CSC236 Worksheet 3

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Question 1

- Given the statement to prove

$$P(x, y, z) : \text{There are no positive integers } x, y, z \text{ such that } x^3 + 3y^3 = 9z^3$$

Proof. We will prove $P(x, y, z)$ using proof by contradiction.

Assume $\exists x, y, z \in \mathbb{N}^+,\ x^3 + 3y^3 = 9z^3$.

First, we need to show there is smallest element $x_0 \in X$ with $y_0, z_0 \in \mathbb{N}^+$ satisfying $x^3 + 3y^3 = 9z^3$, using well-ordering principle.

The header tells us there are elements $x, y, z \in \mathbb{N}^+$, satisfying $x^3 + 3y^3 = 9z^3$.

Then, we can write the set $X = \{x \mid x \in \mathbb{N}^+, \exists y, z \in \mathbb{N}^+, x^3 + 3y^3 = 9z^3\}$ is not empty.

Then, using principle of well-ordering, we can write that there is smallest positive natural number $x_0 \in X$ along with $y_0, z_0 \in \mathbb{N}^+$ satisfying $x^3 + 3y^3 = 9z^3$.

Second, we need to show that $x_1^3 = 9z_1^3 - 3y_1^3$ is satisfied, given $x_0 > x_1$.

We will do so in parts.

Part 1 (Showing $x_0 = 3 \cdot x_1$):

We know that

$$x_0^3 + 3y_0^3 = 9z_0^3 \tag{1}$$
$$x_0^3 = 9z_0^3 - 3y_0^3 \tag{2}$$

Since $3 \mid 9z_0^3 - 3y_0^3$, we can write $3 \mid x_0^3$.

Then, since 3 is a prime number, by using the hint provided in question 3 of assignment 1, we can write there is $x_1 \in \mathbb{Z}$, $x_0 = 3 \cdot x_1$.

Then, because we know $x_0, 3 \in \mathbb{N}^+$, we can conclude $x_1 \in \mathbb{N}^+$.

Part 2 (Showing $y_0 = 3 \cdot y_1$):

We know that

$$x_0^3 + 3y_0^3 = 9z_0^3 \tag{3}$$
$$3y_0^3 = 9z_0^3 - x_0^3 \tag{4}$$

Then, using the fact $x_0 = 3 \cdot x_1$ from part 1, we can calculate

$$3y_0^3 = 9z_0^3 - 3^3x_1^3 \tag{5}$$
$$y_0^3 = 3z_0^3 - 3^2x_1^3 \tag{6}$$

Since $3 \mid 3z_0^3 - 3^2x_1^3$, we can write that $3 \mid y_0^3$.

Then, since 3 is a prime number, by using the hint provided in question 3 of assignment 1, we can write there is $y_1 \in \mathbb{Z}$, $y_0 = 3 \cdot y_1$.

Then, because we know $y_0, 3 \in \mathbb{N}^+$, we can conclude $y_1 \in \mathbb{N}^+$.

Part 3 (Showing $z_0 = 3 \cdot z_1$):

We know that

$$9z_0^3 = x_0^3 + 3y_0^3 \tag{7}$$

Then, using the fact $x_0 = 3 \cdot x_1$ from part 1, and $y_0 = 3 \cdot y_1$ from part 2, we can calculate

$$9z_0^3 = 3^3 x_1^3 + 3^4 y_1^3 \tag{8}$$
$$z_0^3 = 3x_1^3 + 3^2 y_1^3 \tag{9}$$

Since $3 \mid 3x_1^3 + 3^2 y_1^3$, we can write that $3 \mid z_0^3$.

Then, since 3 is a prime number, by using the hint provided in question 3 of assignment 1, we can write there is $z_1 \in \mathbb{Z}$, $z_0 = 3 \cdot z_1$.

Then, because we know $z_0, 3 \in \mathbb{N}^+$, we can conclude $z_1 \in \mathbb{N}^+$.

