

Worksheet 12 Solution

March 21, 2020

Question 1

- a. $c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq c$, where $g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$
- b. Let $c = \frac{277}{2}$, $n_0 = 1$, $n \in \mathbb{N}$, $f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$, $g(n) = 100 + \frac{77}{n+1}$, $f(n) = 1$.
Assume $n \geq n_0$

Then,

$$g(n) = 100 + \frac{77}{n+1} \leq 100 + \frac{77}{n+1} \quad (1)$$

$$\leq 100 + \frac{77}{2} \quad (2)$$

$$\leq \frac{277}{2} \quad (3)$$

$$\leq c \cdot 1 \quad (4)$$

$$\leq cf(x) \quad (5)$$

The, it follows from the definition of Big-Oh that the statement $100 + \frac{77}{n+1} \in \mathcal{O}(1)$ is true.

Question 2

- **Expanded Statement:** $f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$, $(\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq cf(n)) \Rightarrow (\exists d, m_0 \in \mathbb{R}^+, \forall m \in \mathbb{N}, m \geq m_0 \Rightarrow dg(n) \leq f(n))$.

Let $f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$, $n_0 = 1$, $c = \frac{1}{d}$, $n \in \mathbb{N}$, $m_0 = 1$. Assume $n \geq n_0$, $g(n) \leq cf(n)$ and $m \geq m_0$.

Then,

$$g(n) \leq cf(n) \tag{1}$$

$$g(n) \leq \frac{1}{d}f(n) \tag{2}$$

$$dg(n) \leq f(n) \tag{3}$$

Then,

$$dg(m) \leq f(m) \tag{4}$$

by changing variable from n to m .

Then, it follows from the definition of Ω that the statement $f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$, $g \in \mathcal{O}(f) \Rightarrow \Omega(g)$ is true.

Question 3

- Let $g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$, $a \in \mathbb{R}^{\geq 0}$, $m \in \mathbb{N}$, $c_2 \gg a$, $c_1 = \frac{1}{c_2}$. Assume $g \in \Omega(1)$, $m \geq m_0$.

Then

$$a + g \leq a + c_2g \tag{1}$$

$$< c_2g \tag{2}$$

and,

$$a + g \geq g \tag{3}$$

$$> c_1g \tag{4}$$

Then, by the definition of Θ , the statement $\forall g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$, and $a \in \mathbb{R}^{\geq 0}$, $g \in \Omega(1) \Rightarrow a + g \in \Theta(g)$ is true.

Question 4