'$ and $e = e' + 1$, $P(n)$ follows from $H(n)$ in this step.