Worksheet 11 Review

March 30, 2020

Question 1

a. $\forall a, b \in \mathbb{R}^+,\ a \leq b \Rightarrow (\exists c, n_0 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \leq cf(n))$

> **Correct Solution:**
>
> $\forall a, b \in \mathbb{R}^+,\ a \leq b \Rightarrow (\exists c, n_0 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow n^a \leq cn^b)$

b. *Proof.* Let $a, b \in \mathbb{R}^+$, $n \in \mathbb{N}$, $c = 1$, and $n_0 = 1$. Assume $a \leq b$ and $n > n_0$.

We will prove the statement by showing $n^a \leq cn^b$.

Because we know $n \geq 1$, we can conclude that

$$n^a \leq n^b \tag{1}$$

Then, it follows from the fact $c = 1$ that

$$n^a \leq cn^b \tag{2}$$

□

Attempt 2:

Let $a, b \in \mathbb{R}^+$, $n \in \mathbb{N}$, $c = 1$, and $n_0 = 1$. Assume $a \leq b$ and $n > n_0$.

We will prove the statement by showing $n^a \leq cn^b$.

Because we know $n \geq 1$, we can conclude

$$n^a \geq 1^a \tag{1}$$
$$n^a \geq 1 \tag{2}$$

Then, because we know $\frac{b}{a} \geq 1$, we can conclude

$$n^a \leq [n^a]^{\frac{b}{a}} \tag{3}$$
$$n^a \leq n^b \tag{4}$$

Then, it follows from the fact $c = 1$ that

$$n^a \leq cn^b \tag{5}$$

Notes:

- Professor used $\forall a, b \in \mathbb{R}^+, a \leq b \Rightarrow n^a \leq n^b$ as a fact given $n \geq 1$.
- I don't feel comfortable using the above fact with $a, b \in \mathbb{R}^+$.
- What facts can be used intuitively?
- Given $a \in \mathbb{R}^+$, is $1 \leq n \Rightarrow [1]^a \leq n^a$ also true? Can this be used in proof as a fact?

Question 2

- **Predicate Logic:** $\forall a, b \in \mathbb{R}^+,\ a > 1 \wedge\ b > 1 \Rightarrow (\exists c, n_0 \in \mathbb{R}^+,\ n \geq n_0 \Rightarrow \log_a n \leq \log_b n)$

  Proof. Let $a, b \in \mathbb{R}^+$, $c = 2\log_a b$, and $n_0 = 1$. Assume $a > 1$, $b > 1$, and $n \geq n_0$.

  We will prove that given n_0 and c, $\log_a n \leq c \cdot \log_b n$.

  It follows from the change of base rule $\log_b n = \frac{\log_a n}{\log_a b}$ that

  $$\begin{aligned} \log_a n \cdot 1 &= \log_a n \cdot \frac{\log_a b}{\log_a b} && (1) \\ &= \log_b n \cdot \log_a b && (2) \\ &\leq 2\log_a b \cdot \log_b n && (3) \end{aligned}$$

  Then, since $c = 2 \cdot \log_a b$,

  $$\log_a n \leq c \cdot \log_b n \qquad (4)$$

  □

  Attempt 2:

  Let $a, b \in \mathbb{R}^+$. Assume $a > 1$, $b > 1$. Let $c = 2\log_a b$, and $n_0 = 1$. Assume $n \geq n_0$.

  We will prove that given n_0 and c, $\log_a n \leq c \cdot \log_b n$.

  Change of base rule fact tells us the following

  $$\forall a, b \in \mathbb{R}^+, \forall n \in \mathbb{N}, a \neq 1 \wedge b \neq 1 \Rightarrow \log_b n = \frac{\log_a n}{\log_a b} \qquad (1)$$

Using this fact, we can write

$$\begin{align}
\log_a n \cdot 1 &= \log_a n \cdot \frac{\log_a b}{\log_a b} \tag{1}\\
&= \log_b n \cdot \log_a b \tag{2}\\
&\leq 2 \log_a b \cdot \log_b n \tag{3}
\end{align}$$

Then, since $c = 2 \cdot \log_a b$,

$$\log_a n \leq c \cdot \log_b n \tag{4}$$

Notes:

- Change of base rule

$$\forall a, b, n \in \mathbb{R}^+, a \neq 1 \land b \neq 1 \Rightarrow \log_b n = \frac{\log_a n}{\log_a b} \tag{5}$$

- Noticed professor uses 'Let' and 'Assume' twice to introduce headers for the statement and $\log_a n \in \mathcal{O}(\log_b n)$ separately.

