CSC236 Assignment 1

Hyungmo Gu

May 6, 2020

Question 1

a. Yes. We can prove $P(235)$ follows from $P(234)$.

Proof. Let b be the bipartite graph with 235 vertices where 117 vertices are in one partition and 118 vertices in the other partition (Note this is the configuration where maximum number of edges form).

The bipartite graph with 117 vertices on both sides of partition has $\frac{234^2}{4}$ edges, and the assumption tells us this is the maximum number of edges the bipartite graph could form.

Since we know b has 117 more edges than the bipartite graph with 117 vertices on both sides, using these facts, we can conclude the upper bound number of edges for the bipartite graph with 235 vertices is

$$\begin{aligned}
\frac{234^2}{4} + 117 &= \frac{234^2}{4} + \frac{4 \cdot 117}{4} &(1)\\
&= \frac{234^2 + 2 \cdot 234}{4} &(2)\\
&\leq \frac{234^2 + 2 \cdot 234 + 1}{4} &(3)\\
&= \frac{(234+1)^2}{4} &(4)\\
&= \frac{(235)^2}{4} &(5)
\end{aligned}$$

□

Attempt #2:

Assume $P(234)$. That is, every bipartite graph on 234 vertices has no more than $\frac{234^2}{4}$ edges.

We need to prove $P(235)$ follows. That is, every bipartite graph on 235 vertices has no more than $\frac{235^2}{4}$ edges.

Let b be the bipartite graph with 235 vertices. Let b' be the bipartite graph with a vertex removed from the larger of two partitions in b along with its edges.

Since we know the maximum number of edges occur in b' when there are 117 vertices in both sides of the partitions, and since we know the edges of removed vertex forms edges with partition with bigger number of vertices, we can conclude the removed vertex forms at most 117 edges.

Since the assumption tells us b' has at most $\frac{234^2}{4}$ edges, we can conclude the upper bound number of edges for the bipartite graph with 235 vertices is

$$\frac{234^2}{4} + 117 = \frac{234^2}{4} + \frac{4 \cdot 117}{4} \tag{6}$$

$$= \frac{234^2 + 2 \cdot 234}{4} \tag{7}$$

$$\leq \frac{234^2 + 2 \cdot 234 + 1}{4} \tag{8}$$

$$= \frac{(234+1)^2}{4} \tag{9}$$

$$= \frac{(235)^2}{4} \tag{10}$$

So $P(235)$ follows.

Notes:

- I have a stuck feeling as to how I can formulate this kind of proof.
- 5+ hours spent on this problem
- I feel I wronged the proof by using existential quantifier
- Noticed professor generalized his statement instead of using bipartite with x number of vertices in one partition and y vertices in other partition example

proof: Assume $P(234)$ and let G be an arbitrary bipartite graph with 235 vertices. Remove a vertex, together with its edges from the larger of the two partitions to form a new bipartite graph, G' with 234 vertices. Notice that, since all edges must run from the removed vertex to the other (smaller) partition, there are no more than 117 edges.
By $P(234)$ G' has no more than $234^2/4$ edges, so G has no more than:

$$\frac{234^2}{4} + 117 = \frac{234^2 + 4(117)}{4} = \frac{234^2 + 2(234)}{4} < \frac{234^2 + 2(234) + 1}{4} = \frac{235^2}{4}$$

So $P(235)$ follows.

b. No. $P(236)$ doesn't follow from $P(235)$.

Proof. Assume $P(235)$. That is, every bipartite graph with 235 vertices has no more than $\frac{235^2}{4}$ edges.

We need to show $P(236)$ doesn't follow. That is, there is a bipartite graph b with 236 vertices that has more than $\frac{236^2}{4}$ edges.

Let b be bipartite graph with 118 vertices in V_1 and 118 vertices in V_2 (Notice that this forms the most number of edges).

The assumption tells us that every bipartite graph with 235 vertices has $\frac{235^2}{4}$ edges.

Since we know that by removing a vertex with 118 edges from either one of the partition results in bipartite graph with 235 vertices, using above fact, we can write that b has

$$\frac{235^2}{4} + 118 = 13924.25 \tag{1}$$

edges.

