CSC236 Worksheet 1 Solution

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Question 1

a. *Proof.* Assume the statement $P(115)$ is true. That is, $\sum_{i=0}^{i=115} 2^i = 2^{115+1}$.

We need to prove $\sum_{i=0}^{i=116} 2^i = 2^{116+1}$.

Starting from the left, we can write

$$\sum_{i=0}^{i=116} 2^i = \sum_{i=0}^{i=115} 2^i + 2 \tag{1}$$

Then, using the assumption $\sum_{i=0}^{i=115} 2^i = 2^{115+1}$, we can conclude

$$\begin{align}
\sum_{i=0}^{i=116} 2^i &= 2^{115+1} + 2^{116} \tag{2}\\
&= 2^{116} + 2^{116} \tag{3}\\
&= 2^{116}(1+1) \tag{4}\\
&= 2 \cdot 2^{116} \tag{5}\\
&= 2^{116+1} \tag{6}
\end{align}$$

$\square$

b. *Proof.* No. The statement is not true for every natural natural number.

We will prove this by counter example. That is, $\exists n \in \mathbb{N}, \sum_{i=0}^{i=n} 2^i \neq 2^{n+1}$.

Let $n = 0$.

Then, starting from the left hand side, it follows from the fact $n = 0$ that

$$
\begin{align}
\sum_{i=0}^{i=n} 2^i &= \sum_{i=0}^{i=0} 2^i \tag{1} \\
&= 0 \tag{2}
\end{align}
$$

Now, for the right hand side, using the same fact, we can write

$$
\begin{align}
2^{0+1} &= 2^1 \tag{3} \\
&= 2 \tag{4}
\end{align}
$$

$\square$

Question 2

- **Statement:** $\forall n \in \mathbb{N},\ \exists d \in \mathbb{Z},\ 8^n - 1 = 7d$

Proof. We will prove this statement by induction on n.

Base Case:

Let $n = 0$.

We need to prove $8^n - 1 = 7 \cdot 0$.

Starting from the left hand side, using the fact $n = 0$, we can conclude,

$$
\begin{align}
8^0 - 1 &= 1 - 1 \tag{1} \\
&= 0 \tag{2} \\
&= 7 \cdot 0 \tag{3}
\end{align}
$$

Inductive Step:

Let $n \in \mathbb{N}$. Assume there is an integer d such that $8^n - 1 = 7d$.

We need to prove there is an integer $\tilde{d}$ such that $8^{n+1} - 1 = 7\tilde{d}$.

Let $\tilde{d} = 8^n + d$.

Starting from the left hand side, we can write

$$
\begin{aligned}
8^{n+1} - 1 &= 8^n + 8^n + 8^n + 8^n + 8^n + 8^n + 8^n + 8^n - 1 && (4)\\
&= 8^n + 8^n + 8^n + 8^n + 8^n + 8^n + 8^n + (8^n - 1) && (5)
\end{aligned}
$$

Then, using inductive hypothesis, i.e. $8^n - 1 = 7d$, we can conclude

$$
\begin{aligned}
8^{n+1} - 1 &= 8^n + 8^n + 8^n + 8^n + 8^n + 8^n + 8^n + 7d && (6)\\
&= 7 \cdot 8^n + 7d && (7)\\
&= 7 \cdot (8^n + d) && (8)\\
&= 7 \cdot \tilde{d} && (9)
\end{aligned}
$$

□

Question 3

- **Statement:** $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}$, the units digit of 7^n is the same as the units digit of 3^m.

Rough Work:

We will prove this statement by induction on n.

1. Base Case

Base Case:

Let $n = 0$.

We need to prove there is a natural number m such that the units digit of $7^0 = 1$ is the same as the units digit of 3^m. That is, the ones place of the number 3^m is 1.

Let $m = 0$.

Then, using this fact, we can conclude

$$3^0 = 1 \tag{1}$$

2. Inductive Step

   Let $n \in \mathbb{N}$. Assume there exists a natural number m such that the units digit of 7^n is the same as the units digit of 3^m.

   We need to prove there is a natural number $\tilde{m}$ such that the units digit of 7^{n+1} is the same as the units digit of $3^{\tilde{m}}$. That is, the ones digit of 7^{n+1} is the same as the ones digit of $3^{\tilde{m}}$.

   Let $\tilde{m} = m + 9$.

   - Show $7^{n+1} = 7 \cdot 7^n$.

     Starting with 7^{n+1}, we can write

     $$7^{n+1} = 7 \cdot 7^n. \tag{2}$$

   - Show the ones digit of 7^{n+1} is 7 times the ones digit of 7^n.

     Then, it follows from above fact that the ones digit of 7^{n+1} is 7 times the ones digit of 7^n.

   - Show $3^{\tilde{m}}$ also has 7 times the ones digit of 7^n

Now, for $3^{\tilde{m}}$, using the fact $\tilde{m} = m + 9$, we can write

$$\begin{align} 3^{\tilde{m}} &= 3^{m+9} \tag{3} \\ &= 3^9 \cdot 3^m \tag{4} \\ &= 27 \cdot 3^m \tag{5} \end{align}$$

Then, it follows from above fact that that the ones digit of $3^{\tilde{m}}$ is 7 times the ones digit of 3^m.

Then, using inductive hypothesis, we can conclude the ones digit of $3^{\tilde{m}}$ is 7 times the ones digit of 7^n.

Inductive Step:

Let $n \in \mathbb{N}$. Assume there exists a natural number m such that the units digit of 7^n is the same as the units digit of 3^m.

We need to prove there is a natural number $\tilde{m}$ such that the units digit of 7^{n+1} is the same as the units digit of $3^{\tilde{m}}$. That is, the ones digit of 7^{n+1} is the same as the ones digit of $3^{\tilde{m}}$.

Let $\tilde{m} = m + 9$.

Starting with 7^{n+1}, we can write

$$7^{n+1} = 7 \cdot 7^n. \tag{6}$$

Then, it follows from above fact that the ones digit of 7^{n+1} is 7 times the ones digit of 7^n.

Now, for $3^{\tilde{m}}$, using the fact $\tilde{m} = m + 9$, we can write

$$\begin{aligned} 3^{\tilde{m}} &= 3^{m+9} && (7) \\ &= 3^9 \cdot 3^m && (8) \\ &= 27 \cdot 3^m && (9) \end{aligned}$$

Then, it follows from above fact that that the ones digit of $3^{\tilde{m}}$ is 7 times the ones digit of 3^m.

Then, using inductive hypothesis, we can conclude the ones digit of $3^{\tilde{m}}$ is 7 times the ones digit of 7^n.

Question 4