CSC236 Worksheet 7 Solution

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Question 1

- First, I need to find the value of k.

  The definition tells us k is the non-recursive cost.

  Since the non-recursive part of call occurs when $len(s) < 2$ and it returns the input as output, it has cost of 1.

  Second, I need to find the value of b.

  The definition tells us b is the number of almost-equal parts the input is divided into.

  Since the input s is divided into three roughly equal parts, we can conclude $b = 3$.

  Third, I need to find the value of a.

  The definition tells us a is the number of recursive calls.

  Since the recursive calls in this problem are $r(s_1)$, $r(s_2)$ and $r(s_3)$, there are three of them, so $a = 3$.

  Fourth, I need to find the value of f.

  The definition tells us f is the cost of splitting and recombining

  Since the cost of splitting and recombining is $\text{len}(s_3) + \text{len}(s_2) + \text{len}(s_1) = n$, the value of f is n.

  Fifth, I need to evaluate asymptotic time complexity of function r using master's theorem.

Since $f \in \Theta(n^d)$ where $d = 1$ and $a = 3 = 3^d = b^d$, the master's theorem tells us $r(s) \in \Theta(\text{len}(s) \log_3 \text{len}(s))$.

Finally, I need to compare its time complexity to copying the string elements in reverse order, using loop.

In comparison to $\Theta(\text{len}(s) \log_3 \text{len}(s))$ by divide and conquer method, copying the string elements has cost of $\Theta(n)$.

Notes:

- **Divide and Conquer:** Partitions problem into b roughly equal subproblems, solve, and recombine:

$$T(n) = \begin{cases} k & \text{if } n \leq B \\ a_1 T(\lceil n/b \rceil) + a_2 T(\lfloor n/b \rfloor) + f(n) & \text{if } n > B \end{cases} \tag{1}$$

$$T(n) = \begin{cases} k & \text{if } n \leq B \\ aT(n/b) + f(n) & \text{if } n > B \end{cases} \tag{2}$$

where $b, k > 0$, $a_1, a_2 \geq 0$, and $a = a_1 + a_2 > 0$. $f(n)$ is the cost of slptting and recombining.

Note:

k : non-recursive cost, when $n < b$

b : number of almost-equal parts we divide problem into

a_1 : number of recursive calls to ceiling

a_2 : number of recursive calls to floor

a : number of recursive calls

f : cost of splittig and later recombining (should be n^d for master theorem)

- **Divide and Conquer Master Theorem:**

If $f \in \Theta(n^d)$, then

$$T(n) \in \begin{cases} \Theta(n^d) & \text{if } a \leq b^d \\ \Theta(n^d \log_b n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases} \tag{3}$$

- The master theorem is for master method.
- The master method provides a cookbook method for solving recurrences of the form

$$T(n) = aT(n/b) + f(n) \tag{4}$$

where $a \geq 1$ and $b > 1$.

Question 2

- The recurrence of a ternary version of merge sort is

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ T(\lceil n/3 \rceil) + T(\lfloor n/3 \rfloor) + n & \text{if } n > 1 \end{cases} \tag{1}$$

Thus, I know the algorithm divides the "problem" into 3 equal parts, calls the function recursively on those input once on the ceiling and once on the floor with the total of 2, takes time proportional to n to split and later recombine the problem.

Thus, $b = 3$, $a = 2$, $f = n$, and $n \in \mathcal{O}(n = n^1 = n^d)$.

So, since $a < b = b^1 = b^d$, the master's theorem tells us the alogirhtmic time complexity of a ternary version of merge sort is $\Theta(n)$.

Correct Solution:

A ternary version of merge sort works as follows

a. If $n < 3$, then the function terminatees, and a constant time is taken.
b. If $n \geq 3$, then the input A is divided into roughly 3 equal parts: A_1, A_2, A_3.
c. Then, the recursion is performed on each of A_1, A_2 and A_3
d. Time proportional to $n = \text{len}(A)$ is taken to divide, sort, and recombine the result
e. The final sorted segments are merged at the end to re-form A and return as output.

Thus, I know the algorithm divides the "problem" into 3 equal parts, calls the function recursively 3 times on those input, takes time proportional to $n = \text{len}(A)$ to split and later recombine the problem.

Thus, $b = 3$, $a = 3$, $f = n$, and $n \in \mathcal{O}(n = n^1 = n^d)$.

Since $a = b = b^1 = b^d$, the master's theorem tells us the alogirhtmic time complexity of a ternary version of merge sort is $\Theta(n \log_3 n)$.

Notes:

- I feel I jumped to conclusion.
- Realized I should first examine the algorithm before replacing b in recurrence $T(n)$.

Question 3

- The algorithm divides "problem" into 2 roughly equal parts, calls function recursively 1 time, i.e *return* $bis(f, a, (a+b)/2, \gamma, \delta)$, takes time proportional to $n = \lceil |b-a| \rceil / \gamma$ for the splitting.

  Thus, $a = 1$, $b = 2$, $f = n$, $n \in \mathcal{O}(n = n^1 = n^d)$.

  Since $a < b = b^1 = b^d$, the master's theorem tells us the time complexity of *bis* is $\Theta(n)$

Correct Solution:

The algorithm divides "problem" into 2 roughly equal parts, calls function recursively 1 time, i.e *return* $bis(f, a, (a+b)/2, \gamma, \delta)$, and the function performs 1 recursive call.

Thus, $a = 1$, $b = 2$, $f = n$, $n \in \mathcal{O}(1 = n^0 = n^d)$.

Since $a = b^0 = b^d$, the master's theorem tells us the time complexity of *bis* is $\Theta(\lg n)$.