

CSC236 Worksheet 3

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Question 1

- Given the statement to prove

$P(x, y, z)$: There are no positive integers x, y, z such that $x^3 + 3y^3 = 9z^3$

Proof. We will prove $P(x, y, z)$ using proof by contradiction.

Assume $\exists x, y, z \in \mathbb{N}^+, x^3 + 3y^3 = 9z^3$.

First, we need to show there is smallest element $x_0 \in X$ with $y_0, z_0 \in \mathbb{N}^+$ satisfying $x^3 + 3y^3 = 9z^3$, using well-ordering principle.

The header tells us there are elements $x, y, z \in \mathbb{N}^+$, satisfying $x^3 + 3y^3 = 9z^3$.

Then, we can write the set $X = \{x \mid x \in \mathbb{N}^+, \exists y, z \in \mathbb{N}^+, x^3 + 3y^3 = 9z^3\}$ is not empty.

Then, using principle of well-ordering, we can write that there is smallest positive natural number $x_0 \in X$ along with $y_0, z_0 \in \mathbb{N}^+$ satisfying $x^3 + 3y^3 = 9z^3$.

Second, we need to show that $x_1^3 = 9z_1^3 - 3y_1^3$ is satisfied, given $x_0 > x_1$.

We will do so in parts.

Part 1 (Showing $x_0 = 3 \cdot x_1$):

We know that

$$x_0^3 + 3y_0^3 = 9z_0^3 \quad (1)$$

$$x_0^3 = 9z_0^3 - 3y_0^3 \quad (2)$$

Since $3 \mid 9z_0^3 - 3y_0^3$, we can write $3 \mid x_0^3$.

Then, since 3 is a prime number, by using the hint provided in question 3 of assignment 1, we can write there is $x_1 \in \mathbb{Z}$, $x_0 = 3 \cdot x_1$.

Then, because we know $x_0, 3 \in \mathbb{N}^+$, we can conclude $x_1 \in \mathbb{N}^+$.

Part 2 (Showing $y_0 = 3 \cdot y_1$):

We know that

$$x_0^3 + 3y_0^3 = 9z_0^3 \quad (3)$$

$$3y_0^3 = 9z_0^3 - x_0^3 \quad (4)$$

Then, using the fact $x_0 = 3 \cdot x_1$ from part 1, we can calculate

$$3y_0^3 = 9z_0^3 - 3^3x_1^3 \quad (5)$$

$$y_0^3 = 3z_0^3 - 3^2x_1^3 \quad (6)$$

Since $3 \mid 3z_0^3 - 3^2x_1^3$, we can write that $3 \mid y_0^3$.

Then, since 3 is a prime number, by using the hint provided in question 3 of assignment 1, we can write there is $y_1 \in \mathbb{Z}$, $y_0 = 3 \cdot y_1$.

Then, because we know $y_0, 3 \in \mathbb{N}^+$, we can conclude $y_1 \in \mathbb{N}^+$.

Part 3 (Showing $z_0 = 3 \cdot z_1$):

We know that

$$9z_0^3 = x_0^3 + 3y_0^3 \quad (7)$$

Then, using the fact $x_0 = 3 \cdot x_1$ from part 1, and $y_0 = 3 \cdot y_1$ from part 2, we can calculate

$$9z_0^3 = 3^3x_1^3 + 3^4y_1^3 \quad (8)$$

$$z_0^3 = 3x_1^3 + 3^2y_1^3 \quad (9)$$

Since $3 \mid 3x_1^3 + 3^2y_1^3$, we can write that $3 \mid z_0^3$.

Then, since 3 is a prime number, by using the hint provided in question 3 of assignment 1, we can write there is $z_1 \in \mathbb{Z}$, $z_0 = 3 \cdot z_1$.

Then, because we know $z_0, 3 \in \mathbb{N}^+$, we can conclude $z_1 \in \mathbb{N}^+$.

Part 4 (Showing $x_1^3 = 9z_1^3 - 3y_1^3$):

We know that

$$9z_0^3 = x_0^3 + 3y_0^3 \quad (10)$$

Then, using the fact $x_0 = 3 \cdot x_1$ from part 1, $y_0 = 3 \cdot y_1$ from part 2, and $z_0 = 3 \cdot z_1$ we can calculate

$$3^5z_1^3 = 3^3x_1^3 + 3^4y_1^3 \quad (11)$$

$$3^2z_1^3 = x_1^3 + 3y_1^3 \quad (12)$$

$$9z_1^3 = x_1^3 + 3y_1^3 \quad (13)$$

Finally, the part 4 tells us

$$9z_1^3 = x_1^3 + 3y_1^3 \quad (14)$$

where $x_1 < x_0$.

Then, because we know x_0 is the smallest number satisfying $x^3 + 3y^3 = 9z^3$, we can conclude above leads to contradiction.

Then, we can conclude the the assumption is false.

