Midterm 2 Version 1 Review

July 18, 2020

1. a) 1100100

   b) $-\sum_{i=0}^{n-1} 3^i$

   Notes:

   - Balanced Ternary
     - is a way of representing numbers
     - balanced ternary is in base 3, and has values 1,0 or -1

$$\sum_{i=0}^{n-1} d_i \cdot 3^i \text{ where } d_i \in \{0, 1, -1\} \tag{1}$$

   c) i. $\underline{f(n) \in \Omega(n)}$

      True (since $n^2 + 10n + 2 \geq cn$)

      ii. $\underline{g(n) \in \Omega(n)}$

      False (Let $c = 100, n_0 = 100$. Then $100 \log_2 n < 100n$)

      iii. $\underline{f(n) \in \mathcal{O}(g(n))}$

      False ($f(n) = n^2 + 10n + 2$ grows faster than $g(n) = 100 \log_2 n$)

      iv. $\underline{f(n) \in \Theta(g(n))}$

      True (Set $c_1 = -1, c_2 = 1, n_1 = 100$. Then $c_1 f(n) \leq g(n) \leq c_2 f(n)$)

      v. $\underline{g(n) \in \Theta(\log_3 n)}$

      True (set $c_1 = -1, c_2 = 1, n_1 = 2$. Then $c_1 g(n) \leq \log_3 n \leq c_2 g(n)$)

vi. $\underline{g(n) \in \Theta(\log_3 n)}$

False (set $c_1 = -1, c_2 = 1, n_1 = 2$. Then $c_1 g(n) \leq \log_3 n \leq c_2 g(n)$)

vii. $\underline{f(n) + g(n) \in \Theta(f(n))}$

True (set $c_1 = -2, c_2 = 2, n_1 = 1$. Then $c_1(f(n)+g(n)) \leq f(n) \leq c_2(f(n)+g(n))$)

Notes:

- $g \in \Omega(f):\ \exists c, n_o \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \geq cf(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$
- $g \in \mathcal{O}(f):\ \exists c, n_o \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \leq cf(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$
- $g \in \Theta(f):\ g \in \mathcal{O}(f) \wedge g \in \Omega(f)$
  or
  $g \in \Theta(f):\ \exists c_1, c_2, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n)$, where $f, g:\ \mathbb{N} \to \mathbb{R}^{\geq 0}$

d) $i = 3^{2^k}$

Since

k	0	1	2
i	3	9	81
	3^1	3^2	3^4

e) $k = \lceil \log_3(\log_2 n) - 1 \rceil$

Since

$$\begin{aligned} i^2 &\geq n && (1)\\ 3^{2^k} &\geq n^{1/2} && (2)\\ 2^k &\geq \log_3(n^{1/2}) && (3)\\ 2^k &\geq (1/2)\log_3(n) && (4)\\ k &\geq \log_2((1/2)\log_3(n)) && (5)\\ &\geq \log_2(\log_3(n)) - 1 && (6) \end{aligned}$$

which gives $k = \lceil \log_2(\log_3(n)) - 1 \rceil$

2. Let $n \in \mathbb{N}$. Assume $n \geq 3$.

I will prove $5^n + 50 < 6^n$ by induction.

Base Step ($n = 3$):

Let $n = 3$.

Then,

$$5^3 + 50 = 715 < 6^3 = 216 \tag{1}$$

So, the base case holds.

Inductive Step

Let $n \in \mathbb{N}$. Assume $(5^n + 50 < 6^n)$.

I need to show $5^{n+1} + 50 < 6^{n+1}$.

Indeed we have

$$\begin{aligned}
5^{n+1} + 50 &= 5^n 5 + 50 && (2)\\
&= 5(5^n + 10) && (3)\\
&< 5(5^n + 50) && (4)\\
&< 56^n && (5)\\
&< 66^n && (6)\\
&< 6^{n=1} && (7)
\end{aligned}$$

3. **Negation(expanded):** $\forall a \in \mathbb{R}, \forall c_1, c_2, n_1 \in \mathbb{R}^+, \exists n \in \mathbb{N}, (n \geq n_1) \wedge (c_1 g(n) > f(n)) \vee (f(n) > c_2 g(n))$

   Proof. Let $a \in \mathbb{R}$.

