Worksheet 14 Solution

April 1, 2020

Question 1

a. **Inner Loop:** n

Outer Loop: $n - 5$

Theta Expressions: $\Theta(n^2)$

Correct Solution:

Inner Loop: n

Outer Loop: $n \cdot \left\lceil \frac{n}{5} \right\rceil$

Theta Expressions: $\Theta(n^2)$

b. **Inner Loop:** $\frac{n}{3} + (n - 2)$

Outer Loop: $n - 4$

Theta Expressions: $\Theta(n^2)$

Correct Solution:

Inner Loop: $\lceil \log_3 n \rceil + \left\lceil \frac{n}{2} \right\rceil$

Outer Loop: $max(0, n - 4) \cdot \left[\lceil \log_3 n \rceil + \left\lceil \frac{n}{2} \right\rceil \right]$

Theta Expressions: $\Theta(n^2)$

c. **Inner Loop #2:** $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Inner Loop #1: $n \cdot \frac{n(n+1)}{2} = \frac{n^3+n^2}{2}$

Outer Loop: $\frac{n^3+n^2}{2} \cdot (n-4) = \frac{n^4-3n^3+4n^2}{2}$

Theta Expressions: $\Theta(n^4)$

Correct Solution:

Inner Loop #2: j

Inner Loop #1: $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$

Outer Loop: $\left\lceil \frac{n}{4} \right\rceil \cdot \frac{n(n+1)}{2}$

Theta Expressions: $\Theta(n^3)$

d. **Inner Loop:** 2^n

Outer Loop: $\displaystyle\sum_{i=0}^{\frac{n}{2}-1} 2^i = 2^{\frac{n}{2}-1}$

Theta Expressions: $\Theta(2^n)$

Correct Solution:

Inner Loop: i

Outer Loop: $\sum_{i=0}^{\log n-1} 2^i = \frac{1-2^{\log n-1+1}}{1-2} = 2^{\log n} - 1 = n - 1$

Theta Expressions: $\Theta(n)$

Question 2

- **Inner Loop #2:** $j - i$

  Inner Loop #1: $\sum_{j=i}^{n-1} (j - i)$

  Outer Loop: $\sum_{i=0}^{n-1} \sum_{j=i}^{n-1} (j - i)$

  Calculation:

$$
\begin{align}
\sum_{i=0}^{n-1} \sum_{j=i}^{n-1} (j - i) &= \sum_{i=0}^{n-1-i} \sum_{j'=0}^{n-1} (j' + i - i) \tag{1} \\
&= \sum_{i=0}^{n-1} \sum_{j'=0}^{n-1-i} j' \tag{2} \\
&= \sum_{i=0}^{n-1} \frac{(n-1-i)(n-i)}{2} \tag{3} \\
&= \sum_{i=0}^{n-1} \frac{(n-1-i)(n-i)}{2} \tag{4} \\
&= \frac{1}{2} \left[\sum_{i=0}^{n-1} n^2 - \sum_{i=0}^{n-1} n - 2n \sum_{i=0}^{n-1} i + \sum_{i=0}^{n-1} i + \sum_{i=0}^{n-1} i^2 \right] \tag{5} \\
&= \frac{1}{2} \left[\sum_{i=0}^{n-1} n^2 - \sum_{i=0}^{n-1} n - 2n \sum_{i=0}^{n-1} i + \sum_{i=0}^{n-1} i + \sum_{i=0}^{n-1} i^2 \right] \tag{6} \\
&= \frac{1}{2} \left[\frac{(n^2 - n)}{2} + \frac{2n^3 - n^2 - 2n^2 + n}{6} \right] \tag{7} \\
&= \frac{1}{2} \left[\frac{3n^2 - 3n}{6} + \frac{2n^3 - 3n^2 + n}{6} \right] \tag{8} \\
&= \frac{1}{2} \left[\frac{2n^3 - 2n}{6} \right] \tag{9} \\
&= \frac{n^3 - n}{6} \tag{10}
\end{align}
$$