Part 4 (Showing $x_1^3 = 9z_1^3 - 3y_1^3$):

We know that

$$9z_0^3 = x_0^3 + 3y_0^3 \tag{10}$$

Then, using the fact $x_0 = 3 \cdot x_1$ from part 1, $y_0 = 3 \cdot y_1$ from part 2, and $z_0 = 3 \cdot z_1$ we can calculate

$$3^5 z_1^3 = 3^3 x_1^3 + 3^4 y_1^3 \tag{11}$$
$$3^2 z_1^3 = x_1^3 + 3y_1^3 \tag{12}$$
$$9z_1^3 = x_1^3 + 3y_1^3 \tag{13}$$

Finally, the part 4 tells us

$$9z_1^3 = x_1^3 + 3y_1^3 \tag{14}$$

where $x_1 < x_0$.

Then, because we know x_0 is the smallest number satisfying $x^3 + 3y^3 = 9z^3$, we can conclude above leads to contradiction.

Then, we can conclude the the assumption is false.

$\square$

Notes:

- **Proof By Contradiction:** $\neg P \Rightarrow \neg Q \wedge Q$ (Assuming we are proving $P \Rightarrow Q$)
- **Principle of Well-Ordering:** Any nonempty subset A of $\mathbb{N}$ contains a minimum element; i.e. for any $A \subseteq \mathbb{N}$ such that $A \neq \emptyset$, there is some $a \in A$ such that for all $a' \in A$, $a \leq a'$.
- examples of well-ordered sets
  1. $\mathbb{N} \cup \{0\}$
  2. $\mathbb{N} \cup \{1, 2\}$
  3. $\{n \in \mathbb{N} : n > 5\}$
- examples of non-well-ordered sets
  1. $\mathbb{R}$ and the open interval $(0, 2)$
  2. $\mathbb{Z}$
- Learned the P in $\neg P \Rightarrow \neg Q \wedge Q$ is the statement

$$P(x, y, z) : \text{There are no positive integers } x, y, z \text{ such that } x^3 + 3y^3 = 9z^3$$

  And learned the Q is the principle of well-ordering on P.

- Learned the goal of contradiction is to show the assumption violates principle of well-ordering. That is, there is $x_1 \in X$ less than x_0 satisfying $x^3 + 3y^3 = 9z^3$
- Noticed professor reduced wordiness of work using short notations.

Then

$$\begin{aligned}
x_0^3 + 3y_0^3 = 9z_0^3 \quad &\Rightarrow \quad x_0^3 = 9z_0^3 - 3y_0^3 \Rightarrow 3 \mid x_0^3 \Rightarrow 3 \mid x_0 \qquad \text{\# by clue for A1 Q3} \\
\text{let } x_1 \in \mathbb{N}^+, 3x_1 = x_0 \quad &\Rightarrow \quad 3^3x_1^3 = 9z_0^3 - 3y_0^3 \Rightarrow 3^2x_1^3 = 3z_0^3 - y_0^3 \qquad \text{\# divide through by 3} \\
&\Rightarrow \quad y_0^3 = 3z_0^3 - 3^2x_1^3 \Rightarrow 3 \mid y_0^3 \Rightarrow 3 \mid y_0 \\
\text{let } y_1 \in \mathbb{N}^+, 3y_1 = y_0 \quad &\Rightarrow \quad 3^3y_1^3 = 3z_0^3 - 3^2x_1^3 \Rightarrow 3^2y_1^3 = z_0^3 - 3x_1^3 \qquad \text{\# divide through by 3} \\
&\Rightarrow \quad 3x_1^3 + 3^2y_1^3 = z_0^3 \Rightarrow 3 \mid z_0^3 \Rightarrow 3 \mid z_0 \\
\text{let } z_1 \in \mathbb{N}^+, 3z_1 = z_0 \quad &\Rightarrow \quad 3x_1^3 + 3^2y_1^3 = 3^3z_1^3 \Rightarrow x_1^3 + 3y_1^3 = 9z_1^3 \qquad \text{\# divide through by 3} \\
&\Rightarrow \quad x_1 \in X
\end{aligned}$$

Question 2

- *Proof.* **Basis:**

  We need to show that the property is true for the simplest members x, y, z.

  There are three cases: $e = x$, $e = y$ and $e = z$. In each of the cases $s_2(e) = 1$ and $s_1(e) = 0$.

