

# Worksheet 14 Solution

March 25, 2020

## Question 1

a. **Inner Loop:**  $n$

**Outer Loop:**  $n - 5$

**Theta Expressions:**  $\Theta(n^2)$

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**Correct Solution:**

**Inner Loop:**  $n$

**Outer Loop:**  $n \cdot \left\lceil \frac{n}{5} \right\rceil$

**Theta Expressions:**  $\Theta(n^2)$

b. **Inner Loop:**  $\frac{n}{3} + (n - 2)$

**Outer Loop:**  $n - 4$

**Theta Expressions:**  $\Theta(n^2)$

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**Correct Solution:**

**Inner Loop:**  $\lceil \log_3 n \rceil + \left\lceil \frac{n}{2} \right\rceil$

**Outer Loop:**  $\max(0, n - 4) \cdot \left[ \lceil \log_3 n \rceil + \left\lceil \frac{n}{2} \right\rceil \right]$

**Theta Expressions:**  $\Theta(n^2)$

c. **Inner Loop #2:**  $\sum_{i=1}^n i = \frac{n(n+1)}{2}$

**Inner Loop #1:**  $n \cdot \frac{n(n+1)}{2} = \frac{n^3 + n^2}{2}$

**Outer Loop:**  $\frac{n^3 + n^2}{2} \cdot (n-4) = \frac{n^4 - 3n^3 + 4n^2}{2}$

**Theta Expressions:**  $\Theta(n^4)$

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**Correct Solution:**

**Inner Loop #2:**  $j$

**Inner Loop #1:**  $\sum_{j=1}^n j = \frac{n(n+1)}{2}$

**Outer Loop:**  $\left\lceil \frac{n}{4} \right\rceil \cdot \frac{n(n+1)}{2}$

**Theta Expressions:**  $\Theta(n^3)$

d. **Inner Loop:**  $2^n$

**Outer Loop:**  $\sum_{i=0}^{\frac{n}{2}-1} 2^i = 2^{\frac{n}{2}-1}$

**Theta Expressions:**  $\Theta(2^n)$

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**Correct Solution:**

**Inner Loop:**  $i$

**Outer Loop:**  $\sum_{i=0}^{\log n - 1} 2^i = \frac{1 - 2^{\log n - 1 + 1}}{1 - 2} = 2^{\log n} - 1 = n - 1$

**Theta Expressions:**  $\Theta(n)$

## Question 2

- Inner Loop #2:  $j - i$

Inner Loop #1:  $\sum_{j=i}^{n-1} (j - i)$

Outer Loop:  $\sum_{i=0}^{n-1} \sum_{j=i}^{n-1} (j - i)$

Calculation:

$$\sum_{i=0}^{n-1} \sum_{j=i}^{n-1} (j - i) = \sum_{i=0}^{n-1} \sum_{j'=0}^{n-1-i} (j' + i - i) \quad (1)$$

$$= \sum_{i=0}^{n-1} \sum_{j'=0}^{n-1-i} j' \quad (2)$$

$$= \sum_{i=0}^{n-1} \frac{(n-1-i)(n-i)}{2} \quad (3)$$

$$= \sum_{i=0}^{n-1} \frac{(n-1-i)(n-i)}{2} \quad (4)$$

$$= \frac{1}{2} \left[ \sum_{i=0}^{n-1} n^2 - \sum_{i=0}^{n-1} n - 2n \sum_{i=0}^{n-1} i + \sum_{i=0}^{n-1} i + \sum_{i=0}^{n-1} i^2 \right] \quad (5)$$

$$= \frac{1}{2} \left[ \sum_{i=0}^{n-1} n^2 - \sum_{i=0}^{n-1} n - 2n \sum_{i=0}^{n-1} i + \sum_{i=0}^{n-1} i + \sum_{i=0}^{n-1} i^2 \right] \quad (6)$$

$$= \frac{1}{2} \left[ \frac{(n^2 - n)}{2} + \frac{2n^3 - n^2 - 2n^2 + n}{6} \right] \quad (7)$$

$$= \frac{1}{2} \left[ \frac{3n^2 - 3n}{6} + \frac{2n^3 - 3n^2 + n}{6} \right] \quad (8)$$

