CSC373 Worksheet 6 Solution

August 13, 2020

1. 1. Multiply objective function by - 1

   Maximize

   $$-2x_1 - 7x_2 - x_3$$

   Subject to

   $$\begin{aligned} x_1 - x_3 &= 7 \\ 3x_1 + x_2 &\geq 7 \\ x_2 &\geq 0 \\ x_3 &\leq 0 \end{aligned}$$

   2. Replace non-nonnegative constraints x_1

   Maximize

   $$-2x_1' + 2x_1'' - 7x_2 - x_3$$

   Subject to

   $$\begin{aligned} x_1' - x_1'' - x_3 &= 7 \\ 3x_1' - 3x_1'' + x_2 &\geq 7 \\ x_1', x_1'', x_2 &\geq 0 \\ x_3 &\leq 0 \end{aligned}$$

   3. Replace non-nonnegative constraints x_3

Maximize

$$-2x_1' + 2x_1'' - 7x_2 - x_3' + x_3''$$

Subject to

$$\begin{aligned} x_1' - x_1'' - x_3' + x_3'' &= 7 \\ 3x_1' - 3x_1'' + x_2 &\geq 7 \\ x_1', x_1'', x_2, x_3', x_3'' &\geq 0 \end{aligned}$$

4. Replace equality constraints with $\geq$ and $\leq$

Maximize

$$-2x_1' + 2x_1'' - 7x_2 - x_3' + x_3''$$

Subject to

$$\begin{aligned} x_1' - x_1'' - x_3' + x_3'' &\leq 7 \\ x_1' - x_1'' - x_3' + x_3'' &\geq 7 \\ 3x_1' - 3x_1'' + x_2 &\geq 7 \\ x_1', x_1'', x_2, x_3', x_3'' &\geq 0 \end{aligned}$$

5. Correct greater-than-or-equal-to inequality constraints

Maximize

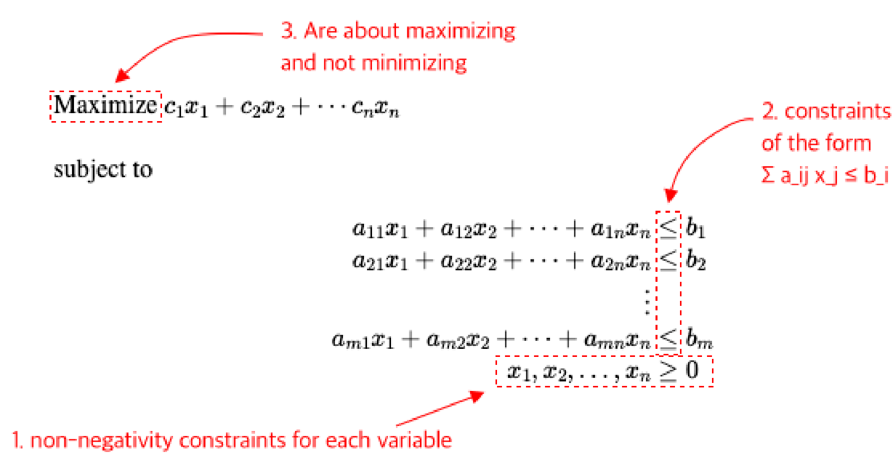
$$-2x_1' + 2x_1'' - 7x_2 - x_3' + x_3''$$

Subject to

$$
\begin{aligned}
x_1' - x_1'' - x_3' + x_3'' &\leq 7 \\
-x_1' + x_1'' + x_3' - x_3'' &\leq -7 \\
-3x_1' + 3x_1'' - x_2 &\leq 7 \\
x_1', x_1'', x_2, x_3', x_3'' &\geq 0
\end{aligned}
$$

Notes:

- **Linear Programming**
  - Is a method to achieve the best outcome (such as maximum profit or lowest cost) in a mathematical model whose requirements are represented by linear relationships. [1]
  - Is named to make it sound cool for government funding
    - Like dynamic programming
  - Applications
    - Microeconomics (maximize profits, minimize costs)
    - Company management
- **Standard Form**
  - Is a form of linear programming
  - Are about maximizing, not minimizing [2]
  - All have a positivity constraint for each variable [2]
  - All other constraints are all of the form "linear combination of variables $\leq$ constant". [2]



