CSC236 Term Test 1 Version 2 Solution

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Question 1

- *Proof.* Define $P(n) : f(n) = 3^n$.

  I will use complete induction to prove that $\forall n \in \mathbb{N}, n > 2 \Rightarrow P(n)$.

  Base Case ($n = 0$):

  Let $n = 0$.

  Then, the definition of $f(n)$ tells us $f(n) = 1$.

  Then, we have

  $$\begin{aligned} f(n) &= 3^0 && (1) \\ &= 3^n && (2) \end{aligned}$$

  Thus, $P(n)$ follows.

  Base Case ($n = 1$):

  Let $n = 1$.

  Then, the definition of $f(n)$ tells us $f(n) = 3$.

  Then, we have

  $$\begin{aligned} f(n) &= 3^1 && (3) \\ &= 3^n && (4) \end{aligned}$$

Thus, $P(n)$ follows.

Base Case ($n = 2$):

Let $n = 2$.

Then, the definition of $f(n)$ tells us $f(n) = 9$.

Then, we have

$$\begin{align} f(n) &= 3^2 \tag{5} \\ &= 3^n \tag{6} \end{align}$$

Thus, $P(n)$ follows.

Case ($n > 2$):

Assume $n > 2$.

Then, since $0 \leq n-1 < n$, $0 \leq n-2 < n$, and $0 \leq n-3 < n$, the complete induction tells us $P(n-1)$, $P(n-2)$, and $P(n-3)$, i.e. $f(n-1) = 3^{n-1}$, $f(n-2) = 3^{n-2}$, and $f(n-3) = 3^{n-3}$, respectively.

Then, using these facts, we can write

$$\begin{align} f(n) &= f(n-1) + 3f(n-2) + 9f(n-3) \tag{7} \\ &= 3^{n-1} + 3 \cdot 3^{n-2} + 3^2 \cdot 3^{n-3} \tag{8} \\ &= 3^{n-1} + 3^{n-1} + 3^{n-1} \tag{9} \\ &= 3^{n-1}(1+1+1) \tag{10} \\ &= 3^{n-1}3 \tag{11} \\ &= 3^n \tag{12} \end{align}$$

Thus, $P(n)$ follows. $\square$

Correct Solution:

Define $P(n) : f(n) = 3^n$.

I will use complete induction to prove that $\forall n \in \mathbb{N},\ P(n)$.

Inductive Step:

Let $n \in \mathbb{N}$. Assume $H(n) : \bigwedge_{i=0}^{n-1} P(i)$. I will prove $P(n)$ follows. That is, $f(n) = 3^n$.

Base Case ($n = 0$):

Let $n = 0$.

Then, the definition of $f(n)$ tells us $f(n) = 1$.

Then, we have

$$f(n) = 3^0 \tag{13}$$
$$= 3^n \tag{14}$$

Thus, $P(n)$ follows.

Base Case ($n = 1$):

Let $n = 1$.

Then, the definition of $f(n)$ tells us $f(n) = 3$.

Then, we have

$$f(n) = 3^1 \tag{15}$$
$$= 3^n \tag{16}$$

Thus, $P(n)$ follows.

Base Case ($n = 2$):

Let $n = 2$.

Then, the definition of $f(n)$ tells us $f(n) = 9$.

Then, we have

$$f(n) = 3^2 \tag{17}$$
$$= 3^n \tag{18}$$

Thus, $P(n)$ follows.

Case $(n > 2)$:

Assume $n > 2$.

Then, since $0 \leq n-1 < n$, $0 \leq n-2 < n$, and $0 \leq n-3 < n$, the complete induction tells us $P(n-1)$, $P(n-2)$, and $P(n-3)$, i.e. $f(n-1) = 3^{n-1}$, $f(n-2) = 3^{n-2}$, and $f(n-3) = 3^{n-3}$, respectively.

