CSC236 Worksheet 3

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Question 1

Rough Work:

Predicate Logic: $\forall A \subseteq \mathbb{N}$, $A \neq \emptyset \Rightarrow (\exists a \in A, \forall x \in A, a \leq x)$

Given the statement to prove

$$P(x, y, z) : \text{There are no positive integers } x, y, z \text{ such that } x^3 + 3y^3 = 9z^3$$

I will prove $P(x, y, z)$ using proof by contradiction.

Assume $\exists x, y, z \in \mathbb{N}^+$, $x^3 + 3y^3 = 9z^3$.

1. State that the set $X = \{x \mid x \in \mathbb{N}^+, \exists y, z \in \mathbb{N}^+, x^3 + 3y^3 = 9z^3\}$ is not empty.

2. State that there is smallest element $x_0 \in X$ with $y_0, z_0 \in \mathbb{N}^+$ satisfying $x^3 + 3y^3 = 9z^3$, using well-ordering principle

3. Show that $x_0 = 3 \cdot k$, using $x_0^3 + 3y_0^3 = 9z_0^3$ and the fact if a prime number p divides a perfect cube n^3, then p also divides n.

4. Show that $x_1^3 = 9z_1^3 - 3y_1^3$ is satisfied, given $x_0 > x_1$

   - First, show that $x_0 = 3 \cdot x_1$, using $x_0^3 + 3y_0^3 = 9z_0^3$ and the fact if a prime number p divides a perfect cube n^3, then p also divides n.
   - Second, show that $y_0 = 3 \cdot y_1$, using $x_0^3 = 3^3 x_1^3 = 9z_0^3 - 3y_0^3$
   - Third, show that $z_0 = 3 \cdot z_1$, using $x_0^3 = 3^3 x_1^3 = 9z_0^3 - 3y_0^3 = 9z_0^3 - 3^4 y_1^3$

- Finally, show $x_1^3 = 9z_1^3 - 3y_1^3$

5. Conclude that this contracts the original claim that x_0 is the smallest non-negative value satisfying $x^3 + 3y^3 = 9z^3$.

Notes:

- **Proof By Contradiction:** $\neg P \Rightarrow \neg Q \wedge Q$ (Assuming we are proving $P \Rightarrow Q$)

- **Principle of Well-Ordering:** Any nonempty subset A of $\mathbb{N}$ contains a minimum element; i.e. for any $A \subseteq \mathbb{N}$ such that $A \neq \emptyset$, there is some $a \in A$ such that for all $a' \in A$, $a \leq a'$.

- examples of well-ordered sets

  1. $\mathbb{N} \cup \{0\}$
  2. $\mathbb{N} \cup \{1, 2\}$
  3. $\{n \in \mathbb{N} : n > 5\}$

- examples of non-well-ordered sets

  1. $\mathbb{R}$ and the open interval $(0, 2)$
  2. $\mathbb{Z}$

Question 2

Question 3