

Worksheet 7 Review 2

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Question 1

- a. In this case assume that $n \leq 1$.

We want to show $n \leq 1$.

Since the assumption tells us $n \leq 1$, we can conclude this is true.

- b. *Proof.* Let $a = d$ and $b = k$. Assume there exists $d \in \mathbb{N}$ where $(\exists k \in \mathbb{Z}, n = dk) \wedge d \neq 1 \wedge d \neq n$. Assume $n > 1$

We need to prove that $n \nmid a$, $n \nmid b$ and $n \mid ab$.

We will do so in parts.

Part 1 (Proving $n \nmid a$):

We need to prove $n \nmid a$.

First, we need to show $n \geq d$.

The fact 2 tells us

$$\forall x, y \in \mathbb{N}, y \geq 1 \wedge x \mid y \Rightarrow 1 \leq x \leq y \quad (1)$$

and we know from headers that $d \mid n$, $n > 1$, and $n, d \in \mathbb{N}$.

Then, by using these facts, we can write

$$1 \leq d \leq n \quad (2)$$

Second, we need to show $n = d$.

The definition of divisibility tells us for n to divide d , there must be some $k_1 \in \mathbb{Z}$ such that d is equal to $k_1 \cdot n$.

Then, since we know $n \geq d$, by using these facts, we can conclude the definition of divisibility is satisfied only when $k_1 = 1$, or when $n = d$.

Finally, since we know from the header that $n \neq d$, we can conclude $n \nmid d$.

Then, since we know $d = a$ from the header, we can conclude $n \nmid a$.

Part 2 (Proving $n \nmid b$):

We need to prove $n \nmid b$.

First, we need to show $k \mid n$.

The assumption tells us $n = kd$.

Then, it follows from the definition of divisibility that $k \mid d$.

Second, we need to show $k \geq 1$.

The header tells us $n > 1$ $d \geq 0$, and we know from assumption that $n = dk$.

Since the facts tell us $k \leq 0$ results in $n \leq 0$ and this cannot happen, we can conclude $k \geq 1$.

Third, we need to show $n \geq k$ using the fact $k \geq 1$ and $k \mid n$.

The fact 2 tells us

$$\forall x, y \in \mathbb{N}, y \geq 1 \wedge x \mid y \Rightarrow 1 \leq x \leq y \quad (3)$$

Since we know $k \mid n$, $n > 1$ and $k, n \in \mathbb{N}$, we can conclude $k \leq n$.

Fourth, we need to show $n = k$.

The definition of divisibility tells us for n to divide k , there must be some $k_1 \in \mathbb{Z}$ such that k is equal to $k_1 \cdot n$.

Then, using the fact $n \geq k$, we can conclude the definition of divisibility is satisfied only when $k_1 = 1$, or $n = k$.

Finally, since we know from the header that $n \neq k$, we can conclude $n \nmid k$.

Then, it follows from the fact $k = b$, we can conclude $n \nmid b$.

Part 3 (Proving $n \mid ab$):

We need to prove $n \mid ab$.

The fact 1 tells us

$$\forall x \in \mathbb{Z}, x \mid x \tag{4}$$

Since we know $n \in \mathbb{N}$, we can conclude $n \mid n$.

Then, since we know $n = dk$, $d = a$ and $k = b$, we can conclude $n \mid ab$.

□

Pseudoproof:

Let $a = d$ and $b = k$. Assume there exists $d \in \mathbb{N}$ where $(\exists k \in \mathbb{Z}, n = dk) \wedge d \neq 1 \wedge d \neq n$.
Assume $n > 1$

We need to prove that $n \nmid a$, $n \nmid b$ and $n \mid ab$.

We will do so in parts.

1. Show $n \nmid a$.

First, we need to show $n \nmid a$.

1. Show $n \geq d$.

The fact 2 tells us

$$\forall x, y \in \mathbb{N}, y \geq 1 \wedge x \mid y \Rightarrow 1 \leq x \leq y \quad (5)$$

and we know from headers that $d \mid n$, $n > 1$, and $n, d \in \mathbb{N}$.

Then, by using these facts, we can write

$$1 \leq d \leq n \quad (6)$$

2. Show that for n to divide d , $n = d$.

Now, the definition of divisibility tells us for n to divide d , there must be some $k_1 \in \mathbb{Z}$ such that d is equal to $k_1 \cdot n$.

Then, since we know $n \geq d$, by using these facts, we can conclude the definition of divisibility is satisfied when $k_1 = 1$, or when $n = d$.

3. Conclude $n \nmid a$.

Then, since we know from header that $n \neq d$, we can conclude $n \nmid d$.

Part 1 (Proving $n \nmid a$):

We need to prove $n \nmid a$.

First, we need to show $n \geq d$.

The fact 2 tells us

$$\forall x, y \in \mathbb{N}, y \geq 1 \wedge x \mid y \Rightarrow 1 \leq x \leq y \quad (7)$$

and we know from headers that $d \mid n$, $n > 1$, and $n, d \in \mathbb{N}$.

