CSC373 Worksheet 5 Solution

August 9, 2020

1. *Proof.* Assume that a flow network $G = (V, E)$ violates the assumption that the network contains a path $s \rightsquigarrow v \rightsquigarrow t$ for all vertices $v \in V$. Let u be a vertex for which there is no path $s \rightsquigarrow u \rightsquigarrow t$.

   I must show such that there is no flow at vertex u. That is, there exists a maximum flow f in G such that $f(u, v) = f(v, u) = 0$ for all vertices $v \in V$.

   Assume for the sake of contradiction that there is some vertex u with flow f. That is, there exists some vertices $v \in V$ such that $f(u, v) > 0$ or $f(v, u) > 0$.

   I see that three cases follows, and I will prove each separately.

   1. **Cases 1:** $f(u, v) = 0$ and $f(v, u) > 0$

      Here, assume that $f(u, v) = 0$ for all $v \in V$ and $f(v, u) > 0$ for some $v \in V$.

      Then, we can write $\sum\limits_{v \in V} f(u, v) = 0$ and $\sum\limits_{v \in V} f(v, u) > 0$

      But this violates the flow conservation property (i.e $\sum\limits_{v \in V} f(u, v) = \sum\limits_{v \in V} f(v, u)$)

      Thus, by proof by contradiction, $f(u, v) = 0$ and $f(v, u) = 0$ for all $v \in V$ and all $u \in V$ with no path $s \rightsquigarrow u \rightsquigarrow t$.

   2. **Cases 2:** $f(u, v) > 0$ and $f(v, u) = 0$

      Here, assume that $f(u, v) > 0$ for some $v \in V$ and $f(v, u) = 0$ for all $v \in V$.

      Then, by similar work as case 1, the same result follows.

3. **Cases 3:** $f(u, v) > 0$ and $f(v, u) > 0$

   Here, assume that $f(u, v) > 0$ and $f(v, u) > 0$ for some $v \in V$.

   Since $s \rightsquigarrow v \rightsquigarrow t$ and u is connected by some vertices v, we can write $s \rightsquigarrow u \rightsquigarrow t$.

   Then, this violates the fact in header that the vertex u has no path $s \rightsquigarrow u \rightsquigarrow t$.

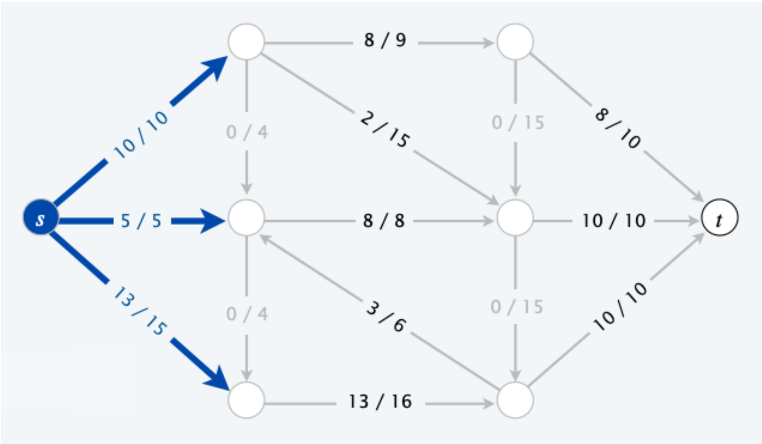
   Thus, by proof by contradiction, $f(u, v) = 0$ and $f(v, u) = 0$ for all $v \in V$ and all $u \in V$ with no path $s \rightsquigarrow u \rightsquigarrow t$.

