CSC373 Worksheet 0 Solution

July 19, 2020

1. Recurrence: $T(n) = T(n-1) + n$

   Guess: $T(n) = \mathcal{O}(n^2)$.

   I need to show $T(n) \leq c \cdot n^2$.

$$\begin{aligned}
T(n) &\leq c(n-1)^2 + n && (1)\\
&= c(n^2 - 2n + 1) + n && (2)\\
&= cn^2 - c2n + c + n && (3)\\
&\leq cn^2 - c2n + cn + n && (4)\\
&= cn^2 - cn + n && (5)\\
&\leq cn^2 - cn + cn && (6)\\
&= cn^2 && (7)
\end{aligned}$$

   Notes:

   - Substitution method
     - Solves recurrences
       - Recurrence characters the running time of divide-and-conquer algorithm
     - How it works:
       1. Make a guess for the solution
       2. Use mathematical induction to prove the guess is correct or incorrect.

       Example:

       Recurrence: $T(n) = 2T(\lfloor n/2 \rfloor) + n$

       Guess: $T(n) = \mathcal{O}(n \log n)$,

       We need to show $T(n) \leq cn \lg n$.

1. Assume the bound holds for all positive $m < n$, in particular $m = \lfloor n/2 \rfloor$
2. Find the upper bound of $T(m)$

   $T(\lfloor n/2 \rfloor) \leq c\lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)$

3. Show $T(n) = 2T(\lfloor n/2 \rfloor) + n$ leads to $T(n) \leq cn \lg n$

$$
\begin{aligned}
T(n) &\leq 2(c\lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)) + n && (8)\\
&\leq cn \lg(n/2) + n && (9)\\
&= cn \lg(n) - cn \lg 2 + n && (10)\\
&= cn \lg(n) - cn + n && (11)\\
&\leq cn \lg(n) - cn + cn && (12)\\
&\leq cn \lg(n) && (13)
\end{aligned}
$$

4. Show that the boundary holds using mathematical induction

   Doesn't have information in detail. Skipping this for now.

- Making good guess
  - Three suggestions
  1. Using recursion tree
  2. Through practice
  3. prove loose upper and lower bounds on the recurrence and then reduce the range of uncertainty

2. Recurrence: $T(n) = T(\lceil n/2 \rceil) + 1$

   Guess: $T(n) = \mathcal{O}(\lg n)$.

   I need to show $T(n) \leq c \cdot \lg n$.

$$
\begin{aligned}
T(n) &\leq c \lg(\lceil n/2 \rceil) + 1 && (1)\\
&\leq c \lg(n/2) + 1 && (2)\\
&= c(\lg n - \lg 2) + 1 && (3)\\
&= c(\lg n - 1) + 1 && (4)\\
&= c \lg n - c + 1 && (5)\\
&\leq c \lg n - c + c && (6)
\end{aligned}
$$

Correct Solution:

Recurrence: $T(n) = T(\lceil n/2 \rceil) + 1$

Guess: $T(n) = \mathcal{O}(\lg n)$.

I need to show $T(n) \leq c \cdot \lg n$.

$$\begin{aligned}
T(n) &\leq c \lg(\lceil n/2 \rceil) + 1 && (1)\\
&\leq c \lg(n/2) + 1 && (2)\\
&= c(\lg n - \lg 2) + 1 && (3)\\
&= c(\lg n - 1) + 1 && (4)\\
&= c \lg n - c + 1 && (5)\\
&\leq c \lg n - c + c && (6)
\end{aligned}$$

The solution holds for $c \geq 1$.

3. Recurrence: $T(n) = 2T(\lfloor n/2 \rfloor) + n$

   Guess (Upperbound): $T(n) = \mathcal{O}(n \lg n)$.

   I first need to show $T(n) \leq c \cdot n \lg n$.

$$\begin{aligned}
T(n) &= 2T(\lfloor n/2 \rfloor) + n && (1)\\
&= 2c\lfloor n/2 \rfloor \lg \lfloor n/2 \rfloor + n && (2)\\
&\leq 2c \cdot (n/2) \lg(n/2) + n && (3)\\
&= c \cdot n(\lg n - 1) + n && (4)\\
&= cn \lg n - cn + n && (5)\\
&\leq cn \lg n - cn + cn && (6)\\
&\leq cn \lg n && (7)
\end{aligned}$$

   The above inequality holds for $c \geq 1$.

   Guess (Lowerbound): $T(n) = \Omega(n \lg n)$.

