

# CSC373 Worksheet 1 Solution

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1. Strassen_Algorithm(A,B):
2     n = A.rows
3     let C be a new n x n matrix
4
5     if n == 1
6         C_11 = A_11 * B_11
7
8     else partition as in step 3 of strassen's algorithm
9
10        p1 = Strassen_Algorithm(A_11, B_12) -
11             Strassen_Algorithm(A_11, B_22)
12
13        p2 = Strassen_Algorithm(A_11, B_22) +
14             Strassen_Algorithm(A_12, B_22)
15
16        p3 = Strassen_Algorithm(A_21, B_11) +
17             Strassen_Algorithm(A_22, B_11)
18
19        p4 = Strassen_Algorithm(A_22, B_21) -
20             Strassen_Algorithm(A_22, B_11)
21
22        p5 = Strassen_Algorithm(A_11, B_11) +
23             Strassen_Algorithm(A_11, B_22) +
24             Strassen_Algorithm(A_22, B_11) +
25             Strassen_Algorithm(A_22, B_22)
26
27        p6 = Strassen_Algorithm(A_12, B_21) +
28             Strassen_Algorithm(A_12, B_22) -
29             Strassen_Algorithm(A_22, B_21) -
30             Strassen_Algorithm(A_22, B_22)
31
32        p7 = Strassen_Algorithm(A_11, B_11) +
33             Strassen_Algorithm(A_11, B_12) -
34             Strassen_Algorithm(A_21, B_11) -
35             Strassen_Algorithm(A_21, B_12)
36
37        C_11 = p5 + p4 - p2 + p6
38        C_12 = p1 + p2
39        C_21 = p3 + p4
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40         C_22 = p5 + p1 - p3 - p7
41
42     return C
43

```

**Notes:**

- Strassen's method for matrix multiplication
  - Reduces the time complexity of matrix multiplication from  $O(n^3)$  to  $O(n^{\log_2 7}) = O(n^{2.81})$
  - Has four steps

- 1) Divide the input matrices  $A$  and  $B$  and output matrix  $C$  into  $n/2 \times n/2$  submatrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$

- 2) Create 10 matrices,  $S_1, S_2, \dots, S_{10}$  each of which is  $n/2 \times n/2$  and is the sum or difference of two matrices created in step 1

$$\begin{aligned}
S_1 &= B_{12} - B_{22} \\
S_2 &= A_{11} + A_{12} \\
S_3 &= A_{21} + A_{22} \\
S_4 &= B_{21} - B_{11} \\
S_5 &= A_{11} + A_{22} \\
S_6 &= B_{11} + B_{22} \\
S_7 &= A_{12} - A_{22} \\
S_8 &= B_{21} + B_{22} \\
S_9 &= A_{11} - A_{21} \\
S_{10} &= B_{11} + B_{12}
\end{aligned}$$

- 3) Recursively multiply  $n/2 \times n/2$  matrices seven times to compute the following  $n/2 \times n/2$  matrices

$$\begin{aligned}
P_1 &= A_{11} \cdot S_1 = A_{11} \cdot B_{12} - A_{11} \cdot B_{22} \\
P_2 &= S_2 \cdot B_{22} = A_{11} \cdot B_{22} + A_{12} \cdot B_{22} \\
P_3 &= S_3 \cdot B_{11} = A_{21} \cdot B_{11} + A_{22} \cdot B_{11} \\
P_4 &= A_{22} \cdot S_4 = A_{21} \cdot B_{11} + A_{22} \cdot B_{11} \\
P_5 &= S_5 \cdot S_6 = A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} \cdot B_{11} + A_{22} \cdot B_{22} \\
P_6 &= S_7 \cdot S_8 = A_{12} \cdot B_{21} + A_{12} \cdot B_{22} - A_{22} \cdot B_{21} - A_{22} \cdot B_{22} \\
P_7 &= S_9 \cdot S_{10} = A_{11} \cdot B_{11} + A_{11} \cdot B_{12} - A_{21} \cdot B_{11} - A_{21} \cdot B_{12}
\end{aligned}$$

