CSC373 Worksheet 0 Solution

July 22, 2020

1. Recurrence: $T(n) = T(n-1) + n$

   Guess: $T(n) = \mathcal{O}(n^2)$.

   I need to show $T(n) \leq c \cdot n^2$.

$$
\begin{aligned}
T(n) &\leq c(n-1)^2 + n && (1)\\
&= c(n^2 - 2n + 1) + n && (2)\\
&= cn^2 - c2n + c + n && (3)\\
&\leq cn^2 - c2n + cn + n && (4)\\
&= cn^2 - cn + n && (5)\\
&\leq cn^2 - cn + cn && (6)\\
&= cn^2 && (7)
\end{aligned}
$$

   Notes:

   - Substitution method
     - Solves recurrences
       - Recurrence characters the running time of divide-and-conquer algorithm
     - How it works:
       1. Make a guess for the solution
       2. Use mathematical induction to prove the guess is correct or incorrect.

     Example:

     Recurrence: $T(n) = 2T(\lfloor n/2 \rfloor) + n$

     Guess: $T(n) = \mathcal{O}(n \log n)$,

     We need to show $T(n) \leq cn \lg n$.

1. Assume the bound holds for all positive $m < n$, in particular $m = \lfloor n/2 \rfloor$
2. Find the upper bound of $T(m)$

   $T(\lfloor n/2 \rfloor) \leq c\lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)$

3. Show $T(n) = 2T(\lfloor n/2 \rfloor) + n$ leads to $T(n) \leq cn \lg n$

$$
\begin{aligned}
T(n) &\leq 2(c\lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)) + n && (8)\\
&\leq cn \lg(n/2) + n && (9)\\
&= cn \lg(n) - cn \lg 2 + n && (10)\\
&= cn \lg(n) - cn + n && (11)\\
&\leq cn \lg(n) - cn + cn && (12)\\
&\leq cn \lg(n) && (13)
\end{aligned}
$$

4. Show that the boundary holds using mathematical induction

   Doesn't have information in detail. Skipping this for now.

- Making good guess
  * Three suggestions
  1. Using recursion tree
  2. Through practice
  3. prove loose upper and lower bounds on the recurrence and then reduce the range of uncertainty

2. Recurrence: $T(n) = T(\lceil n/2 \rceil) + 1$

   Guess: $T(n) = \mathcal{O}(\lg n)$.

   I need to show $T(n) \leq c \cdot \lg n$.

$$
\begin{aligned}
T(n) &\leq c \lg(\lceil n/2 \rceil) + 1 && (1)\\
&\leq c \lg(n/2) + 1 && (2)\\
&= c(\lg n - \lg 2) + 1 && (3)\\
&= c(\lg n - 1) + 1 && (4)\\
&= c \lg n - c + 1 && (5)\\
&\leq c \lg n - c + c && (6)
\end{aligned}
$$

Correct Solution:

Recurrence: $T(n) = T(\lceil n/2 \rceil) + 1$

Guess: $T(n) = \mathcal{O}(\lg n)$.

I need to show $T(n) \leq c \cdot \lg n$.

$$\begin{align}
T(n) &\leq c\lg(\lceil n/2 \rceil) + 1 \tag{1}\\
&\leq c\lg(n/2) + 1 \tag{2}\\
&= c(\lg n - \lg 2) + 1 \tag{3}\\
&= c(\lg n - 1) + 1 \tag{4}\\
&= c\lg n - c + 1 \tag{5}\\
&\leq c\lg n - c + c \tag{6}
\end{align}$$

The solution holds for $c \geq 1$.

3. Recurrence: $T(n) = 2T(\lfloor n/2 \rfloor) + n$

Guess (Upperbound): $T(n) = \mathcal{O}(n \lg n)$.

I first need to show $T(n) \leq c \cdot n \lg n$.

$$\begin{align}
T(n) &= 2T(\lfloor n/2 \rfloor) + n \tag{1}\\
&= 2c\lfloor n/2 \rfloor \lg\lfloor n/2 \rfloor + n \tag{2}\\
&\leq 2c \cdot (n/2)\lg(n/2) + n \tag{3}\\
&= c \cdot n(\lg n - 1) + n \tag{4}\\
&= cn\lg n - cn + n \tag{5}\\
&\leq cn\lg n - cn + cn \tag{6}\\
&\leq cn\lg n \tag{7}
\end{align}$$

The above inequality holds for $c \geq 1$.

Guess (Lowerbound): $T(n) = \Omega(n \lg n)$.

I first need to show $d \cdot (n-2)\lg(n-2) \le T(n)$.

$$
\begin{aligned}
T(n) &= 2T(\lfloor (n-2)/2 \rfloor) + n && (8)\\
&\ge 2d\lfloor (n-2)/2 \rfloor \lg \lfloor (n-2)/2 \rfloor + n && (9)\\
&\ge 2d \cdot ((n-2)/2)\lg((n-2)/2) + n && (10)\\
&= d \cdot (n-2)(\lg(n-2) - 1) + n && (11)\\
&= d \cdot (n-2)\lg(n-2) - d \cdot (n-2) + n && (12)\\
&\ge d \cdot (n-2)\lg(n-2) - d \cdot (n-2) + (n-2) && (13)\\
&\ge d \cdot (n-2)\lg(n-2) - d \cdot (n-2) + d \cdot (n-2) && (14)\\
&= d \cdot (n-2)\lg(n-2) && (15)
\end{aligned}
$$

The above inequality holds for $0 \le d < 1$.

Notes:

- Both upper bound and lower bound don't need to be the same

4.3-3

> We saw that the solution of $T(n) = 2T(\lfloor n/2 \rfloor) + n$ is $O(n \lg n)$. Show that the solution of this recurrence is also $\Omega(n \lg n)$. Conclude that the solution is $\Theta(n \lg n)$.

First, we guess $\boxed{T(n) \le cn \lg n,}$ upper bound

$$
\begin{aligned}
T(n) &\le 2c\lfloor n/2 \rfloor \lg \lfloor n/2 \rfloor + n \\
&\le cn \lg(n/2) + n \\
&= cn \lg n - cn \lg 2 + n \\
&= cn \lg n + (1-c)n \\
&\le cn \lg n,
\end{aligned}
$$

where the last step holds for $c \ge 1$.

lower bound

Next, we guess $\boxed{T(n) \ge c(n+2)\lg(n+2),}$

$$
\begin{aligned}
T(n) &\ge 2c(\lfloor n/2 \rfloor + 2)(\lg(\lfloor n/2 \rfloor + 2) + n \\
&\ge 2c(n/2 - 1 + 2)(\lg(n/2 - 1 + 2) + n \\
&= 2c\frac{n+2}{2} \lg \frac{n+2}{2} + n \\
&= c(n+2)\lg(n+2) - c(n+2)\lg 2 + n \\
&= c(n+2)\lg(n+2) + (1-c)n - 2c \\
&\ge c(n+2)\lg(n+2),
\end{aligned}
$$

where the last step holds for $n \ge \frac{2c}{1-c}$, $0 \le c < 1$.

