Problem Set 4 Solution

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Question 1

a. **Statement:** $\forall f, g : \mathbb{N} \to \mathbb{R}^+,\ b \in \mathbb{R}^+,\ (g(n) \in \Theta(f(n))) \wedge (n_0 \in \mathbb{N},\ n \geq n_0 \Rightarrow f(n) \geq b \wedge g(n) \geq b) \wedge (b > 1) \Rightarrow \log_b(g(n)) \in \Theta(\log_b(f(n)))$

Statement Expanded: $\forall f, g : \mathbb{N} \to \mathbb{R}^+,\ b \in \mathbb{R}^+,\ \Big(\exists c_1, c_2, n_0 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)\Big) \wedge \Big(\exists n_1 \in \mathbb{N},\ n \geq n_1 \Rightarrow f(n) \geq b \wedge g(n) \geq b\Big) \wedge \Big(b > 1\Big) \Rightarrow \Big(\exists d_1, d_2, n_2 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_2 \Rightarrow d_1 \cdot \log_b(g(n)) \leq \log_b(f(n)) \leq d_2 \cdot \log_b(g(n)\Big)$

Proof. Let $f, g : \mathbb{N} \to \mathbb{R}^+$, and $b \in \mathbb{R}^+$. Assume $c_1 = 1$, $c_2 = b$, and $n_0 = 1$, and $n \in \mathbb{N}$ such that $n \geq n_0$ and $c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$. Assume $f(n)$ and $g(n)$ are eventually $\geq b$. Assume $b > 1$. Let $d_1 = 1$, $d_2 = 2$, and $n_2 = n_0$. Assume $n \geq n_2$.

We need to show $d_1 \cdot \log_b g(n) \leq \log_b f(n) \leq d_2 \cdot \log_b g(n)$.

We will do so in two parts. One for $(d_1 \cdot \log_b g(n) \leq \log_b f(n))$ and the other for $(\log_b f(n) \leq d_2 \cdot \log_b g(n))$.

Part 1 ($d_1 \cdot \log_b g(n) \leq \log_b f(n)$):

The assumption tell us

$$c_1 \cdot g(n) \leq f(n) \tag{1}$$

Then, it follows from the fact $\forall x, y \in \mathbb{R}^+, x \geq y \Leftrightarrow \log x \geq \log y$

$$\log(c_1 \cdot g(n)) \leq \log(f(n)) \tag{2}$$

Then, using the fact $b > 1$, we can calculate

$$\frac{\log(c_1 \cdot g(n))}{\log b} \leq \frac{\log(f(n))}{\log b} \tag{3}$$
$$\frac{\log(c_1) + \log(g(n))}{\log b} \leq \frac{\log(f(n))}{\log b} \tag{4}$$

Then,

$$\frac{\log(g(n))}{\log b} \leq \frac{\log(f(n))}{\log b} \tag{5}$$

by the fact $c_1 = 1$ and $\log c_1 = 0$.

Then, since $\frac{\log f(x)}{\log b} = \log_b f(x)$,

$$\log_b(g(n)) \leq \log_b(f(n)) \tag{6}$$

Then, because we know $d_1 = 1$, we can conclude

$$\log_b(g(n)) \leq d_1 \cdot \log_b(f(n)) \tag{7}$$

Part 2 ($\log_b f(n) \leq d_2 \cdot \log_b g(n)$):

The assumption tells us

$$f(n) \leq c_2 \cdot g(n) \tag{8}$$

Then, it follows from the fact $\forall x, y \in \mathbb{R}^+, x \geq y \Leftrightarrow \log x \geq \log y$

$$\log(f(n)) \leq \log(c_2 \cdot g(n)) \tag{9}$$

Then, using the fact $b > 1$, we can calculate

$$\frac{\log(f(n))}{\log b} \leq \frac{\log(c_2 \cdot g(n))}{\log b} \tag{10}$$

$$\frac{\log(f(n))}{\log b} \leq \frac{\log(c_2) + \log(g(n))}{\log b} \tag{11}$$

Then, since $c_2 = b$,

$$\frac{\log(f(n))}{\log b} \leq \frac{\log(b) + \log(g(n))}{\log b} \tag{12}$$

Then, using the fact $g(n)$ is eventually $\geq b$, we can write

$$\frac{\log(f(n))}{\log b} \leq \frac{\log(g(n)) + \log(g(n))}{\log b} \tag{13}$$

$$\frac{\log(f(n))}{\log b} \leq \frac{2 \cdot \log(g(n))}{\log b} \tag{14}$$

Then, since $\frac{\log f(x)}{\log b} = \log_b f(x)$,

$$\log_b(f(n)) \leq 2 \cdot \log_b(g(n)) \tag{15}$$

Then, because we know $d_2 = 2$, we can conclude

$$\log_b(f(n)) \leq d_2 \cdot \log_b(g(n)) \tag{16}$$

$\square$

Notes:

- $\forall x, y \in \mathbb{R}^+, x \geq y \Leftrightarrow \log x \geq \log y$
- $\exists c_1, c_2, n_0 \in \mathbb{R}^+$, $\forall n \in \mathbb{N}, n \geq n_0 \Rightarrow c_1 \cdot g(n) \leq f(n) \leq c2 \cdot g(n)$
- **Definition of Eventually:** $\exists n_0 \in \mathbb{N}, n \geq n_0 \Rightarrow P$, where $P : \mathbb{N} \to \{\text{True}, \text{False}\}$

b. *Proof.* Let $k \in \mathbb{N}$.

First, we will analyze the cost of loop 2 over iteration of loop 1.