Since $\frac{236^2}{4} = 13924.0$, we can conclude $P(236)$ doesn't follow.

$\square$

Notes:

- Noticed that professor tried to show the bipartite graph with 234 vertices with 117 vertices in V_1 and 117 vertices in V_2 is the one and the only bipartite graph with 234 vertices that form the most number of edges.

c. *Proof.* For convenience, define

$$H(n') : \text{ Every bipartite graph with } n' \text{ vertices has no more than } \frac{n'^2 - 1}{4} \text{ edges}$$
$$\text{when } n' \text{ is odd, or } \frac{n'^2}{4} \text{ edges when } n' \text{ is even.}$$

I must prove $\forall n \in \mathbb{N}$, $H(n)$.

Base Case ($n = 0$):

The definition of edge tells us that to form, it requires a pair of two distinct vertices.

Since the graph has 0 vertices, we can conclude no edge can be formed.

Inductive Step:

Let $n \in \mathbb{N}$. And assume $H(n)$.

We must prove that $H(n+1)$ is true. There are two cases: $n+1$ is odd, or $n+1$ is even.

Case 1 ($n+1$ is even):

Assume $n+1$ is even. That is, $\exists k \in \mathbb{Z}$ such that $n+1 = 2k$.

Let b be an arbitrary bipartite graph with $n+1$ vertices. Let v be the vertex in larger of two partitions.

We must prove $H(n+1)$ is true. That is, every bipartite graph with $n+1$ vertices has no more than $\frac{(n+1)^2}{4} = \frac{(2k)^2}{4}$ edges.

First, we need to show k is the most number of edges v could have in b.

The definition of bipartite graph tells us the edges of vertex v runs from v to each vertex in the smaller partition.

Since we know the smaller partition has at most $\frac{n+1}{2} = k$, vertices, we can write the vertex v has no more than k edges.

Second, we need to show the bipartite graph with n vertices has at most $\frac{4k^2-4k}{4}$ edges.

The assumption tells us every bipartite graph with n' vertices has no more than $\frac{n'^2-1}{4}$ edges when n' is odd, or $\frac{n'^2}{4}$ edges when n' is even.

Since we know $n = 2k-1$ is odd, we can write the bipartite graph with n vertices has no more than

$$\begin{aligned} \frac{n^2-1}{4} &= \frac{(2k-1)^2-1}{4} && (2) \\ &= \frac{4k^2-4k+1-1}{4} && (3) \\ &= \frac{4k^2-4k}{4} && (4) \end{aligned}$$

edges.

Finally, by combining the two together, and using the fact $k \geq 1$ (note this is required for n to stay positive), we can conclude the bipartite graph has no more than

$$\begin{aligned}
\frac{n^2-1}{4} + k &= \frac{4k^2-4k}{4} + k && (5)\\
&= \frac{4k^2-4k}{4} + \frac{4k}{4} && (6)\\
&= \frac{4k^2}{4} && (7)\\
&= \frac{(2k)^2}{4} && (8)\\
&= \frac{(2k)^2}{4} && (9)\\
&= \frac{(n+1)^2}{4} && (10)
\end{aligned}$$

edges.

Case 2 ($n+1$ is odd):

Assume $n+1$ is odd. That is, $\exists k \in \mathbb{Z}$ such that $n+1 = 2k-1$.

Let b be an arbitrary bipartite graph with $n+1$ vertices. Let v be the vertex in larger of two partitions.

We must prove $H(n+1)$ is true. That is, every bipartite graph with $n+1$ vertices has no more than $\frac{(n+1)^2-1}{4} = \frac{n(n+2)}{4} = \frac{2k(2k-2)}{4}$ edges.

First, we need to show $k-1$ is the most number of edges v could have in b.

The definition of bipartite graph tells us the edges of vertex v runs from v to each vertex in the smaller partition.

Since we know the smaller partition has at most $\frac{n}{2} = \frac{2k-2}{2} = k-1$, vertices, we can write the vertex v has no more than $k-1$ edges.

