CSC236 Term Test 1 Version 2 Review

Hyungmo Gu

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Question 1

- *Proof.* Define $P(n) : f(n) = 3^n$.

  I will use complete induction to prove that $\forall n \in \mathbb{N}, P(n)$.

  Base Case ($n = 0$):

  Let $n = 0$.

  Then,

$$\begin{aligned} f(n) &= 1 && \text{[By def.]} && (1) \\ &\leq 3^0 && && (2) \\ &= 3^n && && (3) \end{aligned}$$

  Thus, $P(n)$ follows in this step.

  Base Case ($n = 1$):

  Let $n = 1$.

  Then,

$$\begin{aligned} f(n) &= 1 && \text{[By def., since } n = 1\text{]} && (4) \\ &\leq 3^1 && && (5) \\ &= 3^n && && (6) \end{aligned}$$

  Thus, $P(n)$ follows in this step.

Base Case ($n = 2$):

Let $n = 2$.

Then,

$$
\begin{aligned}
f(n) &= 9 && \text{[By def., since } n = 2\text{]} && (7)\\
&\leq 3^2 && && (8)\\
&= 3^n && && (9)
\end{aligned}
$$

Thus, $P(n)$ follows in this step.

Base Case ($n = 3$):

Let $n = 3$.

Then,

$$
\begin{aligned}
f(n) &= f(n-1) + 3f(n-2) + 9f(n-3) && \text{[By def., since } n = 2\text{]} && (10)\\
&= 9 + 3 \cdot 3 + 9 \cdot 1 && \text{[By def., since } n-1 = 2,\ n-2 = 1,\ n-3 = 0\text{]} && (11)\\
&= 3^2 + 3^2 + 3^2 && && (12)\\
&= 3^3 && && (13)\\
&= 3^n && && (14)\\
&\leq 3^n && && (15)
\end{aligned}
$$

Thus, $P(n)$ follows in this step.

Case ($n > 3$):

Let $n > 3$.

Then, since $0 \leq n-3 < n-2 < n-1 < n$, $P(n-3)$, $P(n-2)$, $P(n-1)$ holds by induction hypothesis. That is, $P(n-3) \leq 3^{n-3}$, $P(n-2) \leq 3^{n-2}$, $P(n-1) \leq 3^{n-1}$.

Thus,

$$
\begin{aligned}
f(n) &= f(n-1) + 3f(n-2) + 9f(n-3) && \text{[By def., since } n > 2\text{]} \quad (16)\\
&\leq 3^{n-1} + 3 \cdot 3^{n-2} + 9 \cdot 3^{n-3} && \text{[By header]} \quad (17)\\
&= 3^{n-1} + 3^{n-1} + 3^{n-1} && (18)\\
&= 3^n && (19)
\end{aligned}
$$

So, $P(n)$ follows from $H(n)$ in this step.

$\square$

Question 2

- *Proof.* Define for convenience

  $P(x, y, z, w)$: There are no positive integers x, y, z, w such that $x^4 + 3y^4 + 9z^4 = 27w^4$.

  I will prove $P(x, y, z, w)$ by contradiction.

  Assume $\neg P(x, y, z, w)$. That is, $\exists x, y, z, w \in \mathbb{N}^+$, $x^4 + 3y^4 + 9z^4 = 27w^4$.

  Then, $X = \{x \in \mathbb{N}^+ \mid \exists y, z, w \in \mathbb{N}^+, x^4 + 3y^4 + 9z^4 = 27w^4\}$ is not empty.

  Then, by the principle of well-ordering, X has smallest element.

  Let $x_0 \in X$ be its smallest element, and let $y_0, z_0, w_0 \in \mathbb{N}^+, x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4$.

