# Worksheet 9 Solution

March 18, 2020

## Question 1

a. Every set $S$ of size 0 has $\frac{0(0-1)}{2} = 0$ subsets of size 2

b. Let $n = 0$, and S be an arbitrary set. Assume $S$ has size 0.

   Then, $S$ only has empty subsets by the fact that $S$ has size 0.

   Since empty subset has size 0, there are 0 subsets with size 2.

c. **Section 1:**

   Every set of size $k$ has $\frac{k(k-1)}{2}$ subsets of size 2.

   **Section 2:**

   Every set of sie $k+1$ has $\frac{(k+1)k}{2}$ subsets of size 2.

   **Section 3.1:**

   Because we know

| Index | Set | # of subsets of size 2 containing last element |
|---|---|---|
| 2 | $\{s_1, s_2\}$ | has 1 subset containing $s_2$ |
| 3 | $\{s_1, s_2, s_3\}$ | has 2 subsets containing $s_3$ |
| 4 | $\{s_1, s_2, s_3, s_4\}$ | has 3 subsets containing $s_4$ |

, we can deduce from above that the number of subsets of size 2 containing $s_{k+1}$ is $k$.

**Section 3.2:**

P(n): $\forall n \in \mathbb{N}$, every set of size $n$ has $\frac{n(n-1)}{2}$ subsets of size 2

Let $k \in \mathbb{N}$, and assume P(k).

Then, the number of subsets of $S$ of size 2 that don't contain $s_{k+1}$ is $\frac{k(k-1)}{2}$.

**Section 3.3:**

$$
\begin{align}
\frac{k(k-1)}{2} + k &= \frac{k(k-1)}{2} + \frac{2k}{2} \tag{1} \\
&= \frac{k}{2}\left[(k-1) + 2\right] \tag{2} \\
&= \frac{k(k+1)}{2} \tag{3}
\end{align}
$$

Then, it follows from the proof of induction that the statement $\forall n \in \mathbb{N}$, every set of size $n$ has $\frac{n(n-1)}{2}$ is true.

# Question 2

a. P(n): Every finite set $S$ of size $n$ has exactly $\frac{n(n-1)(n-2)}{6}$ subsets of size 3.
   **Base Case ($n = 0$):**

   Let the size of $S$ be 0.

   Then, $S$ only contains empty subsets.

   Since an empty subset has size 0, $S$ has 0 subsets of size 3.

**Inductive Step:**

Let $n \in \mathbb{N}$.

By the table below

| Index | Set | # of subsets of size 3 containing last element |
|---|---|---|
| 0 | $\{\}$ | 0 |
| 1 | $\{s_1\}$ | 0 |
| 2 | $\{s_1, s_2\}$ | 0 |
| 3 | $\{s_1, s_2, s_3\}$ | 1 |
| 4 | $\{s_1, s_2, s_3, s_4\}$ | 3 |
| 5 | $\{s_1, s_2, s_3, s_4, s_5\}$ | 6 |

,we can deduce that the number of subsets of size 3 containing $s_{k+1}$ is $\frac{k(k-1)}{2}$.

Since the number of subsets of $S$ of size 3 that doesn't contain $s_{k+1}$ is $\frac{(k)(k-1)(k-2)}{6}$,

$$\frac{k(k-1)(k-2)}{6} + \frac{k(k-1)}{2} = \frac{k(k-1)(k-2)}{6} + \frac{3k(k-2)}{6} \tag{1}$$

$$= \frac{k(k-1)}{6}(k-2+3) \tag{2}$$

$$= \frac{(k+1)k(k-1)}{6} \tag{3}$$

Then, it follows from the proof of induction that the statement every finite set $S$ of size $n$ has exactly $\frac{n(n-1)(n-2)}{6}$ subsets of size 3 is true.

# Question 3

a. **Part 1 (The subset of $S$ that contain the element 3):**

$\{1,3\}, \{2,3\}, \{3\}, \{1,2,3\}$

**Part 2 (The subset of $S$ that do not contain the element 3):**

$\{\}, \{1\}, \{2\}, \{1,2\}$

b. P(n): For every natural number $n$, every set $S$ of size $n$ satisfies $|\mathcal{P}(S)| = 2^n$

**Base Case ($n = 0$):**

Let $n = 0$. Then, $\mathcal{P}(\{\}) = \{\{\}\}$.

Then, $|\mathcal{P}(\{\})| = 1$

Since $2^0 = 1$, the base case holds.

**Inductive Case:**

Let $n \in \mathbb{N}$. Assume $P(n)$.

Since the number of subsets containing last element in $S$ of size $n + 1$ is $2^n$ by the table below,

| Index | Set | # of subsets containing last element |
|---|---|---|
| 0 | $\{\}$ | 0 |
| 1 | $\{1\}$ | 1 |
| 2 | $\{1, 2\}$ | 2 |
| 3 | $\{1, 2, 3\}$ | 4 |
| 4 | $\{1, 2, 3, 4\}$ | 8 |

$$
\begin{aligned}
|\mathcal{P}(n + 1)| &= 2^n + 2^n && (1) \\
&= 2^{n+1} && (2)
\end{aligned}
$$

Then, it follows from the proof of induction that the statement 'for every natural number $n$, every set $S$ of size $n$ satisfies $|\mathcal{P}(S)| = 2^n$' is true.