

CSC236 Midterm 2 Version 1 Solution

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May 11, 2020

Question 1

- Let $n, q \in \mathbb{N}$. Let $r \in \{0, 1\}$

Assume $n > 2$, and $n = 2q + r$.

I need to find a closed form for $T(2q + r)$, using repeated substitution.

Starting from $T(n)$, we have

$$T(n) = n + T(n - 2) \quad [\text{By def. since } n > 2] \quad (1)$$

$$T(2q + r) = 2q + r + T(2q + r - 2) \quad [\text{By replacing } n \text{ for } 2q + r] \quad (2)$$

$$= 2q + r + T(2(q - 1) + r) \quad (3)$$

$$\vdots \quad (4)$$

$$= \sum_{i=0}^{i=q-1} (2(q - i) + r) + T(r) \quad [\text{After } q - 1 \text{ repeatitions}] \quad (5)$$

$$= 2 \sum_{i=0}^{i=q-1} (q - i) + \sum_{i=0}^{i=q-1} r + T(r) \quad (6)$$

$$= 2 \sum_{i=0}^{i=q-1} (q - i) + \sum_{i=0}^{i=q-1} r \quad [\text{Since } T(r) = 0] \quad (7)$$

$$= 2 \sum_{i'=1}^{i'=q} i' + \sum_{i=0}^{i=q-1} r \quad (8)$$

$$= 2 \sum_{i'=1}^{i'=q} i' + \sum_{i=0}^{i=q-1} r \quad (9)$$

$$= 2 \sum_{i'=1}^{i'=q} i' + \sum_{i=0}^{i=q-1} r \quad (10)$$

$$= 2(q(q + 1))/2 + \sum_{i=0}^{i=q-1} r \quad [\text{By using } \sum_{i=1}^{i=n} i = (n(n + 1))/2] \quad (11)$$

$$= q(q + 1) + rq \quad (12)$$

$$= q(q + 1 + r) \quad (13)$$

- *Proof.* For convenience, define $H(q) : q(q + r + 1) = T(2q + r)$.

I will use simple induction to prove $\forall q \in \mathbb{N}, H(q)$.

Base Case ($q = 0$):

Let $q = 0$.

Then,

$$q(q + r + 1) = 0 \quad (1)$$

$$= T(2 \cdot 0 + r) \quad [\text{By def.}] \quad (2)$$

$$= T(2q + r) \quad (3)$$

Thus, $T(2q + r)$ verifies in this step.

Inductive Step:

Let $q \in \mathbb{N}$. Assume $H(q)$.

I need to show $H(q + 1)$ follows. That is, $(q + 1) \left[(q + 1) + r + 1 \right] = T(2(q + 1) + r)$.

Starting with $(q + 1) \left[(q + 1) + r + 1 \right]$, we have

$$(q + 1) \left[(q + 1) + r + 1 \right] = (q + 1)(q + 1) + (q + 1)r + (q + 1) \quad (4)$$

$$= q^2 + 2q + 1 + (qr + r) + (q + 1) \quad (5)$$

$$= (q^2 + qr + q) + (2q + r + 2) \quad (6)$$

$$= q(q + r + 1) + (2(q + 1) + r) \quad (7)$$

$$= T(2q + r) + 2(q + 1) + r \quad [\text{By I.H}] \quad (8)$$

$$= T(2(q + 1) + r) \quad [\text{By def.}] \quad (9)$$

Thus, $H(q + 1)$ follows from $H(q)$ in this step.

□

- *Proof.* Define for convenience

$$H(n) : \bigwedge_{i=0}^{n-1} T(n) - T(i) \geq 0 \quad (1)$$

I will use complete induction to prove that $\forall n \in \mathbb{N}, H(n)$.

Inductive Step:

Let $n \in \mathbb{N}$. Assume $\bigwedge_{i=0}^{n-1} H(i)$. I will show $H(n)$ follows.

Base Case($n < 2$):

Assume $n < 2$.

Then, all $T(n)$ and $T(n - 1)$ are 0 by definition.

So, $T(n) - T(n - 1) \geq 0$.

Thus, $C(n)$ follows in this step.

Case ($n \geq 2$):

Assume $n \geq 2$.

Then,

$$\begin{aligned} T(n) - T(n - 1) &= n + T(n - 2) && \text{[By def.]} && (2) \\ &\quad - \left[(n - 1) + T(n - 3) \right] \\ &= 1 + T(n - 2) - T(n - 3) && (3) \\ &\geq 1 && \text{[By I.H, since } 0 \leq n - 2 < n] && (4) \\ &> 0 && (5) \end{aligned}$$

Thus, $C(n)$ follows from $H(n)$ in this step. \square