Worksheet 5 Solution

March 15, 2020

Question 1

- $\forall n, p \in \mathbb{N}, Odd(n) \wedge Odd(p) \Rightarrow Odd(n \times p)$

  Let $n, p \in \mathbb{Z}$, and assume n, p are odd numbers.

  Then, $\exists k, m \in \mathbb{Z}$, $n = 2k - 1$, $p = 2m - 1$ by the definition of odd numbers

  Then,

$$\begin{aligned} n \times p &= (2k-1)(2m-1) && (1) \\ &= 2k2m - 2k - 2m + 1 && (2) \\ &= (2k2m - 2k - 2m + 2) - 1 && (3) \\ &= 2(2km - k - m + 1) - 1 && (4) \\ &= 2l - 1 && (5) \end{aligned}$$

  where $l = 2km - k - m + 1$.

  Since $l \in \mathbb{Z}$, it follows from the definition of odd number that the product of two odd numbers is odd.

Question 2

a. $\forall m, n \in \mathbb{Z},\ Even(m) \wedge Odd(n) \Rightarrow m^2 - n^2 = m + n$

b. The flaw is in the same value k. This implies that the statement is true only when n is 1 less than m. This doesn't mean it's true for all even and odd numbers.

Question 3

a. $Dom(f, g) : \forall n \in \mathbb{N}, g(n) \leq f(n)$, where $f, g :\ \mathbb{N} \to \mathbb{R}^{\geq 0}$

b. Let $n \in \mathbb{R}^{\geq 0}$, $f(n) = 3n$ and $g(n) = n$.

Then,

$$\begin{aligned} g(n) = n &\leq n + n + n && (6) \\ &\leq 3n && (7) \\ &\leq f(n) && (8) \end{aligned}$$

Then, it follows from the definition that f dominates g.

c. Predicate Logic: $\exists n \in \mathbb{N},\ g(n) > f(n)$.

Let $n = 1$.

Then,

$$\begin{aligned} g(1) = (1) + 165 = 166 &> 1 && (9) \\ &> f(1) && (10) \end{aligned}$$

Then, it follows from negation of the definition that f does not dominate g.

d. Predicate Logic: $\exists x \in \mathbb{R}^{\geq 0}, \, \exists n \in \mathbb{N}, \, g(n) > f(n)$

Let $x = 1$ and $n = 1$.

Then,

$$\begin{align} g(1) = (1) + 1 = 2 > 1 \tag{11} \\ > f(1) \tag{12} \end{align}$$

Then, it follows from negation of the definition that f does not dominate g.

Question 4

a. Let $x \in \mathbb{R}^{\geq 0}$, $\epsilon = x - \lfloor x \rfloor$, and assume $x \geq 4$.

Then,

$$(\lfloor x \rfloor)^2 = (x - \epsilon)^2 \tag{1}$$

by the fact that ϵ can be rewritten as $\lfloor x \rfloor = x - \epsilon$.

Then,

$$\begin{align} (x - \epsilon)^2 &= x^2 - 2x\epsilon + \epsilon^2 \tag{2} \\ &\geq x^2 \tag{3} \\ &\geq \frac{1}{2}x^2 \tag{4} \end{align}$$

Conclusion can be made from the above fact that ϵ is sufficiently small and $\epsilon^2 - 2x\epsilon \geq 0$

b. Let $x \in \mathbb{R}^{\geq 0}$, and assume $x \geq 4$.

Then,

$$
\begin{align}
x &\geq 4 \tag{1}\\
x^2 &\geq 4x \tag{2}\\
\frac{1}{2}x^2 &\geq 2x \tag{3}
\end{align}
$$

Then, it follows from the above that the statement $\forall x \in \mathbb{R}^{\geq 0},\ x \geq 4 \Rightarrow \frac{1}{2}x^2 \geq 2x$ is true.