

# Worksheet 12 Solution

March 21, 2020

## Question 1

- a.  $c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq c$ , where  $g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$
- b. Let  $c = \frac{277}{2}$ ,  $n_0 = 1$ ,  $n \in \mathbb{N}$ ,  $f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$ ,  $g(n) = 100 + \frac{77}{n+1}$ ,  $f(n) = 1$ .  
Assume  $n \geq n_0$

Then,

$$g(n) = 100 + \frac{77}{n+1} \leq 100 + \frac{77}{n+1} \quad (1)$$

$$\leq 100 + \frac{77}{2} \quad (2)$$

$$\leq \frac{277}{2} \quad (3)$$

$$\leq c \cdot 1 \quad (4)$$

$$\leq cf(x) \quad (5)$$

The, it follows from the definition of Big-Oh that the statement  $100 + \frac{77}{n+1} \in \mathcal{O}(1)$  is true.

## Question 2

- **Expanded Statement:**  $f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$ ,  $(\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq cf(n)) \Rightarrow (\exists d, m_0 \in \mathbb{R}^+, \forall m \in \mathbb{N}, m \geq m_0 \Rightarrow dg(n) \leq f(n))$ .

Let  $f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$ ,  $n_0 = 1$ ,  $c = \frac{1}{d}$ ,  $n \in \mathbb{N}$ ,  $m_0 = 1$ . Assume  $n \geq n_0$ ,  $g(n) \leq cf(n)$  and  $m \geq m_0$ .

Then,

$$g(n) \leq cf(n) \tag{1}$$

$$g(n) \leq \frac{1}{d}f(n) \tag{2}$$

$$dg(n) \leq f(n) \tag{3}$$

Then,

$$dg(m) \leq f(m) \tag{4}$$

by changing variable from  $n$  to  $m$ .

Then, it follows from the definition of  $\Omega$  that the statement  $f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$ ,  $g \in \mathcal{O}(f) \Rightarrow \Omega(g)$  is true.

### Question 3

- Let  $g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$ ,  $a \in \mathbb{R}^{\geq 0}$ ,  $m \in \mathbb{N}$ ,  $c_2 \gg a$ ,  $c_1 = \frac{1}{c_2}$ . Assume  $g \in \Omega(1)$ ,  $m \geq m_0$ .

Then

$$a + g \leq a + c_2g \tag{1}$$

$$< c_2g \tag{2}$$

and,

$$a + g \geq g \tag{3}$$

$$> c_1g \tag{4}$$

Then, by the definition of  $\Theta$ , the statement  $\forall g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$ , and  $a \in \mathbb{R}^{\geq 0}$ ,  $g \in \Omega(1) \Rightarrow a + g \in \Theta(g)$  is true.

## Question 4

1.  $g \notin \mathcal{O}(f) : c, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N}, (n \geq n_0) \wedge (g(n) > cf(n))$