CSC373 Worksheet 1 Solution

July 23, 2020

1.
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Strassen_Algorithm(A,B):
    n = A.rows
    let C be a new n x n matrix

    if n == 1
        C_11 = A_11 * B_11

    else partition as in step 3 of strassen's algorithm

        p1 = Strassen_Algorithm(A_11, B_12) -
             Strassen_Algorithm(A_11, B_22)

        p2 = Strassen_Algorithm(A_11, B_22) +
             Strassen_Algorithm(A_12, B_22)

        p3 = Strassen_Algorithm(A_21, B_11) +
             Strassen_Algorithm(A_22, B_11)

        p4 = Strassen_Algorithm(A_22, B_21) -
             Strassen_Algorithm(A_22, B_11)

        p5 = Strassen_Algorithm(A_11, B_11) +
             Strassen_Algorithm(A_11, B_22) +
             Strassen_Algorithm(A_22, B_11) +
             Strassen_Algorithm(A_22, B_22)

        p6 = Strassen_Algorithm(A_12, B_21) +
             Strassen_Algorithm(A_12, B_22) -
             Strassen_Algorithm(A_22, B_21) -
             Strassen_Algorithm(A_22, B_22)

        p7 = Strassen_Algorithm(A_11, B_11) +
             Strassen_Algorithm(A_11, B_12) -
             Strassen_Algorithm(A_21, B_11) -
             Strassen_Algorithm(A_21, B_12)

        C_11 = p5 + p4 - p2 + p6
        C_12 = p1 + p2
        C_21 = p3 + p4
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            C_22 = p5 + p1 - p3 - p7

        return C

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Notes:

- Strassen's method for matrix multiplication
  - Reduces the time complexity of matrix multiplication from $O(n^3)$ to $O(n^{\log_2 7}) = O(n^{2.81})$
  - Has four steps

    1) Divide the input matrics A and B and output matrix C into $n/2 \times n/2$ submatrices

    $$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$

    2) Create 10 matrices, $S_1, S_2, ..., S_{10}$ each of which is $n/2 \times n/2$ and is the sum or difference of two matrices created in step 1

    $S_1 = B_{12} - B_{22}$
    $S_2 = A_{11} + A_{12}$
    $S_3 = A_{21} + A_{22}$
    $S_4 = B_{21} - B_{11}$
    $S_5 = A_{11} + A_{22}$
    $S_6 = B_{11} + B_{22}$
    $S_7 = A_{12} - A_{22}$
    $S_8 = B_{21} + B_{22}$
    $S_9 = A_{11} - A_{21}$
    $S_{10} = B_{11} + B_{12}$

    3) Recursively multiply $n/2 \times n/2$ matrices seven times to compute the following $n/2 \times n/2$ matrices

    $P_1 = A_{11} \cdot S_1 = A_{11} \cdot B_{12} - A_{11} \cdot B_{22}$
    $P_2 = S_2 \cdot B_{22} = A_{11} \cdot B_{22} + A_{12} \cdot B_{22}$
    $P_3 = S_3 \cdot B_{11} = A_{21} \cdot B_{11} + A_{22} \cdot B_{11}$
    $P_4 = A_{22} \cdot S_4 = A_{21} \cdot B_{11} + A_{22} \cdot B_{11}$
    $P_5 = S_5 \cdot S_6 = A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} \cdot B_{11} + A_{22} \cdot B_{22}$
    $P_6 = S_7 \cdot S_8 = A_{12} \cdot B_{21} + A12 \cdot B_{22} - A_{22} \cdot B_{21} - A_{22} \cdot B_{22}$
    $P_7 = S_9 \cdot S_{10} = A_{11} \cdot B_{11} + A_{11} \cdot B_{12} - A_{21} \cdot B_{11} - A_{21} \cdot B_{12}$