  Using this fact, starting from the left hand side, we can conclude

  $$\begin{aligned} s_1(e) = 0 = 3 \cdot 0 & \quad (1) \\ = 3 \cdot (1 - 1) & \quad (2) \\ = 3 \cdot (s_2(e) - 1) & \quad (3) \end{aligned}$$

  Inductive Step:

  Let e_1 and e_2 be arbitrary elements of ε. Assume $H(e_1, e_2) : P(e_1)$ and $P(e_2)$. That is, e_1 and e_2 have the property $s_1(e_1) = 3 \cdot (s_2(e) - 1)$ and $s_1(e_2) = 3 \cdot (s_2(e_2) - 1)$.

  We need to show all possible combinations of e_1 and e_2 have the property. That is, $P((e_1 + e_2))$, $P((e_1 \times e_2))$.

  There are two cases, depending on how e is constructed from e_1 and e_2: $e = (e_1 + e_2)$, $e = (e_1 \times e_2)$. In each case we have

  $$\begin{aligned} s_1(e) &= s_1(e_1) + s_1(e_2) + 3 & (4) \\ s_2(e) &= s_2(e_2) + s_2(e_2) & (5) \end{aligned}$$

  Then, using above fact, we can conclude

  $$\begin{aligned} s_1(e) &= s_1(e_1) + s_1(e_2) + 3 && \text{[by 4]} \quad (6) \\ &= 3 \cdot (s_2(e_1) - 1) + 3 \cdot (s_2(e_2) - 1) + 3 && \text{[by induction hypothesis]} \quad (7) \\ &= 3 \cdot s_2(e_1) - 3 + 3 \cdot s_2(e_2) - 3 + 3 && (8) \\ &= 3 \cdot s_2(e_1) + 3 \cdot s_2(e_2) - 6 + 3 && (9) \\ &= 3 \cdot (s_2(e_1) + s_2(e_2)) - 3 && (10) \\ &= 3 \cdot s_2(e) - 3 && \text{[by 5]} \quad (11) \end{aligned}$$

  □

Notes:

- **Structural Induction**
  - is a proof method used in mathematical logic, computer science, graph theory.
  - is a generalization of mathematical induction over natural numbers.
  - is a recursion method
  - **Example:**

    Define ε: The smallest set such that

    * $x, y, z \in \varepsilon$ # variables
    * $e_1, e_2 \in \varepsilon \Rightarrow (e_1 + e_2), (e_1 - e_2), (e_1 \times e_2), (e_1 \div e_2) \in \varepsilon$ # operators

    (steps omitted). Prove $P(e)$: $\mathbf{vr}(e) = \mathbf{op}(e) + 1$ # **vr** means number of variable, **op** means number of operators

    to prove above using structural induction:

    1. **Verify Base Case(s):** Show that the property is true for the simplest members, x,y,z. That is show $P(x)$, $P(y)$, and $P(z)$.

       There are three cases: $e = x$, $e = y$, and $e = z$. In each case $\mathbf{vr}(e) = 1$ and $\mathbf{op}(e) = 0$, so $P(e)$ holds for the basis.

    2. **Inductive Step:** Let e_1 and e_2 be arbitrary elements of ε. Assume $H(e_1, e_2)$: $P(e_1)$ and $P(e_2)$. That is, e_1 and e_2 have the property $\mathbf{vr}(e_1) = \mathbf{op}(e_1) + 1$ and $\mathbf{vr}(e_2) = \mathbf{op}(e_2) + 1$.

       We need to show all possible combinations of e_1 and e_2 have the property. That is, $P((e_1 + e_2))$, $P((e_1 - e_2))$, $P((e_1 \times e_2))$, and $P((e_1 \div e_2))$.

       There are four cases, depending on how e is constructed from e_1 and e_2: $e = (e_1 + e_2)$, $e = (e_1 - e_2)$, $e = (e_1 \times e_2)$ and $e = (e_1 \div e_2)$. In each case we have

       $$\mathbf{vr}(e) = \mathbf{vr}(e_1) + \mathbf{vr}(e_2) \tag{12}$$
       $$\mathbf{op}(e) = \mathbf{op}(e_1) + \mathbf{op}(e_2) + 1 \text{\# } +1 \text{ is from } + \text{ in } e_1 + e_2 \tag{13}$$

