CSC236 Worksheet 6 Solution

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Question 1

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Rough Work:

Assume that for all $k \in \mathbb{N}$, $R(3^k) = k3^k$.

1. Prove that $R \in \mathcal{O}(n \lg n)$

   Define $n^* = 3^{\lceil \log_3 n \rceil}$. Then, we have,

$$\lceil \log_3 n \rceil - 1 < \log_3 n \leq \lceil \log_3 n \rceil \Rightarrow n^*/3 < n \leq n^* \tag{1}$$

   I will also use the assumption (proved last week) that R is non-decreasing.

   Let $d = 6$. Then $d \in \mathbb{R}^+$. Let $B = 3$. Then $B \in \mathbb{N}^+$. Let n be an arbitrary natural number no smaller than B. Then,

$$\begin{aligned}
R(n) &\leq R(n*) && (2)\\
&= k3^k && \text{[By assumption]} \quad (3)\\
&\leq n^* \log_3 n^* && \text{[By replacing } n^* \text{ for } 3^k\text{]} \quad (4)\\
&\leq 3n \log_3 3n && \text{[Since } n^*/3 < n \leq n^* \Rightarrow n^* < 3n < 3n^*\text{]} \quad (5)
\end{aligned}$$

   So $R \in \mathcal{O}(n \lg n)$, since $\log_3 n$ differs from $\lg n$ by a constant factor.

2. Prove $R \in \Omega(n \log n)$

Notes:

- $g \in \Theta(f):\ g \in \mathcal{O}(f) \wedge g \in \Omega(f)$

  or

  $g \in \Theta(f):\ \exists c_1, c_2, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n)$, where $f, g:\ \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$

- $g \in \Omega(f):\ \exists c, n_o \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \geq cf(n)$, where $f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$

- $g \in \mathcal{O}(f):\ \exists c, n_o \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \leq cf(n)$, where $f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$