4. Recurrence (Merge sort):

$$
T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \Theta(n) & \text{if } n > 1 \end{cases}
$$

Guess (upper bound): $T(n) \leq c \cdot (n-2) \cdot \lg(n-2)$

$$
\begin{aligned}
T(n) &\leq c(\lceil n/2 \rceil - 2)\lg(\lceil n/2 \rceil - 2) + c(\lfloor n/2 \rfloor - 2)\lg(\lfloor n/2 \rfloor - 2) + dn && (1)\\
&= c(n/2 + 1 - 2)\lg(n/2 + 1 - 2) + c(n/2 + 1 - 2)\lg(n/2 + 1 - 2) + dn && (2)\\
&= c((n-2)/2)\lg((n-2)/2) + c((n-2)/2)\lg((n-2)/2) + dn && (3)\\
&= c(n-2)\lg((n-2)/2) + dn && (4)\\
&= c(n-2)\lg(n-2) - c(n-2) + dn && (5)\\
&= c(n-2)\lg(n-2) - (d-c)n + 2c && (6)\\
&= c(n-2)\lg(n-2) && (7)
\end{aligned}
$$

The bound holds as long as $c > d$.

Guess (lower bound): $c \cdot (n-2) \cdot \lg(n-2) \leq T(n)$

$$
\begin{aligned}
T(n) &\leq c(\lceil n/2 \rceil + 1)\lg(\lceil n/2 \rceil + 1) + c(\lfloor n/2 \rfloor + 1)\lg(\lfloor n/2 \rfloor + 1) + dn && (8)\\
&\leq c(n/2 - 1 + 1)\lg(n/2 - 1 + 1) + c(n/2 - 1 + 1)\lg(n/2 - 1 + 1) + dn && (9)\\
&= c(n/2)\lg(n/2) + c(n/2)\lg(n/2) + dn && (10)\\
&= cn\lg(n/2) + dn && (11)\\
&= cn\lg(n) - cn + dn && (12)\\
&= cn\lg(n) + (d-c)n && (13)\\
&\leq c(n-1)\lg(n-1) && (14)
\end{aligned}
$$

The bound holds as long as $d > c$, and $0 \leq c < 1$

Notes:

- the n here is asymptotically large

5. Recurrence: $T(n) = 2T(\lfloor n/2 \rfloor + 17) + n$

   Guess (upper bound): $cn \lg n$

$$
\begin{aligned}
T(n) &\le 2c(\lfloor n/2 \rfloor + 17)\lg(\lfloor n/2 \rfloor + 17) + n && (15)\\
&\le 2c((n/2) + 17)\lg((n/2) + 17) + n && (16)\\
&= 2c(n/2)\lg(n/2) + n && (17)\\
&= cn(\lg(n) - 1) + n && (18)\\
&= cn\lg(n) - cn + n && (19)\\
&\le cn\lg(n) - cn + cn && (20)\\
&= cn\lg(n) && (21)
\end{aligned}
$$

6.

$$
\begin{aligned}
T(n) &= 4T(n/3) + n && (1)\\
&\le 4c(n/3)^{\log_3 4} + n && (2)\\
&\le 4c(1/3)^{\log_3 4} n^{\log_3 4} + n && (3)\\
&\le (4/4)cn^{\log_3 4} + n && (4)\\
&\le cn^{\log_3 4} + n && (5)
\end{aligned}
$$

We cannot advance further since n in $cn^{\log_3 4} + n$ cannot be eliminated.

With the new guess $T(n) \le cn^{\log_3 4} - dn$, we have

$$
\begin{aligned}
T(n) &= 4T(n/3) + n && (6)\\
&\le 4c(n/3)^{\log_3 4} - d(n/3) + n && (7)\\
&= 4c(n/3)^{\log_3 4} - d(n/3) + n && (8)\\
&= (4/3^{\log_3 4})cn^{\log_3 4} - d(n/3) + n && (9)\\
&= (4/4)cn^{\log_3 4} - d(n/3) + n && (10)\\
&= cn^{\log_3 4} - d(n/3) + n && (11)\\
&\le cn^{\log_3 4} - d(n/3) + n && (12)\\
&\le cn^{\log_3 4} && (13)
\end{aligned}
$$

The bound holds as long as $d \ge 3$ and $c \ge 1$.

Correct Solution:

Recurrence: $T(n) = 2T(\lfloor n/2 \rfloor + 17) + n$

Guess (upper bound): $cn \lg n$

$$
\begin{aligned}
T(n) &\le 2c(\lfloor n/2 \rfloor + 17)\lg(\lfloor n/2 \rfloor + 17) + n && (14)\\
&\le 2c((n/2) + 17)\lg((n/2) + 17) + n && (15)\\
&= 2c(n/2)\lg(n/2) + n && (16)\\
&= cn(\lg(n) - 1) + n && (17)\\
&= cn\lg(n) - cn + n && (18)\\
&\le cn\lg(n) - cn + cn && (19)\\
&= cn\lg(n) && (20)
\end{aligned}
$$

$$
\begin{aligned}
T(n) &= 4T(n/3) + n && (1)\\
&\le 4c(n/3)^{\log_3 4} + n && (2)\\
&\le 4c(1/3)^{\log_3 4} n^{\log_3 4} + n && (3)\\
&\le (4/4)cn^{\log_3 4} + n && (4)\\
&\le cn^{\log_3 4} + n && (5)
\end{aligned}
$$

We cannot advance further since n in $cn^{\log_3 4} + n$ cannot be eliminated.

With the new guess $T(n) \le cn^{\log_3 4} - dn$, we have

$$
\begin{aligned}
T(n) &= 4T(n/3) + n && (6)\\
&\le 4c(n/3)^{\log_3 4} - 4d(n/3)4d(n/3) + n && (7)\\
&= 4d(n/3) = 4c(n/3)^{\log_3 4} - 4d(n/3) + n && (8)\\
&= 4d(n/3) = (4/3^{\log_3 4})cn^{\log_3 4} - 4d(n/3) + n && (9)\\
&= (4/4)cn^{\log_3 4} - 4d(n/3) + n && (10)\\
&= cn^{\log_3 4} - 4d(n/3) + n && (11)\\
&\le cn^{\log_3 4} - 4d(n/3) + n && (12)\\
&\le cn^{\log_3 4} - 4d(n/2) + n && (13)\\
&\le cn^{\log_3 4} - 2dn + n && (14)\\
&\le cn^{\log_3 4} - 2dn + dn && (15)\\
&\le cn^{\log_3 4} - dn && (16)
\end{aligned}
$$

7. I need to show $T(n) \le cn^2$

$$T(n) = 4T(n/2) + n \tag{17}$$
$$\leq 4c(n/2)^2 + n \tag{18}$$
$$= (4/4)cn^2 + n \tag{19}$$
$$= cn^2 + n \tag{20}$$

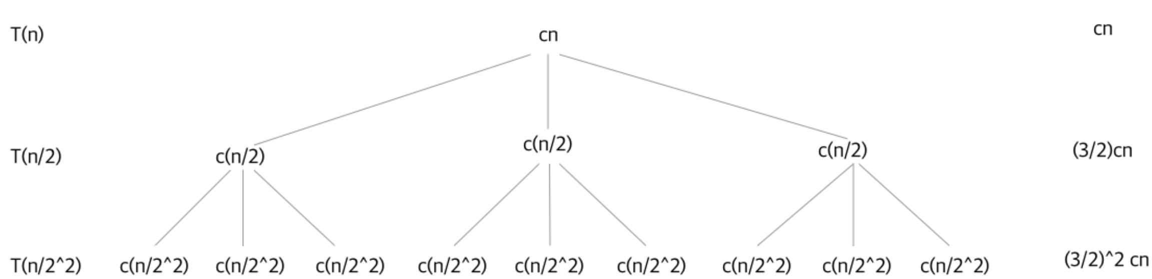
We cannot advance further since n in $cn^2 + n$ cannot be eliminated.