       Thus,

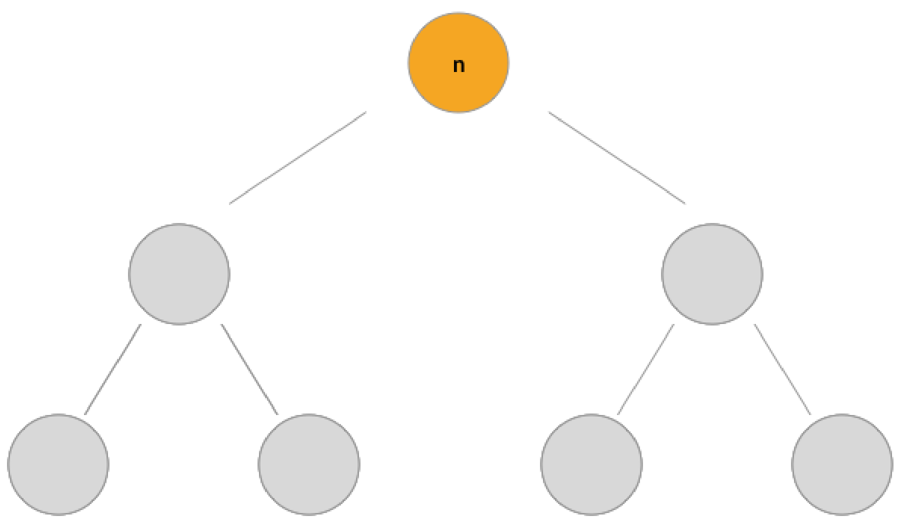
$$
\begin{aligned}
\mathbf{vr}(e) &= \mathbf{vr}(e_1) + \mathbf{vr}(e_2) && \text{[by (4.1)]} && (14)\\
&= (\mathbf{op}(e_1) + 1) + (\mathbf{op}(e_2) + 1) && \text{[by induction hypothesis]} && (15)\\
&= (\mathbf{op}(e_1) + \mathbf{op}(e_2)) + 2 && && (16)\\
&= (\mathbf{op}(e) - 1) + 2 && \text{[by (4.2)]} && (17)\\
&= \mathbf{op}(e) + 1 && && (18)
\end{aligned}
$$

Question 3

- Define the set of non-empty full binary trees, $\mathcal{T}$, as the smallest set such that:

  a. Any single node is an element of $\mathcal{T}$

  b. If $t_1, t_2 \in \mathcal{T}$, n is a node that belongs to neither t_1 nor t_2, and t_1, t_2 have no nodes in common, then n together with edges to the **root nodes** t_1 and t_2 is also an element of $\mathcal{T}$.



Prove $P(t)$: $\mathbf{leaf}(t) = \mathbf{internal}(t) + 1$

Proof. **Basis:**

There is one case, where t is the binary tree with one node. In this case, the node is leaf node. So, $\mathbf{leaf}(t) = 1$. So, $P(t)$ holds for the case.

Inductive Step:

Let t_1 and t_2 be arbitrary element of $\mathcal{T}$. Assume $H(t_1, t_2)$: $P(t_1)$ and $P(t_2)$. That is, t_1 and t_2 have the property $\mathbf{leaf}(t_1) = \mathbf{internal}(t_1) + 1$ and $\mathbf{leaf}(t_2) = \mathbf{internal}(t_2) + 1$.

Let n be a node that belongs to neither t_1 nor t_2. Assume t_1 and t_2 have no nodes in common. Let t be n together with edges to the root node of t_1 and t_2.

We need to show $P(t)$ follows. That is, $\mathbf{leaf}(t) = \mathbf{internal}(t) + 1$.

In this case, we have

$$\begin{align} \mathbf{internal}(t) &= \mathbf{internal}(t_1) + \mathbf{internal}(t_2) + 1 \tag{1} \\ \mathbf{leaf}(t) &= \mathbf{leaf}(t_1) + \mathbf{leaf}(t_2) \tag{2} \end{align}$$

Thus,

$$\begin{align} \mathbf{leaf}(t) &= \mathbf{leaf}(t_1) + \mathbf{leaf}(t_2) && \text{[by 2]} \tag{3} \\ &= \mathbf{internal}(t_1) + 1 + \mathbf{internal}(t_2) + 1 && \text{[by I.H]} \tag{4} \\ &= (\mathbf{internal}(t_1) + \mathbf{internal}(t_2) + 1) + 1 && \tag{5} \\ &= \mathbf{internal}(t) + 1 && \text{[by 1]} \tag{6} \end{align}$$

So, $P(t)$ follows.

$\square$