Worksheet 7 Review 2

Hyungmo Gu

April 17, 2020

Question 1

a. In this case assume that $n \leq 1$.

We want to show $n \leq 1$.

Since the assumption tells us $n \leq 1$, we can conclude this is true.

b. *Proof.* Let $a = d$ and $b = k$. Assume there exists $d \in \mathbb{N}$ where $(\exists k \in \mathbb{Z}, n = dk) \land d \neq 1 \land d \neq n$. Assume $n > 1$

We need to prove that $n \nmid a$, $n \nmid b$ and $n \mid ab$.

We will do so in parts.

Part 1 (Proving $n \nmid a$):

We need to prove $n \nmid a$.

First, we need to show $n \geq d$.

The fact 2 tells us

$$\forall x, y \in \mathbb{N}, y \geq 1 \land x \mid y \Rightarrow 1 \leq x \leq y \tag{1}$$

and we know from headers that $d \mid n$, $n > 1$, and $n, d \in \mathbb{N}$.

Then, by using these facts, we can write

$$1 \leq d \leq n \tag{2}$$

Second, we need to show $n = d$.

The definition of divisibility tells us for n to divide d, there must be some $k_1 \in \mathbb{Z}$ such that d is equal to $k_1 \cdot n$.

Then, since we know $n \geq d$, by using these facts, we can conclude the definition of divisibility is satisfied only when $k_1 = 1$, or when $n = d$.

Finally, since we know from the header that $n \neq d$, we can conclude $n \nmid d$.

Then, since we know $d = a$ from the header, we can conclude $n \nmid a$.

Part 2 (Proving $n \nmid b$):

We need to prove $n \nmid b$.

First, we need to show $k \mid n$.

The assumption tells us $n = kd$.

Then, it follows from the definition of divisibility that $k \mid d$.

Second, we need to show $k \geq 1$.

The header tells us $n > 1$ $d \geq 0$, and we know from assumption that $n = dk$.

Since the facts tell us $k \leq 0$ results in $n \leq 0$ and this cannot happen, we can conclude $k \geq 1$.

Third, we need to show $n \geq k$ using the fact $k \geq 1$ and $k \mid n$.

The fact 2 tells us

$$\forall x, y \in \mathbb{N}, y \geq 1 \wedge x \mid y \Rightarrow 1 \leq x \leq y \tag{3}$$

Since we know $k \mid n$, $n > 1$ and $k, n \in \mathbb{N}$, we can conclude $k \leq n$.

Fourth, we need to show $n = k$.

The definition of divisibility tells us for n to divide k, there must be some $k_1 \in \mathbb{Z}$ such that k is equal to $k_1 \cdot n$.

Then, using the fact $n \geq k$, we can conclude the definition of divisibility is satisfied only when $k_1 = 1$, or $n = k$.

Finally, since we know from the header that $n \neq k$, we can conclude $n \nmid k$.

Then, it follows from the fact $k = b$, we can conclude $n \nmid b$.

Part 3 (Proving $n \mid ab$):

We need to prove $n \mid ab$.

The fact 1 tells us

$$\forall x \in \mathbb{Z}, x \mid x \tag{4}$$

Since we know $n \in \mathbb{N}$, we can conclude $n \mid n$.

Then, since we know $n = dk$, $d = a$ and $k = b$, we can conclude $n \mid ab$.

□

Pseudoproof:

Let $a = d$ and $b = k$. Assume there exists $d \in \mathbb{N}$ where $(\exists k \in \mathbb{Z}, n = dk) \wedge d \neq 1 \wedge d \neq n$. Assume $n > 1$

We need to prove that $n \nmid a$, $n \nmid b$ and $n \mid ab$.

We will do so in parts.

1. Show $n \nmid a$.

   First, we need to show $n \nmid a$.

   1. Show $n \geq d$.

The fact 2 tells us

$$\forall x, y \in \mathbb{N}, y \geq 1 \wedge x \mid y \Rightarrow 1 \leq x \leq y \tag{5}$$

and we know from headers that $d \mid n$, $n > 1$, and $n, d \in \mathbb{N}$.

