CSC236 Worksheet 2 Solution

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Question 1

- **Statement:** Any full binary tree with at least 1 node has more leaves than internal nodes.

Proof. Let n be the total number of nodes in a full binary tree.

We will prove the statement by complete induction on n.

Base Case ($n = 1$):

Let $n = 1$.

We need to prove the full binary tree with 1 total number of nodes has more leaves than internal nodes.

There is only one full binary tree exists with 1 total number of node. That is, the full binary tree with 1 child node and 0 internal nodes.

Then, since we know a node is a leaf if it has no children, and since we know the child node has 0 children, we can write the child node is a leaf node.

Then, because we know the full binary tree has 1 leaf node, and 0 internal node, we can conclude it has more leaves than internal nodes.

Base Case ($n = 2$):

Let $n = 2$.

We need to prove the full binary tree with 2 total number of nodes has more leaves than internal nodes.

The definition of full binary tree tells us that a binary tree is a full binary tree if every internal node has two children.

Since we know the tree with 2 total number of nodes has 1 leaf and 1 internal node, using above fact, we can write the tree is not a full binary tree.

Then, by vacuous truth, we can conclude the tree has more leaves than internal nodes.

Base Case ($n = 3$):

Let $n = 3$.

We need to prove the full binary tree with 3 total number of nodes has more leaves than internal nodes.

The definition of full binary tree tells us that a binary tree is a full binary tree if every internal node has two children.

Because we know there are two types of binary trees possible, one is the tree with 2 internal nodes and 1 child and the other is 1 internal node and 2 children, using above fact, we can write the only possible full binary tree with 3 total nodes is 1 internal node and 2 children.

Now, the definition of leaf tells us leaf is a node that has no children.

Because we know by observation that the 2 child nodes don't have children, we can write the full binary tree has 2 leaves.

So, because we know the full binary tree has 1 internal node and 2 leaves, we can conclude the full binary tree has more leaves than internal node.

Inductive Step:

Let $k \geq 1$ be an arbitrary natural number. Assume that for all natural number i satisfying $1 \leq i \leq k$, any full binary trees with i total number of nodes has more leaves than internal nodes.

Let T be an arbitrary full binary tree with $k + 1$ nodes. Let T' be the binary tree obtained by removing 2 leaves from the same parent node.

Let ℓ be the number of leaves of T, and m be the number of internal nodes of T. Similarly, let ℓ' be the number of leaves of T' and m' be the number of internal nodes of T'. We must prove $l > m$.

First, we need to show $\ell' > m'$.

The header tells us that T' is a full binary tree as a result of removing 2 leaves from the parent node of T.

Using this fact, we can calculate T' has

$$k + 1 - 2 = k - 1 \tag{1}$$

nodes.

Then, because we know $1 \leq k - 1 \leq k$, using induction hypothesis, we can write

$$\ell' > m' \tag{2}$$

Second, we need to show $\ell = \ell' + 1$ and $m = m' + 1$.

The total number of both the leaf nodes and the internal nodes increase by 1 when 2 nodes are added to the same leaf node in a full binary tree.

Since 2 nodes added to the same leaf node of T' is T, we can write $\ell = \ell' + 1$ and $m = m' + 1$.

Finally, putting together, because we know $\ell' > m'$, $\ell = \ell' + 1$ and $m = m' + 1$, we can conclude

$$\ell' + 1 > m' + 1 \tag{3}$$
$$\ell > m \tag{4}$$

$\square$

Notes:

- Complete Induction
  * **Statement:** $\forall i \in \mathbb{N},\ \forall n \in \mathbb{N},\ n < i \Rightarrow A(n) \Rightarrow \forall i \in \mathbb{N},\ A(i)$
  * **Statement Alt.:** $\left(\forall n \in \mathbb{N},\ \left[\bigwedge_{k=0}^{k=n-1} P(k)\right] \Rightarrow P(n)\right) \Rightarrow \forall n \in \mathbb{N}, P(n)$

* **Simple Example 1:**

<u>**Statement:**</u> $\forall n \in \mathbb{N},\ n \geq 0 \Rightarrow 10 \mid (n^5 - n)$

We will prove the statement by strong induction on n.

1. Base Case ($n = 0$)

   Let $n = 0$.

   We need to prove $10 \mid (n^5 - n)$ is true when $n = 0$. That is, there exists $k \in \mathbb{Z}$ such that $(n^5 - n) = 10k$.