> Let $a, b \in \mathbb{R}^+$. Assume that $a > 1$ and $b > 1$. Let $n_0 = 1$, and let $c = \frac{1}{\log_b a}$. Let $n \in \mathbb{N}$, and assume that $n \geq n_0$. We want to show that $\log_a n \leq c \cdot \log_b n$.

- Noticed if $\log_a n = c \cdot \log_b n$ is true, then the following is also true
  1. $\log_a n \leq c \cdot \log_b n$
  2. $\log_a n \geq c \cdot \log_b n$
- Noticed professor uses the phrase

_____ fact tells us the following $$\{\dots\}$$ Using this rule, we can write $$\{\dots\}$$

to introduce an external fact to a proof.

- $g \in \mathcal{O}(f) :\ \exists c, n_o \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \leq cf(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

Question 3

- **Predicate Logic:** $\forall f, g : \mathbb{N} \to \mathbb{R}^{\geq 0},\ (\exists c_0, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \leq c_0 f(n)) \Rightarrow (\exists d_0, m_0 \in \mathbb{R}^+,\ \forall m \in \mathbb{N},\ m \geq m_0 \Rightarrow (f+g)(m) \leq d_0 f(m))$

Proof. Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, $c_0 = 1, n_0 = 1$. Assume $n \geq n_0$, and $g(n) \leq c_0 f(n)$. Let $d_0 = c_0 + 1$ and $m_0 = n_0$. Assume $m \geq m_0$.

We will prove the statement by starting from the assumption $g(n) \leq c_0 f(n)$, and show that $(f+g)(n) \leq d_0 f(n)$.

It follows from the assumption $g(n) \leq c_0 f(n)$ that we can write

$$g(n) \leq c_0 f(n) \tag{1}$$
$$g(n) + f(n) \leq c_0 f(n) + f(n) \tag{2}$$
$$g(n) + f(n) \leq f(n)(c_0 + 1) \tag{3}$$

Then, since $d_0 = c_0 + 1$,

$$f(n) + g(n) \leq d_0 f(n) \tag{4}$$

The **sum of f and g** fact tells us the following

$$\forall n \in \mathbb{N}, (f+g)(n) = f(n) + g(n) \tag{5}$$

Using this fact, we can write

$$(f+g)(n) \leq d_0 f(n) \tag{6}$$

□

Attempt 2:

Predicate Logic: $\forall f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, $(\exists c_0, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \leq c_0 f(n)) \Rightarrow (\exists d_0, m_0 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq m_0 \Rightarrow (f+g)(n) \leq d_0 f(n))$

Proof. Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, $c_0 = 1, n_0 = 1$. Assume $n \geq n_0$, and $g(n) \leq c_0 f(n)$. Let $d_0 = c_0 + 1$ and $m_0 = n_0$. Assume $m \geq m_0$.

We will prove the statement by starting from the assumption $g(n) \leq c_0 f(n)$, and show that $(f+g)(n) \leq d_0 f(n)$.

It follows from the assumption $g(n) \leq c_0 f(n)$ that we can write

$$g(n) \leq c_0 f(n) \tag{1}$$
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$$g(n) + f(n) \leq f(n)(c_0 + 1) \tag{3}$$

Then, since $d_0 = c_0 + 1$,

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The **sum of** f **and** g fact tells us the following

$$\forall n \in \mathbb{N}, (f+g)(n) = f(n) + g(n) \tag{5}$$

Using this fact, we can write

$$(f+g)(n) \leq d_0 f(n) \tag{6}$$

□

Notes:

- Noticed professor uses the same variable n in predicate logic for consistency