    4) Construct the four $n/2 \times n/2$ submatrices of the product C

    $C_{11} = P_5 + P_4 - P_2 + P_6 = A_{11} \cdot B_{11} + A_{12} + B_{12}$
    $C_{12} = P_1 + P_2 = A_{11} \cdot B_{12} + A_{12} \cdot B_{22}$
    $C_{21} = P_3 + P_4 = A_21 \cdot B_11 + A_{22} \cdot B_{21}$
    $C_{22} = P_5 + P_1 - P_3 - P_7 = A_{22} \cdot B_{22} + A_{21} \cdot B_{12}$

Example: Use Strassen's algorithm to compute the matrix product

$$\begin{pmatrix} 1 & 3 \\ 7 & 5 \end{pmatrix} \begin{pmatrix} 6 & 8 \\ 4 & 2 \end{pmatrix}$$

* **STEP 1**

  $A_{11} = 1, A_{12} = 3, A_{21} = 7, A_{22} = 5$

  $B_{11} = 6, B_{12} = 8, B_{21} = 4, B_{22} = 2$

* **STEP 2**

  $S_1 = B_{12} - B_{22} = 4 - 2 = 2$

  $S_2 = A_{11} + A_{12} = 1 + 3 = 4$

  $S_3 = A_{21} + A_{22} = 7 + 5 = 12$

  $S_4 = B_{21} - B_{11} = 4 - 6 = -2$

  $S_5 = A_{11} + A_{22} = 1 + 5 = 6$

  $S_6 = B_{11} + B_{22} = 6 + 2 = 8$

  $S_7 = A_{12} - A_{22} = 3 - 5 = -2$

  $S_8 = B_{21} + B_{22} = 8 + 2 = 10$

  $S_9 = A_{11} - A_{21} = 3 - 5 = -2$

  $S_{10} = B_{11} + B_{12} = 6 + 4 = 10$

* **STEP 3**

  $P_1 = A_{11} \cdot S_1 = A_{11} \cdot B_{12} - A_{11} \cdot B_{22} = 1 \cdot 4 - 1 \cdot 2 = 2$

  $P_2 = S_2 \cdot B_{22} = A_{11} \cdot B_{22} + A_{12} \cdot B_{22} = 1 \cdot 2 + 3 \cdot 2 = 8$

  $P_3 = S_3 \cdot B_{11} = A_{21} \cdot B_{11} + A_{22} \cdot B_{11} = 6 \cdot 7 + 6 \cdot 5 = 72$

  $P_4 = A_{22} \cdot S_4 = A_{22} \cdot B_{21} - A_{22} \cdot B_{11} = 5 \cdot 4 - 5 \cdot 6 = -10$

  $P_5 = S_5 \cdot S_6 = A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} \cdot B_{11} + A_{22} \cdot B_{22} = 48$

  $P_6 = S_7 \cdot S_8 = A_{12} \cdot B_{21} + A_{12} \cdot B_{22} - A_{22} \cdot B_{21} - A_{22} \cdot B_{22} = -20$

  $P_7 = S_9 \cdot S_{10} = A_{11} \cdot B_{11} + A_{11} \cdot B_{12} - A_{21} \cdot B_{11} - A_{21} \cdot B_{12} = -20$

* **STEP 4**

  $C_{11} = P_5 + P_4 - P_2 + P_6 = 48 - 10 - 8 - 20 = 10$

  $C_{12} = P_1 + P_2 = 10$

  $C_{21} = P_3 + P_4 = 62$

  $C_{22} = P_5 + P_1 - P_3 - P_7 = 48 + 2 - 72 + 20 = -2$

– Is not preferred in practical purposes

  1) The constants used in Strassen's method are high and for a typical application Naive method works better.
  2) For Sparse matrices, there are better methods especially designed for them.
  3) The submatrices in recursion take extra space.
  4) Because of the limited precision of computer arithmetic on noninteger values, larger errors accumulate in Strassen's algorithm than in Naive Method

References:

1) GeeksForGeeks, Divide and Conquer — Set 5 (Strassen's Matrix Multiplication), link