$\square$

Notes

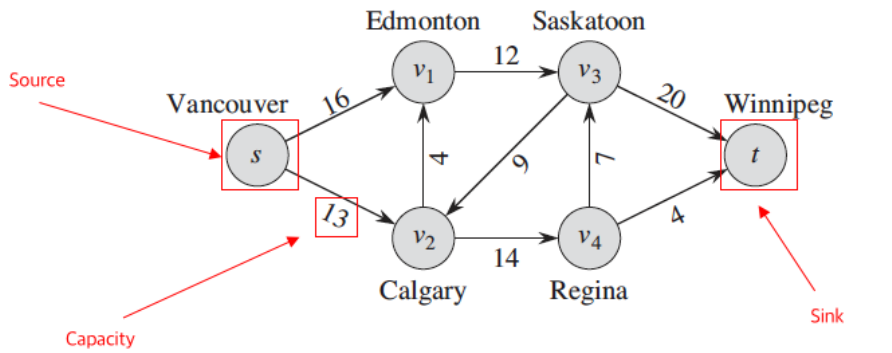
- **Maximum Flow:**
  - Finds a flow of maximum value [1]

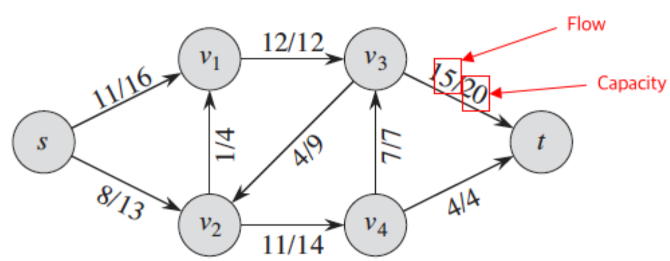
    Example



    Here, the maximum flow is $10 + 5 + 13 = 28$

- **Flow Network:**
  - $G = (V, E)$ is a directed graph in which each edge $(u, v) \in E$ has a nonnegative capacity $c(u, v) \geq 0$.
  - Two vertices must exist: **source** s and **sink** t
  - **path** from source s to vertax v to sink t is represented by $s \rightsquigarrow v \rightsquigarrow t$





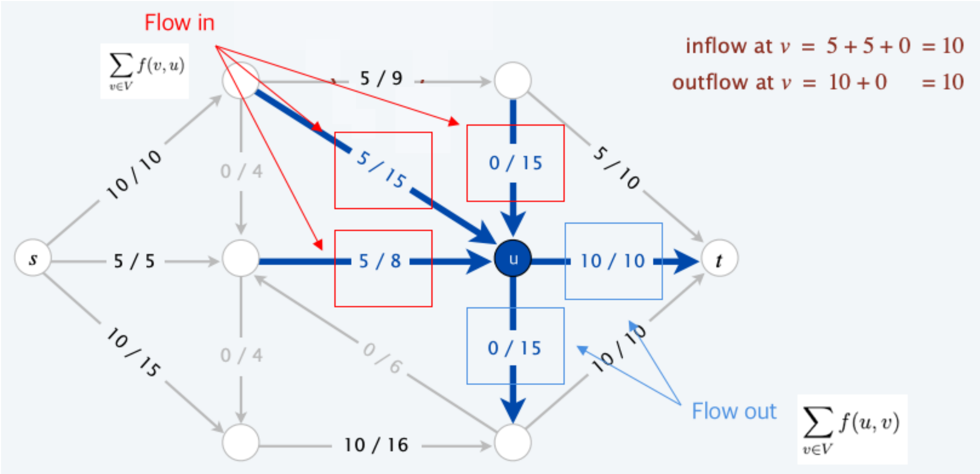
- **Capacity:**
  - Is a non-negative function $f : V \times V \to \mathbb{R}_{\geq 0}$
  - Has **capacity constraint** where for all $u, v \in V$ $0 \leq f(u, v) \leq c(u, v)$
    * Means flow cannot be above capacity constraint
- **Flow:**
  - Is a real valued function $f : V \times V \to \mathbb{R}$ in G
  - Satisfies **capacity constraint** (i.e for all $u, v \in V$, $0 \leq f(u, v) \leq c(u, v)$)
  - Satisfies **flow conservation**

    For all $u \in V - \{s, t\}$, we require

$$\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v) \tag{1}$$

    * Means flow into vertex u is the same as flow going out of vertex u. [1]
    * $\sum_{v \in V} f(u, v)$ means flow <u>out of</u> vertex u
    * $\sum_{v \in V} f(v, u)$ means flow <u>into</u> vertex u
    * $v \in V$ in $\sum_{v \in V} f(u, v)$ means all vertices that are an edge away from vertex u