Second, we need to show the bipartite graph with n vertices has at most $\frac{(2k-2)^2}{4}$ edges

The assumption tells us every bipartite graph with n' vertices has no more than $\frac{n'^2-1}{4}$ edges when n' is odd, or $\frac{n'^2}{4}$ edges when n' is even.

Since we know $n = 2k - 2$ is even, we can write the bipartite graph with n vertices has no more than

$$\frac{n^2}{4} = \frac{(2k-2)^2}{4} \tag{11}$$

edges.

Finally, by combining the two together, we can conclude the bipartite graph has no more than

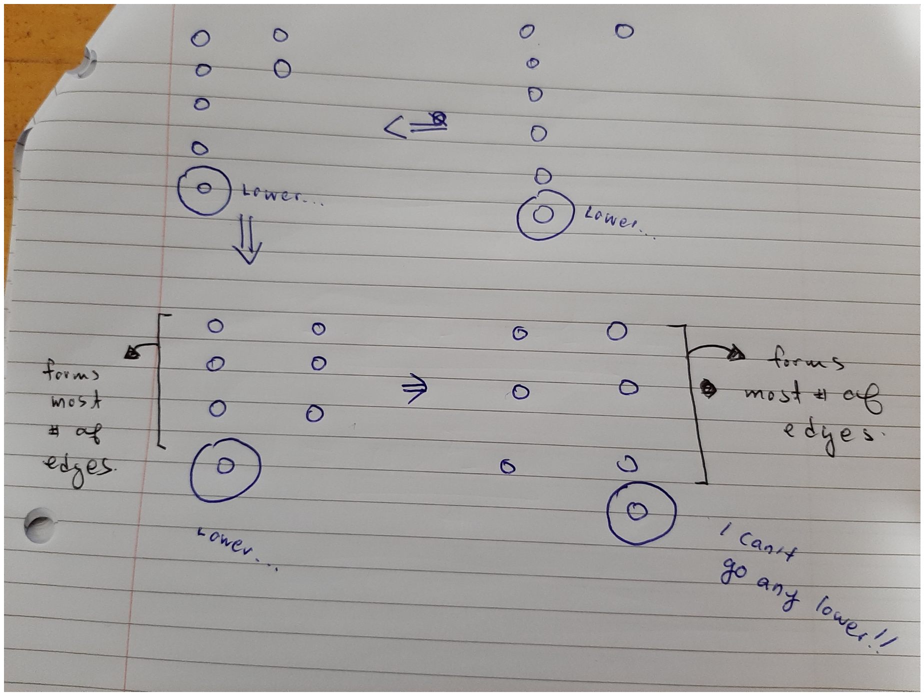
$$\begin{aligned}
\frac{n^2}{4} + k - 1 &= \frac{(2k-2)^2}{4} + k - 1 && (12)\\
&= \frac{(2k-2)^2}{4} + \frac{4k-4}{4} && (13)\\
&= \frac{4k^2 - 8k + 4}{4} + \frac{4k-4}{4} && (14)\\
&= \frac{2k(k-2)}{4} && (15)\\
&= \frac{(n+2) \cdot n}{4} && (16)\\
&= \frac{(n+1)^2 - 1}{4} && (17)
\end{aligned}$$

edges.

$\square$

Notes:

- Learned the line 'Remove a vertex, together with its edges from the larger of the two partitions to form a new bipartite graph ... Notice that, since all edges must run from the removed vertex to other smaller partition, there are no more than...' means the following



- This was tricky. And the trick is sneaky :).

Question 2

a. Yes. $P(29)$ follows from $P(3)$.

Proof. Assume $P(3)$. That is $\exists k \in \mathbb{Z},\ f(3) = 4k$.

I must prove $P(29)$ follows. That is, $\exists d \in \mathbb{Z},\ f(29) = 4d$.

Let $d = 4k^2 + k$.

