Midterm 2 Version 1 Solution

April 4, 2020

Question 1

a.

$$\begin{aligned} 100 \div 2 &= 50, \text{Remainders } \mathbf{0} \\ 50 \div 2 &= 25, \text{Remainders } \mathbf{0} \\ 25 \div 2 &= 12, \text{Remainders } \mathbf{1} \\ 12 \div 2 &= 6, \text{Remainders } \mathbf{0} \\ 6 \div 2 &= 3, \text{Remainders } \mathbf{0} \\ 3 \div 2 &= 1, \text{Remainders } \mathbf{1} \\ 1 \div 2 &= 0, \text{Remainders } \mathbf{1} \end{aligned}$$

Then, it follows from above that the binary representation of 100 is $(1100100)_2$.

b. The smallest number that can be expressed by an n-digit balanced ternary representation is

$$\sum_{i=0}^{n-1} d_i \cdot 3^i, \text{ where } d_i \in \{0, 1, 2\} \tag{1}$$

Correct Solution:

The smallest number that can be expressed by an n-digit balanced ternary representation is

$$-\left[\sum_{i=0}^{n-1} 3^i\right] \tag{1}$$

Notes:

- Realized professor is asking for an example of the smallest number.
- Ternary representation of a number

$$\sum_{i=0}^{n-1} d_i \cdot 3^i, \text{ where } d_i \in \{0, 1, 2\}$$

- Learned a negative number could be expressed in in ternary or binary representation of numbers.

c.

$f(n) \in \Omega(n)$	True	$g(n) \in \Omega(n)$	False	$f(n) \in \mathcal{O}(g(n))$	False
$f(n) \in \Theta(g(n))$	False	$g(n) \in \Theta(\log_3 n)$	True	$f(n) + g(n) \in \Theta(f(n))$	True

Notes:

- $\forall g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, and all numbers $a \in \mathbb{R}^{\geq 0}$, if $g \in \mathcal{O}(f)$, then $f + g \in \mathcal{O}(f)$
- $g \in \Theta(f) : \ g \in \mathcal{O}(f) \wedge g \in \Omega(f)$
  or
  $g \in \Theta(f) : \ \exists c_1, c_2, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n)$, where $f, g : \ \mathbb{N} \to \mathbb{R}^{\geq 0}$
- $g \in \Omega(f) : \ \exists c, n_o \in \mathbb{R}^+, \ \forall n \in \mathbb{N}, \ n \geq n_0 \Rightarrow g(n) \geq cf(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

- $g \in \mathcal{O}(f) : \exists c, n_o \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq cf(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

d.

k	0	1	2
i_k	$3 = 3^1$	$9 = 3^2$	$81 = 3^4$

The value of i_k is

$$3^{2^k} \tag{1}$$

Notes:

- Realized we are only concerned with the lines **i = i * i** and **i = 3**

e. The number of iterations the function's loop will run is

$$\lceil \log_2 \log_3 n \rceil - 1 \tag{1}$$

Notes:

- The loop terminates when $3^{2^{(k+1)}} = i_{k+1} = i_k \cdot i_k \geq n$.
- $\forall x \in \mathbb{Z}, \forall y \in \mathbb{R}, \lfloor x + y \rfloor = x + \lfloor y \rfloor$
- Feel more confident there is no need to add an extra **+1**. Done by playing with examples (i.e is $\lceil \log \log_3(82) \rceil - 1$ true? Would the loop run only once?)

Question 2

- **Predicate Logic:** $\forall n \in \mathbb{N}, n \geq 3 \Rightarrow 5^n + 50 < 6^n$

Proof. Let $n \in \mathbb{N}$.

We will prove the statement by induction on n.

Base Case ($n = 3$):

Let $n = 3$.

We want to show $5^3 + 50 < 6^3$.

Starting from $5^3 + 50$, we can calculate

$$\begin{aligned} 5^3 + 50 &= 125 + 50 && (1)\\ &= 175 && (2)\\ &< 216 && (3)\\ &< 6^3 && (4) \end{aligned}$$

Inductive Case:

Let $n \in \mathbb{N}$. Assume $n \geq 3$ and $5^n + 50 < 6^n$.