But with the new guess $T(n) \leq cn^2 - dn$, we have

$$T(n) = 4T(n/2) + n \tag{21}$$
$$\leq 4c(n/2)^2 - 4d(n/2) + n \tag{22}$$
$$= (4/4)cn^2 - 2dn + n \tag{23}$$
$$\leq cn^2 - 2dn + dn \tag{24}$$
$$= cn^2 - dn \tag{25}$$

The bound holds as long as $d \geq 1$ and $c \geq 1$.

8. **<u>Solution:</u>**



1. Finding number of levels in recursion tree

$$1 = n/2^i \tag{1}$$
$$2^i = n \tag{2}$$
$$i = \log_2 n \tag{3}$$

2. Finding the total cost of recursion tree

   The tree has $n^{\lg 3}$ leaves. So, we have

$$
\begin{aligned}
T(n) &= n \cdot \sum_{i=0}^{\lg_2(n)-1} (3/2)^i + \Theta(n^{\lg 3}) &(4)\\
&= n \cdot \Big(\frac{(3/2)^{\lg_2(n)} - 1}{(3/2) - 1}\Big) + \Theta(n^{\lg 3}) &(5)\\
&= 2n \cdot \Big((3/2)^{\lg_2(n)} - 1\Big) + \Theta(n^{\lg 3}) &(6)\\
&= 2n \cdot \Big(n^{\lg(3/2)} - 1\Big) + \Theta(n^{\lg 3}) &(7)\\
&= 2n \cdot \Big(n^{\lg(3/2)} - 1\Big) + \Theta(n^{\lg 3}) &(8)\\
&= 2 \cdot \Big(n^{\lg 3 - 1 + 1} - n\Big) + \Theta(n^{\lg 3}) &(9)\\
&= 2 \cdot \Big(n^{\lg 3} - n\Big) + \Theta(n^{\lg 3}) &(10)\\
&= 2 \cdot \Big(n^{\lg 3} - n\Big) + \Theta(n^{\lg 3}) &(11)
\end{aligned}
$$

   Thus, the guess for the upper bound is $T(n) = \mathcal{O}(n^{\lg 3})$

3. Verifying the correct guess using the subtitution method

   <u>Guess:</u> $T(n) \leq cn^{\lg 3} - dn$

   I need to show the guess holds in the recurrence $T(n) = 3T(\lfloor n/2 \rfloor) + n$.

   Indeed we have

$$
\begin{aligned}
T(n) &= 3T(\lfloor n/2 \rfloor) + n &(12)\\
&\leq 3(c\lfloor n/2 \rfloor^{\lg 3}) - d(\lfloor n/2 \rfloor) + n &(13)\\
&= 3\Big(\frac{cn^{\lg 3}}{3} - d(\frac{n}{2} + 1)\Big) + n &(14)\\
&= 3\Big(\frac{cn^{\lg 3}}{3} - \frac{3dn}{2}\Big) + n &(15)\\
&\leq 3\Big(\frac{cn^{\lg 3}}{3} - \frac{3dn}{3}\Big) + n &(16)\\
&= cn^{\lg 3} - 3dn + n &(17)\\
&\leq cn^{\lg 3} - 3dn + 2dn &(18)\\
&= cn^{\lg 3} - dn &(19)
\end{aligned}
$$

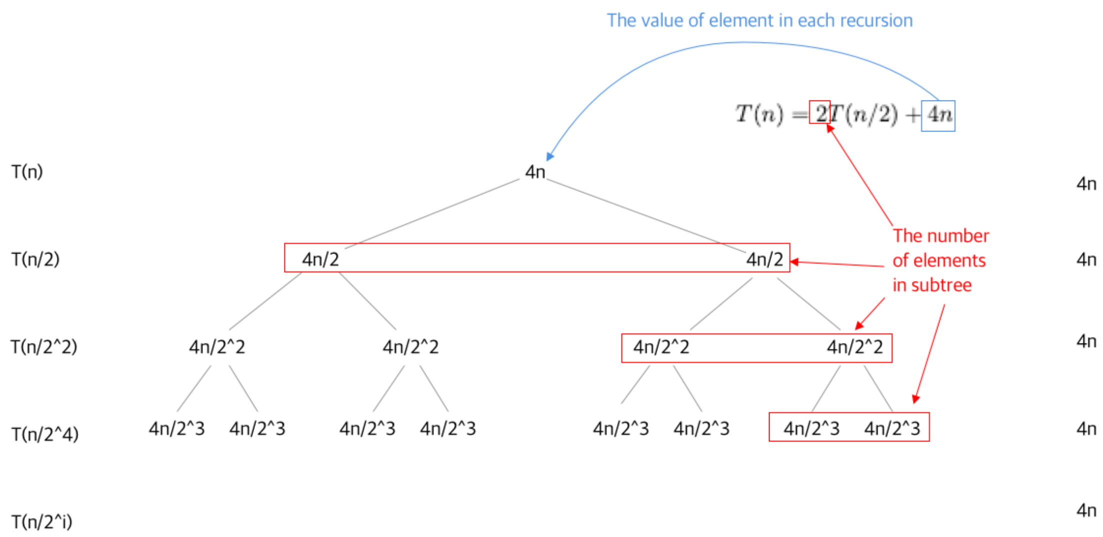
And the boundary holds as long as $c \geq 0$ and $d \geq 1$.

Notes:

- Recursion Tree
  - Provides a straightforward way to provide a good guess.
  - Is then verified using subtitution method

Example:

Recurrence: $T(n) = 2T(n/2) + 4n, T(1) = 4$



1. Finding number of levels in recursion tree

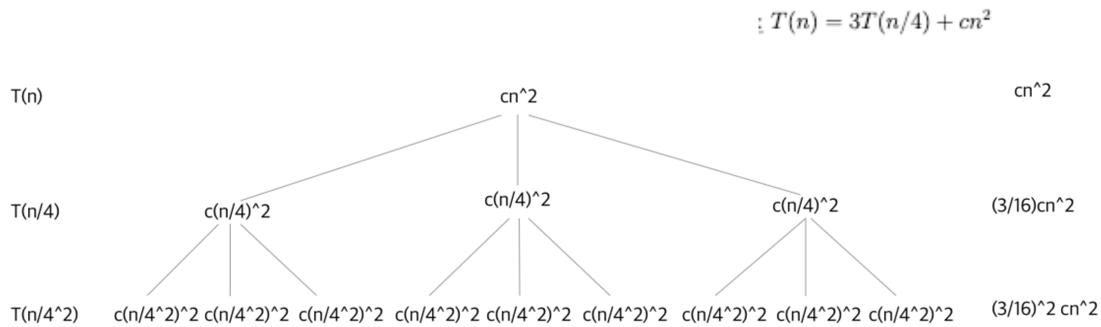
$$1 = n/2^i \tag{20}$$
$$2^i = n \tag{21}$$
$$i = \log_2 n \tag{22}$$

