CSC236 Worksheet 6 Solution

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Question 1

- *Proof.* Assume that for all $k \in \mathbb{N}$, $R(3^k) = k3^k$.

  I need to prove $R \in \mathcal{O}(n \lg n)$ and $R \in \Omega(n \lg n)$.

  I will do so in parts.

  Part 1 (Proving $R \in \mathcal{O}(n \lg n)$):

  Define $n^* = 3^{\lceil \log_3 n \rceil}$. Then, we have,

$$\lceil \log_3 n \rceil - 1 < \log_3 n \leq \lceil \log_3 n \rceil \Rightarrow n^*/3 < n \leq n^* \tag{1}$$

  I will also use the assumption (proved last week) that R is non-decreasing.

  Let $d = 6$. Then $d \in \mathbb{R}^+$. Let $B = 3$. Then $B \in \mathbb{N}^+$. Let n be an arbitrary natural number no smaller than B. Then,

$$\begin{aligned}
R(n) &\leq R(n^*) && \text{[Since } n < n^*\text{, and } R \text{ is non-decreasing]} && (2)\\
&= n^* \log_3 n^* && \text{[By assumption, and replacing } n^* \text{ for } 3^k\text{]} && (3)\\
&\leq 3n \log_3 3n && \text{[Since } n \leq n^* \Rightarrow 3n \leq 3n^*\text{]} && (4)\\
&\leq 3n(\log_3 n + 1) && && (5)\\
&\leq 3n(\log_3 n + \log_3 n) && \text{[Since } n \geq 3 \Rightarrow \log_3 n \geq 1\text{]} && (6)\\
&= 6n \log_3 n && && (7)\\
&\leq (6n \lg n)/\lg 3 && \text{[By change of basis to lg]} && (8)\\
&< 6n \lg n && && (9)\\
&= dn \lg n && \text{[Since } d = 6\text{]} && (10)
\end{aligned}$$

So $R \in \mathcal{O}(n \lg n)$, since $\log_3 n$ differs from $\lg n$ by a constant factor.

Part 2 (Proving $R \in \Omega(n \lg n)$):

Define $n^* = 3^{\lceil \log_3 n \rceil}$. Then, we have,

$$\lceil \log_3 n \rceil - 1 < \log_3 n \leq \lceil \log_3 n \rceil \Rightarrow n^*/3 < n \leq n^* \tag{11}$$

I will also use the assumption (proved last week) that R is non-decreasing.

Let $d = 1/(6 \lg 3)$. Then $d \in \mathbb{R}^+$. Let $B = 9$. Then $B \in \mathbb{N}^+$. Let n be an arbitrary natural number no smaller than B. Then,

$$\begin{aligned}
R(n) &\geq R(n^*/3) && \text{[Since } n^*/3 < n \text{, and } R \text{ is non-decreasing]} && (12)\\
&= (n^*/3) \cdot \log_3(n^*/3) && \text{[By assumption, and replacing } n^* \text{ for } 3^k\text{]} && (13)\\
&\geq (n/3) \cdot \log_3(n/3) && \text{[Since } n^* \leq n \Rightarrow n^*/3 \leq n/3\text{]} && (14)\\
&= (n/3) \cdot (\log_3 n - 1) && && (15)\\
&\geq (n/3) \cdot (\log_3 n - (\log_3 n)/2) && \text{[Since } n \geq 9 \Rightarrow (\log_3 n)/2 \geq 1\text{]} && (16)\\
&= (n/6) \cdot \log_3 n && && (17)\\
&= (n/6) \cdot (\lg n / \lg 3) && && (18)\\
&= (n/(6 \lg 3)) \cdot \lg n && && (19)\\
&= dn \cdot \lg n && \text{[Since } d = 1/(6 \lg 3)\text{]} && (20)
\end{aligned}$$

So, $R \in \Omega(n \lg n)$.

$\square$

Correct Solution:

Assume that for all $k \in \mathbb{N}$, $R(3^k) = k3^k$.

I need to prove $R \in \mathcal{O}(n \lg n)$ and $R \in \Omega(n \lg n)$.

I will do so in parts.

Part 1 (Proving $R \in \mathcal{O}(n \lg n)$):

Define $n^* = 3^{\lceil \log_3 n \rceil}$. Then, we have,

$$\lceil \log_3 n \rceil - 1 < \log_3 n \leq \lceil \log_3 n \rceil \Rightarrow n^*/3 < n \leq n^* \tag{21}$$

I will also use the assumption (proved last week) that R is non-decreasing.

