Midterm 1 Version 3 Solution

March 20, 2020

Question 1

a. Since $S_1 = \{ab, ba, aab, bba, baa, \dots\}$ and $S_2 = \{aaa, aab, aba, baa, abb, bab, bba\}$,

$S_2 \setminus S_1 = \{aaa, aab, aba, bab\}$

Correct Solution:

Since $S_1 = \{ab, ba, aab, abb, bba, baa, \dots\}$ and $S_2 = \{aaa, aab, aba, baa, abb, bab, bba, bbb\}$,
$S_2 \setminus S_1 = \{aaa, aba, bab, bbb\}$

b. See table below

p	q	r	$\neg r$	$p \Rightarrow q$	$(p \Rightarrow q) \Leftrightarrow \neg r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	F	T
F	T	T	F	T	F
T	F	F	T	F	F
F	T	F	T	T	T
F	F	T	F	T	F
F	F	F	T	T	T

c. Let $x =$ _______, and $y \in \mathbb{N}$.

We will prove that $P(x)$ is true and $Q(x, y)$ or $Q(x, y+1)$ is false.

Correct Solution:

Negation: $\exists x \in \mathbb{N}, \forall y \in \mathbb{N}, P(x) \wedge (\neg Q(x,y) \wedge \neg Q(x, y+1))$

Let $x =$ _______ and $y \in \mathbb{N}$.

We will prove that $P(x)$ is true, and both $Q(x,y)$ and $Q(x, y+1)$ are false.

Question 2

a. $\forall x \in T, Canadian(x) \wedge Star(x)$

Correct Solution:

$\forall x \in T, Canadian(x) \Rightarrow Star(x)$

b. $\forall x \in T, Canadian(x) \Rightarrow \forall y \in T,\ \neg Canadian(y) \wedge Defeated(x,y)$

Correct Solution:

$\forall x \in T, Canadian(x) \Rightarrow (\forall y \in T,\ \neg Canadian(y) \Rightarrow Defeated(x,y))$

c. $\exists x \in T, Canadian(x) \wedge Star(x) \Rightarrow \forall y \in T, \exists z \in T, y \neq z \wedge Canadian(y) \wedge Defeated(y,z)$

Correct Solution:

$\exists x \in T, Canadian(x) \wedge Star(x) \Rightarrow (\forall y \in T,\ Canadian(y) \Rightarrow \exists z \in T, y \neq z \wedge Defeated(y,z))$

d. $\exists x \in T, Canadian(x) \wedge Star(x) \wedge (\forall y \in T, x \neq y \wedge Canadian(y) \wedge \neg Star(y))$

Correct Solution:

$\exists x \in T,\ Canadian(x) \wedge Star(x) \wedge (\forall y \in T,\ x \neq y \wedge Canadian(y) \Rightarrow \neg Star(y))$

Question 3

a. $\forall n \in \mathbb{N},\ \exists k \in \mathbb{N},\ n > 1 \wedge n = 2k+1 \Rightarrow \exists p, q \in \mathbb{Z}^+,\ n = p^2 - q^2$

b. Let $n \in \mathbb{N}$. Assume $n > 1$, and that there exists $k \in \mathbb{Z}$ such that $n = 2k + 1$.

Also, let $p = k + 1$ and $q = k$.

Then,

$$\begin{aligned} p^2 - q^2 &= (k+1)^2 - k^2 && (1) \\ &= k^2 + 2k + 1 - k^2 && (2) \\ &= 2k + 1 && (3) \\ &= n && (4) \end{aligned}$$

Then, it follows from above that the statement $\forall n \in \mathbb{N}, \exists k \in \mathbb{N}, n > 1 \wedge n = 2k + 1 \Rightarrow \exists p, q \in \mathbb{Z}^+, \ n = p^2 - q^2$ is true.

Question 4

- Let $d, n \in \mathbb{N}$. Assume $d \mid n$ and $d \neq n$.

  Then, $\exists k \in \mathbb{N}$,

$$n = kd \qquad (1)$$

  Since $k \in \mathbb{N}$, there are two cases of divisors. One is when $k = 1$, and the other is $k \geq 2$.

  Since $n \neq d$, $k \geq 2$.

  Then,

$$\begin{aligned} n &= kd && (2) \\ &\geq 2d && (3) \end{aligned}$$

  Then,

$$\frac{n}{2} \geq d \tag{4}$$

Then, it follows from above that the statement $\forall d, n \in \mathbb{N},\ d \mid n \land d \neq n \Rightarrow d \leq \frac{n}{2}$ is true.