

# CSC236 Midterm 2 Version 1 Solution

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## Question 1

- Let  $n, q \in \mathbb{N}$ . Let  $r \in \{0, 1\}$

Assume  $n > 2$ , and  $n = 2q + r$ .

I need to find a closed form for  $T(2q + r)$ , using repeated substitution.

Starting from  $T(n)$ , we have

$$T(n) = n + T(n - 2) \quad [\text{By def. since } n > 2] \quad (1)$$

$$T(2q + r) = 2q + r + T(2q + r - 2) \quad [\text{By replacing } n \text{ for } 2q + r] \quad (2)$$

$$= 2q + r + T(2(q - 1) + r) \quad (3)$$

$$\vdots \quad (4)$$

$$= \sum_{i=0}^{i=q-1} (2(q - i) + r) + T(r) \quad [\text{After } q - 1 \text{ repeatitions}] \quad (5)$$

$$= 2 \sum_{i=0}^{i=q-1} (q - i) + \sum_{i=0}^{i=q-1} r + T(r) \quad (6)$$

$$= 2 \sum_{i=0}^{i=q-1} (q - i) + \sum_{i=0}^{i=q-1} r \quad [\text{Since } T(r) = 0] \quad (7)$$

$$= 2 \sum_{i'=1}^{i'=q} i' + \sum_{i=0}^{i=q-1} r \quad (8)$$

$$= 2 \sum_{i'=1}^{i'=q} i' + \sum_{i=0}^{i=q-1} r \quad (9)$$

$$= 2 \sum_{i'=1}^{i'=q} i' + \sum_{i=0}^{i=q-1} r \quad (10)$$

$$= 2(q(q + 1))/2 + \sum_{i=0}^{i=q-1} r \quad [\text{By using } \sum_{i=1}^{i=n} i = (n(n + 1))/2] \quad (11)$$

$$= q(q + 1) + rq \quad (12)$$

$$= q(q + 1 + r) \quad (13)$$

- *Proof.* For convenience, define  $H(q) : q(q + r + 1) = T(2q + r)$ .

I will use simple induction to prove  $\forall q \in \mathbb{N}, H(q)$ .

**Base Case ( $q = 0$ ):**

Let  $q = 0$ .

Then,

$$q(q + r + 1) = 0 \quad (1)$$

$$= T(2 \cdot 0 + r) \quad [\text{By def.}] \quad (2)$$

$$= T(2q + r) \quad (3)$$

Thus,  $T(2q + r)$  verifies in this step.

**Inductive Step:**

Let  $q \in \mathbb{N}$ . Assume  $H(q)$ .

I need to show  $H(q + 1)$  follows. That is,  $(q + 1) \left[ (q + 1) + r + 1 \right] = T(2(q + 1) + r)$ .

Starting with  $(q + 1) \left[ (q + 1) + r + 1 \right]$ , we have

$$(q + 1) \left[ (q + 1) + r + 1 \right] = (q + 1)(q + 1) + (q + 1)r + (q + 1) \quad (4)$$

$$= q^2 + 2q + 1 + (qr + r) + (q + 1) \quad (5)$$

$$= (q^2 + qr + q) + (2q + r + 2) \quad (6)$$

$$= q(q + r + 1) + (2(q + 1) + r) \quad (7)$$

$$= T(2q + r) + 2(q + 1) + r \quad [\text{By I.H}] \quad (8)$$

$$= T(2(q + 1) + r) \quad [\text{By def.}] \quad (9)$$

Thus,  $H(q + 1)$  follows from  $H(q)$  in this step.

□