Problem Set 1 Solution

March 15, 2020

Question 1

a. $\forall t \in T, Canadian(t) \Rightarrow \neg Stanley(t)$

b. $\forall t \in T,\ \exists d \in D,\ \neg Canadian(t) \wedge BelongsTo(t, d)$

c. $\forall t \in T,\ \exists d \in D,\ Stanley(t) \wedge BelongsTo(t, d)$

d. $\forall t \in T, \exists d \in D,\ BelongsTo(t, d) \Rightarrow \forall d' \in D, d' \neq d \wedge \neg BelongsTo(t, d')$

e. $\forall t_1 \in T, \exists d \in D, \exists t_2 \in T, t_1 \neq t_2 \wedge (BelongsTo(t_1, d) \wedge BelongsTo(t_2, d)) \Rightarrow \forall t_3 \in T, t_3 \neq t_1 \wedge t_3 \neq t_2 \wedge \neg BelongsTo(t_3, d)$

Question 2

a. $\forall x \in \mathbb{R}, f(-x) = f(x)$

$\forall x \in \mathbb{R}, -f(-x) = f(x)$

b. $\forall g, f : \mathbb{R} \to \mathbb{R},\ \exists h : \mathbb{R} \to \mathbb{R},\ Odd(f) \wedge Odd(g) \Rightarrow Odd(f) \times Odd(g) = Even(h)$

c. $f = 0$ is a solution, since $-f(-x) = -(-0) = 0 = f(x)$ and $f(-x) = -0 = 0 = f(x)$

d. $\forall f : \mathbb{R} \to \mathbb{R}, \exists f_{1,f} 2 : \mathbb{R} \to \mathbb{R}, Odd(f_1) \wedge Even(f_2) \wedge f = Odd(f_1) + Even(f_2)$

e. A solution is $f = x^2 + x$ with $f_1 = x^2$ and $f_2 = x$.

   $f = x^2 + x$ is the summation $\sum_{i=0}^{2n}$ with $n = 1, a_0 = 0, a_1 = 1, a_2 = 1$. f_1 is odd since $-f(-x) = -(-x) = x = f(x)$, and f_2 is even since $f(-x) = (-x)^2 = x^2 = f(x)$

f. A solution is $g_1(x) = \frac{2^x+2^{-x}}{2}$ and $g_2(x) = \frac{2^x-2^{-x}}{2}$.

   $g_1 + g_2$ gives g since $\frac{2^x+2^{-x}}{2} + \frac{2^x-2^{-x}}{2} = 2^x$. Also, $g_1(-x) = \frac{2^{-x}+2^{-(-x)}}{2} = \frac{2^{-x}+2^{x}}{2} = g_1(x)$ is even, and $-g_2(-x) = -(\frac{2^{-x}-2^{-(-x)}}{2}) = \frac{-2^{x}-2^{-x}}{2}) = g_2(x)$

Question 3

a. One solution is $\forall x \in \mathbb{N},\ \exists y \in \mathbb{N}, x > y$.

   With the above as predicate, the first statement is true, because $\forall x, x > 165$ is always greater than one.

   Also, the second statement is false, because on the rhs, $x = 1$ can be chosen, and $1 \ngtr 1$

b. One solution is $\forall x \in \mathbb{N},\ \exists y \in \mathbb{N}, x > y$.

   With the above as predicate, the first statement is true, because $\forall x, x > 165$ is always greater than one.

   Also, the second statement is false, because on the rhs, $x = 1$ can be chosen, and $1 \ngtr 1$

c. One solution is $x \in \mathbb{R}, y \in \mathbb{N}, Q(x) : x \notin \mathbb{R}, P(x, y) : x \geq y, S = \mathbb{R}, T = \mathbb{R}$.

   With the above, the second statement is vacuous truth because we know by choosing an arbitrary real number for y, $x = y - 2$ can be chosen that makes the predicate P false.

   Also with the above, the first statement is false because we know by choosing $y = x$, lhs of the statement becomes true, and because x is always a real number, the rhs of the statement is false

Question 4

a. $\exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, n \leq n_0 \wedge \neg Prime(n)$

b. $\forall n_0 \in \mathbb{N},\ \exists n, a \in \mathbb{N},\ n > n_0 \wedge prime(n) \wedge prime(n+a) \wedge (\forall b \in \{x \mid x \in \mathbb{N},\ n < x < n+a\}, \neg Prime(b))$

c. $\forall n_0 \in \mathbb{N},\ \exists n \in \mathbb{N},\ n > n_0 \Rightarrow \forall a \in \mathbb{N},\ \exists b \in \{x \mid x \in \mathbb{N},\ n < x < n+a\},\ \neg Prime(n) \wedge \neg Prime(n+a) \wedge Prime(b)$