CSC373 Worksheet 0 Solution

July 19, 2020

1. Recurrence: $T(n) = T(n-1) + n$

   Guess: $T(n) = \mathcal{O}(n^2)$.

   I need to show $T(n) \leq c \cdot n^2$.

$$
\begin{aligned}
T(n) &\leq c(n-1)^2 + n && (1)\\
&= c(n^2 - 2n + 1) + n && (2)\\
&= cn^2 - c2n + c + n && (3)\\
&\leq cn^2 - c2n + cn + n && (4)\\
&= cn^2 - cn + n && (5)\\
&\leq cn^2 - cn + cn && (6)\\
&= cn^2 && (7)
\end{aligned}
$$

Notes:

- Substitution method
  - Solves recurrences
    - Recurrence characters the running time of divide-and-conquer algorithm
  - How it works:
    1. Make a guess for the solution
    2. Use mathematical induction to prove the guess is correct or incorrect.

    Example:

    Recurrence: $T(n) = 2T(\lfloor n/2 \rfloor) + n$

    Guess: $T(n) = \mathcal{O}(n \log n)$,

    We need to show $T(n) \leq cn \lg n$.

1. Assume the bound holds for all positive $m < n$, in particular $m = \lfloor n/2 \rfloor$
2. Find the upper bound of $T(m)$

   $T(\lfloor n/2 \rfloor) \leq c\lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)$

3. Show $T(n) = 2T(\lfloor n/2 \rfloor) + n$ leads to $T(n) \leq cn \lg n$

$$
\begin{aligned}
T(n) &\leq 2(c\lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)) + n && (8)\\
&\leq cn \lg(n/2) + n && (9)\\
&= cn \lg(n) - cn \lg 2 + n && (10)\\
&= cn \lg(n) - cn + n && (11)\\
&\leq cn \lg(n) - cn + cn && (12)\\
&\leq cn \lg(n) && (13)
\end{aligned}
$$

4. Show that the boundary holds using mathematical induction

   Doesn't have information in detail. Skipping this for now.

- Making good guess
  * Three suggestions
  1. Using recursion tree
  2. Through practice
  3. prove loose upper and lower bounds on the recurrence and then reduce the range of uncertainty

2. Recurrence: $T(n) = T(\lceil n/2 \rceil) + 1$

   Guess: $T(n) = \mathcal{O}(\lg n)$.

   I need to show $T(n) \leq c \cdot \lg n$.

$$
\begin{aligned}
T(n) &\leq c \lg(\lceil n/2 \rceil) + 1 && (1)\\
&\leq c \lg(n/2) + 1 && (2)\\
&= c(\lg n - \lg 2) + 1 && (3)\\
&= c(\lg n - 1) + 1 && (4)\\
&= c \lg n - c + 1 && (5)\\
&\leq c \lg n - c + c && (6)
\end{aligned}
$$

Correct Solution:

Recurrence: $T(n) = T(\lceil n/2 \rceil) + 1$

Guess: $T(n) = \mathcal{O}(\lg n)$.

I need to show $T(n) \leq c \cdot \lg n$.

$$\begin{align}
T(n) &\leq c \lg(\lceil n/2 \rceil) + 1 \tag{1}\\
&\leq c \lg(n/2) + 1 \tag{2}\\
&= c(\lg n - \lg 2) + 1 \tag{3}\\
&= c(\lg n - 1) + 1 \tag{4}\\
&= c \lg n - c + 1 \tag{5}\\
&\leq c \lg n - c + c \tag{6}
\end{align}$$

The solution holds for $c \geq 1$.

3. Recurrence: $T(n) = 2T(\lfloor n/2 \rfloor) + n$

   Guess (Upperbound): $T(n) = \mathcal{O}(n \lg n)$.

   I first need to show $T(n) \leq c \cdot n \lg n$.

$$\begin{align}
T(n) &= 2T(\lfloor n/2 \rfloor) + n \tag{1}\\
&= 2c\lfloor n/2 \rfloor \lg\lfloor n/2 \rfloor + n \tag{2}\\
&\leq 2c \cdot (n/2) \lg(n/2) + n \tag{3}\\
&= c \cdot n(\lg n - 1) + n \tag{4}\\
&= cn \lg n - cn + n \tag{5}\\
&\leq cn \lg n - cn + cn \tag{6}\\
&\leq cn \lg n \tag{7}
\end{align}$$

   The above inequality holds for $c \geq 1$.

   Guess (Lowerbound): $T(n) = \Omega(n \lg n)$.

I first need to show $d \cdot n \lg n \leq T(n)$.

$$
\begin{align}
T(n) &= 2T(\lfloor n/2 \rfloor) + n \tag{8}\\
&\leq 2d\lfloor n/2 \rfloor \lg\lfloor n/2 \rfloor + n \tag{9}\\
&\geq 2c \cdot (n/2)\lg(n/2) + n \tag{10}\\
&= c \cdot n(\lg n - 1) + n \tag{11}\\
&= cn\lg n - cn + n \tag{12}\\
&\leq cn\lg n - cn + cn \tag{13}\\
&\leq cn\lg n \tag{14}
\end{align}
$$

The above inequality holds for $c \geq 1$.

<u>Notes:</u>

- Both upper bound and lower bound don't need to be the same