- **Converting Linear Programming to Standard Form**

1) The objective function might be a minimization rather than a maximization
   - Negate coefficients of the objective function

multiply by -1

$$\begin{array}{ll} \boxed{\text{minimize} \quad -2x_1 + 3x_2} & \\ \text{subject to} & \\ & x_1 + x_2 = 7 \\ & x_1 - 2x_2 \le 4 \\ & x_1 \ge 0 \end{array} \qquad \begin{array}{ll} \boxed{\text{maximize} \quad 2x_1 - 3x_2} & \\ \text{subject to} & \\ & x_1 + x_2 = 7 \\ & x_1 - 2x_2 \le 4 \\ & x_1 \ge 0 \end{array}$$

2) There might be variables without nonnegativity constraints
   - Replace each non-nonnegative variable x_i with x_i' and x_i''
   - Modify linear program

Replace x_i with x_i' and x_i''

$$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2 \\ \text{subject to} & \\ & x_1 + x_2 = 7 \\ & x_1 - 2x_2 \le 4 \\ & x_1 \ge 0 \end{array} \qquad \begin{array}{ll} \text{maximize} & 2x_1 - 3x_2' + 3x_2'' \\ \text{subject to} & \\ & x_1 + x_2' - x_2'' = 7 \\ & x_1 - 2x_2' + 2x_2'' \le 4 \\ & x_1, x_2', x_2'' \ge 0 \end{array}$$

x_2 is not nonnegative :(

They are now nonnegative :)! Yayy!!

3) There might be **equality constraints**, which have an equal sign rather than a less-than-or-equal-to sign
   - Replace equality constraint $f(x_1, x_2, ..., x_n) = b$ with $f(x_1, x_2, ..., x_n) \le b$ and $f(x_1, x_2, ..., x_n) \ge b$

Multiply incorrect constraints by -1

$$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2' + 3x_2'' \\ \text{subject to} & \\ & x_1 + x_2' - x_2'' \le 7 \\ & x_1 + x_2' - x_2'' \ge 7 \\ & x_1 - 2x_2' + 2x_2'' \le 4 \\ & x_1, x_2', x_2'' \ge 0\ . \end{array} \qquad \begin{array}{ll} \text{maximize} & 2x_1 - 3x_2 + 3x_3 \\ \text{subject to} & \\ & x_1 + x_2 - x_3 \le 7 \\ & -x_1 - x_2 + x_3 \le -7 \\ & x_1 - 2x_2 + 2x_3 \le 4 \\ & x_1, x_2, x_3 \ge 0 \end{array}$$

4) There might be **inequality constraints**, but instead of having a less-than-or-equal-to-sign
   - Multiply incorrect inequality constraints by -1

Replace = with ≤ and ≥

$$\begin{array}{llcrcrcl}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & = & 7 \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& x_1, x_2', x_2'' & & & & & \geq & 0
\end{array}
\qquad
\begin{array}{llcrcrcl}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
& x_1 & + & x_2' & - & x_2'' & \geq & 7 \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& x_1, x_2', x_2'' & & & & & \geq & 0\ .
\end{array}$$

References:

1) Wikipedia, Linear Programming, link
2) Instituto de Mathematicas, Standard form for Linear Programs, link

2.

$$\begin{aligned}
z &= 2x_1 - 6x_3 \\
x_4 &= 7 - x_1 - x_2 + x_3 \\
x_5 &= -8 + 3x_1 - x_2 \\
x_6 &= -x_1 + 2x_2 + 2x_3
\end{aligned}$$

The basic variables are variables on the lhs (i.e $B = 4, 5, 6$), and the non-basic variables are the variables on the rhs of the expressions (i.e. $N = 1, 2, 3$).

$$\begin{aligned}
z &= 2x_1 - 6x_3 \\
x_4 &= 7 - x_1 - x_2 + x_3 \\
x_5 &= -8 + 3x_1 - x_2 \\
x_6 &= -x_1 + 2x_2 + 2x_3
\end{aligned}$$

