CSC236 Worksheet 3

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Question 1

- Given the statement to prove

$$P(x, y, z) : \text{There are no positive integers } x, y, z \text{ such that } x^3 + 3y^3 = 9z^3$$

Proof. We will prove $P(x, y, z)$ using proof by contradiction.

Assume $\exists x, y, z \in \mathbb{N}^+$, $x^3 + 3y^3 = 9z^3$.

First, we need to show there is smallest element $x_0 \in X$ with $y_0, z_0 \in \mathbb{N}^+$ satisfying $x^3 + 3y^3 = 9z^3$, using well-ordering principle.

The header tells us there are elements $x, y, z \in \mathbb{N}^+$, satisfying $x^3 + 3y^3 = 9z^3$.

Then, we can write the set $X = \{x \mid x \in \mathbb{N}^+, \exists y, z \in \mathbb{N}^+, x^3 + 3y^3 = 9z^3\}$ is not empty.

Then, using principle of well-ordering, we can write that there is smallest positive natural number $x_0 \in X$ along with $y_0, z_0 \in \mathbb{N}^+$ satisfying $x^3 + 3y^3 = 9z^3$.

Second, we need to show that $x_1^3 = 9z_1^3 - 3y_1^3$ is satisfied, given $x_0 > x_1$.

We will do so in parts.

Part 1 (Showing $x_0 = 3 \cdot x_1$):

We know that

$$x_0^3 + 3y_0^3 = 9z_0^3 \tag{1}$$
$$x_0^3 = 9z_0^3 - 3y_0^3 \tag{2}$$

Since $3 \mid 9z_0^3 - 3y_0^3$, we can write $3 \mid x_0^3$.

Then, since 3 is a prime number, by using the hint provided in question 3 of assignment 1, we can write there is $x_1 \in \mathbb{Z}$, $x_0 = 3 \cdot x_1$.

Then, because we know $x_0, 3 \in \mathbb{N}^+$, we can conclude $x_1 \in \mathbb{N}^+$.

Part 2 (Showing $y_0 = 3 \cdot y_1$):

We know that

$$x_0^3 + 3y_0^3 = 9z_0^3 \tag{3}$$
$$3y_0^3 = 9z_0^3 - x_0^3 \tag{4}$$

Then, using the fact $x_0 = 3 \cdot x_1$ from part 1, we can calculate

$$3y_0^3 = 9z_0^3 - 3^3x_1^3 \tag{5}$$
$$y_0^3 = 3z_0^3 - 3^2x_1^3 \tag{6}$$

Since $3 \mid 3z_0^3 - 3^2x_1^3$, we can write that $3 \mid y_0^3$.

Then, since 3 is a prime number, by using the hint provided in question 3 of assignment 1, we can write there is $y_1 \in \mathbb{Z}$, $y_0 = 3 \cdot y_1$.

Then, because we know $y_0, 3 \in \mathbb{N}^+$, we can conclude $y_1 \in \mathbb{N}^+$.

Part 3 (Showing $z_0 = 3 \cdot z_1$):

We know that

$$9z_0^3 = x_0^3 + 3y_0^3 \tag{7}$$

Then, using the fact $x_0 = 3 \cdot x_1$ from part 1, and $y_0 = 3 \cdot y_1$ from part 2, we can calculate

$$9z_0^3 = 3^3x_1^3 + 3^4y_1^3 \tag{8}$$
$$z_0^3 = 3x_1^3 + 3^2y_1^3 \tag{9}$$

Since $3 \mid 3x_1^3 + 3^2y_1^3$, we can write that $3 \mid z_0^3$.

Then, since 3 is a prime number, by using the hint provided in question 3 of assignment 1, we can write there is $z_1 \in \mathbb{Z}$, $z_0 = 3 \cdot z_1$.

Then, because we know $z_0, 3 \in \mathbb{N}^+$, we can conclude $z_1 \in \mathbb{N}^+$.

Part 4 (Showing $x_1^3 = 9z_1^3 - 3y_1^3$):

We know that

$$9z_0^3 = x_0^3 + 3y_0^3 \tag{10}$$

Then, using the fact $x_0 = 3 \cdot x_1$ from part 1, $y_0 = 3 \cdot y_1$ from part 2, and $z_0 = 3 \cdot z_1$ we can calculate

$$3^5z_1^3 = 3^3x_1^3 + 3^4y_1^3 \tag{11}$$
$$3^2z_1^3 = x_1^3 + 3y_1^3 \tag{12}$$
$$9z_1^3 = x_1^3 + 3y_1^3 \tag{13}$$

Finally, the part 4 tells us

$$9z_1^3 = x_1^3 + 3y_1^3 \tag{14}$$

where $x_1 < x_0$.

Then, because we know x_0 is the smallest number satisfying $x^3 + 3y^3 = 9z^3$, we can conclude above leads to contradiction.

Then, we can conclude the the assumption is false.

$\square$

Notes:

- **Proof By Contradiction:** $\neg P \Rightarrow \neg Q \wedge Q$ (Assuming we are proving $P \Rightarrow Q$)
- **Principle of Well-Ordering:** Any nonempty subset A of $\mathbb{N}$ contains a minimum element; i.e. for any $A \subseteq \mathbb{N}$ such that $A \neq \emptyset$, there is some $a \in A$ such that for all $a' \in A$, $a \leq a'$.
- examples of well-ordered sets
  1. $\mathbb{N} \cup \{0\}$
  2. $\mathbb{N} \cup \{1, 2\}$
  3. $\{n \in \mathbb{N} : n > 5\}$
- examples of non-well-ordered sets
  1. $\mathbb{R}$ and the open interval $(0, 2)$
  2. $\mathbb{Z}$

Question 2

Question 3