

CSC236 Worksheet 1 Solution

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Question 1

a. *Proof.* Assume the statement $P(115)$ is true. That is, $\sum_{i=0}^{115} 2^i = 2^{115+1}$.

We need to prove $\sum_{i=0}^{116} 2^i = 2^{116+1}$.

Starting from the left, we can write

$$\sum_{i=0}^{116} 2^i = \sum_{i=0}^{115} 2^i + 2 \quad (1)$$

Then, using the assumption $\sum_{i=0}^{115} 2^i = 2^{115+1}$, we can conclude

$$\sum_{i=0}^{116} 2^i = 2^{115+1} + 2^{116} \quad (2)$$

$$= 2^{116} + 2^{116} \quad (3)$$

$$= 2^{116}(1 + 1) \quad (4)$$

$$= 2 \cdot 2^{116} \quad (5)$$

$$= 2^{116+1} \quad (6)$$

□

b. *Proof.* No. The statement is not true for every natural number.

We will prove this by counter example. That is, $\exists n \in \mathbb{N}, \sum_{i=0}^n 2^i \neq 2^{n+1}$.

Let $n = 0$.

Then, starting from the left hand side, it follows from the fact $n = 0$ that

$$\sum_{i=0}^{i=n} 2^i = \sum_{i=0}^{i=0} 2^i \quad (1)$$

$$= 0 \quad (2)$$

Now, for the right hand side, using the same fact, we can write

$$2^{0+1} = 2^1 \quad (3)$$

$$= 2 \quad (4)$$

□

Question 2

- **Statement:** $\forall n \in \mathbb{N}, \exists d \in \mathbb{Z}, 8^n - 1 = 7d$

Proof. We will prove this statement by induction on n .

Base Case:

Let $n = 0$.

We need to prove $8^n - 1 = 7 \cdot 0$.

Starting from the left hand side, using the fact $n = 0$, we can conclude,

$$8^0 - 1 = 1 - 1 \quad (1)$$

$$= 0 \quad (2)$$

$$= 7 \cdot 0 \quad (3)$$

Inductive Step:

Let $n \in \mathbb{N}$. Assume there is an integer d such that $8^n - 1 = 7d$.

We need to prove there is an integer \tilde{d} such that $8^{n+1} - 1 = 7\tilde{d}$.

Let $\tilde{d} = 8^n + d$.

Starting from the left hand side, we can write

$$8^{n+1} - 1 = 8^n + 8^n + 8^n + 8^n + 8^n + 8^n + 8^n + 8^n - 1 \quad (4)$$

$$= 8^n + 8^n + 8^n + 8^n + 8^n + 8^n + 8^n + (8^n - 1) \quad (5)$$

Then, using inductive hypothesis, i.e. $8^n - 1 = 7d$, we can conclude

$$8^{n+1} - 1 = 8^n + 8^n + 8^n + 8^n + 8^n + 8^n + 8^n + 7d \quad (6)$$

$$= 7 \cdot 8^n + 7d \quad (7)$$

$$= 7 \cdot (8^n + d) \quad (8)$$

$$= 7 \cdot \tilde{d} \quad (9)$$

□

Question 3

- **Statement:** $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}$, the units digit of 7^n is the same as the units digit of 3^m .

Proof. We will prove this statement by induction on n .

Base Case:

Let $n = 0$.

We need to prove there is a natural number m such that the units digit of $7^0 = 1$ is the same as the units digit of 3^m . That is, the ones place of the number 3^m is 1.

Let $m = 0$.

Then, using this fact, we can conclude

$$3^0 = 1 \quad (1)$$

Inductive Step:

Let $n \in \mathbb{N}$. Assume there exists a natural number m such that the units digit of 7^n is the same as the units digit of 3^m .

We need to prove there is a natural number \tilde{m} such that the units digit of 7^{n+1} is the same as the units digit of $3^{\tilde{m}}$. That is, the ones digit of 7^{n+1} is the same as the ones digit of $3^{\tilde{m}}$.

Let $\tilde{m} = m + 9$.

Starting with 7^{n+1} , we can write

$$7^{n+1} = 7 \cdot 7^n. \quad (2)$$

Then, it follows from above fact that the ones digit of 7^{n+1} is 7 times the ones digit of 7^n .

Now, for $3^{\tilde{m}}$, using the fact $\tilde{m} = m + 9$, we can write

$$3^{\tilde{m}} = 3^{m+9} \quad (3)$$

$$= 3^9 \cdot 3^m \quad (4)$$

$$= 27 \cdot 3^m \quad (5)$$

Then, it follows from above fact that the ones digit of $3^{\tilde{m}}$ is 7 times the ones digit of 3^m .

Then, using inductive hypothesis, we can conclude the ones digit of $3^{\tilde{m}}$ is 7 times the ones digit of 7^n .

□

Correct Solution:

Statement: $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}$, the units digit of 7^n is the same as the units digit of 3^m .

Proof. We will prove this statement by induction on n .

Base Case:

Let $n = 0$.

We need to prove there is a natural number m such that the units digit of $7^0 = 1$ is the same as the units digit of 3^m . That is, the ones place of the number 3^m is 1.

Let $m = 0$.

Then, using this fact, we can conclude

$$3^0 = 1 \tag{1}$$

Inductive Step:

Let $n \in \mathbb{N}$. Assume there exists a natural number m such that the units digit of 7^n is the same as the units digit of 3^m .

We need to prove there is a natural number \tilde{m} such that the units digit of 7^{n+1} is the same as the units digit of $3^{\tilde{m}}$. That is, the ones digit of 7^{n+1} is the same as the ones digit of $3^{\tilde{m}}$.

Let $\tilde{m} = m + 3$.

Starting with 7^{n+1} , we can write

$$7^{n+1} = 7 \cdot 7^n. \tag{2}$$

Then, it follows from above fact that the ones digit of 7^{n+1} is 7 times the ones digit of 7^n .

Now, for $3^{\tilde{m}}$, using the fact $\tilde{m} = m + 3$, we can write

$$3^{\tilde{m}} = 3^{m+3} \tag{3}$$

$$= 3^3 \cdot 3^m \tag{4}$$

$$= 27 \cdot 3^m \tag{5}$$

Then, it follows from above fact that that the ones digit of $3^{\tilde{m}}$ is 7 times the ones digit of 3^m .

Then, using inductive hypothesis, we can conclude the ones digit of $3^{\tilde{m}}$ is 7 times the ones digit of 7^n .

□

Question 4

- Rough Work:

Let $m = 7$. Let $n \in \mathbb{N}$. Assume $n \geq m$.

We need to prove $4^n \geq 5n^4 + 6$.

We will do so by induction on n .

1. Base Case ($m = 7$)

Let $m = 7$.

We need to prove $4^n \geq 5n^4 + 6$.

Starting with the left hand side, using the fact $m = 7$, we can calculate

$$4^7 = 16384 \tag{1}$$

Now, for the right hand side, using the same fact, we can conclude

$$5 \cdot 7^4 + 6 = 12011 \tag{2}$$

2. Inductive Step