Worksheet 20 Solution

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Question 1

a. *Proof.* Let $V = \{1, 2, 3, 4, 5, 6\}$, $E = \{(1, 2), (1, 6), (2, 3), (3, 4), (4, 5), (5, 6)\})$.

We need to prove the graph $G = (V, E)$ is bipartite by proving the following properties:

1. There exists subsets $V_1, V_2 \subset V$ such that $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V.
2. Every edge in E has exactly one endpoint in V_1 and one in V_2.

We will prove the properties in parts.

Part 1 (Proving $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V):

Let $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

We need to prove $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V, i.e $V_1 \cup V_2 = V \wedge V_1 \cap V_2 = \emptyset$.

First, we need to show the subsets V_1 and V_2 are non-empty.

The header tells us both subsets V_1 and V_2 have more than 1 elements.

Then, using these facts, we can conclude $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$.

Finally, we need to show $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$.

The header tells us $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

Then, we can calculate

$$V_1 \cup V_2 = \{1, 2, 3, 4, 5, 6\} = V \tag{1}$$
$$V_1 \cap V_2 = \emptyset \tag{2}$$

Part 2 (Proving every edge in E has exactly one endpoint in V_1 and one in V_2):

Let $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

We need to show every edge in E has exactly one endpoint in V_1 and one in V_2.

The header tells us $V_1 = \{1, 3, 5\}$, $V_2 = \{2, 4, 6\}$, and $E = \{(1, 2), (1, 6), (2, 3), (3, 4), (4, 5), (5, 6)\})$.

Using these facts, we can generate the following table.

Edge (1,2)	- 1 is in V_1 - 2 is in V_2	Edge (3,4)	- 3 is in V_1 - 4 is in V_2
Edge (1,6)	- 1 is in V_1 - 6 is in V_2	Edge (4,5)	- 4 is in V_2 - 6 is in V_1
Edge (2,3)	- 2 is in V_2 - 3 is in V_1	Edge (5,6)	- 5 is in V_1 - 6 is in V_2

Then, it follows from observation that every edge in E has one endpoint in V_1 and one in V_2.

□

Pseudoproof:

Let $V = \{1, 2, 3, 4, 5, 6\}$, $E = \{(1, 2), (1, 6), (2, 3), (3, 4), (4, 5), (5, 6)\})$.

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1. There exists subsets $V_1, V_2 \subset V$ such that $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V.
2. Every edge in E has exactly one endpoint in V_1 and one in V_2.

We will prove the properties in parts.

1. Show there exists subsets $V_1, V_2 \subset V$ such that $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V

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We need to prove $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V, i.e $V_1 \cup V_2 = V \wedge V_1 \cap V_2 = \emptyset$.

1. Show $V_1 \neq \emptyset, V_2 \neq \emptyset$
   First, we need to show the subsets V_1 and V_2 are non-empty.

   The header tells us both subsets V_1 and V_2 have more than 1 elements.

   Then, using these facts, we can conclude $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$.

2. Show $V_1 \cup V_2 = V \wedge V_1 \cap V_2 = \emptyset$

   Second, we need to show $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$.

   The header tells us $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

   Then, we can calculate

$$V_1 \cup V_2 = \{1, 2, 3, 4, 5, 6\} = V \tag{3}$$
$$V_1 \cap V_2 = \emptyset \tag{4}$$

Part 1:

Let $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

We need to prove $V_1 \neq \emptyset, V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V, i.e $V_1 \cup V_2 = V \wedge V_1 \cap V_2 = \emptyset$.

First, we need to show the subsets V_1 and V_2 are non-empty.

The header tells us both subsets V_1 and V_2 have more than 1 elements.

Then, using these facts, we can conclude $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$.

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The header tells us $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

Then, we can calculate

$$V_1 \cup V_2 = \{1, 2, 3, 4, 5, 6\} = V \tag{5}$$
$$V_1 \cap V_2 = \emptyset \tag{6}$$

2. Show every edge in E has exactly one endpoint in V_1 and one in V_2.

   Let $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

   We need to show every edge in E has exactly one endpoint in V_1 and one in V_2.

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Using these facts, we can generate the following table.

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Edge (2,3)	- 2 is in V_2 - 3 is in V_1	Edge (5,6)	- 5 is in V_1 - 6 is in V_2

Then, it follows from observation that every edge in E has one endpoint in V_1 and one in V_2.

<u>Part 2:</u>

Let $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$.

We need to show every edge in E has exactly one endpoint in V_1 and one in V_2.

The header tells us $V_1 = \{1, 3, 5\}$, $V_2 = \{2, 4, 6\}$, and $E = \{(1, 2), (1, 6), (2, 3), (3, 4), (4, 5), (5, 6)\})$.

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Then, it follows from observation that every edge in E has one endpoint in V_1 and one in V_2.

b. Let $G = (V, E)$ be a complete bipartite graph.

Then, by property 3, we can conclude each vertex in V_1 is adjacent to all verticies in V_2.

Since there are n many edges for each vertex in V_1, and since there are m many verticies in V_1, we can calculate that the vertices in V_1 has

$$nm \tag{1}$$

edges.