The definition of $f(n)$ tells us

$$f(29) = [f(\lfloor \log_3 29 \rfloor)]^2 + f(\lfloor \log_3 29 \rfloor) \tag{1}$$

Using above fact, we can calculate

$$f(29) = [f(3)]^2 + f(3) \tag{2}$$

Then, since we know from the header that $f(3) = 4k$, we can write

$$\begin{aligned} f(29) &= [4k]^2 + 4k && (3) \\ &= 4 \cdot (4k^2 + k) && (4) \end{aligned}$$

Then, since we know from the header that $d = 4k^2 + k$, we can conclude

$$f(29) = 4d \tag{5}$$

$\square$

b. Yes. $P(29)$ follows from $P(4)$.

Proof. Assume $P(4)$. That is $\exists k \in \mathbb{Z},\ f(4) = 4k$.

I must prove $P(29)$ follows. That is, $\exists d \in \mathbb{Z},\ f(29) = 4d$.

Let $d = k \cdot (4k + 1)$.

The definition of $f(n)$ tells us

$$f(29) = [f(\lfloor \log_3 29 \rfloor)]^2 + f(\lfloor \log_3 29 \rfloor) \tag{1}$$

Using above fact, we can calculate

$$f(29) = [f(3)]^2 + f(3) \tag{2}$$

Then, since we know from the definition of $f(n)$ that $f(4) = f(3) = f(1)^2 + f(1)$, we can write

$$\begin{aligned} f(29) &= [f(1)^2 + f(1)]^2 + [f(1)^2 + f(1)] && (3) \\ &= [f(4)]^2 + f(4) && (4) \end{aligned}$$

Then, since we know from the header that $f(4) = 4k$, we can write

$$\begin{aligned} f(29) &= [4k]^2 + 4k && (5) \\ &= 4k \cdot (4k+1) && (6) \end{aligned}$$

Then, since we know from the header that $d = k \cdot (4k+1)$, we can conclude

$$f(29) = 4d \qquad (7)$$

$\square$

c. *Proof.* Define

$$P(n) : f(n) \text{ is a multiple of } 4$$

I must prove by complete induction that $\forall n \in \mathbb{N} - \{0\}$, $P(n)$.

Inductive Step:

Let $n \in \mathbb{N} - \{0\}$. Assume $H(n) : \bigwedge_{i=1}^{i=n-1} P(i)$.

I will show that $P(n)$ follows, that $f(n)$ is a multiple of 4. In other words, $\exists d \in \mathbb{Z}$, $f(n) = 4d$.

Base Case ($n = 1$):

Let $n = 1$. Let $d = 3$.

Then,

$$\begin{aligned} f(1) &= [f(\lfloor \log_3 1 \rfloor)]^2 + f(\lfloor \log_3 1 \rfloor) && (1) \\ &= f(0)^2 + f(0) && (2) \\ &= (3)^2 + 3 && (3) \\ &= 12 && (4) \\ &= 4 \cdot 3 && (5) \end{aligned}$$

Then, since $d = 3$,

$$f(1) = 4d \tag{6}$$

<u>Base Case ($n = 2$):</u>

Let $n = 2$. Let $d = 3$.

Then,

$$\begin{align}
f(1) &= [f(\lfloor \log_3 1 \rfloor)]^2 + f(\lfloor \log_3 1 \rfloor) \tag{7}\\
&= f(0)^2 + f(0) \tag{8}\\
&= (3)^2 + 3 \tag{9}\\
&= 12 \tag{10}\\
&= 4 \cdot 3 \tag{11}
\end{align}$$

Then, since $d = 3$,

$$f(1) = 4d \tag{12}$$

<u>Case ($n \geq 3$):</u>

Let $n \geq 3$, and $d = 4k^2 + k$

Then, since $n > 0$,

$$f(n) = f(\lfloor \log_3 n \rfloor)^2 + f(\lfloor \log_3 n \rfloor) \tag{13}$$

Then, since $1 \leq \lfloor \log_3 n \rfloor < n$, and $P(\lfloor \log_3 n \rfloor)$, $\exists k \in \mathbb{Z}$,

$$\begin{align}
f(n) &= [4k]^2 + 4k \tag{14}\\
&= 4(4k^2 + k) \tag{15}
\end{align}$$

Then, since $d = 4k^2 + k$,

$$f(n) = 4d \tag{16}$$

$\square$

Question 3

- Given the statement to prove

$$P(x, y, z) : \text{There are no positive integers } x, y, z \text{ such that } 5x^3 + 50y^3 = 3z^3$$

Proof. We will prove $P(x, y, z)$ using proof by contradiction.

