CSC373 Worksheet 5 Solution

August 11, 2020

1. *Proof.* Assume that a flow network $G = (V, E)$ violates the assumption that the network contains a path $s \rightsquigarrow v \rightsquigarrow t$ for all vertices $v \in V$. Let u be a vertex for which there is no path $s \rightsquigarrow u \rightsquigarrow t$.

   I must show such that there is no flow at vertex u. That is, there exists a maximum flow f in G such that $f(u, v) = f(v, u) = 0$ for all vertices $v \in V$.

   Assume for the sake of contradiction that there is some vertex u with flow f. That is, there exists some vertices $v \in V$ such that $f(u, v) > 0$ or $f(v, u) > 0$.

   I see that three cases follows, and I will prove each separately.

   1. **Cases 1:** $f(u, v) = 0$ and $f(v, u) > 0$

      Here, assume that $f(u, v) = 0$ for all $v \in V$ and $f(v, u) > 0$ for some $v \in V$.

      Then, we can write $\sum\limits_{v \in V} f(u, v) = 0$ and $\sum\limits_{v \in V} f(v, u) > 0$

      But this violates the flow conservation property (i.e $\sum\limits_{v \in V} f(u, v) = \sum\limits_{v \in V} f(v, u)$)

      Thus, by proof by contradiction, $f(u, v) = 0$ and $f(v, u) = 0$ for all $v \in V$ and all $u \in V$ with no path $s \rightsquigarrow u \rightsquigarrow t$.

   2. **Cases 2:** $f(u, v) > 0$ and $f(v, u) = 0$

      Here, assume that $f(u, v) > 0$ for some $v \in V$ and $f(v, u) = 0$ for all $v \in V$.

      Then, by similar work as case 1, the same result follows.

3. **Cases 3:** $f(u, v) > 0$ and $f(v, u) > 0$

   Here, assume that $f(u, v) > 0$ and $f(v, u) > 0$ for some $v \in V$.

   Since $s \rightsquigarrow v \rightsquigarrow t$ and u is connected by some vertices v, we can write $s \rightsquigarrow u \rightsquigarrow t$.

   Then, this violates the fact in header that the vertex u has no path $s \rightsquigarrow u \rightsquigarrow t$.

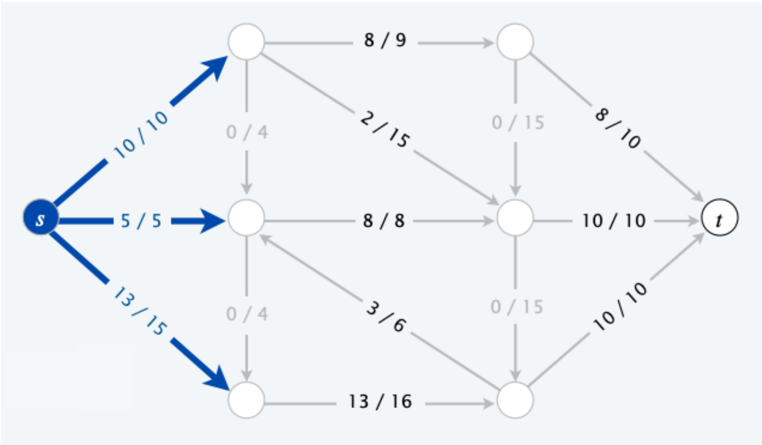
   Thus, by proof by contradiction, $f(u, v) = 0$ and $f(v, u) = 0$ for all $v \in V$ and all $u \in V$ with no path $s \rightsquigarrow u \rightsquigarrow t$.

$\square$

Notes

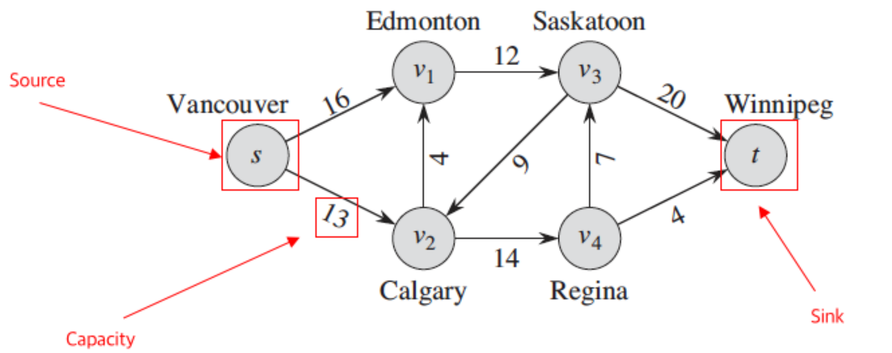
- **Maximum Flow:**
  - Finds a flow of maximum value [1]

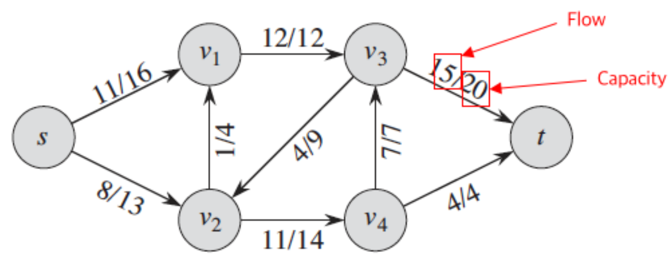
    Example



    Here, the maximum flow is $10 + 5 + 13 = 28$

- **Flow Network:**
  - $G = (V, E)$ is a directed graph in which each edge $(u, v) \in E$ has a nonnegative capacity $c(u, v) \geq 0$.
  - Two vertices must exist: **source** s and **sink** t
  - **path** from source s to vertax v to sink t is represented by $s \rightsquigarrow v \rightsquigarrow t$





