CSC236 Term Test 1 Version 2 Review

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Question 1

- *Proof.* Define $P(n) : f(n) = 3^n$.

  I will use complete induction to prove that $\forall n \in \mathbb{N}, P(n)$.

  Base Case ($n = 0$):

  Let $n = 0$.

  Then,

$$
\begin{align}
f(n) &= 1 && \text{[By def.]} \tag{1}\\
&\leq 3^0 \tag{2}\\
&= 3^n \tag{3}
\end{align}
$$

  Thus, $P(n)$ follows in this step.

  Base Case ($n = 1$):

  Let $n = 1$.

  Then,

$$
\begin{align}
f(n) &= 1 && \text{[By def., since } n = 1\text{]} \tag{4}\\
&\leq 3^1 \tag{5}\\
&= 3^n \tag{6}
\end{align}
$$

  Thus, $P(n)$ follows in this step.

Base Case ($n = 2$):

Let $n = 2$.

Then,

$$\begin{aligned} f(n) &= 9 && \text{[By def., since } n = 2\text{]} && (7) \\ &\leq 3^2 && && (8) \\ &= 3^n && && (9) \end{aligned}$$

Thus, $P(n)$ follows in this step.

Base Case ($n = 3$):

Let $n = 3$.

Then,

$$\begin{aligned} f(n) &= f(n-1) + 3f(n-2) + \\ &\quad 9f(n-3) && \text{[By def., since } n = 2\text{]} && (10) \\ &= 9 + 3 \cdot 3 + 9 \cdot 1 && \text{[By def., since } n-1 = 2,\ n-2 = 1,\ n-3 = 0\text{]} && (11) \\ &= 3^2 + 3^2 + 3^2 && && (12) \\ &= 3^3 && && (13) \\ &= 3^n && && (14) \\ &\leq 3^n && && (15) \end{aligned}$$

Thus, $P(n)$ follows in this step.

Case ($n > 3$):

Let $n > 3$.

Then, since $0 \leq n-3 < n-2 < n-1 < n$, $P(n-3)$, $P(n-2)$, $P(n-1)$ holds by induction hypothesis. That is, $P(n-3) \leq 3^{n-3}$, $P(n-2) \leq 3^{n-2}$, $P(n-1) \leq 3^{n-1}$.

Thus,

$$
\begin{aligned}
f(n) &= f(n-1) + 3f(n-2) + 9f(n-3) && \text{[By def., since } n > 2\text{]} && (16)\\
&\leq 3^{n-1} + 3 \cdot 3^{n-2} + 9 \cdot 3^{n-3} && \text{[By header]} && (17)\\
&= 3^{n-1} + 3^{n-1} + 3^{n-1} && && (18)\\
&= 3^n && && (19)
\end{aligned}
$$

So, $P(n)$ follows from $H(n)$ in this step.

$\square$