CSC373 Worksheet 6 Solution

August 13, 2020

1. 1. Multiply objective function by - 1

    Maximize

    $$-2x_1 - 7x_2 - x_3$$

    Subject to

    $$x_1 - x_3 = 7$$
    $$3x_1 + x_2 \geq 7$$
    $$x_2 \geq 0$$
    $$x_3 \leq 0$$

    2. Replace non-nonnegative constraints x_1

    Maximize

    $$-2x_1' + 2x_1'' - 7x_2 - x_3$$

    Subject to

    $$x_1' - x_1'' - x_3 = 7$$
    $$3x_1' - 3x_1'' + x_2 \geq 7$$
    $$x_1', x_1'', x_2 \geq 0$$
    $$x_3 \leq 0$$

    3. Replace non-nonnegative constraints x_3

Maximize

$$-2x_1' + 2x_1'' - 7x_2 - x_3' + x_3''$$

Subject to

$$x_1' - x_1'' - x_3' + x_3'' = 7$$
$$3x_1' - 3x_1'' + x_2 \geq 7$$
$$x_1', x_1'', x_2, x_3', x_3'' \geq 0$$

4. Replace equality constraints with $\geq$ and $\leq$

Maximize

$$-2x_1' + 2x_1'' - 7x_2 - x_3' + x_3''$$

Subject to

$$x_1' - x_1'' - x_3' + x_3'' \leq 7$$
$$x_1' - x_1'' - x_3' + x_3'' \geq 7$$
$$3x_1' - 3x_1'' + x_2 \geq 7$$
$$x_1', x_1'', x_2, x_3', x_3'' \geq 0$$

5. Correct greater-than-or-equal-to inequality constraints

Maximize

$$-2x_1' + 2x_1'' - 7x_2 - x_3' + x_3''$$

Subject to

$$x_1' - x_1'' - x_3' + x_3'' \leq 7$$
$$-x_1' + x_1'' + x_3' - x_3'' \leq -7$$
$$-3x_1' + 3x_1'' - x_2 \leq 7$$
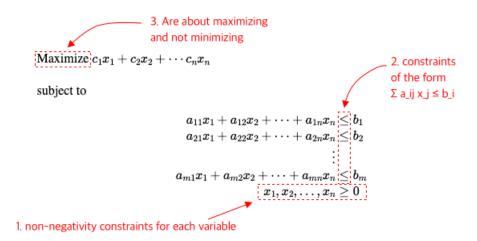$$x_1', x_1'', x_2, x_3', x_3'' \geq 0$$

Notes:

- **Linear Programming**
  - Is a method to achieve the best outcome (such as maximum profit or lowest cost) in a mathematical model whose requirements are represented by linear relationships. [1]
  - Is named to make it sound cool for government funding
    * Like dynamic programming
  - Applications
    * Microeconomics (maximize profits, minimize costs)
    * Company management
- **Standard Form**
  - Is a form of linear programming
  - Are about <u>maximizing</u>, not minimizing [2]
  - All have a positivity constraint for each variable [2]
  - All other constraints are all of the form "linear combination of variables $\leq$ constant". [2]

3. Are about maximizing and not minimizing

$\text{Maximize } c_1 x_1 + c_2 x_2 + \cdots c_n x_n$

subject to

2. constraints of the form $\Sigma\, a\_ij\, x\_j \leq b\_i$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m$$
$$x_1, x_2, \ldots, x_n \geq 0$$

1. non-negativity constraints for each variable

- **Converting Linear Programming to Standard Form**

1) The objective function might be a minimization rather than a maximization
   – Negate coefficients of the objective function
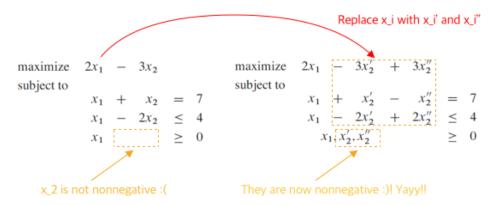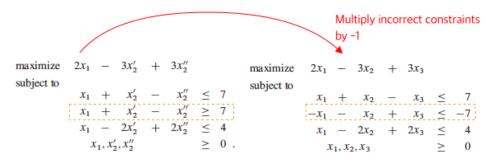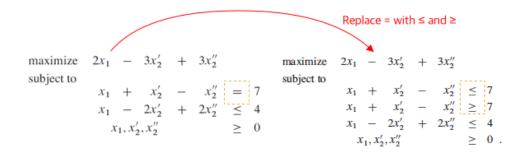
multiply by –1