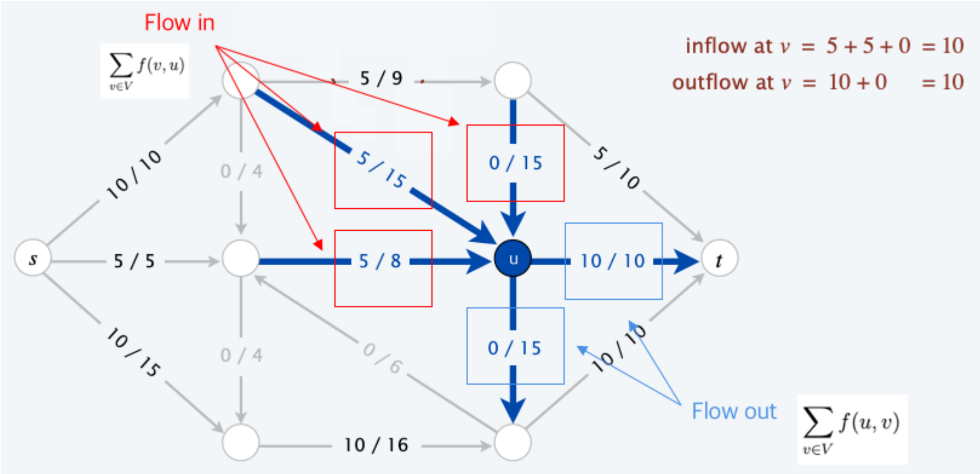
- **Capacity:**
  - Is a non-negative function $f : V \times V \to \mathbb{R}_{\geq 0}$
  - Has **capacity constraint** where for all $u, v \in V$ $0 \leq f(u, v) \leq c(u, v)$
    - Means flow cannot be above capacity constraint
- **Flow:**
  - Is a real valued function $f : V \times V \to \mathbb{R}$ in G
  - Satisfies **capacity constraint** (i.e for all $u, v \in V$, $0 \leq f(u, v) \leq c(u, v)$)
  - Satisfies **flow conservation**

    For all $u \in V - \{s, t\}$, we require

$$\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v) \tag{1}$$

    - Means flow into vertex u is the same as flow going out of vertex u. [1]
    - $\sum\limits_{v \in V} f(u, v)$ means flow <u>out of</u> vertex u
    - $\sum\limits_{v \in V} f(v, u)$ means flow <u>into</u> vertex u
    - $v \in V$ in $\sum\limits_{v \in V} f(u, v)$ means all vertices that are an edge away from vertex u

<u>**Example:**</u>



References

1) Princeton University, Network Flow 1, link

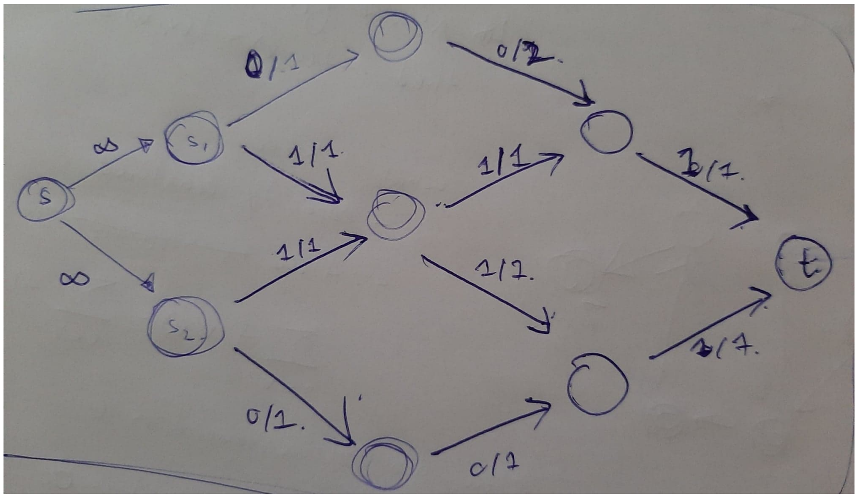
2. I need to formulate the problem of determining whether both of professor Adam's two children can go to the same school as maximum-flow problem.

   The problem statement tells us the following:

   1. There is 1 supersource (location of home)
   2. There is 1 sink (location of school)
   3. There are two sources (s_1 as child 1, s_2 as child 2)
   4. Edge (u, v) has capacity of 0 or more (0 representing unavailable sidewalk, 1 for sidewalk with capacity of 1, 2 for street with capacity of 2 and so on)
   5. Each vertex represents corner of intersection, and two children can have their paths crossing here.
   6. Has flow of 2, 1 or 0 (1 is where one of the two children walking on the road. 0 is none.)

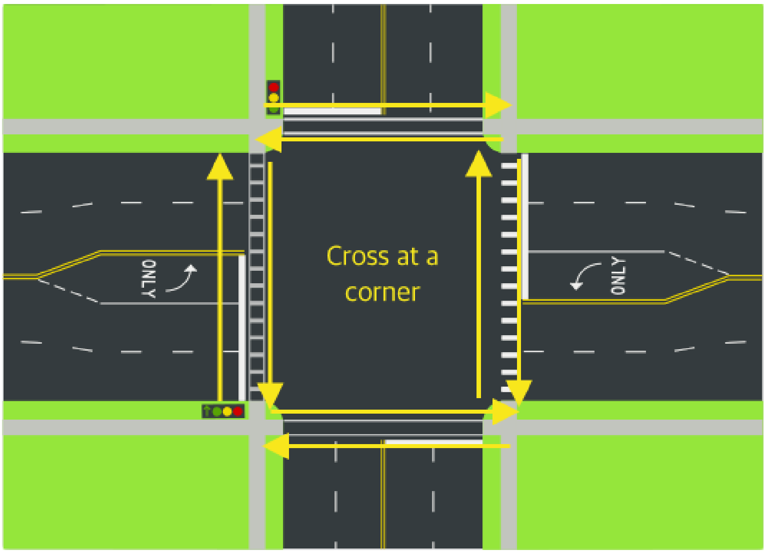
   Here we are to find whether children must go on to a vertex and out to the same edge with the flow of 2, or determine whether there is only edge to school with capacity of 1 or less.