Then, since there are no new edges for each vertex in V_2, we can conclude the graph has nm edges.

c. **Conjecture:** The length of every cycle in a bipartite graph is even (i.e. $\forall G = (V, E)$, $Bipartite(G) \Rightarrow \forall k \in \mathbb{N}, C = v_0, \ldots, v_k \wedge Cycle(C, G) \Rightarrow \exists d \in \mathbb{Z}, k = 2d$)

Proof. Let $G = (V, E)$, and assume G is bipartite, with bipartition V_1, V_2. Let $C = v_0, ..., v_k$ form a cycle in G. Without loss of generality, assume $v_0 \in V_1$, $v_i \in V_1$ if $Even(i)$, and $v_i \in V_2$ if $Odd(i)$.

We will prove that a cycle that forms in G has even value of k by using induction on k.

Case 1 (Base case):

Let $k = 3$.

We need to show the sequence of verticles $C = v_1, v_2, v_3$ in G do not form a cycle. That is, there is a consecutive pair of vertices that's not adjacent.

Assume $v_3 = v_0$.

First, we need to show v_2 is in V_1.

The header tells us all even vertices in C are in V_1.

Since 2 is even, we can conclude v_2 is in V_1.

Second, we need to show v_3 is in V_1.

The header tells us $v_0 \in V_1$ and $v_3 = v_0$.

Using these facts, we can conclude $v_3 \in V_1$.

Finally, we need to show v_2, v_3 are not adjacent.

The second property of bipartite graph tells us that no two verticies in V_1 are adjacent

Since v_2, v_3 are in V_1, we can conclude v_2, v_3 are not adjacent.

Case 2 (Inductive Case):

Let $k \in \mathbb{N}$. Assume $C = v_0, v_1, \ldots, v_k$ forms a cycle in G, and $\exists d \in \mathbb{Z}, k = 2d$.

We need to prove the sequence of verticies $C = v_1, \ldots, v_{k+1}$ do not form a cycle in G. That is, there is a consecutive pair of vertices that's not adjacent.

Assume $v_0 = v_{k+1}$.

First, we need to show v_k is in V_1.

The header tells us all even verticies in C are in V_1.

Since we know from assumption that k is even, we can conclude v_k is in V_1.

Second, we need to show v_{k+1} is in V_1.

The assumption tells us $v_0 \in V_1$ and $v_0 = v_{k+1}$.

Using these facts, we can conclude v_{k+1} is in V_1.

Finally, we need to show v_2, v_3 are not adjacent.

The second property of bipartite graph tells us that no two verticies in V_1 are adjacent

Since v_k, v_{k+1} are in V_1, we can conclude v_k, v_{k+1} are not adjacent.

$\square$

Pseudoproof:

Let $G = (V, E)$, and assume G is bipartite, with bipartition V_1, V_2. Let $C = v_0, ..., v_k$ form a cycle in G. Without loss of generality, assume $v_0 \in V_1$, $v_i \in V_1$ if $Even(i)$, and $v_i \in V_2$ if $Odd(i)$.

We will prove that a cycle that forms in G has even value of k by using induction on k.

1. Case 1 (Base case):

   Let $k = 3$.

We need to show the sequence of verticles $C = v_1, v_2, v_3$ in G do not form a cycle. That is, there is a consecutive pair of vertices that's not adjacent.

Assume $v_3 = v_0$.

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- Show v_3 is in V_1

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Second, we need to show v_3 is in V_1. The header tells us $v_0 \in V_1$ and $v_3 = v_0$. Using these facts, we can conclude $v_3 \in V_1$.

- Conclude v_2, v_3 are not adjacent using the properties of bipartite that no two vertices in V_1 are adjacent.

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Since v_2, v_3 are in V_1, we can conclude v_2, v_3 are not adjacent.

2. Case 2 (Inductive case):

   Let $k \in \mathbb{N}$. Assume $C = v_0, v_1, \ldots, v_k$ forms a cycle in G, and $\exists d \in \mathbb{Z}, k = 2d$.

   We need to prove the sequence of verticies $C = v_1, \ldots, v_{k+1}$ do not form a cycle in G. That is, there is a consecutive pair of vertices that's not adjacent.

   Assume $v_0 = v_{k+1}$.

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The assumption tells us $v_0 \in V_1$ and $v_0 = v_{k+1}$.

Using these facts, we can conclude v_{k+1} is in V_1.

- Conclude v_k, v_{k+1} are not adjacent using the properties of bipartite that no two vertices in V_1 are adjacent.

  Finally, we need to show v_k, v_{k+1} are not adjacent.

Finally, we need to show v_2, v_3 are not adjacent.

The second property of bipartite graph tells us that no two verticies in V_1 are adjacent

Since v_k, v_{k+1} are in V_1, we can conclude v_k, v_{k+1} are not adjacent.

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The second property of bipartite graph tells us that no two verticies in V_1 are adjacent

Since v_k, v_{k+1} are in V_1, we can conclude v_k, v_{k+1} are not adjacent.

Notes:

- Cycle with odd number of verticies - Not bipartite
- Cycle with even number of verticies - Bipartite
- 뚜퍼맨!! 영차! 영차! 형모 풀뚜있쪄!!
- 할뚜있다 형모야!!
- 형모 많이 틀렸쬬
- 형모 틀리면 틀리면서 배우면 되느니라. 흠허허허허!!
- 형모 화이팅!!
- 파이팅 파이팅!!
- 형모 해낼 수 있쬬!!!
- 형모야. 한걸음 더.
- 고마워요