# CSC373 Worksheet 5 Solution

August 8, 2020

1. **Rough Works:**

Assume that a flow network $G = (V, E)$ violates the assumption that the network contains a path $s \rightsquigarrow v \rightsquigarrow t$ for all vertices $v \in V$. Let $u$ be a vertex for which there is no path $s \rightsquigarrow u \rightsquigarrow t$.

I must show such that there is no flow at vertex $u$. That is, there exists a maximum flow $f$ in $G$ such that $f(u, v) = f(v, u) = 0$ for all vertices $v \in V$.

Assume for the sake of contradiction that there is some vertex $u$ with flow $f$. That is, there exists some vertices $v \in V$ such that $f(u, v) > 0$ or $f(v, u) > 0$.

I see that three cases follows, and I will prove each separately.

1. **Cases 1:** $f(u, v) = 0$ and $f(v, u) > 0$

   Here, assume that $f(u, v) = 0$ for all $v \in V$ and $f(v, u) > 0$ for some $v \in V$.

   - Show that $\sum_{v \in V} f(u, v) = 0$ and $\sum_{v \in V} f(v, u) > 0$

     Then, we can write $\sum_{v \in V} f(u, v) = 0$ and $\sum_{v \in V} f(v, u) > 0$

   - Show that this violates flow conservation [contradiction]

     But this violates the flow conservation property (i.e $\sum_{v \in V} f(u, v) = \sum_{v \in V} f(v, u)$)

   - Conclude that $f(u, v) = 0$ and $f(v, u) = 0$

Thus, by proof by contradiction, $f(u,v) = 0$ and $f(v,u) = 0$ for all $v \in V$ and all $u \in V$ with no path $s \rightsquigarrow u \rightsquigarrow t$.

Here, assume that $f(u,v) = 0$ for all $v \in V$ and $f(v,u) > 0$ for some $v \in V$.

Then, we can write $\sum_{v \in V} f(u,v) = 0$ and $\sum_{v \in V} f(v,u) > 0$

But this violates the flow conservation property (i.e $\sum_{v \in V} f(u,v) = \sum_{v \in V} f(v,u)$)

Thus, by proof by contradiction, $f(u,v) = 0$ and $f(v,u) = 0$ for all $v \in V$ and all $u \in V$ with no path $s \rightsquigarrow u \rightsquigarrow t$.

2. **Cases 2:** $f(u,v) > 0$ and $f(v,u) = 0$

By similar work as case 1, the same result follows.

3. **Cases 3:** $f(u,v) > 0$ and $f(v,u) > 0$

Here, assume that $f(u,v) > 0$ and $f(v,u) > 0$ for some $v \in V$.

- Write that the path $s \rightsquigarrow u \rightsquigarrow t$ exists

Since $s \rightsquigarrow v \rightsquigarrow t$ and $u$ is connected by some vertices $v$, we can write $s \rightsquigarrow u \rightsquigarrow t$.

- Write that this results in contradiction to the header that a vertex $u$ has no path $s \rightsquigarrow u \rightsquigarrow t$.

Then, this violates the fact in header that the vertex $u$ has no path $s \rightsquigarrow u \rightsquigarrow t$.

- Conclude that $f(u,v) = 0$ and $f(v,u) = 0$

Thus, by proof by contradiction, $f(u,v) = 0$ and $f(v,u) = 0$ for all $v \in V$ and all $u \in V$ with no path $s \rightsquigarrow u \rightsquigarrow t$.

Here, assume that $f(u,v) > 0$ and $f(v,u) > 0$ for some $v \in V$.

Since $s \rightsquigarrow v \rightsquigarrow t$ and $u$ is connected by some vertices $v$, we can write $s \rightsquigarrow u \rightsquigarrow t$.

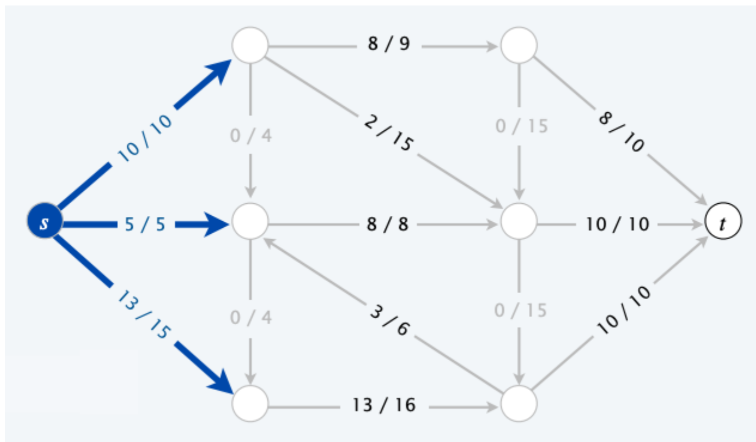
Then, this violates the fact in header that the vertex $u$ has no path $s \rightsquigarrow u \rightsquigarrow t$.