   Let $k = 0$.

   Starting from the left hand side, using the fact $n = 0$, we can write

   $$(n^5 - n) = 0 \tag{5}$$

   Then, because we know $10k = 0$, we can conclude

   $$(n^5 - n) = 10k \tag{6}$$

2. Base Case ($n = 1$)

Let $n = 1$.

We need to prove $10 \mid (n^5 - n)$ is true when $n = 1$. That is, there exists $k \in \mathbb{Z}$ such that $(n^5 - n) = 10k$.

Let $k = 0$.

Starting from the left hand side, using the fact $n = 0$, we can write

$$(n^5 - n) = 1 - 1 \tag{7}$$
$$= 0 \tag{8}$$

Then, because we know $10k = 0$, we can conclude

$$(n^5 - n) = 10k \tag{9}$$

3. Inductive Step

Assume $k \geq 1$. Assume that for all natural number i satisfying $0 \leq i \leq k$, $10 \mid (i^5 - i)$. That is, $\exists d \in \mathbb{Z}$, $(i^5 - i) = 10d$.

We need to prove $\exists \tilde{d} \in \mathbb{Z}$ such that $((k+1)^5 - (k+1)) = 10\tilde{d}$.

Let $\tilde{d} = c + (k-1)^4 + 4 \cdot (k-1)^3 + 8 \cdot (k-1)^2 + 8 \cdot (k-1) + 3$.

Starting from $((k+1)^5 - (k+1))$, using binominal theorem, we can write,

$$
\begin{aligned}
(k+1)^5 - (k+1) &= \Big[(k-1) + 2\Big]^5 - \Big[(k-1) + 2\Big] && (10)\\
&= \sum_{b=0}^{5} \binom{5}{b} (k-1)^{5-b} \cdot 2^b && (11)\\
&= (k-1)^5 + 10 \cdot (k-1)^4 + 40 \cdot (k-1)^3 + \\
&\quad 80 \cdot (k-1)^2 + 80 \cdot (k-1) + 32 - \Big[(k-1) + 2\Big] && (12)\\
&= \Big[(k-1)^5 - (k-1)\Big] + 10 \cdot (k-1)^4 + \\
&\quad 40 \cdot (k-1)^3 + 80 \cdot (k-1)^2 + 80 \cdot (k-1) + 30 && (13)
\end{aligned}
$$

(The reason why $k-1$ is chosen instead of $k-2$ and $k-3$ is because of the last term $2^5 = 32$, i.e $32 - 2 = 30$)

Then, because we know $0 \leq k-1 \leq k$ and $10 \mid (k-1)^5 - (k-1)$ from the header, we can write $\exists c \in \mathbb{Z}$ such that $(k-1)^5 - (k-1) = 10c$, and

$$
\begin{aligned}
(k+1)^5 - (k+1) &= 10c + 10 \cdot (k-1)^4 + 40 \cdot (k-1)^3 + \\
&\quad 80 \cdot (k-1)^2 + 80 \cdot (k-1) + 30 && (14)\\
(k+1)^5 - (k+1) &= 10 \cdot \Big[c + \cdot (k-1)^4 + 4 \cdot (k-1)^3 + \\
&\quad 8 \cdot (k-1)^2 + 8 \cdot (k-1) + 3\Big] && (15)\\
& && (16)
\end{aligned}
$$

Then, because we know $\tilde{d} = c + (k-1)^4 + 4 \cdot (k-1)^3 + 8 \cdot (k-1)^2 + 8 \cdot (k-1) + 3$ from the header, we can conclude

$$
(k+1)^5 - (k+1) = 10\tilde{d} \tag{17}
$$

Question 2

- *Proof.* Let $P(n)$ be the predicate defined as follows

  $P(n)$: Postage of exactly n cents can be made using only 3-cent and 4-cent stamps

  We will use complete induction to prove the statement holds for $n \geq 13$.

  Base Case ($n = 13$):

  Let $n = 13$.

  We need to prove the statement is true for $n = 13$. That is, the postage of exactly 13 cents can be made using only 3-cent and 4-cent.

  Because we know $(3 \cdot 3) + (1 \cdot 4) = 13$, we can conclude the statement holds.

  Base Case ($n = 14$):

  Let $n = 14$.

  We need to prove the statement is true for $n = 14$. That is, the postage of exactly 14 cents can be made using only 3-cent and 4-cent.