We want to show $5^{n+1} + 50 < 6^{n+1}$.

Starting from $5^{n+1} + 50$, we can calculate

$$\begin{aligned} 50^{n+1} + 50 &= 5^n \cdot 5 + 50 && (5)\\ &< 5^n \cdot 5 + 50 \cdot 5 && (6)\\ &< 5(5^n + 50) && (7) \end{aligned}$$

Then,

$$\begin{aligned} 50^{n+1} + 5 &< 5 \cdot 6^n && (8)\\ &< 6 \cdot 6^n && (9)\\ &< 6^{n+1} && (10) \end{aligned}$$

by using inductive hypothesis (i.e $5^n + 50 < 6^n$)

□

Correct Solution:

Let $n \in \mathbb{N}$.

We will prove the statement by induction on n.

Base Case ($n = 3$):

Let $n = 3$.

We want to show $5^3 + 50 < 6^3$.

Starting from $5^3 + 50$, we can calculate

$$\begin{aligned} 5^3 + 50 &= 125 + 50 && (1)\\ &= 175 && (2)\\ &< 216 && (3)\\ &< 6^3 && (4) \end{aligned}$$

Inductive Case:

Let $n \in \mathbb{N}$. Assume $n \geq 3$ and $5^n + 50 < 6^n$.

We want to show $5^{n+1} + 50 < 6^{n+1}$.

Starting from $5^{n+1} + 50$, we can calculate

$$\begin{aligned} 50^{n+1} + 50 &= 5^n \cdot 5 + 50 && (5)\\ &= 5^n \cdot 5 + 50 \cdot 5 && (6)\\ &< 5(5^n + 50) && (7) \end{aligned}$$

Then,

$$50^{n+1} + 5 < 5 \cdot 6^n \tag{8}$$
$$< 6 \cdot 6^n \tag{9}$$
$$= 6^{n+1} \tag{10}$$

by using inductive hypothesis (i.e $5^n + 50 < 6^n$)

Notes:

- Noticed professor uses '=' sign if the expression's value remains unchanged from the one before

  See equation 5 and 6 for example.

Question 3

- **Statement:** $\exists a \in \mathbb{R}^+,\ an + 1 \in \Theta(n^3)$

  Negation of Statement: $\forall a \in \mathbb{R}^+,\ \forall c_1, c_2, n_0 \in \mathbb{R}^+,\ \exists n \in \mathbb{N},\ (n \geq n_0) \wedge \big((an + 1 < c_1 n^3) \vee (an + 1 > c_2 n^3)\big)$

  Proof. Let $n = \left\lceil max(n_0, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{1}{c_1}}) \right\rceil + 1$.

  We will disprove the statement by showing $n \geq n_0$ and $an + 1 < c_1 n^3$

  Part 1 (Showing $n \geq n_0$):

  Using the fact that $\left\lceil max(n_0, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{1}{c_1}}) \right\rceil$ will result in a value greater than or equal to n_0, we can calculate

$$n_0 \leq \left\lceil max(n_0, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{1}{c_1}}) \right\rceil \tag{1}$$

$$\leq \left\lceil max(n_0, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{1}{c_1}}) \right\rceil + 1 \tag{2}$$

Then, because we know $n = \left\lceil max(n_0, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{1}{c_1}}) \right\rceil + 1$, we can conclude

$$n_0 \leq n \tag{3}$$

□

Part 2 (Showing $an + 1 < c_1 n^3$):

We will prove $an + 1 < c_1 n^3$ by showing $an < \frac{c_1}{2} n^3$ and $1 < \frac{c_1}{2} n^3$, and then combining the two together.