   I need to show $an + 1 \notin \Theta(n^3)$. That is, $an + 1 \notin \mathcal{O}(n^3) \vee an + 1 \notin \Omega(n^3)$. In other words, $\forall c, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N}, (n \geq n_0) \wedge (an + 1 > c \cdot n^3)$ or $\forall c_1, n_1 \in \mathbb{R}^+, \exists n \in \mathbb{N}, (n \geq n_1) \wedge (an + 1 < c_1 \cdot n^3)$.

   Let $c_1, c_2, n_1 \in \mathbb{R}^+$, and let $n = \lceil max(n_1, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{2}{c_1}})\rceil + 1$.

   Then, we can write

$$
\begin{align}
n = \lceil max(n_1, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{2}{c_1}})) \rceil + 1 > \sqrt{\frac{2a}{c_1}} \tag{1}\\
n^2 > \frac{2a}{c_1} \tag{2}\\
\frac{c_1 n^3}{2} > an \tag{3}
\end{align}
$$

And

$$
\begin{align}
n = \lceil max(n_1, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{2}{c_1}})) \rceil + 1 > \sqrt[3]{\frac{2}{c_1}} \tag{4}\\
\frac{c_1 n^3}{2} > 1 \tag{5}
\end{align}
$$

Thus, we can conclude

$$
\begin{align}
\frac{c_1 n^3}{2} + \frac{c_1 n^3}{2} > an + 1 \tag{6}\\
c_1 \cdot n^3 > an + 1 \tag{7}
\end{align}
$$

$\square$

Notes:

- $g \in \Omega(f): \exists c, n_o \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \geq cf(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$
- $g \in \mathcal{O}(f): \exists c, n_o \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq cf(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$
- $g \in \Theta(f): g \in \mathcal{O}(f) \land g \in \Omega(f)$
  or
  $g \in \Theta(f): \exists c_1, c_2, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n)$, where $f, g: \mathbb{N} \to \mathbb{R}^{\geq 0}$

4. a) I need to evaluate the total number of iterations of loop 2.

   First, I need to evaluate the number of iterations of loop 2 per iteration of loop 1.

   The code tells us that the value of j increases by 3 per iteration k. That is, $j = 3k$.

Since the inner loop ends when $j \geq i$, the earliest iteration at which the loop ends per iteration of the outerloop is $k = \lceil \frac{i}{3} \rceil$.

Finally, the outer loop starts from $i = 0$ to $i = n^2$.

Thus, the total number of iterations in loop 2 is:

$$\sum_{i=0}^{n^2} \frac{i}{3} = \frac{n^2(n^2+1)}{6} \tag{1}$$

Correct Solution:

I need to evaluate the total number of iterations of loop 2.

First, I need to evaluate the number of iterations of loop 2 per iteration of loop 1.

The code tells us that the value of j increases by 3 per iteration k. That is, $j = 3k$.

Since the inner loop ends when $j \geq i$, the earliest iteration at which the loop ends per iteration of the outerloop is $k = \lceil \frac{i}{3} \rceil$.

Finally, the outer loop starts from $i = 0$ to $i = n^2$.

Thus, the total number of iterations in loop 2 is:

$$\sum_{i=0}^{n^2-1} \frac{i}{3} = \frac{(n^2-1)n^2}{6} \tag{1}$$

b) **Finding upperbound:**

The code tells us the worst case in loop occurs when there are odd numbers or no odd numbers at all.

In both of the cases, the loop runs from $i = 0$ to $i = n$.

So, the upperbound of my_alg is $\mathcal{O}(n)$