Assume $\exists x, y, z \in \mathbb{N}^+, 5x^3 + 50y^3 = 3z^3$.

Then, we can write that the set $X = \{x \mid x \in \mathbb{N}^+,\ \exists y, z \in \mathbb{N}^+,\ 5x^3 + 50y^3 = 3z^3\}$ is not empty.

Then, using the well-ordering principle, we can write there is the smallest positive integer $x_0 \in X$, and $y_0, z_0 \in \mathbb{N}^+$ such that $5x_0^3 + 50y_0^3 = 3z_0^3$.

Then,

$$50x_0^3 + 50y_0^3 = 3z_0^3 \Rightarrow 5 \mid 3z_0^3 \Rightarrow 5 \mid z_0 \qquad \text{[by hint]}$$

$$\begin{aligned}\text{Let } \exists z_1 \in \mathbb{N}^+, z_0 = 5z_1 &\Rightarrow 5x_0^3 + 50y_0^3 = 3z_0^3 \Rightarrow 5x_0^3 = 5^3 3z_1^3 - 50y_0^3 \\ &\Rightarrow x_0^3 = 5^2 \cdot 3z_1^3 - 10y_0^3 \Rightarrow 5 \mid x_0^3 \Rightarrow 5 \mid x_0\end{aligned} \qquad \text{[by hint]}$$

$$\begin{aligned}\text{Let } \exists x_1 \in \mathbb{N}^+, x_0 = 5x_1 &\Rightarrow 5x_0^3 + 50y_0^3 = 3z_0^3 \Rightarrow 50y_0^3 = 5^3 3z_1^3 - 5^4 x_1^3 \\ &\Rightarrow 2y_0^3 = 5^3 3z_1^3 - 5^2 x_1^3 \Rightarrow 5 \mid y_0^3 \Rightarrow 5 \mid y_0\end{aligned} \qquad \text{[by hint]}$$

$$\begin{aligned}\text{Let } \exists y_1 \in \mathbb{N}^+, y_0 = 5y_1 &\Rightarrow 5x_0^3 + 50y_0^3 = 3z_0^3 \Rightarrow 5^3 5x_1^3 + 5^3 50y_1^3 = 5^3 3z_1^3 \\ &\Rightarrow 5x_1^3 + 50y_1^3 = 3z_1^3 \Rightarrow x_1 \in X\end{aligned}$$

Then, since $x_1 < x_0$, $x_1 \in X$, but x_0 must be the smallest number in X, this leads to contradiction.

Then, we can conclude the assumption is false.

$\square$

Question 4

a. Define $\mathcal{T}$ as the smallest set of strings that satisfies:

- "$*$" $\in \mathcal{T}$
- $t_1, t_2 \in \mathcal{T}$ then their parenthesized concatenation $(t_1t_2) \in \mathcal{T}$.

Prove $\forall t \in \mathcal{T}$, $\text{left\_count}(t) \leq 2^{\text{max\_left\_surplus}(t)} - 1$

Proof. **Basis:**

Let $t =$ "$*$" $\in \mathcal{T}$.

Then, the code tells us $\text{left\_count}(t) = 0$, and $\text{max\_left\_surplus}(t) = 0$.

Then, we can conclude

$$
\begin{aligned}
\text{left\_count}(t) = 0 &\leq 0 && (1)\\
&\leq 1 - 1 && (2)\\
&\leq 2^0 - 1 && (3)\\
&\leq 2^{\text{max\_left\_surplus}(t)} - 1 && (4)\\
& && (5)
\end{aligned}
$$

Inductive Step:

Let t_1 and t_2 be arbitrary string of $\mathcal{T}$. Assume $H(t_1, t_2) : P(t_1)$ and $P(t_2)$. That is, t_1 and t_2 satisfies the property $\text{left\_count}(t_1) \leq 2^{\text{max\_left\_surplus}(t_1)} - 1$ and $\text{left\_count}(t_2) \leq 2^{\text{max\_left\_surplus}(t_2)} - 1$, respectively.

