# Midterm 2 Version 1 Solution

April 3, 2020

## Question 1

a.

$$\begin{aligned}
100 \div 2 &= 50, \text{Remainders } \mathbf{0} \\
50 \div 2 &= 25, \text{Remainders } \mathbf{0} \\
25 \div 2 &= 12, \text{Remainders } \mathbf{1} \\
12 \div 2 &= 6, \text{Remainders } \mathbf{0} \\
6 \div 2 &= 3, \text{Remainders } \mathbf{0} \\
3 \div 2 &= 1, \text{Remainders } \mathbf{1} \\
1 \div 2 &= 0, \text{Remainders } \mathbf{1}
\end{aligned}$$

Then, it follows from above that the binary representation of 100 is $(1100100)_2$.

b. The smallest number that can be expressed by an n-digit balanced ternary representation is

$$\sum_{i=0}^{n-1} d_i \cdot 3^i, \text{ where } d_i \in \{0, 1, 2\} \tag{1}$$

**Correct Solution:**

The smallest number that can be expressed by an n-digit balanced ternary representation is

$$-\left[\sum_{i=0}^{n-1} 3^i\right] \tag{1}$$

**Notes:**

- Realized professor is asking for an example of the smallest number.
- Ternary representation of a number

$$\sum_{i=0}^{n-1} d_i \cdot 3^i, \text{ where } d_i \in \{0, 1, 2\}$$

- Learned a negative number could be expressed in in ternary or binary representation of numbers.

c.

| | | | | | |
|---|---|---|---|---|---|
| $f(n) \in \Omega(n)$ | True | $g(n) \in \Omega(n)$ | False | $f(n) \in \mathcal{O}(g(n))$ | False |
| $f(n) \in \Theta(g(n))$ | False | $g(n) \in \Theta(\log_3 n)$ | True | $f(n) + g(n) \in \Theta(f(n))$ | True |

**Notes:**

- $\forall g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, and all numbers $a \in \mathbb{R}^{\geq 0}$, if $g \in \mathcal{O}(f)$, then $f + g \in \mathcal{O}(f)$
- $g \in \Theta(f) : \; g \in \mathcal{O}(f) \wedge g \in \Omega(f)$

  or

  $g \in \Theta(f) : \; \exists c_1, c_2, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$
- $g \in \Omega(f) : \; \exists c, n_o \in \mathbb{R}^+, \; \forall n \in \mathbb{N}, \; n \geq n_0 \Rightarrow g(n) \geq cf(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

- $g \in \mathcal{O}(f)$ : $\exists c, n_o \in \mathbb{R}^+$, $\forall n \in \mathbb{N}$, $n \geq n_0 \Rightarrow g(n) \leq cf(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

d.

| k | 0 | 1 | 2 |
|---|---|---|---|
| $i_k$ | $3 = 3^1$ | $9 = 3^2$ | $81 = 3^4$ |

The value of $i_k$ is

$$3^{2^k} \tag{1}$$

**Notes:**

- Realized we are only concerned with the lines **i = i * i** and **i = 3**

e. The number of iterations the function's loop will run is

$$\lceil \log_2 \log_3 n \rceil - 1 \tag{1}$$

**Notes:**

- The loop terminates when $3^{2^{(k+1)}} = i_{k+1} = i_k \cdot i_k \geq n$.
- $\forall x \in \mathbb{Z}$, $\forall y \in \mathbb{R}$, $\lfloor x + y \rfloor = x + \lfloor y \rfloor$
- Feel more confident there is no need to add an extra **+1**. Done by playing with examples (i.e is $\lceil \log \log_3(82) \rceil - 1$ true? Would the loop run only once?)

# Question 2

- **Predicate Logic:** $\forall n \in \mathbb{N}, n \geq 3 \Rightarrow 5^n + 50 < 6^n$

  *Proof.* Let $n \in \mathbb{N}$.

  We will prove the statement by induction on $n$.

**Base Case ($n = 3$):**

Let $n = 3$.

We want to show $5^3 + 50 < 6^3$.

Starting from $5^3 + 50$, we can calculate

$$\begin{aligned} 5^3 + 50 &= 125 + 50 && (1) \\ &= 175 && (2) \\ &< 216 && (3) \\ &< 6^3 && (4) \end{aligned}$$

**Inductive Case:**

Let $n \in \mathbb{N}$. Assume $n \geq 3$ and $5^n + 50 < 6^n$.

We want to show $5^{n+1} + 50 < 6^{n+1}$.

