

CSC373 Worksheet 1 Solution

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1. Notes:

- Strassen's method for matrix multiplication

- Reduces the time complexity of matrix multiplication from $O(n^3)$ to $O(n^{\log_2 7}) = O(n^{2.81})$
- Has four steps

- 1) Divide the input matrices A and B and output matrix C into $n/2 \times n/2$ submatrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$

- 2) Create 10 matrices, S_1, S_2, \dots, S_{10} each of which is $n/2 \times n/2$ and is the sum or difference of two matrices created in step 1

$$S_1 = B_{12} - B_{22}$$

$$S_2 = A_{11} + A_{12}$$

$$S_3 = A_{21} + A_{22}$$

$$S_4 = B_{21} - B_{11}$$

$$S_5 = A_{11} + A_{22}$$

$$S_6 = B_{11} + B_{22}$$

$$S_7 = A_{12} - A_{22}$$

$$S_8 = B_{21} + B_{22}$$

$$S_9 = A_{11} - A_{21}$$

$$S_{10} = B_{11} + B_{12}$$

- 3) Recursively multiply $n/2 \times n/2$ matrices seven times to compute the following $n/2 \times n/2$ matrices

$$P_1 = A_{11} \cdot S_1 = A_{11} \cdot B_{12} - A_{11} \cdot B_{22}$$

$$P_2 = S_2 \cdot B_{22} = A_{11} \cdot B_{22} + A_{12} \cdot B_{22}$$

$$P_3 = S_3 \cdot B_{11} = A_{21} \cdot B_{11} + A_{22} \cdot B_{11}$$

$$\begin{aligned}
P_4 &= A_{22} \cdot S_4 = A_{21} \cdot B_{11} + A_{22} \cdot B_{11} \\
P_5 &= S_5 \cdot S_6 = A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} \cdot B_{11} + A_{22} \cdot B_{22} \\
P_6 &= S_7 \cdot S_8 = A_{12} \cdot B_{21} + A_{12} \cdot B_{22} - A_{22} \cdot B_{21} - A_{22} \cdot B_{22} \\
P_7 &= S_9 \cdot S_{10} = A_{11} \cdot B_{11} + A_{11} \cdot B_{12} - A_{21} \cdot B_{11} - A_{21} \cdot B_{12}
\end{aligned}$$

4) Construct the four $n/2 \times n/2$ submatrices of the product C

$$\begin{aligned}
C_{11} &= P_5 + P_4 - P_2 + P_6 = A_{11} \cdot B_{11} + A_{12} + B_{12} \\
C_{12} &= P_1 + P_2 = A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\
C_{21} &= P_3 + P_4 = A_{21} \cdot B_{11} + A_{22} \cdot B_{21} \\
C_{22} &= P_5 + P_1 - P_3 - P_7 = A_{22} \cdot B_{22} + A_{21} \cdot B_{12}
\end{aligned}$$

Example: Use Strassen's algorithm to compute the matrix product

$$\begin{pmatrix} 1 & 3 \\ 7 & 5 \end{pmatrix} \begin{pmatrix} 6 & 8 \\ 4 & 2 \end{pmatrix}$$

* **STEP 1**

$$A_{11} = 1, A_{12} = 3, A_{21} = 7, A_{22} = 5$$

$$B_{11} = 6, B_{12} = 8, B_{21} = 4, B_{22} = 2$$

* **STEP 2**

$$S_1 = B_{12} - B_{22} = 4 - 2 = 2$$

$$S_2 = A_{11} + A_{12} = 1 + 3 = 4$$

$$S_3 = A_{21} + A_{22} = 7 + 4 = 12$$

$$S_4 = B_{21} - B_{11} = 4 - 6 = -2$$

$$S_5 = A_{11} + A_{22} = 1 + 5 = 6$$

$$S_6 = B_{11} + B_{22} = 6 + 2 = 8$$

$$S_7 = A_{12} - A_{22} = 3 - 5 = -2$$

$$S_8 = B_{21} + B_{22} = 4 + 2 = 6$$

$$S_9 = A_{11} - A_{21} = 1 - 7 = -6$$

$$S_{10} = B_{11} + B_{12} = 6 + 4 = 10$$

* **STEP 3**

$$P_1 = A_{11} \cdot S_1 = A_{11} \cdot B_{12} - A_{11} \cdot B_{22} = 1 \cdot 4 - 1 \cdot 2 = 2$$

$$P_2 = S_2 \cdot B_{22} = A_{11} \cdot B_{22} + A_{12} \cdot B_{22} = 1 \cdot 2 + 3 \cdot 2 = 8$$

* **STEP 4**

- Is not preferred in practical purposes
 - 1) The constants used in Strassen's method are high and for a typical application Naive method works better.
 - 2) For Sparse matrices, there are better methods especially designed for them.
 - 3) The submatrices in recursion take extra space.
 - 4) Because of the limited precision of computer arithmetic on noninteger values, larger errors accumulate in Strassen's algorithm than in Naive Method

References:

- 1) GeeksForGeeks, Divide and Conquer — Set 5 (Strassen's Matrix Multiplication), [link](#)
- Regular matrix multiplication

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