Worksheet 8 Review

March 27, 2020

Question 1

a. $\forall n \in \mathbb{N}, (0 \leq 1) \land (n \leq 2^n) \Rightarrow (n+1) \leq 2^{n+1}$

Note:

- **Induction:** $\forall n \in \mathbb{N},\ P(0) \land P(n) \Rightarrow P(n+1)$

b. We will prove this statement by induction on n.

Base Case:

Let $n = 0$.

Then,

$$0 \leq 2^0 \tag{1}$$
$$0 \leq 1 \tag{2}$$

Since the above inequality is true, the base case holds.

Inductive Case:

Let $n \in \mathbb{N}$. Assume $P(n)$.

Then,

$$n \leq 2^n \tag{3}$$
$$n + 1 \leq 2^n + 1 \tag{4}$$
$$n + 1 \leq 2^n + 2^n \tag{5}$$
$$n + 1 \leq 2^{n+1} \tag{6}$$

by the fact $2^k + 2^k = 2^{k+1}$.

Then, it follows from proof by induction that the statement $n \leq 2^n$ is true for all n.

Correct Solution:

We will prove this statement by induction on n.

Base Case:

Let $n = 0$.

Then,

$$0 \leq 2^0 \tag{7}$$
$$0 \leq 1 \tag{8}$$

Since the above inequality is true, the base case holds.

Inductive Case:

Let $n \in \mathbb{N}$. Assume $P(n)$.

We want to show $n + 1 \leq 2^{n+1}$.

Then,

$$
\begin{aligned}
n &\le 2^n && (9)\\
n+1 &\le 2^n + 1 && (10)\\
n+1 &\le 2^n + 2^n && (11)\\
n+1 &\le 2^{n+1} && (12)
\end{aligned}
$$

by the fact $2^n + 2^n = 2^{n+1}$.

Then, it follows from proof by induction that the statement $n \le 2^n$ is true for all n.

Notes:

- professor specifically states what we want to show in inductive case part of the proof. I thought it was obvious, and not necessary.
- When are the times 'we want to show x' in proof can be omitted?

Question 2

- We will prove the statement by induction on natural number n.

  Base Case:

  Let $n = 1$.

  Then,

$$
\begin{aligned}
\sum_{j=1}^{1} T_j &= 1\frac{\cdot(1+1)(1+2)}{6} && (1)\\
&= 1 && (2)
\end{aligned}
$$

  Since the data also shows value 1 at $n = 1$, the base case holds.

Inductive Case:

Let $n \in \mathbb{N}$. Assume $\sum\limits_{j=0}^{n} T_j = \frac{n\cdot(n+1)(n+2)}{6}$.

We want to show $\sum\limits_{j=0}^{n+1} T_j = \frac{(n+1)(n+2)(n+3)}{6}$.

It follows from the following table

n	1	2	3	4	5
$T_i = \frac{n\cdot(n+1)}{2}$	1	3	6	10	15
$\sum\limits_{j=1}^{n} T_j$	1	4	10	20	35

that $n+1^{th}$ value of the summation is $\frac{(n+1)(n+2)}{2}$ more than the n^{th} sum.

Then,

$$\begin{aligned}
\sum_{j=0}^{n+1} T_j &= \frac{n\cdot(n+1)(n+2)}{6} + \frac{(n+1)(n+2)}{2} && (3)\\
&= \frac{n\cdot(n+1)(n+2)}{6} + \frac{3(n+1)(n+2)}{6} && (4)\\
&= \frac{(n+1)(n+2)(n+3)}{6} && (5)
\end{aligned}$$

<u>**Correct Solution:**</u>

We will prove the statement by induction on natural number n.

Base Case:

Let $n = 0$.

Then,

$$\sum_{j=0}^{1} T_j = \frac{0 \cdot (0+1)(0+2)}{6} \tag{1}$$
$$= 0 \tag{2}$$

Since

$$\sum_{j=0}^{0} T_j = T_0 \tag{3}$$

and,

$$T_0 = \frac{0 \cdot (0+1)}{2} \tag{4}$$
$$= 0 \tag{5}$$

, the base case holds.

Inductive Case:

Let $n \in \mathbb{N}$. Assume $\sum_{j=0}^{n} T_j = \frac{n \cdot (n+1)(n+2)}{6}$.

We want to show $\sum_{j=0}^{n+1} T_j = \frac{(n+1)(n+2)(n+3)}{6}$.

It follows from the following table

n	1	2	3	4	5
$T_i = \frac{n \cdot (n+1)}{2}$	1	3	6	10	15
$\sum_{j=1}^{n} T_j$	1	4	10	20	35

that $n+1^{th}$ value of the summation is $\frac{(n+1)(n+2)}{2}$ more than the n^{th} sum.

Then,

$$\sum_{j=0}^{n+1} T_j = \frac{n \cdot (n+1)(n+2)}{6} + \frac{(n+1)(n+2)}{2} \tag{6}$$

$$= \frac{n \cdot (n+1)(n+2)}{6} + \frac{3(n+1)(n+2)}{6} \tag{7}$$

$$= \frac{(n+1)(n+2)(n+3)}{6} \tag{8}$$

Notes:

– I wasn't explicit about where the value 1 in data came from.

Question 3

a. Let $x \in \mathbb{R}$.

Correct Solution:

Let $x \in \mathbb{R}$.

We will prove the statement $\forall n \in \mathbb{N}, (1+x)^n \geq 1 + nx$ using induction on n.

Notes:

- Professor separately introduced 'Let $x \in \mathbb{R}$' from the rest of the statement.

