

# Worksheet 15 Solution

March 26, 2020

## Question 1

a. **Inner Loop Iterations (upper bound):**  $n$

**Inner Loop Step Size:** 1

**Inner Loop Steps Total:**  $n$

**Outer Loop Iterations (upper bound):**  $n$

**Outer Loop Step Size:** 1

**Outer Loop Steps Total:**  $n$

**Steps Total:**  $n \cdot n = n^2$

### Correct Solution:

Since the inner loop starts at  $i+1$  and ends at  $n-1$ , where  $i$  represents the variable in outer loop, the inner loop has  $(n-1-(i+1)+1) = n-i-1$  iterations.

Since each iteration takes 1 step, the total steps taken by inner loop is:

$$(n-i-1) \cdot 1 = (n-i-1) \tag{1}$$

Now, we will evaluate total steps taken by outer loop.

Since the outer loop starts at  $i = 0$ , and ends at  $n - 1$ , the loop runs at most  $n$  iterations.

Since each iteration takes  $(n - i - 1)$  steps, the total steps of outer loop is:

$$\sum_{i=0}^{n-1} (n - i + 1) = \sum_{i=0}^{n-1} [(n - 1) - i] \quad (2)$$

$$= \sum_{i=0}^{n-1} (n - 1) - \sum_{i=0}^{n-1} i \quad (3)$$

$$= n(n - 1) - \frac{n(n - 1)}{2} \quad (4)$$

$$= \frac{n^2 - n}{2} \quad (5)$$

Then, since the last **return** statement takes 1 step, it follows that the total number of steps of this algorithm is at most  $\frac{n^2 - n}{2} + 2$ , or  $\mathcal{O}(n^2)$ .

- b. Consider the input family where none of the values in a list are the same (i.e.  $[1, 2, 3, 4, 5, 6, 7, 8, 9]$ ).

Since all values in the input list are not matching, both the inner and the outer loop will run, giving the loops the total number of steps of  $\frac{n^2 - n}{2}$ .

Since the last **return** statement takes 1 step, the total number of steps of this algorithm is  $\frac{n^2 - n}{2} + 1$ , or  $\Omega(n^2)$ .

### Correct Solution:

Let  $n \in \mathbb{N}$  and  $lst = [1, 2, 3, \dots, n - 1, n - 1]$ .

Since the inner loop will run without interruptions until the end, the inner loop has

$$n - 1 - (i + 1) + 1 = n - i - 1 \quad (1)$$

iterations.

Then, since the inner loop takes 1 step per iteration, the total steps taken by the inner loop is

$$(n - i - 1) \cdot 1 = (n - i - 1) \quad (2)$$

Since the **if condition**  $lst[i] == lst[j]$  and the **return** statement are activated when  $i = n - 2$ , the outer loop will run until  $i = n - 2$ , where  $j$  is the variable of the inner loop and  $i$  is the variable of the outer loop.

Since the outer loop starts at 0 and ends at  $n - 2$ , it has

$$n - 2 + 1 = n - 1 \quad (3)$$

iterations.

Since each iteration in the outer loop takes  $(n - i - 1)$  steps, the outer loop has total cost of

$$\sum_{i=0}^{n-2} (n - i - 1) = \sum_{i=0}^{n-2} (n - 1) + \sum_{i=0}^{n-2} i \quad (4)$$

$$= (n - 1)(n - 1) - \frac{(n - 2)(n - 1)}{2} \quad (5)$$

$$= \frac{(n - 1)n}{2} \quad (6)$$

Since each of the **if condition** and **return** statement has cost of 1, the total cost of algorithm is  $\frac{n(n-1)}{2} + 2$ , or  $\Omega(n^2)$

c. Let  $n \in \mathbb{N}$ , and  $lst = [1, 2, 3, \dots, n - 1, 1]$

Since the inner loop will run from  $j = i + 1$  until the end without interruptions, the loop has

$$(n - 1) - (i + 1) + 1 = n - i - 1 \quad (1)$$

iterations.

Since the inner loop takes 1 step per iteration, the loop takes total of

$$(n - i - 1) \cdot 1 = (n - i - 1) \quad (2)$$

steps.

Now, because we know that the **if condition** and **return** statement will occur at  $i = 0$ , the outer loop has at most 1 iteration.

