Worksheet 1 Review

Hyungmo Gu

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Question 1

a. $A^c = \{1, 3, 4, 6\}$

b. $A = U \setminus A$

c. $A^c \cap B^c = \{x \mid x \in U,\ x \leq 0 \text{ and } x \geq 4\}$
$A^c \cup B^c = \{x \mid x \in U,\ x < 1 \text{ and } x > 2\}$
$(A \cap B)^c = \{x \mid x \in U,\ x < 1 \text{ and } x > 2\}$
$(A \cup B)^c = \{x \mid x \in U,\ x \leq 0 \text{ and } x \geq 4\}$

Correct Solution:

$A^c \cap B^c = \{x \mid x \in U,\ x \leq 0 \text{ or } x \geq 4\}$
$A^c \cup B^c = \{x \mid x \in U,\ x < 1 \text{ or } x > 2\}$
$(A \cap B)^c = \{x \mid x \in U,\ x < 1 \text{ or } x > 2\}$
$(A \cup B)^c = \{x \mid x \in U,\ x \leq 0 \text{ or } x \geq 4\}$
It follows from above that $A^c \cap B^c = (A \cup B)^c$ and $A^c \cup B^c = (A \cap B)^c$

Question 2

a. $T_0 = \{3, 6, 9, \dots\}$
$T_1 = \{1, 4, 7, \dots\}$
$T_2 = \{2, 5, 8, \dots\}$
$T_3 = \{6, 12, 18, \dots\}$

b. A partition of $\mathbb{Z}$ is $\{T_0, T_1, T2\}$.

All four sets can't be used because elements in T_3 overlaps with T_0. A partition cannot have any elements in common.

Notes:

- **Definition of Partition:** Let A be a set. A (finite or infinite) collection of nonempty sets $\{A_1, A_2, A_3\}$ is called a **partition** of A when (1) A is the union of all of the A_i, and (2) the sets $A_1, A_2, A_3, \ldots$ do not have any element in common.

Question 3

a. All strings over the alphabet $\{0, 1\}$ of length three are

$$000, 100, 010, 001, 110, 101, 011, 111$$

b. $S_1 = \{aa, ab, ac, ba, bb, bc, ca, cb, cc\}$

$S_2 = \{a, b, c, aa, bb, cc, \ldots\}$

$S_1 \cap S_2 = \{aa, bb, cc\}$

$S_1 \setminus S_2 = \{ab, ac, ba, bc, ca, cb\}$

c. $S_1 = (S_1 \cap S_2) \cup (S_1 \setminus S_2)$

Question 4

a.

	$\lfloor x \rfloor$	$\lceil x \rceil$
$\frac{25}{4}$	6	7
0.99	0	1
-2.01	-3.0	-2.0

Notes:

- floor of a negative number: ceiling but with negative sign
- ceiling of a negative number: floor but with negative sign

b. **Domain of the floor & ceiling function:** $\mathbb{R}$

Codomain of the floor & ceiling function: $\mathbb{N}$

c. The statement is false. Consider example $x = -0.5$ and $y = 0.5$.

Then, $\lfloor x + y \rfloor = 0$ and $\lfloor x \rfloor + \lfloor y \rfloor = -1 + 0 = -1$.

Question 5

a. $\sum_{k=1}^{3}(k+1) = (1+1)+(2+1)+(3+1)$

$\sum_{m=0}^{1} \frac{1}{2^m} = \frac{1}{2^0}$

$\sum_{k=-1}^{2}(k^2+3) = ((-1)^2+3)+(0^2+3)+(1^2+3)+(2^2+3)$

$\sum_{j=0}^{4}(-1)^j \frac{j}{j+1} = (-1)^0 \cdot \frac{0}{0+1} + (-1) \cdot \frac{1}{1+1} + (-1)^2 \cdot \frac{2}{2+1} + (-1)^3 \cdot \frac{3}{3+1} + (-1)^4 \cdot \frac{4}{4+1}$

$\sum_{k=1}^{5}(2 \cdot k) = (2 \cdot 1)+(2 \cdot 2)+(2 \cdot 3)+(2 \cdot 4)+(2 \cdot 5)$

$\prod_{i=2}^{4} \frac{i(i+2)}{(i-1)(i+1)} = \left(\frac{0\cdot(0+2)}{(0-1)(0+1)}\right)\left(\frac{1\cdot(1+2)}{(1-1)(1+1)}\right)\left(\frac{2\cdot(2+2)}{(2-1)(2+1)}\right)\left(\frac{3\cdot(3+2)}{(3-1)(3+1)}\right)\left(\frac{4\cdot(4+2)}{(4-1)(4+1)}\right)$

b. $3+6+12+24+48+96 = \sum_{i=0}^{6} 3 \cdot 2^i$

$\frac{1}{3}+\frac{4}{9}+\frac{9}{27}+\frac{16}{81}+\frac{25}{243}+\frac{36}{729} = \sum_{i=1}^{6} \frac{i^2}{3^i}$

$0+1-2+3-4+5 = \sum_{i=0}^{5}(-1)^{i+1} \cdot i$

$\left(\frac{1}{1+1}\right) \times \left(\frac{2}{2+1}\right) \times \left(\frac{3}{3+1}\right) \times \left(\frac{k}{k+1}\right) = \prod_{i=1}^{k}\left(\frac{i}{i+1}\right)$

$\left(\frac{1\cdot2}{3\cdot4}\right) \times \left(\frac{2\cdot3}{4\cdot5}\right) \times \left(\frac{3\cdot4}{5\cdot6}\right) = \prod_{i=1}^{3} \frac{i\cdot(i+1)}{(i+2)\cdot(i+3)}$

Question 6

a. $3 \cdot \sum_{i=1}^{n}(2i-3) + \sum_{i=1}^{n}(4-5i)$

By the pulling of constant,

$$\begin{aligned}
3 \cdot \sum_{i=1}^{n}(2i-3) + \sum_{i=1}^{n}(4-5i) &= \sum_{i=1}^{n} 3(2i-3) + \sum_{i=1}^{n}(4-5i) \\
&= \sum_{i=1}^{n}(6i-9) + \sum_{i=1}^{n}(4-5i)
\end{aligned}$$

Then, by the separating sums

$$\sum_{i=1}^{n}(6i-9)+\sum_{i=1}^{n}(4-5i)=\sum_{i=1}^{n}(6i-9)+(4-5i)$$
$$=\sum_{i=1}^{n}(i-5)$$

b. $\left(\prod_{i=1}^{n}\frac{i}{i+1}\right)\left(\prod_{i=1}^{n}\frac{i+1}{i+2}\right)$

By the separating products,

$$\left(\prod_{i=1}^{n}\frac{i}{i+1}\right)\left(\prod_{i=1}^{n}\frac{i+1}{i+2}\right)=\prod_{i=1}^{n}\left(\frac{i}{i+1}\right)\left(\frac{i+1}{i+2}\right)$$

c. $\sum_{i=10}^{1}52i+\sum_{i=101}^{106}(i-1)$

By the changing index,

$$\sum_{i=10}^{1}52i+\sum_{i=101}^{106}(i-1)=\sum_{i'=0}^{5}2(i'+10)+\sum_{i'=0}^{5}(i'+101-1)$$
$$=\sum_{i'=0}^{5}(2i'+20)+\sum_{i'=0}^{5}(i'+100)$$

Then, by the separating sums,

$$\sum_{i'=0}^{5}(2i'+20)+\sum_{i'=0}^{5}(i'+100)=\sum_{i'=0}^{5}(2i'+20)+(i'+100)$$
$$=\sum_{i'=0}^{5}(3i'+120)$$