Then, using these facts, we can write

$$\begin{aligned} f(n) &= f(n-1) + 3f(n-2) + 9f(n-3) && \text{[By definition, since } n > 2\text{]} && (19) \\ &= 3^{n-1} + 3 \cdot 3^{n-2} + 3^2 \cdot 3^{n-3} && && (20) \\ &= 3^{n-1} + 3^{n-1} + 3^{n-1} && && (21) \\ &= 3^{n-1}(1+1+1) && && (22) \\ &= 3^{n-1}3 && && (23) \\ &= 3^n && && (24) \end{aligned}$$

Thus, $P(n)$ follows.

Notes:

1. Learned that $n > i$ in $\forall n \in \mathbb{N}, n > i \Rightarrow P(n)$ is used when $P(n)$ is true starting $i+1$.

   If $P(n)$ is true for all natural numbers, then $\forall n \in \mathbb{N}, P(n)$ is used.

2. Learned that 'Assume $n > 2$' in 'Let $n \in \mathbb{N}$. Assume $n > 2$. Assume $H(n) : \bigwedge_{i=0}^{n-1} P(i)$'. is used when $P(i)$ is true starting $n = 3$.

   If $P(i)$ is true for all natural numbers, then, 'Let $n \in \mathbb{N}$. Assume $H(n) : \bigwedge_{i=0}^{n-1} P(i)$' is used.

Question 2

- Given the statement to prove

  $P(x, y, z, w)$: There are no positive integers x, y, z, w such that $x^4 + 3y^4 + 9z^4 = 27w^4$.

Proof. I will prove $P(x, y, z, w)$ using proof by contradiction.

Assume $\exists x, y, z, w \in \mathbb{N}^+$, $x^4 + 3y^4 + 9z^4 = 27w^4$.

Then, the set $X = \{x \in \mathbb{N}^+ \mid \exists y, z, w \in \mathbb{N}^+, x^4 + 3y^4 + 9z^4 = 27w^4\}$ is not empty.

Then, by principle of well-ordering, there is smallest positive integer $x_0 \in X$, and postive integers $y_0, z_0, w_0 \in \mathbb{N}^+$ such that $x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4$.

Then,

$$\begin{aligned} x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 &\Rightarrow x_0^4 = 27w_0^4 - 3y_0^4 - 9z_0^4 \\ &\Rightarrow 3 \mid x_0^4 \Rightarrow 3 \mid x_0 \end{aligned} \qquad \text{[By hint]} \qquad (1)$$

$$\begin{aligned} \text{Let } \exists x_1 \in \mathbb{N}^+,\ x_0 = 3x_1 &\Rightarrow x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \\ &\Rightarrow 3y_0^4 = 27w_0^4 - 9z_0^4 - x_0^4 \\ &\Rightarrow 3y_0^4 = 27w_0^4 - 9z_0^4 - 3^4x_1^4 \\ &\Rightarrow y_0^4 = 9w_0^4 - 3z_0^4 - 3^3x_1^4 \\ &\Rightarrow 3 \mid y_0^4 \Rightarrow 3 \mid y_0 \end{aligned} \qquad \text{[By hint]} \qquad (2)$$

$$\begin{aligned} \text{Let } \exists y_1 \in \mathbb{N}^+,\ y_0 = 3y_1 &\Rightarrow x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \\ &\Rightarrow 9z_0^4 = 27w_0^4 - 3y_0^4 - x_0^4 \\ &\Rightarrow 9z_0^4 = 27w_0^4 - 3^5y_1^4 - 3^4x_1^4 \\ &\Rightarrow z_0^4 = 3w_0^4 - 3^3y_1^4 - 3^2x_1^4 \\ &\Rightarrow 3 \mid z_0^4 \Rightarrow 3 \mid z_0 \end{aligned} \qquad \text{[By hint]} \qquad (3)$$

$$\begin{aligned} \text{Let } \exists w_1 \in \mathbb{N}^+,\ w_0 = 3w_1 &\Rightarrow x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \\ &\Rightarrow 3^4x_1^4 + 3^5y_1^4 + 3^6z_1^4 = 3^7w_1^4 \\ &\Rightarrow x_1^4 + 3y_1^4 + 3^2z_1^4 = 3^3w_1^4 \\ &\Rightarrow x_1^4 + 3y_1^4 + 9z_1^4 = 27w_1^4 \\ &\Rightarrow x_1 \in X \end{aligned} \qquad (4)$$

Then, this leads to contradiction, because we know $x_1 < x_0$, $x_1 \in X$, but x_0 is the smallest number in X.