I first need to show $d \cdot (n-2)\lg(n-2) \leq T(n)$.

$$
\begin{aligned}
T(n) &= 2T(\lfloor (n-2)/2 \rfloor) + n && (8)\\
&\geq 2d\lfloor (n-2)/2 \rfloor \lg\lfloor (n-2)/2 \rfloor + n && (9)\\
&\geq 2d \cdot ((n-2)/2)\lg((n-2)/2) + n && (10)\\
&= d \cdot (n-2)(\lg(n-2) - 1) + n && (11)\\
&= d \cdot (n-2)\lg(n-2) - d \cdot (n-2) + n && (12)\\
&\geq d \cdot (n-2)\lg(n-2) - d \cdot (n-2) + (n-2) && (13)\\
&\geq d \cdot (n-2)\lg(n-2) - d \cdot (n-2) + d \cdot (n-2) && (14)\\
&= d \cdot (n-2)\lg(n-2) && (15)
\end{aligned}
$$

The above inequality holds for $0 \leq d < 1$.

Notes:

- Both upper bound and lower bound don't need to be the same

4.3-3

> We saw that the solution of $T(n) = 2T(\lfloor n/2 \rfloor) + n$ is $O(n \lg n)$. Show that the solution of this recurrence is also $\Omega(n \lg n)$. Conclude that the solution is $\Theta(n \lg n)$.

First, we guess $\boxed{T(n) \leq cn\lg n,}$ upper bound

$$
\begin{aligned}
T(n) &\leq 2c\lfloor n/2 \rfloor \lg \lfloor n/2 \rfloor + n\\
&\leq cn\lg(n/2) + n\\
&= cn\lg n - cn\lg 2 + n\\
&= cn\lg n + (1-c)n\\
&\leq cn\lg n,
\end{aligned}
$$

where the last step holds for $c \geq 1$.

lower bound

Next, we guess $\boxed{T(n) \geq c(n+2)\lg(n+2),}$

$$
\begin{aligned}
T(n) &\geq 2c(\lfloor n/2 \rfloor + 2)(\lg(\lfloor n/2 \rfloor + 2) + n\\
&\geq 2c(n/2 - 1 + 2)(\lg(n/2 - 1 + 2) + n\\
&= 2c\frac{n+2}{2}\lg\frac{n+2}{2} + n\\
&= c(n+2)\lg(n+2) - c(n+2)\lg 2 + n\\
&= c(n+2)\lg(n+2) + (1-c)n - 2c\\
&\geq c(n+2)\lg(n+2),
\end{aligned}
$$

where the last step holds for $n \geq \frac{2c}{1-c}$, $0 \leq c < 1$.

4. Recurrence (Merge sort):

$$
T(n) = \begin{cases} \Theta(1) & \text{if } n = 1\\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \Theta(n) & \text{if } n > 1 \end{cases}
$$

Guess (upper bound): $T(n) \leq c \cdot (n-2) \cdot \lg(n-2)$

$$\begin{align}
T(n) &\leq c(\lceil n/2 \rceil - 2)\lg(\lceil n/2 \rceil - 2) + c(\lfloor n/2 \rfloor - 2)\lg(\lfloor n/2 \rfloor - 2) + dn \tag{1}\\
&= c(n/2 + 1 - 2)\lg(n/2 + 1 - 2) + c(n/2 + 1 - 2)\lg(n/2 + 1 - 2) + dn \tag{2}\\
&= c((n-2)/2)\lg((n-2)/2) + c((n-2)/2)\lg((n-2)/2) + dn \tag{3}\\
&= c(n-2)\lg((n-2)/2) + dn \tag{4}\\
&= c(n-2)\lg(n-2) - c(n-2) + dn \tag{5}\\
&= c(n-2)\lg(n-2) - (d-c)n + 2c \tag{6}\\
&= c(n-2)\lg(n-2) \tag{7}
\end{align}$$

The bound holds as long as $c > d$.

Guess (lower bound): $c \cdot (n-2) \cdot \lg(n-2) \leq T(n)$

$$\begin{align}
T(n) &\leq c(\lceil n/2 \rceil + 1)\lg(\lceil n/2 \rceil + 1) + c(\lfloor n/2 \rfloor + 1)\lg(\lfloor n/2 \rfloor + 1) + dn \tag{8}\\
&\leq c(n/2 - 1 + 1)\lg(n/2 - 1 + 1) + c(n/2 - 1 + 1)\lg(n/2 - 1 + 1) + dn \tag{9}\\
&= c(n/2)\lg(n/2) + c(n/2)\lg(n/2) + dn \tag{10}\\
&= cn\lg(n/2) + dn \tag{11}\\
&= cn\lg(n) - cn + dn \tag{12}\\
&= cn\lg(n) + (d-c)n \tag{13}\\
&\leq c(n-1)\lg(n-1) \tag{14}
\end{align}$$

The bound holds as long as $d > c$, and $0 \leq c < 1$

Notes:

- the n here is asymptotically large

5. Recurrence: $T(n) = 2T(\lfloor n/2 \rfloor + 17) + n$

Guess (upper bound): $cn \lg n$

$$\begin{align}
T(n) &\leq 2c(\lfloor n/2 \rfloor + 17)\lg(\lfloor n/2 \rfloor + 17) + n \tag{15}\\
&\leq 2c((n/2) + 17)\lg((n/2) + 17) + n \tag{16}\\
& \tag{17}
\end{align}$$