CSC236 Midterm 2 Version 1 Solution

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Question 1

- Let $n, q \in \mathbb{N}$. Let $r \in \{0, 1\}$

  Assume $n > 2$, and $n = 2q + r$.

  I need to find a closed form for $T(2q + r)$, using repeated subtitution.

  Starting from $T(n)$, we have

$$
\begin{aligned}
T(n) &= n + T(n-2) && \text{[By def. since } n > 2\text{]} && (1)\\
T(2q+r) &= 2q + r + T(2q+r-2) && \text{[By replacing } n \text{ for } 2q+r\text{]} && (2)\\
&= 2q + r + T(2(q-1)+r) && && (3)\\
&\vdots && && (4)\\
&= \sum_{i=0}^{i=q-1} (2(q-i)+r) + T(r) && \text{[After } q-1 \text{ repeatitions]} && (5)\\
&= 2\sum_{i=0}^{i=q-1} (q-i) + \sum_{i=0}^{i=q-1} r + T(r) && && (6)\\
&= 2\sum_{i=0}^{i=q-1} (q-i) + \sum_{i=0}^{i=q-1} r && \text{[Since } T(r) = 0\text{]} && (7)\\
&= 2\sum_{i'=1}^{i=q} i' + \sum_{i=0}^{i=q-1} r && && (8)\\
&= 2\sum_{i'=1}^{i'=q} i' + \sum_{i=0}^{i=q-1} r && && (9)\\
&= 2\sum_{i'=1}^{i'=q} i' + \sum_{i=0}^{i=q-1} r && && (10)\\
&= 2(q(q+1))/2 + \sum_{i=0}^{i=q-1} r && \text{[By using } \sum_{i=1}^{i=n} i = (n(n+1))/2\text{]} && (11)\\
&= q(q+1) + rq && && (12)\\
&= q(q+1+r) && && (13)
\end{aligned}
$$

- *Proof.* For convenience, define $H(q) : q(q+r+1) = T(2q+r)$.

  I will use simple induction to prove $\forall q \in \mathbb{N}$, $H(q)$.

  Base Case ($q = 0$):

  Let $q = 0$.

  Then,

$$
\begin{aligned}
q(q+r+1) &= 0 && && (1)\\
&= T(2 \cdot 0 + r) && \text{[By def.]} && (2)\\
&= T(2q+r) && && (3)
\end{aligned}
$$

Thus, $T(2q+r)$ verifies in this step.

Inductive Step:

Let $q \in \mathbb{N}$. Assume $H(q)$.

I need to show $H(q+1)$ follows. That is, $(q+1)\Big[(q+1)+r+1\Big] = T(2(q+1)+r)$.

Starting with $(q+1)\Big[(q+1)+r+1\Big]$, we have

$$\begin{aligned}
(q+1)\Big[(q+1)+r+1\Big] &= (q+1)(q+1)+(q+1)r+(q+1) && (4)\\
&= q^2+2q+1+(qr+r)+(q+1) && (5)\\
&= (q^2+qr+q)+(2q+r+2) && (6)\\
&= q(q+r+1)+(2(q+1)+r) && (7)\\
&= T(2q+r)+2(q+1)+r \qquad \text{[By I.H]} && (8)\\
&= T(2(q+1)+r) \qquad \text{[By def.]} && (9)
\end{aligned}$$

Thus, $H(q+1)$ follows from $H(q)$ in this step.

$\square$

- *Proof.* Define for convenience

$$H(n) : \bigwedge_{i=0}^{n-1} T(n) - T(i) \geq 0 \tag{1}$$

I will use complete induction to prove that $\forall n \in \mathbb{N}, H(n)$.

Inductive Step:

Let $n \in \mathbb{N}$. Assume $\bigwedge_{i=0}^{n-1} H(i)$. I will show $H(n)$ follows.

Base Case($n < 2$):

Assume $n < 2$.

Then, all $T(n)$ and $T(n-1)$ are 0 by definition.

So, $T(n) - T(n-1) \geq 0$.

Thus, $C(n)$ follows in this step.

Case $(n \geq 2)$:

Assume $n \geq 2$.

Then,

$$
\begin{aligned}
T(n) - T(n-1) &= n + T(n-2) \\
&\quad - \Big[(n-1) + T(n-3)\Big] && \text{[By def.]} \quad (2) \\
&= 1 + T(n-2) - T(n-3) && (3) \\
&\geq 1 && \text{[By I.H, since } 0 \leq n-2 < n] \quad (4) \\
&> 0 && (5)
\end{aligned}
$$

Thus, $C(n)$ follows from $H(n)$ in this step. □

Correct Solution:

Proof. Define for convenience

$$H(n) : \bigwedge_{i=1}^{n-1} T(n) - T(i) \geq 0 \tag{1}$$

I will use complete induction to prove that $\forall n \in \mathbb{N}, H(n)$.

Inductive Step:

Let $n \in \mathbb{N}$. Assume $\bigwedge_{i=1}^{n-1} H(i)$. I will show $H(n)$ follows.

Base Case$(n < 2)$:

Assume $n < 2$.

Then, all $T(n)$ and $T(n-1)$ are 0 by definition.

So, $T(n) - T(n-1) \geq 0$.

Thus, $C(n)$ follows in this step.

Case ($n = 2$):

Let $n = 2$.

Then,

$$\begin{aligned}
T(n) - T(n-1) &= (2 + T(1)) - (1 + T(0)) && \text{[By def.]} \quad (2)\\
&= 1 && \text{[Since T(1) = T(0) = 0 by def.]} \quad (3)\\
&> 0 && (4)
\end{aligned}$$

Thus, $C(n)$ follows in this step.

Case ($n > 2$):

Assume $n > 2$.

Then,

$$\begin{aligned}
T(n) - T(n-1) &= n + T(n-2) \\
&\quad - \Big[(n-1) + T(n-3)\Big] && \text{[By def.]} \quad (5)\\
&= 1 + T(n-2) - T(n-3) && (6)\\
&\geq 1 && \text{[By I.H, since } 0 \leq n-3 < n-2 < n\text{]} \quad (7)\\
&> 0 && (8)
\end{aligned}$$

Thus, $C(n)$ follows from $H(n)$ in this step. □