Worksheet 6 Review 2

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Question 1

a. $\forall x \in \mathbb{N},\ P(123) \land P(x) \Rightarrow x \leq 123$

Correct Solution:

$P(123) \land (\forall x \in \mathbb{N}, P(x) \Rightarrow x \leq 123)$

b. $IsCD(x, y, d):\ d \mid x \land d \mid y$, where $x, y, d \in \mathbb{Z}$

$IsGCD(x, y, d):\ \forall n \in \mathbb{N},\ IsCD(x, y, n) \Rightarrow \exists d \in \mathbb{N},\ IsCD(x, y, d) \land n \leq d$

Correct Solution:

$IsCD(x, y, d):\ d \mid x \land d \mid y$, where $x, y, d \in \mathbb{Z}$

$IsGCD(x, y, d): (x = 0 \land y = 0 \Rightarrow d = 0) \land (x \neq 0 \land y \neq 0 \Rightarrow IsCD(x, y, d) \land (\forall d_1 \in \mathbb{Z},\ IsCD(x, y, d_1) \Rightarrow d_1 \leq d))$, where $x, y, d \in \mathbb{Z}$

Notes:

- Realized the definition of $IsGCD$ extends from previous question
- Noticed professor defines if...else conditions in a predicate logic the following way

(case 1 $\Rightarrow$ statement 1) $\land$ (case 2 $\Rightarrow$ statement 2)

- Hm... I feel puzzled about $\land$ operator used in between cases (i.e. $(x = 0 \land y = 0 \Rightarrow d = 0) \land (x \neq 0 ...)$). At glimpse, I felt $\lor$ is more appropriate since if this case is not true, then we want other case should be true.

c. **Statement:** $IsCD(x, 0, x) \wedge (\forall d_1 \in \mathbb{Z}, IsCD(x, 0, d_1) \Rightarrow d_1 \leq x)$

Proof. Let $x \in \mathbb{Z}^+$

We need to prove x is a common divisor to both 0 and x, and we need to prove all common divisors d_1 of 0 and x is less than or equal to x.

First, we need to show there is $k_1 \in \mathbb{Z}$ such that $x = k_1 \cdot x$ and we need to show $k_2 \in \mathbb{Z}$ such that $0 = k_2 \cdot x$.

Let $k_1 = 1$ and $k_2 = 0$.

Then, we can calculate that

$$x = 1 \cdot x = k_1 \cdot x \tag{1}$$
$$0 = 0 \cdot x = k_2 \cdot x \tag{2}$$

Now, we need to show all integers d_1 that is a common divisor to both 0 and x is less than equal to x.

Let $d_1 \in \mathbb{Z}$ and assume $d_1 \mid x$ and $d_1 \mid 0$.

We need to show $d_1 \leq x$.

The hint tells us

$$\forall n \in \mathbb{Z}^+,\ \forall d \in \mathbb{Z},\ d \mid n \Rightarrow d \leq n \tag{3}$$

Because we know from assumption that $d_1 \mid x$, by using the hint, we can conclude

$$d_1 \leq x \tag{4}$$

□

Pseudoproof:

Let $x \in \mathbb{Z}^+$

We need to prove x is a common divisor to both 0 and x, and we need to prove all common divisors d_1 of 0 and x is less than or equal to x.

1. Show $IsCD(x, 0, x)$

   We need to show there is $k_1 \in \mathbb{Z}$ such that $x = k_1 \cdot x$ and we need to show $k_2 \in \mathbb{Z}$ such that $0 = k_2 \cdot x$.

   Let $k_1 = 1$ and $k_2 = 0$.

   - Show $x = k_1 \cdot x$ and $0 = k_2 \cdot 0$

     Then, we can calculate that

     $$x = 1 \cdot x = k_1 \cdot x \tag{5}$$
     $$0 = 0 \cdot x = k_2 \cdot x \tag{6}$$

2. Show $\forall d_1 \in \mathbb{Z}, IsCD(x, 0, d_1) \Rightarrow d_1 \leq x$

   Let $d_1 \in \mathbb{Z}$ and assume $d_1 \mid x$ and $d_1 \mid 0$.

   We need to show $d_1 \leq x$.

   1. Use fact '$\forall n \in \mathbb{Z}^+,\ \forall d \in \mathbb{Z},\ d \mid n \Rightarrow d \leq n$' to show $d_1 \leq x$.

   The hint tells us

   $$\forall n \in \mathbb{Z}^+,\ \forall d \in \mathbb{Z},\ d \mid n \Rightarrow d \leq n \tag{7}$$

   Because we know from assumption that $d_1 \mid x$, by using the hint, we can conclude

   $$d_1 \leq x \tag{8}$$

d. $\forall a, b \in \mathbb{Z},\ (a \neq 0) \vee (b \neq 0) \Rightarrow \exists p, q \in \mathbb{Z},\ pa + qb = gcd(a, b)$

Question 2

a. *Proof.* Assume $Even(n)$. That is $\exists k \in \mathbb{Z},\ n = 2k$.

We need to show there is an integer k_1 such that $n^2 - 3n = 2k_1$.

