

Problem Set 3 Solution

March 22, 2020

Question 1

1. Let $x \in \mathbb{R}$.

Base Case ($n = 0$):

Let $n = 0$.

Then,

$$a_0 = 0 \tag{1}$$

Then it follows from above that the base case holds.

Inductive Case ($n > 0$):

Let $k \in \mathbb{N}$, and assume $a_n = x \prod_{i=0}^{n-1} a_i$.

Then,

$$x \prod_{i=0}^{n-1} a_i \cdot a_n = x \prod_{i=0}^n a_i \tag{1}$$

$$= a_{n+1} \tag{2}$$

Then it follows from above that the recursive sequence of numbers is true for all natural numbers.

2. From the following table

| String Length | Number of Even (Digit Sum) | Number of Odd (Digit Sum) | Total |
|---------------|----------------------------|---------------------------|-------|
| 1 | 2 | 1 | 3 |
| 2 | 5 | 4 | 9 |
| 3 | 14 | 13 | 27 |

we see that $E_n = \frac{3^n+1}{2}$ and $O_n = \frac{3^n-1}{2}$.

As well, we see that the number of new elements in E_{n+1} is 3^n .

Now, we will prove that E_n and O_n are true for all natural numbers using the induction hypothesis.

Base Case (n = 1):

Let $n = 1$.

Then, $E_n = \frac{4}{2} = 2$ and $O_n = \frac{2}{2} = 1$.

Since the result matches to data in table, the base case holds.

Inductive Case:

Let $n \in \mathbb{N}$. Assume $E_n = \frac{3^n+1}{2}$ and $O_n = \frac{3^n-1}{2}$.

Then,

$$E_{n+1} = \frac{3^n + 1}{2} + 3^n \quad (1)$$

$$= \frac{3^n + 1}{2} + \frac{2 \cdot 3^n}{2} \quad (2)$$

$$= \frac{3 \cdot 3^n + 1}{2} \quad (3)$$

$$= \frac{3^{n+1} + 1}{2} \quad (4)$$

Then, it follows from above that the inductive step for E_n holds.

Similarly, for O_n ,

$$O_{n+1} = \frac{3^n - 1}{2} + 3^n \quad (5)$$

$$= \frac{3^n - 1}{2} + \frac{2 \cdot 3^n}{2} \quad (6)$$

$$= \frac{3 \cdot 3^n - 1}{2} \quad (7)$$

$$= \frac{3^{n+1} - 1}{2} \quad (8)$$

Then, it follows from above that the inductive step for O_n holds.

Then, it follows from the definition of induction hypothesis that the value of E_n and O_n are true for all n .

Question 2

a. $\forall n \in \mathbb{Z}^+, 4 \mid n \Rightarrow 0.2 - z_n = 3 \left(\frac{1 - \frac{1}{16} \frac{n}{4}}{1 - (\frac{1}{16})} \right)$

Question 3

Question 4