

# CSC373 Worksheet 6 Solution

August 13, 2020

1. Multiply objective function by - 1

Maximize

$$-2x_1 - 7x_2 - x_3$$

Subject to

$$x_1 - x_3 = 7$$

$$3x_1 + x_2 \geq 7$$

$$x_2 \geq 0$$

$$x_3 \leq 0$$

2. Replace non-nonnegative constraints  $x_1$

Maximize

$$-2x'_1 + 2x''_1 - 7x_2 - x_3$$

Subject to

$$x'_1 - x''_1 - x_3 = 7$$

$$3x'_1 - 3x''_1 + x_2 \geq 7$$

$$x'_1, x''_1, x_2 \geq 0$$

$$x_3 \leq 0$$

3. Replace non-nonnegative constraints  $x_3$

Maximize

$$-2x'_1 + 2x''_1 - 7x_2 - x'_3 + x''_3$$

Subject to

$$x'_1 - x''_1 - x'_3 + x''_3 = 7$$

$$3x'_1 - 3x''_1 + x_2 \geq 7$$

$$x'_1, x''_1, x_2, x'_3, x''_3 \geq 0$$

4. Replace equality constraints with  $\geq$  and  $\leq$

Maximize

$$-2x'_1 + 2x''_1 - 7x_2 - x'_3 + x''_3$$

Subject to

$$x'_1 - x''_1 - x'_3 + x''_3 \leq 7$$

$$x'_1 - x''_1 - x'_3 + x''_3 \geq 7$$

$$3x'_1 - 3x''_1 + x_2 \geq 7$$

$$x'_1, x''_1, x_2, x'_3, x''_3 \geq 0$$

5. Correct greater-than-or-equal-to inequality constraints

Maximize

$$-2x'_1 + 2x''_1 - 7x_2 - x'_3 + x''_3$$

Subject to

$$\begin{aligned}
 x'_1 - x''_1 - x'_3 + x''_3 &\leq 7 \\
 -x'_1 + x''_1 + x'_3 - x''_3 &\leq -7 \\
 -3x'_1 + 3x''_1 - x_2 &\leq 7 \\
 x'_1, x''_1, x_2, x'_3, x''_3 &\geq 0
 \end{aligned}$$

**Notes:**

- **Linear Programming**

- Is a method to achieve the best outcome (such as maximum profit or lowest cost) in a mathematical model whose requirements are represented by linear relationships. <sup>[1]</sup>
- Is named to make it sound cool for government funding
  - \* Like dynamic programming
- Applications
  - \* Microeconomics (maximize profits, minimize costs)
  - \* Company management

- **Standard Form**

- Is a form of linear programming
- Are about maximizing, not minimizing <sup>[2]</sup>
- All have a positivity constraint for each variable <sup>[2]</sup>
- All other constraints are all of the form “linear combination of variables  $\leq$  constant”. <sup>[2]</sup>



- **Converting Linear Programming to Standard Form**

- 1) The objective function might be a minimization rather than a maximization
- Negate coefficients of the objective function

multiply by -1

$\begin{array}{ll} \text{minimize} & -2x_1 + 3x_2 \\ \text{subject to} & \end{array}$ $\begin{array}{rcl} x_1 + x_2 & = & 7 \\ x_1 - 2x_2 & \leq & 4 \\ x_1 & \geq & 0 \end{array}$	$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2 \\ \text{subject to} & \end{array}$ $\begin{array}{rcl} x_1 + x_2 & = & 7 \\ x_1 - 2x_2 & \leq & 4 \\ x_1 & \geq & 0 \end{array}$
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- 2) There might be variables without nonnegativity constraints
- Replace each non-nonnegative variable  $x_i$  with  $x'_i$  and  $x''_i$
  - Modify linear program

Replace  $x_i$  with  $x'_i$  and  $x''_i$

$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2 \\ \text{subject to} & \end{array}$ $\begin{array}{rcl} x_1 + x_2 & = & 7 \\ x_1 - 2x_2 & \leq & 4 \\ x_1 & \geq & 0 \end{array}$ <p style="text-align: center; color: orange;">x_2 is not nonnegative :(</p>	$\begin{array}{ll} \text{maximize} & 2x_1 - 3x'_2 + 3x''_2 \\ \text{subject to} & \end{array}$ $\begin{array}{rcl} x_1 + x'_2 - x''_2 & = & 7 \\ x_1 - 2x'_2 + 2x''_2 & \leq & 4 \\ x_1, x'_2, x''_2 & \geq & 0 \end{array}$ <p style="text-align: center; color: orange;">They are now nonnegative :) Yayy!!</p>
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- 3) There might be **equality constraints**, which have an equal sign rather than a less-than-or-equal-to sign
- Replace equality constraint  $f(x_1, x_2, \dots, x_n) = b$  with  $f(x_1, x_2, \dots, x_n) \leq b$  and  $f(x_1, x_2, \dots, x_n) \geq b$

