Worksheet 5 Review 2

April 13, 2020

Question 1

- **Statement:** $\forall m, n \in \mathbb{Z},\ (\exists k_1 \in \mathbb{Z},\ m = 2k_1 + 1) \wedge (\exists k_2 \in \mathbb{Z},\ n = 2k_2 + 1) \Rightarrow (\exists k_3 \in \mathbb{Z},\ mn = 2k_3 + 1)$

  Proof. Let $m, n \in \mathbb{Z}$. Assume there is an integer k_1 such that $m = 2k_1 + 1$. Assume there is an integer k_2 such that $n = 2k_2 + 1$. Let $k_3 = (2k_1k_2) + k_1 + k_2$.

  We need to prove $mn = 2k_3 + 1$.

  The assumption tells us $m = 2k_1 + 1$ and $n = 2k_2 + 1$.

  By using these facts and then multiplying them together, we can conclude

$$
\begin{aligned}
mn &= (2k_1 + 1)(2k_2 + 1) && (1)\\
&= 4k_1k_2 + 2k_1 + 2k_2 + 1 && (2)\\
&= 2[(2k_1k_2) + k_1 + k_2] + 1 && (3)\\
&= 2k_3 + 1 && (4)
\end{aligned}
$$

$\square$

  Notes:

- Noticed professor pre-calculates the value of k_3 as roughwork before writing proof
- Noticed professor uses 'That is...' when expanding definition in writing

> ... and assume they are both odd. That is, we assume there exists $k_1, k_2 \in \mathbb{Z}$ such that $m = 2k_1 - 1$ and $n = 2k_2 - 1$.

- Noticed professor uses 'i.e. ...' when expanding definition in writing.

> We need to prove that mn is odd, i.e. there exists k_3 such that $mn = 2k_3 + 1$.

- Noticed professor defines the header for R.H.S of $\Rightarrow$ operator after 'We need to prove that ...'

> We need to prove that mn is odd, i.e. there exists k_3 such that $mn = 2k_3 + 1$.
>
> **Let** $k_3 = 2k_1k_2 - k_1 - k_2 + 1$

Question 2

a. **Predicate Logic:** $\forall m, n \in \mathbb{Z},\ Even(m) \wedge Odd(n) \Rightarrow m^2 - n^2 = m + n$

   Predicate Logic Expanded: $\forall m, n \in \mathbb{Z},\ (\exists k_1 \in \mathbb{Z},\ m = 2k_1) \wedge (\exists k_2 \in \mathbb{Z},\ n = 2k_2 + 1) \Rightarrow m^2 - n^2 = m + n$

b. The value of k used for m and n must not be under the same variable.

Question 3

a. $Dom(f, g) : \ \forall n \in \mathbb{N},\ g(n) \leq f(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

Notes:

- **Definition of is Dominated By:** Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. We say that g is **is dominated by** f (or f **dominates** g) when for every natural number n, $g(n) \leq f(n)$.

b. *Proof.* Let $f(n) = 3n$ and $g(n) = n$.

We need to prove that g is dominated by f, i.e. for every natural number n, $g(n) \leq f(n)$.

The header tells us $g(n) = n$ and $f(n) = 3n$.

Starting from $g(n)$, we can conclude

$$
\begin{aligned}
g(n) = n &\leq 3n && (1)\\
&= f(n) && (2)
\end{aligned}
$$

□

<u>Correct Solution:</u>

Let $n \in \mathbb{N}$, $f(n) = 3n$ and $g(n) = n$.

We need to prove that g is dominated by f, i.e. for every natural number n, $g(n) \leq f(n)$.

The header tells us $g(n) = n$ and $f(n) = 3n$.

Since $n \geq 0$ from the fact $n \in \mathbb{N}$, starting from $g(n)$, we can conclude

$$g(n) = n \leq 3n \tag{1}$$
$$= f(n) \tag{2}$$

Notes:

- Are there proof equivalent of program compliers or unit testing program? Is there a quick proof checklist one can go through to make sure the author avoids common mistakes?

c. **Negation of is dominated by:** $\neg Dom(f, g)$: $\exists n \in \mathbb{N},\ g(n) > f(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

Proof. Let $f(n) = n^2$ and $g(n) = n + 165$.

We need to prove g is not dominated by f. That is, there is a natural number n such that $g(n) > f(n)$.

Let $n = 0$.

Then, we can conclude

$$g(n) = 165 + n = 165 \tag{1}$$
$$> 0 \tag{2}$$
$$= (0)^2 \tag{3}$$
$$= (n)^2 \tag{4}$$
$$= f(n) \tag{5}$$

□

d. **Statement:** $\forall x \in \mathbb{R}^+, \exists n \in \mathbb{N},\ g(n) = n + x > n^2 = f(n)$.

Proof. Let $x \in \mathbb{R}^+$, $g(n) + n + x$ and $f(n) = n^2$.

We need to prove $g(n)$ is not dominated by $f(n)$. That is, there is a natural number n such that $g(n) = n + x > n^2 = f(n)$.

Let $n = 0$.

Then, we can conclude

$$\begin{aligned} g(n) = n + x &= (0) + x && (1)\\ &= x && (2)\\ &> 0 && (3)\\ &= 0^2 && (4)\\ &= n^2 && (5)\\ &= f(n) && (6) \end{aligned}$$

□

Question 4