2. Finding the value of guess

$$\sum_{i=0}^{\log_2 n} 4n = 4n \cdot \sum_{i=0}^{\log_2 n} 1 \tag{23}$$
$$= 4n(\log_2 n + 1) \tag{24}$$

Example 2:

Recurrence: $T(n) = 3T(n/4) + cn^2$



Steps:

1. Finding number of levels in recursion tree

$$1 = n/4^i \tag{25}$$
$$4^i = n \tag{26}$$
$$i = \log_4 n \tag{27}$$

2. Finding the cost of entire tree

$$T(n) = \sum_{i=0}^{\log_4 n-1} c(3/16)^i n^2 + \Theta(n^{\log_4 3}) \tag{28}$$
$$= cn^2 \cdot \sum_{i=0}^{\log_4 n-1} (3/16)^i + \Theta(n^{\log_4 3}) \tag{29}$$
$$< cn^2 \cdot \sum_{i=0}^{\infty} (3/16)^i + \Theta(n^{\log_4 3}) \qquad \text{[since } n \text{ is asympt. large]} \tag{30}$$
$$= cn^2\left(\frac{1}{1-(3/16)}\right) + \Theta(n^{\log_4 3}) \qquad \text{[Since } \sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}\text{]} \tag{31}$$

   - **Note:** ($\log_4(n-1)$ because in $i = 0, ...i = \log_4(n-1)$ there are $\log_4(n)$ elements

3. Finding the upper bound of $T(n)$

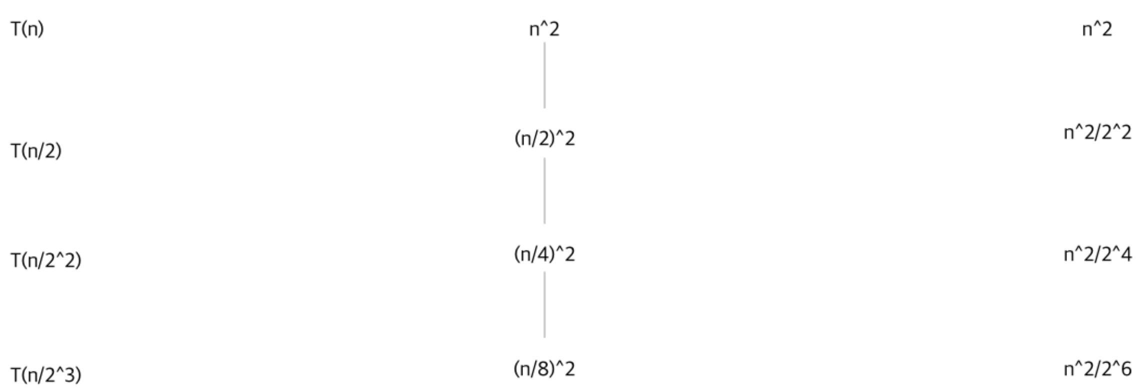
   Since the total cost is $T(n) = cn^2\left(\frac{1}{1-(3/16)}\right) + \Theta(n^{\log_4 3})$, we have $\mathcal{O}(n^2)$

4. Verify the correctness of guess using subtitution method

$$T(n) \le 3T(\lfloor n/4 \rfloor) + cn^2 \tag{32}$$
$$\le 3d\lfloor n/4 \rfloor^2 + cn^2 \tag{33}$$
$$\le 3d(n/4)^2 + cn^2 \tag{34}$$
$$= (3/16)dn^2 + cn^2 \tag{35}$$
$$\le dn^2 \tag{36}$$

where the last step holds as long as $d \ge (16/13)c$.

9. **Solution:**



1. Finding number of levels in recursion tree

$$1 = \frac{n}{2^i} \tag{1}$$
$$2^i = n \tag{2}$$
$$i = \lg n \tag{3}$$

2. Finding the upper bound of $T(n)$

$$
\begin{aligned}
T(n) &= n^2 \cdot \sum_{i=0}^{\lg n-1} \frac{1}{2^{2i}} + \Theta(1) && && (4)\\
&= n^2 \cdot \sum_{i=0}^{\infty} \frac{1}{2^{2i}} + \Theta(1) && \text{[since } n \text{ is asympt. large]} && (5)\\
&= n^2 \cdot \left(\frac{1}{1-\frac{1}{4}}\right) + \Theta(1) && && (6)\\
&= \frac{4n^2}{3} + \Theta(1) && && (7)
\end{aligned}
$$

Thus, we we can conclude $T(n) = \mathcal{O}(n^2)$

3. Verify the correctness of guess using subtitution method

   Guess: $T(n) \leq cn^2$

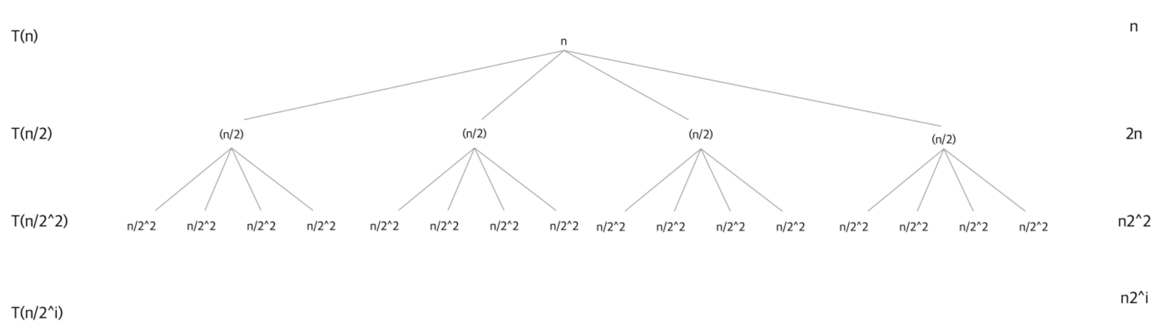
   I need to show the guess holds for the recurrence $T(n) = T(\frac{n}{2}) + n$.

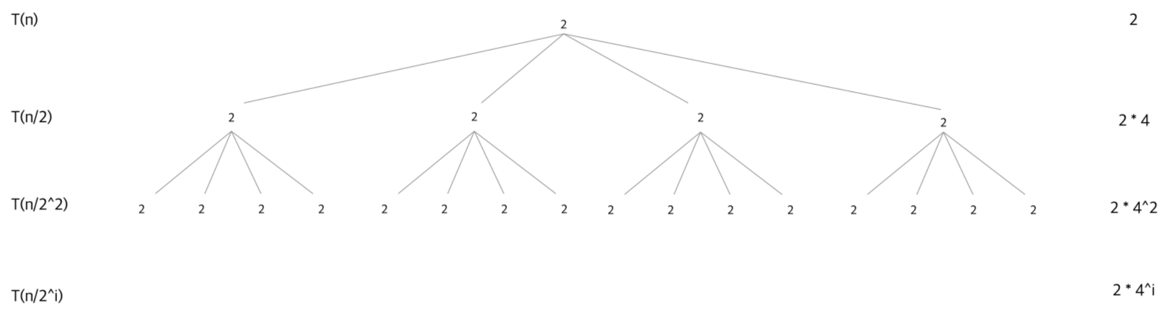
   And, indeed we have

$$
\begin{aligned}
T(n) &= T(\frac{n}{2}) + n^2 && (8)\\
&\leq \frac{cn^2}{4} + n^2 && (9)\\
&= (\frac{c}{4} + 1) \cdot n^2 && (10)\\
&\leq cn^2 && (11)
\end{aligned}
$$

   THe boundary holds when $c \geq \frac{4}{3}$.