Multiply incorrect constraints by -1

$\begin{array}{ll} \text{maximize} & 2x_1 - 3x'_2 + 3x''_2 \\ \text{subject to} & \end{array}$ $\begin{array}{rcl} x_1 + x'_2 - x''_2 & \leq & 7 \\ x_1 + x'_2 - x''_2 & \geq & 7 \\ x_1 - 2x'_2 + 2x''_2 & \leq & 4 \\ x_1, x'_2, x''_2 & \geq & 0 \end{array}$	$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2 + 3x_3 \\ \text{subject to} & \end{array}$ $\begin{array}{rcl} x_1 + x_2 - x_3 & \leq & 7 \\ -x_1 - x_2 + x_3 & \leq & -7 \\ x_1 - 2x_2 + 2x_3 & \leq & 4 \\ x_1, x_2, x_3 & \geq & 0 \end{array}$
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- 4) There might be **inequality constraints**, but instead of having a less-than-or-equal-to-sign
- Multiply incorrect inequality constraints by -1

Replace = with  $\leq$  and  $\geq$

<p>maximize <math>2x_1 - 3x'_2 + 3x''_2</math></p> <p>subject to</p> $\begin{array}{rclcl} x_1 + x'_2 - x''_2 & = & 7 \\ x_1 - 2x'_2 + 2x''_2 & \leq & 4 \\ x_1, x'_2, x''_2 & \geq & 0 \end{array}$	<p>maximize <math>2x_1 - 3x'_2 + 3x''_2</math></p> <p>subject to</p> $\begin{array}{rclcl} x_1 + x'_2 - x''_2 & \leq & 7 \\ x_1 + x'_2 - x''_2 & \geq & 7 \\ x_1 - 2x'_2 + 2x''_2 & \leq & 4 \\ x_1, x'_2, x''_2 & \geq & 0 \end{array}$
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### References:

- 1) Wikipedia, Linear Programming, [link](#)
  - 2) Instituto de Mathematicas, Standard form for Linear Programs, [link](#)
- 2.

$$\begin{aligned} z &= 2x_1 - 6x_3 \\ x_4 &= 7 - x_1 - x_2 + x_3 \\ x_5 &= -8 + 3x_1 - x_2 \\ x_6 &= -x_1 + 2x_2 + 2x_3 \end{aligned}$$

The basic variables are variables on the lhs (i.e.  $B = 4, 5, 6$ ), and the non-basic variables are the variables on the rhs of the expressions (i.e.  $N = 1, 2, 3$ ).

$\begin{aligned} z &= 2x_1 - 6x_3 \\ x_4 &= 7 - x_1 - x_2 + x_3 \\ x_5 &= -8 + 3x_1 - x_2 \\ x_6 &= -x_1 + 2x_2 + 2x_3 \end{aligned}$
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The basic  
variables

The non-basic  
variables

### Notes:

- Slack Form

- Is a form of linear programming
- Is for efficient solving of linear programming problem using simplex algorithm

Slack variables

The value of objective function

$$\begin{array}{rcl}
 z & = & 2x_1 - 3x_2 + 3x_3 \\
 x_4 & = & 7 - x_1 - x_2 + x_3 \\
 x_5 & = & -7 + x_1 + x_2 - x_3 \\
 x_6 & = & 4 - x_1 + 2x_2 - 2x_3
 \end{array}$$

### • Converting Linear Programs into Slack Form

- 1) Start from the standard form of linear programming
- 2) Shift objective functions to right

maximize  
subject to

$$\begin{array}{rcl}
 2x_1 & - & 3x_2 + 3x_3 \\
 x_1 + x_2 - x_3 & \leq & 7 \\
 -x_1 - x_2 + x_3 & \leq & -7 \\
 x_1 - 2x_2 + 2x_3 & \leq & 4 \\
 x_1, x_2, x_3 & \geq & 0
 \end{array}$$

Introduce slack variables

maximize  
subject to

$$\begin{array}{rcl}
 2x_1 & - & 3x_2 + 3x_3 \\
 x_4 & = & 7 - x_1 - x_2 + x_3 \\
 x_5 & = & -7 + x_1 + x_2 - x_3 \\
 x_6 & = & 4 - x_1 + 2x_2 - 2x_3 \\
 x_1, x_2, x_3, x_4, x_5, x_6 & \geq & 0
 \end{array}$$

Shift objective functions to right

- 3) Introduce slack variable  $x_i$  to lhs and move expressions  $\sum_{j=1}^n a_{ij}x_j$  to rhs

maximize  
subject to

$$\begin{array}{rcl}
 2x_1 & - & 3x_2 + 3x_3 \\
 x_1 + x_2 - x_3 & \leq & 7 \\
 -x_1 - x_2 + x_3 & \leq & -7 \\
 x_1 - 2x_2 + 2x_3 & \leq & 4 \\
 x_1, x_2, x_3 & \geq & 0
 \end{array}$$