- By using 'the standard proof structure to introduce x', does it include the line up to 'we will prove the statement x by induction'?
- **Proof by Induction:** $\forall k \in \mathbb{N},\ P(k) \Rightarrow P(k+1)$

b. **Base Case:**

Let $n = 0$.

Then,

$$1 = (1+x)^0 \geq 1 + (0)x \tag{1}$$
$$\geq 1 \tag{2}$$

Because we know the inequality is true, we can conclude that the base case holds.

Inductive Case:

Let $n \in \mathbb{N}$. Assume $(1+x)^n \geq 1 + nx$.

We want to show $(1+x)^{n+1} \geq 1 + (n+1)x$.

Because we know $(1+x)^{n+1} = (1+x)^n(1+x)$ and $(1+x)^n \geq 1 + nx$, we can write

$$(1+x)^{n+1} = (1+x)^n(1+x) \tag{3}$$
$$\geq (1+nx)(1+x) \tag{4}$$
$$\geq 1 + x + nx + nx^2 \tag{5}$$

Then,

$$1 + x + nx + nx^2 \geq 1 + x + nx \tag{6}$$

by the fact that $nx^2 \geq 0$.

Then,

$$1 + x + nx \geq 1 + x(n+1) \tag{7}$$

Since $(1+x)^{n+1} \geq 1 + x(n+1)$ is true, it follows from proof by induction that the statement $(1+x)^n \geq 1 + xn$ is true for all n.

Correct Solution:

Base Case:

Let $n = 0$.

Since $(1+x)^0 = 1$ and $1 + (0)x = 1$, we know $(1+x)^0 \geq 1 + (0)x$ is true.

Because we know the inequality is true, we can conclude that the base case holds.

Inductive Case:

Let $n \in \mathbb{N}$. Assume $(1+x)^n \geq 1 + nx$.

We want to show $(1+x)^{n+1} \geq 1 + (n+1)x$.

Because we know $(1+x)^{n+1} = (1+x)^n(1+x)$ and $(1+x)^n \geq 1 + nx$, we can write

$$\begin{aligned} (1+x)^{n+1} &= (1+x)^n(1+x) && (8) \\ &\geq (1+nx)(1+x) && (9) \\ &\geq 1 + x + nx + nx^2 && (10) \end{aligned}$$

Then,

$$1 + x + nx + nx^2 \geq 1 + x + nx \tag{11}$$

by the fact that $nx^2 \geq 0$.

Then,

$$1 + x + nx \geq 1 + x(n+1) \tag{12}$$

Since $(1+x)^{n+1} \geq 1 + x(n+1)$ is true, it follows from proof by induction that the statement $(1+x)^n \geq 1 + xn$ is true for all n.

Notes:

- Realized professor evaluates lhs and rhs before validating the inequality for the base case
- Can values can be compared directly from inequality? i.e

$$\begin{aligned} 1 = (1+x)^0 &\geq 1 + (0)x && (13) \\ &\geq 1 && (14) \end{aligned}$$

- 'Assume $P(n)$' is called **inductive hypothesis**
- $\mathbb{N} = \{0, 1, 2, \dots\}$
- $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$

Question 4

a. $\forall n \in \mathbb{N}, n \geq 8 \Rightarrow 30n \leq 2^n$

b. We will prove the statement $\forall n \in \mathbb{N}, n \geq 8 \Rightarrow 30n \leq 2^n$ by using induction on n.

Base Case:

Let $n = 8$.

Since

$$\begin{align} 30n &= 30(8) \tag{1} \\ &= 240 \tag{2} \end{align}$$

,and

$$\begin{align} 2^n &= 2^8 \tag{3} \\ &= 256 \tag{4} \end{align}$$

the inequality $30(8) \leq 2^{(8)}$ is true, and the base case holds.

Inductive Case:

Let $n \in \mathbb{N}$. Assume $30n \leq 2^n$.

We want to show $30(n+1) \leq 2^{(n+1)}$.

It follows from inductive hypothesis that

$$\begin{align} 30(n+1) &= 30n + 30 \tag{5} \\ &\leq 2^n + 30 \tag{6} \end{align}$$

Then,

$$30(n+1) < 2^n + 2^n \tag{7}$$

by the fact $n \geq 8$ and $2^n > 30$.

Then, since $2^n + 2^n = 2^{n+1}$

$$30(n+1) < 2^{(n+1)} \tag{8}$$

Correct Solution:

$\forall n \in \mathbb{N}, n \geq 8 \Rightarrow 30n \leq 2^n$

We will prove the statement $\forall n \in \mathbb{N},\ n \geq 8 \Rightarrow 30n \leq 2^n$ by using induction on n.

Base Case:

Let $n = 8$.

Since

$$\begin{aligned} 30n &= 30(8) && (1) \\ &= 240 && (2) \end{aligned}$$

,and

$$\begin{aligned} 2^n &= 2^8 && (3) \\ &= 256 && (4) \end{aligned}$$

the inequality $30(8) \leq 2^{(8)}$ is true, and the base case holds.

Inductive Case:

Let $n \in \mathbb{N}$. Assume $30n \leq 2^n$.

We want to show $30(n+1) \leq 2^{(n+1)}$.

It follows from inductive hypothesis that

$$\begin{aligned} 30(n+1) &= 30n + 30 && (5) \\ &\leq 2^n + 30 && (6) \end{aligned}$$

Then,

$$30(n+1) \leq 2^n + 2^n \tag{7}$$

by the fact $n \geq 8$ and $2^n \geq 30$.

Then, since $2^n + 2^n = 2^{n+1}$

$$30(n+1) \leq 2^{(n+1)} \tag{8}$$

Notes:

- $n \geq 8 \wedge 2^n \geq 30$ is true.
- $2^{(8)} \geq 30$ is true.
- $2^{(8)} > 30$ is also true.