<u>Example:</u>



<u>**References**</u>

1) Princeton University, Network Flow 1, link

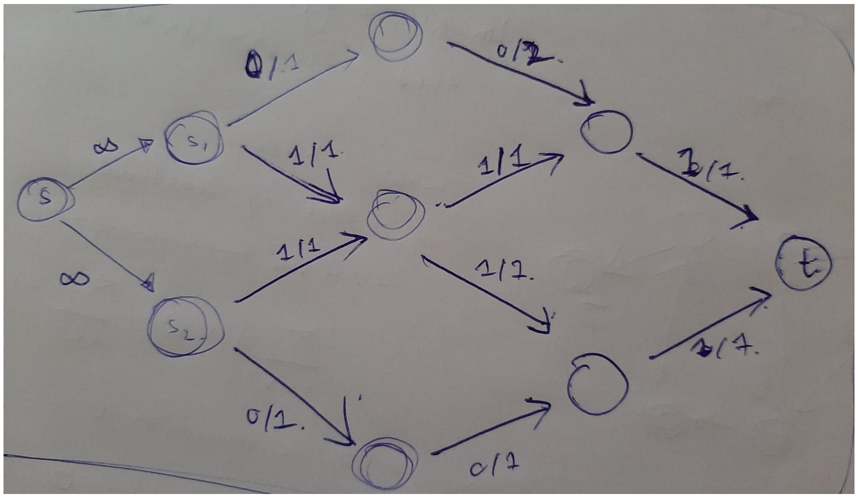
2. I need to formulate the problem of determining whether both of professor Adam's two children can go to the same school as maximum-flow problem.

   The problem statement tells us the following:

   1. There is 1 supersource (location of home)
   2. There is 1 sink (location of school)
   3. There are two sources (s_1 as child 1, s_2 as child 2)
   4. Edge (u, v) has capacity of 0 or more (0 representing unavailable sidewalk, 1 for sidewalk with capacity of 1, 2 for street with capacity of 2 and so on)
   5. Each vertex represents corner of intersection, and two children can have their paths crossing here.
   6. Has flow of 2, 1 or 0 (1 is where one of the two children walking on the road. 0 is none.)

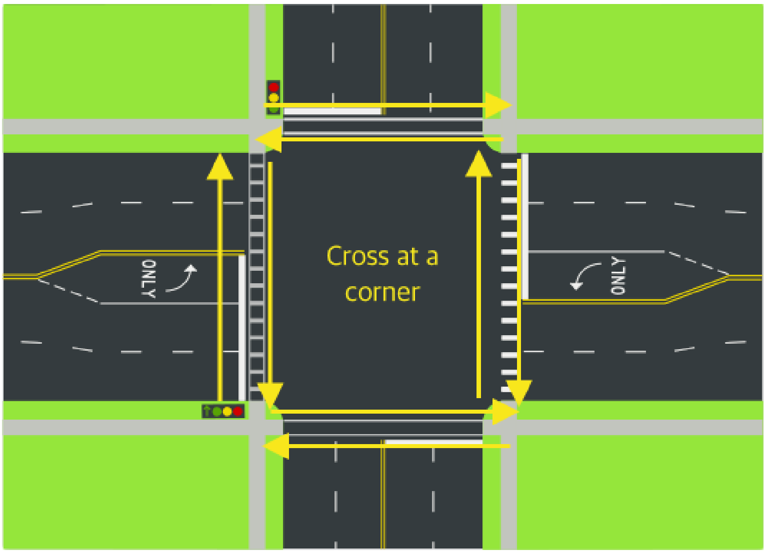
   Here we are to find whether children must go on to a vertex and out to the same edge with the flow of 2, or determine whether there is only edge to school with capacity of 1 or less.

   If none, then both children can safely go to school.