The non-basic variables

The basic variables

Notes:

- **Slack Form**

- Is a form of linear programming
- Is for efficient solving of liner programming problem using simplex algorithm

The value of objective function

Slack variables

$$\begin{array}{rcrcrcrcr}
\boxed{z} & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 .
\end{array}$$

- **Converting Linear Programs into Slack Form**
  1) Start from the standard form of linear programming
  2) Shift objective functions to right

$$\begin{array}{ll}
\text{maximize} & \boxed{2x_1 - 3x_2 + 3x_3} \\
\text{subject to} & \\
& x_1 + x_2 - x_3 \le 7 \\
& -x_1 - x_2 + x_3 \le -7 \\
& x_1 - 2x_2 + 2x_3 \le 4 \\
& x_1, x_2, x_3 \ge 0
\end{array}$$

Shift objective functions to right

Introduce slack variables

$$\begin{array}{ll}
\text{maximize} & \boxed{\quad 2x_1 - 3x_2 + 3x_3} \\
\text{subject to} & \\
& x_4 = 7 - x_1 - x_2 + x_3 \\
& x_5 = -7 + x_1 + x_2 - x_3 \\
& x_6 = 4 - x_1 + 2x_2 - 2x_3 \\
& x_1, x_2, x_3, x_4, x_5, x_6 \ge 0
\end{array}$$

  3) Introduce slack variable x_i to lhs and move expressions $\sum_{j=1}^{n} a_{ij}x_j$ to rhs

$$\begin{array}{ll}
\text{maximize} & 2x_1 - 3x_2 + 3x_3 \\
\text{subject to} & \\
& \boxed{x_1 + x_2 - x_3} \le 7 \\
& \boxed{-x_1 - x_2 + x_3} \le -7 \\
& \boxed{x_1 - 2x_2 + 2x_3} \le 4 \\
& x_1, x_2, x_3 \ge 0
\end{array}$$

Move expressions to rhs

Introduce slack variables

$$\begin{array}{ll}
\text{maximize} & 2x_1 - 3x_2 + 3x_3 \\
\text{subject to} & \\
& \boxed{x_4} = \boxed{7 - x_1 - x_2 + x_3} \\
& \boxed{x_5} = \boxed{-7 + x_1 + x_2 - x_3} \\
& \boxed{x_6} = \boxed{4 - x_1 + 2x_2 - 2x_3} \\
& x_1, x_2, x_3, x_4, x_5, x_6 \ge 0
\end{array}$$

Introduce slack variables

4) Change inequalities in linear programming to equality

$$\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \le & 7 \\
& -x_1 & - & x_2 & + & x_3 & \le & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \le & 4 \\
& & & x_1, x_2, x_3 & & & \ge & 0
\end{array}$$

Change inequality sign to equality

Introduce slack variables

$$\begin{array}{lrcrcrcrcrcr}
\text{maximize} & & & & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} & & & & & & & & & & \\
& x_4 & = & 7 & - & & x_1 & - & x_2 & + & x_3 \\
& x_5 & = & -7 & + & & x_1 & + & x_2 & - & x_3 \\
& x_6 & = & 4 & - & & x_1 & + & 2x_2 & - & 2x_3 \\
& \multicolumn{6}{c}{x_1, x_2, x_3, x_4, x_5, x_6} & \ge & 0 & &
\end{array}$$

5) Use Variable z to denote objective function

$$\begin{array}{lrcrcrcrcr}
\text{maximize} & & & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} & & & & & & & & & \\
& x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
& x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
& x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
& \multicolumn{5}{c}{x_1, x_2, x_3, x_4, x_5, x_6} & \ge & 0 & &
\end{array}$$

Use variable z to denote objective function

Use variable z to denote objective function and omit the nonnegativity constraints.

$$\begin{array}{rcrcrcrcr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}$$