- Regular matrix multiplication
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- The master method for solving recurrences
  – provides 'cookbook' method for solving recurrences of the form

$$T(n) = aT(n/b) + f(n)$$

  – depends on the following theorem

  * Let $a \leq 1$ and $b > 1$ be constants, let $f(n)$ be a function and let $T(n)$ be defined on the nonnegative integers by the recurrence

    $T(n) = aT(n/b) + f(n)$

    where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ has the following asymptotic bounds:

    1. If $f(n) = O(n^{\log_b a - \epsilon}) for some constant \epsilon > 0, then T(n) = \Theta(n^{\log_b a})$
    2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$
    3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$, and all sufficiently large n, then $T(n) = \Theta(f(n))$.

Example:

$T(n) = 9T(n/3) + n$

Here, $a = 9$, $b = 3$, and $f(n) = n = O(n^{\log_3 9-1})$ where $\epsilon = 1$.

Thus, $T(n) = \Theta(n^{\log_3 9})$ or $T(n) = \Theta(n^2)$

Example 2:

$T(n) = T(2n/3) + 1$

Here, $a = 1$, $b = 3/2$, $f(n) = 1 = \Theta(n^{\log_{3/2} 1})$.

Thus, $T(n) = \theta(\lg n)$

Example 3:

$T(n) = T(n/4) + n \lg n$

Here $a = 1$, $b = 4$, and $f(n) = n \lg n$ has asymptotic lowerbound of $f(n) = \Omega(n^{\log_4 3+\epsilon}) = \Omega(n)$ where $\epsilon \approx 0.2$

Furthermore,

$$\begin{aligned} af(n/b) &= (3n/4) \lg n/4 \\ &= (3/4)n \lg n/4 \\ &= (3/4)n \lg n/4 \\ &= 3/4n \lg n - \lg 4 \\ &< 3/4n \lg n \\ &= cf(n) \end{aligned}$$

where $c = 3/4$.

Thus, $T(n) = \Theta(n \lg n)$

Example 4:

$T(n) = 2T(n/2) + n \lg n$

Here, $a = 2$, $b = 2$, $f(n) = n \lg n$.

2. Let $n = 3^m$ where m is an element of $\mathbb{Z}^+ \cup \{0\}$

   Then we know the time it takes to multiply $n \times n$ matrices in 3×3 matrices is $T(n) = kT(\frac{n}{3}) + \Theta(n^2)$.

   Now, I need to look for the upper bound of k in $T(n) = \Theta(n^{\log_3 k})$ satisfying $O(n^{\lg 7}) \approx O(n^{2.81})$.

   And using master's master's theorem, we can write that the upper limit of k is 21.

Improved Solution:

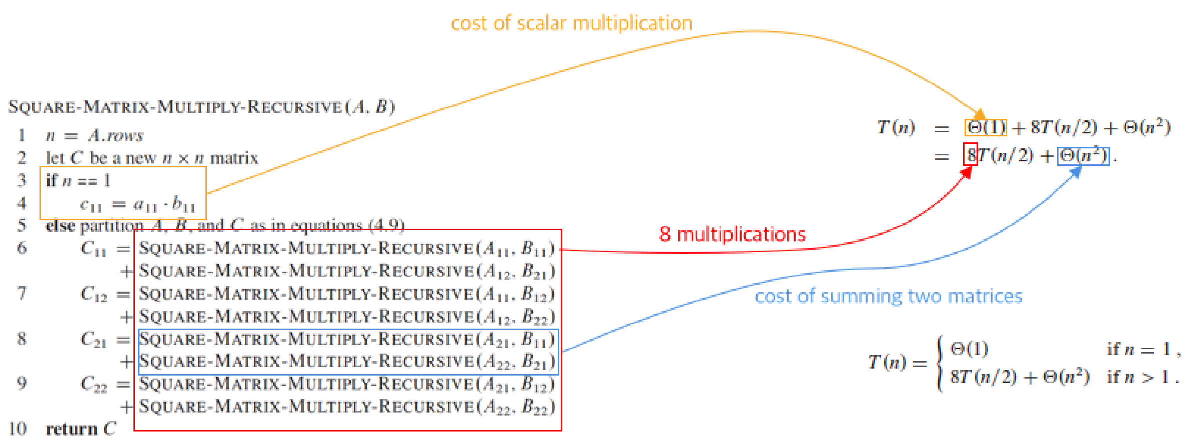
Let $n = 3^m$ where m is an element of $\mathbb{Z}^+ \cup \{0\}$

