Worksheet 12 Review

March 31, 2020

Question 1

a. $g \in \mathcal{O}(1) : \exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq c$, where $g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

Notes:

- $g \in \mathcal{O}(f) : \exists c, n_o \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq cf(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

b. **Predicate Logic** $\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq c$, where $g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

Proof. Let $n_0 = 1$, $c = 200$ and $g(n) = 100 + \frac{77}{n+1}$. Assume $n \geq n_0$.

We will prove the statement by showing

$$100 + \frac{77}{n+1} \leq c \tag{1}$$

It follows from the fact $n_0 \geq 1$ that we can write

$$
\begin{aligned}
100 + \frac{77}{n+1} &\leq 100 + \frac{77}{1+1} && (2)\\
&\leq 100 + \frac{77}{2} && (3)\\
&\leq 100 + 77 && (4)\\
&\leq 100 + 100 && (5)\\
&\leq 200 && (6)
\end{aligned}
$$

Then,

$$100 + \frac{77}{n+1} \leq c \tag{7}$$

by the fact that $c = 200$. $\square$

Question 2

- **Predicate Logic:** $\forall f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, $(\exists c_0, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \leq c_0 f(n)) \Rightarrow (\exists d_0, m_0 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq m_0 \Rightarrow f(n) \geq dg(n))$

  Proof. Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. Let $c = 2$, $n_0 = 1$ and $n \in \mathbb{N}$. Assume $n \geq m_0$. Let $d = \frac{1}{c}$ and $m_0 = n_0$. Assume $n \geq m_0$.

  We will prove that $d_0 g(n) \leq f(n)$ given $g(n) \leq c_0 f(n))$.

  It follows from the assumption $g(n) \leq f(n)$ that we can write

$$
\begin{aligned}
g(n) &\leq cf(n) && (1)\\
\frac{1}{2}g(n) &\leq f(n) && (2)\\
\frac{1}{2}g(n) &\leq f(n) && (3)
\end{aligned}
$$

Then since $d = \frac{1}{2}$,

$$d \cdot g(n) \leq f(n) \tag{4}$$

$\square$

Question 3

- **Predicate Logic:** $\forall g : \mathbb{N} \to \mathbb{R}^{\geq 0},\ \forall a \in \mathbb{R}^{\geq 0},\ (\exists c, n_0 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \geq c) \Rightarrow (\exists c_1, c_2, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow c_1 g(n) \leq a + g(n) \leq c_2 g(n))$

  Notes:

  - $g \in \Theta(f) :\ g \in \mathcal{O}(f) \wedge g \in \Omega(f)$
    or
    $g \in \Theta(f) :\ \exists c_1, c_2, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n)$, where $f, g :\ \mathbb{N} \to \mathbb{R}^{\geq 0}$
  - $g \in \Omega(f) :\ \exists c, n_o \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \geq cf(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$
  - $g \in \mathcal{O}(f) :\ \exists c, n_o \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \leq cf(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

Question 4