

Worksheet 6 Review

March 24, 2020

Question 1

a. $\forall n \in \mathbb{N}, P(123) \wedge \neg(n > 123 \Rightarrow P(n))$

Correct Solution:

$$P(123) \wedge (\forall n \in \mathbb{N}, P(n) \Rightarrow n \leq 123)$$

b. $IsCD(x, y) : \exists x, y, d \in \mathbb{Z}, d \mid x \wedge d \mid y$

$$IsGCD(x, y) : \exists x, y, d \in \mathbb{Z}, d \mid x \wedge d \mid y \wedge (\forall n \in \mathbb{N}, n > d \Rightarrow n \nmid x \vee n \nmid y)$$

Correct Solution:

$$IsGCD(x, y, d) : \exists x, y, d \in \mathbb{Z}, (x = 0 \wedge y = 0 \Rightarrow d = 0) \wedge (x \neq 0 \vee y \neq 0 \Rightarrow IsCD(x, y, d) \Rightarrow \forall d' \in \mathbb{Z}, IsCD(x, y, d') \Rightarrow d' \leq d)$$

c. Let $a = x$, $b = 0$, $d = x$ and $d' \in \mathbb{Z}$. Assume $IsCD(x, y, d')$.

Because we know $x \mid x$ and $x \mid 0$, we can conclude that d is a common divisor to a and b .

Since $d' \mid a$ and $d' \mid b$, and since $\forall n \in \mathbb{Z}^+, \forall d \in \mathbb{Z}, d \mid n \Rightarrow d \leq n$, we can conclude that

$$d' \leq a \tag{1}$$

Then,

$$d' \leq d \tag{2}$$

by the fact that $d = a$.

Then it follows from above that the statement $\forall x \in \mathbb{Z}^+, IsGCD(x, 0, x)$ is true.

d. **Attempt 1:**

$$a, b \in \mathbb{Z}, a \neq 0 \vee b \neq 0 \Rightarrow (\exists d \in \mathbb{Z}, d = GCD(a, b) \Rightarrow \forall d' \in \mathbb{Z}^+, \exists p, q \in \mathbb{Z})$$

Attempt 2:

$$a, b \in \mathbb{Z}, \exists d \in \mathbb{Z}, (a \neq 0 \vee b \neq 0) \wedge d = GCD(a, b) \Rightarrow \exists p, q \in \mathbb{Z}, d = ap + bq \wedge d > 0 \wedge (\forall d' \in \mathbb{Z}^+, d' = ap + bq \Rightarrow d' \geq d)$$

Question 2

a. Let $n \in \mathbb{Z}$. Assume that $\exists k \in \mathbb{Z}, n = 2k$.

Then,

$$n^2 - 3n = 4n^2 - 6n \tag{1}$$

$$= 2(n^2 - 3n) \tag{2}$$

$$= 2m \tag{3}$$

where $m = n^2 - 3n \in \mathbb{Z}$.

Then, by definition of even number, $n^2 - 3n$ is even.

b. Let $n \in \mathbb{Z}$. Assume $\exists k \in \mathbb{Z}, n = 2k - 1$.

Then,

$$n^2 - 3n = (2k - 1)^2 - 3 \cdot (2k - 1) \tag{1}$$

$$= 4k^2 - 4k + 1 - 6k + 3 \tag{2}$$

$$= 4k^2 - 10k + 4 \tag{3}$$

$$= 2(2k^2 - 5k + 2) \tag{4}$$

$$= 2m \tag{5}$$

where $m = 2k^2 - 5k + 2 \in \mathbb{Z}$.

Then, it follows from the definition of even number that $n^2 - 3n$ is even.

Question 3

- a. $\forall a, b \in \mathbb{N}, \text{Prime}(b) \Rightarrow 1 \geq \gcd(a, b) \vee \gcd(a, b) \geq b$
- b. Let $a, b \in \mathbb{N}$. Assume $\text{Prime}(b)$.

We will prove the statement by considering two cases, when $b \mid a$, and when $b \nmid a$.

Case 1 ($b \mid a$):

Assume $b \mid a$.

Since b is a prime number, there are two possible divisors b and 1.

Since $b \mid a$ and $b \mid b$, $b = \gcd(a, b)$.

Since $b \mid \gcd(a, b)$, by the fact $\forall n \in \mathbb{Z}^+, d \in \mathbb{Z}, d \mid n \Rightarrow d \leq n$, we can conclude that

$$b \leq \gcd(a, b) \tag{1}$$