   If none, then both children can safely go to school.



Notes:

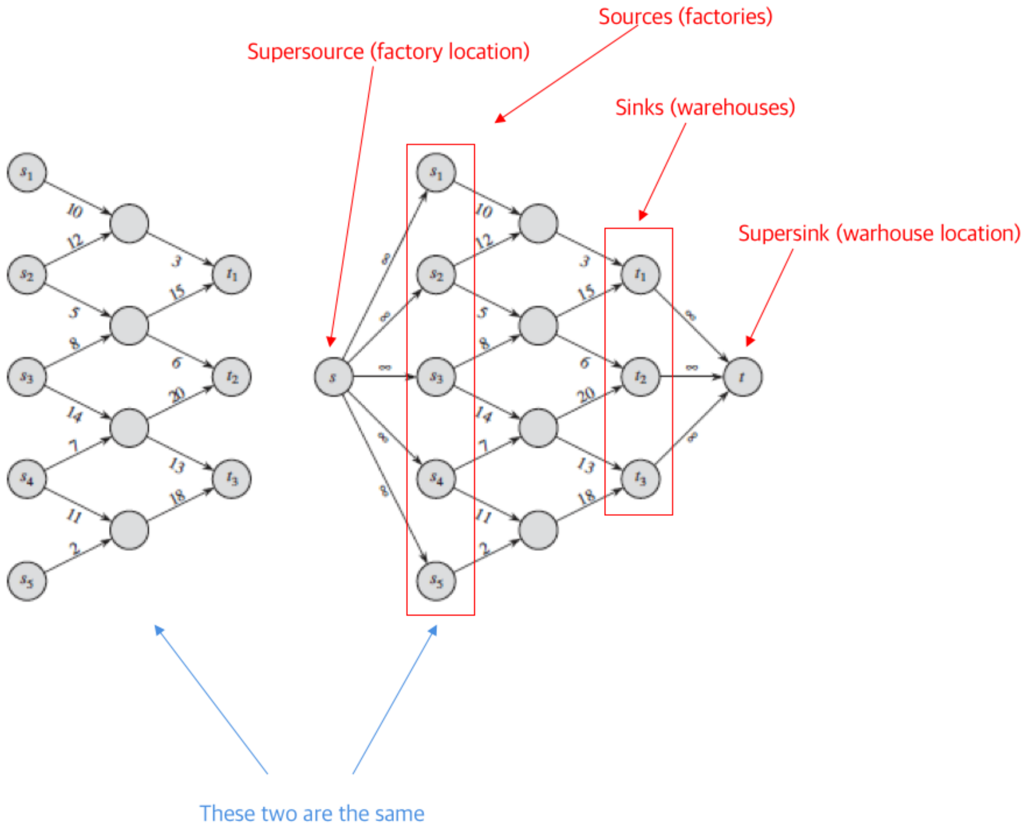
- **Cross at a Corner**
  - Means to walk across the street at a corner of the intersection.



- **Multiple Sources and Sinks**
  - Has edges (s, s_i) where $i = 1...n$ and (t_j, t) where $j = 1...n$ with capacity of ∞

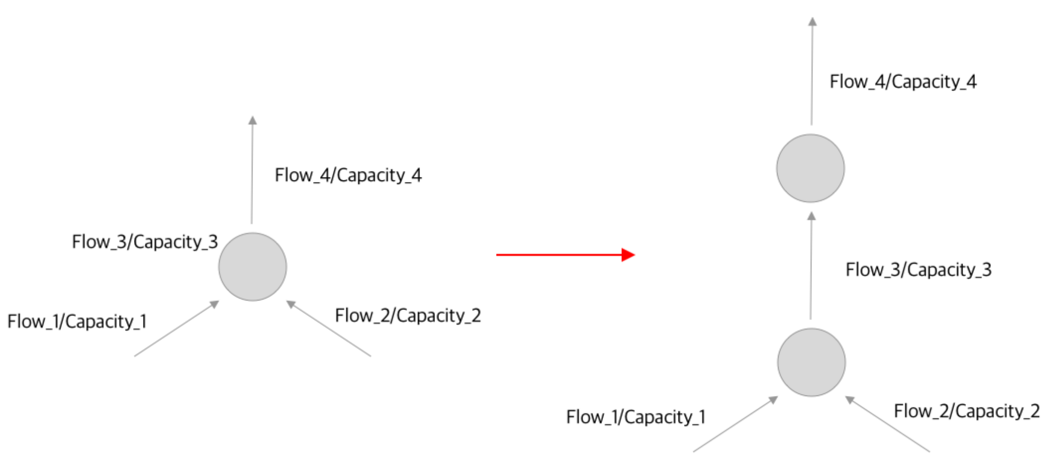
  Example:

  Lucky Puck Company having a set of m factories $\{s_1, s_2, ..., s_m\}$, and a set of n warehourses and n warehouses $\{t_1, t_2, ..., t_n\}$



3. I need to show how to transform a flow network $G = (V, E)$ with vertex capacities into an equivalent flow network $G' = (V', E')$ without vertex capacities.

   For each vertex capacities, change as follows.