  Then,

$$
\begin{aligned}
x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 &\Rightarrow x_0 = 27w_0^4 - 3y_0^4 + 9z_0^4 && (20)\\
&\Rightarrow x \mid x_0^4 && (21)\\
&\Rightarrow 3 \mid x_0 \qquad \text{[By hint, since 3 is prime]} && (22)\\[1em]
\text{Let } x_1 \in \mathbb{N}^+,\ x_0 = 3x_1 &\Rightarrow 3^4x_1^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 && (23)\\
&\Rightarrow 3y_0^4 = 27w_0^4 - 9z_0^4 - 3^4x_1^4 && (24)\\
&\Rightarrow y_0^4 = 9w_0^4 - 3z_0^4 - 3^3x_1^4 && (25)\\
&\Rightarrow 3 \mid y_0^4 && (26)\\
&\Rightarrow 3 \mid y_0 \qquad \text{[By hint, since 3 is prime]} && (27)\\[1em]
\text{Let } y_1 \in \mathbb{N}^+,\ y_0 = 3y_1 &\Rightarrow 3^4x_1^4 + 3^5y_1^4 + 9z_0^4 = 27w_0^4 && (28)\\
&\Rightarrow 9z_0^4 = 27w_0^4 - 3^5y_1^4 - 3^4x_1^4 && (29)\\
&\Rightarrow z_0^4 = 3w_0^4 - 3^3y_1^4 - 3^2x_1^4 && (30)\\
&\Rightarrow 3 \mid z_0^4 && (31)\\
&\Rightarrow 3 \mid z_0 \qquad \text{[By hint, since 3 is prime]} && (32)\\[1em]
\text{Let } z_1 \in \mathbb{N}^+,\ z_0 = 3z_1 &\Rightarrow 3^4x_1^4 + 3^5y_1^4 + 3^6z_1^4 = 27w_0^4 && (33)\\
&\Rightarrow 3x_1^4 + 3^2y_1^4 + 3^3z_1^4 = w_0^4 && (34)\\
&\Rightarrow 3 \mid w_0^4 && (35)\\
&\Rightarrow 3 \mid w_0 \qquad \text{[By hint, since 3 is prime]} && (36)\\[1em]
\text{Let } w_1 \in \mathbb{N}^+,\ w_0 = 3w_1 &\Rightarrow 3^4x_1^4 + 3^5y_1^4 + 3^6z_1^4 = 3^7w_1^4 && (37)\\
&\Rightarrow x_1^4 + 3y_1^4 + 3^2z_1^4 = 3^3w_1^4 && (38)\\
&\Rightarrow x_1 \in X && (39)
\end{aligned}
$$

Then, since $x_1 < x_0$ and $x_1 \in X$, but x_0 is be the smallest element in X, this leads to contradiction.

Thus, the assumption is false, and $P(x, y, z, w)$ holds. $\square$

Question 3

- *Proof.* For convenience, define $P(t)$: left(t) is odd.

  I will use structural induction to prove that $\forall t \in \mathcal{T}, P(t)$.

  <u>Basis:</u>

  Let $t =$ '$()'$.

  Then, left$(t) = 1$, which is odd.

  Thus, $P(t)$ follows in this step.

  <u>Inductive Step:</u>

  Let $t_1, t_2 \in \mathcal{T}$ be arbitrary elements. Assume $P(t_1)$ and $P(t_2)$. That is left(t_1) and left(t_2) are odd. In other words, $\exists k_1, k_2 \in \mathbb{Z}$, left$(t_1) = 2k_1 + 1$ and left$(t_2) = 2k_2 + 1$. Let $(t_1 t_2)$ be parenthesized concatenated string in $\mathcal{T}$.

  I need to show $P((t_1 t_2))$ follows. That is, left$((t_1 t_2))$ is odd. In other words, $\exists k \in \mathbb{Z}$, left$((t_1 t_2)) = 2k + 1$.

  Let $k = k_1 + k_2 + 1$.

  Then, starting from left$((t_1 t_2))$, we have

  $$\begin{aligned} \text{left}((t_1 t_2)) &= \text{left}((t_1)) + \text{left}((t_2)) + 1 && \text{[Adding initial `(' plus } t_1 t_2 \text{ to left\_count]} \quad (1) \\ &= (2k_1 + 1) + (2k_2 + 1) + 1 && \text{[By header]} \quad (2) \\ &= 2(k_1 + k_2 + 1) + 1 && (3) \\ &= 2k + 1 && \text{[Since } k = k_1 + k_2 + 1\text{]} \quad (4) \end{aligned}$$

  Thus, $P((t_1 t_2))$ follows in this step. $\square$