CSC236 Worksheet 1 Solution

Hyungmo Gu

April 28, 2020

Question 1

a. *Proof.* Assume the statement $P(115)$ is true. That is, $\sum_{i=0}^{i=115} 2^i = 2^{115+1}$.

We need to prove $\sum_{i=0}^{i=116} 2^i = 2^{116+1}$.

Starting from the left, we can write

$$\sum_{i=0}^{i=116} 2^i = \sum_{i=0}^{i=115} 2^i + 2 \tag{1}$$

Then, using the assumption $\sum_{i=0}^{i=115} 2^i = 2^{115+1}$, we can conclude

$$\begin{aligned}
\sum_{i=0}^{i=116} 2^i &= 2^{115+1} + 2^{116} && (2)\\
&= 2^{116} + 2^{116} && (3)\\
&= 2^{116}(1+1) && (4)\\
&= 2 \cdot 2^{116} && (5)\\
&= 2^{116+1} && (6)
\end{aligned}$$

□

b. *Proof.* No. The statement is not true for every natural natural number.

We will prove this by counter example. That is, $\exists n \in \mathbb{N}$, $\sum_{i=0}^{i=n} 2^i \neq 2^{n+1}$.

Let $n = 0$.

Then, starting from the left hand side, it follows from the fact $n = 0$ that

$$\sum_{i=0}^{i=n} 2^i = \sum_{i=0}^{i=0} 2^i \tag{1}$$
$$= 0 \tag{2}$$

Now, for the right hand side, using the same fact, we can write

$$2^{0+1} = 2^1 \tag{3}$$
$$= 2 \tag{4}$$

□

Question 2

- **Statement:** $\forall n \in \mathbb{N}, \exists d \in \mathbb{Z}, 8^n - 1 = 7d$

Proof. We will prove this statement by induction on n.

Base Case:

Let $n = 0$.

We need to prove $8^n - 1 = 7 \cdot 0$.

Starting from the left hand side, using the fact $n = 0$, we can conclude,

$$8^0 - 1 = 1 - 1 \tag{1}$$
$$= 0 \tag{2}$$
$$= 7 \cdot 0 \tag{3}$$

Inductive Step:

Let $n \in \mathbb{N}$. Assume there is an integer d such that $8^n - 1 = 7d$.

We need to prove there is an integer $\tilde{d}$ such that $8^{n+1} - 1 = 7\tilde{d}$.

Let $\tilde{d} = 8^n + d$.

Starting from the left hand side, we can write

$$\begin{aligned} 8^{n+1} - 1 &= 8^n + 8^n + 8^n + 8^n + 8^n + 8^n + 8^n + 8^n - 1 && (4) \\ &= 8^n + 8^n + 8^n + 8^n + 8^n + 8^n + 8^n + (8^n - 1) && (5) \end{aligned}$$

Then, using inductive hypothesis, i.e. $8^n - 1 = 7d$, we can conclude

$$\begin{aligned} 8^{n+1} - 1 &= 8^n + 8^n + 8^n + 8^n + 8^n + 8^n + 8^n + 7d && (6) \\ &= 7 \cdot 8^n + 7d && (7) \\ &= 7 \cdot (8^n + d) && (8) \\ &= 7 \cdot \tilde{d} && (9) \end{aligned}$$

$\square$

Question 3

- **<u>Statement:</u>** $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}$, the units digit of 7^n is the same as the units digit of 3^m.

Proof. We will prove this statement by induction on n.

<u>Base Case:</u>

Let $n = 0$.

We need to prove there is a natural number m such that the units digit of $7^0 = 1$ is the same as the units digit of 3^m. That is, the ones place of the number 3^m is 1.

Let $m = 0$.

Then, using this fact, we can conclude

$$3^0 = 1 \qquad (1)$$

Inductive Step:

Let $n \in \mathbb{N}$. Assume there exists a natural number m such that the units digit of 7^n is the same as the units digit of 3^m.

We need to prove there is a natural number $\tilde{m}$ such that the units digit of 7^{n+1} is the same as the units digit of $3^{\tilde{m}}$. That is, the ones digit of 7^{n+1} is the same as the ones digit of $3^{\tilde{m}}$.

Let $\tilde{m} = m + 9$.

Starting with 7^{n+1}, we can write

$$7^{n+1} = 7 \cdot 7^n. \tag{2}$$

Then, it follows from above fact that the ones digit of 7^{n+1} is 7 times the ones digit of 7^n.