$$= \frac{1}{2} \left[ \frac{2n^3 - 2n}{6} \right] \quad (9)$$

$$= \frac{n^3 - n}{6} \quad (10)$$

Theta Expressions:  $\Theta(n^3)$

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**Correct Solution:**

**Inner Loop # 2:**  $j - i + 1$

**Inner Loop # 1:**  $\sum_{j=i}^{n-1} (j - i + 1)$

**Outer Loop:**  $\sum_{i=0}^{n-1} \sum_{j=i}^{n-1} (j - i + 1)$

**Calculation:**

$$\sum_{i=0}^{n-1} \sum_{j=i}^{n-1} (j - i + 1) = \sum_{i=0}^{n-1} \sum_{j'=1}^{n-i} j' \quad (1)$$

by setting  $j' = j - i + 1$ , and also by replacing the inner summation notation from  $\sum_{i=0}^{b-a} f(i + a)$  to  $\sum_{i=a}^b f(i')$

Then,

$$\sum_{i=0}^{n-1} \sum_{j'=1}^{n-i} j' = \sum_{i=0}^{n-1} \frac{(n-i)(1+n-i)}{2} \quad (2)$$

by using the arithmetic sum  $\sum_{i=1}^n a_i = \left(\frac{n}{2}\right) (a_1 + a_n)$ .

Then,

$$= \frac{1}{2} [n + n^2 - 2in - i + i^2] \quad (3)$$

$$= \frac{1}{2} \sum_{i=0}^{n-1} [(n + n^2) - i(2n + 1) + i^2] \quad (4)$$

$$= \frac{1}{2} \left[ \sum_{i=0}^{n-1} (n + n^2) - \sum_{i=0}^{n-1} i(2n + 1) + \sum_{i=0}^{n-1} i^2 \right] \quad (5)$$

$$= \frac{1}{2} \left[ \sum_{i=0}^{n-1} (n + n^2) - \frac{3(2n + 1)n(n - 1)}{6} + \frac{n(n - 1)(2n - 1)}{6} \right] \quad (6)$$

$$= \frac{1}{2} \left[ n(n + n^2) - \frac{3(2n + 1)n(n - 1)}{6} + \frac{n(n - 1)(2n - 1)}{6} \right] \quad (7)$$

$$= \frac{1}{2} \left[ (n^2 + n^3) - \frac{4n(n - 1)(n + 1)}{6} \right] \quad (8)$$

$$= \frac{1}{2} \left[ (n^2 + n^3) - \frac{4n - 4n^3}{6} \right] \quad (9)$$

$$= \frac{1}{2} \left[ n^2 + \frac{6n^3}{6} - \frac{4n - 4n^3}{6} \right] \quad (10)$$

$$= \frac{1}{2} \left[ n^2 + \frac{n^3}{3} + \frac{2n}{3} \right] \quad (11)$$

$$= \frac{n^2}{2} + \frac{n^3}{6} + \frac{n}{3} \quad (12)$$

**Theta Expressions:**  $\Theta(n^3)$

**Note**

- forgot that it starts at 0, has total of  $n + 1$  many iterations.
- must be grouped in terms of variables before expanding

$$\frac{1}{2} \sum_{i=0}^{n-1} [n^2 + n - 2in - i + i^2]$$

$$\text{NO: } \frac{1}{2} \left[ \sum_{i=0}^{n-1} n^2 + \sum_{i=0}^{n-1} n - \sum_{i=0}^{n-1} 2in - \sum_{i=0}^{n-1} i + \sum_{i=0}^{n-1} i^2 \right] \quad (13)$$

$$\text{YES: } \frac{1}{2} \left[ \sum_{i=0}^{n-1} (n^2 + n) + \sum_{i=0}^{n-1} (2n - 1)i + \sum_{i=0}^{n-1} i^2 \right] \quad (14)$$

- replace whole  $(j - i + 1)$  in  $\sum_{j=i}^{n-1} (j - i + 1)$  by setting  $j' = j - i + 1$ ,  
and adding  $-i + 1$  to  $j = i$
- the formula for arithmetic sum starting at  $i = 1$  is

$$\sum_{i=1}^n a_i = \left( \frac{n}{2} \right) (a_1 + a_n) \quad (15)$$