

# CSC236 Term Test 1 Version 2 Solution

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## Question 1

- *Proof.* Define  $P(n) : f(n) = 3^n$ .

I will use complete induction to prove that  $\forall n \in \mathbb{N}, n > 2 \Rightarrow P(n)$ .

Base Case ( $n = 0$ ):

Let  $n = 0$ .

Then, the definition of  $f(n)$  tells us  $f(n) = 1$ .

Then, we have

$$\begin{aligned} f(n) &= 3^0 & (1) \\ &= 3^n & (2) \end{aligned}$$

Thus,  $P(n)$  follows.

Base Case ( $n = 1$ ):

Let  $n = 1$ .

Then, the definition of  $f(n)$  tells us  $f(n) = 3$ .

Then, we have

$$\begin{aligned} f(n) &= 3^1 & (3) \\ &= 3^n & (4) \end{aligned}$$

Thus,  $P(n)$  follows.

**Base Case ( $n = 2$ ):**

Let  $n = 2$ .

Then, the definition of  $f(n)$  tells us  $f(n) = 9$ .

Then, we have

$$f(n) = 3^2 \tag{5}$$

$$= 3^n \tag{6}$$

Thus,  $P(n)$  follows.

**Case ( $n > 2$ ):**

Assume  $n > 2$ .

Then, since  $0 \leq n - 1 < n$ ,  $0 \leq n - 2 < n$ , and  $0 \leq n - 3 < n$ , the complete induction tells us  $P(n - 1)$ ,  $P(n - 2)$ , and  $P(n - 3)$ , i.e.  $f(n - 1) = 3^{n-1}$ ,  $f(n - 2) = 3^{n-2}$ , and  $f(n - 3) = 3^{n-3}$ , respectively.

Then, using these facts, we can write

$$f(n) = f(n - 1) + 3f(n - 2) + 9f(n - 3) \tag{7}$$

$$= 3^{n-1} + 3 \cdot 3^{n-2} + 3^2 \cdot 3^{n-3} \tag{8}$$

$$= 3^{n-1} + 3^{n-1} + 3^{n-1} \tag{9}$$

$$= 3^{n-1}(1 + 1 + 1) \tag{10}$$

$$= 3^{n-1}3 \tag{11}$$

$$= 3^n \tag{12}$$

Thus,  $P(n)$  follows. □

**Correct Solution:**

Define  $P(n) : f(n) = 3^n$ .

I will use complete induction to prove that  $\forall n \in \mathbb{N}, P(n)$ .

**Inductive Step:**

Let  $n \in \mathbb{N}$ . Assume  $H(n) : \bigwedge_{i=0}^{n-1} P(i)$ . I will prove  $P(n)$  follows. That is,  $f(n) = 3^n$ .

**Base Case ( $n = 0$ ):**

Let  $n = 0$ .

Then, the definition of  $f(n)$  tells us  $f(n) = 1$ .

Then, we have

$$f(n) = 3^0 \tag{13}$$

$$= 3^n \tag{14}$$

Thus,  $P(n)$  follows.

**Base Case ( $n = 1$ ):**

Let  $n = 1$ .

Then, the definition of  $f(n)$  tells us  $f(n) = 3$ .

Then, we have

$$f(n) = 3^1 \tag{15}$$

$$= 3^n \tag{16}$$

Thus,  $P(n)$  follows.

**Base Case ( $n = 2$ ):**

Let  $n = 2$ .

Then, the definition of  $f(n)$  tells us  $f(n) = 9$ .

Then, we have

$$f(n) = 3^2 \tag{17}$$

$$= 3^n \tag{18}$$

Thus,  $P(n)$  follows.

### Case ( $n > 2$ ):

Assume  $n > 2$ .

Then, since  $0 \leq n-1 < n$ ,  $0 \leq n-2 < n$ , and  $0 \leq n-3 < n$ , the complete induction tells us  $P(n-1)$ ,  $P(n-2)$ , and  $P(n-3)$ , i.e.  $f(n-1) = 3^{n-1}$ ,  $f(n-2) = 3^{n-2}$ , and  $f(n-3) = 3^{n-3}$ , respectively.