Now, for $3^{\tilde{m}}$, using the fact $\tilde{m} = m + 9$, we can write

$$\begin{aligned} 3^{\tilde{m}} &= 3^{m+9} && (3)\\ &= 3^9 \cdot 3^m && (4)\\ &= 27 \cdot 3^m && (5) \end{aligned}$$

Then, it follows from above fact that that the ones digit of $3^{\tilde{m}}$ is 7 times the ones digit of 3^m.

Then, using inductive hypothesis, we can conclude the ones digit of $3^{\tilde{m}}$ is 7 times the ones digit of 7^n.

$\square$

Correct Solution:

Statement: $\forall n \in \mathbb{N}$, $\exists m \in \mathbb{N}$, the units digit of 7^n is the same as the units digit of 3^m.

Proof. We will prove this statement by induction on n.

Base Case:

Let $n = 0$.

We need to prove there is a natural number m such that the units digit of $7^0 = 1$ is the same as the units digit of 3^m. That is, the ones place of the number 3^m is 1.

Let $m = 0$.

Then, using this fact, we can conclude

$$3^0 = 1 \tag{1}$$

Inductive Step:

Let $n \in \mathbb{N}$. Assume there exists a natural number m such that the units digit of 7^n is the same as the units digit of 3^m.

We need to prove there is a natural number $\tilde{m}$ such that the units digit of 7^{n+1} is the same as the units digit of $3^{\tilde{m}}$. That is, the ones digit of 7^{n+1} is the same as the ones digit of $3^{\tilde{m}}$.

Let $\tilde{m} = m + 3$.

Starting with 7^{n+1}, we can write

$$7^{n+1} = 7 \cdot 7^n. \tag{2}$$

Then, it follows from above fact that the ones digit of 7^{n+1} is 7 times the ones digit of 7^n.

Now, for $3^{\tilde{m}}$, using the fact $\tilde{m} = m + 3$, we can write

$$\begin{aligned} 3^{\tilde{m}} &= 3^{m+3} && (3)\\ &= 3^3 \cdot 3^m && (4)\\ &= 27 \cdot 3^m && (5) \end{aligned}$$

Then, it follows from above fact that that the ones digit of $3^{\tilde{m}}$ is 7 times the ones digit of 3^m.

Then, using inductive hypothesis, we can conclude the ones digit of $3^{\tilde{m}}$ is 7 times the ones digit of 7^n.

$\square$

Question 4

- *Proof.* Let $m = 7$. Let $n \in \mathbb{N}$. Assume $n \geq m$.

  We need to prove $4^n \geq 5n^4 + 6$.

  We will do so by induction on n.

  Base Case ($n = 7$):

  Let $n = 7$.

  We need to prove $4^n \geq 5n^4 + 6$.

  Starting with the left hand side, using the fact $n = 7$, we can calculate

  $$4^7 = 16384 \tag{1}$$

  Now, for the right hand side, using the same fact, we can conclude

  $$5 \cdot 7^4 + 6 = 12011 \tag{2}$$

  Inductive Step:

  Let $n \in \mathbb{N}$. Assume that $4^n \geq 5n^4 + 6$.

  We need to prove $4^{n+1} \geq 5 \cdot (n+1)^4 + 6$.

  Starting from the left hand side, we can write

  $$4^{n+1} = 4^n + 4^n + 4^n + 4^n \tag{3}$$

Then, by inductive hypothesis, i.e. $4^n \geq 5n^4 + 6$,

$$\begin{align}
4^{n+1} &\geq (5n^4+6)+(5n^4+6)+(5n^4+6)+(5n^4+6) \tag{4}\\
&= 5n^4+5n^4+5n^4+5n^4+24 \tag{5}\\
&= (5n^4+5\cdot n\cdot n^3+5\cdot n^2\cdot n^2+5\cdot n^3\cdot n)+24 \tag{6}
\end{align}$$

Then, because we know $n \geq 7$ from the header, we can conclude

$$\begin{align}
4^{n+1} &\geq (5n^4+5\cdot 7\cdot n^3+5\cdot 7^2\cdot n^2+5\cdot 7^3\cdot n)+24 \tag{7}\\
&\geq (5n^4+5\cdot 4\cdot n^3+5\cdot 6\cdot n^2+5\cdot 4\cdot n)+5+19 \tag{8}\\
&\geq 5\cdot(n^4+4n^3+6n^2+4n+1)+19 \tag{9}\\
&= 5\cdot\left((n+1)^2\cdot(n+1)^2\right)+19 \tag{10}\\
&= 5\cdot(n+1)^4+19 \tag{11}\\
&> 5\cdot(n+1)^4+6 \tag{12}
\end{align}$$

$\square$