10. **Solution:**





- Find the total cost of the recursion tree

$$1 = \frac{n}{2^i} \tag{1}$$
$$2^i = n \tag{2}$$
$$i = \lg n \tag{3}$$

- Finding the upper bound of $T(n)$

$$T(n) = \sum_{i=0}^{\lg n - 1} (2 \cdot 4^i + n2^i) + \Theta(n^2) \tag{4}$$
$$= \sum_{i=0}^{\lg n - 1} 2 \cdot 4^i + \sum_{i=0}^{\lg n - 1} n2^i + \Theta(n^2) \tag{5}$$
$$= 2 \cdot \sum_{i=0}^{\lg n - 1} 4^i + n \cdot \sum_{i=0}^{\lg n - 1} 2^i + \Theta(n^2) \tag{6}$$
$$= 2 \cdot \left(\frac{4^{\lg n} - 1}{4 - 1}\right) + n \cdot (n-1) + \Theta(n^2) \quad [\text{Since } \sum_{i=0}^{n-1} ar^i = a \cdot \frac{r^n - 1}{r - 1}, \text{ where } r \neq 1] \tag{7}$$
$$= \frac{2}{3} \cdot (n^2 - 1) + n \cdot (n-1) + \Theta(n^2) \tag{8}$$
$$= \mathcal{O}(n^2) + \Theta(n^2) \tag{9}$$
$$\tag{10}$$

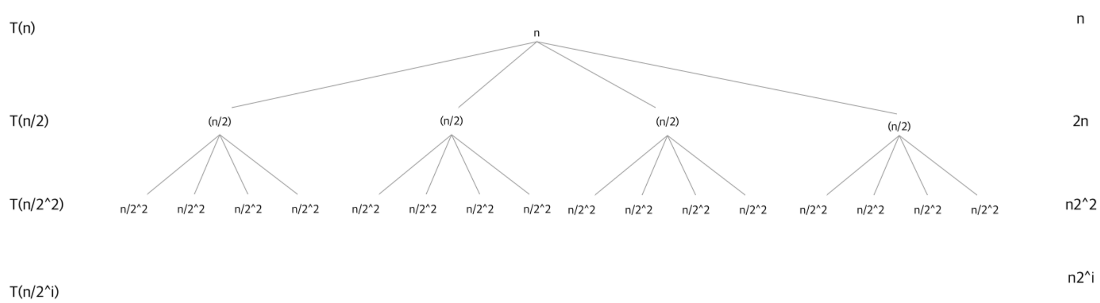
- Verify the correctness of guess using subtitution method

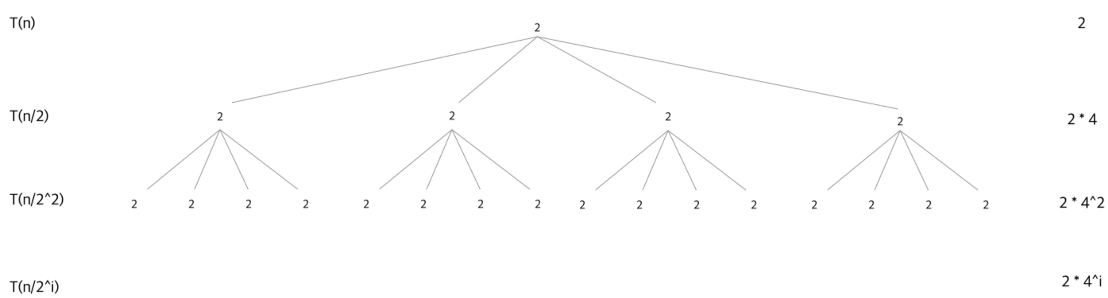
  <u>Guess:</u> $T(n) \leq cn^2 - dn$.

  I need to show the guess holds for the recurrence $T(n) = 4T(\frac{n}{2} + 2) + n$

$$\begin{aligned}
T(n) &= 4T(\frac{n}{2}+2)+n && (11)\\
&\leq 4c(\frac{n}{2}+2)^2 - 4dn + n && (12)\\
&= 4c(\frac{n^2}{4}+2n+4) - 4dn + n && (13)\\
&\leq cn^2 - 4dn + n && \text{[Since } n^2 \text{ dominates } n \text{ asymptotically]} \quad (14)\\
&\leq cn^2 - 4dn + 3dn && (15)\\
&= cn^2 - dn && (16)
\end{aligned}$$

Correct Solution:





- Find the total cost of the recursion tree

$$\begin{aligned}
1 &= \frac{n}{2^i} && (17)\\
2^i &= n && (18)\\
i &= \lg n && (19)
\end{aligned}$$

- Finding the upper bound of $T(n)$

$$
\begin{aligned}
T(n) &= \sum_{i=0}^{\lg n-1} (2 \cdot 4^i + n2^i) + \Theta(n^2) && (20)\\
&= \sum_{i=0}^{\lg n-1} 2 \cdot 4^i + \sum_{i=0}^{\lg n-1} n2^i + \Theta(n^2) && (21)\\
&= 2 \cdot \sum_{i=0}^{\lg n-1} 4^i + n \cdot \sum_{i=0}^{\lg n-1} 2^i + \Theta(n^2) && (22)\\
&= 2 \cdot \left(\frac{4^{\lg n} - 1}{4-1}\right) + n \cdot (n-1) + \Theta(n^2) \quad \left[\text{Since } \sum_{i=0}^{n-1} ar^i = a \cdot \frac{r^n - 1}{r-1}, \text{ where } r \neq 1\right] && (23)\\
&= \frac{2}{3} \cdot (n^2 - 1) + n \cdot (n-1) + \Theta(n^2) && (24)\\
&= \Theta(n^2) && (25)
\end{aligned}
$$

- Verify the correctness of guess using subtitution method

  Guess: $T(n) \leq cn^2 - dn$.

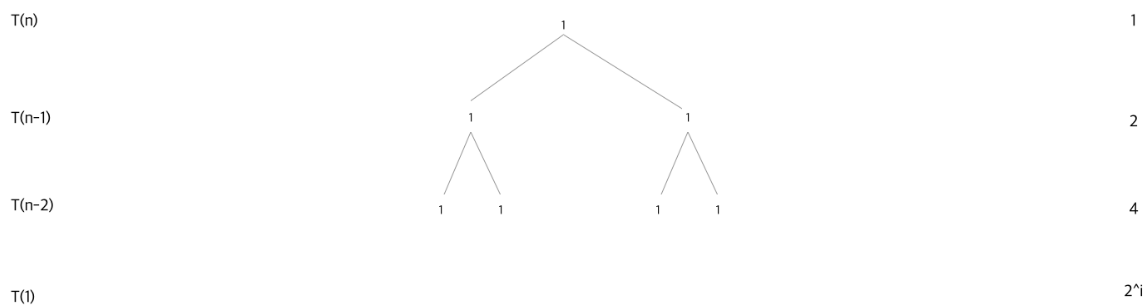
  I need to show the guess holds for the recurrence $T(n) = 4T(\frac{n}{2} + 2) + n$

$$
\begin{aligned}
T(n) &= 4T(\frac{n}{2} + 2) + n && (26)\\
&\leq 4c(\frac{n}{2} + 2)^2 - 4dn + n && (27)\\
&= 4c(\frac{n^2}{4} + 2n + 4) - 4dn + n && (28)\\
&\leq cn^2 - 4dn + n \quad [\text{Since } n^2 \text{ dominates } n \text{ asymptotically}] && (29)\\
&\leq cn^2 - 4dn + 3dn && (30)\\
&= cn^2 - dn && (31)
\end{aligned}
$$

Notes:

- The solution has $4^{\lg n} = n^2$. I noticed the same for $3^{\lg n} = n^3$. I had trouble looking for relevant formulas. Is this true in general? I can I replace variables in powers with the base?
- Noticed that in solution, the total cost is found for each term in $T(\frac{n}{2} + 2)$ (i.e. first for $\frac{n}{2}$ and second for 2). and then combined together in the end.