   After transformation, there will be m more edges and verticies, where m represents the number of vertex capacities in G.

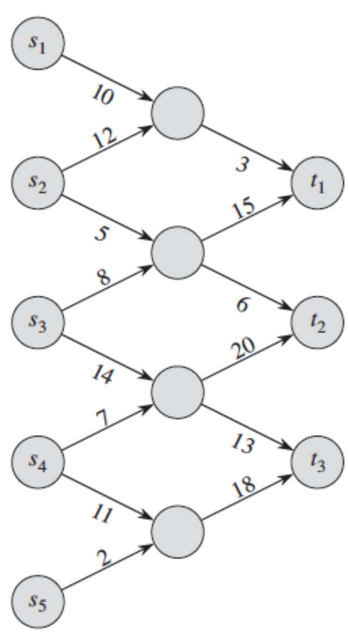
   Notes:

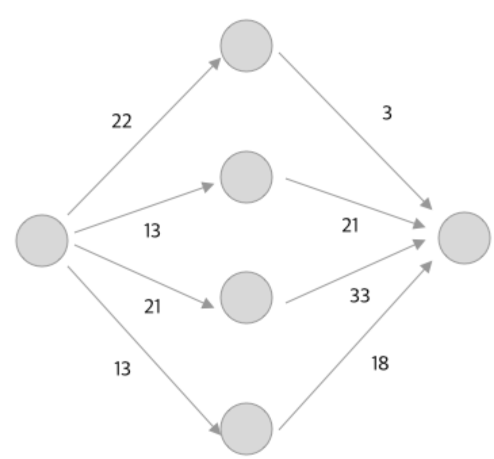
- **Vertex Capacities**
  - Each vertex v has limit $l(v)$ on how much flow can pass through v

4. I need to show how to convert the problem of finding a flow f that obeys the constraints into the problem of finding a maximum flow in a single source, single-sink flow network

   The steps are as follows:

   - Combine all sources s_i into a single source s
   - Combine all sinks t_j into a single sink t
   - Connect source s to each adjacent vertex v with edge weight $\sum_i f(s_i, v) = p_i$
     - The total edge weight from s should be $\sum_i p_i$
   - Connect each adjacent vertex v of t to t with edge weight $\sum_j f(v, t_j) = q_j$
     - The total edge weight to t should be $\sum_j q_j$
   - Find a simple path from s to t with the maximum amount of total flow





Correct Solution:

I need to show how to convert the problem of finding a flow f that obeys the constraints into the problem of finding a maximum flow in a single source, single-sink flow network

The steps are as follows:

- Combine all sources s_i into a single source s
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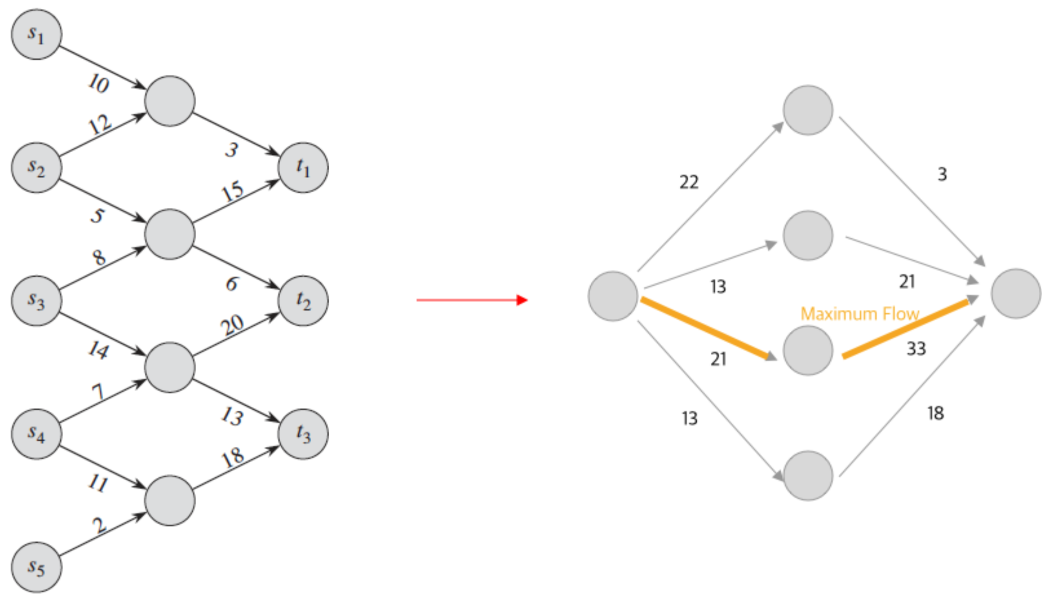
- Connect source s to each adjacent vertex v with edge weight $\sum_i f(s_i, v) = p_i$
  - The total edge weight from s should be $\sum_i p_i$
- Connect each adjacent vertex v of t to t with edge weight $\sum_j f(v, t_j) = q_j$
  - The total edge weight to t should be $\sum_j q_j$
- Find a simple path from s to t with the maximum amount of total flow

Example



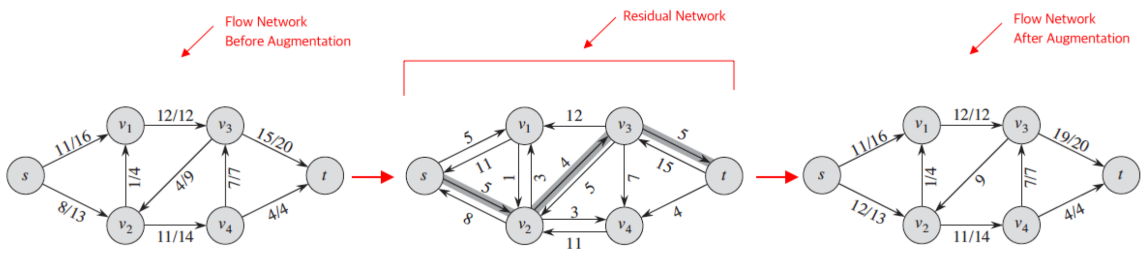
Notes:

- **Ford-Fulkerson Method**
  - Is a greedy algorithm that solves the maximum-flow problem
    - Determines maximum flow from start vertex to sink vertex in a graph
  - Called method (not algorithm) because several different implementations with different running time is used

FORD-FULKERSON-METHOD(G, s, t)

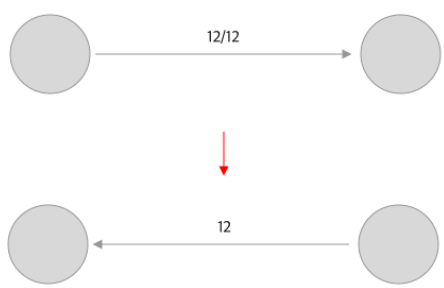
initialize flow f to 0
while there exists an augmenting path p in the residual network G_f
    augment flow f along p
return f

- **Residual Network**
  - Indicates how muh more flow is allowed in each edge in the network graph [1]
  - Consists of edges with capacities that represents how we can change the flow on edges of G.
  - Provides roadmap for adding flow to the original flow network

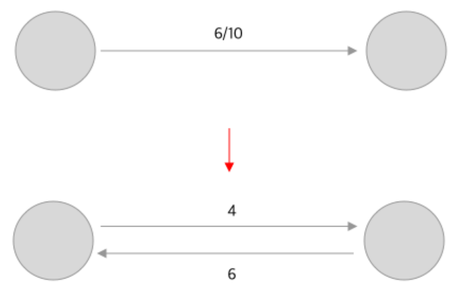


Steps

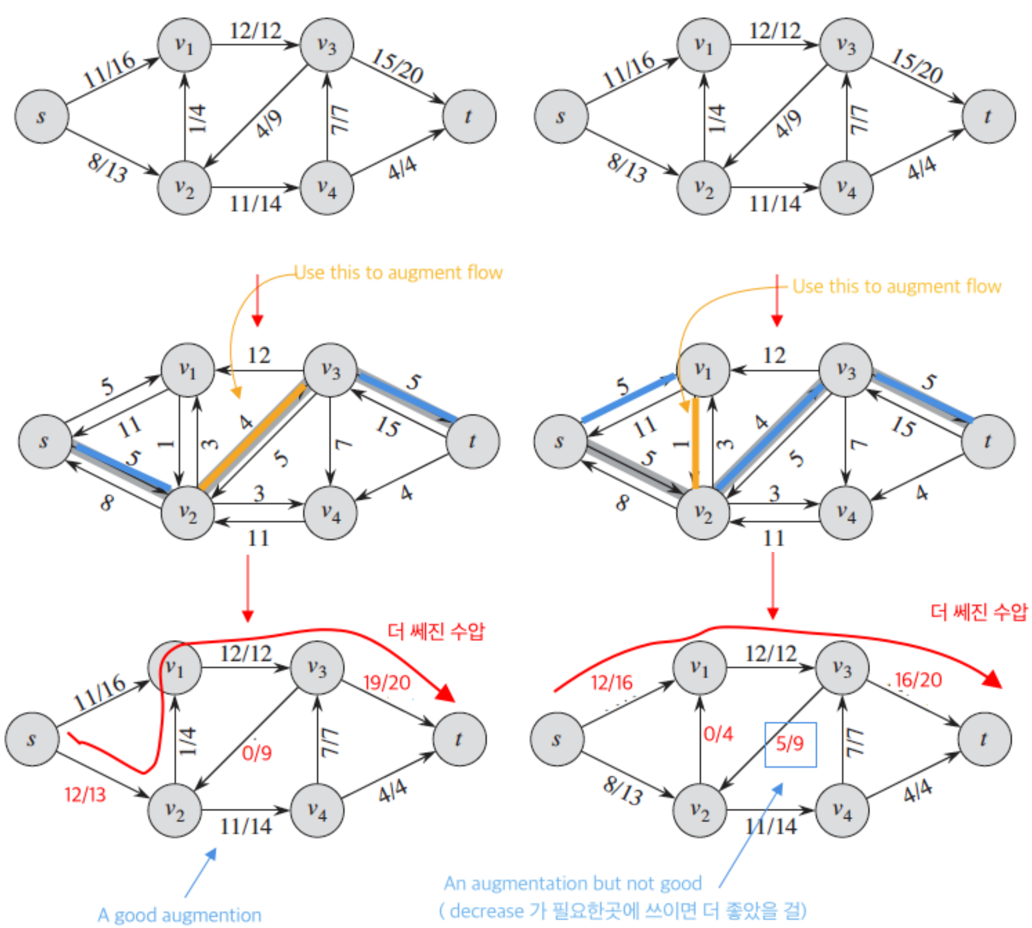
1) $Flow = Capacity$: Opposite arrow



2) $Flow < Capacity$:
   - $Flow$: Oppisite Arrow
   - $Capacity - Flow$: Current Arrow



- **Augmenting Path**
  - Is a path from source S to sink T where you can increase the amount of flow
  - Is a path that doesn't contain cycle (simple path) [2]



  - Edge (u, v) of an augmented path can be increased by upto $c_f(u, v)$ withhout violating the capacity constraint
- **Augmentation**
  - 한국어로 '불필요한 수압 decrease 해서 앞으로 가는 수압 더 쎄게 만들기'
  - Is symbolized by $f \uparrow f'$

