

CSC236 Worksheet 5 Solution

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Question 1

- a. *Proof.* For convenience, define $H(k) : R(3^k) = 3^k k$. Notice that when $n = 3^k$, $3^k k$ is the same as $n \log_3 n$. I will use simple induction to prove $\forall k \in \mathbb{N}, H(k)$.

Base Case ($k = 0$):

Let $k = 0$.

Then,

$$3^k \cdot k = 3^0 \cdot 0 \tag{1}$$

$$= 0 \tag{2}$$

$$= R(n) \tag{3} \quad \text{[By def.]}$$

Thus, $H(0)$ is verified.

Inductive Step:

Let $k \in \mathbb{N}$. Assume $H(k)$. That is $R(3^k) = 3^k k$.

I will show that $H(k+1)$ follows. That is, $R(3^{k+1}) = (k+1)3^{k+1}$.

The definition of $R(3^{k+1})$ tells us,

$$R(3^{k+1}) = 3^{k+1} + 3R(\lceil 3^{k+1}/3 \rceil) \tag{4}$$

$$= 3^{k+1} + 3R(3^k) \tag{5}$$

Then, using this fact, we can conclude

$$R(3^{k+1}) = 3^{k+1} + 3 \cdot 3^k k \quad [\text{By I.H}] \quad (6)$$

$$= 3^{k+1} + 3^{k+1} k \quad (7)$$

$$= 3^{k+1} (k + 1) \quad (8)$$

□

Correct Solution:

For convenience, define $H(k) : R(3^k) = 3^k k$. Notice that when $n = 3^k$, $3^k k$ is the same as $n \log_3 n$. I will use simple induction to prove $\forall k \in \mathbb{N}, H(k)$.

Base Case ($k = 0$):

Let $k = 0$.

Then,

$$3^k \cdot k = 3^0 \cdot 0 \quad (9)$$

$$= 0 \quad (10)$$

$$= R(n) \quad [\text{By def.}] \quad (11)$$

Thus, $H(0)$ is verified.

Inductive Step:

Let $k \in \mathbb{N}$. Assume $H(k)$. That is $R(3^k) = 3^k k$.

I will show that $H(k + 1)$ follows. That is, $R(3^{k+1}) = (k + 1)3^{k+1}$.

Since $k + 1 > 0$, $3^{k+1} > 1$.

So the definition of $R(3^{k+1})$ tells us,

$$R(3^{k+1}) = 3^{k+1} + 3R(\lceil 3^{k+1}/3 \rceil) \quad (12)$$

$$= 3^{k+1} + 3R(3^k) \quad (13)$$

Then, using this fact, we can conclude

$$R(3^{k+1}) = 3^{k+1} + 3 \cdot 3^k k \quad [\text{By I.H}] \quad (14)$$

$$= 3^{k+1} + 3^{k+1} k \quad (15)$$

$$= 3^{k+1}(k+1) \quad (16)$$

Notes:

- Noticed that professor used the phrase ‘Notice that when $n = 3^k$, $3^k k$ is the same as $n \log_3 n$.’ to express n in terms of 3^k .
- I feel I should review this problem to make sure I understood.

b. **Rough Works:**

Define $P(k)$ as

$$P(k) : \bigwedge_{m=1}^{m=k} R(m) \leq R(k)$$

I will prove $\forall n \in \mathbb{N}^+, P(n)$ using complete induction. I will assume without proof, that if n is a natural number greater than 1, then $n > \lceil n/3 \rceil \geq 1$.

1. Inductive Step

Let $n \in \mathbb{N}$. Assume $n \geq 1$. Assume $H(n) : \bigwedge_{i=1}^{n-1} P(i)$. I will prove that $P(n)$ follows.

2. Base Case ($n = 1$)

Let $n = 1$.

Then, $\bigwedge_{m=1}^{m=1} R(m) = R(1)$.

So, $P(1)$ follows.

3. Base Case ($n = 2$)

Let $n = 2$.

I need to show $P(2)$ holds. That is, $R(1) \leq R(2)$ and $R(2) \leq R(2)$.

I will do so in parts.

Part 1 ($R(1) \leq R(2)$):

In this part, $R(1) = 0$ and $R(2) = 2 + R(\lceil 2/3 \rceil) = 2 + R(1) = 2$.

Since $0 \leq 2$, we can conclude $R(1) \leq R(2)$.

Part 2 ($R(2) \leq R(2)$):

In this part, $R(2) = R(2)$, so we can conclude $R(2) \leq R(2)$.

4. Case ($n > 2$)

- Show that only $R(n-1) \leq R(n)$ needs proving
- Show $R(n)$ holds, starting from $R(n-1)$

$$R(n-1) = (n-1) + R(\lceil (n-1)/3 \rceil) \quad [\text{By def.}] \quad (17)$$

$$\leq n + R(\lceil (n-1)/3 \rceil) \quad (18)$$

$$\leq n + R(\lceil n/3 \rceil) \quad [\text{By I.H, Since } 1 \leq \lceil (n-1)/3 \rceil \leq \lceil n/3 \rceil < n] \quad (19)$$

$$= R(n) \quad (20)$$

Thus, $P(n)$ holds.