Then, using these facts, we can write

$$f(n) = f(n-1) + 3f(n-2) + 9f(n-3) \quad [\text{By definition, since } n > 2] \quad (19)$$

$$= 3^{n-1} + 3 \cdot 3^{n-2} + 3^2 \cdot 3^{n-3} \quad (20)$$

$$= 3^{n-1} + 3^{n-1} + 3^{n-1} \quad (21)$$

$$= 3^{n-1}(1 + 1 + 1) \quad (22)$$

$$= 3^{n-1}3 \quad (23)$$

$$= 3^n \quad (24)$$

Thus,  $P(n)$  follows.

### Notes:

1. Learned that  $n > i$  in  $\forall n \in \mathbb{N}, n > i \Rightarrow P(n)$  is used when  $P(n)$  is true starting  $i + 1$ .

If  $P(n)$  is true for all natural numbers, then  $\forall n \in \mathbb{N}, P(n)$  is used.

2. Learned that ‘Assume  $n > 2$ ’ in ‘Let  $n \in \mathbb{N}$ . Assume  $n > 2$ . Assume  $H(n) : \bigwedge_{i=0}^{n-1} P(i)$ ’. is used when  $P(i)$  is true starting  $n = 3$ .

If  $P(i)$  is true for all natural numbers, then, ‘Let  $n \in \mathbb{N}$ . Assume  $H(n) : \bigwedge_{i=0}^{n-1} P(i)$ ’ is used.

## Question 2

- Given the statement to prove

$P(x, y, z, w) : \text{There are no positive integers } x, y, z, w \text{ such that } x^4 + 3y^4 + 9z^4 = 27w^4.$

*Proof.* I will prove  $P(x, y, z, w)$  using proof by contradiction.

Assume  $\exists x, y, z, w \in \mathbb{N}^+, x^4 + 3y^4 + 9z^4 = 27w^4$ .

Then, the set  $X = \{x \in \mathbb{N}^+ \mid \exists y, z, w \in \mathbb{N}^+, x^4 + 3y^4 + 9z^4 = 27w^4\}$  is not empty.

Then, by principle of well-ordering, there is smallest positive integer  $x_0 \in X$ , and positive integers  $y_0, z_0, w_0 \in \mathbb{N}^+$  such that  $x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4$ .

Then,

$$\begin{aligned} x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 &\Rightarrow x_0^4 = 27w_0^4 - 3y_0^4 - 9z_0^4 \\ &\Rightarrow 3 \mid x_0^4 \Rightarrow 3 \mid x_0 \end{aligned} \quad \begin{array}{l} \text{[By hint]} \\ (1) \end{array}$$

$$\begin{aligned} \text{Let } \exists x_1 \in \mathbb{N}^+, x_0 = 3x_1 &\Rightarrow x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \\ &\Rightarrow 3y_0^4 = 27w_0^4 - 9z_0^4 - x_0^4 \\ &\Rightarrow 3y_0^4 = 27w_0^4 - 9z_0^4 - 3^4x_1^4 \\ &\Rightarrow y_0^4 = 9w_0^4 - 3z_0^4 - 3^3x_1^4 \\ &\Rightarrow 3 \mid y_0^4 \Rightarrow 3 \mid y_0 \end{aligned} \quad \begin{array}{l} \text{[By hint]} \\ (2) \end{array}$$

$$\begin{aligned} \text{Let } \exists y_1 \in \mathbb{N}^+, y_0 = 3y_1 &\Rightarrow x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \\ &\Rightarrow 9z_0^4 = 27w_0^4 - 3y_0^4 - x_0^4 \\ &\Rightarrow 9z_0^4 = 27w_0^4 - 3^5y_1^4 - 3^4x_1^4 \\ &\Rightarrow z_0^4 = 3w_0^4 - 3^3y_1^4 - 3^2x_1^4 \\ &\Rightarrow 3 \mid z_0^4 \Rightarrow 3 \mid z_0 \end{aligned} \quad \begin{array}{l} \text{[By hint]} \\ (3) \end{array}$$

$$\begin{aligned} \text{Let } \exists w_1 \in \mathbb{N}^+, w_0 = 3w_1 &\Rightarrow x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \\ &\Rightarrow 3^4x_1^4 + 3^5y_1^4 + 3^6z_1^4 = 3^7w_1^4 \\ &\Rightarrow x_1^4 + 3y_1^4 + 3^2z_1^4 = 3^3w_1^4 \\ &\Rightarrow x_1^4 + 3y_1^4 + 9z_1^4 = 27w_1^4 \\ &\Rightarrow x_1 \in X \end{aligned} \quad (4)$$

Then, this leads to contradiction, because we know  $x_1 < x_0$ ,  $x_1 \in X$ , but  $x_0$  is the smallest number in  $X$ .

