Learning Objectives

By the end of this worksheet, you will:

- Translate statements between natural English and predicate logic.
- Determine the truth value of sentences on small domains.
- Define a property of a mathematical function using predicate logic.
- Express conditions on domains in predicate logic.

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1. **Alternating quantifiers**. Consider the following table that shows the employees of a (very small) company:[1]

Employee	Salary	Department
Aizah	60,000	Sales
Betty	15,000	Sales
Carlos	40,000	HR
Doug	30,000	Sales
Ellen	50,000	Design
Flo	20,000	Design

Let E be the set of the six employees listed above. We'll define two predicates on this set:

$$Rich(x): \text{“}x \text{ earns more than 35,000,”} \qquad \text{where } x \in E$$
$$SameDept(x, y): \text{“}x \text{ and } y \text{ are in the same department,”} \qquad \text{where } x, y \in E$$

(a) Consider the statement

$$\exists x, y \in E,\ Rich(x) \land SameDept(x, y).$$

Give one *example* to show that this statement is *True*. Is there more than one possible answer? Note: just as in programming and math, the two variables x and y can have the same value (i.e., refer to the same employee).

(b) Here's the same statement, but with a universal quantifier instead:

$$\forall x, y \in E,\ Rich(x) \land SameDept(x, y).$$

Give one *counter-example* to show that this statement is *False*. Is there more than one possible answer?

(c) Now let's look at alternating quantifiers.

$$\forall y \in E,\ \exists x \in E,\ Rich(x) \land SameDept(x, y).$$

Is this True or False? How do you know?

[1] This example is adapted from an old set of CSC165 course notes.

(d) With the quantifiers switched:

$$\exists x \in E,\ \forall y \in E,\ \mathit{Rich}(x) \land \mathit{SameDept}(x, y).$$

Explain why this statement is *False*.

2. **A property of functions**. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Recall that the *codomain* of f is the set after the $\to$ symbol (in this case, $\mathbb{R}$), which contains the possible output values of f. However, in general the codomain of a function might contain values that cannot possibly be output (e.g., if $f : \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = x^2$). We say that $f : \mathbb{R} \to \mathbb{R}$ is **onto** when its codomain $\mathbb{R}$ *only* contains values that could possibly be output by f (and no "impossible" values). For example, $f(x) = x + 1$ is onto, but $f(x) = x^2$ is not onto.

(a) Let $f : \mathbb{R} \to \mathbb{R}$. Express the statement "f outputs 10 (for at least one input)" in symbolic form. You may use an expression like $f(x) = 10$ in your statement.

(b) Now express the definition of onto as a predicate in symbolic form by filling in the following blank.

$\mathit{Onto}(f)$: , where $f : \mathbb{R} \to \mathbb{R}$.

Space for rough work:

(c) Let $f(x) = x^2$. Using your definition, give a *counter-example* to show that f is not onto.

3. **Expressing conditions**. Often when we want to express statements using predicate logic, the common domains like $\mathbb{N}$ and $\mathbb{R}$ do not quite fit. For example, consider the statement:

Every natural number greater than 3 is greater than 1.

The use of "every" suggests a universal quantification, but using the domain of $\mathbb{N}$ in the following way does not capture the meaning of the statement:

$$\forall n \in \mathbb{N},\ n > 1$$

In this question, you'll explore two equivalent ways of modifying the above formula to capture the condition "greater than 3".

(a) One simple way is to change the domain over which we're quantifying. Write down the definition of a set S such that the following formula is equivalent to the original statement:

$$\forall n \in S,\ n > 1$$

(b) We've hinted at another way to express this statement by using the word "condition" to describe the role of "greater than 3". The original statement only talks about natural numbers greater than 3, and ignores all others—the same is true of the implication operator. This suggests that we can keep the original domain, but use an implication to narrow the scope of what we're talking about.
Define a predicate $P(n)$ such that the following formula is equivalent to the original statement:

$$\forall n \in \mathbb{N},\ P(n) \Rightarrow n > 1$$

(c) Using what you've learned, express each of the following statements in *two different* ways:

Every integer that is greater than 10 or less than -40 is not equal to 0.

Every employee who is in the same department as Doug is rich. *(See pg. 1.)*