

# Worksheet 9 Review

March 28, 2020

## Question 1

- a. For every set of size 0 has 0 subsets of size 2.
- b. Let  $n = 0$ . Let  $S$  be an arbitrary set. Assume  $S$  has size 0.

Since  $S$  has size 0, empty subsets are the **only** subsets that can be included in  $S$ .

Then, because we know empty subsets have size 0, we can conclude there are 0 subsets of size 2.

It follows from above that the base case holds.

### Attempt #2:

**We want to show every set  $S$  of size 0 has 0 subsets of size 2.**

Since  $S$  has size 0, empty subsets are the **only** subsets that can be included in  $S$ .

Then, because we know an empty subset have size 0, we can conclude there are 0 subsets of size 2.

**Notes:**

- Professor specifically mentions **We want to show every set  $S$  of size  $0$  has  $0$  subsets of size  $2$**
- Professor doesn't include conclusion at the end of proof
- Under which cases conclusion to a proof are included.

c. Now we will prove inductive step.

Let  $k \in \mathbb{N}$ . Assume every set of size  $k$  has  $\frac{k(k-1)}{2}$  subsets of size  $2$ .

We want to show a set of size  $k+1$  has  $\frac{(k+1)k}{2}$  subsets of size  $2$ .

### Part 1: counting subsets of $S$ of size $2$ that contain $s_{k+1}$

It follows from the table below,

k	Sets	Subsets of Size 2 with $s_{k+1}$
0	$\{0, 1\}$	1
1	$\{0, 1, 2\}$	2
2	$\{0, 1, 2, 3\}$	3
2	$\{0, 1, 2, 3, 4\}$	4

that the number of subsets of size  $2$  that contain  $s_{k+1}$  is  $k+1$ .

### Part 2: counting subsets of $S$ of size $2$ that doesn't contain $s_{k+1}$

Because we know the subset of  $S$  that doesn't contain  $s_{k+1}$  is a set  $S$  of size  $k$ , we can conclude using induction hypothesis that there are

$$\frac{k(k-1)}{2} \tag{1}$$

subsets of size  $2$ .

### Part 3: Putting the counts together

Then,

$$\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2} \quad (2)$$

Then, it follows from proof by induction that the statement ' $\forall n \in \mathbb{N}$ , every set of size  $n$  has  $\frac{n(n-1)}{2}$  subsets of size 2' is true for all natural numbers  $n$ .

### Attempt #2:

#### Part 1: counting subsets of $S$ of size 2 that contain $s_{k+1}$

It follows from the table below,

k	Sets	Subsets of Size 2 with $s_{k+1}$
2	$\{0, 1\}$	1
3	$\{0, 1, 2\}$	2
4	$\{0, 1, 2, 3\}$	3
5	$\{0, 1, 2, 3, 4\}$	4

that the number of subsets of size 2 that contain  $s_{k+1}$  is  $k$ .

#### Part 2: counting subsets of $S$ of size 2 that doesn't contain $s_{k+1}$

Because we know the subset of  $S$  that doesn't contain  $s_{k+1}$  is a set  $S$  of size  $k$ , we can conclude using induction hypothesis that there are

$$\frac{k(k-1)}{2} \quad (3)$$

subsets of size 2.

#### Part 3: Putting the counts together

Then,

$$\frac{k(k-1)}{2} + k = \frac{(k+1)k}{2} \quad (4)$$

Then, it follows from proof by induction that the statement ' $\forall k \in \mathbb{N}$ , every set of size  $k$  has  $\frac{k(k-1)}{2}$  subsets of size 2' is true for all natural numbers  $k$ .

**Notes:**

- I forgot that  $k$  represents number of elements in a set.

## Question 2

- **Statement:** For every  $n \in \mathbb{N}$ , every finite set  $S$  of size  $n$ , has

$$\frac{n(n-1)(n-2)}{6} \quad (1)$$

subsets of size 3.

We will prove this statement by using induction on  $n$ .

**Base Case:**

Let  $n = 0$ .

