CSC236 Term Test 1 Version 2 Review

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Question 1

- *Proof.* Define $P(n) : f(n) = 3^n$.

  I will use complete induction to prove that $\forall n \in \mathbb{N}, P(n)$.

  Base Case ($n = 0$):

  Let $n = 0$.

  Then,

  $$\begin{align} f(n) &= 1 && \text{[By def.]} \tag{1}\\ &\leq 3^0 \tag{2}\\ &= 3^n \tag{3}\end{align}$$

  Thus, $P(n)$ follows in this step.

  Base Case ($n = 1$):

  Let $n = 1$.

  Then,

  $$\begin{align} f(n) &= 1 && \text{[By def., since } n = 1\text{]} \tag{4}\\ &\leq 3^1 \tag{5}\\ &= 3^n \tag{6}\end{align}$$

  Thus, $P(n)$ follows in this step.

Base Case ($n = 2$):

Let $n = 2$.

Then,

$$\begin{aligned} f(n) &= 9 && \text{[By def., since } n = 2\text{]} && (7) \\ &\leq 3^2 && && (8) \\ &= 3^n && && (9) \end{aligned}$$

Thus, $P(n)$ follows in this step.

Base Case ($n = 3$):

Let $n = 3$.

Then,

$$\begin{aligned} f(n) &= f(n-1) + 3f(n-2) + \\ &\quad 9f(n-3) && \text{[By def., since } n = 2\text{]} && (10) \\ &= 9 + 3 \cdot 3 + 9 \cdot 1 && \text{[By def., since } n-1 = 2,\ n-2 = 1,\ n-3 = 0\text{]} && (11) \\ &= 3^2 + 3^2 + 3^2 && && (12) \\ &= 3^3 && && (13) \\ &= 3^n && && (14) \\ &\leq 3^n && && (15) \end{aligned}$$

Thus, $P(n)$ follows in this step.

Case ($n > 3$):

Let $n > 3$.

Then, since $0 \leq n - 3 < n - 2 < n - 1 < n$, $P(n-3)$, $P(n-2)$, $P(n-1)$ holds by induction hypothesis. That is, $P(n-3) \leq 3^{n-3}$, $P(n-2) \leq 3^{n-2}$, $P(n-1) \leq 3^{n-1}$.

Thus,

$$
\begin{aligned}
f(n) &= f(n-1) + 3f(n-2) + 9f(n-3) && \text{[By def., since } n > 2\text{]} \quad (16)\\
&\leq 3^{n-1} + 3 \cdot 3^{n-2} + 9 \cdot 3^{n-3} && \text{[By header]} \quad (17)\\
&= 3^{n-1} + 3^{n-1} + 3^{n-1} && (18)\\
&= 3^n && (19)
\end{aligned}
$$

So, $P(n)$ follows from $H(n)$ in this step.

$\square$

Question 2

- **Rough Works:**

Define for convenience

$$P(x, y, z, w) : \text{There are no positive integers } x, y, z, w \text{ such that } x^4 + 3y^4 + 9z^4 = 27w^4.$$

I will prove $P(x, y, z, w)$ by contradiction.

Assume $\neg P(x, y, z, w)$. That is, $\exists x, y, z, w \in \mathbb{N}^+$, $x^4 + 3y^4 + 9z^4 = 27w^4$.

Then, $X = \{x \in \mathbb{N}^+ \mid \exists y, z, w \in \mathbb{N}^+, x^4 + 3y^4 + 9z^4 = 27w^4\}$ is not empty.

Then, by the principle of well-ordering, X has smallest element.

Let $x_0 \in X$ be its smallest element, and let $y_0, z_0, w_0 \in \mathbb{N}^+, x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4$.

Then,