

Learning Objectives

By the end of this worksheet, you will:

- Prove statements using the technique of simple induction.
 - Write proofs using simple induction with different starting base cases.
 - Write proofs using simple induction within the scope of a larger proof.
-

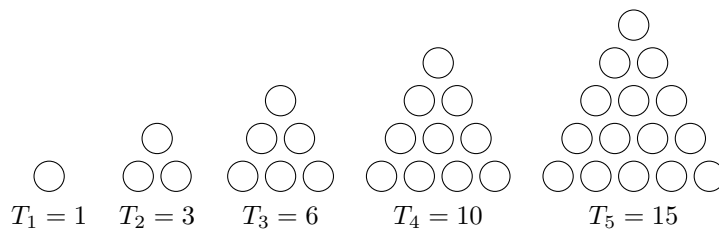
1. **Induction.** Consider the following statement:

$$\forall n \in \mathbb{N}, n \leq 2^n$$

- (a) Suppose we want to prove this statement using induction. Write down the full statement we'll prove (it should be an **AND** of the base case and induction step). Consult your notes if you aren't sure about this!
- (b) Prove the statement using induction. We strongly recommend reviewing the induction proof template from lecture before working on a proof here.

Hint: $2^{k+1} = 2^k + 2^k$.

2. **Induction (summations).** If marbles are arranged to form an equilateral triangle shape, with n marbles on each side, a total of $\sum_{i=1}^n i$ marbles will be required. (For convenience, we also define $T_0 = 0$.)



In the course notes, we prove that $\sum_{i=1}^n i = n(n+1)/2$. For each $n \in \mathbb{N}$, let $T_n = n(n+1)/2$; these numbers are usually called the *triangular numbers*. Use induction to prove that

$$\forall n \in \mathbb{N}, \quad \sum_{j=0}^n T_j = \frac{n(n+1)(n+2)}{6}$$

3. Induction (inequalities). Consider the statement:

For every positive real number x and every natural number n , $(1+x)^n \geq (1+nx)$.

We can express the statement using the notation of predicate logic as:

$$\forall x \in \mathbb{R}^+, \forall n \in \mathbb{N}, (1+x)^n \geq 1+nx$$

Notice that in this statement, there are two universally-quantified variables: n and x .¹ Prove this statement is True using the following approach:

- (a) Use the standard proof structure to introduce x .
- (b) When proving the $(\forall n \in \mathbb{N}, (1+x)^n \geq 1+nx)$, do induction on n .²

¹ For extra practice, think about the following questions. First, would the statement still be True with the order of the quantifiers reversed: $\forall n \in \mathbb{N}, \forall x \in \mathbb{R}^+, (1+x)^n \geq 1+nx$? Second, if this variation is correct, how would this change the proof?

² Your predicate $P(n)$ that you want to prove will also contain the variable x —that's okay, since when we do the induction proof, x has already been defined.

4. **Changing the starting number.** Recall that you previously proved that $\forall n \in \mathbb{N}, n \leq 2^n$ using induction.

- (a) First, use trial and error to fill in the blank to make the following statement true—try finding the *smallest natural number* that works!

$$\forall n \in \mathbb{N}, n \geq \text{_____} \Rightarrow 30n \leq 2^n$$

- (b) Now, prove the completed statement using induction. Be careful about how you choose your base case!