Notes:

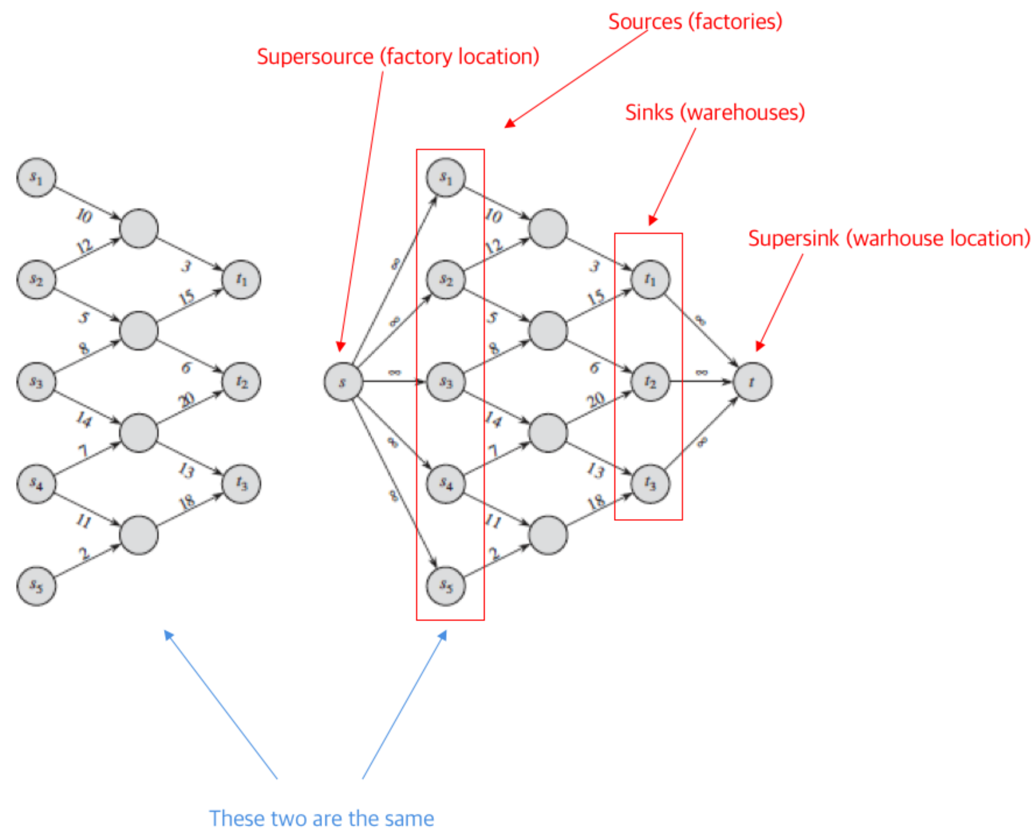
- **Cross at a Corner**
  - Means to walk across the street at a corner of the intersection.



- **Multiple Sources and Sinks**
  - Has edges (s, s_i) where $i = 1...n$ and (t_j, t) where $j = 1...n$ with capacity of ∞

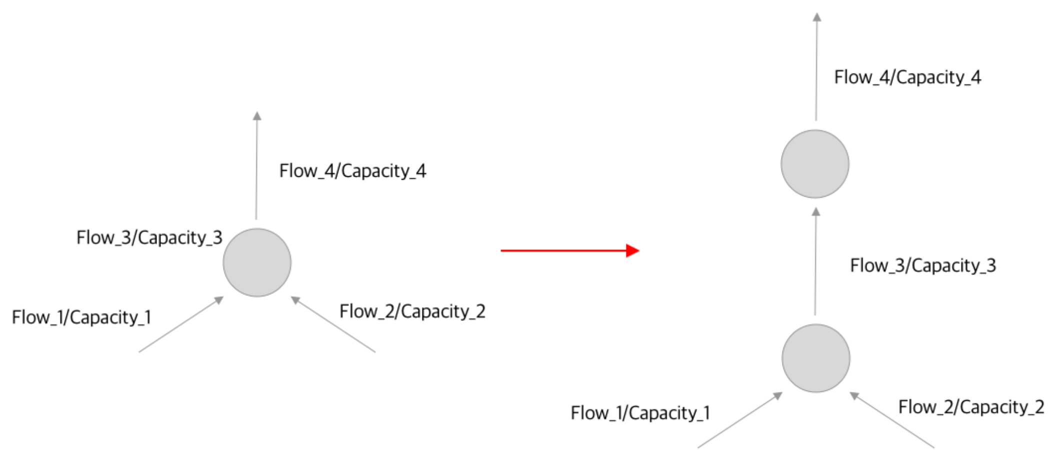
Example:

Lucky Puck Company having a set of m factories $\{s_1, s_2, ..., s_m\}$, and a set of n warehourses and n warehouses $\{t_1, t_2, ..., t_n\}$



3. I need to show how to transform a flow network $G = (V, E)$ with vertex capacities into an equivalent flow network $G' = (V', E')$ without vertex capacities.