Let $d = 6$. Then $d \in \mathbb{R}^+$. Let $B = 3$. Then $B \in \mathbb{N}^+$. Let n be an arbitrary natural number no smaller than B. Then,

$$
\begin{aligned}
R(n) &\leq R(n^*) && \text{[Since } n < n^*\text{, and } R \text{ is non-decreasing]} && (22)\\
&= n^* \log_3 n^* && \text{[By assumption, and replacing } n^* \text{ for } 3^k\text{]} && (23)\\
&\leq 3n \log_3 3n && \text{[Since } n \leq n^* \Rightarrow 3n \leq 3n^*\text{]} && (24)\\
&= 3n(\log_3 n + 1) && && (25)\\
&\leq 3n(\log_3 n + \log_3 n) && \text{[Since } n \geq 3 \Rightarrow \log_3 n \geq 1\text{]} && (26)\\
&= 6n \log_3 n && && (27)\\
&= (6n \lg n)/\lg 3 && \text{[By change of basis to lg]} && (28)\\
&< 6n \lg n && && (29)\\
&= dn \lg n && \text{[Since } d = 6\text{]} && (30)
\end{aligned}
$$

So $R \in \mathcal{O}(n \lg n)$, since $\log_3 n$ differs from $\lg n$ by a constant factor.

Part 2 (Proving $R \in \Omega(n \lg n)$):

Define $n^* = 3^{\lceil \log_3 n \rceil}$. Then, we have,

$$\lceil \log_3 n \rceil - 1 < \log_3 n \leq \lceil \log_3 n \rceil \Rightarrow n^*/3 < n \leq n^* \quad (31)$$

I will also use the assumption (proved last week) that R is non-decreasing.

Let $d = 1/(6 \lg 3)$. Then $d \in \mathbb{R}^+$. Let $B = 9$. Then $B \in \mathbb{N}^+$. Let n be an arbitrary natural number no smaller than B. Then,

$$
\begin{aligned}
R(n) &\geq R(n^*/3) && \text{[Since } n^*/3 < n\text{, and } R \text{ is non-decreasing]} \quad (32)\\
&= (n^*/3) \cdot \log_3(n^*/3) && \text{[By assumption, and replacing } n^* \text{ for } 3^k\text{]} \quad (33)\\
&\geq (n/3) \cdot \log_3(n/3) && \text{[Since } n^* \leq n \Rightarrow n^*/3 \leq n/3\text{]} \quad (34)\\
&= (n/3) \cdot (\log_3 n - 1) && (35)\\
&\geq (n/3) \cdot (\log_3 n - (\log_3 n)/2) && \text{[Since } n \geq 9 \Rightarrow (\log_3 n)/2 \geq 1\text{]} \quad (36)\\
&= (n/6) \cdot \log_3 n && (37)\\
&= (n/6) \cdot (\lg n / \lg 3) && (38)\\
&= (n/(6 \lg 3)) \cdot \lg n && (39)\\
&= dn \cdot \lg n && \text{[Since } d = 1/(6 \lg 3)\text{]} \quad (40)
\end{aligned}
$$

So, $R \in \Omega(n \lg n)$, since $\log_3 n$ differs from $\lg n$ by a constant factor.

Notes:

- Learned that if there is trouble going from $\log_3 n - 1$ to $dn \lg n$, a good approach is to increase the value of B.
- Noticed that professor used 'Let $d =$ _____. Then $d \in \mathbb{R}^+$' to define variable's value as well as its type.
- $g \in \Theta(f): \; g \in \mathcal{O}(f) \land g \in \Omega(f)$

  or

  $g \in \Theta(f): \; \exists c_1, c_2, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n)$, where $f, g: \mathbb{N} \to \mathbb{R}^{\geq 0}$
- $g \in \Omega(f): \exists c, n_o \in \mathbb{R}^+, \; \forall n \in \mathbb{N}, \; n \geq n_0 \Rightarrow g(n) \geq cf(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$
- $g \in \mathcal{O}(f): \exists c, n_o \in \mathbb{R}^+, \; \forall n \in \mathbb{N}, \; n \geq n_0 \Rightarrow g(n) \leq cf(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$