Then, only the empty subsets can be included in  $S$ .

Because an empty subset has size 0, there are 0 subsets of size 3 in  $S$ .

Then, since

$$\frac{0 \cdot (0 - 1)(0 - 2)}{6} = 0 \quad (2)$$

the base case holds.

### Inductive Case:

Let  $k \in \mathbb{N}$ . Assume every finite set  $S$  of size  $k$  has exactly  $\frac{k(k-1)(k-2)}{6}$  subsets of size 3.

We want to show finite set  $S$  of size  $k + 1$  contains  $\frac{(k+1)k(k-1)}{6}$  subsets of size 3.

It follows from the table below

k	Sets	# of Subsets of Size 3	# of Subsets of Size 2
0	$\{\}$	0	0
1	$\{s_0\}$	0	0
2	$\{s_0, s_1\}$	0	1
3	$\{s_0, s_1, s_2\}$	1	3
4	$\{s_0, s_1, s_2, s_3\}$	4	6
5	$\{s_0, s_1, s_2, s_3, s_4\}$	10	10

we can deduce that given a set  $S$  size  $k + 1$ , the number of subsets of size 3 containing  $s_{k+1}$  is the sum of # of subsets of size 3 that doesn't contain  $s_{k+1}$  and # of subsets of size 2 that doesn't contain  $s_{k+1}$ .

Then,

$$\frac{k(k-1)(k-2)}{6} + \frac{k(k-1)}{2} = \frac{k(k-1)(k-2)}{6} + \frac{3k(k-1)}{6} \quad (3)$$

$$= \frac{k(k-1)(k-2+3)}{6} \quad (4)$$

$$= \frac{k(k-1)(k+1)}{6} \quad (5)$$

**Attempt #2:**

**Statement:** For every  $n \in \mathbb{N}$ , every finite set  $S$  of size  $n$ , has

$$\frac{n(n-1)(n-2)}{6} \tag{1}$$

subsets of size 3.

We will prove this statement by using induction on  $n$ .

**Base Case:**

Let  $n = 0$ .

Then, only the empty subsets can be included in  $S$ .

Because an empty subset has size 0, there are 0 subsets of size 3 in  $S$ .

Then, since

$$\frac{0 \cdot (0-1)(0-2)}{6} = 0 \tag{2}$$

the base case holds.

**Inductive Case:**

Let  $k \in \mathbb{N}$ . Assume every finite set  $S$  of size  $k$  has exactly  $\frac{k(k-1)(k-2)}{6}$  subsets of size 3.

We want to show finite set  $S$  of size  $k+1$  contains  $\frac{(k+1)k(k-1)}{6}$  subsets of size 3. We will do that by determining the number of subsets of size 3 including  $s_{k+1}$ , and the number of subsets of size 3 not including  $s_{k+1}$ , and then combining them together.

First, we will show the number of subsets of size 3 including  $s_{k+1}$  is  $\frac{n(n-1)}{2}$ .

Because we know the number of subsets of size 3 (i.e  $\{a_1, a_2, s_{k+1}\}$ ) containing  $s_{k+1}$  depends on the unique combination of first two elements  $a_1$  and  $a_2$ , and because we know

1.  $a_1 \neq a_2$
2.  $a_1, a_2 \in \{s_0, s_1, \dots, s_k\}$

, we can conclude that the number of subsets of size 3 containing  $s_{k+1}$  is exactly the number of subsets of size 2 in a set  $S$  of size  $k$  or

$$\frac{n(n-1)}{2} \tag{3}$$

Second, we will show the number of subsets of size 3 not including  $s_{k+1}$  is  $\frac{n(n-1)(n-2)}{6}$ .

Since the number of subsets of size 3 not containing  $s_{k+1}$  must contain 3 elements from  $\{s_0, \dots, s_k\}$ , this is exactly number of subsets of size 3 in a set  $S$  of size  $k$ , or

$$\frac{n(n-1)(n-2)}{6} \tag{4}$$

by induction hypothesis.

