Problem Set 2 Solution

March 18, 2020

Question 1

a.

b. **Predicate Logic:** $\forall k, n \in \mathbb{Z}^+$, $\forall p \in \mathbb{N}$, $Prime(p) \wedge p^k < n < p^k + p \Rightarrow gcd(p^k, n) = 1$

Let $k, n \in \mathbb{Z}^+$, and $p \in \mathbb{N}$. Assume $Prime(p)$, and $p^k < n < p^k + p$.

Then, p^k can either be divided by 1 or p by fact 3.

Since, $p^k < n < p^k + p$, n cannot be written in multiples of p.

Then, it follows from the definition of divisibility that $p \nmid n$.

Since $p \nmid n$, but $1 \mid p^k$ and $1 \mid n$, $gcd(p^k, n) = 1$.

c. **Predicate Logic:** $\forall m \in \mathbb{Z}$, $\forall n_0 \in \mathbb{N}$, $\exists n \in \mathbb{N}\ n > n_0 \wedge gcd(n, n + m) = 1$

Since there are infinitely many primes by fact 4, let $Prime(n)$ and $n > m$.

Since $Prime(n)$, by fact 3, n can either be divided by 1 or n.

Since $n \mid n$, but $n \nmid m$, $n \nmid (n + m)$, and n can't be chosen as the greatest common divisor of n and $n + m$.

Since $gcd(n, n+m) \neq n$ but $1 \mid n$ and $1 \mid (n+m)$, $gcd(n, n+m) = 1$.

Then, it follows from above that the statement $\forall m \in \mathbb{Z}, \forall n_0 \in \mathbb{N}, \exists n \in \mathbb{N}$ $n > n_0 \wedge gcd(n, n+m) = 1$ is true.

d. **Definition of Primary Gap:** Let $a \in \mathbb{N}$. We say that a is a prime gap when there exists a prime p such that $p + a$ is also prime, and none of the numbers between p and $p + a$ (exclusive) are prime.

Case 1 ($a > 2$):

Let $a, p \in \mathbb{Z}^+$. Assume $PrimaryGap(a)$, $Primary(p)$, and $a > 2$.

Then, $2 \nmid p$ and $2 \nmid p + a$.

Then,

$$2 \mid (p + a) - a \tag{1}$$
$$2 \mid a \tag{2}$$

by fact 1.

Then it follows from above that in case $a > 2$, primary gap is divisible by 2.

Case 2 ($a \leq 2$):

Let $a, p \in \mathbb{Z}^+$. Assume $PrimaryGap(a)$, $Primary(p)$, and $a \leq 2$.

Then, only two primary numbers in $\mathbb{Z}^+$ exist - 1 and 2.

Then,

$$a = 2 - 1 \tag{1}$$
$$a = 1 \tag{2}$$

Then, it follows from above that in case $a \leq 2$, the value of primary gap is 1.

Question 2

a. Let $n \in \mathbb{N}$, and $x \in \mathbb{R}$.

Because we know $\forall x \in \mathbb{R}$, $0 \leq x - \lfloor x \rfloor < 1$ from fact 1, we can conclude $\lfloor x \rfloor \leq x < 1 + \lfloor x \rfloor$.

Then,

$$\begin{aligned}
\lfloor nx \rfloor - n\lfloor x \rfloor &\leq nx - n\lfloor x \rfloor && (1)\\
&\leq n(x - \lfloor x \rfloor) && (2)
\end{aligned}$$

by using the above.

Then,

$$\begin{aligned}
\lfloor nx \rfloor - n\lfloor x \rfloor &\leq n(x - \lfloor x \rfloor) && (3)\\
&< n && (4)\\
&< k && (5)
\end{aligned}$$

by using fact 1 and choosing $k = n$.

Then, it follows that the statement the statement $\forall n \in \mathbb{N}$, $\exists k \in \mathbb{N}$, $x \in \mathbb{R}$, $\lfloor nx \rfloor - n\lfloor x \rfloor \leq k$ is true.

b. **Negation of statement:** $\forall k \in \mathbb{N}$, $\exists m \in \mathbb{N}$, $\exists x \in \mathbb{R}$, $\lfloor nx \rfloor - n\lfloor x \rfloor > k$

Let $x = 0.5$ and $n = 2(k+1)$.

Then,

$$\begin{aligned}
\lfloor nx \rfloor - n\lfloor x \rfloor &= \lfloor \frac{2(k+1)}{2} \rfloor - n\lfloor 0.5 \rfloor && (1)\\
&= k + 1 - 0 && (2)\\
&= k + 1 && (3)\\
&> k && (4)
\end{aligned}$$

Then it follows that the statement $\exists k \in \mathbb{N}, \forall n \in \mathbb{N}, \forall x \in \mathbb{R}, \lfloor nx \rfloor - n\lfloor x \rfloor \leq k$ is false.

c. **Proof of $0 \leq \epsilon < 1$:**

Let $y \in \mathbb{R}^{\geq 0}$, and $n \in \mathbb{Z}$. Assume $n > y$, and assume $y = (n+\epsilon)^2 - n^2$.

