CSC236 Worksheet 4 Solution

Hyungmo Gu

May 7, 2020

Question 1

- Let $n \in \mathbb{N}$ and assume that $\exists k \in \mathbb{N}^+$, $n = 3^k$, so $k = \log_3 n$.

  Then, since $n = 3^k$ and $3 \mid n$, we have $\lceil n/3 \rceil = n/3$.

  Then,

$$
\begin{aligned}
T(n) &= 2n + T(n/3) && \text{[By def.]} && (1)\\
&= 2(n/3) + (2(n/3) + T(n/3^2)) && \text{[By subtituting n/3 for n in def.]} && (2)\\
&= 2^2(n/3) + T(n/3^2) && && (3)\\
&= 2^3(n/3^2) + T(n/3^3) && \text{[By subtituting n/3 for n in def.]} && (4)\\
&\vdots && && (5)\\
&= 2^k(n/3^{k-1}) + T(n/3^k) && \text{[After } k \text{ applications]} && (6)\\
&= 2^{\log_3 n}(n/3^{log_3 n-1}) + T(n/3^{\log_3 n}) && \text{[By replacing } k = \log_3 n] && (7)\\
&= 2^{\log_3 n}(n(3)/n) + T(n/n) && && (8)\\
&= 3 \cdot 2^{\log_3 n} + T(1) && && (9)\\
&= 3 \cdot 2^{\log_3 n} + 2 && && (10)
\end{aligned}
$$

Correct Solution:

Let $n \in \mathbb{N}$ and assume that $\exists k \in \mathbb{N}^+$, $n = 3^k$, so $k = \log_3 n$.

Then, since $n = 3^k$ and $3 \mid n$, we have $\lceil n/3 \rceil = n/3$.

Then,

$$
\begin{aligned}
T(n) &= 2n + T(n/3) && \text{[By def.]} && (11)\\
&= 2n + 2(n/3) + T(n/3^2) && \text{[By subtituting n/3 for n in def.]} && (12)\\
&= 2n + 2(n/3) + 2(n/3^2) + T(n/3^3) && \text{[By subtituting n/3 for n in def.]} && (13)\\
&\vdots && && (14)\\
&= 2\sum_{i=0}^{k-1} n/3^i + T(n/3^k) && && (15)\\
&= 2 \cdot 3^k \Big(\frac{1-(1/3)^k}{1-1/3}\Big) + T(n/3^k) && \text{[By using geometric series]} && (16)\\
&= 2 \cdot 3^k \cdot 3/2\Big(1-(1/3)^k\Big) + T(n/n) && && (17)\\
&= 3(3^k - 1) + T(1) && && (18)\\
&= 3^{k+1} - 1 && && (19)
\end{aligned}
$$

Notes:

- **Repeated Subtitution:**
  - Is a technique used to find a closed form formula
  - **closed form formula** is a simple formula that allows evaluation of $T(n)$ without the need to evaluate, say $T(n/2)$

    i.e. from

$$
T(n) = \begin{cases} c & \text{if } n = 1 \\ 2T(n/2) + dn & \text{if } n > 1 \end{cases} \tag{20}
$$

    to

    $T(n) = cn + dn \log_2 n$

Example:

Consider the recurrence

$$
T(n) = \begin{cases} c & \text{if } n = 1 \\ 2T(n/2) + dn & \text{if } n > 1 \end{cases} \tag{1}
$$

Find closed form formula for $T(n)$, where n is an arbitrary power of 2. That is

$\exists k \in \mathbb{N}, n = 2^k$.

Let $n \in \mathbb{N}$ and assume that $\exists k \in \mathbb{N}^+$, $n = 2^k$, so $k = \log_2 n$.