Starting from $5^{n+1} + 50$, we can calculate

$$\begin{aligned} 50^{n+1} + 50 &= 5^n \cdot 5 + 50 && (5) \\ &< 5^n \cdot 5 + 50 \cdot 5 && (6) \\ &< 5(5^n + 50) && (7) \end{aligned}$$

Then,

$$\begin{aligned} 50^{n+1} + 5 &< 5 \cdot 6^n && (8) \\ &< 6 \cdot 6^n && (9) \\ &< 6^{n+1} && (10) \end{aligned}$$

by using inductive hypothesis (i.e $5^n + 50 < 6^n$)

□

**Correct Solution:**

Let $n \in \mathbb{N}$.

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Then,

$$50^{n+1} + 5 < 5 \cdot 6^n \tag{8}$$
$$< 6 \cdot 6^n \tag{9}$$
$$= 6^{n+1} \tag{10}$$

by using inductive hypothesis (i.e $5^n + 50 < 6^n$)

**Notes:**

- Noticed professor uses '=' sign if the expression's value remains unchanged from the one before

  See equation 5 and 6 for example.

# Question 3

- **Statement:** $\exists a \in \mathbb{R}^+$, $an + 1 \in \Theta(n^3)$

  **Negation of Statement:** $\forall a \in \mathbb{R}^+$, $\forall c_1, c_2, n_0 \in \mathbb{R}^+$, $\exists n \in \mathbb{N}$, $(n \geq n_0) \wedge \big((an + 1 < c_1 n^3) \vee (an + 1 > c_2 n^3)\big)$

  *Proof.* Let $n = \left\lceil max(n_0, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{1}{c_1}}) \right\rceil + 1$.

  We will disprove the statement by showing $n \geq n_0$ and $an + 1 < c_1 n^3$

  **Part 1 (Showing $n \geq n_0$):**

  Using the fact that $\left\lceil max(n_0, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{1}{c_1}}) \right\rceil$ will result in a value greater than or equal to $n_0$, we can calculate

$$
\begin{aligned}
n_0 &\le \left\lceil max(n_0, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{1}{c_1}}) \right\rceil && (1)\\
&\le \left\lceil max(n_0, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{1}{c_1}}) \right\rceil + 1 && (2)
\end{aligned}
$$

Then, because we know $n = \left\lceil max(n_0, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{1}{c_1}}) \right\rceil + 1$, we can conclude

$$
n_0 \le n \qquad (3)
$$

$\square$

**Part 2 (Showing $an + 1 < c_1 n^3$):**

We will prove $an + 1 < c_1 n^3$ by showing $an < \frac{c_1}{2} n^3$ and $1 < \frac{c_1}{2} n^3$, and then combining the two together.

For the first inequality, because we know $n = \left\lceil max(n_0, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{1}{c_1}}) \right\rceil + 1 > \sqrt{\frac{2a}{c_1}}$, we can conclude

$$
\begin{aligned}
\sqrt{\frac{2a}{c_1}} &< n && (4)\\
\frac{2a}{c_1} &< n^2 && (5)\\
a &< \frac{c_1}{2} n^2 && (6)\\
an &< \frac{c_1}{2} n^3 && (7)
\end{aligned}
$$

For the second inequality, because we know $n = \left\lceil max(n_0, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{1}{c_1}}) \right\rceil + 1 > \sqrt[3]{\frac{1}{c_1}}$, we can conclude

$$\sqrt[3]{\frac{1}{c_1}} < n \tag{8}$$
$$\frac{1}{c_1} < n^3 \tag{9}$$
$$1 < n^3 \tag{10}$$

Then,

$$an + 1 < \frac{c_1}{2} \cdot n^3 + \frac{c_1}{2} \cdot n^3 \tag{11}$$
$$an + 1 < c_1 n^3 \tag{12}$$

**Notes:**

- I struggled on this question.
- Learned **+1** in $\left\lceil max(n_0, \sqrt{\frac{2a}{c_1}}, \sqrt[3]{\frac{1}{c_1}}) \right\rceil + 1 > \sqrt[3]{\frac{1}{c_1}}$ is to allow the use of inequality sign '$<$'.
- Learned that when $c_1$ is in inequality, with multiple terms like $an+1$ on the other side, and is asking to disprove it, I should first divide them up, find valid n for each term, and then recombine to create a valid $n$.