Then,

$$\begin{align} y + n^2 &= (n+\epsilon)^2 \tag{1}\\ \pm\sqrt{y+n^2} - n &= \epsilon \tag{2}\\ \sqrt{y+n^2} - n &= \epsilon \tag{3}\\ & \tag{4} \end{align}$$

by assuming ϵ is in $\mathbb{R}^{\geq 0}$.

Then,

$$\sqrt{n^2} - n \leq \epsilon \tag{5}$$

by using inequality $y \geq 0$.

Then,

$$\begin{align} \sqrt{n^2} - n &\leq \epsilon \tag{6}\\ n - n &\leq \epsilon \tag{7}\\ 0 &\leq \epsilon \tag{8} \end{align}$$

And for $\epsilon < 1$, let $y \in \mathbb{R}^{\geq 0}$, and $n \in \mathbb{Z}$. Assume $n > y$, and assume $y = (n+\epsilon)^2 - n^2$.

$$
\begin{aligned}
y + n &= (\epsilon + n)^2 && (9)\\
\pm\sqrt{y + n^2} - n &= \epsilon && (10)\\
\sqrt{y + n^2} - n &= \epsilon && (11)\\
& && (12)
\end{aligned}
$$

by assuming ϵ is in $\mathbb{R}^{\geq 0}$.

Then,

$$
\begin{aligned}
\sqrt{y + n^2} - n &= \epsilon && (13)\\
\frac{(\sqrt{y - n^2})^2 - n^2}{\sqrt{y + n^2} + n} &= \epsilon && (14)
\end{aligned}
$$

by using hint 2 ($x - y = \frac{x^2 - y^2}{x + y}$).

Then,

$$
\begin{aligned}
\frac{(\sqrt{y - n^2})^2 - n^2}{\sqrt{y + n^2} + n} &= \epsilon && (15)\\
\frac{y}{\sqrt{y + n^2} + n} &= \epsilon && (16)\\
\frac{y}{\sqrt{y + y^2} + y} &> \epsilon && (17)
\end{aligned}
$$

by using assumption $n > y$.

Then,

$$\frac{y}{\sqrt{y+y^2}+y} > \epsilon \tag{18}$$

$$\frac{y}{\sqrt{y^2}+y} > \epsilon \tag{19}$$

$$\frac{y}{y+y} > \epsilon \tag{20}$$

$$\frac{y}{2y} > \epsilon \tag{21}$$

$$\frac{y}{y} > \epsilon \tag{22}$$

$$1 > \epsilon \tag{23}$$

Then, it follows from above that the statement $\forall y \in \mathbb{R}^{\geq 0}$, $\forall n \in \mathbb{Z}^+$, $n > y \Rightarrow (\exists \epsilon \in \mathbb{R}^{\geq 0},\ 0 \leq \epsilon < 1 \wedge y = (n+\epsilon)^2 - n^2)$ is true given $n > y$ and $y = (n+\epsilon)^2 - n^2$.

Proof of $y = (n+\epsilon)^2 - n^2$:

Let $y \in \mathbb{R}^{\geq 0}$, and $n \in \mathbb{Z}$. Assume $n > y$, and assume $0 \leq \epsilon < 1$.

Then,

$$\begin{aligned}
(n+\epsilon)^2 &= n^2 + 2n\epsilon + \epsilon^2 && (1)\\
(n+\epsilon)^2 - n^2 &= 2n\epsilon + \epsilon^2 && (2)\\
(n+\epsilon)^2 - n^2 &= 2n(\sqrt{y+n^2} - n) + (\sqrt{y+n^2} - n)^2 && (3)
\end{aligned}$$

by assuming $\epsilon = \sqrt{y+n^2} - n$ (since $0 \leq \sqrt{y+n^2} - n < 1$).

Then,

$$\begin{aligned}
(n+\epsilon)^2 - n^2 &= 2n(\sqrt{y+n^2} - n) + (\sqrt{y+n^2} - n)^2 && (4)\\
&= 2(\sqrt{y+n^2} - n)(2n + \sqrt{y+n^2} - n) && (5)\\
&= (\sqrt{y+n^2} - n)(\sqrt{y+n^2} + n) && (6)\\
&= (\sqrt{y+n^2})^2 - n^2 && (7)\\
&= y && (8)
\end{aligned}$$

Then, it follows from above that the statement $\forall y \in \mathbb{R}^{\geq 0}$, $\forall n \in \mathbb{Z}^{+}$, $n > y \Rightarrow (\exists \epsilon \in \mathbb{R}^{\geq 0},\ 0 \leq \epsilon < 1 \wedge y = (n+\epsilon)^2 - n^2)$ is true given $n > y$ and $0 \leq \epsilon < 1$.

d. **Predicate Logic:** $\exists f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0},\ f(x) = x^2 - (\lfloor x \rfloor)^2 \wedge Onto(f)$

Definition of Onto: $f : \mathbb{R} \to \mathbb{R}$ is **onto** when its codomain $\mathbb{R}$ only contains values that could possibly be output by f (and no impossible values).

