# Midterm 2 Version 3 Solution

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## Question 1

a.

$$165 \div 2 = 82, \text{remainders } \mathbf{1}$$
$$82 \div 2 = 41, \text{remainders } \mathbf{0}$$
$$41 \div 2 = 20, \text{remainders } \mathbf{1}$$
$$20 \div 2 = 10, \text{remainders } \mathbf{0}$$
$$10 \div 2 = 5, \text{remainders } \mathbf{0}$$
$$5 \div 2 = 2, \text{remainders } \mathbf{1}$$
$$2 \div 2 = 1, \text{remainders } \mathbf{0}$$
$$1 \div 2 = 0, \text{remainders } \mathbf{1}$$

From the above, we can conclude the binary representation of the decimal number 165 is $(10100101)_2$

b. The largest number that can be expressed by an $n$-digit balanced ternary representation is

$$\sum_{i=0}^{n-1} 3^i = \frac{1}{2} \cdot (3^n - 1) \tag{1}$$

**Notes:**

- Geometric Series

$$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}, \text{ where } |r| > 1$$

c.

| $f(n) \in \mathcal{O}(n)$ | True | $g(n) \in \Omega(n)$ | True | $f(n) \in \Omega(g(n))$ | True |
|---|---|---|---|---|---|
| $f(n) \in \Theta(g(n))$ | False | $g(n) \in \Theta(n)$ | False | $f(n) + g(n) \in \Theta(g(n))$ | False |

**Notes:**

- Note that for $f(n) + g(n) \in \Theta(g(n))$, large values of $n$ causes $g(n) = n^{\log_2 n}$ to dominate $f(n) = \frac{3n}{\log_2 n + 8}$. This causes the inequality to be simplified to

$$c_1 \cdot n^{\log_2 n} \leq n^{\log_2 n} \leq c_2 \cdot n^{\log_2 n} \tag{1}$$

It follows from above the answer is True.

d.

| $k$ | 0 | 1 | 2 |
| --- | --- | --- | --- |
| $i * i$ | $3 = 3^{2^0}$ | $9 = 3^{2^1}$ | $81 = 3^{2^4}$ |

From the rough work, we can deduce the value of $i$ after $k$ iterations is

$$3^{2^k} \tag{1}$$

e. Loop termination occurs when $i_k \geq n^3$.

We need to find the smallest value of $k$, and the value is

$$\lceil \log_2 3 \log_3 n \rceil \tag{1}$$

# Question 2

- **Statement:** $\forall n \in \mathbb{N}, \ n \geq 1 \Rightarrow \sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n} - 1$

*Proof.* Let $n \in \mathbb{N}$. Assume $n \geq 1$.

We will prove the statement using induction on $n$.

**Base Case ($n = 1$):**

Let $n = 1$.

We want to show $\sum\limits_{i=1}^{1} \frac{1}{\sqrt{i}} > \sqrt{1} - 1$

Starting from the left hand side of the inequality, we can calculate

$$\sum_{i=1}^{1} \frac{1}{\sqrt{i}} = \frac{1}{\sqrt{1}} \tag{1}$$
$$= 1 \tag{2}$$
$$> 0 \tag{3}$$
$$= \sqrt{1} - 1 \tag{4}$$

**Inductive Case:**

Let $n \in mathbbN$. Assume $\sum\limits_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n} - 1$.

We want to show $\sum\limits_{i=1}^{n+1} \frac{1}{\sqrt{i}} > \sqrt{n+1} - 1$.

Starting from the left hand side of the inequality, we can conclude

$$\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} = \sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}} \tag{5}$$

Then, it follows from induction hypothesis (i.e. $\sum\limits_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n} - 1$) that

$$\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} > \sqrt{n+1} - 1 + \frac{1}{\sqrt{n+1}} \tag{6}$$

The hint tells us the following

$$\forall n \in \mathbb{Z}^{+}, \ \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}}, \text{ and } \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n+1}} \tag{7}$$

Using the hint, we can calculate

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$$\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} > (\sqrt{n} - 1) + \frac{1}{\sqrt{n+1} + \sqrt{n}} \tag{8}$$

$$= (\sqrt{(n)} - 1) + (\sqrt{n+1} - \sqrt{n}) \tag{9}$$

$$= \sqrt{n+1} - 1 \tag{10}$$

$\square$

## Question 3

## Question 4