- 4) Construct the four  $n/2 \times n/2$  submatrices of the product  $C$

$$\begin{aligned}
C_{11} &= P_5 + P_4 - P_2 + P_6 = A_{11} \cdot B_{11} + A_{12} \cdot B_{12} \\
C_{12} &= P_1 + P_2 = A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\
C_{21} &= P_3 + P_4 = A_{21} \cdot B_{11} + A_{22} \cdot B_{21} \\
C_{22} &= P_5 + P_1 - P_3 - P_7 = A_{22} \cdot B_{22} + A_{21} \cdot B_{12}
\end{aligned}$$

**Example:** Use Strassen's algorithm to compute the matrix product

$$\begin{pmatrix} 1 & 3 \\ 7 & 5 \end{pmatrix} \begin{pmatrix} 6 & 8 \\ 4 & 2 \end{pmatrix}$$

\* **STEP 1**

$$A_{11} = 1, A_{12} = 3, A_{21} = 7, A_{22} = 5$$

$$B_{11} = 6, B_{12} = 8, B_{21} = 4, B_{22} = 2$$

\* **STEP 2**

$$S_1 = B_{12} - B_{22} = 4 - 2 = 2$$

$$S_2 = A_{11} + A_{12} = 1 + 3 = 4$$

$$S_3 = A_{21} + A_{22} = 7 + 5 = 12$$

$$S_4 = B_{21} - B_{11} = 4 - 6 = -2$$

$$S_5 = A_{11} + A_{22} = 1 + 5 = 6$$

$$S_6 = B_{11} + B_{22} = 6 + 2 = 8$$

$$S_7 = A_{12} - A_{22} = 3 - 5 = -2$$

$$S_8 = B_{21} + B_{22} = 4 + 2 = 6$$

$$S_9 = A_{11} - A_{21} = 1 - 7 = -6$$

$$S_{10} = B_{11} + B_{12} = 6 + 4 = 10$$

\* **STEP 3**

$$P_1 = A_{11} \cdot S_1 = A_{11} \cdot B_{12} - A_{11} \cdot B_{22} = 1 \cdot 4 - 1 \cdot 2 = 2$$

$$P_2 = S_2 \cdot B_{22} = A_{11} \cdot B_{22} + A_{12} \cdot B_{22} = 1 \cdot 2 + 3 \cdot 2 = 8$$

$$P_3 = S_3 \cdot B_{11} = A_{21} \cdot B_{11} + A_{22} \cdot B_{11} = 7 \cdot 6 + 5 \cdot 6 = 72$$

$$P_4 = A_{22} \cdot S_4 = A_{22} \cdot B_{21} - A_{22} \cdot B_{11} = 5 \cdot 4 - 5 \cdot 6 = -10$$

$$P_5 = S_5 \cdot S_6 = A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} \cdot B_{11} + A_{22} \cdot B_{22} = 48$$

$$P_6 = S_7 \cdot S_8 = A_{12} \cdot B_{21} + A_{12} \cdot B_{22} - A_{22} \cdot B_{21} - A_{22} \cdot B_{22} = -20$$

$$P_7 = S_9 \cdot S_{10} = A_{11} \cdot B_{11} + A_{11} \cdot B_{12} - A_{21} \cdot B_{11} - A_{21} \cdot B_{12} = -20$$

\* **STEP 4**

$$C_{11} = P_5 + P_4 - P_2 + P_6 = 48 - 10 - 8 - 20 = 10$$

$$C_{12} = P_1 + P_2 = 10$$

$$C_{21} = P_3 + P_4 = 62$$

$$C_{22} = P_5 + P_1 - P_3 - P_7 = 48 + 2 - 72 + 20 = -2$$

– Is not preferred in practical purposes

- 1) The constants used in Strassen's method are high and for a typical application Naive method works better.
- 2) For Sparse matrices, there are better methods especially designed for them.
- 3) The submatrices in recursion take extra space.
- 4) Because of the limited precision of computer arithmetic on noninteger values, larger errors accumulate in Strassen's algorithm than in Naive Method

**References:**

- 1) GeeksForGeeks, Divide and Conquer — Set 5 (Strassen's Matrix Multiplication), [link](#)
- Regular matrix multiplication
    -
  - The master method for solving recurrences
    - provides 'cookbook' method for solving recurrences of the form

$$T(n) = aT(n/b) + f(n)$$

– depends on the following theorem

- \* Let  $a \leq 1$  and  $b > 1$  be constants, let  $f(n)$  be a function and let  $T(n)$  be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n)$$