Problem Set 3 Solution

March 23, 2020

Question 1

1. Let $x \in \mathbb{R}$.

   Base Case (n = 0):

   Let $n = 0$.

   Then,

   $$a_0 = 0 \tag{1}$$

   Then it follows from above that the base case holds.

   Inductive Case ($n > 0$):

   Let $k \in \mathbb{N}$, and assume $a_n = x \prod_{i=0}^{n-1} a_i$.

   Then,

   $$\begin{aligned} x \prod_{i=0}^{n-1} a_i \cdot a_n &= x \prod_{i=0}^{n} a_i && (1) \\ &= a_{n+1} && (2) \end{aligned}$$

Then it follows from above that the recursive sequence of numbers is true for all natural numbers.

2. From the following table

String Length	Number of Even (Digit Sum)	Number of Odd (Digit Sum)	Total
1	2	1	3
2	5	4	9
3	14	13	27

we see that $E_n = \frac{3^n+1}{2}$ and $O_n = \frac{3^n-1}{2}$.

As well, we see that the number of new elements in E_{n+1} is 3^n.

Now, we will prove that E_n and O_n are true for all natural numbers using the induction hypothesis.

Base Case (n = 1):

Let $n = 1$.

Then, $E_n = \frac{4}{2} = 2$ and $O_n = \frac{2}{2} = 1$.

Since the result matches to data in table, the base case holds.

Inductive Case:
Let $n \in \mathbb{N}$. Assume $E_n = \frac{3^n+1}{2}$ and $O_n = \frac{3^n-1}{2}$.

Then,

$$\begin{aligned}
E_{n+1} &= \frac{3^n+1}{2} + 3^n && (1)\\
&= \frac{3^n+1}{2} + \frac{2 \cdot 3^n}{2} && (2)\\
&= \frac{3 \cdot 3^n + 1}{2} && (3)\\
&= \frac{3^{n+1}+1}{2} && (4)
\end{aligned}$$

Then, it follows from above that the inductive step for E_n holds.

Similarly, for O_n,

$$\begin{aligned} O_{n+1} &= \frac{3^n - 1}{2} + 3^n && (5) \\ &= \frac{3^n - 1}{2} + \frac{2 \cdot 3^n}{2} && (6) \\ &= \frac{3 \cdot 3^n - 1}{2} && (7) \\ &= \frac{3^{n+1} - 1}{2} && (8) \end{aligned}$$

Then, it follows from above that the inductive step for O_n holds.

Then, it follows from the definition of induction hypothesis that the value of E_n and O_n are true for all n.

Question 2

a. Since first 1 repeats every $4i - 1$ times and the second 1 repeats every $4i$ times,

$$\begin{aligned} (0.\overline{0011})_2 &= \sum_{i=1}^{\frac{n}{4}} \left(\frac{1}{2}\right)^{4i} + \sum_{i=1}^{\frac{n}{4}} \left(\frac{1}{2}\right)^{4i-1} && (1) \\ &= \sum_{i=1}^{\frac{n}{4}} \left(\frac{1}{16}\right)^{i} + 2 \cdot \sum_{i=1}^{\frac{n}{4}} \left(\frac{1}{16}\right)^{i} && (2) \\ &= \frac{1}{16} \cdot \sum_{i=0}^{\frac{n}{4}-1} \left(\frac{1}{16}\right)^{i} + \sum_{i=0}^{\frac{n}{4}-1} \left(\frac{1}{16}\right)^{i} && (3) \\ &= \frac{3}{16} \cdot \sum_{i=0}^{\frac{n}{4}-1} \left(\frac{1}{16}\right)^{i} && (4) \end{aligned}$$

Then,

$$\frac{3}{16} \cdot \sum_{i=0}^{\frac{n}{4}-1} \left(\frac{1}{16}\right)^i = \frac{3}{16} \cdot \left(\frac{1 - \frac{1}{16}^{\frac{n}{4}}}{1 - (\frac{1}{16})}\right) \tag{5}$$

by using the formula $\forall n \in \mathbb{Z}^+ and r \in \mathbb{R},\ r \neq 1 \Rightarrow \sum\limits_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$.

Then,

$$\frac{3}{16} \cdot \left(\frac{1 - \frac{1}{16}^{\frac{n}{4}}}{1 - (\frac{1}{16})}\right) = (\frac{1 - \frac{1}{2}^n}{\frac{15}{16}}) \tag{6}$$

$$= \frac{1}{5} \cdot \left(1 - \frac{1}{2}^n\right) \tag{7}$$

$$= \frac{1}{5} \cdot \left(\frac{2^n - 1}{2^n}\right) \tag{8}$$

Then,

$$0.2 - \frac{1}{5} \cdot \left(\frac{2^n - 1}{2^n}\right) = \frac{1}{5} - \frac{1}{5} \cdot \left(\frac{2^n - 1}{2^n}\right) \tag{9}$$

$$= \frac{2^n}{5 \cdot 2^n} - \frac{1}{5} \cdot \left(\frac{2^n - 1}{2^n}\right) \tag{10}$$

$$= \frac{1}{5 \cdot 2^n} \tag{11}$$

Then, it follows from above that $\forall n \in \mathbb{Z}^+,\ 4 \mid n \Rightarrow \frac{1}{5 \cdot 2^n}$

b. Let $n \in \mathbb{Z}^+$, and $x \in \{x \mid x \in \mathbb{R}^+, 0 \leq x < 1\}$.