Then, by using these facts, we can write

$$1 \leq d \leq n \quad (8)$$

Second, we need to show $n = d$.

The definition of divisibility tells us for n to divide d , there must be some $k_1 \in \mathbb{Z}$ such that d is equal to $k_1 \cdot n$.

Then, since we know $n \geq d$, by using these facts, we can conclude the definition of divisibility is satisfied only when $k_1 = 1$, or when $n = d$.

Finally, since we know from the header that $n \neq d$, we can conclude $n \nmid d$.

Then, since we know $d = a$ from the header, we can conclude $n \nmid a$.

2. Show $n \nmid b$

- Show $k \mid n$

First, we need to show $k \mid n$.

- State $n = kd$.

The assumption tells us $n = kd$.

- Show $k \mid n$ by using the definition of divisibility

Then, it follows from the definition of divisibility that $k \mid d$.

First, we need to show $k \mid n$.

The assumption tells us $n = kd$.

Then, it follows from the definition of divisibility that $k \mid d$.

- Show $k \geq 1$.

Second, we need to show $k \geq 1$.

Second, we need to show $k \geq 1$.

The header tells us $n > 1$ $d \geq 0$, and we know from assumption that $n = dk$.

Since the facts tell us $k \leq 0$ results in $n \leq 0$ and this cannot happen, we can conclude $k \geq 1$.

- Show $n \geq k$ using the fact $k \mid n$ and $k \geq 1$.

Third, we need to show $n \geq k$.

Third, we need to show $n \geq k$.

The fact 2 tells us

$$\forall x, y \in \mathbb{N}, y \geq 1 \wedge x \mid y \Rightarrow 1 \leq x \leq y \quad (9)$$

Since we know $k \mid n$, $n > 1$ and $k, n \in \mathbb{N}$, we can conclude $k \leq n$.

- Show that for n to divide k , $n = k$.

Fourth, we need to show $n = k$.

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The definition of divisibility tells us for n to divide k , there must be some $k_1 \in \mathbb{Z}$ such that k is equal to $k_1 \cdot n$.

Then, using the fact $n \geq k$, we can conclude the definition of divisibility is satisfied only when $k_1 = 1$, or $n = k$.

- Conclude $n \nmid a$.

Finally, since we know from the header that $n \neq k$, we can conclude $n \nmid k$.

It follows from the fact $k = b$, we can conclude $n \nmid b$.

Part 2 (Proving $n \nmid b$):

We need to show $n \nmid b$.

First, we need to show $k \mid n$.

The assumption tells us $n = kd$.

Then, it follows from the definition of divisibility that $k \mid d$.

Second, we need to show $k \geq 1$.

The header tells us $n > 1$ $d \geq 0$, and we know from assumption that $n = dk$.

Since the facts tell us $k \leq 0$ results in $n \leq 0$ and this cannot happen, we can conclude $k \geq 1$.

Third, we need to show $n \geq k$ using the fact $k \geq 1$ and $k \mid n$.

The fact 2 tells us

$$\forall x, y \in \mathbb{N}, y \geq 1 \wedge x \mid y \Rightarrow 1 \leq x \leq y \quad (10)$$

Since we know $k \mid n$, $n > 1$ and $k, n \in \mathbb{N}$, we can conclude $k \leq n$.

Fourth, we need to show $n = k$.

The definition of divisibility tells us for n to divide k , there must be some $k_1 \in \mathbb{Z}$ such that k is equal to $k_1 \cdot n$.

Then, using the fact $n \geq k$, we can conclude the definition of divisibility is satisfied only when $k_1 = 1$, or $n = k$.

Finally, since we know from the header that $n \neq k$, we can conclude $n \nmid k$.

Then, it follows from the fact $k = b$, we can conclude $n \nmid b$.

3. Show $n \mid ab$

We need to show $n \mid ab$.

- State fact 1

The fact 1 tells us

$$\forall x \in \mathbb{Z}, x \mid x \quad (11)$$

- Show $n \mid n$

Since we know $n \in \mathbb{N}$, we can conclude $n \mid n$.

- Show $n \mid ab$ using the fact $n = dk$ where $a = d$ and $b = k$.

Then, since we know $n = dk$, $d = a$ and $k = b$, we can conclude $n \mid ab$.

Part 3 (Proving $n \mid ab$):

We need to show $n \mid ab$.

The fact 1 tells us

$$\forall x \in \mathbb{Z}, x \mid x \quad (12)$$

Since we know $n \in \mathbb{N}$, we can conclude $n \mid n$.

Then, since we know $n = dk$, $d = a$ and $k = b$, we can conclude $n \mid ab$.

Notes:

- Made some serious errors (i.e. show $n = a$ or $n = b$) :(.
- How can a proof be organized so it's structurally clear so moe 3 months from now can say I understand this proof? I used first, second and third to show steps involved but I still feel something is missing...
- Can I write a predicate logic for proving $n \nmid b$ or $n \nmid a$? (i.e. $\dots \Rightarrow n \nmid b$)?