11. **Solution:**



- Finding the depth of tree

$$n - 1 \tag{1}$$

- Finding the number of leaves in the tree

$$\text{number of branchings}^{\text{depth of tree}} = 2^{n-1} \tag{2}$$

- Finding the upper bound of $T(n)$

$$\begin{aligned}
T(n) &\leq \sum_{i=0}^{n-1} 2^i + \Theta(2^n) && (3)\\
&= \left(\frac{2^n - 1}{2 - 1}\right) + \Theta(2^n) && (4)\\
&= (2^n - 1) + \Theta(2^n) && (5)\\
&= \Theta(2^n) && (6)
\end{aligned}$$

- Verify the correctness of guess using subtitution method

  Guess: $T(n) \geq c2^n$

  I need to show the bound holds for $T(n) = 2T(n-1) + 1$.

  Indeed we have

$$\begin{aligned}
T(n) &= 2T(n-1) + 1 && (7)\\
&< 2c2^{n-1} + 1 && (8)\\
&= c2^n + 1 && (9)\\
&= c2^n && \text{[Since } n \text{ is asympt. large]} \quad (10)
\end{aligned}$$

And the boundary holds when $c \geq 1$.

Notes:

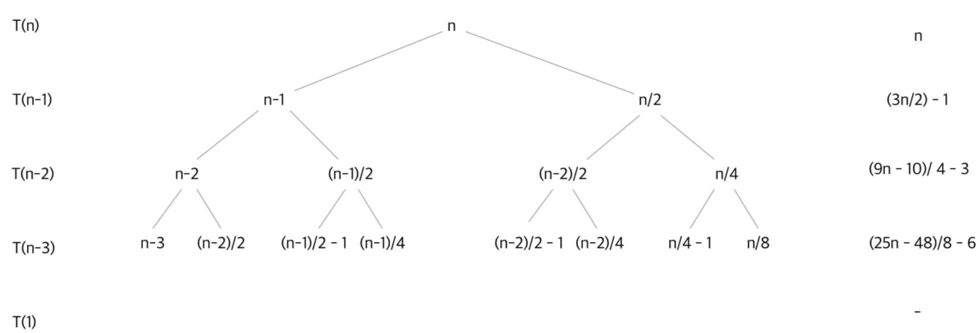
- If constant term in T exists, but The term after $T()$ is constant, then it's ignored. It is considered when it's in terms of n.
- Calculating the number of leaves

$$\text{number of branchings}^{\text{depth of tree}} \tag{11}$$

Example:
2^{n-1} (in above example)

12. **Solution:**

I will solve only the upper bound for now.



1. Find the depth of longest simple path in recursion tree

   The longest simple path is created by $T(n-1)$ and has depth of 2^{n-1}.

2. Find the number of leaves expecting a full binary tree of the same depth

   Here, the number of leaves is at most:

$$\text{number of branchings}^{\text{depth of tree}} = 2^{2^{n-1}} \tag{1}$$

3. Find the upper bound of $T(n)$ that produces most depth

$$\begin{align}
\sum_{i=0}^{n-1} 2^i + ... &= \left(\frac{2^n - 1}{2 - 1}\right) + ... \tag{2}\\
&= (2^n - 1) + ... \tag{3}\\
&= \mathcal{O}(2^n) \tag{4}
\end{align}$$

4. Valiate the upper bound using the subtitution method

   <u>Guess:</u> $T(n) \le c2^n - 2dn$

   I need to show the guess holds for the recurrence $T(n) = T(n-1) + T(\frac{n}{2}) + n$.

   And indeed we have

$$\begin{align}
T(n) &= T(n-1) + T(\frac{n}{2}) + n \tag{5}\\
&\le c2^{n-1} - 2d(n-1) + c2^{\frac{n}{2}} - 2\left(\frac{dn}{2}\right) + n \tag{6}\\
&= c2^{n-1} - 2d(n-1) + c2^{\frac{n}{2}} - dn + n \tag{7}\\
&\le c2^{n-1} - 2d(n-1) + c2^{\frac{n}{2}} - dn + dn \tag{8}\\
&= c2^{n-1} - 2d(n-1) + c2^{\frac{n}{2}} \tag{9}\\
&= c2^{n-1} - 2d(n-1) && \text{[Since } c2^{n-1} \text{ dominates } c2^{\frac{n}{2}}\text{]} \tag{10}\\
&= c2^n - 2dn && \text{[Since } n \text{ dominates } -1\text{]} \tag{11}\\
&\le c2^n - dn \tag{12}
\end{align}$$

   And the bound holds when $c \ge 1$ (not too sure) and $d \ge 1$.

Notes:

- Solving recurrence with uneven recursion tree

  Example: $T(n) = T(\frac{n}{3}) + T(\frac{2n}{3}) + \mathcal{O}(n)$

  1. Find the depth of longest simple path in recursion tree

     The longest simple path is created in $T(\frac{2n}{3})$. With the depth of $i = \log_{3/2} n$.

  2. Find the number of leaves expecting a full binary tree of the same depth

     Here, the number of leaves is $\mathcal{O}(n)$.

  3. Find the upper bound of $T(...)$ that produces most depth

     $$\mathcal{O}(\text{cost at depth} \times \text{depth}) = \mathcal{O}(cn \log_{3/2} n) = \mathcal{O}(n \lg n)$$

     – $\mathcal{O}(cn \log_{3/2} n) \to \mathcal{O}(n \lg n)$ since $\frac{3}{2} < 2$ (There seems to be a lot of sloppiness)

  4. Valiate the upper bound using the subtitution method

     $$\begin{aligned}
T(n) &\leq T(\frac{n}{3}) + T(\frac{2n}{3}) + cn && (13)\\
&\leq d(\frac{n}{3}) \cdot \lg(\frac{n}{3}) + d(\frac{2n}{3}) \lg(\frac{2n}{3}) + cn && (14)\\
&= (d(\frac{n}{3}) \lg n - d(\frac{n}{3}) \cdot \lg 3) + (d(\frac{2n}{3}) \lg n - d(\frac{2n}{3}) \lg(\frac{3}{2}) + cn) && (15)\\
&= dn \lg n - d((\frac{n}{3} \lg 3) + (\frac{2n}{3}) \lg(3/2)) + cn && (16)\\
&= dn \lg n - d((\frac{n}{3}) \lg 3 + (\frac{2n}{3}) \lg(3) - (\frac{2n}{3}) \lg(2)) + cn && (17)\\
&= dn \lg n - dn(\lg 3 - \frac{2}{3}) + cn && (18)\\
&\leq dn \lg n && (19)
\end{aligned}$$