□

Notes:

- **Proof By Contradiction:** $\neg P \Rightarrow \neg Q \wedge Q$ (Assuming we are proving $P \Rightarrow Q$)
- **Principle of Well-Ordering:** Any nonempty subset A of \mathbb{N} contains a minimum element; i.e. for any $A \subseteq \mathbb{N}$ such that $A \neq \emptyset$, there is some $a \in A$ such that for all $a' \in A$, $a \leq a'$.
- examples of well-ordered sets
 1. $\mathbb{N} \cup \{0\}$
 2. $\mathbb{N} \cup \{1, 2\}$
 3. $\{n \in \mathbb{N} : n > 5\}$
- examples of non-well-ordered sets
 1. \mathbb{R} and the open interval $(0, 2)$
 2. \mathbb{Z}
- Learned the P in $\neg P \Rightarrow \neg Q \wedge Q$ is the statement

$P(x, y, z)$: There are no positive integers x, y, z such that $x^3 + 3y^3 = 9z^3$

And learned the Q is the principle of well-ordering on P .

- Learned the goal of contradiction is to show the assumption violates principle of well-ordering. That is, there is $x_1 \in X$ less than x_0 satisfying $x^3 + 3y^3 = 9z^3$
- Noticed professor reduced wordiness of work using short notations.

Then

$$\begin{aligned} x_0^3 + 3y_0^3 = 9z_0^3 &\Rightarrow x_0^3 = 9z_0^3 - 3y_0^3 \Rightarrow 3 \mid x_0^3 \Rightarrow 3 \mid x_0 && \# \text{ by clue for A1 Q3} \\ \text{let } x_1 \in \mathbb{N}^+, 3x_1 = x_0 &\Rightarrow 3^3 x_1^3 = 9z_0^3 - 3y_0^3 \Rightarrow 3^2 x_1^3 = 3z_0^3 - y_0^3 && \# \text{ divide through by 3} \\ &\Rightarrow y_0^3 = 3z_0^3 - 3^2 x_1^3 \Rightarrow 3 \mid y_0^3 \Rightarrow 3 \mid y_0 \\ \text{let } y_1 \in \mathbb{N}^+, 3y_1 = y_0 &\Rightarrow 3^3 y_1^3 = 3z_0^3 - 3^2 x_1^3 \Rightarrow 3^2 y_1^3 = z_0^3 - 3x_1^3 && \# \text{ divide through by 3} \\ &\Rightarrow 3x_1^3 + 3^2 y_1^3 = z_0^3 \Rightarrow 3 \mid z_0^3 \Rightarrow 3 \mid z_0 \\ \text{let } z_1 \in \mathbb{N}^+, 3z_1 = z_0 &\Rightarrow 3x_1^3 + 3^2 y_1^3 = 3^3 z_1^3 \Rightarrow x_1^3 + 3y_1^3 = 9z_1^3 && \# \text{ divide through by 3} \\ &\Rightarrow x_1 \in X \end{aligned}$$

Question 2

Notes:

- **Structural Induction**

- is a proof method used in mathematical logic, computer science, graph theory.
- is a generalization of mathematical induction over natural numbers.
- is a recursion method
- **Example:**

Define ε : The smallest set such that

- * $x, y, z \in \varepsilon$ # variables
- * $e_1, e_2 \in \varepsilon \Rightarrow (e_1 + e_2), (e_1 - e_2), (e_1 \times e_2), (e_1 \div e_2) \in \varepsilon$ # operators

(steps omitted). Prove $P(e) : \mathbf{vr}(e) = \mathbf{op}(e) + 1$ # **vr** means number of variable, **op** means number of operators

to prove above using structural induction:

1. **Verify Base Case(s):** Show that the property is true for the simplest members, x, y, z . That is show $P(x)$, $P(y)$, and $P(z)$.

There are three cases: $e = x$, $e = y$, and $e = z$. In each case $\mathbf{vr}(e) = 1$ and $\mathbf{op}(e) = 0$, so $P(e)$ holds for the basis.

2. **Inductive Step:** Let e_1 and e_2 be arbitrary elements of ε . Assume $H(e_1, e_2) : P(e_1)$ and $P(e_2)$. That is, e_1 and e_2 have the property $\mathbf{vr}(e_1) = \mathbf{op}(e_1) + 1$ and $\mathbf{vr}(e_2) = \mathbf{op}(e_2) + 1$.

We need to show all possible combinations of e_1 and e_2 have the property. That is, $P((e_1 + e_2))$, $P((e_1 - e_2))$, $P((e_1 \times e_2))$, and $P((e_1 \div e_2))$.

There are four cases, depending on how e is constructed from e_1 and e_2 : $e = (e_1 + e_2)$, $e = (e_1 - e_2)$, $e = (e_1 \times e_2)$ and $e = (e_1 \div e_2)$. In each case we have

$$\mathbf{vr}(e) = \mathbf{vr}(e_1) + \mathbf{vr}(e_2) \quad (15)$$

$$\mathbf{op}(e) = \mathbf{op}(e_1) + \mathbf{op}(e_2) + 1 \text{ # } +1 \text{ is from } + \text{ in } e_1 + e_2 \quad (16)$$

Thus,

$$\mathbf{vr}(e) = \mathbf{vr}(e_1) + \mathbf{vr}(e_2) \quad [\text{by (4.1)}] \quad (17)$$

$$= (\mathbf{op}(e_1) + 1) + (\mathbf{op}(e_2) + 1) \quad [\text{by induction hypothesis}] \quad (18)$$

$$= (\mathbf{op}(e_1) + \mathbf{op}(e_2)) + 2 \quad (19)$$

$$= (\mathbf{op}(e) - 1) + 2 \quad [\text{by (4.2)}] \quad (20)$$

$$= \mathbf{op}(e) + 1 \quad (21)$$

Question 3