Thus, we can conclude the assumption is false.

$\square$

Correct Solution:

Given the statement to prove

$P(x, y, z, w)$: There are no positive integers x, y, z, w such that $x^4 + 3y^4 + 9z^4 = 27w^4$.

Proof. I will prove $P(x, y, z, w)$ using proof by contradiction.

Assume $\exists x, y, z, w \in \mathbb{N}^+$, $x^4 + 3y^4 + 9z^4 = 27w^4$.

Then, the set $X = \{x \in \mathbb{N}^+ \mid \exists y, z, w \in \mathbb{N}^+, x^4 + 3y^4 + 9z^4 = 27w^4\}$ is not empty.

Then, since X is subset of $\mathbb{N}$, by principle of well-ordering, there is smallest positive integer $x_0 \in X$. Furthermore, there are postive integers $y_0, z_0, w_0 \in \mathbb{N}^+$ such that $x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4$.

Then,

$$\begin{aligned} x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 &\Rightarrow x_0^4 = 27w_0^4 - 3y_0^4 - 9z_0^4 \\ &\Rightarrow 3 \mid x_0^4 \Rightarrow 3 \mid x_0 \end{aligned} \qquad \text{[By hint]} \qquad (1)$$

$$\begin{aligned} \text{Let } \exists x_1 \in \mathbb{N}^+,\ x_0 = 3x_1 &\Rightarrow x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \\ &\Rightarrow 3y_0^4 = 27w_0^4 - 9z_0^4 - x_0^4 \\ &\Rightarrow 3y_0^4 = 27w_0^4 - 9z_0^4 - 3^4x_1^4 \\ &\Rightarrow y_0^4 = 9w_0^4 - 3z_0^4 - 3^3x_1^4 \\ &\Rightarrow 3 \mid y_0^4 \Rightarrow 3 \mid y_0 \end{aligned} \qquad \text{[By hint]} \qquad (2)$$

$$\begin{aligned} \text{Let } \exists y_1 \in \mathbb{N}^+,\ y_0 = 3y_1 &\Rightarrow x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \\ &\Rightarrow 9z_0^4 = 27w_0^4 - 3y_0^4 - x_0^4 \\ &\Rightarrow 9z_0^4 = 27w_0^4 - 3^5y_1^4 - 3^4x_1^4 \\ &\Rightarrow z_0^4 = 3w_0^4 - 3^3y_1^4 - 3^2x_1^4 \\ &\Rightarrow 3 \mid z_0^4 \Rightarrow 3 \mid z_0 \end{aligned} \qquad \text{[By hint]} \qquad (3)$$

$$\begin{aligned} \text{Let } \exists w_1 \in \mathbb{N}^+,\ w_0 = 3w_1 &\Rightarrow x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \\ &\Rightarrow 3^4x_1^4 + 3^5y_1^4 + 3^6z_1^4 = 3^7w_1^4 \\ &\Rightarrow x_1^4 + 3y_1^4 + 3^2z_1^4 = 3^3w_1^4 \\ &\Rightarrow x_1^4 + 3y_1^4 + 9z_1^4 = 27w_1^4 \\ &\Rightarrow x_1 \in X \end{aligned} \qquad (4)$$

Then, this leads to contradiction, because we know $x_1 < x_0$, $x_1 \in X$, but x_0 is the smallest number in X.

Thus, we can conclude the assumption is false.

