CSC373 Worksheet 5 Solution

August 8, 2020

1. **Rough Works:**

Assume that a flow network $G = (V, E)$ violates the assumption that the network contains a path $s \rightsquigarrow v \rightsquigarrow t$ for all vertices $v \in V$. Let u be a vertex for which there is no path $s \rightsquigarrow u \rightsquigarrow t$.

I must show such that there is no flow at vertex u. That is, there exists a maximum flow f in G such that $f(u, v) = f(v, u) = 0$ for all vertices $v \in V$.

I will do so in cases

1. **Case 1:** When u only has no flow going in and out

- Show

Assume u only has no flow going in and out.

Then, we know that $(u, v) \notin E$ for all $v \in V$.

Then, we know by the edge case of flow conservation that $f(u, v) = 0$ for all $v \in V$.

Then, we can write

$$\sum_{v \in V} f(u, v) = \sum_{v \in V} f(v, u) = 0 \qquad \text{[By flow conservation]} \tag{1}$$

Then, using the capacity constraint ($\forall a, b \in V,\ 0 \leq f(a, b)$), we can write $f(v, u)$ in $\sum_{v \in V} f(v, u)$ must be 0 for all $v \in V$.

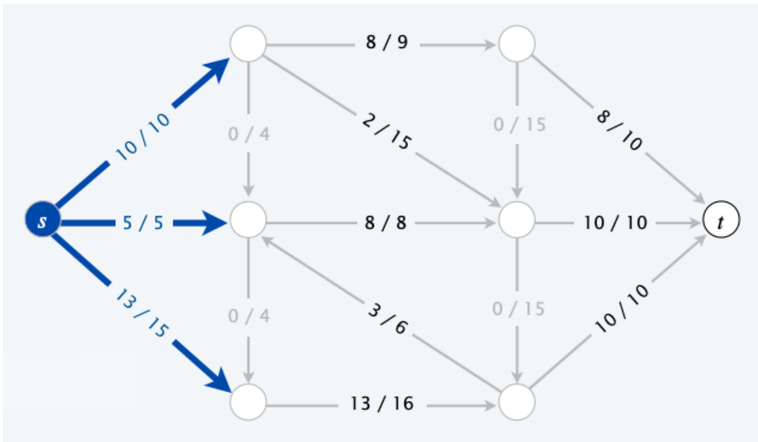
So, $f(v, u) = 0$

This applies to all $u \in V$.

Notes

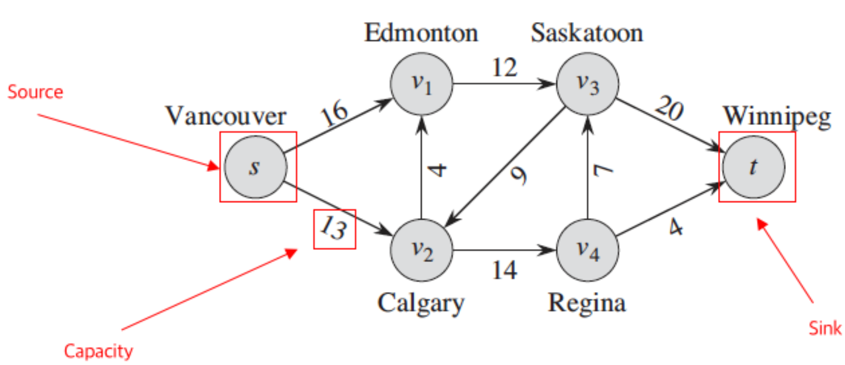
- **Maximum Flow:**
  - Finds a flow of maximum value [1]

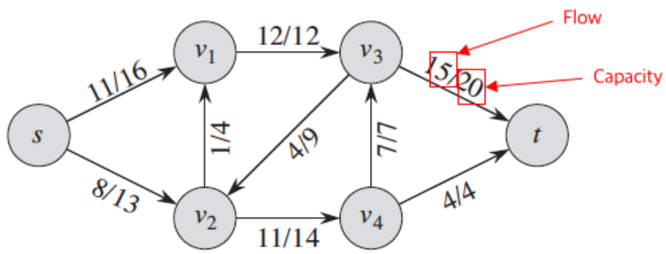
    Example



    Here, the maximum flow is $10 + 5 + 13 = 28$

- **Flow Network:**
  - $G = (V, E)$ is a directed graph in which each edge $(u, v) \in E$ has a nonnegative capacity $c(u, v) \geq 0$.
  - Two vertices must exist: **source** s and **sink** t
  - **path** from source s to vertax v to sink t is represented by $s \rightsquigarrow v \rightsquigarrow t$





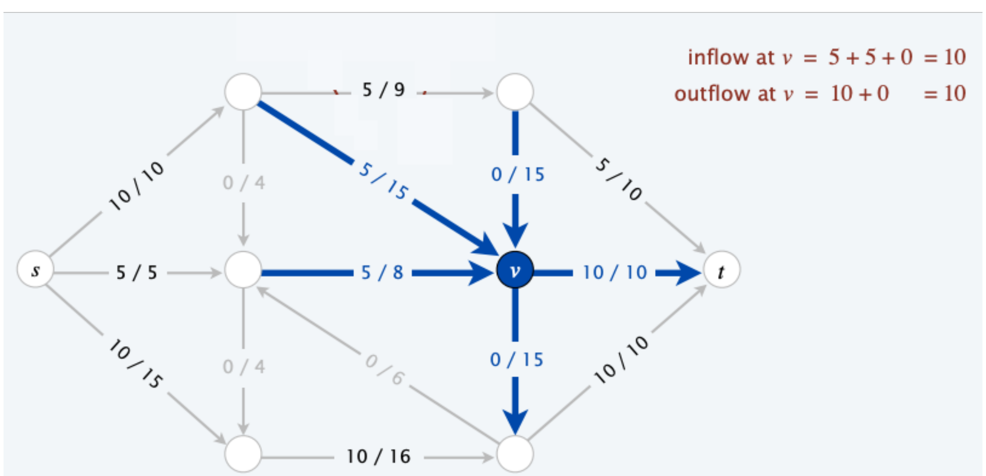
- **Capacity:**
  - Is a non-negative function $f : V \times V \to \mathbb{R}_{\geq 0}$
  - Has **capacity constraint** where for all $u, v \in V$ $0 \leq f(u, v) \leq c(u, v)$
    * Means flow cannot be above capacity constraint
- **Flow:**
  - Is a real valued function $f : V \times V \to \mathbb{R}$ in G
  - Satisfies **capacity constraint** (i.e for all $u, v \in V$, $0 \leq f(u, v) \leq c(u, v)$)
  - Satisfies **flow conservation**

  For all $u \in V - \{s, t\}$, we require

$$\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v) \tag{2}$$

  Means flow into vertex u is the same as flow going out of vertex u. [1]

  <u>**Example:**</u>



References

1) Princeton University, Network Flow 1, link