  See figure 1 for example

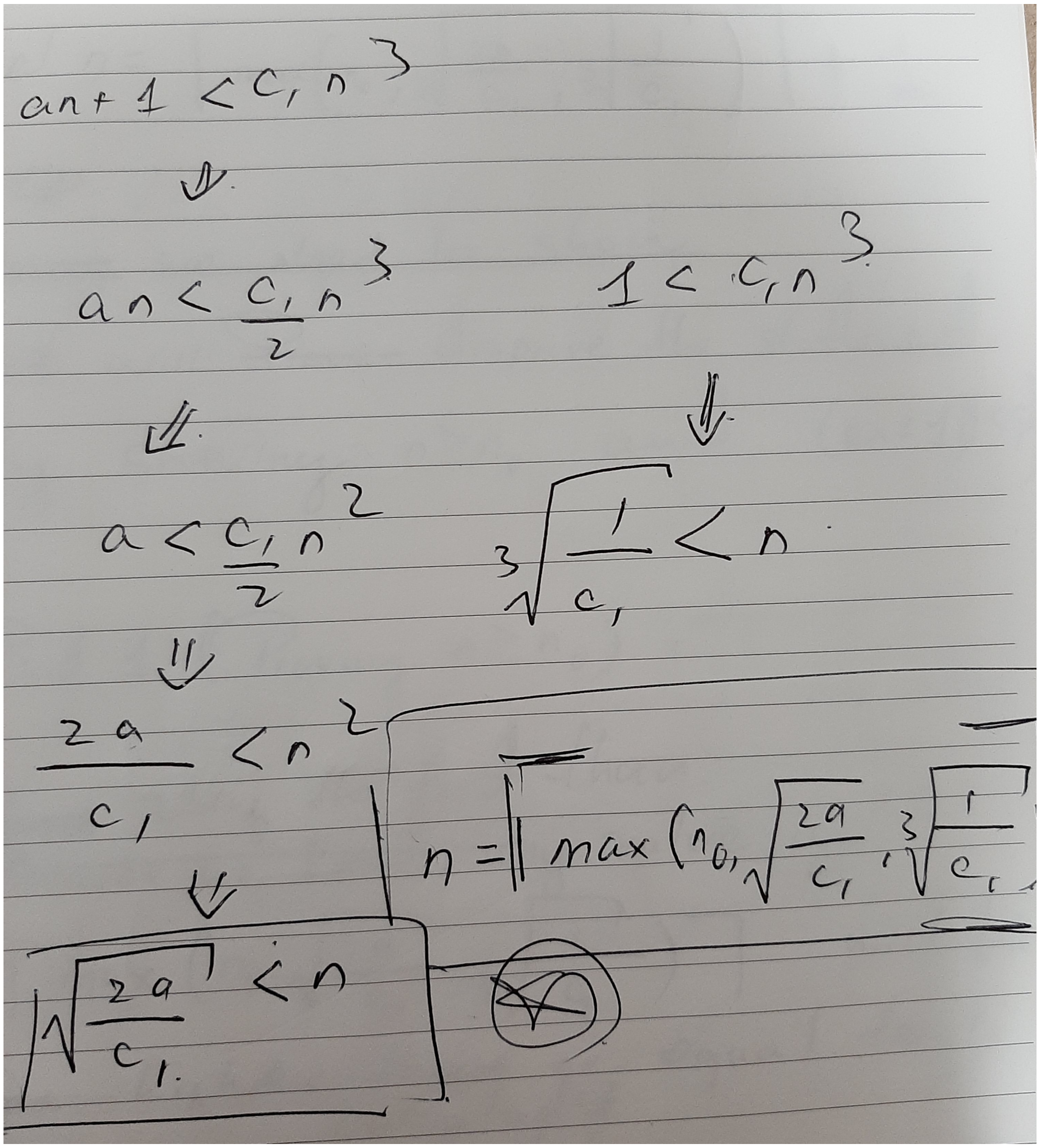


Figure 1: A sample work for question 3

# Question 4

a. *Proof.* Let $n \in \mathbb{N}$.

We will determine the exact number of iterations by first evaluating the number of iterations of loop 2, and then evaluating the number of iterations of loop 1.

For loop 2, because we know it starts at $j = 0$, and ends at $j = i - 1$, with $j$ increasing by 3 per iteration, we can conclude the loop has

$$\left\lceil \frac{i - 1 - 0 + 1}{3} \right\rceil = \left\lceil \frac{i}{3} \right\rceil \tag{1}$$

iterations.

For loop 1, because we know it starts at $i = 0$ and ends at $i = n^2 - 1$, with each iteration taking $\lceil \frac{i}{3} \rceil$ steps, we can conclude the loop takes total of

$$\sum_{i=0}^{n^2-1} \left\lceil \frac{i}{3} \right\rceil \tag{2}$$

iterations.

Then, because we know the floor and ceilings signs can be ignored, we can conclude the exact total number of iterations is

$$\begin{align}
\sum_{i=0}^{n^2-1} \frac{i}{3} &= \frac{1}{3} \cdot \sum_{i=0}^{n^2-1} i \tag{3} \\
&= \frac{1}{3} \cdot \frac{(n^2 - 1)(n^2 - 1 + 1)}{2} \tag{4} \\
&= \frac{(n^2 - 1) \cdot n^2}{6} \tag{5}
\end{align}$$

□