$\square$

Note:

- Noticed professor wrote ‘Divide by 3’ as a reasoning in calculation.

$$
\begin{aligned}
\text{Let } z_1 \in \mathbb{N}^+, 3z_1 = z_0 \quad &\Rightarrow \quad 81z_1^4 = 3w_0^4 - (9x_1^4 + 27y_1^4) \\
&\Rightarrow \quad 27z_1^4 = w_0^4 - (3x_1^4 + 9y_1^4) \Rightarrow 3x_1^4 + 9y_1^4 + 27z_1^4 = w_0^4 \qquad \text{\# divide by 3} \\
&\Rightarrow \quad 3 \mid w_0^4 \Rightarrow 3 \mid w_0 \qquad \text{\# since 3 divides LHS and allowed assumption}
\end{aligned}
$$

Question 3

- *Proof.* Define $\mathcal{T}$ as the smallest set such that

  a. $() \in \mathcal{T}$

  b. If $t_1, t_2 \in \mathcal{T}$, $(t_1 t_2) \in \mathcal{T}$

  I need to prove $\forall t \in \mathcal{T}, P(t)$. That is, left(t) is odd.

  Basis:

  Let $() \in \mathcal{T}$.

  Then, since there is only 1 left parenthesis and 1 is odd, $P(t)$ holds.

  Inductive Step:

  Let t_1, t_2 be arbitrary string in $\mathcal{T}$. Assume $H(t_1, t_2) : P(t_1)$ and $P(t_2)$. That is left(t_1) and left(t_2) are odd. Let $(t_1 t_2) \in \mathcal{T}$.

  I need to prove $P((t_1 t_2))$ follows. That is, left$((t_1 t_2))$ is odd.

  The initial left parenthesis in $(t_1 t_2)$ increases the total number of left parenthesis by 1. This means, we have

$$\text{left}((t_1 t_2)) = \text{left}(t_1) + \text{left}(t_2) + 1 \tag{1}$$

Then, since we know from induction hypothesis that $\text{left}(t_1)$ and $\text{left}(t_2)$ are odd, we can write $\text{left}(t_1) + \text{left}(t_2)$ is even.

Then, we can write $\text{left}(t_1) + \text{left}(t_2) + 1$ is odd.

Then, we can conclude $\text{left}((t_1 t_2))$ is odd.

Thus, $P((t_1 t_2))$ follows.

$\square$

Correct Solution:

Define $\mathcal{T}$ as the smallest set such that

a. $() \in \mathcal{T}$

b. If $t_1, t_2 \in \mathcal{T}$, $(t_1 t_2) \in \mathcal{T}$

I need to prove $\forall t \in \mathcal{T}, P(t)$. That is, left(t) is odd.

Basis:

Let $() \in \mathcal{T}$.

Then, since there is only 1 left parenthesis and 1 is odd, $P(t)$ holds.

Inductive Step:

Let t_1, t_2 be arbitrary string in $\mathcal{T}$. Assume $H(t_1, t_2) : P(t_1)$ and $P(t_2)$. That is $\text{left}(t_1)$ and $\text{left}(t_2)$ are odd. In other words, $\exists k_1, k_2 \in \mathbb{Z}$ such that $\text{left}(t_1) = 2k_1 + 1$, $\text{left}(t_2) = 2k_2 + 1$. Let $(t_1 t_2) \in \mathcal{T}$.

I need to prove $P((t_1 t_2))$ follows. That is, $\text{left}((t_1 t_2))$ is odd. In other words, $\exists k_3 \in \mathbb{Z}$, $\text{left}((t_1 t_2)) = 2k_3 + 1$.

Let $k_3 = k_1 + k_2 + 1$.

The initial left parenthesis in $(t_1 t_2)$ increases the total number of left parenthesis by 1. This means, we have

$$\text{left}((t_1 t_2)) = \text{left}(t_1) + \text{left}(t_2) + 1 \tag{1}$$

Then,

$$\begin{aligned}
\text{left}((t_1 t_2)) &= 2k_1 + 1 + 2k_2 + 1 + 1 && \text{[By I.H]} && (2)\\
&= 2(k_1 + k_2 + 1) + 1 && && (3)\\
&= 2k_3 + 1 && && (4)
\end{aligned}$$

Thus, $P((t_1 t_2))$ follows.

Notes:

1. Realized that the professor wanted to test whether students can unfold definitions twice using terms like 'In other words'.