Then,

$$\frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{6} = \frac{3n(n-1)}{6} + \frac{n(n-1)(n-2)}{6} \quad (5)$$

$$= \frac{n(n-1)}{6} \cdot (3+n-2) \quad (6)$$

$$= \frac{n(n-1)(n+1)}{6} \quad (7)$$

### Notes:

- I wonder if table like above can be used in a proof. If not, why can't it be done? If yes, when is the use of table not valid? How should it be constructed that it's valid?
- Can I put a subheader like 'Part 1: counting subsets of  $S$  of size 2 that contain  $s_{k+1}$ ' in a proof? If so, are there anything that I should be aware/be careful of?
- Is it alright to play with examples when I don't know how to proceed? What's the danger to this approach?
- Noticed that professor lays out the big ideas of proof (proof by induction evaluating number of  $s_{k+1}$  and  $s_k$ ) and then fills in the missing detail

## Question 3

a. Subsets that contain 3:

$$\{\{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Subsets that doesn't contain 3:

$$\{\{\}, \{1\}, \{2\}, \{1, 2\}\}$$



b. Let  $n \in \mathbb{N}$ .

We will prove the statement using induction on  $n$ .

**Base Case:**

Let  $n = 0$ .

We will show  $|\mathcal{P}(\{\})|$  and  $2^0$  are equal.

Because we know  $S$  with size 0 is an empty set and  $\mathcal{P}(\{\}) = \{\{\}\}$ , we can conclude

$$\begin{aligned} |\mathcal{P}(\{\})| &= |\{\{\}\}| & (1) \\ &= 1 & (2) \end{aligned}$$

Because we know  $2^{(0)} = 1$ , we can conclude  $|\mathcal{P}(\{\})| = 2^0$

**Inductive Case:**

Let  $n \in \mathbb{N}$ . Assume  $|\mathcal{P}(\{s_0, \dots, s_n\})| = 2^n$ .

We will show  $|\mathcal{P}(\{s_0, \dots, s_n, s_{n+1}\})| = 2^{n+1}$  by finding the number of subsets containing  $s_{n+1}$  and the number of subsets not containing  $s_{n+1}$ , and then combining the result together.

**Part 1 Determining number of subsets containing  $s_{n+1}$ :**

Because we know a subset containing  $s_{n+1}$  can be decomposed into two parts  $S'_n \cup \{s_{n+1}\}$  where  $S'_n$  is a subset with combination of elements in  $\{\phi, s_0, \dots, s_n\}$ , we can conclude that the number of subsets containing  $s_{n+1}$  is exactly  $|\mathcal{P}(\{s_0, \dots, s_n\})|$ , or

$$2^n \tag{3}$$

using induction hypothesis.

**Part 2 Determining number of subsets not containing  $s_{n+1}$ :**

Because we know the subsets don't have  $s_{n+1}$ , we can conclude that the subsets are combination of elements in  $\{\phi, s_0, \dots, s_n\}$ .

Since this is exactly,  $|\mathcal{P}(\{s_0, \dots, s_n\})|$ , using induction hypothesis, we can conclude that the number of subsets not containing  $s_{n+1}$  is

$$2^n \tag{4}$$

**Part 3 Combining together:**

Since  $|\mathcal{P}(s_0, \dots, s_n, s_{n+1})|$  is the sum of the number of subsets containing  $s_{n+1}$  and the number of subsets not containing  $s_{n+1}$ , we have

$$|\mathcal{P}(s_0, \dots, s_n, s_{n+1})| = 2^n + 2^n \tag{5}$$

Then, it follows from the fact  $2^n + 2^n = 2^{n+1}$  that

$$|\mathcal{P}(s_0, \dots, s_n, s_{n+1})| = 2^{n+1} \tag{6}$$

**Notes:**

- How can the phrase 'every set  $S$  of size  $n$ ' be described using predicate logic?
- How we know if we are over doing/over thinking it? How do we know if a proof is not enough or has missing detail or jumping to conclusion?