Let $(t_1t_2) \in \mathcal{T}$ be the parenthesized concatenation of $t_1, t_2 \in \mathcal{T}$.

We need to prove $P((t_1t_2))$ follows. That is, $\text{left\_count}((t_1t_2)) \leq 2^{\text{max\_left\_surplus}((t_1t_2))} - 1$.

The code tells us

$$\text{left\_count}((t_1t_2)) = \text{left\_count}(t_1) + \text{left\_count}(t_2) + 1 \qquad (6)$$

Then, using this fact, we have

$$
\begin{aligned}
\text{left\_count}((t_1t_2)) &= \text{left\_count}(t_1) + \text{left\_count}(t_2) + 1 && (7)\\
&\leq 2^{\text{max\_left\_surplus}(t_1)} - 1 + 2^{\text{max\_left\_surplus}(t_2)} - 1 + 1 \qquad \text{[By I.H]} && (8)\\
&\leq 2^{\text{max\_left\_surplus}(t_1)} + 2^{\text{max\_left\_surplus}(t_2)} - 1 && (9)\\
&\leq 2^{\text{max\_left\_surplus}(t_1)} + 2^{\text{max\_left\_surplus}(t_2)} - 1 && (10)\\
&\leq 2^{\text{max\_left\_surplus}(t_1)} + 2^{\text{max\_left\_surplus}(t_2)} - 1 && (11)\\
&\leq 2 \cdot 2^{max(\text{max\_left\_surplus}(t_1),\text{max\_left\_surplus}(t_2))} - 1 && (12)\\
&\leq 2^{max(\text{max\_left\_surplus}(t_1),\text{max\_left\_surplus}(t_2))+1} - 1 && (13)\\
&\leq 2^{\text{max\_left\_surplus}((t_1t_2))} - 1 \qquad \text{[By hint]} && (14)\\
& && (15)
\end{aligned}
$$

Thus, $P((t_1t_2))$ follows.

□

b. Define $\mathcal{T}$ as the smallest set of strings that satisfies:

- "$*''$ $\in \mathcal{T}$
- $t_1, t_2 \in \mathcal{T}$ then their parenthesized concatenation $(t_1t_2) \in \mathcal{T}$.

Prove $\forall t \in \mathcal{T}$,

$$
\text{double\_count}(t) = \begin{cases} 0 & \text{if } t = `*' \\ \text{left\_count(t)} - 1 & \text{otherwise} \end{cases}
$$

Basis:

Let $t =$ "$*'' \in \mathcal{T}$.

Then, since it doesnt have parenthesis, double_count$(t) = 0$.

Thus, $P(t)$ holds.

Inductive Step:

Let t_1 and t_2 be arbitrary string of $\mathcal{T}$. Assume $H(t_1, t_2) : P(t_1)$ and $P(t_2)$.

Let $(t_1t_2) \in \mathcal{T}$ be the parenthesized concatenation of $t_1, t_2 \in \mathcal{T}$.

We need to prove $P((t_1t_2))$ follows. That is, $\text{double\_count}((t_1t_2)) = \text{left\_count}((t_1t_2)) - 1$. There are four cases to consider.

Case 1 ($t_1, t_2 =$ '$*'$):

Assume $t_1, t_2 =$ '$*'$.

Then, since t_1t_2 are surrounded by single parenthesis,

$$\begin{align}\text{double\_count}((t_1t_2)) &= 0 \tag{1}\\ \text{left\_count}((t_1t_2)) &= 1 \tag{2}\end{align}$$

Then, using above facts, we can conclude

$$\begin{align}\text{double\_count}((t_1t_2)) &= 1 - 1 \tag{3}\\ &= \text{left\_count}((t_1t_2)) - 1 \tag{4}\end{align}$$

So, $P((t_1t_2))$ follows.