Thus, we can conclude the assumption is false.

□

**Correct Solution:**

Given the statement to prove

$P(x, y, z, w)$  : There are no positive integers  $x, y, z, w$  such that  $x^4 + 3y^4 + 9z^4 = 27w^4$ .

*Proof.* I will prove  $P(x, y, z, w)$  using proof by contradiction.

Assume  $\exists x, y, z, w \in \mathbb{N}^+, x^4 + 3y^4 + 9z^4 = 27w^4$ .

Then, the set  $X = \{x \in \mathbb{N}^+ \mid \exists y, z, w \in \mathbb{N}^+, x^4 + 3y^4 + 9z^4 = 27w^4\}$  is not empty.

Then, **since  $X$  is subset of  $\mathbb{N}$** , by principle of well-ordering, there is smallest positive integer  $x_0 \in X$ . **Furthermore**, there are positive integers  $y_0, z_0, w_0 \in \mathbb{N}^+$  such that  $x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4$ .

Then,

$$\begin{aligned} x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 &\Rightarrow x_0^4 = 27w_0^4 - 3y_0^4 - 9z_0^4 \\ &\Rightarrow 3 \mid x_0^4 \Rightarrow 3 \mid x_0 \end{aligned} \quad \begin{array}{l} \text{[By hint]} \\ (1) \end{array}$$

$$\begin{aligned} \text{Let } \exists x_1 \in \mathbb{N}^+, x_0 = 3x_1 &\Rightarrow x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \\ &\Rightarrow 3y_0^4 = 27w_0^4 - 9z_0^4 - x_0^4 \\ &\Rightarrow 3y_0^4 = 27w_0^4 - 9z_0^4 - 3^4x_1^4 \\ &\Rightarrow y_0^4 = 9w_0^4 - 3z_0^4 - 3^3x_1^4 \\ &\Rightarrow 3 \mid y_0^4 \Rightarrow 3 \mid y_0 \end{aligned} \quad \begin{array}{l} \text{[By hint]} \\ (2) \end{array}$$

$$\begin{aligned} \text{Let } \exists y_1 \in \mathbb{N}^+, y_0 = 3y_1 &\Rightarrow x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \\ &\Rightarrow 9z_0^4 = 27w_0^4 - 3y_0^4 - x_0^4 \\ &\Rightarrow 9z_0^4 = 27w_0^4 - 3^5y_1^4 - 3^4x_1^4 \\ &\Rightarrow z_0^4 = 3w_0^4 - 3^3y_1^4 - 3^2x_1^4 \\ &\Rightarrow 3 \mid z_0^4 \Rightarrow 3 \mid z_0 \end{aligned} \quad \begin{array}{l} \text{[By hint]} \\ (3) \end{array}$$

$$\begin{aligned} \text{Let } \exists w_1 \in \mathbb{N}^+, w_0 = 3w_1 &\Rightarrow x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \\ &\Rightarrow 3^4x_1^4 + 3^5y_1^4 + 3^6z_1^4 = 3^7w_1^4 \\ &\Rightarrow x_1^4 + 3y_1^4 + 3^2z_1^4 = 3^3w_1^4 \\ &\Rightarrow x_1^4 + 3y_1^4 + 9z_1^4 = 27w_1^4 \\ &\Rightarrow x_1 \in X \end{aligned} \quad (4)$$

Then, this leads to contradiction, because we know  $x_1 < x_0$ ,  $x_1 \in X$ , but  $x_0$  is the smallest number in  $X$ .

