

CSC236 Worksheet 2 Review

Hyungmo Gu

May 7, 2020

Question 3

- *Proof.* For convenience, define $P(n) : f(n) \leq 3^n$. I will use complete induction to prove that $\forall n \in \mathbb{N}, P(n)$.

Inductive Step:

Let $n \in \mathbb{N}$. Assume $H(n) : \bigwedge_{i=0}^{n-1} P(i)$. I will show $P(n)$ follows. That is $f(n) \leq 3^n$.

Base Case ($n = 0$):

Let $n = 0$.

Then,

$$\begin{aligned} f(n) &= 1 && \text{[By def.]} && (1) \\ &= 3^0 && && (2) \\ &\leq 3^0 && && (3) \\ &= 3^n && && (4) \end{aligned}$$

Thus, $P(n)$ follows.

Base Case ($n = 1$):

Let $n = 1$.

Then,

$$f(n) = 3 \quad [\text{By def.}] \quad (5)$$

$$= 3^1 \quad (6)$$

$$\leq 3^1 \quad (7)$$

$$= 3^n \quad (8)$$

Thus, $P(n)$ follows.

Case ($n > 1$):

Let $n \in \mathbb{N} \setminus \{0\}$.

Then, we have

$$f(n) = 2(f(n-2) + f(n-1)) + 1 \quad [\text{By def., since } 1 < n] \quad (9)$$

$$\leq 2(3^{n-2} + 3^{n-1}) + 1 \quad [\text{By I.H, since } 1 \leq n-2 < n-1 < n] \quad (10)$$

$$= 2 \cdot 3^{n-2}(1 + 3) + 1 \quad (11)$$

$$= 8 \cdot 3^{n-2} + 1 \quad (12)$$

$$\leq 8 \cdot 3^{n-2} + 3^{n-2} \quad [\text{Since } 1 < n \text{ and } 0 \leq 3^{n-2}] \quad (13)$$

$$= 9 \cdot 3^{n-2} \quad (14)$$

$$= 3^n \quad (15)$$

Thus, $P(n)$ follows. □

Correct Solution:

For convenience, define $P(n) : f(n) \leq 3^n$. I will use complete induction to prove that $\forall n \in \mathbb{N}, P(n)$.

Inductive Step:

Let $n \in \mathbb{N}$. Assume $H(n) : \bigwedge_{i=0}^{n-1} P(i)$. I will show $P(n)$ follows. That is $f(n) \leq 3^n$.

Base Case ($n = 0$):

Let $n = 0$.

Then,

$$f(n) = 1 \quad [\text{By def.}] \quad (16)$$

$$= 3^0 \quad (17)$$

$$\leq 3^0 \quad (18)$$

$$= 3^n \quad (19)$$

Thus, $P(n)$ follows in this case.

Base Case ($n = 1$):

Let $n = 1$.

Then,

$$f(n) = 3 \quad [\text{By def.}] \quad (20)$$

$$= 3^1 \quad (21)$$

$$\leq 3^1 \quad (22)$$

$$= 3^n \quad (23)$$

Thus, $P(n)$ follows in this case.

Case ($n > 1$):

Let $n \in \mathbb{N} \setminus \{0\}$.

Then, we have

$$f(n) = 2(f(n-2) + f(n-1)) + 1 \quad [\text{By def., since } 1 < n] \quad (24)$$

$$\leq 2(3^{n-2} + 3^{n-1}) + 1 \quad [\text{By I.H., since } 1 \leq n-2 < n-1 < n] \quad (25)$$

$$= 2 \cdot 3^{n-2}(1 + 3) + 1 \quad (26)$$

$$= 8 \cdot 3^{n-2} + 1 \quad (27)$$

$$\leq 8 \cdot 3^{n-2} + 3^{n-2} \quad [\text{Since } 1 < n \text{ and } 1 \leq 3^{n-2}] \quad (28)$$

$$= 9 \cdot 3^{n-2} \quad (29)$$

$$= 3^n \quad (30)$$

Thus, $P(n)$ follows from $H(n)$ in this case.

Notes:

- Learned $n \in \mathbb{N} \setminus \{0, \dots, k\}$ is used to express $n > k$, where $n \in \mathbb{N}$.
- Noticed professor wrote ‘ \dots in this case.’ at the end of each case.