Then,

$$
\begin{align}
T(n) &= 2T(n/2) + dn && \text{[By 1]} \tag{2}\\
&= 2\Big(2T(n/2^2) + dn/2\Big) + dn && \text{[By subtituting } n/2 \text{ for } n \text{ in 1]} \tag{3}\\
&= 2^2T(n/2^2) + 2dn \tag{4}\\
&= 2^2\Big(2T(n/2^3) + dn/2^2\Big) + 2dn && \text{[By subtituting } n/2^2 \text{ for } n \text{ in 1]} \tag{5}\\
&= 2^3T(n/2^3) + 3dn && \text{[By subtituting } n/2^2 \text{ for } n \text{ in 1]} \tag{6}\\
&\vdots \tag{7}\\
&= 2^kT(n/2^k) + kdn && \text{[After } k \text{ applications]} \tag{8}\\
&= 2^{\log_2 n}T(n/2^{\log_2 n}) + (\log_2 n)dn && \text{[By replacing } k = \log_2 n\text{]} \tag{9}\\
&= nT(1) + (\log_2 n)dn \tag{10}\\
&= cn + (\log_2 n)dn \tag{11}
\end{align}
$$

Question 2

- Let $n \in \mathbb{N}$. Assume $\exists k \in \mathbb{N}^+$, $n = 3^k$, so $\log_3 n = k$.

  Then, because we know $3 \mid 3^k$, we can write $\lceil n/3 \rceil = n/3$.

  Then,

$$
\begin{aligned}
R(n) &= n + 3R(n/3) && \text{[By def.]} \quad (12)\\
&= n + (n/3 + 3R(n/3^2)) && \text{[By subtituting } n/3 \text{ for } n \text{ in def.]} \quad (13)\\
&= n + n/3 + (n/3^2 + 3R(n/3^3)) && \text{[By subtituting } n/3 \text{ for } n \text{ in def.]} \quad (14)\\
&\vdots && (15)\\
&= \sum_{i=0}^{k-1} n/3^i + 3R(n/3^k)) && \text{[After k repeatitons]} \quad (16)\\
&= n\Big(\frac{1-1/3^k}{1-1/3}\Big) + 3R(n/3^k) && (17)\\
&= (3n)/2\Big(1-1/3^k\Big) + 3R(n/3^k) && \text{[By using geometric series]} \quad (18)\\
&= (3\cdot 3^k)/2(1-1/3^k) + 3R(3^k/3^k) && \text{[By subtituting } 3^k \text{ for } n\text{]} \quad (19)\\
&= 3/2(3^k-1) + 3R(1) && (20)\\
&= 3/2(3^k-1) + 3\cdot 0 && \text{[By def.]} \quad (21)\\
&= 3/2(3^k-1) && (22)
\end{aligned}
$$

Correct Solution:

Let $n \in \mathbb{N}$. Assume $\exists k \in \mathbb{N}^+$, $n = 3^k$, so $\log_3 n = k$.

Then, because we know $3 \mid 3^k$, we can write $\lceil n/3 \rceil = n/3$.

Then,

$$
\begin{aligned}
R(n) &= n + 3R(n/3) && \text{[By def.]} && (23)\\
&= n + 3(n/3 + 3R(n/3^2)) && \text{[By subtituting } n/3 \text{ for } n \text{ in def.]} && (24)\\
&= 2n + 3^2 R(n/3^2) && && (25)\\
&= 2n + 3^2(n/3^2 + 3R(n/3^3)) && \text{[By subtituting } n/3 \text{ for } n \text{ in def.]} && (26)\\
&= 3n + 3^3 R(n/3^3) && && (27)\\
&\vdots && && (28)\\
&= kn + 3^k R(n/3^k) && \text{[After k repeatitons]} && (29)\\
&= kn + 3^k R(1) && \text{[By subtituting } 3^k \text{ for } n \text{ in } R(n/3^k)\text{]} && (30)\\
&= kn + 3^k \cdot 0 && \text{[By def.]} && (31)\\
&= kn && && (32)\\
&= n \log_3 n && \text{[By subtituting } \log_3 n \text{ for } k\text{]} && (33)
\end{aligned}
$$