Worksheet 20 Solution

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Question 1

a. *Proof.* Let $V = \{1, 2, 3, 4, 5, 6\}$, $E = \{(1, 2), (1, 6), (2, 3), (3, 4), (4, 5), (5, 6)\})$.

We need to prove the graph $G = (V, E)$ is bipartite by proving the following properties:

1. There exists subsets $V_1, V_2 \subset V$ such that $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V.
2. Every edge in E has exactly one endpoint in V_1 and one in V_2.

We will prove the properties in parts.

Part 1 (Proving $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V):

Let $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

We need to prove $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V, i.e $V_1 \cup V_2 = V \wedge V_1 \cap V_2 = \emptyset$.

First, we need to show the subsets V_1 and V_2 are non-empty.

The header tells us both subsets V_1 and V_2 have more than 1 elements.

Then, using these facts, we can conclude $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$.

Finally, we need to show $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$.

The header tells us $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

Then, we can calculate

$$V_1 \cup V_2 = \{1, 2, 3, 4, 5, 6\} = V \tag{1}$$
$$V_1 \cap V_2 = \emptyset \tag{2}$$

Part 2 (Proving every edge in E has exactly one endpoint in V_1 and one in V_2):

Let $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

We need to show every edge in E has exactly one endpoint in V_1 and one in V_2.

The header tells us $V_1 = \{1, 3, 5\}$, $V_2 = \{2, 4, 6\}$, and $E = \{(1, 2), (1, 6), (2, 3), (3, 4), (4, 5), (5, 6)\})$.

Using these facts, we can generate the following table.

Edge (1,2)	- 1 is in V_1 - 2 is in V_2	Edge (3,4)	- 3 is in V_1 - 4 is in V_2
Edge (1,6)	- 1 is in V_1 - 6 is in V_2	Edge (4,5)	- 4 is in V_2 - 6 is in V_1
Edge (2,3)	- 2 is in V_2 - 3 is in V_1	Edge (5,6)	- 5 is in V_1 - 6 is in V_2

Then, it follows from observation that every edge in E has one endpoint in V_1 and one in V_2.

□

Pseudoproof:

Let $V = \{1, 2, 3, 4, 5, 6\}$, $E = \{(1, 2), (1, 6), (2, 3), (3, 4), (4, 5), (5, 6)\})$.

We need to prove the graph $G = (V, E)$ is bipartite by proving the following properties:

1. There exists subsets $V_1, V_2 \subset V$ such that $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V.
2. Every edge in E has exactly one endpoint in V_1 and one in V_2.

We will prove the properties in parts.

1. Show there exists subsets $V_1, V_2 \subset V$ such that $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V

Let $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

We need to prove $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V, i.e $V_1 \cup V_2 = V \wedge V_1 \cap V_2 = \emptyset$.

1. Show $V_1 \neq \emptyset, V_2 \neq \emptyset$
   First, we need to show the subsets V_1 and V_2 are non-empty.

   The header tells us both subsets V_1 and V_2 have more than 1 elements.

   Then, using these facts, we can conclude $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$.

2. Show $V_1 \cup V_2 = V \wedge V_1 \cap V_2 = \emptyset$

   Second, we need to show $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$.

   The header tells us $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

   Then, we can calculate

$$V_1 \cup V_2 = \{1, 2, 3, 4, 5, 6\} = V \tag{3}$$
$$V_1 \cap V_2 = \emptyset \tag{4}$$

Part 1:

Let $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

We need to prove $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V, i.e $V_1 \cup V_2 = V \wedge V_1 \cap V_2 = \emptyset$.

First, we need to show the subsets V_1 and V_2 are non-empty.

The header tells us both subsets V_1 and V_2 have more than 1 elements.

Then, using these facts, we can conclude $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$.

Finally, we need to show $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$.

The header tells us $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

Then, we can calculate

$$V_1 \cup V_2 = \{1, 2, 3, 4, 5, 6\} = V \tag{5}$$
$$V_1 \cap V_2 = \emptyset \tag{6}$$

2. Show every edge in E has exactly one endpoint in V_1 and one in V_2.

   Let $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

   We need to show every edge in E has exactly one endpoint in V_1 and one in V_2.

The header tells us $V_1 = \{1,3,5\}$, $V_2 = \{2,4,6\}$, and $E = \{(1,2),(1,6),(2,3),(3,4),(4,5),(5,6)\}$).

Using these facts, we can generate the following table.

Edge (1,2)	- 1 is in V_1 - 2 is in V_2	Edge (3,4)	- 3 is in V_1 - 4 is in V_2
Edge (1,6)	- 1 is in V_1 - 6 is in V_2	Edge (4,5)	- 4 is in V_2 - 6 is in V_1
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Then, it follows from observation that every edge in E has one endpoint in V_1 and one in V_2.

<u>Part 2:</u>

Let $V_1 = \{1,3,5\}$ and $V_2 = \{2,4,6\}$.

We need to show every edge in E has exactly one endpoint in V_1 and one in V_2.

The header tells us $V_1 = \{1,3,5\}$, $V_2 = \{2,4,6\}$, and $E = \{(1,2),(1,6),(2,3),(3,4),(4,5),(5,6)\}$).

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Then, it follows from observation that every edge in E has one endpoint in V_1 and one in V_2.

b. Let $G = (V, E)$ be a complete bipartite graph.

Then, by property 3, we can conclude each vertex in V_1 is adjacent to all verticies in V_2.

Since there are n many edges for each vertex in V_1, and since there are m many verticies in V_1, we can calculate that the vertices in V_1 has

$$nm \tag{1}$$

edges.

Then, since there are no new edges for each vertex in V_2, we can conclude the graph has nm edges.