minimize	$-2x_1$	$+$	$3x_2$

subject to

$$x_1 + x_2 = 7$$
$$x_1 - 2x_2 \leq 4$$
$$x_1 \qquad \geq 0$$

maximize	$2x_1$	$-$	$3x_2$

subject to

$$x_1 + x_2 = 7$$
$$x_1 - 2x_2 \leq 4$$
$$x_1 \qquad \geq 0$$

2) There might be variables without nonnegativity constraints
   – Replace each non-nonnegative variable x_i with x_i' and x_i''
   – Modify linear program

Replace x_i with x_i' and x_i"

maximize $2x_1 - 3x_2$

subject to

$$x_1 + x_2 = 7$$
$$x_1 - 2x_2 \leq 4$$
$$x_1 \qquad \geq 0$$

maximize $2x_1 - 3x_2' + 3x_2''$

subject to

$$x_1 + x_2' - x_2'' = 7$$
$$x_1 - 2x_2' + 2x_2'' \leq 4$$
$$x_1, x_2', x_2'' \qquad \geq 0$$

x_2 is not nonnegative :(

They are now nonnegative :)! Yayy!!

3) There might be **equality constraints**, which have an equal sign rather than a less-than-or-equal-to sign
   – Replace equality constraint $f(x_1, x_2, ..., x_n) = b$ with $f(x_1, x_2, ..., x_n) \leq b$ and $f(x_1, x_2, ..., x_n) \geq b$

Multiply incorrect constraints by –1

maximize $2x_1 - 3x_2' + 3x_2''$

subject to

$$x_1 + x_2' - x_2'' \leq 7$$
$$x_1 + x_2' - x_2'' \geq 7$$
$$x_1 - 2x_2' + 2x_2'' \leq 4$$
$$x_1, x_2', x_2'' \qquad \geq 0 .$$

maximize $2x_1 - 3x_2 + 3x_3$

subject to

$$x_1 + x_2 - x_3 \leq 7$$
$$-x_1 - x_2 + x_3 \leq -7$$
$$x_1 - 2x_2 + 2x_3 \leq 4$$
$$x_1, x_2, x_3 \qquad \geq 0$$

4) There might be **inequality constraints**, but instead of having a less-than-or-equal-to-sign
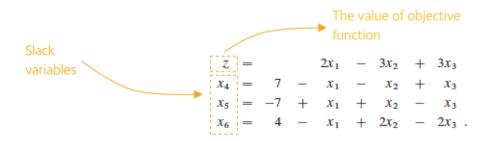   – Multiply incorrect inequality constraints by -1

Replace = with ≤ and ≥

$$
\begin{array}{llllll}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' \\
\text{subject to} & & & & & \\
& x_1 & + & x_2' & - & x_2'' & = & 7 \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& x_1, x_2', x_2'' & & & & & \geq & 0
\end{array}
$$

$$
\begin{array}{llllll}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' \\
\text{subject to} & & & & & \\
& x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
& x_1 & + & x_2' & - & x_2'' & \geq & 7 \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& x_1, x_2', x_2'' & & & & & \geq & 0 \;.
\end{array}
$$

References:

1) Wikipedia, Linear Programming, link

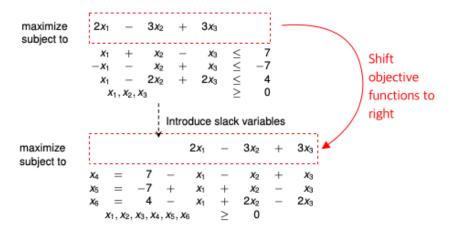2) Instituto de Mathematicas, Standard form for Linear Programs, link

