CSC236 Worksheet 6 Solution

Hyungmo Gu

May 9, 2020

Question 1

•

Rough Work:

Assume that for all $k \in \mathbb{N}$, $R(3^k) = k3^k$.

I need to prove $R \in \mathcal{O}(n \lg n)$ and $R \in \Omega(n \lg n)$.

I will do so in parts.

1. Prove that $R \in \mathcal{O}(n \lg n)$.

   Define $n^* = 3^{\lceil \log_3 n \rceil}$. Then, we have,

$$\lceil \log_3 n \rceil - 1 < \log_3 n \leq \lceil \log_3 n \rceil \Rightarrow n^*/3 < n \leq n^* \tag{1}$$

   I will also use the assumption (proved last week) that R is non-decreasing.

   Let $d = 6$. Then $d \in \mathbb{R}^+$. Let $B = 3$. Then $B \in \mathbb{N}^+$. Let n be an arbitrary natural number no smaller than B. Then,

$$
\begin{aligned}
R(n) &\le R(n^*) && \text{[Since } n < n^*\text{, and } R \text{ is non-decreasing]} && (2)\\
&= n^* \log_3 n^* && \text{[By assumption, and replacing } n^* \text{ for } 3^k\text{]} && (3)\\
&\le 3n \log_3 3n && \text{[Since } n^*/3 < n \le n^* \Rightarrow n^* < 3n < 3n^*\text{]} && (4)\\
&\le 3n(\log_3 n + 1) && && (5)\\
&\le 3n(\log_3 n + \log_3 n) && \text{[Since } n \ge 3 \Rightarrow \log_3 n \ge 1\text{]} && (6)\\
&= 6n \log_3 n && && (7)\\
&\le (6n \lg n)/\lg 3 && \text{[By change of basis to lg]} && (8)\\
&< 6n \lg n && && (9)\\
&= dn \lg n && \text{[Since } d = 6\text{]} && (10)
\end{aligned}
$$

So $R \in \mathcal{O}(n \lg n)$, since $\log_3 n$ differs from $\lg n$ by a constant factor.

2. Prove that $R \in \Omega(n \log n)$

Define $n^* = 3^{\lceil \log_3 n \rceil}$. Then, we have,

$$\lceil \log_3 n \rceil - 1 < \log_3 n \le \lceil \log_3 n \rceil \Rightarrow n^*/3 < n \le n^* \tag{11}$$

I will also use the assumption (proved last week) that R is non-decreasing.

Let $d =$ _____. Then $d \in \mathbb{R}^+$. Let $B = 3$. Then $B \in \mathbb{N}^+$. Let n be an arbitrary natural number no smaller than B. Then,

$$
\begin{aligned}
R(n) &\ge R(n^*/3) && \text{[Since } n^*/3 < n\text{, and } R \text{ is non-decreasing]} && (12)\\
&= (n^*/3) \cdot \log_3(n^*/3) && \text{[By assumption, and replacing } n^* \text{ for } 3^k\text{]} && (13)\\
& && && (14)
\end{aligned}
$$

Notes:

- $g \in \Theta(f) :\ g \in \mathcal{O}(f) \wedge g \in \Omega(f)$

  or

  $g \in \Theta(f) :\ \exists c_1, c_2, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \ge n_1 \Rightarrow c_1 g(n) \le f(n) \le c_2 g(n)$, where $f, g :\ \mathbb{N} \to \mathbb{R}^{\ge 0}$
- $g \in \Omega(f) :\ \exists c, n_o \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \ge n_0 \Rightarrow g(n) \ge cf(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\ge 0}$
- $g \in \mathcal{O}(f) :\ \exists c, n_o \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \ge n_0 \Rightarrow g(n) \le cf(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\ge 0}$