  Theta Expressions: $\Theta(n^3)$

Correct Solution:

Inner Loop # 2: $j - i + 1$

Inner Loop # 1: $\sum_{j=i}^{n-1} (j - i + 1)$

Outer Loop: $\sum_{i=0}^{n-1} \sum_{j=i}^{n-1} (j - i + 1)$

Calculation:

$$\sum_{i=0}^{n-1} \sum_{j=i}^{n-1} (j - i + 1) = \sum_{i=0}^{n-1} \sum_{j'=1}^{n-i} j' \tag{1}$$

by setting $j' = j - i + 1$, and also by replacing the inner summation notation from $\sum_{i=0}^{b-a} f(i + a)$ to $\sum_{i=a}^{b} f(i')$

Then,

$$\sum_{i=0}^{n-1} \sum_{j'=1}^{n-i} j' = \sum_{i=0}^{n-1} \frac{(n-i)(1+n-i)}{2} \tag{2}$$

by using the arithmetic sum $\sum_{i=1}^{n} a_i = \left(\frac{n}{2}\right)(a_1 + a_n)$.

Then,

$$= \frac{1}{2}\left[n + n^2 - 2in - i + i^2\right] \tag{3}$$

$$= \frac{1}{2}\sum_{i=0}^{n-1}\left[(n + n^2) - i(2n+1) + i^2\right] \tag{4}$$

$$= \frac{1}{2}\left[\sum_{i=0}^{n-1}(n + n^2) - \sum_{i=0}^{n-1} i(2n+1) + \sum_{i=0}^{n-1} i^2\right] \tag{5}$$

$$= \frac{1}{2}\left[\sum_{i=0}^{n-1}(n + n^2) - \frac{3(2n+1)n(n-1)}{6} + \frac{n(n-1)(2n-1)}{6}\right] \tag{6}$$

$$= \frac{1}{2}\left[n(n + n^2) - \frac{3(2n+1)n(n-1)}{6} + \frac{n(n-1)(2n-1)}{6}\right] \tag{7}$$

$$= \frac{1}{2}\left[(n^2 + n^3) - \frac{4n(n-1)(n+1)}{6}\right] \tag{8}$$

$$= \frac{1}{2}\left[(n^2 + n^3) - \frac{4n - 4n^3}{6}\right] \tag{9}$$

$$= \frac{1}{2}\left[n^2 + \frac{6n^3}{6} - \frac{4n - 4n^3}{6}\right] \tag{10}$$

$$= \frac{1}{2}\left[n^2 + \frac{n^3}{3} + \frac{2n}{3}\right] \tag{11}$$

$$= \frac{n^2}{2} + \frac{n^3}{6} + \frac{n}{3} \tag{12}$$

Theta Expressions: $\Theta(n^3)$

Note

- forgot that if starts at 0, has total of n + 1 many iterations.
- must be grouped in terms of variables before expanding
  $\frac{1}{2}\sum_{i=0}^{n-1}[n^2 + n - 2in - i + i^2]$

$$\text{NO:}\frac{1}{2}\left[\sum_{i=0}^{n-1} n^2 + \sum_{i=0}^{n-1} n - \sum_{i=0}^{n-1} 2in - \sum_{i=0}^{n-1} i + \sum_{i=0}^{n-1} i^2\right] \tag{13}$$

$$\text{YES:}\frac{1}{2}\left[\sum_{i=0}^{n-1}(n^2 + n) + \sum_{i=0}^{n-1}(2n-1)i + \sum_{i=0}^{n-1} i^2\right] \tag{14}$$

- replace whole (j - i + 1) in $\sum_{j=i}^{n-1}(j-i+1)$ by setting $j' = j - i + 1$, and adding $-i+1$ to $j = i$
- the formula for arithematic sum starting at $i = 1$ is

$$\sum_{i=1}^{n} a_i = \left(\frac{n}{2}\right)(a_1 + a_n) \tag{15}$$