Let $f(x) = x^2 - (\lfloor x \rfloor)^2$.

Then,

$$
\begin{aligned}
(x^2 - \lfloor x \rfloor^2) &\geq (x - \lfloor x \rfloor)(x + \lfloor x \rfloor) && (1)\\
&\geq (x - \lfloor x \rfloor) \cdot 0 && (2)\\
&\geq 0 && (3)
\end{aligned}
$$

by fact 1 (i.e. $\forall x \in \mathbb{R},\ 0 \leq x + \lfloor x \rfloor < 1$).

Since f has lower bound of 0, and covers all of its codomain, f is onto.

Question 3

a. **Predicate Logic**: $\forall f : \mathbb{R} \to \mathbb{R},\ f(x) = f(-x) \wedge -f(-x) = f(x) \Leftrightarrow f = 0$

Part 1: Proving in $\Rightarrow$ direction

Let $f : \mathbb{R} \to \mathbb{R}$. Assume $f(x) = f(-x) \wedge -f(-x) = f(x)$.

Then,

$$
\begin{aligned}
f(-x) - f(-x) &= 2f(x) && (1)\\
0 &= 2f(x) && (2)
\end{aligned}
$$

by adding $f(x) = f(-x)$ and$-f(-x) = f(x)$ together.

Then,

$$0 = f(x) \tag{3}$$

Then it follows that the statement $\forall f : \mathbb{R} \to \mathbb{R}, f(x) = f(-x) \wedge -f(-x) = f(x) \Rightarrow f = 0$ is true.

Part 2: Proving in $\Leftarrow$ direction

Let $f : \mathbb{R} \to \mathbb{R}$. Assume $f(x) = 0$.

Then,

$$\begin{aligned} -f(-x) &= -(-0) && (1)\\ &= 0 && (2)\\ &= f(x) && (3) \end{aligned}$$

It follows from above that $f(x) = 0$ is an odd function.

Also,

$$\begin{aligned} f(-x) &= (-0) && (4)\\ &= 0 && (5)\\ &= f(x) && (6) \end{aligned}$$

It follows from above that $f(x) = 0$ is an odd function.

Because we know $f(x) = 0$ is both even and odd, we can conclude that the statement $\forall f : \mathbb{R} \to \mathbb{R},\ f = 0 \Rightarrow f(x) = f(-x) \wedge -f(-x) = f(x)$ is true.

b. **Predicate Logic:** $\forall f : \mathbb{R} \to \mathbb{R}, \exists f_1, f_2 : \mathbb{R} \to \mathbb{R}, -f_1(x) = f_1(x) \wedge f_2(-x) = f_2(fx) \wedge f(x) = f_1(x) + f_2(x)$

Negation: $\exists f : \mathbb{R} \to \mathbb{R},\ \forall f_1, f_2 : \mathbb{R} \to \mathbb{R},\ -f_1(-x) \neq f_1(x) \lor f_2(x) \neq f_2(x) \lor f(x) \neq f_1(x) + f_2(x)$

Let f be an even function. Assume $Even(f_1)$ and $Odd(f_2)$.

Then,

$$\begin{align} f(-x) &= (f_1(-x) + f_2(-x)) \tag{1}\\ &= f_1(x) - f_2(x) \tag{2}\\ &\neq f(x) \tag{3} \end{align}$$

Then, it follows from negation of the statement that every function cannot be written as a sum of an even function and an odd function.

c. **Predicate Logic:** $\forall f : \mathbb{R} \to \mathbb{R},\ \exists f_1, f_2 : \mathbb{R} \to \mathbb{R},\ Even(f_1) \land Odd(f_2) \land f(x) = f_1(x) + f_2(x)$.

Negation: $\exists f : \mathbb{R} \to \mathbb{R}, \forall f_1, f_2 : \mathbb{R} \to \mathbb{R}, \neg Even(f_1) \lor \neg Odd(f_2) \lor f(x) \neq f_1(x) f_2(x)$

Let f be an even function. Assume $Even(f_1)$ and $Odd(f_2)$.

Then,

$$\begin{align} f(-x) &= f_1(-x) f_2(-x) \tag{1}\\ &= -f_1(x) f_2(x) \tag{2}\\ &\neq f(x) \tag{3} \end{align}$$

Then, it follows from negation of the statement that every function cannot be written as a product of an even function and an odd function.