Worksheet 16 Review

April 2, 2020

Question 1

a. Let $k \in \mathbb{N}$.

Here, the minimum possible change occurs for the loop variable in a single iteration when $i = i + 1$.

The maximum possible change occurs for the loop variable in a single iteration when $i = i + 6$.

The exact upper bound of the variable after k iteration is

$$i_k \leq 6k \tag{1}$$

The exact lower bound of the variable after k iteration is

$$k \leq i_k \tag{2}$$

Using the fact that the termination occurs when $i_k = n$, we can calculate that for the upper bound, the loop terminates when

$$6k \geq n \tag{3}$$
$$k \geq \frac{n}{6} \tag{4}$$

Because we know $\frac{n}{6}$ may be a decimal, we can conclude the closest value at which the loop terminates is when

$$k = \left\lceil \frac{n}{6} \right\rceil \tag{5}$$

Using the same fact, we can calculate that for the lower bound, the loop terminates when

$$k \geq n \tag{6}$$

It follows from above that for the lower bound, the smallest value of k at which the loop termination occurs is when

$$k = n \tag{7}$$

Then, we can conclude the function has asymptotic lower bound of $\Omega(n)$, and asymptotic upper bound of $\mathcal{O}(n)$.

Then, since both Ω and $\mathcal{O}$ have the same value, $\Theta(n)$ is also true.

Correct Solution:

Here, the minimum possible change occurs for the loop variable in a single iteration when $i = i + 1$.

The maximum possible change occurs for the loop variable in a single iteration when $i = i + 6$.

The exact upper bound of the variable after k iteration is

$$i_k \leq 6k \tag{8}$$

The exact lower bound of the variable after k iteration is

$$k \leq i_k \tag{9}$$

Using the fact that the termination occurs when $i_k = n$, we can calculate that for the upper bound, the loop terminates when

$$6k \geq n \tag{10}$$
$$k \geq \frac{n}{6} \tag{11}$$

Because we know $\frac{n}{6}$ may be a decimal, we can conclude the closest value at which the loop terminates is when

$$k = \left\lceil \frac{n}{6} \right\rceil + 1 \tag{12}$$

Using the same fact, we can calculate that for the lower bound, the loop terminates when

$$k \geq n \tag{13}$$

It follows from above that for the lower bound, the smallest value of k at which the loop termination occurs is when

$$k = n + 1 \tag{14}$$

Then, we can conclude the function has asymptotic lower bound of $\Omega(n)$, and asymptotic upper bound of $\mathcal{O}(n)$.

Since both Ω and $\mathcal{O}$ have the same value, $\Theta(n)$ is also true.

Notes:

- Realized $\mathbf{+\ 1}$ required after thinking there are n iterations between $i = 0$ and $i = n - 1$ when $i = i + 1$.
- Realized $\mathbf{+\ 1}$ in $\frac{n}{6}$ is required after playing with the following example [0,1,2,3,4,5] and [0,1,2,3,4,5,6].
- Realized $\lceil \frac{5}{6} \rceil = 1$ and $\lceil \frac{6}{6} \rceil = 1$, and indeed $\mathbf{+\ 1}$ is required to reach loop termination.
- Perhaps the choice of ceiling and floor can be determined by playing with examples.

b. Let $k \in \mathbb{N}$.

Part 1 (Determining maximum and minimum possible change in a single iteration):

It follows from observation that the minimum possible change occurs when $i = i \cdot 2$, and the maximum possible change when $i = i \cdot 3$.

Part 2 (Determining lower bound and upper bound of loop iteration):

Because we know the smallest possible change occurs when $i = i \cdot 2$ occurs repeatedly, we can conclude that at k^{th} iteration i_k has the lower bound of 2^k.

Similarly, because we know largest possible change occurs when $i = i \cdot 3$ occurs repeatedly, we can conclude that at k^{th} iteration, i_k has the upper bound of 3^k.

Then, by putting together, we can conclude that

$$2^k \leq i_k \leq 3^k \tag{1}$$

Part 3 (Determining exact number of iterations for the lower bound and upper bound):

Because we know the loop runs until $i_k < n$, we can conclude that at lower bound, termination occurs when

$$i_k \geq n \tag{2}$$
$$2^k \geq n \tag{3}$$
$$\log_2 2^k \geq \log_2 n \tag{4}$$
$$k \geq \log_2 n \tag{5}$$

Using the fact that we are looking for smallest value of k, we can calculate that for lower bound

$$k = \lceil \log_2 n \rceil + 1 \tag{6}$$

Similarly, for the upper bound, loop terminates when

$$i_k \geq n \tag{7}$$
$$3^k \geq n \tag{8}$$
$$\log_3 3^k \geq \log_3 n \tag{9}$$
$$k \geq \log_3 n \tag{10}$$

Using the fact, we can calculate that for upper bound,

$$k = \lceil \log_3 n \rceil + 1 \tag{11}$$

Part 4 (Determining Big-Oh and Omega):

Because we know $\log_2 n$ dominates $\log_3 n$, we can conclude $\log_2 n$ is the asymptotic upper bound, and $\log_3 n$ is the asymptotic lower bound.

Then, we can conclude the algorithm has $\mathcal{O}(\log_2 n)$ and $\Omega(\log_3 n)$.

Question 2

Question 3