CSC236 Worksheet 2 Solution

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Question 1

- **Statement:** Any full binary tree with at least 1 node has more leaves than internal nodes.

  Rough Work:

  Let n be the total number of nodes in a full binary tree.

  We will prove the statement by complete induction on n.

  1. Base Case $(n = 1)$

Let $n = 1$.

We need to prove the full binary tree with 1 total number of nodes has more leaves than internal nodes.

The definition of leaf tells us that a node that has no children is a leaf.

Since the full binary tree with 1 node has no children, we can conclude the node is a leaf.

Using this fact, we can write the full binary tree has 1 leaf.

Now, the definition of an internal node tells us that a node is an internal node if it is not a leaf.

Since there is 1 leaf and 1 total node in the full binary tree, we can write there is 0 internal node.

So, since there is 1 leaf node and 0 internal node, we can conclude the full binary tree has more leaves than internal nodes.

2. Base Case ($n = 2$)

Let $n = 2$.

We need to prove the full binary tree with 2 total number of nodes has more leaves than internal nodes.

The definition of full binary tree tells us that a binary tree is a full binary tree if every internal node has two children.

Since we know by observation that the tree with 2 total number of nodes has 1 leaf and 1 internal node, using above fact, we can write the tree is not a full binary tree.

Then, by vacuous truth, we can conclude the tree has more leaves than internal nodes.

3. Base Case ($n = 3$)

Let $n = 3$.

We need to prove the full binary tree with 3 total number of nodes has more leaves than internal nodes.

1. Show the three forms a full binary tree
   - State the definition of full binary tree

   > The definition of full binary tree tells us that a binary tree is a full binary tree if every internal node has two children.

   - Show that with 3 as the total number of nodes, the only full binary tree that can be formed is 1 internal node and 2 children.

   > Because we know there are two types of binary trees possible, one is the tree with 2 internal nodes and 1 child and the other is 1 internal node and 2 children, using above fact, we can write the only possible full binary tree with 3 total nodes is 1 internal node and 2 children.

   - Conclude that the children are leaves, and there are more leaves than internal nodes

   > Now, the definition of leaf tells us leaf is a node that has no children.
   >
   > Because we know by observation that the 2 child nodes don't have children, we can write the full binary tree has 2 leaves.
   >
   > So, because we know the full binary tree has 1 internal node and 2 leaves, we can conclude the full binary tree has more leaves than internal node.

Base Case ($n = 3$)**:**

Let $n = 3$.

We need to prove the full binary tree with 3 total number of nodes has more leaves than internal nodes.

The definition of full binary tree tells us that a binary tree is a full binary tree if every internal node has two children.

Because we know there are two types of binary trees possible, one is the tree with 2 internal nodes and 1 child and the other is 1 internal node and 2 children, using above fact, we can write the only possible full binary tree with 3 total nodes is 1 internal node and 2 children.

Now, the definition of leaf tells us leaf is a node that has no children.

Because we know by observation that the 2 child nodes don't have children, we can write the full binary tree has 2 leaves.

So, because we know the full binary tree has 1 internal node and 2 leaves, we can conclude the full binary tree has more leaves than internal node.

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4. Inductive Step

   Let $k \geq 1$ be an arbitrary natural number. Assume that for all natural number i satisfying $1 \leq i \leq k$, any full binary trees with i total number of nodes has more leaves than internal nodes.

   Let T be an arbitrary full binary tree with $k + 1$ nodes. Let T' be the binary tree obtained by removing 2 leaves from the same parent node.

   Let ℓ be the number of leaves of T, and m be the number of internal nodes of T. Similarly, let ℓ' be the number of leaves of T' and m' be the number of internal nodes of T'. We must prove $l > m$.

   – Show $\ell' > m'$

   First, we need to show $\ell' > m'$.

   * State that T'.

   The header tells us that T' is a full binary tree as a result of removing 2 leaves from the parent node of T.

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* Show that T' has $k - 1$ nodes.

> Using this fact, we can calculate T' has
>
> $$k + 1 - 2 = k - 1 \tag{1}$$
>
> nodes.

* Show that $\ell' > m'$, using induction hypothesis.

> Then, because we know $1 \leq k-1 \leq k$, using induction hypothesis, we can write
>
> $$\ell' > m' \tag{2}$$

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> The header tells us that T' is a full binary tree as a result of removing 2 leaves from the parent node of T.
>
> Using this fact, we can calculate T' has
>
> $$k + 1 - 2 = k - 1 \tag{3}$$
>
> nodes.
>
> Then, because we know $1 \leq k - 1 \leq k$, using induction hypothesis, we can write
>
> $$\ell' > m' \tag{4}$$

– Show that $\ell = \ell' + 1$ and $m = m' + 1$ using the fact that when 2 nodes are added to a leaf node of T', the number of leaf nodes increase by 1 and internal nodes increase by 1

Second, we need to show $\ell = \ell' + 1$ and $m = m' + 1$.

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The total number of both the leaf nodes and the internal nodes increase by 1 when 2 nodes are added to the same leaf node in a full binary tree.

Since 2 nodes added to the same leaf node of T' is T, we can write $\ell = \ell' + 1$ and $m = m' + 1$.

– Conclude $\ell > m$.

Finally, putting together, because we know $\ell' > m'$, $\ell = \ell' + 1$ and $m = m' + 1$, we can conclude

$$\ell' + 1 > m' + 1 \tag{5}$$
$$\ell > m \tag{6}$$

Inductive Step:

Let $k \geq 1$ be an arbitrary natural number. Assume that for all natural number i satisfying $1 \leq i \leq k$, any full binary trees with i total number of nodes has more leaves than internal nodes.