  Because we know $(2 \cdot 3) + (2 \cdot 4) = 14$, we can conclude the statement holds.

  Base Case ($n = 15$):

  Let $n = 15$.

  We need to prove the statement is true for $n = 15$. That is, the postage of exactly 15 cents can be made using only 3-cent and 4-cent.

  Because we know $(1 \cdot 3) + (3 \cdot 4) = 15$, we can conclude the statement holds.

  Base Case ($n = 16$):

  Let $n = 16$.

We need to prove the statement is true for $n = 16$. That is, the postage of exactly 16 cents can be made using only 3-cent and 4-cent.

Because we know $(4 \cdot 3) + (1 \cdot 4) = 16$, we can conclude the statement holds.

Base Case ($n = 17$):

Let $n = 17$.

We need to prove the statement is true for $n = 17$. That is, the postage of exactly 17 cents can be made using only 3-cent and 4-cent.

Because we know $(3 \cdot 3) + (2 \cdot 4) = 17$, we can conclude the statement holds.

Inductive Step:

Let $i \in \mathbb{N}$ such that $i \geq 13$. Suppose that $P(i)$ holds. That is, the postage of exactly i cents can be made using only 3-cent and 4-cent stamps. In other words, $\exists k, \ell \in \mathbb{N}$, $k \cdot 3 + \ell \cdot 4 = i$.

We need to prove the statement is true for $P(i+1)$. That is, the postage of exactly $i+1$ cents can be made using only 3-cent and 4-cent stamps. In other words, we need to prove $\exists k', \ell' \in \mathbb{N}$, $3k' + 4\ell' = i + 1$. There are two cases: $\ell > 0$ or $\ell = 0$.

We will use proof by cases.

Case 1 ($\ell > 0$):

Assume $\ell > 0$.

We need to prove $\exists k',\ \ell' \in \mathbb{N}$, $3k' + 4\ell' = i + 1$.

Let $k' = k + 3$ and $\ell' = \ell - 2$ (where $\ell - 2$ is possible since $\ell > 0$).

Starting from the left hand side, using the facts $k' = k + 3$ and $\ell' = \ell - 2$, we can write

$$\begin{aligned}
3k' + 4\ell' &= (k+3) \cdot 3 + (\ell - 2) \cdot 4 && (1)\\
&= 3 \cdot k + 9 + 4 \cdot \ell - 8 && (2)\\
&= 3 \cdot k + 4 \cdot \ell + 1 && (3)\\
&= (3 \cdot k + 4 \cdot \ell) + 1 && (4)
\end{aligned}$$

Then, using induction hypothesis, i.e. $k \cdot 3 + \ell \cdot 4 = i$, we can conclude

$$3k' + 4\ell' = i + 1 \tag{5}$$

Case 2 ($\ell = 0$):

First, we need to choose the value of k'.

The header tells us

$$3 \cdot k + 4 \cdot \ell = i \tag{6}$$

Using the fact $\ell = 0$, we can write

$$\begin{align} 3 \cdot k &= i \tag{7} \\ k &= \frac{i}{3} \tag{8} \end{align}$$

Then, because we know $i \geq 18$, we can write $k \geq 6$.

Then, since k' must be a natural number and $k \geq 6$, let $k' = k - 5$.

Second, we need to choose the value of ℓ'.

Since we know $\ell = 0$, and since we want the total to increase from i by 1 in $3 \cdot k' + 4 \cdot \ell$, let $\ell' = 4$.

Finally, starting from the left, using the facts $k' = k - 5$ and $\ell' = 4$, we can write

$$\begin{align} 3k' + 4\ell' &= (k - 5) \cdot 3 + 4 \cdot 4 \tag{9} \\ &= 3k - 15 + 16 \tag{10} \\ &= 3k + 1 \tag{11} \end{align}$$

Then, by the fact $4\ell = \ell = 0$, we can write

$$3k' + 4\ell' = 3k + 4\ell + 1 \tag{12}$$
$$= (3k + 4\ell) + 1 \tag{13}$$

Then, by using inductive hypothesis, $3k + 4\ell = i$, we can conclude

$$3k' + 4\ell' = i + 1 \tag{14}$$

$\square$

Notes:

- Noticed professor's solution is much shorter
- Noticed professor's solution uses inductive step before base case

  inductive step: Let $n \in \mathbb{N}$ and assume $n \geq 6$. Assume $H(n) : \bigwedge_{i=6}^{n-1} C(i)$. I will show that $C(n)$ follows, that postage of n cents can be made using only 3- and 4- cent stamps.

  base case $n = 6$: Use two 3-cent stamps. So $C(n)$ follows in this case.

  base case $n = 7$: Use one 3-cent and one 4-cent stamps. So $C(n)$ follows in this case.

  base case $n = 8$: Use two 4-cent stamps. So $C(n)$ follows in this case.