6) Omit the nonnegativivty constraints

$$\begin{array}{lrcrcrcrcr}
\text{maximize} & & & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} & & & & & & & & & \\
& x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
& x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
& x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
& \multicolumn{5}{c}{x_1, x_2, x_3, x_4, x_5, x_6} & \ge & 0 & &
\end{array}$$

remove nonnegative constraints

Use variable z to denote objective function and omit the nonnegativity constraints.

$$\begin{array}{rcrcrcrcr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}$$

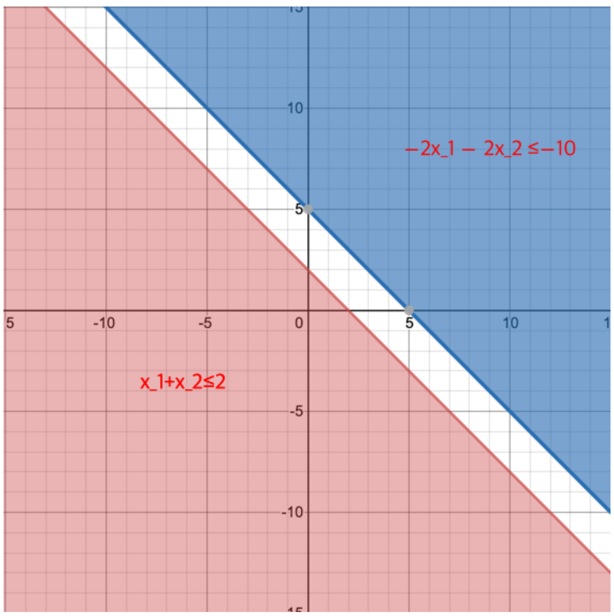
References:

1) Cambridge University, Linear Programming, link

3. Multiplying the first expression (under subject to) by 2, and summing the inequality constraints, we have

$$0 \leq -6 \tag{1}$$

which is impossible.



Notes

- I noticed that infeasible solution has non-overlapping region
- **Infeasible**
  - A Linear Program is infeasible if there is no solution that satisfies all of the constraints

4. Let$x_1 = 3r$ and $x_2 = r$ where $r \geq 0$. Then the inequality constraints become

$$\begin{aligned} -2(3r) + (r) = -5r &\leq -1 \\ -(3r) - 2(r) = -5r &\leq -2 \\ 3r, r &\geq 0 \end{aligned}$$

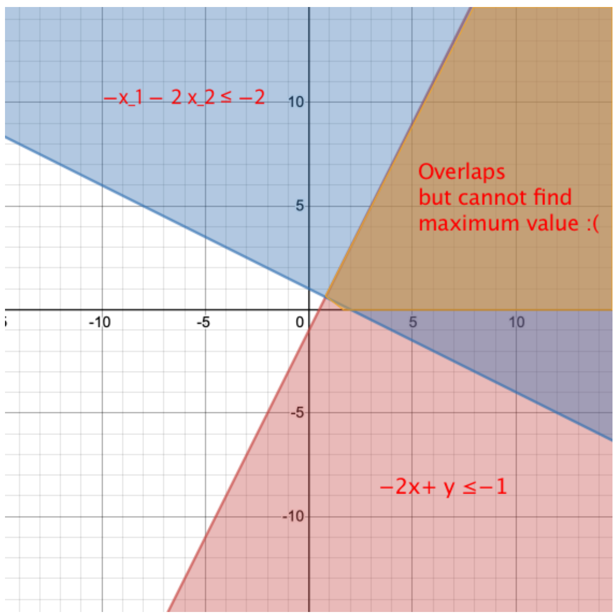
and are valid.

Now, looking at the objective functions, with $x_1 = 3r$ and $x_2 = r$, it becomes

$$3r - r = 2r$$

which increases without bound.

Thus, there is no maximum, and the linear program is unboudned.



Notes

- I learned that to show an LP is unbounded, I first have to subtitute x_i with a common variable r (e.g. $x_1 = 3r$, $x_2 = r$), check inequality constraints, and then look at objective functions and see if I can get max/min.
- **Unbounded**
  - A Linear Program is unbounded if it has some feasible solutions but does not have a finite optimal objective value

References:

1) CLRS Solutions, 29.1 Standard and slack forms, link