# CSC236 Worksheet 6 Solution

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May 9, 2020

## Question 1

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**Rough Work:**

Assume that for all $k \in \mathbb{N}$, $R(3^k) = k3^k$.

1. Prove that $R \in \mathcal{O}(n \lg n)$.

   Define $n^* = 3^{\lceil \log_3 n \rceil}$. Then, we have,

$$\lceil \log_3 n \rceil - 1 < \log_3 n \leq \lceil \log_3 n \rceil \Rightarrow n^*/3 < n \leq n^* \tag{1}$$

   I will also use the assumption (proved last week) that $R$ is non-decreasing.

   Let $d = 6$. Then $d \in \mathbb{R}^+$. Let $B = 3$. Then $B \in \mathbb{N}^+$. Let $n$ be an arbitrary natural number no smaller than $B$. Then,

$$
\begin{aligned}
R(n) &\le R(n*) && (2)\\
&= k3^k && \text{[By assumption]} \quad (3)\\
&\le n^* \log_3 n^* && \text{[By replacing } n^* \text{ for } 3^k\text{]} \quad (4)\\
&\le 3n \log_3 3n && \text{[Since } n^*/3 < n \le n^* \Rightarrow n^* < 3n < 3n^*\text{]} \quad (5)\\
&\le 3n(\log_3 n + 1) && (6)\\
&\le 3n(\log_3 n + \log_3 n) && \text{[Since } n \ge 3 \Rightarrow \log_3 n \ge 1\text{]} \quad (7)\\
&= 6n \log_3 n && (8)\\
&\le (6n \lg n)/\lg 3 && \text{[By change of basis to lg]} \quad (9)\\
&< 6n \lg n && (10)\\
&= dn \lg n && \text{[Since } d = 6\text{]} \quad (11)
\end{aligned}
$$

So $R \in \mathcal{O}(n \lg n)$, since $\log_3 n$ differs from $\lg n$ by a constant factor.

2. Prove $R \in \Omega(n \log n)$

**Notes:**

- $g \in \Theta(f): \; g \in \mathcal{O}(f) \wedge g \in \Omega(f)$

  or

  $g \in \Theta(f): \; \exists c_1, c_2, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \ge n_1 \Rightarrow c_1 g(n) \le f(n) \le c_2 g(n)$, where $f, g: \mathbb{N} \to \mathbb{R}^{\ge 0}$
- $g \in \Omega(f): \; \exists c, n_o \in \mathbb{R}^+, \; \forall n \in \mathbb{N}, \; n \ge n_0 \Rightarrow g(n) \ge cf(n)$, where $f, g: \mathbb{N} \to \mathbb{R}^{\ge 0}$
- $g \in \mathcal{O}(f): \; \exists c, n_o \in \mathbb{R}^+, \; \forall n \in \mathbb{N}, \; n \ge n_0 \Rightarrow g(n) \le cf(n)$, where $f, g: \mathbb{N} \to \mathbb{R}^{\ge 0}$