For the first inequality, because we know $n = \left\lceil max(n_0, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{1}{c_1}}) \right\rceil + 1 > \sqrt{\frac{2a}{c_1}}$, we can conclude

$$\sqrt{\frac{2a}{c_1}} < n \tag{4}$$

$$\frac{2a}{c_1} < n^2 \tag{5}$$

$$a < \frac{c_1}{2} n^2 \tag{6}$$

$$an < \frac{c_1}{2} n^3 \tag{7}$$

For the second inequality, because we know $n = \left\lceil max(n_0, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{1}{c_1}}) \right\rceil + 1 > \sqrt[3]{\frac{1}{c_1}}$, we can conclude

$$\sqrt[3]{\frac{1}{c_1}} < n \tag{8}$$

$$\frac{1}{c_1} < n^3 \tag{9}$$

$$1 < n^3 \tag{10}$$

Then,

$$an + 1 < \frac{c_1}{2} \cdot n^3 + \frac{c_1}{2} \cdot n^3 \tag{11}$$

$$an + 1 < c_1 n^3 \tag{12}$$

Notes:

- I struggled on this question.
- Learned **+1** in $\left\lceil max(n_0, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{1}{c_1}}) \right\rceil + 1 > \sqrt[3]{\frac{1}{c_1}}$ is to allow the use of inequality sign '<'.
- Learned that when c_1 is in inequality, with multiple terms like $an+1$ on the other side, and is asking to disprove it, I should first divide them up, find valid n for each term, and then recombine to create a valid n.