     And the above is true as long as $d \geq \frac{c}{\lg 3 - \frac{2}{3}}$

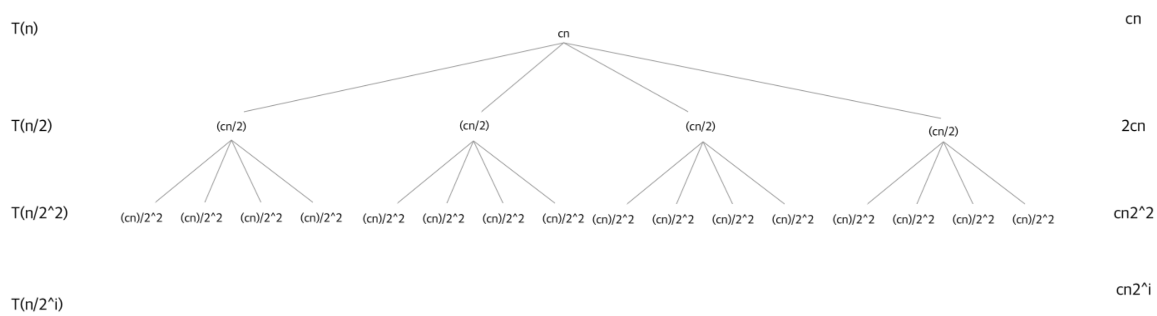
- I don't feel too sure about how to calculate the number of leaf nodes.

13. The shortest simple path from the root occurs in $T(\frac{n}{3})$ with the value of $i = \log_3 n$.

    The figure 4.6 tells us each level in the recurrence tree has cost of cn.

    Since the solution to the recurrence is at least the number of levels times the cost of each level, the solution is $\Omega(cn \log_3 n) = \Omega(\frac{cn \lg n}{\lg 3}) = \Omega(n \lg n)$.

14. **Solution:**



1. Finding the depth of tree

The longest simple path from the root to a leaf is $n \to \frac{n}{2} \to \frac{n}{4} \to \cdots \to 1$.

Since $(\frac{n}{2^i} = 1)$ when $i = \lg n$, the height of the tree is $\lg n$.

2. Finding the cost at each level in the tree

Each level has four times more nodes than the level above.

So, the number of nodes at depth i is 4^i.

Now, each node at $i = 0, ..., \lg(n) - 1$ has cost of $\frac{cn}{2^i}$.

So, by multiplying together, the cost of all nodes at depth i is $cn2^i$

3. Finding the cost of leaf nodes

The bottom level, at depth $\lg n$ has $n^{\lg 4} = n^2$ nodes with the cost of $n^2 T(1)$ or $\Theta(n^2)$.

4. Finding the total cost of $T(n)$, or the tight asymptotic bound

$$\begin{aligned} T(n) &\leq \sum_{i=0}^{\lg n - 1} cn2^i + \Theta(n^2) && (1)\\ &= cn\Big(\frac{2^{\lg n} - 1}{2 - 1}\Big) + \Theta(n^2) && (2)\\ &= cn(n-1) + \Theta(n^2) && (3)\\ &= \Theta(n^2) && (4) \end{aligned}$$

Thus, the tight asymptotic bound is $\Theta(n^2)$.

5. Verifying the upper bound

Let the guess be $T(n) \leq dn^2 - en$.

I need to show the guess holds for the recurrence $T(n) = 4T(\lfloor n/2 \rfloor) + cn$.

Indeed we have

$$\begin{aligned}
T(n) &= 4T(\lfloor n/2 \rfloor) + cn && (5)\\
&\leq 4d\lfloor \frac{n}{2} \rfloor^2 - 4e\lfloor \frac{n}{2} \rfloor + cn && (6)\\
&\leq 4d\Big(\frac{n}{2}\Big)^2 - 4e\Big(\frac{n}{2} - 1\Big) + cn && (7)\\
&= 4d\Big(\frac{n^2}{4}\Big) - e\Big(2n - 4\Big) + cn && (8)\\
&= dn^2 - e\Big(2n - 4\Big) + cn && (9)\\
&= dn^2 - e\Big(2n - 4\Big) + cn && \text{[since } n \text{ is asympt. large]} \quad (10)\\
&= dn^2 - e2n + cn && (11)\\
&= dn^2 - n(e2 - c) && (12)\\
&\leq dn^2 - ne && (13)
\end{aligned}$$

as long as $c \geq e$ and $d \geq 1$.

6. Verifying the lower bound

Let the guess be $d(n+2)^2 \leq T(n)$.

I need to show the guess holds for the recurrence $T(n) = 4T(\lfloor n/2 \rfloor) + cn$.

Indeed we have

$$
\begin{aligned}
T(n) &= 4T(\lfloor n/2 \rfloor) + cn && (14)\\
&\geq 4d(\lfloor \frac{n}{2} \rfloor + 2)^2 + cn && (15)\\
&\geq 4d(\frac{n}{2} - 1 + 2)^2 + cn && (16)\\
&= 4d\Big(\frac{n}{2} + 1\Big)^2 + cn && (17)\\
&= d\Big(n + 2\Big)^2 + cn && (18)\\
&\geq d(n+2)^2 && (19)
\end{aligned}
$$

as long as $c \geq 0$ and $d \geq 1$.

15. a) Here we have $a = 2, b = 4, f(n) = 1$. Since $1 = n^0 = n^{\log_4(2) - \log_4(2)}$ where $\epsilon = \log_4(2)$, the case 1 of master's theorem applies, and $T(n) = \Theta(n^{\log_4 2})$

Notes:

- Master method
  - The master method provides a "cookbook" method for solving recurrences of the form

$$T(n) = aT(\frac{n}{b}) + f(n) \tag{1}$$

    where $a \geq 1$ and $b > 1$ are constants and $f(n)$ is asymptotically positive function.
  - Allows to solve problems without pencil or paper
- Master Theorem
  - Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

    $T(n) = aT(\frac{n}{b}) + f(n)$

    where we interpret $\frac{n}{b}$ to be either $\lfloor \frac{n}{b} \rfloor$ or $\lceil \frac{n}{b} \rceil$. Then $T(n)$ has the following asymptotic bounds:
    1. if $f(n) = \mathcal{O}(n^{\log_b(a-\epsilon)})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
    2. if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$
    3. if $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$ and if $af(\frac{n}{b}) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n = \Theta(f(n)))$.

- Comparing $f(n)$ with the function $n^{\log_b a}$, the larger of two functions determine the solution to the recurrence.

  Example:

  $T(n) = 9T(\frac{n}{3}) + n$

  Here $a = 9, b = 3, f(n) = n$. Since $f(n) = \mathcal{O}(n^{\log_3 9-\epsilon})$ where $\epsilon = 1$, the case 1 of master theorem tells us $T(n) = \Theta(n^2)$.

  Example #2:

  $T(n) = T(\frac{2n}{3}) + 1$

  Here, $a = 1, b = \frac{3}{2}, f(n) = 1$. Since $f(n) = 1 = n^0 = n^{\log_{3/2} 1}$, the case 2 of master theorem applies and $T(n) = \Theta(\lg n)$.

b) Here $a = 2, b = 4$ and $f(n) = n^{\frac{1}{2}}$. Since $f(n) = n^{\frac{1}{2}} = n^{\log_b a} = n^{\log_4 2}$, the case 2 of master theorem applies and $T(n) = \Theta(n^{\log_4 2}) = \Theta(\sqrt{n})$.