Move expressions to rhs

Introduce slack variables

maximize  
subject to

$$\begin{array}{rcl}
 2x_1 & - & 3x_2 + 3x_3 \\
 x_4 & = & 7 - x_1 - x_2 + x_3 \\
 x_5 & = & -7 + x_1 + x_2 - x_3 \\
 x_6 & = & 4 - x_1 + 2x_2 - 2x_3 \\
 x_1, x_2, x_3, x_4, x_5, x_6 & \geq & 0
 \end{array}$$

4) Change inequalities in linear programming to equality

maximize  $2x_1 - 3x_2 + 3x_3$   
 subject to

$$\begin{array}{rclclcl} x_1 & + & x_2 & - & x_3 & \leq & 7 \\ -x_1 & - & x_2 & + & x_3 & \leq & -7 \\ x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \end{array}$$

$x_1, x_2, x_3 \geq 0$

Change inequality sign to equality

Introduce slack variables

maximize  $2x_1 - 3x_2 + 3x_3$   
 subject to

$$\begin{array}{rclclclcl} x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\ x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\ x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \end{array}$$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

5) Use Variable  $z$  to denote objective function

maximize  $2x_1 - 3x_2 + 3x_3$   
 subject to

$$\begin{array}{rclclclcl} x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\ x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\ x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \end{array}$$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Use variable  $z$  to denote objective function

Use variable  $z$  to denote objective function and omit the nonnegativity constraints.

$$\begin{array}{rclclclcl} z & = & 2x_1 & - & 3x_2 & + & 3x_3 \\ x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\ x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\ x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \end{array}$$

6) Omit the nonnegativity constraints

maximize  $2x_1 - 3x_2 + 3x_3$   
 subject to

$$\begin{array}{rclclclcl} x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\ x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\ x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \end{array}$$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

remove nonnegative constraints

Use variable  $z$  to denote objective function and omit the nonnegativity constraints.

$$\begin{array}{rclclclcl} z & = & 2x_1 & - & 3x_2 & + & 3x_3 \\ x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\ x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\ x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \end{array}$$

References:

1) Cambridge University, Linear Programming, [link](#)

3. Multiplying the first expression (under subject to) by 2, and summing the inequality constraints, we have

$$0 \leq -6 \quad (1)$$

which is impossible.



### Notes

- I noticed that infeasible solution has non-overlapping region
- **Infeasible**
  - A Linear Program is infeasible if there is no solution that satisfies all of the constraints

4. Let  $x_1 = 3r$  and  $x_2 = r$  where  $r \geq 0$ . Then the inequality constraints become

$$\begin{aligned} -2(3r) + (r) &= -5r \leq -1 \\ -(3r) - 2(r) &= -5r \leq -2 \\ 3r, r &\geq 0 \end{aligned}$$



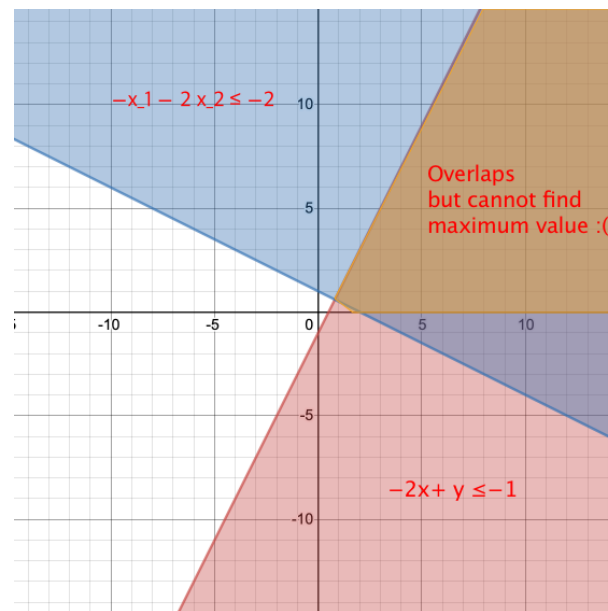
and are valid.

Now, looking at the objective functions, with  $x_1 = 3r$  and  $x_2 = r$ , it becomes

$$3r - r = 2r$$

which increases without bound.

Thus, there is no maximum, and the linear program is unbounded.



### Notes

- I learned that to show an LP is unbounded, I first have to substitute  $x_i$  with a common variable  $r$  (e.g.  $x_1 = 3r$ ,  $x_2 = r$ ), check inequality constraints, and then look at objective functions and see if I can get max/min.
- **Unbounded**
  - A Linear Program is unbounded if it has some feasible solutions but does not have a finite optimal objective value

### References:

- 1) CLRS Solutions, 29.1 Standard and slack forms, [link](#)

### 5. Notes

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