Because we know that the outer loop terminates at  $i = 0$ , the total cost of inner loop can be simplified to

$$(n - i - 1) = n - 1 \quad (3)$$

Since the outer loop has 1 iteration and takes  $n - 1$  steps, the loop has total cost of  $n - 1$ .

Lastly, since each of the **if condition** and **return** statement has cost of 1, the total cost of the algorithm is

$$n - 1 + 2 = n + 1 \quad (4)$$

steps, or  $\Theta(n)$ .

### Note

- What's the lower/upper bound of this input family? How can I find them?
- $[1, 2, 3, \dots, 1, n - 1]$  returns total cost of algorithm of  $n$ . Does it imply  $[1, 2, 3, \dots, 1, n - 1]$  is in different input family than  $[1, 2, 3, \dots, n - 1, 1]$ ?
- $g \in \mathcal{O}(f) : \exists c, n_o \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_o \Rightarrow g(n) \leq cf(n)$ , where  $f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$
- $g \in \Omega(f) : \exists c, n_o \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_o \Rightarrow g(n) \geq cf(n)$ , where  $f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$
- $g \in \Theta(f) : g \in \mathcal{O}(f) \wedge g \in \Omega(f)$

## Question 2

a. Since  $j = \text{len}(lst) = n$  and  $i = 0$  initially, the initial value of  $r$  is

$$r = j - i \quad (1)$$

$$= n - 0 \quad (2)$$

$$= n \quad (3)$$

b. The loop terminates when  $r \leq 0$ .

c. Let  $k \in \mathbb{N}$ , and  $j, i \in \mathbb{Z}$ . Assume  $j > i$ .

We will prove the statement by separating into two cases, and combining them at the end.

**Case1** ( $lst[mid] < x$ ):

Assume  $lst[mid] < x$ .

Then, it follows from the fact  $i = \lfloor \frac{i+j}{2} \rfloor + 1$  and  $j = j$  that the value of  $r$  at  $k + 1^{th}$  step is

$$r_{k+1} = j - \left( \left\lfloor \frac{i+j}{2} \right\rfloor + 1 \right) \quad (1)$$

Then,

$$r_{k+1} \leq j - \left( \left\lfloor \frac{i+j}{2} \right\rfloor \right) \quad (2)$$

$$\leq \frac{2j}{2} - \left( \left\lfloor \frac{i+j}{2} \right\rfloor \right) \quad (3)$$

$$\leq - \left( \left\lfloor \frac{i+j}{2} \right\rfloor + \frac{(-2j)}{2} \right) \quad (4)$$

$$\leq - \left( \left\lfloor \frac{i+j}{2} \right\rfloor + \frac{(-2j)}{2} \right) \quad (5)$$

$$\leq - \left( \left\lfloor \frac{i+j}{2} + \frac{(-2j)}{2} \right\rfloor \right) \quad (6)$$

by using the fact  $\forall x \in \mathbb{Z}, \forall y \in \mathbb{R}, \lfloor x + y \rfloor = x + \lfloor y \rfloor$ .

Then,

$$-\left(\left\lfloor \frac{i+j}{2} + \frac{(-2j)}{2} \right\rfloor\right) \leq -\left(\left\lfloor \frac{i-j}{2} \right\rfloor\right) \quad (7)$$

$$\leq -\left(\frac{i-j}{2}\right) \quad (8)$$

$$\leq \left(\frac{j-i}{2}\right) \quad (9)$$

$$\leq \frac{1}{2}r_k \quad (10)$$

**Case2** ( $lst[mid] > x$ ):

**Notes:**

- External properties of ceiling and floor

1.  $\forall x \in \mathbb{R}, 0 \leq x - \lfloor x \rfloor < 1$
2.  $\forall x \in \mathbb{R}^{\geq 0}, x \geq 4 \Rightarrow (\lfloor x \rfloor)^2 \geq \frac{1}{2}x^2$
3.  $\forall x \in \mathbb{R}^{\geq 0}, x \geq 4 \Rightarrow \frac{1}{2}x^2 \geq 2x$
4.  $\forall x \in \mathbb{Z}, \forall y \in \mathbb{R}, \lfloor x + y \rfloor = x + \lfloor y \rfloor$