CSC236 Worksheet 5 Review

Hyungmo Gu

May 8, 2020

Question 1

a. *Proof.* Define $P(k) : R(3^k) = k3^k$. Note that when $n = 3^k$, this is equivalent to $R(n) = n \log_3 n$. I will use simple induction to prove $P(k)$.

Base Case ($k = 0$):

Let $k = 0$.

Then,

$$
\begin{aligned}
R(3^k) &= 0 && \text{[By def., since } n = 3^0 = 1\text{]} \quad (1)\\
&= 0 \cdot 3^0 && (2)\\
&= k \cdot 3^k && (3)
\end{aligned}
$$

Thus, $P(k)$ is verified in this step.

Inductive Step:

Let $k \in \mathbb{N}$. Assume $P(k)$. That is, $R(3^k) = k \cdot 3^k$. I need to prove $P(k+1)$ follows. That is, $R(3^{k+1}) = (k+1) \cdot 3^{k+1}$

Starting from $R(3^{k+1})$, we have

$$
\begin{aligned}
R(3^{k+1}) &= 3^{k+1} + 3R(\lceil 3^{k+1}/3 \rceil) && \text{[By def., since } 0 < k+1 \text{, and } 1 < 3^{k+1}\text{]} \quad (4)\\
&= 3^{k+1} + 3R(\lceil 3^k \rceil) && (5)\\
&= 3^{k+1} + 3R(3^k) && \text{[Since } \lceil 3^k \rceil = 3^k\text{]} \quad (6)\\
&= 3^{k+1} + 3(k \cdot 3^k) && \text{[By I.H]} \quad (7)\\
&= 3^{k+1} + (k \cdot 3^{k+1}) && (8)\\
&= (k+1) \cdot 3^{k+1} && (9)
\end{aligned}
$$

□

b. **Rough Work:**

For convenience, define $P(n) : \bigwedge_{i=1}^{i=n} R(i) \leq R(n)$.

I will use complete induction to prove that $\forall n \in \mathbb{N}, 0 < n \Rightarrow P(n)$.

1. Inductive Step

   Let $n \in \mathbb{N} \setminus \{0\}$. Assume $\bigwedge_{i=1}^{i=n-1} P(i)$. I will show that $P(n)$ follows.

2. Base Case ($n = 1$)

   Let $n = 1$.

   Then, $\bigwedge_{i=1}^{i=n} R(i) = R(n)$.

   Thus, $P(n)$ follows in this step.

3. Base Case ($n = 2$)

   Let $n = 2$.

   In this step, I need to prove $P(n)$. That is, $R(1) \leq R(2)$ and $R(2) \leq R(2)$.

   I will do so in part.

   Part 1 (Proving $R(1) \leq R(2)$):

   In this part, the definition tells us $R(1) = 1$ and $R(2) = 2 + 3R(\lceil 2/3 \rceil) = 2 + 3R(1) = 2$.

   Since $R(1) = 1 < R(2) = 2$, we can conclude $R(1) \leq R(2)$ holds.

   Part 2 (Proving $R(2) \leq R(2)$):

   In this part, since $R(2) = R(2)$, $R(2) \leq R(2)$ holds.

4. Inductive Step