CSC236 Worksheet 4 Solution

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Question 1

- Let $n \in \mathbb{N}$ and assume that $\exists k \in \mathbb{N}^+$, $n = 3^k$, so $k = \log_3 n$.

  Then, since $n = 3^k$ and $3 \mid n$, we have $\lceil n/3 \rceil = n/3$.

  Then,

$$
\begin{aligned}
T(n) &= 2n + T(n/3) && \text{[By def.]} && (1)\\
&= 2(n/3) + (2(n/3) + T(n/3^2)) && \text{[By subtituting n/3 for n in def.]} && (2)\\
&= 2^2(n/3) + T(n/3^2) && && (3)\\
&= 2^3(n/3^2) + T(n/3^3) && \text{[By subtituting n/3 for n in def.]} && (4)\\
&\vdots && && (5)\\
&= 2^k(n/3^{k-1}) + T(n/3^k) && \text{[After } k \text{ applications]} && (6)\\
&= 2^{\log_3 n}(n/3^{log_3 n-1}) + T(n/3^{\log_3 n}) && \text{[By replacing } k = \log_3 n\text{]} && (7)\\
&= 2^{\log_3 n}(n(3)/n) + T(n/n) && && (8)\\
&= 3 \cdot 2^{\log_3 n} + T(1) && && (9)\\
&= 3 \cdot 2^{\log_3 n} + 2 && && (10)
\end{aligned}
$$

<u>**Notes:**</u>

- **Repeated Subtitution:**
  - Is a technique used to find a closed form formula
  - **closed form formula** is a simple formula that allows evaluation of $T(n)$ without the need to evaluate, say $T(n/2)$

    i.e. from

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ 2T(n/2) + dn & \text{if } n > 1 \end{cases} \tag{11}$$

to

$T(n) = cn + dn \log_2 n$

Example:

Consider the recurrence

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ 2T(n/2) + dn & \text{if } n > 1 \end{cases} \tag{1}$$

Find closed form formula for $T(n)$, where n is an arbitrary power of 2. That is $\exists k \in \mathbb{N}, n = 2^k$.

Let $n \in \mathbb{N}$ and assume that $\exists k \in \mathbb{N}^+$, $n = 2^k$, so $k = \log_2 n$.

Then,

$$\begin{aligned}
T(n) &= 2T(n/2) + dn && \text{[By 1]} && (2)\\
&= 2\Big(2T(n/2^2) + dn/2\Big) + dn && \text{[By subtituting } n/2 \text{ for } n \text{ in 1]} && (3)\\
&= 2^2T(n/2^2) + 2dn && && (4)\\
&= 2^2\Big(2T(n/2^3) + dn/2^2\Big) + 2dn && \text{[By subtituting } n/2^2 \text{ for } n \text{ in 1]} && (5)\\
&= 2^3T(n/2^3) + 3dn && \text{[By subtituting } n/2^2 \text{ for } n \text{ in 1]} && (6)\\
&\vdots && && (7)\\
&= 2^kT(n/2^k) + kdn && \text{[After } k \text{ applications]} && (8)\\
&= 2^{\log_2 n}T(n/2^{\log_2 n}) + (\log_2 n)dn && \text{[By replacing } k = \log_2 n\text{]} && (9)\\
&= nT(1) + (\log_2 n)dn && && (10)\\
&= cn + (\log_2 n)dn && && (11)
\end{aligned}$$

Question 2