The code tells us loop 2 starts at $j_k = 1$ with j_k increasing by a factor of 3 per iteration until $j_k \geq 1$.

Using these facts, we can calculate that the terminating condition occurs when

$$\begin{aligned} 3^k &\geq i && (1) \\ k &\geq \log_3 i && (2) \end{aligned}$$

Because we know the number of iterations is the smallest value of k satisfying the above inequality, we can conclude loop 2 has

$$\lceil \log_3 i \rceil \tag{3}$$

iterations.

Next, we need to determine the total number of iterations of loop 2 over all iterations of loop 1.

The code tells us loop 1 starts at $i = 1$ and ends at $i = n$ with each i increasing by 1 per iteration.

Using these facts, we can conclude loop 2 has total of

$$\lceil \log_3 1 \rceil + \lceil \log_3 2 \rceil + \cdots + \lceil \log_3 n \rceil = \sum_{i=1}^{n} \lceil \log_3 i \rceil \tag{4}$$

iterations. □

c. After scratching head and looking at solution many times, I realized that there are many things I do not yet understand, and it's the best to write what I have and learn from the solution. Here is my best attempt :).

Proof. Let $n \in \mathbb{N}$.

The previous answer tells us the exact cost of the algorithm is

$$\sum_{i=1}^{n} \lceil \log_3 i \rceil \tag{1}$$

Then, it follows by changing the variable i to $i' = \log_3 i$ we can write

$$\sum_{i'=0}^{\lceil \log_3 n \rceil} i' \tag{2}$$

Then, because we know $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$, we can conclude

$$\sum_{i'=0}^{\lceil \log_3 n \rceil} i' = \frac{(\lceil \log_3 n \rceil)(\lceil \log_3 n \rceil + 1)}{2} \tag{3}$$

$$= \frac{\lceil \log_3 n \rceil^2 + \lceil \log_3 n \rceil}{2} \tag{4}$$

Then, we can conclude the runtime of the algorithm is $\Theta(\log_3^2 n)$. □

Correct Solution:

We need to determne Θ of the algorithm.

We will prove that the Θ of the algorithm is $\Theta(n \log n)$.

The answer to previous question tells us the total exact cost of the algorithm is

$$\sum_{i=1}^{n} \lceil \log_3 i \rceil \tag{5}$$

Then, by using fact 1 $\forall x \in \mathbb{R}, x \leq \lceil x \rceil \leq x + 1$, we can calculate

$$\sum_{i=1}^{n} \log_3 i \leq \sum_{i=1}^{n} \lceil \log_3 i \rceil \leq \sum_{i=1}^{n} \Big(\log_3 i + 1 \Big) \tag{6}$$

$$\sum_{i=1}^{n} \log_3 i \leq \sum_{i=1}^{n} \lceil \log_3 i \rceil \leq \Big(\sum_{i=1}^{n} \log_3 i + \sum_{i=1}^{n} 1 \Big) \tag{7}$$

$$\sum_{i=1}^{n} \log_3 i \leq \sum_{i=1}^{n} \lceil \log_3 i \rceil \leq \sum_{i=1}^{n} \log_3 i + n \tag{8}$$

Then,

$$\log_3 \Big(\prod_{i=1}^{n} i \Big) \leq \sum_{i=1}^{n} \lceil \log_3 i \rceil \leq \log_3 \Big(\prod_{i=1}^{n} i \Big) + n \tag{9}$$

$$\log_3(n!) \leq \sum_{i=1}^{n} \lceil \log_3 i \rceil \leq \log_3(n!) + n \tag{10}$$

by the fact $\forall a, b \in \mathbb{R}^+, \log(a) + \log(b) = \log(ab)$.

Then,

$$\frac{\ln n!}{\ln 3} \leq \sum_{i=1}^{n} \lceil \log_3 i \rceil \leq \frac{\ln(n!)}{\ln 3} + n \tag{11}$$

by changing the base to e using the formula $\log_3 n! = \frac{\log_e n!}{\log_e 3} = \frac{\ln n!}{\ln 3}$.

Now, the fact 2 tells us $n! \in \Theta(e^{n \ln n - n + \frac{1}{2} \ln n})$.

Because we know from fact 3 that $n \ln n - n + \frac{1}{2} \ln n$ is eventually ≥ 1, we can conclude $e^{n \ln n - n \frac{1}{2} \ln n}$ is eventually $\geq e$.

Since $n!$ is also eventually $\geq e$, by using solution to problem 1.a with $g(n) = n!$ and $f(n) = e^{n \ln n - n + \frac{1}{2} \ln}$ and $b = e$, we can write

$$\ln(n!) \in \Theta(\ln(e^{n \ln n - n + \frac{1}{2}\ln n})) \tag{12}$$

$$\ln(n!) \in \Theta(n \ln n - n + \frac{1}{2}\ln n) \tag{13}$$

Then,

$$\ln(n!) \in \Theta(n \ln n) \tag{14}$$

by the fact $n \ln n - n + \frac{1}{2}\ln n \in \Theta(n \ln n)$.

So, since the algorithm runs at least $\frac{\ln n!}{\ln 3}$, we can conclude it has asymptotic lower bound of $\Omega(n \ln n)$, and since the algorithm runs at most $\frac{\ln n!}{\ln 3} + n$, we can conclude it has upper bound running time of $\mathcal{O}(n \ln n)$.

Since the value of Ω and $\mathcal{O}$ are the same, we can conclude the algorithm has running time of $\Theta(n \ln n)$ or $\Theta(n \log n)$.

Question 2

Question 3

Question 4