

CSC236 Term Test 1 Version 2 Solution

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Question 1

- *Proof.* Define $P(n) : f(n) = 3^n$.

I will use complete induction to prove that $\forall n \in \mathbb{N}, n > 2 \Rightarrow P(n)$.

Base Case ($n = 0$):

Let $n = 0$.

Then, the definition of $f(n)$ tells us $f(n) = 1$.

Then, we have

$$\begin{aligned} f(n) &= 3^0 & (1) \\ &= 3^n & (2) \end{aligned}$$

Thus, $P(n)$ follows.

Base Case ($n = 1$):

Let $n = 1$.

Then, the definition of $f(n)$ tells us $f(n) = 3$.

Then, we have

$$\begin{aligned} f(n) &= 3^1 & (3) \\ &= 3^n & (4) \end{aligned}$$

Thus, $P(n)$ follows.

Base Case ($n = 2$):

Let $n = 2$.

Then, the definition of $f(n)$ tells us $f(n) = 9$.

Then, we have

$$f(n) = 3^2 \tag{5}$$

$$= 3^n \tag{6}$$

Thus, $P(n)$ follows.

Case ($n > 2$):

Assume $n > 2$.

Then, since $0 \leq n - 1 < n$, $0 \leq n - 2 < n$, and $0 \leq n - 3 < n$, the complete induction tells us $P(n - 1)$, $P(n - 2)$, and $P(n - 3)$, i.e. $f(n - 1) = 3^{n-1}$, $f(n - 2) = 3^{n-2}$, and $f(n - 3) = 3^{n-3}$, respectively.

Then, using these facts, we can write

$$f(n) = f(n - 1) + 3f(n - 2) + 9f(n - 3) \tag{7}$$

$$= 3^{n-1} + 3 \cdot 3^{n-2} + 3^2 \cdot 3^{n-3} \tag{8}$$

$$= 3^{n-1} + 3^{n-1} + 3^{n-1} \tag{9}$$

$$= 3^{n-1}(1 + 1 + 1) \tag{10}$$

$$= 3^{n-1}3 \tag{11}$$

$$= 3^n \tag{12}$$

Thus, $P(n)$ follows. □

Correct Solution:

Define $P(n) : f(n) = 3^n$.

I will use complete induction to prove that $\forall n \in \mathbb{N}, P(n)$.

Inductive Step:

Let $n \in \mathbb{N}$. Assume $H(n) : \bigwedge_{i=0}^{n-1} P(i)$. I will prove $P(n)$ follows. That is, $f(n) = 3^n$.

Base Case ($n = 0$):

Let $n = 0$.

Then, the definition of $f(n)$ tells us $f(n) = 1$.

Then, we have

$$f(n) = 3^0 \tag{13}$$

$$= 3^n \tag{14}$$

Thus, $P(n)$ follows.

Base Case ($n = 1$):

Let $n = 1$.

Then, the definition of $f(n)$ tells us $f(n) = 3$.

Then, we have

$$f(n) = 3^1 \tag{15}$$

$$= 3^n \tag{16}$$

Thus, $P(n)$ follows.

Base Case ($n = 2$):

Let $n = 2$.

Then, the definition of $f(n)$ tells us $f(n) = 9$.

Then, we have

$$f(n) = 3^2 \tag{17}$$

$$= 3^n \tag{18}$$

Thus, $P(n)$ follows.

Case ($n > 2$):

Assume $n > 2$.

Then, since $0 \leq n-1 < n$, $0 \leq n-2 < n$, and $0 \leq n-3 < n$, the complete induction tells us $P(n-1)$, $P(n-2)$, and $P(n-3)$, i.e. $f(n-1) = 3^{n-1}$, $f(n-2) = 3^{n-2}$, and $f(n-3) = 3^{n-3}$, respectively.

Then, using these facts, we can write

$$f(n) = f(n-1) + 3f(n-2) + 9f(n-3) \quad [\text{By definition, since } n > 2] \quad (19)$$

$$= 3^{n-1} + 3 \cdot 3^{n-2} + 3^2 \cdot 3^{n-3} \quad (20)$$

$$= 3^{n-1} + 3^{n-1} + 3^{n-1} \quad (21)$$

$$= 3^{n-1}(1 + 1 + 1) \quad (22)$$

$$= 3^{n-1}3 \quad (23)$$

$$= 3^n \quad (24)$$

Thus, $P(n)$ follows.

Notes:

1. Learned that $n > i$ in $\forall n \in \mathbb{N}, n > i \Rightarrow P(n)$ is used when $P(n)$ is true starting $i + 1$.

If $P(n)$ is true for all natural numbers, then $\forall n \in \mathbb{N}, P(n)$ is used.

2. Learned that ‘Assume $n > 2$ ’ in ‘Let $n \in \mathbb{N}$. Assume $n > 2$. Assume $H(n) : \bigwedge_{i=0}^{n-1} P(i)$ ’. is used when $P(i)$ is true starting $n = 3$.

If $P(i)$ is true for all natural numbers, then, ‘Let $n \in \mathbb{N}$. Assume $H(n) : \bigwedge_{i=0}^{n-1} P(i)$ ’ is used.

Question 2

Question 3