

Worksheet 20 Solution

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April 15, 2020

Question 1

a. *Proof.* Let $V = \{1, 2, 3, 4, 5, 6\}$, $E = \{(1, 2), (1, 6), (2, 3), (3, 4), (4, 5), (5, 6)\}$.

We need to prove the graph $G = (V, E)$ is bipartite by proving the following properties:

1. There exists subsets $V_1, V_2 \subset V$ such that $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V .
2. Every edge in E has exactly one endpoint in V_1 and one in V_2 .

We will prove the properties in parts.

Part 1 (Proving $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V):

Let $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

We need to prove $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V , i.e $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$.

First, we need to show the subsets V_1 and V_2 are non-empty.

The header tells us both subsets V_1 and V_2 have more than 1 elements.

Then, using these facts, we can conclude $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$.

Finally, we need to show $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$.

The header tells us $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

Then, we can calculate

$$V_1 \cup V_2 = \{1, 2, 3, 4, 5, 6\} = V \quad (1)$$

$$V_1 \cap V_2 = \emptyset \quad (2)$$

Part 2 (Proving every edge in E has exactly one endpoint in V_1 and one in V_2):

Let $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

We need to show every edge in E has exactly one endpoint in V_1 and one in V_2 .

The header tells us $V_1 = \{1, 3, 5\}$, $V_2 = \{2, 4, 6\}$, and $E = \{(1, 2), (1, 6), (2, 3), (3, 4), (4, 5), (5, 6)\}$.

Using these facts, we can generate the following table.

Edge (1,2)	- 1 is in V_1 - 2 is in V_2	Edge (3,4)	- 3 is in V_1 - 4 is in V_2
Edge (1,6)	- 1 is in V_1 - 6 is in V_2	Edge (4,5)	- 4 is in V_2 - 5 is in V_1
Edge (2,3)	- 2 is in V_2 - 3 is in V_1	Edge (5,6)	- 5 is in V_1 - 6 is in V_2

Then, it follows from observation that every edge in E has one endpoint in V_1 and one in V_2 .

□

Pseudoproof:

Let $V = \{1, 2, 3, 4, 5, 6\}$, $E = \{(1, 2), (1, 6), (2, 3), (3, 4), (4, 5), (5, 6)\}$.

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We will prove the properties in parts.

1. Show there exists subsets $V_1, V_2 \subset V$ such that $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V

Let $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

We need to prove $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V , i.e $V_1 \cup V_2 = V \wedge V_1 \cap V_2 = \emptyset$.

1. Show $V_1 \neq \emptyset, V_2 \neq \emptyset$

First, we need to show the subsets V_1 and V_2 are non-empty.

The header tells us both subsets V_1 and V_2 have more than 1 elements.

Then, using these facts, we can conclude $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$.

2. Show $V_1 \cup V_2 = V \wedge V_1 \cap V_2 = \emptyset$

Second, we need to show $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$.

The header tells us $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

Then, we can calculate

$$V_1 \cup V_2 = \{1, 2, 3, 4, 5, 6\} = V \quad (3)$$

$$V_1 \cap V_2 = \emptyset \quad (4)$$

Part 1:

Let $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

We need to prove $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V , i.e $V_1 \cup V_2 = V \wedge V_1 \cap V_2 = \emptyset$.

First, we need to show the subsets V_1 and V_2 are non-empty.

The header tells us both subsets V_1 and V_2 have more than 1 elements.

Then, using these facts, we can conclude $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$.

Finally, we need to show $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$.

The header tells us $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

Then, we can calculate

$$V_1 \cup V_2 = \{1, 2, 3, 4, 5, 6\} = V \quad (5)$$

$$V_1 \cap V_2 = \emptyset \quad (6)$$

2. Show every edge in E has exactly one endpoint in V_1 and one in V_2 .

Let $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

We need to show every edge in E has exactly one endpoint in V_1 and one in V_2 .

The header tells us $V_1 = \{1, 3, 5\}$, $V_2 = \{2, 4, 6\}$, and $E = \{(1, 2), (1, 6), (2, 3), (3, 4), (4, 5), (5, 6)\}$.

Using these facts, we can generate the following table.

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Then, it follows from observation that every edge in E has one endpoint in V_1 and one in V_2 .

Part 2:

Let $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

We need to show every edge in E has exactly one endpoint in V_1 and one in V_2 .

The header tells us $V_1 = \{1, 3, 5\}$, $V_2 = \{2, 4, 6\}$, and $E = \{(1, 2), (1, 6), (2, 3), (3, 4), (4, 5), (5, 6)\}$.

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Then, it follows from observation that every edge in E has one endpoint in V_1 and one in V_2 .

b. Let $G = (V, E)$ be a complete bipartite graph.

Then, by property 3, we can conclude each vertex in V_1 is adjacent to all vertices in V_2 .

Since there are n many edges for each vertex in V_1 , and since there are m many vertices in V_1 , we can calculate that the vertices in V_1 has

$$nm \tag{1}$$

edges.

Then, since there are no new edges for each vertex in V_2 , we can conclude the graph has nm edges.

- c. **Conjecture:** The length of every cycle in a bipartite graph is even (i.e. $\forall G = (V, E)$, $Bipartite(G) \Rightarrow \forall k \in \mathbb{N}, C = \{v_0, \dots, v_k\} \subseteq V \wedge Cycle(C, G) \Rightarrow \exists d \in \mathbb{Z}, k = 2d$)

Pseudoproof:

Let $G = (V, E)$, and assume G is bipartite, with bipartition V_1, V_2 . Let $C = v_0, \dots, v_k$ be a cycle in G . Without loss of generality, assume $v_0 \in V_1$. [Needs improvement*]

We will prove that k is even by using induction on k .

1. Case 1 (Base case):

Let $k = 3$.

We need to show the sequence of vertices $C = v_1, v_2, v_3$ in G do not form a cycle. That is, there is a consecutive pair of vertices that's not adjacent.

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2. Case 2 (Inductive case):

Let $k \in \mathbb{N}$. Assume $C = v_0, v_1, \dots, v_k$ is a cycle in G , and $\exists d \in \mathbb{Z}, k = 2d$.

We need to prove the cycle $C = v_1, \dots, v_{k+1}$ that forms in G has even length.

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Notes:

- Cycle with odd number of vertices - Not bipartite
- Cycle with even number of vertices - Bipartite
- 푸퍼맨!! 영차! 영차! 형모 풀뚜있찌!!
- 할뚜있다 형모야!!

- 형모 많이 틀렸쪄
- 형모 틀리면 틀리면서 배우면 되느니라. 흠허허허허!!
- 형모 화이팅!!
- 파이팅 파이팅!!