   For each vertex capacities, change as follows.



   After transformation, there will be m more edges and verticies, where m represents the number of vertex capacities in G.

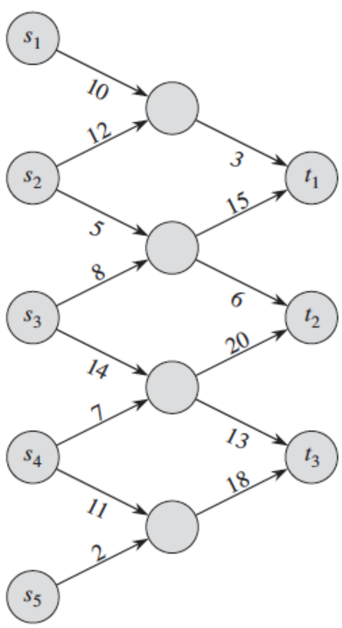
   Notes:

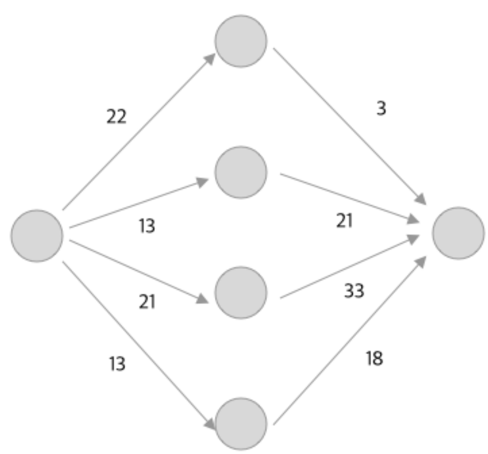
- **Vertex Capacities**
  - Each vertex v has limit $l(v)$ on how much flow can pass through v

4. I need to show how to convert the problem of finding a flow f that obeys the constraints into the problem of finding a maximum flow in a single source, single-sink flow network

   The steps are as follows:

   - Combine all sources s_i into a single source s
   - Combine all sinks t_j into a single sink t
   - Connect source s to each adjacent vertex v with edge weight $\sum_i f(s_i, v) = p_i$
     - The total edge weight from s should be $\sum_i p_i$
   - Connect each adjacent vertex v of t to t with edge weight $\sum_j f(v, t_j) = q_j$
     - The total edge weight to t should be $\sum_j q_j$
   - Find a simple path from s to t with the maximum amount of total flow





Correct Solution:

I need to show how to convert the problem of finding a flow f that obeys the constraints into the problem of finding a maximum flow in a single source, single-sink flow network

The steps are as follows:

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  - The total edge weight from s should be $\sum_i p_i$
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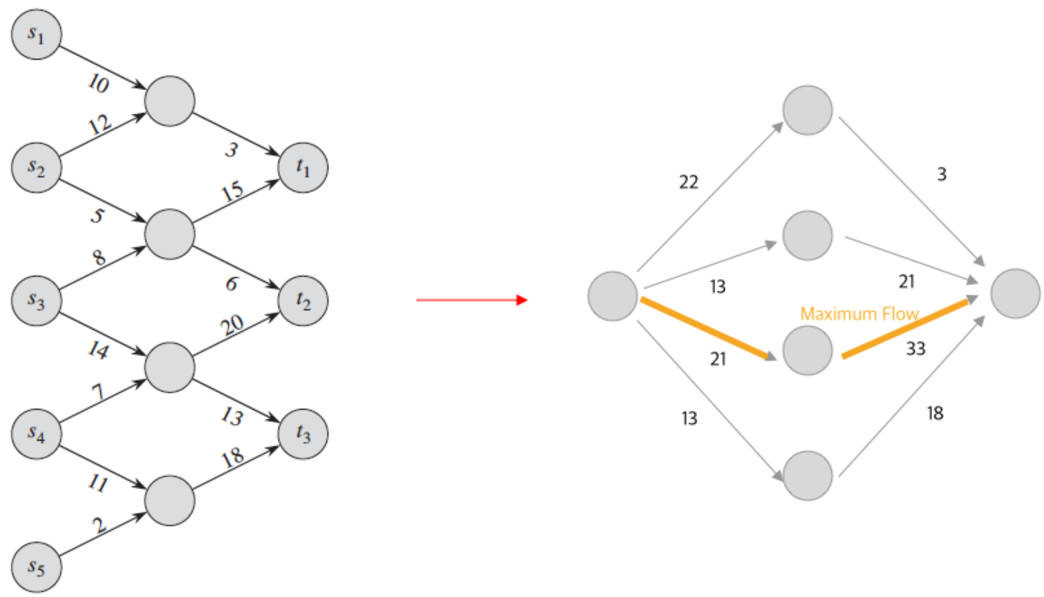
- Find a simple path from s to t with the maximum amount of total flow

Example

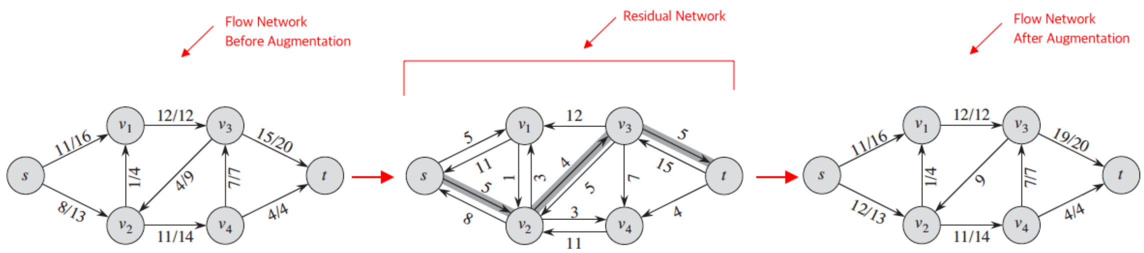


Notes:

- **Ford-Fulkerson Method**
  - Is a greedy algorithm that solves the maximum-flow problem
    - Determines maximum flow from start vertex to sink vertex in a graph
  - Called method (not algorithm) because several different implementations with different running time is used

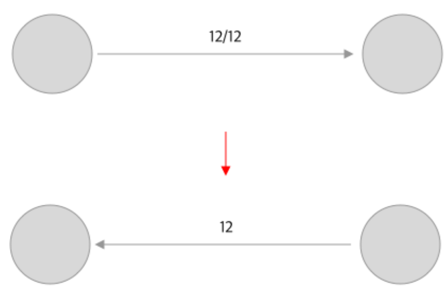
```
FORD-FULKERSON-METHOD(G, s, t)
initialize flow f to 0
while there exists an augmenting path p in the residual network G_f
    augment flow f along p
return f
```

- **Residual Network**
  - Indicates how muh more flow is allowed in each edge in the network graph [1]
  - Consists of edges with capacities that represents how we can change the flow on edges of G.
  - Provides roadmap for adding flow to the original flow network

