Worksheet 9 Review

March 28, 2020

Question 1

a. For every set of size 0 has 0 subsets of size 2.

b. Let $n = 0$. Let S be an arbitrary set. Assume S has size 0.

   Since S has size 0, empty subsets are the **only** subsets that can be included in S.

   Then, because we know empty subsets have size 0, we can conclude there are 0 subsets of size 2.

   It follows from above that the base case holds.

   Attempt #2:

   We want to show every set S of size 0 has 0 subsets of size 2.

   Since S has size 0, empty subsets are the **only** subsets that can be included in S.

   Then, because we know an empty subset have size 0, we can conclude there are 0 subsets of size 2.

   Notes:

- Professor specifically mentions **We want to show every set S of size 0 has 0 subsets of size 2**
- Professor doesn't include conclusion at the end of proof
- Under which cases conclusion to a proof are included.

c. Now we will prove inductive step.

Let $k \in \mathbb{N}$. Assume every set of size k has $\frac{k(k-1)}{2}$ subsets of size 2.

We want to show a set of size $k+1$ has $\frac{(k+1)k}{2}$ subsets of size 2.

Part 1: counting subsets of S of size 2 that contain s_{k+1}

It follows from the table below,

k	Sets	Subsets of Size 2 with s_{k+1}
0	$\{0,1\}$	1
1	$\{0,1,2\}$	2
2	$\{0,1,2,3\}$	3
2	$\{0,1,2,3,4\}$	4

that the number of subsets of size 2 that contain s_{k+1} is $k+1$.

Part 2: counting subsets of S of size 2 that doesn't contain s_{k+1}

Because we know the subset of S that doesn't contain s_{k+1} is a set S of size k, we can conclude using induction hypothesis that there are

$$\frac{k(k+1)}{2} \tag{1}$$

subsets of size 2.

Part 3: Putting the counts together

Then,

$$\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2} \tag{2}$$

Then, it follows from proof by induction that the statement '$\forall n \in \mathbb{N}$, every set of size n has $\frac{n(n-1)}{2}$ subsets of size 2' is true for all natural numbers n.

Attempt #2:

Part 1: counting subsets of S of size 2 that contain s_{k+1}

It follows from the table below,

k	Sets	Subsets of Size 2 with s_{k+1}
2	$\{0,1\}$	1
3	$\{0,1,2\}$	2
4	$\{0,1,2,3\}$	3
5	$\{0,1,2,3,4\}$	4

that the number of subsets of size 2 that contain s_{k+1} is k.

Part 2: counting subsets of S of size 2 that doesn't contain s_{k+1}

Because we know the subset of S that doesn't contain s_{k+1} is a set S of size k, we can conclude using induction hypothesis that there are

$$\frac{k(k-1)}{2} \tag{3}$$

subsets of size 2.

Part 3: Putting the counts together

Then,

$$\frac{k(k-1)}{2} + k = \frac{(k+1)k}{2} \tag{4}$$

Then, it follows from proof by induction that the statement '$\forall k \in \mathbb{N}$, every set of size $k$has $\frac{k(k-1)}{2}$subsets of size 2' is true for all natural numbers k.

Notes:

- I forgot that k represents number of elements in a set.

Question 2

- **Statement:** For every $n \in \mathbb{N}$, every finite set S of size n, has

$$\frac{n(n-1)(n-2)}{6} \tag{1}$$

subsets of size 3.

We will prove this statement by using induction on n.

Base Case:

Let $n = 0$.

Then, only the empty subsets can be included in S.

Because an empty subset has size 0, there are 0 subsets of size 3 in S.

Then, since

$$\frac{0 \cdot (0-1)(0-2)}{6} = 0 \tag{2}$$

the base case holds.

Inductive Case:

Let $k \in \mathbb{N}$. Assume every finite set S of size k has exactly $\frac{k(k-1)(k-2)}{6}$ subsets of size 3.

We want to show finite set S of size $k+1$ contains $\frac{(k+1)k(k-1)}{6}$ subsets of size 3.

It follows from the table below

k	Sets	# of Subsets of Size 3	# of Subsets of Size 2
0	$\{\}$	0	0
1	$\{s_0\}$	0	0
2	$\{s_0, s_1\}$	0	1
3	$\{s_0, s_1, s_2\}$	1	3
4	$\{s_0, s_1, s_2, s_3\}$	4	6
5	$\{s_0, s_1, s_2, s_3, s_4\}$	10	10

we can deduce that given a set S size $k+1$, the number of subsets of size 3 containing s_{k+1} is the sum of # of subsets of size 3 that doesn't contain s_{k+1}) and # of subsets of size 2 that doesn't contain s_{k+1}.

Then,

$$\begin{aligned}
\frac{k(k-1)(k-2)}{6} + \frac{k(k-1)}{2} &= \frac{k(k-1)(k-2)}{6} + \frac{3k(k-1)}{6} && (3)\\
&= \frac{k(k-1)(k-2+3)}{6} && (4)\\
&= \frac{k(k-1)(k+1)}{6} && (5)
\end{aligned}$$

Attempt #2:

Statement: For every $n \in \mathbb{N}$, every finite set S of size n, has

$$\frac{n(n-1)(n-2)}{6} \tag{1}$$

subsets of size 3.

We will prove this statement by using induction on n.

Base Case:

Let $n = 0$.

Then, only the empty subsets can be included in S.

Because an empty subset has size 0, there are 0 subsets of size 3 in S.

