Worksheet 5 Review 2

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Question 1

- **Statement:** $\forall m, n \in \mathbb{Z},\ (\exists k_1 \in \mathbb{Z},\ m = 2k_1 + 1) \land (\exists k_2 \in \mathbb{Z},\ n = 2k_2 + 1) \Rightarrow (\exists k_3 \in \mathbb{Z},\ mn = 2k_3 + 1)$

  Proof. Let $m, n \in \mathbb{Z}$. Assume there is an integer k_1 such that $m = 2k_1 + 1$. Assume there is an integer k_2 such that $n = 2k_2 + 1$. Let $k_3 = (2k_1k_2) + k_1 + k_2$.

  We need to prove $mn = 2k_3 + 1$.

  The assumption tells us $m = 2k_1 + 1$ and $n = 2k_2 + 1$.

  By using these facts and then multiplying them together, we can conclude

$$\begin{aligned} mn &= (2k_1 + 1)(2k_2 + 1) && (1) \\ &= 4k_1k_2 + 2k_1 + 2k_2 + 1 && (2) \\ &= 2[(2k_1k_2) + k_1 + k_2] + 1 && (3) \\ &= 2k_3 + 1 && (4) \end{aligned}$$

  $\square$

  Notes:

  - Noticed professor pre-calculates the value of k_3 as roughwork before writing proof
  - Noticed professor uses 'That is...' when expanding definition in writing

    > ... and assume they are both odd. That is, we assume there exists $k_1, k_2 \in \mathbb{Z}$ such that $m = 2k_1 - 1$ and $n = 2k_2 - 1$.

  - Noticed professor uses 'i.e. ...' when expanding definition in writing.

We need to prove that mn is odd, i.e. there exists k_3 such that $mn = 2k_3 + 1$.

– Noticed professor defines the header for R.H.S of $\Rightarrow$ operator after 'We need to prove that ...'

We need to prove that mn is odd, i.e. there exists k_3 such that $mn = 2k_3 + 1$.

Let $k_3 = 2k_1k_2 - k_1 - k_2 + 1$

Question 2

a. **Predicate Logic:** $\forall m, n \in \mathbb{Z},\ Even(m) \wedge Odd(n) \Rightarrow m^2 - n^2 = m + n$

Predicate Logic Expanded: $\forall m, n \in \mathbb{Z}$, $(\exists k_1 \in \mathbb{Z}, m = 2k_1) \wedge (\exists k_2 \in \mathbb{Z}, n = 2k_2 + 1) \Rightarrow m^2 - n^2 = m + n$

b. The value of k used for m and n must not be under the same variable.

Question 3

a. $Dom(f, g) :\ \forall n \in \mathbb{N},\ g(n) \leq f(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

Notes:

- **Definition of is Dominated By:** Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. We say that g is **is dominated by** f (or f **dominates** g) when for every natural number n, $g(n) \leq f(n)$.

b. *Proof.* Let $f(n) = 3n$ and $g(n) = n$.

We need to prove that g is dominated by f, i.e. for every natural number n, $g(n) \leq f(n)$.

The header tells us $g(n) = n$ and $f(n) = 3n$.

Starting from $g(n)$, we can conclude

$$\begin{aligned} g(n) = n &\leq 3n && (1) \\ &= f(n) && (2) \end{aligned}$$

$\square$

Correct Solution:

Let $n \in \mathbb{N}$, $f(n) = 3n$ and $g(n) = n$.

We need to prove that g is dominated by f, i.e. for every natural number n, $g(n) \leq f(n)$.

The header tells us $g(n) = n$ and $f(n) = 3n$.

Since $n \geq 0$ from the fact $n \in \mathbb{N}$, starting from $g(n)$, we can conclude

$$
\begin{aligned}
g(n) = n &\leq 3n && (1)\\
&= f(n) && (2)
\end{aligned}
$$

Notes:

- Are there proof equivalent of program compliers or unit testing program? Is there a quick proof checklist one can go through to make sure the author avoids common mistakes?

c. **Negation of is dominated by:** $\neg Dom(f, g) :\ \exists n \in \mathbb{N},\ g(n) > f(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

Proof. Let $f(n) = n^2$ and $g(n) = n + 165$.

We need to prove g is not dominated by f. That is, there is a natural number n such that $g(n) > f(n)$.

Let $n = 0$.

Then, we can conclude

$$
\begin{aligned}
g(n) = 165 + n &= 165 && (1)\\
&> 0 && (2)\\
&= (0)^2 && (3)\\
&= (n)^2 && (4)\\
&= f(n) && (5)
\end{aligned}
$$

□

d. **Statement:** $\forall x \in \mathbb{R}^+, \exists n \in \mathbb{N}, g(n) = n + x > n^2 = f(n)$.

Proof. Let $x \in \mathbb{R}^+$, $g(n) + n + x$ and $f(n) = n^2$.

We need to prove $g(n)$ is not dominated by $f(n)$. That is, there is a natural number n such that $g(n) = n + x > n^2 = f(n)$.

Let $n = 0$.

Then, we can conclude

$$\begin{aligned} g(n) = n + x &= (0) + x && (1)\\ &= x && (2)\\ &> 0 && (3)\\ &= 0^2 && (4)\\ &= n^2 && (5)\\ &= f(n) && (6) \end{aligned}$$

□

Question 4

- **Statement:** $\forall x \in \mathbb{R}^{\geq 0}, x \geq 4 \Rightarrow (\lfloor x \rfloor)^2 \geq \frac{1}{2}x^2$

  Proof. Let $x \in \mathbb{R}^{\geq 0}$. Assume $x \geq 4$.

  We need to prove $(\lfloor x \rfloor)^2 \geq \frac{1}{2}x^2$.

  Let $\epsilon = x - \lfloor x \rfloor$.

  Starting from $(\lfloor x \rfloor)^2$, it follows from the fact $\lfloor x \rfloor = x - \epsilon$ that we can write

$$\begin{aligned} (\lfloor x \rfloor)^2 &= (x - \epsilon)^2 && (1)\\ &= x^2 - 2x\epsilon + \epsilon^2 && (2) \end{aligned}$$

  Then, by using $0 \leq \epsilon < 1$ from the fact, we can calculate

$$\begin{aligned} x^2 - 2x\epsilon + \epsilon^2 &> x^2 - 2x + \epsilon^2 && (3)\\ &> x^2 - 2x && (4) \end{aligned}$$

Now, since we know $x \geq 4$, we can write

$$\begin{aligned} \frac{1}{2} \cdot x^2 = \frac{1}{2}x \cdot x \geq \frac{1}{2}x \cdot 4 && (5) \\ = 2x && (6) \end{aligned}$$

So, by using this fact into equation 4, we can conclude

$$\begin{aligned} (\lfloor x \rfloor)^2 = x^2 - 2x\epsilon + \epsilon^2 &> x^2 - 2x && (7) \\ &\geq x^2 - \frac{1}{2}x^2 && (8) \\ &= \frac{1}{2}x^2 && (9) \end{aligned}$$

$\square$