* f is a flow in G
* f' is a flow in the residual network G_f

References

1) Hacker Earth, Maximum Flow, link
2) Stack Overflow, What Exactly Is Augmentation Path, link

5. The augmented flow satisfies flow conservation, but not capacity constraint.

Proof. Let $G = (V, E)$ be a flow network with sources s and sink t. Let f, f' be a flow in G. Let (u, v) be an edge in E where $u \in V - \{s, t\}$ and $v \in V$. We note that if $(u, v) \in E$, then $(v, u) \notin E$ and $f(v, u) = 0$. Thus, we can re-write the definition of flow augmentation (equation (26.4)) as

$$(f \uparrow f')(u, v) = \begin{cases} f(u, v) + f'(u, v) & \text{[If } (u, v) \in E\text{]} \\ 0 & \text{[Otherwise]} \end{cases} \tag{1}$$

which implies that the value of the augmentation of flow $f \uparrow f'$ on edge (u, v) is the sum of flow $f(u, v)$ and $f'(u, v)$ in G.

I need to show if the augmented flow of f and $f' \in G$ and satisfy the flow conservation property but not capacity constraint.

I will do so in parts

- **Part 1: Proving that $f \uparrow f'$ satisfies the flow conservation property**

  Here I prove that the augmented flow satisfies flow conservation. That is,

$$\sum_{v \in V} f \uparrow f'(u, v) = \sum_{v \in V} f \uparrow f'(v, u) \tag{2}$$

  .

  And indeed we have,

$$\begin{aligned}
\sum_{v\in V} f \uparrow f'(u,v) &= \sum_{v\in V} f(u,v) + f'(u,v) && \text{[By augmentation def.]} \quad (3)\\
&= \sum_{v\in V} f(u,v) + \sum_{v\in V} f'(u,v) && (4)\\
&= \sum_{v\in V} f(v,u) + \sum_{v\in V} f'(v,u) && \text{[By flow conserv. of } f \text{ and } f'] \quad (5)\\
&= \sum_{v\in V} f(v,u) + f'(v,u) && (6)\\
&= f \uparrow f'(v,u) && (7)
\end{aligned}$$

- **Part 2: Disproving that $f \uparrow f'$ satisfies the capacity constraint**

  Here, I need to disprove that the augmented flow satisfies capacity constraint. That is,

  $$(f \uparrow f')(u,v) > c(u,v) \tag{8}$$

  Let $f(u,v) = f'(u,v) = 10$ and $c(u,v) = 8$.

  Then, we can write $(f \uparrow f')(s,t) = 20$ and $c(u,v) = 8$.

  Thus, we can conclude the augmentation of flow doesn't satisfy capacity constraint.

  □

<u>**Notes:**</u>

- I need clarification from professor about the meaning of $f' \in G$. Is f' a flow from flow network or residual network?
- I feel I am struggling because I am jumping to solution without understanding the problem
- I feel constructing a predicate logic would have helped to better understand this problem
- Noticed that a solution in University of Texas really elaborated on $f \uparrow f'(u,v)$ before moving onto strategizing and constructing a solution

capacity constraint property.

First, we prove that $f \uparrow f'$ satisfies the flow conservation property. We note that if edge $(u, v) \in E$, then $(v, u) \notin E$ and $f(v, u) = 0$. Thus, we can rewrite the definition of flow augmentation (equation (26.4)), when applied to two flows, as

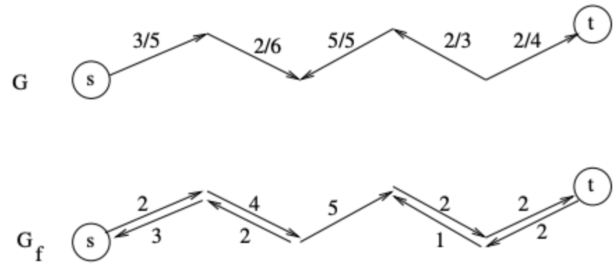
$$f \uparrow f'(u, v) = \begin{cases} f(u, v) + f'(u, v), & if (u, v) \in E \\ 0, & otherwise. \end{cases}$$

The definition implies that the new flow on each edge is simply the sum of the two flows on that edge. We now prove that in $f \uparrow f'$, the net incoming flow for each vertex equals the net outgoing flow. Let $u \notin \{s, t\}$ be any vertex of G. We have

- Noticed that a solution in University of Texas made quick sketches before laying the outline of proof
- **Flow Network (cont'd)** [Important!]
  - Flow network requires that
    1) $G = (V, E)$ is a directed graph
    2) each edge $(u, v) \in E$ has a non-negative capacity $c(u, v) \geq 0$
    3) If E contains an edge (u, v), then there is no edge (v, u) in the reverse direction (no anti-parallel edge)
- **Augmentation (cont'd)**
  - Flow value can be increased by

$$c_f(p) = \min_{(u,v) \in p} c_f(u, v) \tag{9}$$

Example:

