Worksheet 11 Solution

March 21, 2020

Question 1

a. $\forall a, b \in \mathbb{R}^+,\ a \leq b \Rightarrow \exists c, n_0 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow n^a \leq cn^b$

b. Let $a, b \in \mathbb{R}^+$, $n \in \mathbb{N}$, $c = 1$, $n_0 = 1$. Assume $a \leq b$, and $n \geq n_0$.

Then,

$$\begin{align} n^a &\leq [n^a]^k \tag{1}\\ &\leq n^{ak} \tag{2}\\ &\leq n^b \tag{3}\end{align}$$

by the fact that $k = \frac{b}{a}$, and $k \in \mathbb{R}^+$.

Then,

$$\begin{align} n^a &\leq n^b \tag{4}\\ &\leq cn^b \tag{5}\end{align}$$

Then, it follows from above that the statement $\forall a, b \in \mathbb{R}^+,\ a \leq b \Rightarrow n^a \in \mathcal{O}(n^b)$ is true.

Question 2

a. Let $c = \frac{1}{log_b a}$, $n_0 = 1$, $a \in \mathbb{R}^+$, $b \in \mathbb{R}^+$. Assume $a > 1$ and $b > 1$. We want to show that $\log_a n \leq c \log_b n$.

Then,

$$c \log_b n = \frac{1}{\log_b a} log_b n \tag{1}$$
$$= log_a n \tag{2}$$

by change of base rule for logarithms.

Then it follows from the definition of Big-Oh that $\log_a n \in \mathcal{O}(\log_b n)$

Question 3

a. Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, $n \in \mathbb{N}$, $c = 1$, $n_0 = 1$, $d = 2$, $m_0 = 1$. Assume $n \geq n_0$, $g(n) \leq cf(n)$ and $m \geq m_0$.

Then,

$$g(n) \leq cf(n) \tag{1}$$
$$f(n) + g(n) \leq cf(n) + f(n) \tag{2}$$
$$f(n) + g(n) \leq f(n) + f(n) \tag{3}$$
$$f(n) + g(n) \leq 2f(n) \tag{4}$$
$$f(n) + g(n) \leq df(n) \tag{5}$$

Then,

$$f(m) + g(m) \leq df(m) \tag{6}$$

by changing variable from n to m.

Then, by the definition of Big-Oh, $f + g \in \mathcal{O}(f)$

Then, it follows that the statement $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, $g \in \mathcal{O}(f) \Rightarrow f + g \in \mathcal{O}(f)$ is true.