Question 2

- a. *Proof.* Let $m, n \in \mathbb{N}$. Assume $Prime(n)$ and $n \nmid m$.

We need to prove there are some integer numbers r and s such that $rn + sm = 1$.

First, we need to show $\gcd(n, m) = 1$.

The fact 3 tells us

$$\forall n, m \in \mathbb{Z}, \text{Prime}(n) \wedge n \nmid m \Rightarrow \gcd(n, m) = 1 \quad (1)$$

Because we know from assumption that n is prime and $n \nmid m$, we can write $\gcd(n, m) = 1$.

Finally, the fact 6 tells us

$$\forall n, m \in \mathbb{N}, \exists r, s \in \mathbb{Z}, rn + sm = \gcd(n, m) \quad (2)$$

Since $\gcd(n, m) = 1$, we can conclude

$$\gcd(n, m) = rn + sm = 1 \quad (3)$$

□

Pseudoproof:

Let $m, n \in \mathbb{N}$. Assume $\text{Prime}(n)$ and $n \nmid m$.

We need to prove $\exists r, s \in \mathbb{Z}, rn + sm = 1$.

1. Show $\gcd(n, m) = 1$, using fact 3

First, we need to show $\gcd(n, m) = 1$.

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The fact 3 tells us

$$\forall n, m \in \mathbb{Z}, \text{Prime}(n) \wedge n \nmid m \Rightarrow \gcd(n, m) = 1 \quad (4)$$

Because we know from assumption that n is prime and $n \nmid m$, we can write $\gcd(n, m) = 1$.

2. Show $rn + sm = \gcd(n, m) = 1$ using fact 6

Finally, the fact 6 tells us

$$\forall n, m \in \mathbb{N}, \exists r, s \in \mathbb{Z}, rn + sm = \gcd(n, m) \quad (5)$$

Since $\gcd(n, m) = 1$, we can conclude

$$\gcd(n, m) = rn + sm = 1 \quad (6)$$

Notes:

- Noticed that professor doesn't put \exists symbols in 'we need to prove that...'.

Let $n, m \in \mathbb{N}$. Assume that n is prime and that $n - m$. We want to prove there exist $r, s \in \mathbb{Z}, rn + sm = 1$.

- 형모야. 오늘도 사랑하는 내 여보 향해 화이팅 :)
- 오늘 켈거리에 구름이 많은데 날씨가 굉장히 밝구나.
- 오오오오오!!!!

- b. **Contrapositive of Statement:** $\forall n, m \in \mathbb{N}, n \mid m \Rightarrow \neg \text{Prime}(n) \vee (\forall r, s \in \mathbb{Z}, rn + sm \neq 1)$

Pseudoproof:

Let $n, m \in \mathbb{N}$. Assume $n \mid m$ and assume n is prime, i.e $n > 1 \wedge (\forall d \in \mathbb{N}, d \mid n \Rightarrow d = 1 \vee d = n, \text{ where } n \in \mathbb{N})$

We need to show for every $r, s \in \mathbb{Z}, rn + sm \neq 1$.

1. Show $\gcd(n, m) \geq 1$ using fact 4.

First, we need to show $\gcd(n, m) \geq 1$.

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The fact 4 tells us

$$\forall n, m \in \mathbb{N}, n \neq 0 \vee m \neq 0 \Rightarrow \gcd(m, n) \geq 1 \quad (7)$$

Because we know from assumption that $n > 1$, we can write $\gcd(n, m) \geq 1$.

2. Show $\gcd(n, m) = n$

Second, we need to show $\gcd(n, m) = n$.

Second, we need to show $\gcd(n, m) = n$.

The definition of greatest common divisor tells us

$$\forall n, m \in \mathbb{Z}, IsCD(n, m, n) \wedge (\forall d_1 \in \mathbb{Z}, IsCD(n, m, d_1) \Rightarrow d_1 \leq n) \quad (8)$$

Because we know n is a common divisor to both n and m , and n is the highest value that divides n and m , we can conclude $\gcd(n, m) = n$.

3. Show $\forall r, s \in \mathbb{Z}, rn + sm \geq n$.

Third, we need to show $\forall r, s \in \mathbb{Z}, rn + sm \geq n$.

Third, we need to show for every $r, s \in \mathbb{Z}, rn + sm \geq n$.

The fact 5 tells us

$$\forall n, m \in \mathbb{N}, \forall r, s \in \mathbb{Z}, \gcd(n, m) \mid (rn + sm) \quad (9)$$

Using the fact $\gcd(n, m) = n$, we can write $rn + sm \geq n$.

4. Conclude $rn + sm \neq 1$ using the fact $n > 1$.

Finally, because we know $n > 1$ from assumption and $rn + sm \geq n$, we can conclude $rn + sm > 1$, or $rn + sm \neq 1$.

Question 3