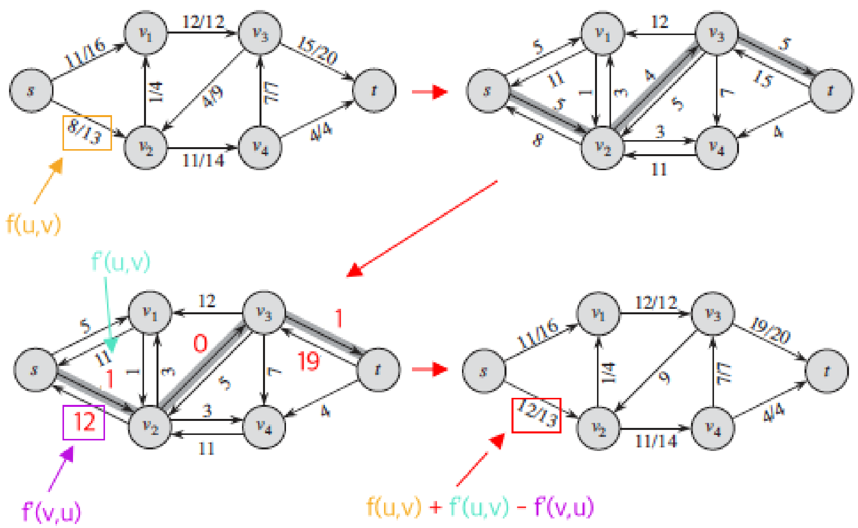


In this example, augmentation by 2.

  - Augmentation of flow f by f' or $f \uparrow f'$ is a function $V \times V \rightarrow \mathbb{R}$ is defined by

$$(f \uparrow f')(u, v) = \begin{cases} f(u, v) + f'(u, v) - f'(v, u) & [\text{If } (u, v) \in E] \\ 0 & [\text{Otherwise}] \end{cases} \tag{10}$$

– Augmentation of flow $f \uparrow f'$ is the sum of flow on edge (u, v) in both flow network G and residual network G'



[NOTE!!] I really need to ask professor about this. I can't seem to understand using simple example why $f \uparrow f'(u, v) = f(u, v) + f'(u, v) - f'(v, u)$.

- **Proof of flow conservation for $f \uparrow f'$ when $f \in G$ and $f' \in G_f$**

Let $G = (V, E)$ be a flow network with sources s and sink t. Let f be a flow in G. Let G_f be a residual network of G induced by f and let f' be a flow in G_f. Let (u, v) be an edge in E where $u \in V - \{s, t\}$ and $v \in V$. We note that if $(u, v) \in E$, then $(v, u) \notin E$ and $f(v, u) = 0$. Thus, the definition

$$(f \uparrow f')(u, v) = \begin{cases} f(u, v) + f'(u, v) - f'(v, u) & [\text{If } (u, v) \in E] \\ 0 & [\text{Otherwise}] \end{cases} \tag{1}$$

implies that the augmented flow $f \uparrow f'(u, v)$ on edge (u, v) is the sum of flow $f(u, v)$ in flow network G and flow $f'(u, v)$ minus its antiparallel flow $-f'(v, u)$ in residual flow network G'.

We now prove that the augmented flow satisfies flow conservation. That is,

$$\sum_{v \in V} f \uparrow f'(u, v) = \sum_{v \in V} f \uparrow f'(v, u) \tag{2}$$

.

And indeed we have

$$
\begin{aligned}
\sum_{v \in V} f \uparrow f'(u,v) &= \sum_{v \in V} f(u,v) + f'(u,v) - f'(v,u) && \text{[By augmentation def.]} \quad (3) \\
&= \sum_{v \in V} f(u,v) + \sum_{v \in V} f'(u,v) - \sum_{v \in V} f'(v,u) && (4) \\
&= \sum_{v \in V} f(v,u) + \sum_{v \in V} f'(v,u) - \sum_{v \in V} f'(u,v) && \text{[By flow conserv. of } f \text{ and } f'] \quad (5) \\
&= \sum_{v \in V} f(v,u) + f'(v,u) - f'(u,v) && (6) \\
&= \sum_{v \in V} f \uparrow f'(v,u) && (7)
\end{aligned}
$$

– Flow in residual network also obey flow conservation

- **Proof of capacity constraint for $f \uparrow f'$ when $f \in G$ and $f' \in G_f$**

  Predicate Logic: $\forall f \in G$, $\forall f' \in G_f$, $\forall (u,v) \in E$ where $u, v \in V$, $0 \leq (f \uparrow f')(u,v) \wedge (f \uparrow f')(u,v) \leq c(u,v)$

  Let $G = (V, E)$ be a flow network with sources s and sink t. Let f be a flow in G. Let G_f be a residual network of G induced by f and let f' be a flow in G_f. Let (u,v) be an edge in E where $u, v \in V$.

  I need to prove that $f \uparrow f'$ satisfies capacity constraint. That is, $0 \leq (f \uparrow f')(u,v) \wedge (f \uparrow f')(u,v) \leq c(u,v)$.

  I see there are two parts. I will prove each parts separately.

  1. **Part 1** $(0 \leq (f \uparrow f')(u,v))$

     Here, I need to show $0 \leq (f \uparrow f')(u,v)$. That is, $0 \leq f(u,v) + f'(u,v) - f'(v,u)$.

     And indeed we have,

$$
\begin{aligned}
(f \uparrow f')(u,v) &= f(u,v) + f'(u,v) - f'(v,u) && (8) \\
&\geq f(u,v) + f'(u,v) - c_f(v,u) && \text{[Since } f'(v,u) \leq c_f(v,u)] \quad (9) \\
&= f(u,v) + f'(u,v) - f(u,v) && \text{[By def. of residual capacity]} \quad (10) \\
&= f'(u,v) && (11) \\
&\geq 0 && \text{[By cap. const. of } f' \text{ in } G_f \text{]} \quad (12) \\
& && (13)
\end{aligned}
$$

- $c_f(v, u) = f(u, v)$ is allowed

2. **Part 2** ($(f \uparrow f')(u, v) \leq c(u, v)$)
Here, I need to show $(f \uparrow f')(u, v) \leq c(u, v)$. That is, $f(u, v) + f'(u, v) - f'(v, u) \leq c(u, v)$.

