Midterm 2 Version 1 Review

July 18, 2020

1. a) 1100100

   b) $-\sum_{i=0}^{n-1} 3^i$

   Notes:

   - Balanced Ternary
     - is a way of representing numbers
     - balanced ternary is in base 3, and has values 1,0 or -1

$$\sum_{i=0}^{n-1} d_i \cdot 3^i \text{ where } d_i \in \{0, 1, -1\} \tag{1}$$

   c)   i. $\underline{f(n) \in \Omega(n)}$

   True (since $n^2 + 10n + 2 \geq cn$)

   ii. $\underline{g(n) \in \Omega(n)}$

   False (Let $c = 100, n_0 = 100$. Then $100 \log_2 n < 100n$)

   iii. $\underline{f(n) \in \mathcal{O}(g(n))}$

   False ($f(n) = n^2 + 10n + 2$ grows faster than $g(n) = 100 \log_2 n$)

   iv. $\underline{f(n) \in \Theta(g(n))}$

   True (Set $c_1 = -1, c_2 = 1, n_1 = 100$. Then $c_1 f(n) \leq g(n) \leq c_2 f(n)$)

   v. $\underline{g(n) \in \Theta(\log_3 n)}$

   True (set $c_1 = -1, c_2 = 1, n_1 = 2$. Then $c_1 g(n) \leq \log_3 n \leq c_2 g(n)$)

    vi. $g(n) \in \Theta(\log_3 n)$

        False (set $c_1 = -1, c_2 = 1, n_1 = 2$. Then $c_1 g(n) \leq \log_3 n \leq c_2 g(n)$)

    vii. $f(n) + g(n) \in \Theta(f(n))$

        True (set $c_1 = -2, c_2 = 2, n_1 = 1$. Then $c_1(f(n) + g(n)) \leq f(n) \leq c_2(f(n) + g(n))$)

Notes:

- $g \in \Omega(f) : \exists c, n_o \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \geq cf(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$
- $g \in \mathcal{O}(f) : \exists c, n_o \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq cf(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$
- $g \in \Theta(f) : g \in \mathcal{O}(f) \wedge g \in \Omega(f)$
  or
  $g \in \Theta(f) : \exists c_1, c_2, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

d) $i = 3^{2^k}$

Since

k	0	1	2
i	3	9	81
	3^1	3^2	3^4

e) $k = \lceil \log_3(\log_2 n) - 1 \rceil$

Since

$$i^2 \geq n \tag{1}$$
$$3^{2^k} \geq n^{1/2} \tag{2}$$
$$2^k \geq \log_3(n^{1/2}) \tag{3}$$
$$2^k \geq (1/2) \log_3(n) \tag{4}$$
$$k \geq \log_2((1/2) \log_3(n)) \tag{5}$$
$$\geq \log_2(\log_3(n)) - 1 \tag{6}$$

which gives $k = \lceil \log_2(\log_3(n)) - 1 \rceil$

f) Let $n \in \mathbb{N}$. Assume $n \geq 3$.

I will prove $5^n + 50 < 6^n$ by induction.

Base Step ($n = 3$):

Let $n = 3$.

Then,

$$5^3 + 50 = 715 < 6^3 = 216 \tag{1}$$

So, the base case holds.

Inductive Step

Let $n \in \mathbb{N}$. Assume $(5^n + 50 < 6^n)$.

I need to show $5^{n+1} + 50 < 6^{n+1}$.

Indeed we have

$$5^{n+1} + 50 = 5^n 5 + 50 \tag{2}$$
$$= 5(5^n + 10) \tag{3}$$
$$< 5(5^n + 50) \tag{4}$$
$$< 56^n \tag{5}$$
$$< 66^n \tag{6}$$
$$< 6^{n=1} \tag{7}$$

2. **Negation(expanded):** $\forall a \in \mathbb{R}, \forall c_1, c_2, n_1 \in \mathbb{R}^+, \exists n \in \mathbb{N}, (n \geq n_1) \wedge (c_1 g(n) > f(n)) \vee (f(n) > c_2 g(n))$

   Proof. Let $a \in \mathbb{R}$.

   I need to show $an + 1 \notin \Theta(n^3)$. That is, $an + 1 \notin \mathcal{O}(n^3) \vee an + 1 \notin \Omega(n^3)$. In other words, $\forall c, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N}, (n \geq n_0) \wedge (an + 1 > c \cdot n^3)$ or $\forall c_1, n_1 \in \mathbb{R}^+, \exists n \in \mathbb{N}, (n \geq n_1) \wedge (an + 1 < c_1 \cdot n^3)$.

   Let $c_1, c_2, n_1 \in \mathbb{R}^+$, and let $n = \left\lceil max\left(n_1, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{2}{c_1}}\right)\right\rceil + 1$.

   Then, we can write

$$n = \left\lceil max\left(n_1, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{2}{c_1}}\right)\right\rceil + 1 > \sqrt{\frac{2a}{c_1}} \tag{1}$$

$$n^2 > \frac{2a}{c_1} \tag{2}$$

$$\frac{c_1 n^3}{2} > an \tag{3}$$

And

$$n = \lceil max\left(n_1, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{2}{c_1}}\right) \rceil + 1 > \sqrt[3]{\frac{2}{c_1}} \tag{4}$$

$$\frac{c_1 n^3}{2} > 1 \tag{5}$$

Thus, we can conclude

$$\frac{c_1 n^3}{2} + \frac{c_1 n^3}{2} > an + 1 \tag{6}$$

$$c_1 \cdot n^3 > an + 1 \tag{7}$$

$$\square$$

Notes:

- $g \in \Omega(f):\ \exists c, n_o \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \geq cf(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$
- $g \in \mathcal{O}(f):\ \exists c, n_o \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \leq cf(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$
- $g \in \Theta(f):\ g \in \mathcal{O}(f) \wedge g \in \Omega(f)$

  or

  $g \in \Theta(f):\ \exists c_1, c_2, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

3. a) I need to evaluate the total number of iterations of loop 2.

   First, I need to evaluate the number of iterations in loop 2 per iteration of loop 1.

   The code tells us that the value of j increases by 3 per iteration k. That is, $j = 3k$.

   Since the inner loop ends when $j \geq i$, the earliest iteration at which the loop ends is $k = \lceil \frac{i}{3} \rceil$.

   The k many iterations in inner loop is triggered per iteration of outerloop.

   Since