Worksheet 11 Review

March 30, 2020

Question 1

a. $\forall a, b \in \mathbb{R}^+,\ a \le b \Rightarrow (\exists c, n_0 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \ge n_0 \Rightarrow g(n) \le cf(n))$

Correct Solution:

$\forall a, b \in \mathbb{R}^+,\ a \le b \Rightarrow (\exists c, n_0 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \ge n_0 \Rightarrow n^a \le cn^b)$

b. *Proof.* Let $a, b \in \mathbb{R}^+$, $n \in \mathbb{N}$, $c = 1$, and $n_0 = 1$. Assume $a \le b$ and $n > n_0$.

We will prove the statement by showing $n^a \le cn^b$.

Because we know $n \ge 1$, we can conclude that

$$n^a \le n^b \tag{1}$$

Then, it follows from the fact $c = 1$ that

$$n^a \le cn^b \tag{2}$$

□

Attempt 2:

Let $a, b \in \mathbb{R}^+$, $n \in \mathbb{N}$, $c = 1$, and $n_0 = 1$. Assume $a \leq b$ and $n > n_0$.

We will prove the statement by showing $n^a \leq cn^b$.

Because we know $n \geq 1$, we can conclude

$$n^a \geq 1^a \tag{1}$$
$$n^a \geq 1 \tag{2}$$

Then, because we know $\frac{b}{a} \geq 1$, we can conclude

$$n^a \leq [n^a]^{\frac{b}{a}} \tag{3}$$
$$n^a \leq n^b \tag{4}$$

Then, it follows from the fact $c = 1$ that

$$n^a \leq cn^b \tag{5}$$

Notes:

- Professor used $\forall a, b \in \mathbb{R}^+, a \leq b \Rightarrow n^a \leq n^b$ as a fact given $n \geq 1$.
- I don't feel comfortable using the above fact with $a, b \in \mathbb{R}^+$.
- What facts can be used intuitively?
- Given $a \in \mathbb{R}^+$, is $1 \leq n \Rightarrow [1]^a \leq n^a$ also true? Can this be used in proof as a fact?

c. **Predicate Logic:** $\forall a, b \in \mathbb{R}^+,\ a > 1 \wedge\ b > 1 \Rightarrow (\exists c, n_0 \in \mathbb{R}^+,\ n \geq n_0 \Rightarrow \log_a n \leq \log_b n)$

Proof. Let $a, b \in \mathbb{R}^+$, $c = 2\log_a b$, and $n_0 = 1$. Assume $a > 1$, $b > 1$, and $n \geq n_0$.

We will prove that given n_0 and c, $\log_a n \leq c \cdot \log_b n$.

It follows from the change of base rule $\log_b n = \frac{\log_a n}{\log_a b}$ that

$$\begin{aligned} \log_a n \cdot 1 &= \log_a n \cdot \frac{\log_a b}{\log_a b} && (1) \\ &= \log_b n \cdot \log_a b && (2) \\ &\leq 2\log_a b \cdot \log_b n && (3) \end{aligned}$$

Then, since $c = 2 \cdot \log_a b$,

$$\log_a n \leq c \cdot \log_b n \tag{4}$$

$\square$

Attempt 2:

Let $a, b \in \mathbb{R}^+$. Assume $a > 1$, $b > 1$. Let $c = 2\log_a b$, and $n_0 = 1$. Assume $n \geq n_0$.

We will prove that given n_0 and c, $\log_a n \leq c \cdot \log_b n$.

Change of base rule fact tells us the following

$$\forall a, b \in \mathbb{R}^+, \forall n \in \mathbb{N}, a \neq 1 \land b \neq 1 \Rightarrow \log_b n = \frac{\log_a n}{\log_a b} \tag{1}$$

Using this fact, we can write

$$\log_a n \cdot 1 = \log_a n \cdot \frac{\log_a b}{\log_a b} \tag{1}$$
$$= \log_b n \cdot \log_a b \tag{2}$$
$$\leq 2 \log_a b \cdot \log_b n \tag{3}$$

Then, since $c = 2 \cdot \log_a b$,

$$\log_a n \leq c \cdot \log_b n \tag{4}$$

Question 2

Question 3