Then we know the time it takes to multiply $n \times n$ matrices in 3×3 matrices is $T(n) = kT(\frac{n}{3}) + \Theta(n^2)$.

Now, I need to look for the upper bound of k in $T(n) = \Theta(n^{\log_3 k})$ satisfying $O(n^{\lg 7}) \approx O(n^{2.81})$.

And using master's master's theorem, we can write that the upper limit of k is 21 (Following the first condition $f(n) = \mathcal{O}(n^{\log_3 k - \epsilon})$ where $\epsilon \approx 0.81$).

Notes:



- $T(n)$ represents the time it takes to multiply two $n \times n$ matrices.
- At base case scalar multiplication is performed. So, $T(1) = \Theta(1)$.
- 8 represents the number of recursive calls on the function SQUARE-MATRIX-MULTIPLY-RECURSIVE
- $\Theta(n^2)$ represents the addition of two $\frac{n}{2} \times \frac{n}{2}$ matrices

3. - 68×68 matrices using $132,464$ multiplications:
     -

4.

5. a) Here, $a = 2, b = 2, f(n) = 4$.

   Since $f(n) = n^{\log_2 2+3} = n^{\log_b a+\epsilon}$ where $\epsilon = 3$, and $af(\frac{n}{b}) = 2\left(\frac{n^4}{16}\right) = \frac{n^4}{8} \leq cn^4$ where $c = \frac{1}{8}$ for sufficiently large n, the case 3 of master's theorem applies.

   Thus, $T(n)$ has upper bound of $\mathcal{O}(n^4)$ and lower bounds of $\Omega(n^4)$, or $\Theta(n^4)$.

   b) Here $a = 1, b = \frac{10}{7}, f(n) = n$.

   Since $f(n) = 1 = n^{0+1} = n^{\log_{10/7}(1)+1} = n^{\log_b(a)+\epsilon}$, where $\epsilon = 1$, and $af(\frac{n}{b}) = \frac{7n}{10} \leq cn^4$ where $c = \frac{7}{10}$ for sufficiently large n, the case 3 of master's theorem applies.

   Thus, $T(n)$ has upper bound of $\mathcal{O}(n)$ and lower bounds of $\Omega(n^4)$, or $\Theta(n)$.

   c) Here we have $a = 16, b = 4, f(n) = n^2$.

   Since $f(n) = n^2 = n^{\log_4 16} = n^{\log_b a}$, case 2 of master's theorem applies.

   Thus, $T(n)$ has upper bound of $\mathcal{O}(n^2 \lg n)$ and lower bounds of $\Omega(n^2 \lg n)$.

   d) Here we have $a = 7, b = 3, f(n) = n^2$.

   Since $n^2 = n^{\log_3 7+\epsilon} = n^{\log_b a+\epsilon}$ where $\epsilon \approx 0.23$, and $af(\frac{n}{b}) \leq cn^2$ where $c = \frac{7}{9}$, the case 3 of master's theorem applies.

   Thus, $T(n)$ has upper bound of $\mathcal{O}(n^2)$ and lower bounds of $\Omega(n^2)$, or $\Theta(n^2)$.

   e) Here we have $a = 7, b = 2, f(n) = n^2$.

   Since $f(n) = n^2 = n^{\log_2(7)-\epsilon}$, where $\epsilon \approx 0.81$, case 1 of master theorem applies.