Then, since

$$\frac{0 \cdot (0-1)(0-2)}{6} = 0 \tag{2}$$

the base case holds.

Inductive Case:

Let $k \in \mathbb{N}$. Assume every finite set S of size k has exactly $\frac{k(k-1)(k-2)}{6}$ subsets of size 3.

We want to show finite set S of size $k+1$ contains $\frac{(k+1)k(k-1)}{6}$ subsets of size 3. We will do that by determining the number of subsets of size 3 including s_{k+1}, and the number of subsets of size 3 not including s_{k+1}, and then combining them together.

First, we will show the number of subsets of size 3 including s_{k+1} is $\frac{n(n-1)}{2}$.

Because we know the number of subsets of size 3 (i.e $\{a_1, a_2, s_{k+1}\}$) containing s_{k+1} depends on the unique combination of first two elements a_1 and a_2, and because we know

1. $a_1 \neq a_2$
2. $a_1, a_2 \in \{s_0, s_1, \ldots, s_k\}$

, we can conclude that the number of subsets of size 3 containing s_{k+1} is exactly the number of subsets of size 2 in a set S of size k or

$$\frac{n(n-1)}{2} \tag{3}$$

Second, we will show the number of subsets of size 3 not including s_{k+1} is $\frac{n(n-1)(n-2)}{6}$.

Since the number of subsets of size 3 not containing s_{k+1} must contain 3 elements from $\{s_0, \ldots, s_k\}$, this is exactly number of subsets of size 3 in a set S of size k, or

$$\frac{n(n-1)(n-2)}{6} \tag{4}$$

by induction hypothesis.

Then,

$$\frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{6} = \frac{3n(n-1)}{6} + \frac{n(n-1)(n-2)}{6} \tag{5}$$
$$= \frac{n(n-1)}{6} \cdot (3+n-2) \tag{6}$$
$$= \frac{n(n-1)(n+1)}{6} \tag{7}$$

Notes:

- I wonder if table like above can be used in a proof. If not, why can't it be done? If yes, when is the use of table not valid? How should it be constructed that it's valid?
- Can I put a subheader like 'Part 1: counting subsets of S of size 2 that contain s_{k+1}' in a proof? If so, are there anything that I should be aware/be careful of?
- Is it alright to play with examples when I don't know how to proceed? What's the danger to this approach?
- Noticed that professor lays out the big ideas of proof (proof by induction evaluating number of s_{k+1} and s_k) and then fills in the missing detail

Question 3

a. **Subsets that contain 3:**

$\{\{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Subsets that doesn't contain 3:

$\{\{\}, \{1\}, \{2\}, \{1,2\}\}$

b. Let $n \in \mathbb{N}$.

We will prove the statement using induction on n.

Base Case:

Let $n = 0$.

We will show $\mid \mathcal{P}(\{\}) \mid$ and 2^0 are equal.

Because we know S with size 0 is an empty set and $\mathcal{P}(\{\}) = \{\{\}\}$, we can conclude

$$\begin{aligned} \mid \mathcal{P}(\{\}) \mid &= \mid \{\{\}\} \mid && (1) \\ &= 1 && (2) \end{aligned}$$

Because we know $2^{(0)} = 1$, we can conclude $\mid \mathcal{P}(\{\}) \mid = 2^0$

Inductive Case:

Let $n \in \mathbb{N}$. Assume $\mid \mathcal{P}(\{s_0, \dots, s_n\}) \mid = 2^n$.

We will show $\mid \mathcal{P}(\{s_0, \dots, s_n, s_{n+1}\}) \mid = 2^{n+1}$ by finding the number of subsets containing s_{n+1} and the number of subsets not containing s_{n+1}, and then combining the result together.

Part 1 Determining number of subsets containing s_{n+1}:

Because we know a subset containing s_{n+1} can be decomposed into two parts $S'_n \cup \{s_{n+1}\}$ where S'_n is a subset with combination of elements in $\{\phi, s_0, \dots, s_n\}$, we can conclude that the number of subsets containing s_{n+1} is exactly $\mid \mathcal{P}(\{s_0, \dots, s_n\}) \mid$, or

$$2^n \qquad (3)$$

using induction hypothesis.

Part 2 Determining number of subsets not containing s_{n+1}:

Because we know the subsets don't have s_{n+1}, we can conclude that the subsets are combination of elements in $\{\phi, s_0, \ldots, s_n\}$.

Since this is exactly, $\mid \mathcal{P}(\{s_0, \ldots, s_n\}) \mid$, using induction hypothesis, we can conclude that the number of subsets not containing s_{n+1} is

$$2^n \tag{4}$$

Part 3 Combining together:

Since $\mid \mathcal{P}(s_0, \ldots, s_n, s_{n+1}) \mid$ is the sum of the number of subsets containing s_{n+1} and the number of subsets not containing s_{n+1}, we have

$$\mid \mathcal{P}(s_0, \ldots, s_n, s_{n+1}) \mid = 2^n + 2^n \tag{5}$$

Then, it follows from the fact $2^n + 2^n = 2^{n+1}$ that

$$\mid \mathcal{P}(s_0, \ldots, s_n, s_{n+1}) \mid = 2^{n+1} \tag{6}$$

Notes:

- How can the phrase 'every set S of size n' be described using predicate logic?
- Noticed professor approaches the problem by unraveling definition just enough to gain more insight and information about how to proceed.
- How we know if we are over doing/over thinking it? How do we know if a proof is not enough or has missing detail or jumping to conclusion?