CSC236 Worksheet 2 Solution

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April 29, 2020

Question 1

- **Statement:** Any full binary tree with at least 1 node has more leaves than internal nodes.

Rough Work:

Let n be the total number of nodes in a full binary tree.

We will prove the statement by complete induction on n.

1. Base Case ($n = 1$)

Let $n = 1$.

We need to prove the full binary tree with 1 total number of nodes has more leaves than internal nodes.

The definition of leaf tells us that a node that has no children is a leaf.

Since the full binary tree with 1 node has no children, we can conclude the node is a leaf.

Using this fact, we can write the full binary tree has 1 leaf.

Now, the definition of an internal node tells us that a node is an internal node if it is not a leaf.

Since there is 1 leaf and 1 total node in the full binary tree, we can write there is 0 internal node.

So, since there is 1 leaf node and 0 internal node, we can conclude the full binary tree has more leaves than internal nodes.

2. Base Case ($n = 2$)

Let $n = 2$.

We need to prove the full binary tree with 2 total number of nodes has more leaves than internal nodes.

The definition of full binary tree tells us that a binary tree is a full binary tree if every internal node has two children.

Since we know by observation that the tree with 2 total number of nodes has 1 leaf and 1 internal node, using above fact, we can write the tree is not a full binary tree.

Then, by vacuous truth, we can conclude the tree has more leaves than internal nodes.

3. Base Case ($n = 3$)

Let $n = 3$.

We need to prove the full binary tree with 3 total number of nodes has more leaves than internal nodes.

1. Show the three forms a full binary tree
   - State the definition of full binary tree

     > The definition of full binary tree tells us that a binary tree is a full binary tree if every internal node has two children.

   - Show that with 3 as the total number of nodes, the only full binary tree that can be formed is 1 internal node and 2 children.

     > Because we know there are two types of binary trees possible, one is the tree with 2 internal nodes and 1 child and the other is 1 internal node and 2 children, using above fact, we can write the only possible full binary tree with 3 total nodes is 1 internal node and 2 children.

   - Conclude that the children are leaves, and there are more leaves than internal nodes

     > Now, the definition of leaf tells us leaf is a node that has no children.
     >
     > Because we know by observation that the 2 child nodes don't have children, we can write the full binary tree has 2 leaves.
     >
     > So, because we know the full binary tree has 1 internal node and 2 leaves, we can conclude the full binary tree has more leaves than internal node.

Base Case ($n = 3$):

Let $n = 3$.

We need to prove the full binary tree with 3 total number of nodes has more leaves than internal nodes.

The definition of full binary tree tells us that a binary tree is a full binary tree if every internal node has two children.

Because we know there are two types of binary trees possible, one is the tree with 2 internal nodes and 1 child and the other is 1 internal node and 2 children, using above fact, we can write the only possible full binary tree with 3 total nodes is 1 internal node and 2 children.

Now, the definition of leaf tells us leaf is a node that has no children.

Because we know by observation that the 2 child nodes don't have children, we can write the full binary tree has 2 leaves.

So, because we know the full binary tree has 1 internal node and 2 leaves, we can conclude the full binary tree has more leaves than internal node.

4. Inductive Step

   Let $k \geq 1$ be an arbitrary natural number. Assume that for all natural number i satisfying $1 \leq i \leq k$, any full binary trees with i total number of nodes has more leaves than internal nodes.

   Let T be an arbitrary full binary tree with $k + 1$ nodes. Let T' be the binary tree obtained by removing 2 leaves from the same parent node.

   Let ℓ be the number of leaves of T, and m be the number of internal nodes of T. Similarly, let ℓ' be the number of leaves of T' and m' be the number of internal nodes of T'. We must prove $l > m$.

   - State that T' is a full binary tree with the number of leaves more than the internal nodes, using induction hypothesis.
   - Show that $\ell' > m'$
   - Show that $\ell = \ell' + 1$ and $m = m' + 1$ using the fact that when 2 nodes are added to a leaf node of T', the number of leaf nodes increase by 1 and internal nodes increase by 1
   - Conclude $\ell > m$.