Let $k_1 = (2k^2 - 3k)$.

The assumption tells us $n = 2k$.

Then, by using this fact, we can write

$$\begin{aligned} n^2 - 3n &= (2k)^2 - 3(2k) && (1) \\ &= 4k^2 - 6k && (2) \\ &= 2(2k^2 - 3k) && (3) \\ &= 2k_1 && (4) \end{aligned}$$

□

Pseudoproof:

Assume $Even(n)$. That is $\exists k \in \mathbb{Z},\ n = 2k$.

We need to show there is an integer k_1 such that $n^2 - 3n = 2k_1$.

Let $k_1 = (2k^2 - 3k)$.

- Show $n^2 - 3n = 2k_1$ by using assumption.

The assumption tells us $n = 2k$.

Then, by using this fact, we can write

$$\begin{aligned} n^2 - 3n &= (2k)^2 - 3(2k) && (5) \\ &= 4k^2 - 6k && (6) \\ &= 2(2k^2 - 3k) && (7) \\ &= 2k_1 && (8) \end{aligned}$$

b. *Proof.* In this case, assume $Odd(n)$. That is $\exists k \in \mathbb{Z},\ n = 2k - 1$.

We need to show there is an integer k_1 such that $n^2 - 3n = 2k_1$.

Let $k_1 = (2k^2 - 5k + 2)$.

The assumption tells us $n = 2k - 1$.

Then, by using this fact, we can write

$$
\begin{aligned}
n^2 - 3n &= (2k-1)^2 - 3(2k-1) && (1)\\
&= 4k^2 - 4k + 1 - 6k + 3 && (2)\\
&= 4k^2 - 10k + 4 && (3)\\
&= 2(2k^2 - 5k + 2) && (4)\\
&= 2k_1 && (5)
\end{aligned}
$$

$\square$

Pseudoproof:

Assume $Odd(n)$. That is $\exists k \in \mathbb{Z},\ n = 2k - 1$.

We need to show there is an integer k_1 such that $n^2 - 3n = 2k_1$.

Let $k_1 = (2k^2 - 5k + 2)$.

- Show $n^2 - 3n = 2k_1$ by using assumption.

The assumption tells us $n = 2k - 1$.

Then, by using this fact, we can write

$$
\begin{aligned}
n^2 - 3n &= (2k-1)^2 - 3(2k-1) && (6)\\
&= 4k^2 - 4k + 1 - 6k + 3 && (7)\\
&= 4k^2 - 10k + 4 && (8)\\
&= 2(2k^2 - 5k + 2) && (9)\\
&= 2k_1 && (10)
\end{aligned}
$$

Notes:

- Noticed professor uses predicate logic when expanding definition in assumption.

Assume that n is odd, i.e. $\exists k \in \mathbb{Z},\ n = 2k - 1$.

Question 3

a. $\forall a, b \in \mathbb{N},\ Prime(b) \Rightarrow 1 \geq gcd(a,b) \vee gcd(a,b) \geq b$

b. **Statement (Contrapositive):** $\forall a, b \in \mathbb{N},\ Prime(b) \Rightarrow 1 \geq gcd(a,b) \vee gcd(a,b) \geq b$

Proof. Let $a, b \in \mathbb{N}$. Assume $Prime(b)$. That is, $p > 1 \wedge (\forall d \in \mathbb{N},\ d \mid p \Rightarrow d = 1 \vee d = p)$.

We will prove $1 \geq gcd(a,b)$ or $gcd(a,b) \geq b$ using proof by cases.

Case 1 ($b \mid a$):

In this case, assume b divides a. That is, $\exists k \in \mathbb{Z},\ a = kb$.

We need to prove $b \geq gcd(a,b)$.

First, we need to show b is the greatest common divisor to both a and b. That is, $IsCD(a,b,b) \wedge (\forall d_1 \in \mathbb{Z},\ IsCD(a,b,d_1) \Rightarrow d_1 \leq b))$

Starting with showing $IsCD(a,b,b)$, the assumption tells us $b \mid a$, and we know $b \mid b$.

Then, it follows from these facts that b is a common divisor to both a and b.

Next for showing $(\forall d_1 \in \mathbb{Z},\ IsCD(a,b,d_1) \Rightarrow d_1 \leq b))$, the definition of prime number tells us b has two non-negative divisors 1 and b.

Because we know $1 \mid a$ and $b \mid a$, we can conclude 1 and b are the only non-negative common divisor to both a and b.

Since $1 < b$, $b = b$ and all other common divisors are less than 0, we can conclude all common divisors to both a and b are less than or equal to b.

Now, we need to show $b \leq gcd(a,b)$.

The hint from question 1.c tells us

$$\forall n \in \mathbb{Z}^+,\ \forall d \in \mathbb{Z},\ d \mid n \Rightarrow d \leq n \tag{1}$$

Since we know b divides b, by using this fact, we can write

$$b \leq b \tag{2}$$

Because we know $b = gcd(a,b)$, we can conclude

$$b \leq gcd(a, b) \tag{3}$$

Case 2 ($b \nmid a$):

In this case, assume b doesn't divide a.