Case 2 ($t_1, t_2 =$ '$*'$):

Assume t_1 is not '*' and $t_2 =$ '$*'$.

Then, the definition tells us t_1 is a parenthesized concatenation of string that satisfies $\mathcal{T}$.

Then, we can write

$$\begin{align}\text{double\_count}((t_1t_2)) &= \text{double\_count}(t_1) + 1 \tag{5}\\ \text{left\_count}((t_1t_2)) &= \text{left\_count}(t_1) + 1 \tag{6}\end{align}$$

Then, using above facts, we can conclude

$$\begin{align}\text{double\_count}((t_1t_2)) &= \text{left\_count}(t_1) - 1 + 1 && \text{[By I.H]} \tag{7}\\ &= \text{left\_count}((t_1t_2)) - 1 && \text{[By 5]} \tag{8}\end{align}$$

So, $P((t_1t_2))$ follows.

Case 3 ($t_1 =$ '$*$' and t_2 is not '*'):

Assume $t_1 =$ '$*$' and t_2 is not '*'.

Then, the definition of $\mathcal{T}$ tells us t_2 is a parenthesized concatenation of string in $\mathcal{T}$.

Then, we can write

$$\text{double\_count}((t_1 t_2)) = \text{double\_count}(t_2) + 1 \tag{9}$$
$$\text{left\_count}((t_1 t_2)) = \text{left\_count}(t_2) + 1 \tag{10}$$

Then, using above facts, we can conclude

$$\begin{aligned} \text{double\_count}((t_1 t_2)) &= \text{left\_count}(t_2) - 1 + 1 && \text{[By I.H]} \quad (11) \\ &= \text{left\_count}((t_1 t_2)) - 1 && \text{[By 9]} \quad (12) \end{aligned}$$

So, $P((t_1 t_2))$ follows.

Case 4 (both t_1, t_2 are not '*'):

Assume both t_1, t_2 are not '*'.

Then, the definition tells us t_1 and t_2 are parenthesized concatentation of string in $\mathcal{T}$.

Then, we can write

$$\text{double\_count}((t_1 t_2)) = \text{double\_count}(t_1) + \text{double\_count}(t_2) + 2 \tag{13}$$
$$\text{left\_count}((t_1 t_2)) = \text{left\_count}(t_1) + \text{left\_count}(t_2) + 1 \tag{14}$$

Then, using above facts, we can conclude

$$\begin{aligned} \text{double\_count}((t_1 t_2)) &= \text{left\_count}(t_1) - 1 + \text{left\_count}(t_2) - 1 + 2 && \text{[By I.H]} \quad (15) \\ &= \text{left\_count}(t_1) + \text{left\_count}(t_2) && (16) \\ &= \text{left\_count}(t) - 1 && \text{[By 13]} \quad (17) \end{aligned}$$

So, $P((t_1 t_2))$ follows.

Correct Solution:

Define $\mathcal{T}$ as the smallest set of strings that satisfies:

- "$*''$ $\in \mathcal{T}$
- $t_1, t_2 \in \mathcal{T}$ then their parenthesized concatenation $(t_1 t_2) \in \mathcal{T}$.

Prove $\forall t \in \mathcal{T}$,

$$\text{double\_count}(t) = \begin{cases} 0 & \text{if } t = \text{`}*\text{'} \\ \text{left\_count(t)} - 1 & \text{otherwise} \end{cases}$$

Basis:

Let $t =$ "$*'' \in \mathcal{T}$.

Then, since it doesnt have parenthesis, double_count$(t) = 0$.

Thus, $P(t)$ holds.

Inductive Step:

Let t_1 and t_2 be arbitrary string of $\mathcal{T}$. Assume $H(t_1, t_2) : P(t_1)$ and $P(t_2)$.

Let $(t_1 t_2) \in \mathcal{T}$ be the parenthesized concatenation of $t_1, t_2 \in \mathcal{T}$.