Then, by using these facts, we can write

$$1 \leq d \leq n \tag{6}$$

2. Show that for n to divide d, $n = d$.

Now, the definition of divisibility tells us for n to divide d, there must be some $k_1 \in \mathbb{Z}$ such that d is equal to $k_1 \cdot n$.

Then, since we know $n \geq d$, by using these facts, we can conclude the definition of divisibility is satisfied when $k_1 = 1$, or when $n = d$.

3. Conclude $n \nmid a$.

Then, since we know from header that $n \neq d$, we can conclude $n \nmid d$.

Part 1 (Proving $n \nmid a$):

We need to prove $n \nmid a$.

First, we need to show $n \geq d$.

The fact 2 tells us

$$\forall x, y \in \mathbb{N}, y \geq 1 \wedge x \mid y \Rightarrow 1 \leq x \leq y \tag{7}$$

and we know from headers that $d \mid n$, $n > 1$, and $n, d \in \mathbb{N}$.

Then, by using these facts, we can write

$$1 \leq d \leq n \tag{8}$$

Second, we need to show $n = d$.

The definition of divisibility tells us for n to divide d, there must be some $k_1 \in \mathbb{Z}$ such that d is equal to $k_1 \cdot n$.

Then, since we know $n \geq d$, by using these facts, we can conclude the definition of divisibility is satisfied only when $k_1 = 1$, or when $n = d$.

Finally, since we know from the header that $n \neq d$, we can conclude $n \nmid d$.

Then, since we know $d = a$ from the header, we can conclude $n \nmid a$.

2. Show $n \nmid b$
   - Show $k \mid n$

     First, we need to show $k \mid n$.

     - State $n = kd$.

       The assumption tells us $n = kd$.

     - Show $k \mid n$ by using the definition of divisibility

Then, it follows from the definition of divisibility that $k \mid d$.

First, we need to show $k \mid n$.

The assumption tells us $n = kd$.

Then, it follows from the definition of divisibility that $k \mid d$.

- Show $k \geq 1$.

  Second, we need to show $k \geq 1$.

Second, we need to show $k \geq 1$.

The header tells us $n > 1$ $d \geq 0$, and we know from assumption that $n = dk$.

Since the facts tell us $k \leq 0$ results in $n \leq 0$ and this cannot happen, we can conclude $k \geq 1$.

- Show $n \geq k$ using the fact $k \mid n$ and $k \geq 1$.

  Third, we need to show $n \geq k$.

Third, we need to show $n \geq k$.

The fact 2 tells us

$$\forall x, y \in \mathbb{N}, y \geq 1 \wedge x \mid y \Rightarrow 1 \leq x \leq y \tag{9}$$

Since we know $k \mid n$, $n > 1$ and $k, n \in \mathbb{N}$, we can conclude $k \leq n$.

- Show that for n to divide k, $n = k$.

  Fourth, we need to show $n = k$.

Fourth, we need to show $n = k$.

The definition of divisibility tells us for n to divide k, there must be some $k_1 \in \mathbb{Z}$ such that k is equal to $k_1 \cdot n$.

Then, using the fact $n \geq k$, we can conclude the definition of divisibility is satisfied only when $k_1 = 1$, or $n = k$.

- Conclude $n \nmid a$.

Finally, since we know from the header that $n \neq k$, we can conclude $n \nmid k$.

It follows from the fact $k = b$, we can conclude $n \nmid b$.

Part 2 (Proving $n \nmid b$):

We need to show $n \nmid b$.

First, we need to show $k \mid n$.

The assumption tells us $n = kd$.

Then, it follows from the definition of divisibility that $k \mid d$.

Second, we need to show $k \geq 1$.

The header tells us $n > 1$ $d \geq 0$, and we know from assumption that $n = dk$.

Since the facts tell us $k \leq 0$ results in $n \leq 0$ and this cannot happen, we can conclude $k \geq 1$.

