

# Problem Set 4 Solution

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## Question 1

- a. **Statement:**  $\forall f, g : \mathbb{N} \rightarrow \mathbb{R}^+, b \in \mathbb{R}^+, (g(n) \in \Theta(f(n))) \wedge (n_0 \in \mathbb{N}, n \geq n_0 \Rightarrow f(n) \geq b \wedge g(n) \geq b) \wedge (b > 1) \Rightarrow \log_b(g(n)) \in \Theta(\log_b(f(n)))$

**Statement Expanded:**  $\forall f, g : \mathbb{N} \rightarrow \mathbb{R}^+, b \in \mathbb{R}^+, \left( \exists c_1, c_2, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n) \right) \wedge \left( \exists n_1 \in \mathbb{N}, n \geq n_1 \Rightarrow f(n) \geq b \wedge g(n) \geq b \right) \wedge (b > 1) \Rightarrow \left( \exists d_1, d_2, n_2 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_2 \Rightarrow d_1 \cdot \log_b(g(n)) \leq \log_b(f(n)) \leq d_2 \cdot \log_b(g(n)) \right)$

*Proof.* Let  $f, g : \mathbb{N} \rightarrow \mathbb{R}^+$ , and  $b \in \mathbb{R}^+$ . Assume  $c_1 = 1$ ,  $c_2 = b$ , and  $n_0 = 1$ , and  $n \in \mathbb{N}$  such that  $n \geq n_0$  and  $c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ . Assume  $f(n)$  and  $g(n)$  are eventually  $\geq b$ . Assume  $b > 1$ . Let  $d_1 = 1$ ,  $d_2 = 2$ , and  $n_2 = n_0$ . Assume  $n \geq n_2$ .

We need to show  $d_1 \cdot \log_b g(n) \leq \log_b f(n) \leq d_2 \cdot \log_b g(n)$ .

We will do so in two parts. One for  $(d_1 \cdot \log_b g(n) \leq \log_b f(n))$  and the other for  $(\log_b f(n) \leq d_2 \cdot \log_b g(n))$ .

**Part 1**  $(d_1 \cdot \log_b g(n) \leq \log_b f(n))$ :

The assumption tell us

$$c_1 \cdot g(n) \leq f(n) \tag{1}$$

Then, it follows from the fact  $\forall x, y \in \mathbb{R}^+, x \geq y \Leftrightarrow \log x \geq \log y$

$$\log(c_1 \cdot g(n)) \leq \log(f(n)) \tag{2}$$

Then, using the fact  $b > 1$ , we can calculate

$$\frac{\log(c_1 \cdot g(n))}{\log b} \leq \frac{\log(f(n))}{\log b} \quad (3)$$

$$\frac{\log(c_1) + \log(g(n))}{\log b} \leq \frac{\log(f(n))}{\log b} \quad (4)$$

Then,

$$\frac{\log(g(n))}{\log b} \leq \frac{\log(f(n))}{\log b} \quad (5)$$

by the fact  $c_1 = 1$  and  $\log c_1 = 0$ .

Then, since  $\frac{\log f(x)}{\log b} = \log_b f(x)$ ,

$$\log_b(g(n)) \leq \log_b(f(n)) \quad (6)$$

Then, because we know  $d_1 = 1$ , we can conclude

$$\log_b(g(n)) \leq d_1 \cdot \log_b(f(n)) \quad (7)$$

**Part 2** ( $\log_b f(n) \leq d_2 \cdot \log_b g(n)$ ):

The assumption tells us

$$f(n) \leq c_2 \cdot g(n) \quad (8)$$

Then, it follows from the fact  $\forall x, y \in \mathbb{R}^+, x \geq y \Leftrightarrow \log x \geq \log y$

$$\log(f(n)) \leq \log(c_2 \cdot g(n)) \quad (9)$$

Then, using the fact  $b > 1$ , we can calculate

$$\frac{\log(f(n))}{\log b} \leq \frac{\log(c_2 \cdot g(n))}{\log b} \quad (10)$$

$$\frac{\log(f(n))}{\log b} \leq \frac{\log(c_2) + \log(g(n))}{\log b} \quad (11)$$

Then, since  $c_2 = b$ ,

$$\frac{\log(f(n))}{\log b} \leq \frac{\log(b) + \log(g(n))}{\log b} \quad (12)$$

Then, using the fact  $g(n)$  is eventually  $\geq b$ , we can write

$$\frac{\log(f(n))}{\log b} \leq \frac{\log(g(n)) + \log(g(n))}{\log b} \quad (13)$$

$$\frac{\log(f(n))}{\log b} \leq \frac{2 \cdot \log(g(n))}{\log b} \quad (14)$$

Then, since  $\frac{\log f(x)}{\log b} = \log_b f(x)$ ,

$$\log_b(f(n)) \leq 2 \cdot \log_b(g(n)) \quad (15)$$

Then, because we know  $d_2 = 2$ , we can conclude

$$\log_b(f(n)) \leq d_2 \cdot \log_b(g(n)) \quad (16)$$

□

**Notes:**

- $\forall x, y \in \mathbb{R}^+, x \geq y \Leftrightarrow \log x \geq \log y$
- $\exists c_1, c_2, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$
- **Definition of Eventually:**  $\exists n_0 \in \mathbb{N}, n \geq n_0 \Rightarrow P$ , where  $P : \mathbb{N} \rightarrow \{\text{True}, \text{False}\}$

b. *Proof.* Let  $k \in \mathbb{N}$ .

First, we will analyze the cost of loop 2 over iteration of loop 1.

The code tells us loop 2 starts at  $j_k = 1$  with  $j_k$  increasing by a factor of 3 per iteration until  $j_k \geq i$ .

Using these facts, we can calculate that the terminating condition occurs when

$$3^k \geq i \tag{1}$$

$$k \geq \log_3 i \tag{2}$$

Because we know the number of iterations is the smallest value of  $k$  satisfying the above inequality, we can conclude loop 2 has

$$\lceil \log_3 i \rceil \tag{3}$$

iterations.

Next, we need to determine the total number of iterations of loop 2 over all iterations of loop 1.

The code tells us loop 1 starts at  $i = 1$  and ends at  $i = n$  with each  $i$  increasing by 1 per iteration.

Using these facts, we can conclude loop 2 has total of

$$\lceil \log_3 1 \rceil + \lceil \log_3 2 \rceil + \cdots + \lceil \log_3 n \rceil = \sum_{i=1}^n \lceil \log_3 i \rceil \tag{4}$$

iterations. □

- c. After scratching head and looking at solution many times, I realized that there are many things I do not yet understand, and it's the best to write what I have and learn from the solution. Here is my best attempt :).

*Proof.* Let  $n \in \mathbb{N}$ .

The previous answer tells us the exact cost of the algorithm is

$$\sum_{i=1}^n \lceil \log_3 i \rceil \tag{1}$$

Then, it follows by changing the variable  $i$  to  $i' = \log_3 i$  we can write

$$\sum_{i'=0}^{\lceil \log_3 n \rceil} i' \tag{2}$$

Then, because we know  $\sum_{i=0}^n i = \frac{n(n+1)}{2}$ , we can conclude

$$\sum_{i'=0}^{\lceil \log_3 n \rceil} i' = \frac{(\lceil \log_3 n \rceil)(\lceil \log_3 n \rceil + 1)}{2} \tag{3}$$

$$= \frac{\lceil \log_3 n \rceil^2 + \lceil \log_3 n \rceil}{2} \tag{4}$$

Then, we can conclude the runtime of the algorithm is  $\Theta(\log_3^2 n)$ . □

**Question 2**

**Question 3**

**Question 4**