Correct Solution:

Here $a = 2, b = 4$ and $f(n) = n^{\frac{1}{2}}$. Since $f(n) = n^{\frac{1}{2}} = n^{\log_b a} = n^{\log_4 2}$, the case 2 of master theorem applies and $T(n) = \Theta(n^{\log_4 2}) = \Theta(\sqrt{n}\lg n)$.

c) Here, we have $a = 2, b = 4, f(n) = n$ and $n^{\log_4 2} = \mathcal{O}(n^{0.5})$.

Since $f(n) = \Omega(n^{\log_4 2+\epsilon})$, where $\epsilon = 0.5$, and $af(\frac{n}{b}) = 2f(\frac{n}{4}) = \frac{n}{2} \leq cn$ where $c = \frac{1}{2}$ for sufficiently large n, the case 3 of master theorem applies.

Thus, we have $T(n) = \Theta(n)$.

d) Here, we have $a = 2, b = 4, f(n) = n^2$ and $n^{\log_4 2} = \mathcal{O}(n^0.5)$.

Since $f(n) = \Omega(n^{\log_4 2+\epsilon})$, where $\epsilon = 1.5$, and $af(\frac{n}{b}) = 2f(\frac{n}{4}) = \frac{n^2}{8} \leq cn$ where $c = \frac{1}{8}$ for sufficiently large n, the case 3 of master theorem applies.

Thus, we have $T(n) = \Theta(n^2)$.

16. Here $a = 1, b = 2$, and $f(n) = 1$.

Since $f(n) = 1 = n^0 = n^{\log_2 1} = n^{\log_b a}$, case 2 of master's theorem applies.

Thus, we have $T(n) = \Theta(n^{\log_2 1}\lg n) = \Theta(\lg n)$.

17. $f(n) = n \lg n$ cannot be represented in the form $n^{\log_b a + \epsilon}$.

Thus, master's method cannot be applied here.

So, to determine asymptotic upper bound, recurrence tree method + subtitution method is used.

Using the recurrence tree method, the recurrence tree has depth of $\lg n$, the level cost of $n^2(\lg n - i)$ where $i = 0, 1, ..., \lg n - 1$, and the leaf cost of $\Theta(n^2)$.

So, the asymptotic upper bound of $T(n)$ is:

$$\begin{aligned}
T(n) &= \sum_{i=0}^{\lg n - 1} n^2(\lg n - i) + \Theta(n^2) && (1)\\
&= \Big[\sum_{i=0}^{\lg n - 1}\Big] n^2 \lg n - n^2 \sum_{i=0}^{\lg n - 1} i + \Theta(n^2) && (2)\\
&= n^2 \lg^2 n - n^2 \cdot \Big(\frac{\lg n(\lg n - 1)}{2}\Big) + \Theta(n^2) && (3)\\
&= n^2 \lg^2 n - \frac{n^2}{2} \cdot (\lg^2 n - \lg n) + \Theta(n^2) && (4)\\
&= \mathcal{O}(n^2 \lg^2 n) && (5)
\end{aligned}$$

And we verify it using subtitution method.

Let the guess be $T(n) \leq cn^2 \lg^2 n$.

I need to show the guess holds in the recurrence $T(n) = 4T(\frac{n}{2}) + n^2 \lg n$.

And indeed, we have

$$\begin{aligned}
T(n) &= T(\frac{n}{2}) + n^2 \lg(n) && (6)\\
&\leq 4c(\frac{n^2}{4}) \lg^2(\frac{n}{2}) + n^2 \lg(n) && (7)\\
&= cn^2 \lg^2(\frac{n}{2}) + n^2 \lg(n) && (8)\\
&\leq cn^2 \lg^2(n) + n^2 \lg(n) && (9)\\
&\leq cn^2 \lg^2(n) + n^2 \lg(n) && (10)\\
&= cn^2 \lg^2(n) && \text{[Since } cn^2 \lg^2(n) \text{ dominates } n^2 \lg(n)\text{]} \quad (11)
\end{aligned}$$

as long as $c \geq 1$.

Correct Solution:

$f(n) = n \lg n$ cannot be represented in the form $n^{\log_b a + \epsilon}$.

Thus, master's method cannot be applied here.

So, to determine asymptotic upper bound, recurrence tree method + subtitution method is used.

Using the recurrence tree method, the recurrence tree has depth of $\lg n$, the level cost of $n^2(\lg n - i)$ where $i = 0, 1, ..., \lg n - 1$, and the leaf cost of $\Theta(n^2)$.

So, the asymptotic upper bound of $T(n)$ is:

$$
\begin{align}
T(n) &= \sum_{i=0}^{\lg n - 1} n^2(\lg n - i) + \Theta(n^2) \tag{1}\\
&= \Big[\sum_{i=0}^{\lg n - 1}\Big] n^2 \lg n - n^2 \sum_{i=0}^{\lg n - 1} i + \Theta(n^2) \tag{2}\\
&= n^2 \lg^2 n - n^2 \cdot \Big(\frac{\lg n(\lg n - 1)}{2}\Big) + \Theta(n^2) \tag{3}\\
&= n^2 \lg^2 n - \frac{n^2}{2} \cdot (\lg^2 n - \lg n) + \Theta(n^2) \tag{4}\\
&= \mathcal{O}(n^2 \lg^2 n) \tag{5}
\end{align}
$$

And I will verify it using subtitution method.

Let the guess be $T(n) \leq cn^2 \lg^2 n$.

I need to show the guess holds in the recurrence $T(n) = 4T(\frac{n}{2}) + n^2 \lg n$.

And indeed, we have

$$
\begin{aligned}
T(n) &= T(\frac{n}{2}) + n^2 \lg(n) && (6)\\
&\leq 4c(\frac{n^2}{4}) \lg^2(\frac{n}{2}) + n^2 \lg(n) && (7)\\
&= cn^2(\lg(n) - 1) \lg(\frac{n}{2}) + n^2 \lg n && (8)\\
&= cn^2 \lg(n) \lg(\frac{n}{2}) - cn^2 \lg(\frac{n}{2}) + n^2 \lg n && (9)\\
&= cn^2 \lg(n) \lg(\frac{n}{2}) - cn^2(\lg n - 1) + n^2 \lg n && (10)\\
&= cn^2 \lg(n) \lg(\frac{n}{2}) - cn^2(\lg n + cn^2 + n^2 \lg n && (11)\\
&= cn^2 \lg(n) \lg(\frac{n}{2}) - n^2 \lg n(c - 1) + cn^2 && (12)\\
&= cn^2(\lg(n) \lg(\frac{n}{2}) + 1) - n^2 \lg n(c - 1) && (13)\\
&= cn^2 \lg(n) \lg(\frac{n}{2}) - n^2 \lg n(c - 1) \qquad \text{[Since } \lg(n) \lg(\tfrac{n}{2}) \text{ over 1]} && (14)\\
&\leq cn^2 \lg^2(n) - n^2 \lg n(c - 1) && (15)\\
&\leq cn^2 \lg^2(n) && (16)
\end{aligned}
$$

as long as $c \geq 1$.