Third, we need to show $n \geq k$ using the fact $k \geq 1$ and $k \mid n$.

The fact 2 tells us

$$\forall x, y \in \mathbb{N}, y \geq 1 \land x \mid y \Rightarrow 1 \leq x \leq y \tag{10}$$

Since we know $k \mid n$, $n > 1$ and $k, n \in \mathbb{N}$, we can conclude $k \leq n$.

Fourth, we need to show $n = k$.

The definition of divisibility tells us for n to divide k, there must be some $k_1 \in \mathbb{Z}$ such that k is equal to $k_1 \cdot n$.

Then, using the fact $n \geq k$, we can conclude the definition of divisibility is satisfied only when $k_1 = 1$, or $n = k$.

Finally, since we know from the header that $n \neq k$, we can conclude $n \nmid k$.

Then, it follows from the fact $k = b$, we can conclude $n \nmid b$.

3. Show $n \mid ab$

   We need to show $n \mid ab$.

   - State fact 1

The fact 1 tells us

$$\forall x \in \mathbb{Z}, x \mid x \tag{11}$$

- Show $n \mid n$

Since we know $n \in \mathbb{N}$, we can conclude $n \mid n$.

- Show $n \mid ab$ using the fact $n = dk$ where $a = d$ and $b = k$.

Then, since we know $n = dk$, $d = a$ and $k = b$, we can conclude $n \mid ab$.

Part 3 (Proving $n \mid ab$):

We need to show $n \mid ab$.

The fact 1 tells us

$$\forall x \in \mathbb{Z}, x \mid x \tag{12}$$

Since we know $n \in \mathbb{N}$, we can conclude $n \mid n$.

Then, since we know $n = dk$, $d = a$ and $k = b$, we can conclude $n \mid ab$.

Notes:

- Made some serious errors (i.e. show n = a or n = b) :(.
- How can a proof be organized so it's structurally clear so moe 3 months from now can say I understand this proof? I used first, second and third to show steps involved but I still feel something is missing...
- Can I write a predicate logic for proving $n \nmid b$ or $n \nmid a$? (i.e. ... $\Rightarrow n \nmid b$)?

Question 2

a. *Proof.* Let $m, n \in \mathbb{N}$. Assume $Prime(n)$ and $n \nmid m$.

We need to prove there are some integer numbers r and s such that $rn + sm = 1$.

First, we need to show $gcd(n, m) = 1$.

The fact 3 tells us

$$\forall n, m \in \mathbb{Z},\ Prime(n) \wedge n \nmid m \Rightarrow gcd(n, m) = 1 \tag{1}$$

Because we know from assumption that n is prime and $n \nmid m$, we can write $gcd(n, m) = 1$.

Finally, the fact 6 tells us

$$\forall n, m \in \mathbb{N},\ \exists r, s \in \mathbb{Z},\ rn + sm = gcd(n, m) \tag{2}$$

Since $gcd(n, m) = 1$, we can conclude

$$gcd(n, m) = rn + sm = 1 \tag{3}$$

□

Pseudoproof:

Let $m, n \in \mathbb{N}$. Assume $Prime(n)$ and $n \nmid m$.

We need to prove $\exists r, s \in \mathbb{Z}, rn + sm = 1$.

1. Show $gcd(n, m) = 1$, using fact 3

   First, we need to show $gcd(n, m) = 1$.

   First, we need to show $gcd(n, m) = 1$.

   The fact 3 tells us

   $$\forall n, m \in \mathbb{Z},\ Prime(n) \wedge n \nmid m \Rightarrow gcd(n, m) = 1 \tag{4}$$

   Because we know from assumption that n is prime and $n \nmid m$, we can write $gcd(n, m) = 1$.

2. Show $rn + sm = gcd(n, m) = 1$ using fact 6

Finally, the fact 6 tells us

$$\forall n, m \in \mathbb{N},\ \exists r, s \in \mathbb{Z},\ rn + sm = gcd(n, m) \tag{5}$$

Since $gcd(n, m) = 1$, we can conclude

$$gcd(n, m) = rn + sm = 1 \tag{6}$$

Notes:

- Noticed that professor doesn't put $\exists$ symbols in 'we need to prove that...'.