Notes:

– Complete Induction

* **Statement:** $\forall i \in \mathbb{N},\ \forall n \in \mathbb{N},\ n < i \Rightarrow A(n) \Rightarrow \forall i \in \mathbb{N},\ A(i)$
* **Statement Alt.:** $\left(\forall n \in \mathbb{N},\ \left[\bigwedge_{k=0}^{k=n-1} P(k)\right] \Rightarrow P(n)\right) \Rightarrow \forall n \in \mathbb{N}, P(n)$

* **Simple Example 1:**

  Statement: $\forall n \in \mathbb{N},\ n \geq 0 \Rightarrow 10 \mid (n^5 - n)$

  We will prove the statement by strong induction on n.

  1. Base Case ($n = 0$)

     Let $n = 0$.

     We need to prove $10 \mid (n^5 - n)$ is true when $n = 0$. That is, there exists $k \in \mathbb{Z}$ such that $(n^5 - n) = 10k$.

     Let $k = 0$.

     Starting from the left hand side, using the fact $n = 0$, we can write

     $$(n^5 - n) = 0 \tag{1}$$

     Then, because we know $10k = 0$, we can conclude

     $$(n^5 - n) = 10k \tag{2}$$

  2. Base Case ($n = 1$)

Let $n = 1$.

We need to prove $10 \mid (n^5 - n)$ is true when $n = 1$. That is, there exists $k \in \mathbb{Z}$ such that $(n^5 - n) = 10k$.

Let $k = 0$.

Starting from the left hand side, using the fact $n = 0$, we can write

$$
\begin{aligned}
(n^5 - n) &= 1 - 1 && (3)\\
&= 0 && (4)
\end{aligned}
$$

Then, because we know $10k = 0$, we can conclude

$$(n^5 - n) = 10k \tag{5}$$

3. Inductive Step

Assume $k \geq 1$. Assume that for all natural number i satisfying $0 \leq i \leq k$, $10 \mid (i^5 - i)$. That is, $\exists d \in \mathbb{Z}$, $(i^5 - i) = 10d$.

We need to prove $\exists \tilde{d} \in \mathbb{Z}$ such that $((k+1)^5 - (k+1)) = 10\tilde{d}$.

Let $\tilde{d} = c + (k-1)^4 + 4 \cdot (k-1)^3 + 8 \cdot (k-1)^2 + 8 \cdot (k-1) + 3$.

Starting from $((k+1)^5 - (k+1))$, using binominal theorem, we can write,

$$\begin{aligned}
(k+1)^5 - (k+1) &= \Big[(k-1)+2\Big]^5 - \Big[(k-1)+2\Big] && (6)\\
&= \sum_{b=0}^{5} \binom{5}{b} (k-1)^{5-b} \cdot 2^b && (7)\\
&= (k-1)^5 + 10 \cdot (k-1)^4 + 40 \cdot (k-1)^3 + \\
&\quad 80 \cdot (k-1)^2 + 80 \cdot (k-1) + 32 - \Big[(k-1)+2\Big] && (8)\\
&= \Big[(k-1)^5 - (k-1)\Big] + 10 \cdot (k-1)^4 + \\
&\quad 40 \cdot (k-1)^3 + 80 \cdot (k-1)^2 + 80 \cdot (k-1) + 30 && (9)
\end{aligned}$$

(The reason why $k-1$ is chosen instead of $k-2$ and $k-3$ is because of the last term $2^5 = 32$, i.e $32 - 2 = 30$)

Then, because we know $0 \leq k-1 \leq k$ and $10 \mid (k-1)^5 - (k-1)$ from the header, we can write $\exists c \in \mathbb{Z}$ such that $(k-1)^5 - (k-1) = 10c$, and

$$\begin{aligned}
(k+1)^5 - (k+1) &= 10c + 10 \cdot (k-1)^4 + 40 \cdot (k-1)^3 + \\
&\quad 80 \cdot (k-1)^2 + 80 \cdot (k-1) + 30 && (10)\\
(k+1)^5 - (k+1) &= 10 \cdot \Big[c + \cdot (k-1)^4 + 4 \cdot (k-1)^3 + \\
&\quad 8 \cdot (k-1)^2 + 8 \cdot (k-1) + 3\Big] && (11)\\
& && (12)
\end{aligned}$$

Then, because we know $\tilde{d} = c + (k-1)^4 + 4 \cdot (k-1)^3 + 8 \cdot (k-1)^2 + 8 \cdot (k-1) + 3$ from the header, we can conclude

$$(k+1)^5 - (k+1) = 10\tilde{d} \tag{13}$$

Question 2

Question 3