Thus, by proof by contradiction, $f(u,v) = 0$ and $f(v,u) = 0$ for all $v \in V$ and all $u \in V$ with no path $s \rightsquigarrow u \rightsquigarrow t$.

## Notes

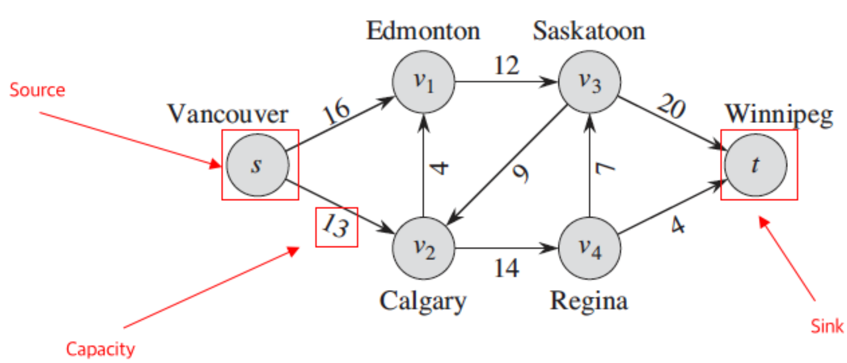
- **Maximum Flow:**
  - Finds a flow of maximum value [1]

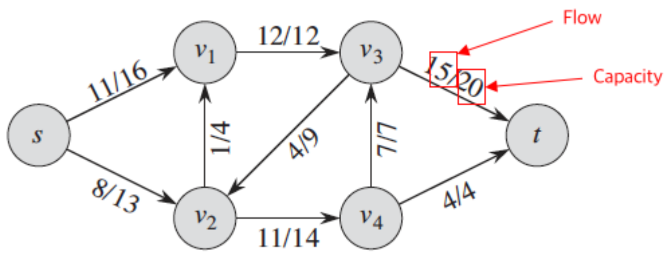
    **Example**




Here, the maximum flow is $10 + 5 + 13 = 28$

- **Flow Network:**
  - $G = (V, E)$ is a directed graph in which each edge $(u, v) \in E$ has a nonnegative capacity $c(u, v) \geq 0$.
  - Two vertices must exist: **source** s and **sink** t
  - **path** from source $s$ to vertax $v$ to sink $t$ is represented by $s \rightsquigarrow v \rightsquigarrow t$





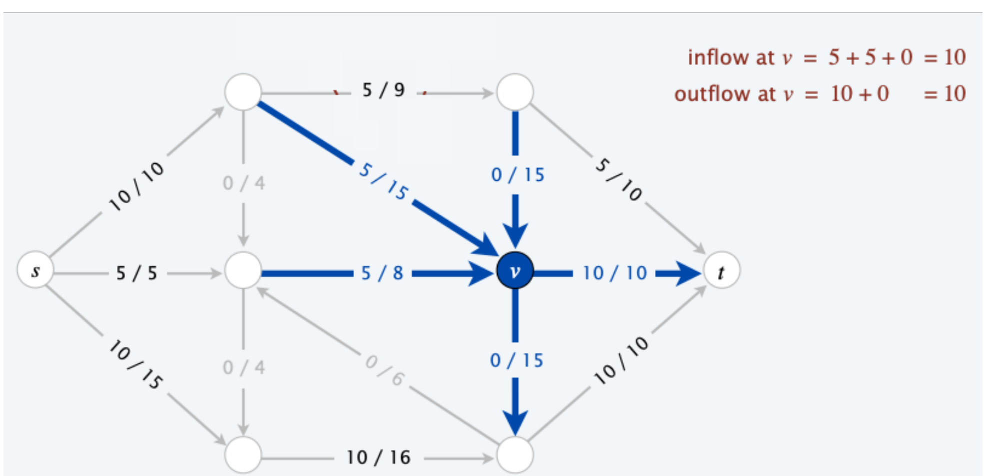

- **Capacity:**
  - Is a non-negative function $f : V \times V \to \mathbb{R}_{\geq 0}$
  - Has **capacity constraint** where for all $u, v \in V$ $0 \leq f(u, v) \leq c(u, v)$
    * Means flow cannot be above capacity constraint
- **Flow:**
  - Is a real valued function $f : V \times V \to \mathbb{R}$ in $G$
  - Satisfies **capacity constraint** (i.e for all $u, v \in V$, $0 \leq f(u, v) \leq c(u, v)$)
  - Satisfies **flow conservation**

  For all $u \in V - \{s, t\}$, we require

$$\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v) \tag{1}$$

  Means flow into vertex $u$ is the same as flow going out of vertex $u$. [1]

  **Example:**



**References**

1) Princeton University, Network Flow 1, link