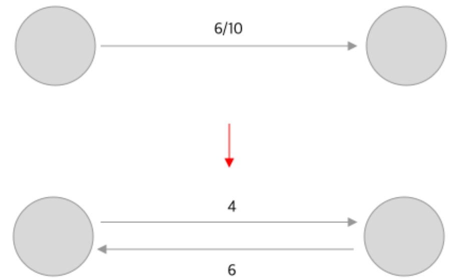


Steps

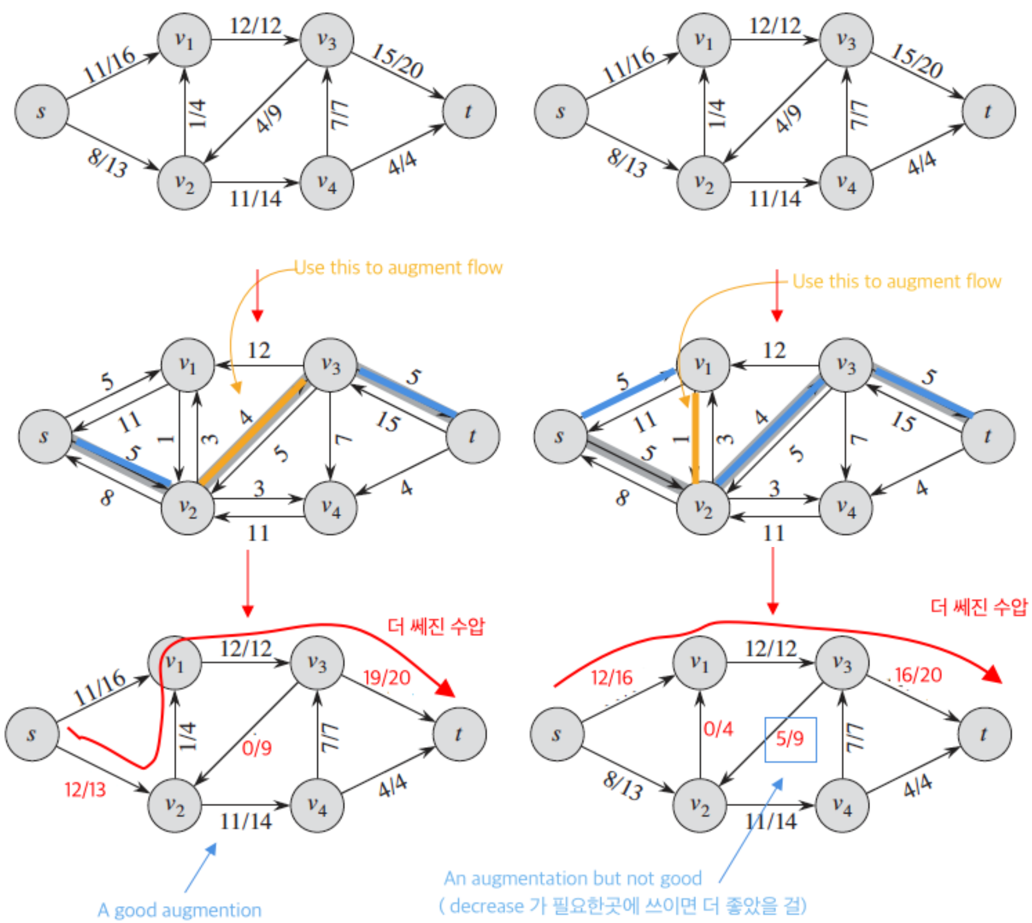
1) $Flow = Capacity$: Opposite arrow



2) $Flow < Capacity$:
   - $Flow$: Oppisite Arrow
   - $Capacity - Flow$: Current Arrow



- **Augmenting Path**
  - Is a path from source S to sink T where you can increase the amount of flow
  - Is a path that doesn't contain cycle (simple path) [2]



  - Edge (u, v) on simple path on an augmenting path can be increased by upto $c_f(u, v)$ withhout violating the capacity constraint
- **Augmentation**
  - 한국어로 '불필요한 수압 decrease 해서 앞으로 가는 수압 더 쎄게 만들기'
  - Is symbolized by $f \uparrow f'$

  * f is a flow in G
  * f' is a flow in the residual network G_f

References

1) Hacker Earth, Maximum Flow, link

2) Stack Overflow, What Exactly Is Augmentation Path, link

5. **Rough Works:**

I need to show if the augmented flow of f and $f' \in G$ and satisfy the flow conservation property and capacity constraint.

- Proving the flow conservation property

  1)

- Proving the capacity constraint

Notes:

- **Augmentation (cont'd)**
  - Augmentation of flow f by f' or $f \uparrow f'$ is a function $V \times \to \mathbb{R}$ is defined by