We will prove that the statement $\forall n \in \mathbb{Z}^+,\ \forall x \in S,\ \exists x_1 \in S,\ FB(n, x_1) \wedge 0 \leq x - x_1 < 1$ is true using induction hypothesis.

Let $n = 1$.

Case 1 ($0 \leq x < 0.5$, from $S = x \mid x \in \mathbb{R}, 0 \leq x < 1$):

Let $x_1 = 0$.

Then,

$$0 = (0.0)_2 \tag{1}$$
$$= \sum_{i=1}^{1} \frac{b_i}{2} \tag{2}$$

by the fact that $b_i = 0$.

Then, it follows from above that $FB(1, x_1)$ is true.

Now we will prove that $0 \leq x - x_1 < \frac{1}{2}$ is true.

Let $x_1 = 0$. Assume $0 \leq x < 0.5$.

Then,

$$0 \leq x < 0.5 \tag{3}$$
$$0 - x_1 \leq x - x_1 < \frac{1}{2} - x_1 \tag{4}$$
$$0 \leq x - x_1 < \frac{1}{2} \tag{5}$$

Then, it follows from above that $FB(n, x_1) \wedge 0 \leq x - x_1 < \frac{1}{2}$ hold for the base case with $0 \leq x < 0.5$.

Case 2 ($0.5 \leq x < 1$ from $S = \{x \mid x \in \mathbb{R}^{\geq 0}, 0 \leq x < 1\}$):

First, we will prove that $FB(1, x_1)$ is true.

Let $x_1 = 0.5$.

Then,

$$0.5 = \frac{1}{2} \tag{6}$$
$$= \sum_{i=1}^{1} \frac{b_i}{2} \tag{7}$$

where $b_i = 1$.

Then, it follows from the definition of finite fractional binary representation that x has fractional binary representation with 1 bits, and $FB(1, x_1)$ is true.

Now, we will prove that $0 \leq x - x_1 < 0.5$.

Let $x_1 = 0.5$. Assume $0.5 \leq x < 1$.

Then,

$$0.5 - x_1 \leq x - x_1 < 1 - x_1 \tag{8}$$
$$0 \leq x - x_1 < 0.5 \tag{9}$$

Then, it follows from above that $0 \leq x - x_1 < 0.5$ is true.

Then, since $0 \leq x - x_1 < 0.5$ is true and $FB(1, x_1)$ is true, $FB(1, x_1) \land 0 \leq x - x_1 < 0.5$ is true for the case $0.5 \leq x < 1$.

Then, by combining results from case 1 and case 2, we can conclude that the statement holds for the base case.

Now, let $n \in \mathbb{Z}^+$, and $x \in S$. Assume $\exists x_1 \in S$, $FB(n, x_1) \land 0 \leq x - x_1 \leq \frac{1}{2^n}$.

Then, we will prove that the statement $\forall n \in \mathbb{Z}^+$, $\forall x \in S$, $FB(n, x_1) \wedge 0 \leq x - x_1 \leq \frac{1}{2^n}$ is true for inductive case by separating $0 \leq x - x_1 \leq \frac{1}{2^n}$ into following cases.

Case 1 ($0 \leq x - x_1 < \frac{1}{2^{n+1}}$):

First, we will prove that $FB(n+1, x_2)$ is true.

Let $x_2 = x_1$.

Then,

$$\begin{aligned} x_2 &= \sum_{i=1}^{n} \frac{b_i}{2} && (10) \\ &= \sum_{i=1}^{n} \frac{b_i}{2} + \frac{b_{i+1}}{2} && (11) \\ &= \sum_{i=1}^{n+1} \frac{b_i}{2} && (12) \end{aligned}$$

by setting $b_{i+1} = 0$.

Then, it follows from above that $FB(n+1, x_2)$ is true.

Now, we will prove that $0 \leq x - x_2 < \frac{1}{2^{n+1}}$ is true.

Let $x_2 = x_1$. Assume $0 \leq x - x_1 < \frac{1}{2^{n+1}}$.

Then, it follows from assumption that $0 \leq x - x_2 < \frac{1}{2^{n+1}}$ is true.

Then, since $FB(n+1, x_2)$ is true and $0 \leq x - x_2 < \frac{1}{2^{n+1}}$ is true, $FB(n+1, x_2) \wedge 0 \leq x - x_2 < \frac{1}{2^{n+1}}$ is true for the case $0 \leq x - x_1 < \frac{1}{2^{n+1}}$.

Question 3

Question 4