And indeed we have,

$$
\begin{aligned}
(f \uparrow f')(u, v) &= f(u, v) + f'(u, v) - f'(v, u) && (14)\\
&\leq f(u, v) + f'(u, v) && \text{[Since } f'(u, v) \geq 0 \text{ by cap. cons. of } f'\text{]} \quad (15)\\
&= f(u, v) + c_f(u, v) && \text{[Since } f'(u, v) \leq c_f(u, v)\text{]} \quad (16)\\
&= f(u, v) + (c(u, v) - f(u, v)) && \text{[By def of res. capacity]} \quad (17)\\
&= c(u, v) && (18)\\
& && (19)
\end{aligned}
$$

References

1) University of Teaxs, CSE 5311 Homework 5 Solution, link

6. **Rough Works:**
Let $G = (V, E)$ be an undirected graph.

I need to show how to determine the edge connectivity of G by running a maximum-flow algorithm.

- Convert undirected graph to directed graph

First, I need to convert the undirected graph G to a directed graph. I do so by assigning G' as a directed graph and transforming each edges in G to two directed edges (u, v) and (v, u).

- Setup directed graph as a flow network

Second, I need to setup graph G' as a flow network. I do so by assigning each edge in G' with capacity of 1 and flow of 1.

- Find the edge connectivity of new directed graph

Third, I need to find the edge connectivity of new directed graph.

We first note that a directed graph is connected if there is a path between every pair of verticies.

Thus, the edge connectivity implies the minimum number of edges required to remove a vertex from graph G'.

So, I now show the minimum number of edges that must be removed to disconnect the graph.

I note that a directed graph is connected if there is a path between every pair of verticies.

So I do so by using maximum flow algorithm to find the vertex u with the minimum amount of flow, and then cutting those edges to make the graph disconnected.

Thus, I claim that

- Prove correctness of Algorithm

  Suppose k is the edge connectivity of the graph and S is the set of k edges such that removal of S will disconnect the graph into 2 non-empty subgraphs G_1 and G_2. WLOG assume the node $u \in G$. Let w be a node in G_2. Since $u/neqw$, the value $f^*(u.w)$ will be computed by the algorithm. By the min-cut theorem $f^*(u,w)$ equals the min cut size between the pair (u,w), which is at most k since S duscibbects u and w. Therefore, we have

  $$c* \leq f^*(u,w) \leq k \tag{20}$$

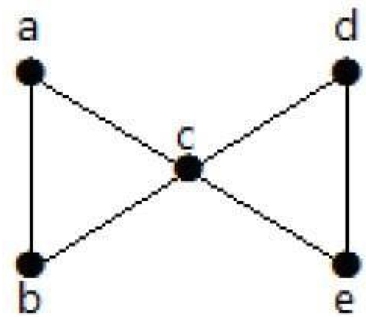
  But c° cannot be smaller than k since that would imply a cut set of size smaller than k, contradicting the fact that k is the edge connectivity. Therefore $c^{\circ} = k$ and the algorithm returns the edge connectivity of the graph correctly.

Notes

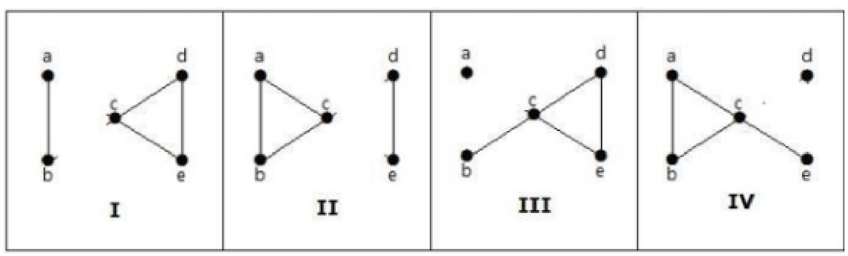
- I feel that this maximum-flow algorithm maximizes flow value in a vertex with edges and antiparallel edges.
- I wonder what does this maximum-flow algorithm do.
- I am wondering how I can
- **Maximum Flow:** Is a vertex in flow network with the maximum value of flow

- **Edge Connectivity:** Is the minimum number k of edges that must be removed to disconnect the graph. That is, the **number of edges in a smallest cut set of G** [2]

  <u>**Example:**</u>

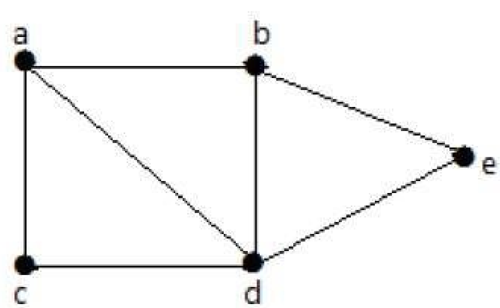


Here are the four ways to disconnect the graph by removing two edges −

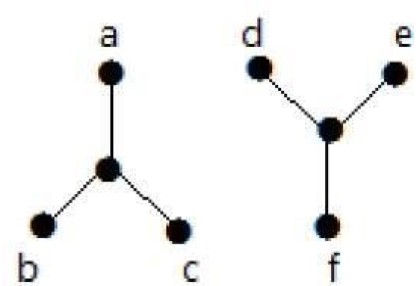


- **Connected:** A graph is said to be connected if there is a path between <u>every pair of vertex</u>

  <u>**Example:**</u>



Connected



Not Connected

- **Cut:**

  - Is denoted (S, T)
  - Is a partition of v into S and $T = V - S$ such that $s \in S$ and $t \in T$

References

1) National Taiwan University, Voluntary Exercise 3, link
2) TutorialsPoint, Graph Theory - Connectivity, link