CSC236 Worksheet 5 Solution

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Question 1

a. Proof. For convenience, define $H(k): R(3^k) = 3^k k$. Notice that when $n = 3^k$, $3^k k$ is the same as $n \log_3 n$. I will use simple induction to prove $\forall k \in \mathbb{N}, \ H(k)$.

Base Case (k = 0):

Let k = 0.

Then,

$$3^k \cdot k = 3^0 \cdot 0 \tag{1}$$

$$=0 (2)$$

$$= R(n)$$
 [By def.] (3)

Thus, H(0) is verified.

Inductive Step:

Let $k \in \mathbb{N}$. Assume H(k). That is $R(3^k) = 3^k k$.

I will show that H(k+1) follows. That is, $R(3^{k+1}) = (k+1)3^{k+1}$.

The definition of $R(3^{k+1})$ tells us,

$$R(3^{k+1}) = 3^{k+1} + 3R(\lceil 3^{k+1}/3 \rceil) \tag{4}$$

$$=3^{k+1} + 3R(3^k) (5)$$

Then, using this fact, we can conclude

$$R(3^{k+1}) = 3^{k+1} + 3 \cdot 3^k k$$
 [By I.H] (6)

$$=3^{k+1}+3^{k+1}k\tag{7}$$

$$=3^{k+1}(k+1) (8)$$

Correct Solution:

For convenience, define $H(k): R(3^k) = 3^k k$. Notice that when $n = 3^k$, $3^k k$ is the same as $n \log_3 n$. I will use simple induction to prove $\forall k \in \mathbb{N}, \ H(k)$.

Base Case (k = 0):

Let k = 0.

Then,

$$3^k \cdot k = 3^0 \cdot 0 \tag{9}$$

$$=0 (10)$$

$$= R(n)$$
 [By def.] (11)

Thus, H(0) is verified.

Inductive Step:

Let $k \in \mathbb{N}$. Assume H(k). That is $R(3^k) = 3^k k$.

I will show that H(k+1) follows. That is, $R(3^{k+1}) = (k+1)3^{k+1}$.

Since k + 1 > 0, $3^{k+1} > 1$.

So the definition of $R(3^{k+1})$ tells us,

$$R(3^{k+1}) = 3^{k+1} + 3R(\lceil 3^{k+1}/3 \rceil)$$
(12)

$$=3^{k+1} + 3R(3^k) (13)$$

Then, using this fact, we can conclude

$$R(3^{k+1}) = 3^{k+1} + 3 \cdot 3^k k$$
 [By I.H] (14)

$$=3^{k+1}+3^{k+1}k\tag{15}$$

$$=3^{k+1}(k+1) (16)$$

Notes:

- Noticed that professor used the phrase 'Notice that when $n = 3^k$, $3^k k$ is the same as $n \log_3 n$.' to express n in terms of 3^k .
- I feel I should review this problem to make sure I understood.

b. Rough Works:

Define P(k) as

$$P(k): \bigwedge_{m=1}^{m=k} R(m) \le R(k)$$

I will prove $\forall n \in \mathbb{N}^+, P(n)$ using complete induction.

1. Inductive Step

Let $n \in \mathbb{N}$. Assume $n \geq 1$. Assume $H(n): \bigwedge_{i=1}^{n-1} P(i)$. I will prove that P(n) follows.

2. Base Case (n = 1)

Let
$$n=1$$
.

Then,
$$\bigwedge_{m=1}^{m=1} R(m) = R(1)$$
.

So, P(1) follows.

3. Base Case (n=2)

Let n=2.

I need to show P(2) holds. That is, $R(1) \leq R(2)$ and $R(2) \leq R(2)$.

I will do so in parts.

Part 1 $(R(1) \le R(2))$:

In this part, R(1) = 0 and $R(2) = 2 + R(\lceil 2/3 \rceil) = 2 + R(1) = 2$.

Since $0 \le 2$, we can conclude $R(1) \le R(2)$.

Part 2 $(R(2) \le R(2))$:

In this part, R(2) = R(2), so we can conclude $R(2) \le R(2)$.

- 4. Case (n > 2)
 - Show that only $R(n-1) \leq R(n)$ needs proving

Since n > 2, $1 \le n - 1 < n$, so we know P(n - 1) holds.

Then, it is suffice to prove $R(n-1) \leq R(n)$.

• Show R(n) holds, starting from R(n-1)

Starting from R(n-1), we have

$$R(n-1) = (n-1) + R(\lceil (n-1)/3 \rceil)$$

$$\leq n + R(\lceil (n-1)/3 \rceil)$$

$$\leq n + R(\lceil (n/3 \rceil))$$
[By def.]
$$(17)$$

$$(18)$$

$$\leq n + R(\lceil (n/3 \rceil))$$
[By I.H, Since $1 \leq \lceil (n-1)/3 \rceil \leq \lceil n/3 \rceil < n$]
$$(19)$$

(20)

Thus, P(n) holds.

= R(n)