Worksheet 12 Solution

March 21, 2020

Question 1

a. $c, n_0 \in \mathbb{R}^+$, $\forall n \in \mathbb{N}$, $n \geq n_0 \Rightarrow g(n) \leq c$, where $g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$

b. Let $c = \frac{277}{2}$, $n_0 = 1$, $n \in \mathbb{N}$, $f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$, $g(n) = 100 + \frac{77}{n+1}$, $f(n) = 1$. Assume $n \geq n_0$

Then,

$$
\begin{align}
g(n) = 100 + \frac{77}{n+1} &\leq 100 + \frac{77}{n+1} \tag{1}\\
&\leq 100 + \frac{77}{2} \tag{2}\\
&\leq \frac{277}{2} \tag{3}\\
&\leq c \cdot 1 \tag{4}\\
&\leq cf(x) \tag{5}
\end{align}
$$

The, it follows from the definition of Big-Oh that the statement $100+\frac{77}{n+1} \in \mathcal{O}(1)$ is true.

Question 2

- **Expanded Statement:** $f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$, $(\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \leq cf(n)) \Rightarrow (\exists d, m_0 \in \mathbb{R}^+, \forall m \in \mathbb{N},\ m \geq m_0 \Rightarrow dg(n) \leq f(n))$.

Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, $n_0 = 1$, $c = \frac{1}{d}$, $n \in \mathbb{N}, m_0 = 1$. Assume $n \geq n_0$, $g(n) \leq cf(n)$ and $m \geq m_0$.

Then,

$$g(n) \leq cf(n) \tag{1}$$
$$g(n) \leq \frac{1}{d} f(n) \tag{2}$$
$$dg(n) \leq f(n) \tag{3}$$

Then,

$$dg(m) \leq f(m) \tag{4}$$

by changing variable from n to m.

Then, it follows from the definition of Omega that the statement , $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, $g \in \mathcal{O}(f) \Rightarrow \Omega(g)$ is true.

Question 3

- Let $g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, $a \in \mathbb{R}^{\geq 0}$, $m \in \mathbb{N}$, $c_2 \gg a$, $c_1 = \frac{1}{c_2}$. Assume $g \in \Omega(1)$, $m \geq m_0$.

  Then

  $$a + g \leq a + c_2 g \tag{1}$$
  $$< c_2 g \tag{2}$$

  and,

  $$a + g \geq g \tag{3}$$
  $$> c_1 g \tag{4}$$

  Then, by the definition of theta, the statement $\forall g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, and $a \in \mathbb{R}^{\geq 0}$, $g \in \Omega(1) \Rightarrow a + g \in \Theta(g)$ is true.

Question 4

1. $g \notin \mathcal{O}(f) : \forall c, n_0 \in \mathbb{R}^+,\ \exists n \in \mathbb{N},\ (n \geq n_0) \wedge (g(n) > cf(n))$

2. Let $c, n_0 \in \mathbb{R}^+$, and $n = n_0 + c^{\frac{1}{a-b}}$. Assume $a > b$.

   Note: Need to ask how $n = n_0 + c^{\frac{1}{a-b}} \in \mathbb{N}$.

   Then, $n \geq n_0$.

   And

$$\begin{aligned} cn^b &< (n_0 + c^{\frac{1}{a-b}})^{a-b} n^b && (1) \\ &< (n_0 + c^{\frac{1}{a-b}})^{a-b} (n_0 + c^{\frac{1}{a-b}})^b && (2) \\ &< (n_0 + c^{\frac{1}{a-b}})^{a-b+b} && (3) \\ &< (n_0 + c^{\frac{1}{a-b}})^a && (4) \\ &< n^a && (5) \end{aligned}$$

   Then, it follows that the statement $\forall a, b \in \mathbb{R}^+, a > b \Rightarrow n^a \notin \mathcal{O}(n^b)$ is true.