CSC236 Worksheet 6 Review

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Question 1

- *Proof.* Assume that $\forall k \in \mathbb{N}$, $R(3^k) = k3^k$.

  I need to prove $R \in \Theta(n \lg n)$. That is, $R \in \mathcal{O}(n \lg n)$ and $R \in \Omega(n \lg n)$.

  I will do so in parts.

  Part 1 (Proving $R \in \mathcal{O}(n \lg n)$):

  Let $n \in \mathbb{N}$. Define $n^* = 3^{\lceil \log_3 n \rceil}$. Then, we have

  $$\lceil \log_3 n \rceil - 1 < \log_3 n \leq \lceil \log_3 n \rceil \Rightarrow n^*/3 < n \leq n^* \tag{1}$$

  I will also use the fact proved in week 7 tutorial exercise that R is non-decreasing.

  Let $d = 6$. Then, $d \in \mathbb{R}^+$. Let $B = 3$. Then, $B \in \mathbb{R}^+$. Assume $n \geq B$.

  I need to show $R(n) \leq dn \lg n$.

  And indeed, we have

  $$\begin{aligned}
  R(n) &\leq R(n^*) && \text{[Since } n \leq n^* \text{ and } R \text{ is non-decreasing]} && (2)\\
  &= n^* \log_3 n^* && \text{[By replacing } 3^k \text{ for } n^*] && (3)\\
  &\leq 3n \log_3 3n && \text{[Since } n \leq n^* \Rightarrow 3n \leq 3n^*] && (4)\\
  &= 3n(\log_3 n + 1) && && (5)\\
  &= 3n(\log_3 n + \log_3 n) && \text{[Since } n \leq B = 3 \Rightarrow \log_3 n \leq 1] && (6)\\
  &\leq 6n \log_3 n && && (7)\\
  &\leq dn \log_3 n && \text{[Since } d = 6] && (8)\\
  &\leq dn \lg n && && (9)
  \end{aligned}$$

Part 2 (Proving $R \in \Omega(n \lg n)$):

Let $n \in \mathbb{N}$. Define $n^* = 3^{\lceil \log_3 n \rceil}$. Then, we have

$$\lceil \log_3 n \rceil - 1 < \log_3 n \leq \lceil \log_3 n \rceil \Rightarrow n^*/3 < n \leq n^* \tag{10}$$

I will also use the fact proved in week 7 tutorial exercise that R is non-decreasing.

Let $d = 1/(6 \lg 3)$. Then, $d \in \mathbb{R}^+$. Let $B = 9$. Then, $B \in \mathbb{R}^+$. Assume $n \geq B$.

I need to show $R(n) \geq dn \lg n$.

And indeed, we have

$$\begin{aligned}
R(n) &\geq R(n/3) && \text{[Since } n^*/3 < n \text{ and } R \text{ is non-decreasing]} && (11)\\
&= (n^*/3) \log_3(n^*/3) && \text{[By replacing } 3^k \text{ for } n^*] && (12)\\
&= (n/3) \log_3(n/3) && \text{[Since } n < n^* \Rightarrow (n/3) \leq (n^*/3)] && (13)\\
&= (n/3)(\log_3 n - 1) && && (14)\\
&\geq (\log_3 n - (\log_3 n)/2) && \text{[Since } n \geq B = 9 \Rightarrow (\log_3 n)/2 \geq 1] && (15)\\
&= (n \log_3 n)/6 && && (16)\\
&= (n \lg n)/(6 \lg 3) && && (17)\\
&= dn \lg n && && (18)
\end{aligned}$$

$\square$