Let T be an arbitrary full binary tree with $k + 1$ nodes. Let T' be the binary tree obtained by removing 2 leaves from the same parent node.

Let ℓ be the number of leaves of T, and m be the number of internal nodes of T. Similarly, let ℓ' be the number of leaves of T' and m' be the number of internal nodes of T'. We must prove $l > m$.

First, we need to show $\ell' > m'$.

The header tells us that T' is a full binary tree as a result of removing 2 leaves from the parent node of T.

Using this fact, we can calculate T' has

$$k + 1 - 2 = k - 1 \tag{1}$$

nodes.

Then, because we know $1 \leq k - 1 \leq k$, using induction hypothesis, we can write

$$\ell' > m' \tag{2}$$

Second, we need to show $\ell = \ell' + 1$ and $m = m' + 1$.

The total number of both the leaf nodes and the internal nodes increase by 1 when 2 nodes are added to the same leaf node in a full binary tree.

Since 2 nodes added to the same leaf node of T' is T, we can write $\ell = \ell' + 1$ and $m = m' + 1$.

Finally, putting together, because we know $\ell' > m'$, $\ell = \ell' + 1$ and $m = m' + 1$, we can conclude

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$$\ell > m \tag{4}$$

7

<u>Notes:</u>

– Complete Induction

  * **<u>Statement:</u>** $\forall i \in \mathbb{N}, \forall n \in \mathbb{N}, n < i \Rightarrow A(n) \Rightarrow \forall i \in \mathbb{N}, A(i)$

  * **<u>Statement Alt.:</u>** $\left(\forall n \in \mathbb{N}, \left[\bigwedge_{k=0}^{k=n-1} P(k) \right] \Rightarrow P(n) \right) \Rightarrow \forall n \in \mathbb{N}, P(n)$

  * **Simple Example 1:**

    <u>Statement:</u> $\forall n \in \mathbb{N}, n \geq 0 \Rightarrow 10 \mid (n^5 - n)$

    We will prove the statement by strong induction on n.

    1. Base Case $(n = 0)$

       Let $n = 0$.

       We need to prove $10 \mid (n^5 - n)$ is true when $n = 0$. That is, there exists $k \in \mathbb{Z}$ such that $(n^5 - n) = 10k$.

       Let $k = 0$.

       Starting from the left hand side, using the fact $n = 0$, we can write

       $$(n^5 - n) = 0 \tag{5}$$

       Then, because we know $10k = 0$, we can conclude

       $$(n^5 - n) = 10k \tag{6}$$

    2. Base Case $(n = 1)$

Let $n = 1$.

We need to prove $10 \mid (n^5 - n)$ is true when $n = 1$. That is, there exists $k \in \mathbb{Z}$ such that $(n^5 - n) = 10k$.

Let $k = 0$.

Starting from the left hand side, using the fact $n = 0$, we can write

$$(n^5 - n) = 1 - 1 \tag{7}$$
$$= 0 \tag{8}$$

Then, because we know $10k = 0$, we can conclude

$$(n^5 - n) = 10k \tag{9}$$

3. Inductive Step

Assume $k \geq 1$. Assume that for all natural number i satisfying $0 \leq i \leq k$, $10 \mid (i^5 - i)$. That is, $\exists d \in \mathbb{Z}$, $(i^5 - i) = 10d$.

We need to prove $\exists \tilde{d} \in \mathbb{Z}$ such that $((k+1)^5 - (k+1)) = 10\tilde{d}$.

Let $\tilde{d} = c + (k-1)^4 + 4 \cdot (k-1)^3 + 8 \cdot (k-1)^2 + 8 \cdot (k-1) + 3$.

Starting from $((k+1)^5 - (k+1))$, using binominal theorem, we can write,

$$(k+1)^5 - (k+1) = \left[(k-1) + 2\right]^5 - \left[(k-1) + 2\right] \tag{10}$$

$$= \sum_{b=0}^{5} \binom{5}{b} (k-1)^{5-b} \cdot 2^b \tag{11}$$

$$= (k-1)^5 + 10 \cdot (k-1)^4 + 40 \cdot (k-1)^3 +$$
$$80 \cdot (k-1)^2 + 80 \cdot (k-1) + 32 - \left[(k-1) + 2\right] \tag{12}$$

$$= \left[(k-1)^5 - (k-1)\right] + 10 \cdot (k-1)^4 +$$
$$40 \cdot (k-1)^3 + 80 \cdot (k-1)^2 + 80 \cdot (k-1) + 30 \tag{13}$$

(The reason why $k - 1$ is chosen instead of $k - 2$ and $k - 3$ is because of the last term $2^5 = 32$, i.e $32 - 2 = 30$)

Then, because we know $0 \leq k - 1 \leq k$ and $10 \mid (k-1)^5 - (k-1)$ from the header, we can write $\exists c \in \mathbb{Z}$ such that $(k-1)^5 - (k-1) = 10c$, and

$$(k+1)^5 - (k+1) = 10c + 10 \cdot (k-1)^4 + 40 \cdot (k-1)^3 +$$
$$80 \cdot (k-1)^2 + 80 \cdot (k-1) + 30 \tag{14}$$

$$(k+1)^5 - (k+1) = 10 \cdot \left[c + \cdot(k-1)^4 + 4 \cdot (k-1)^3 +\right.$$
$$\left. 8 \cdot (k-1)^2 + 8 \cdot (k-1) + 3\right] \tag{15}$$

$$\tag{16}$$

Then, because we know $\tilde{d} = c + (k-1)^4 + 4 \cdot (k-1)^3 + 8 \cdot (k-1)^2 + 8 \cdot (k-1) + 3$ from the header, we can conclude

$$(k+1)^5 - (k+1) = 10\tilde{d} \tag{17}$$

Question 2

Question 3