  Let $n, m \in \mathbb{N}$. Assume that n is prime and that $n - m$. We want to prove there exist $r, s \in \mathbb{Z}$, $rn + sm = 1$.

- 형모야. 오늘도 사랑하는 내 여보 향해 화이팅 :)
- 오늘 캘거리에 구름이 많은데 날씨가 굉장히 밝구나.
- <u>오오오오오</u>!!!!

b. **<u>Contrapositive of Statement:</u>** $\forall n, m \in \mathbb{N}, n \mid m \Rightarrow \neg Prime(n) \vee (\forall r, s \in \mathbb{Z}, rn + sm \neq 1)$

Proof. Let $n, m \in \mathbb{N}$. Assume $n \mid m$, and assume n is prime, i.e $n > 1 \wedge (\forall d \in \mathbb{N}, d \mid n \Rightarrow d = 1 \vee d = n$, where $n \in \mathbb{N})$

We need to prove for every $r, s \in \mathbb{Z}$, $rn + sm \neq 1$.

First, we need to show $gcd(n, m) = n$.

The definition of greatest common divisor tells us

$$\forall n, m \in \mathbb{Z},\ IsCD(n, m, n) \wedge (\forall d_1 \in \mathbb{Z},\ IsCD(n, m, d_1) \Rightarrow d_1 \leq n) \tag{7}$$

Because we know n is a common divisor to both n and m, and n is the highest value that divides n and m, we can conclude $gcd(n, m) = n$.

Second, we need to show for every $r, s \in \mathbb{Z}$, $rn + sm \geq n$.

The fact 5 tells us

$$\forall n, m \in \mathbb{N}, \forall r, s \in \mathbb{Z}, gcd(n, m) \mid (rn + sm) \tag{8}$$

Using the fact $gcd(n, m) = n$, we can write $rn + sm \geq n$.

Finally, because we know $n > 1$ from assumption and $rn + sm \geq n$, we can conclude $rn + sm > 1$, which is $rn + sm \neq 1$. $\square$

Pseudoproof:

Let $n, m \in \mathbb{N}$. Assume $n \mid m$ and assume n is prime, i.e $n > 1 \wedge (\forall d \in \mathbb{N}, d \mid n \Rightarrow d = 1 \vee d = n$, where $n \in \mathbb{N})$

We need to prove for every $r, s \in \mathbb{Z}$, $rn + sm \neq 1$.

1. Show $gcd(n, m) = n$

   First, we need to show $gcd(n, m) = n$.

   First, we need to show $gcd(n, m) = n$.

   The definition of greatest common divisor tells us

   $$\forall n, m \in \mathbb{Z},\ IsCD(n, m, n) \wedge (\forall d_1 \in \mathbb{Z},\ IsCD(n, m, d_1) \Rightarrow d_1 \leq n) \tag{9}$$

   Because we know n is a common divisor to both n and m, and n is the highest value that divides n and m, we can conclude $gcd(n, m) = n$.

2. Show $\forall r, s \in \mathbb{Z}, rn + sm \geq n$.

   Second, we need to show $\forall r, s \in \mathbb{Z},\ rn + sm \geq n$.

Second, we need to show for every $r, s \in \mathbb{Z}$, $rn + sm \geq n$.

The fact 5 tells us

$$\forall n, m \in \mathbb{N}, \forall r, s \in \mathbb{Z}, gcd(n, m) \mid (rn + sm) \tag{10}$$

Using the fact $gcd(n, m) = n$, we can write $rn + sm \geq n$.

3. Conclude $rn + sm \neq 1$ using the fact $n > 1$.

Finally, because we know $n > 1$ from assumption and $rn + sm \geq n$, we can conclude $rn + sm > 1$, or $rn + sm \neq 1$.

Question 3