Thus, we can conclude the assumption is false. □

**Note:**

- Noticed professor wrote ‘Divide by 3’ as a reasoning in calculation.

$$\begin{aligned}
 \text{Let } z_1 \in \mathbb{N}^+, 3z_1 = z_0 &\Rightarrow 81z_1^4 = 3w_0^4 - (9x_1^4 + 27y_1^4) \\
 &\Rightarrow 27z_1^4 = w_0^4 - (3x_1^4 + 9y_1^4) \Rightarrow 3x_1^4 + 9y_1^4 + 27z_1^4 = w_0^4 \quad \# \text{ divide by 3} \\
 &\Rightarrow 3 \mid w_0^4 \Rightarrow 3 \mid w_0 \quad \# \text{ since 3 divides LHS and allowed assumption}
 \end{aligned}$$

### Question 3

- *Proof.* Define  $\mathcal{T}$  as the smallest set such that

a.  $() \in \mathcal{T}$

b. If  $t_1, t_2 \in \mathcal{T}$ ,  $(t_1 t_2) \in \mathcal{T}$

I need to prove  $\forall t \in \mathcal{T}, P(t)$ . That is,  $\text{left}(t)$  is odd.

**Basis:**

Let  $() \in \mathcal{T}$ .

Then, since there is only 1 left parenthesis and 1 is odd,  $P(t)$  holds.

**Inductive Step:**

Let  $t_1, t_2$  be arbitrary string in  $\mathcal{T}$ . Assume  $H(t_1, t_2) : P(t_1)$  and  $P(t_2)$ . That is  $\text{left}(t_1)$  and  $\text{left}(t_2)$  are odd. Let  $(t_1 t_2) \in \mathcal{T}$ .

I need to prove  $P((t_1 t_2))$  follows. That is,  $\text{left}((t_1 t_2))$  is odd.

The initial left parenthesis in  $(t_1 t_2)$  increases the total number of left parenthesis by 1. This means, we have

$$\text{left}((t_1 t_2)) = \text{left}(t_1) + \text{left}(t_2) + 1 \tag{1}$$

Then, since we know from induction hypothesis that  $\text{left}(t_1)$  and  $\text{left}(t_2)$  are odd, we can write  $\text{left}(t_1) + \text{left}(t_2)$  is even.

Then, we can write  $\text{left}(t_1) + \text{left}(t_2) + 1$  is odd.

Then, we can conclude  $\text{left}((t_1t_2))$  is odd.

Thus,  $P((t_1t_2))$  follows.

□

### **Correct Solution:**

Define  $\mathcal{T}$  as the smallest set such that

- a.  $() \in \mathcal{T}$
- b. If  $t_1, t_2 \in \mathcal{T}$ ,  $(t_1t_2) \in \mathcal{T}$

I need to prove  $\forall t \in \mathcal{T}, P(t)$ . That is,  $\text{left}(t)$  is odd.

### **Basis:**

Let  $() \in \mathcal{T}$ .

Then, since there is only 1 left parenthesis and 1 is odd,  $P(t)$  holds.

### **Inductive Step:**

Let  $t_1, t_2$  be arbitrary string in  $\mathcal{T}$ . Assume  $H(t_1, t_2) : P(t_1)$  and  $P(t_2)$ . That is  $\text{left}(t_1)$  and  $\text{left}(t_2)$  are odd. In other words,  $\exists k_1, k_2 \in \mathbb{Z}$  such that  $\text{left}(t_1) = 2k_1 + 1$ ,  $\text{left}(t_2) = 2k_2 + 1$ . Let  $(t_1t_2) \in \mathcal{T}$ .

I need to prove  $P((t_1t_2))$  follows. That is,  $\text{left}((t_1t_2))$  is odd. In other words,  $\exists k_3 \in \mathbb{Z}$ ,  $\text{left}((t_1t_2)) = 2k_3 + 1$ .

Let  $k_3 = k_1 + k_2 + 1$ .

The initial left parenthesis in  $(t_1t_2)$  increases the total number of left parenthesis by 1. This means, we have

$$\text{left}((t_1t_2)) = \text{left}(t_1) + \text{left}(t_2) + 1 \tag{1}$$

Then,



$$\text{left}((t_1 t_2)) = 2k_1 + 1 + 2k_2 + 1 + 1 \quad [\text{By I.H}] \quad (2)$$

$$= 2(k_1 + k_2 + 1) + 1 \quad (3)$$

$$= 2k_3 + 1 \quad (4)$$

Thus,  $P((t_1 t_2))$  follows.