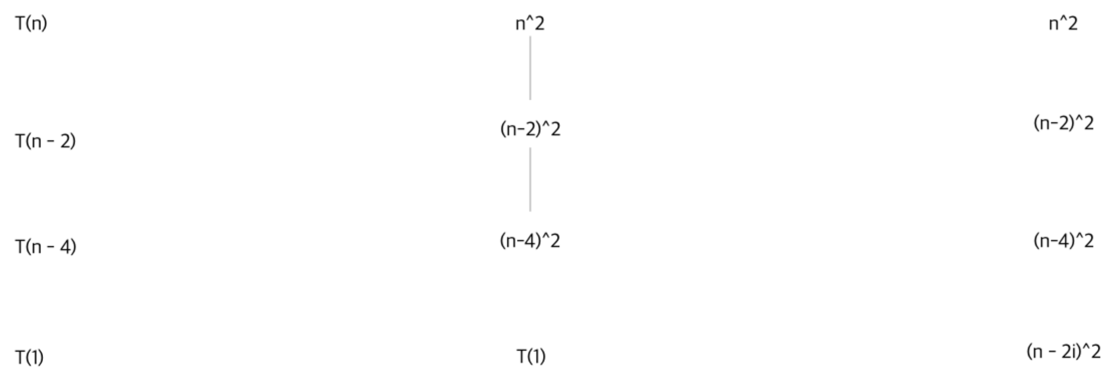
   Thus, $T(n)$ has upper bound of $\mathcal{O}(n^{\log_2 7})$ and lower bound of $\Omega(n^{\log_2 7})$, or $\Theta(n^{\log_2 7})$.

   f) Here we have $a = 2, b = 4, f(n) = \sqrt{(}n)$.

   Since $f(n) = \sqrt{n} = n^{\log_4 2} = n^{\log_b a}$, case 2 of master's theorem applies.

   Thus $T(n)$ has upper bound of $\mathcal{O}(\sqrt{n} \lg n)$, and lower bound of $\Omega(\sqrt{n} \lg n)$, or $\Theta(\sqrt{n} \lg n)$.

   g) **Solution:**



Using recurrence tree method, we can see that the tree has depth of $\frac{n}{2}$, level cost of $(n-2i)^2$ where $i = 0, 1, ..., n-1$, and bottom level cost of $T(1)$ or $\Theta(1)$.

So, the total cost of $T(n)$ is:

$$
\begin{aligned}
T(n) &= \sum_{i=0}^{\frac{n}{2}-1} (n-2i) + \Theta(1) && (1)\\
&= \frac{n^2}{2} - 2\sum_{i=0}^{\frac{n}{2}-1} i + \Theta(1) && (2)\\
&= \frac{n^2}{2} - \left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right) + \Theta(1) && (3)\\
&= \left(\frac{n^2}{2}\right) - \left(\frac{n^2}{4} - \frac{n}{2}\right) + \Theta(1) && (4)\\
&= \frac{n^2}{4} + \frac{n}{2} + \Theta(1) && (5)\\
&= \Theta(n^2) && (6)
\end{aligned}
$$

And to verify $T(n)$, I will use subtitution method.

Let the guess be $T(n) \leq cn^3$.

Then,

$$
\begin{aligned}
T(n) &= T(n-2) + n^2 && (7)\\
&\leq c(n-2)^3 + n^2 && (8)\\
&= c(n^3 - 6n^2 + 12n - 8) + n^2 && (9)\\
&= c(n^3 - 5n^2 + 12n - 8) - n^2(c-1) && (10)\\
&\leq c(n^3 + 12n - 8) - n^2(c-1) && (11)\\
&= cn^3 - n^2(c-1) \quad \text{[Since } n^3 \text{ dominates } n\text{]} && (12)\\
&\leq cn^3 && (13)
\end{aligned}
$$

and the boundary holds as long as $c \geq 1$.