We need to prove $P((t_1 t_2))$ follows. That is, double_count$((t_1 t_2)) =$ left_count$((t_1 t_2)) -$ 1. There are four cases to consider.

Case 1 ($t_1, t_2 =$ '$*'$):

Assume $t_1, t_2 =$ '$*'$.

Then, since there is only single parenthesis in $(t_1 t_2)$, we have

$$\begin{aligned} \text{double\_count}((t_1 t_2)) &= 0 && (1) \\ \text{left\_count}((t_1 t_2)) &= 1 && (2) \end{aligned}$$

Then, using above facts, we can conclude

$$\begin{aligned} \text{double\_count}((t_1 t_2)) &= 1 - 1 && (3)\\ &= \text{left\_count}((t_1 t_2)) - 1 && (4) \end{aligned}$$

So, $P((t_1 t_2))$ follows.

Case 2 ($t_1, t_2 =$ '$*$'):

Assume t_1 is not '*' and $t_2 =$ '$*$'.

Then, the initial left parenthesis in $(t_1 t_2)$ increases the number of left parenthesis by 1, and the number of double left parenthesis increases by 1. Furthermore, since $t_2 =$ '$*$' the final right parenthesis in $(t_1 t_2)$ does not increases the number of double right parenthesis.

Then, it folows from these facts that

$$\begin{aligned} \text{double\_count}((t_1 t_2)) &= \text{double\_count}(t_1) + 1 && (5)\\ \text{left\_count}((t_1 t_2)) &= \text{left\_count}(t_1) + 1 && (6) \end{aligned}$$

Then, using above facts, we can conclude

$$\begin{aligned} \text{double\_count}((t_1 t_2)) &= \text{left\_count}(t_1) - 1 + 1 && \text{[By I.H]} && (7)\\ &= \text{left\_count}((t_1 t_2)) - 1 && \text{[By 5]} && (8) \end{aligned}$$

So, $P((t_1 t_2))$ follows.

Case 3 ($t_1 =$ '$*$' and t_2 is not '*'):

Assume $t_1 =$ '$*$' and t_2 is not '*'.

Then, the initial left parenthesis in $(t_1 t_2)$ increases the number of left parenthesis by 1, but not the number of double left parenthesis. Furthermore, the final right parenthesis in $(t_1 t_2)$ increases the number of double right parenthesis by 1.

Then, it folows from these facts that

$$\text{double\_count}((t_1t_2)) = \text{double\_count}(t_2) + 1 \tag{9}$$
$$\text{left\_count}((t_1t_2)) = \text{left\_count}(t_2) + 1 \tag{10}$$

Then, using above facts, we can conclude

$$\begin{aligned} \text{double\_count}((t_1t_2)) &= \text{left\_count}(t_2) - 1 + 1 && \text{[By I.H]} && (11) \\ &= \text{left\_count}((t_1t_2)) - 1 && \text{[By 9]} && (12) \end{aligned}$$

So, $P((t_1t_2))$ follows.

Case 4 (both t_1, t_2 are not '*'):

Assume both t_1, t_2 are not '*'.

Then, the initial left parenthesis in (t_1t_2) increases the number of left parenthesis by 1, and the number of double left parenthesis by 1. Similarly, the final right parenthesis in (t_1t_2) increases the number of double right parenthesis by 1.

Then, it folows from these facts that

$$\text{double\_count}((t_1t_2)) = \text{double\_count}(t_1) + \text{double\_count}(t_2) + 2 \tag{13}$$
$$\text{left\_count}((t_1t_2)) = \text{left\_count}(t_1) + \text{left\_count}(t_2) + 1 \tag{14}$$

Then, using above facts, we can conclude

$$\begin{aligned} \text{double\_count}((t_1t_2)) &= \text{left\_count}(t_1) - 1 + \text{left\_count}(t_2) - 1 + 2 && \text{[By I.H]} && (15) \\ &= \text{left\_count}(t_1) + \text{left\_count}(t_2) && && (16) \\ &= \text{left\_count}(t) - 1 && \text{[By 13]} && (17) \end{aligned}$$

So, $P((t_1t_2))$ follows.