We need to prove $1 \geq gcd(a, b)$.

First, we need to show 1 is the greatest common divisor to both a and b.

The assumption tells us b is a prime number, and so, from definition, we know b has two non-negative divisors 1 and b.

Because we know $b \nmid a$ from assumption and $1 \mid a$, we can conclude 1 is the only non-negative common divisor to both a and b.

Because we know all common divisors to both a and b are less than or equal to 1, we can conclude $gcd(a, b) = 1$.

Now, we need to show $1 \geq gcd(a, b)$.

The hint from question 1.c tells us

$$\forall n \in \mathbb{Z}^+, \forall d \in \mathbb{Z},\ d \mid n \Rightarrow d \leq n \tag{4}$$

Since we know 1 divides 1, by using this fact, we can write

$$1 \geq 1 \tag{5}$$

Because we know $1 = gcd(a, b)$, we can conclude

$$1 \geq gcd(a, b) \tag{6}$$

$\square$

Pseudoproof:

Let $a, b \in \mathbb{N}$. Assume $Prime(b)$. That is, $p > 1 \land (\forall d \in \mathbb{N},\ d \mid p \Rightarrow d = 1 \lor d = p)$.

We will prove $1 \geq gcd(a, b)$ or $gcd(a, b) \geq b$ using proof by cases.

Case 1 ($b \mid a$):

In this case, assume b divides a. That is, $\exists k \in \mathbb{Z},\ a = kb$.

We need to prove $gcd(a, b) \geq b$.

1. Show $IsGCD(a, b, b)$, i.e. $IsCD(a, b, b) \land (\forall d_1 \in \mathbb{Z},\ IsCD(a, b, d_1) \Rightarrow d_1 \leq b))$

   First, we need to show b is the greatest common divisor to both a and b. That is, $IsCD(a, b, b) \land (\forall d_1 \in \mathbb{Z},\ IsCD(a, b, d_1) \Rightarrow d_1 \leq b))$

   - Show $IsCD(a, b, b)$

     > Starting with showing $IsCD(a, b, b)$, the assumption tells us $b \mid a$, and we know $b \mid b$.
     >
     > Then, it follows from these facts that b is a common divisor to both a and b.

   - Show $\forall d_1 \in \mathbb{Z},\ IsCD(a, b, d_1) \Rightarrow d_1 \leq b$

     > Next for showing $(\forall d_1 \in \mathbb{Z},\ IsCD(a, b, d_1) \Rightarrow d_1 \leq b))$, the definition of prime number tells us b has two non-negative divisors 1 and b.
     >
     > Because we know $1 \mid a$ and $b \mid a$, we can conclude 1 and b are the only non-negative common divisor to both a and b.
     >
     > Since $1 < b$, $b = b$ and all other common divisors are less than 0, we can conclude all common divisors to both a and b are less than or equal to b.

2. Show $b \leq gcd(a, b)$ by using the fact $b \mid b$ and $\forall n \in \mathbb{Z}^{+},\ \forall d \in \mathbb{Z},\ d \mid n \Rightarrow d \leq n$.

Now, we need to show $b \leq gcd(a, b)$.

The hint from question 1.c tells us

$$\forall n \in \mathbb{Z}^+, \forall d \in \mathbb{Z}, \ d \mid n \Rightarrow d \leq n \tag{7}$$

Since we know b divides b, by using this fact, we can write

$$b \leq b \tag{8}$$

Because we know $b = gcd(a, b)$, we can conclude

$$b \leq gcd(a, b) \tag{9}$$

Case 2 ($b \nmid a$):

In this case, assume b doesn't divide a.

We need to prove $1 \geq gcd(a, b)$.

1. Show $IsGCD(a, b, 1)$

   First, we need to show 1 is the greatest common divisor to both a and b.

   - Find all possible common divisors to both a and b.

     The assumption tells us b is a prime number, and so, from definition, we know b has two non-negative divisors 1 and b.

   - Show 1 is the only common divisor to a and b.

     Because we know $b \nmid a$ from assumption and $1 \mid a$, we can conclude 1 is the only non-negative common divisor to both a and b.

   - Conclude $gcd(a, b) = 1$.

Because we know all common divisors to both a and b are less than or equal to 1, we can conclude $gcd(a, b) = 1$.

2. Show $gcd(a, b) \leq 1$ by using the fact $\forall n \in \mathbb{Z}^+, \forall d \in \mathbb{Z}, d \mid n \Rightarrow d \leq n$.

Now, we need to show $1 \geq gcd(a, b)$.

The hint from question 1.c tells us

$$\forall n \in \mathbb{Z}^+, \forall d \in \mathbb{Z}, d \mid n \Rightarrow d \leq n \tag{10}$$

Since we know 1 divides 1, by using this fact, we can write

$$1 \geq 1 \tag{11}$$

Because we know $1 = gcd(a, b)$, we can conclude

$$1 \geq gcd(a, b) \tag{12}$$

Notes:

- $Prime(p) : p > 1 \wedge (\forall d \in \mathbb{N}, d \mid p \Rightarrow d = 1 \vee d = p)$