2. **Rough Works:**

Notes:

- **Slack Form**
  - Is a form of linear programming
  - Is for efficient solving of liner programming problem using simplex algorithm

The value of objective function

Slack variables

$$
\begin{array}{llllll}
z & = & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
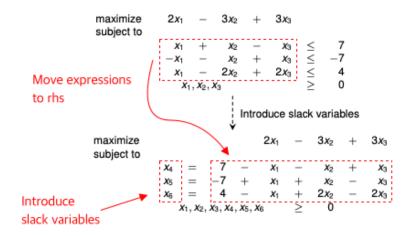x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \;.
\end{array}
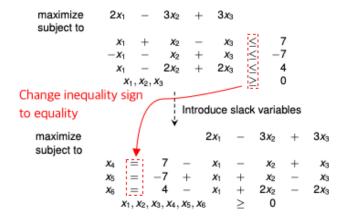$$

- **Converting Linear Programs into Slack Form**
  1) Start from the standard form of linear programming
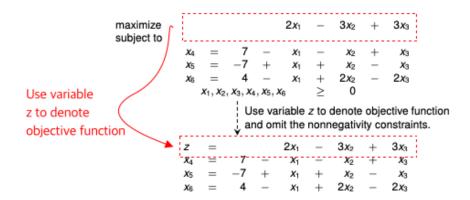 2) Shift objective functions to right

3) Introduce slack variable x_i to lhs and move expressions $\sum\limits_{j=1}^{n} a_{ij}x_j$ to rhs
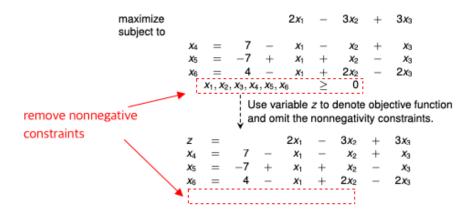

4) Change inequalities in linear programming to equality

5) Use Variable z to denote objective function

maximize $\qquad\qquad 2x_1 \quad - \quad 3x_2 \quad + \quad 3x_3$
subject to

$$
\begin{aligned}
x_4 &= 7 \; - \quad x_1 \; - \quad x_2 \; + \quad x_3 \\
x_5 &= -7 \; + \quad x_1 \; + \quad x_2 \; - \quad x_3 \\
x_6 &= 4 \; - \quad x_1 \; + \; 2x_2 \; - \; 2x_3 \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}
$$

Use variable z to denote objective function

Use variable z to denote objective function and omit the nonnegativity constraints.

$$
\begin{aligned}
z &= \quad 2x_1 \; - \; 3x_2 \; + \; 3x_3 \\
x_4 &= 7 \; - \quad x_1 \; - \quad x_2 \; + \quad x_3 \\
x_5 &= -7 \; + \quad x_1 \; + \quad x_2 \; - \quad x_3 \\
x_6 &= 4 \; - \quad x_1 \; + \; 2x_2 \; - \; 2x_3
\end{aligned}
$$

6) Omit the nonnegativivty constraints

maximize $\qquad\qquad 2x_1 \quad - \quad 3x_2 \quad + \quad 3x_3$
subject to

$$
\begin{aligned}
x_4 &= 7 \; - \quad x_1 \; - \quad x_2 \; + \quad x_3 \\
x_5 &= -7 \; + \quad x_1 \; + \quad x_2 \; - \quad x_3 \\
x_6 &= 4 \; - \quad x_1 \; + \; 2x_2 \; - \; 2x_3 \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}
$$

remove nonnegative constraints

Use variable z to denote objective function and omit the nonnegativity constraints.

$$
\begin{aligned}
z &= \quad 2x_1 \; - \; 3x_2 \; + \; 3x_3 \\
x_4 &= 7 \; - \quad x_1 \; - \quad x_2 \; + \quad x_3 \\
x_5 &= -7 \; + \quad x_1 \; + \quad x_2 \; - \quad x_3 \\
x_6 &= 4 \; - \quad x_1 \; + \; 2x_2 \; - \; 2x_3
\end{aligned}
$$

References:

1) Cambridge University, Linear Programming, link