  $n \geq 9$: Since $9 \leq n$, $6 \leq n - 3 < n$, so we know $C(n-3)$, postage of $n - 3$ cents can be made using 3- and 4-cent stamps. Let k and j be integers such that $n - 3 = 3k + 4j$. Adding 3 to both sides yields $n = 3(k+1) + 4j$, so $C(n)$ follows in this case.
  So $C(n)$ follows from $H(n)$ in all possible cases ■

- Noticed professor's note uses **thus** and **in other words** to unwrap statement further.

  We will prove that $P(i+1)$ holds, i.e., that we can make i + 1 cents of postage using only 4-cent and 7-cent stamps. In other words, we must prove that there are k', $\ell' \in \mathbb{N}$ such that $4 \cdot k' + 7 \cdot \ell' = i + 1$.

Question 3

- *Proof.* Define $C(n) : f(n) \leq 3^n$.

  We will prove by complete induction that $\forall n \in \mathbb{N}, n \geq 2 \Rightarrow C(n)$.

  Base Case $(n = 0)$**:**

Let $n = 0$.

We need to prove $C(0)$ follows. That is, $f(0) \leq 3^0$.

The definition of $f(n)$ tells us that $f(0) = 1$.

Using this fact, we can conclude

$$\begin{aligned} f(0) = 1 &\leq 1 && (1) \\ &\leq 3^0 && (2) \end{aligned}$$

Base Case ($n = 1$):

Let $n = 1$.

We need to prove $C(n)$ follows. That is, $f(1) \leq 3$.

The definition of $f(n)$ tells us that $f(1) = 3$.

Using this fact, we can conclude

$$\begin{aligned} f(1) = 3 &\leq 3 && (3) \\ &\leq 3^1 && (4) \end{aligned}$$

Inductive Step:

Let $n \in \mathbb{N}$ and assume $n \geq 2$. Assume $H(n) : \bigwedge_{i=0}^{n-1} C(i)$.

We need to prove $C(n)$ follows. That is, $f(n) \leq 3^n$.

Since $2 \leq n$, $0 \leq n-1 < n$ and $0 \leq n-2 < n$, we can conclude $C(n-1)$ and $C(n-2)$ is true, i.e. $f(n-1) \leq 3^{n-1}$ and $f(n-2) \leq 3^{n-2}$.

Then, since we know from the definition of $f(n)$ that $f(n) = 2(f(n-1) + f(n-2)) + 1$, using it with above fact, we can write

$$\begin{aligned} f(n) = 2(f(n-1) + f(n-2)) + 1 &\leq 2(3^{n-1} + 3^{n-2}) + 1 && (5) \\ &= 2 \cdot 3^{n-2}(3+1) + 1 && (6) \\ &= 8 \cdot 3^{n-2} + 1 && (7) \end{aligned}$$

Then, since $n \geq 2$ and $3^{n-2} \geq 1$, we can conclude

$$
\begin{align}
f(n) &\leq 8 \cdot 3^{n-2} + 3^{n-2} \tag{8}\\
&= (8+1) \cdot 3^{n-2} \tag{9}\\
&= 9 \cdot 3^{n-2} \tag{10}\\
&= 3^2 \cdot 3^{n-2} \tag{11}\\
&= 3^{n-2+2} \tag{12}\\
&= 3^n \tag{13}
\end{align}
$$

$\square$

Notes:

- Noticed professor defined header for the inductive case on top, before base case

inductive step: Let $n \in \mathbb{N}$. Assume $H(n) : \bigwedge_{i=0}^{i=n-1} C(i)$. I will show that $C(n)$ follows, that is $f(n) \leq 3^n$.
base case $n = 0$: Then $f(n) = 1 \leq 3^0$, so $C(n)$ follows in this case.

- Noticed professor labeled the traditional inductive case as **Case** $(n > 1)$**:**