  See figure 1 for example

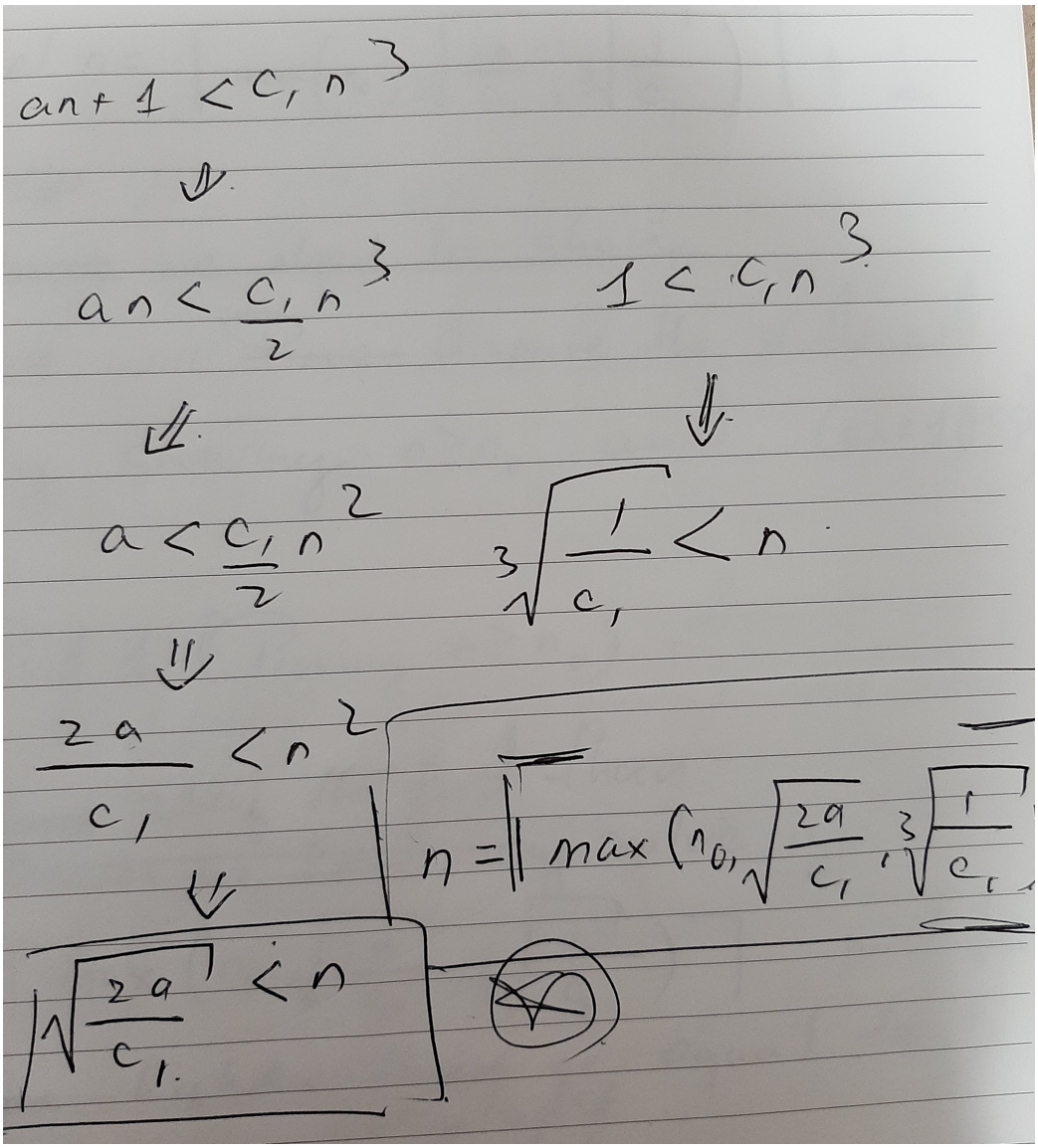

Figure 1: A sample work for question 3

Question 4

a. *Proof.* Let $n \in \mathbb{N}$.

We will determine the exact number of iterations by first evaluating the number of iterations of loop 2, and then evaluating the number of iterations of loop 1.

For loop 2, because we know it starts at $j = 0$, and ends at $j = i - 1$, with j increasing by 3 per iteration, we can conclude the loop has

$$\left\lceil \frac{i - 1 - 0 + 1}{3} \right\rceil = \left\lceil \frac{i}{3} \right\rceil \tag{1}$$

iterations.

For loop 1, because we know it starts at $i = 0$ and ends at $i = n^2 - 1$, with each iteration taking $\lceil \frac{i}{3} \rceil$ steps, we can conclude the loop takes total of

$$\sum_{i=0}^{n^2-1} \left\lceil \frac{i}{3} \right\rceil \tag{2}$$

iterations.

Then, because we know the floor and ceilings signs can be ignored, we can conclude the exact total number of iterations is

$$\sum_{i=0}^{n^2-1} \frac{i}{3} = \frac{1}{3} \cdot \sum_{i=0}^{n^2-1} i \tag{3}$$

$$= \frac{1}{3} \cdot \frac{(n^2 - 1)(n^2 - 1 + 1)}{2} \tag{4}$$

$$= \frac{(n^2 - 1) \cdot n^2}{6} \tag{5}$$

□

b. *Proof.* **Part 1 (Evaluating upper bound of the worst case running time)**

Assume the algorithm runs on worst possible case.

We will evaluate the algorithm's upper bound running time by determining the total cost of the loop, and then the big-oh.

First, we will evaluate the total cost the cost of loop 2.

Because we know loop 2 starts at $j = 1$ and ends at $j = n - 1$ with j increasing by 1 per iteration, we can conclude loop 2 runs at most

$$\left\lceil \frac{n - 1 - 1 + 1}{1} \right\rceil = n - 1 \tag{1}$$

iterations.

Because we know each iteration in loop 2 costs a constant step, we can conclude loop 2 has at most

$$(n - 1) \cdot 1 = n - 1 \tag{2}$$

steps.

Next, we will calculate the cost of loop 1.

Because we know loop 1 starts at $i = 0$ and ends at $i = n - 1$, we can conclude the loop has at most

$$n - 1 - 0 + 1 = n \tag{3}$$

iterations.

Because we know all the elements in lst become even with loop 2 triggered, and because we know loop 2 will not be triggered for all other values of i, we can conclude each iteration of loop 1 will cost 1 step.

Then, we can conclude loop 1 will cost at most

$$n \cdot 1 = n \tag{4}$$

steps.

Now, we will combine the costs together.

Since loop 2 costs n steps, and loop 1 cost n steps, we can conclude the algorithm has total cost of at most

$$n + n = 2n \tag{5}$$

steps.

Then, it follows from above that the algorithm has upper bound worst-case running time of $\mathcal{O}(n)$.

Part 2 (Evaluating lower bound of the worst case running time)

Let $n \in \mathbb{N}$, and $lst = [1, 2, \cdots, n-1]$

Because we know loop 2 will be triggered when $i = 0$, and because we know the rest of elements in lst will become even, we can conclude the algorithm with this input group will run the same as when evaluating the big oh.

Then, we can conclude the algorithm has total cost of $2n$.

Then, we can conclude the algorithm has lower bound worst-case running time of $\Omega(n)$.

Because we know the value of $\mathcal{O}$ and Ω are the same, we can conclude algorithm has running time of $\Theta(n)$. □