Problem Set 4 Solution

Hyungmo Gu

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Question 1

a. **Statement:** $\forall f, g : \mathbb{N} \to \mathbb{R}^+,\ b \in \mathbb{R}^+,\ (g(n) \in \Theta(f(n))) \wedge (n_0 \in \mathbb{N},\ n \geq n_0 \Rightarrow f(n) \geq b \wedge g(n) \geq b) \wedge (b > 1) \Rightarrow \log_b(g(n)) \in \Theta(\log_b(f(n)))$

Statement Expanded: $\forall f, g : \mathbb{N} \to \mathbb{R}^+,\ b \in \mathbb{R}^+,\ \Big(\exists c_1, c_2, n_0 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)\Big) \wedge \Big(\exists n_1 \in \mathbb{N},\ n \geq n_1 \Rightarrow f(n) \geq b \wedge g(n) \geq b\Big) \wedge \Big(b > 1\Big) \Rightarrow \Big(\exists d_1, d_2, n_2 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_2 \Rightarrow d_1 \cdot \log_b(g(n)) \leq \log_b(f(n)) \leq d_2 \cdot \log_b(g(n)\Big)$

Proof. Let $f, g : \mathbb{N} \to \mathbb{R}^+$, and $b \in \mathbb{R}^+$. Assume $c_1 = 1$, $c_2 = b$, and $n_0 = 1$, and $n \in \mathbb{N}$ such that $n \geq n_0$ and $c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$. Assume $f(n)$ and $g(n)$ are eventually $\geq b$. Assume $b > 1$. Let $d_1 = 1$, $d_2 = 2$, and $n_2 = n_0$. Assume $n \geq n_2$.

We need to show $d_1 \cdot \log_b g(n) \leq \log_b f(n) \leq d_2 \cdot \log_b g(n)$.

We will do so in two parts. One for $(d_1 \cdot \log_b g(n) \leq \log_b f(n))$ and the other for $(\log_b f(n) \leq d_2 \cdot \log_b g(n))$.

Part 1 ($d_1 \cdot \log_b g(n) \leq \log_b f(n)$):

The assumption tell us

$$c_1 \cdot g(n) \leq f(n) \tag{1}$$

Then, it follows from the fact $\forall x, y \in \mathbb{R}^+, x \geq y \Leftrightarrow \log x \geq \log y$

$$\log(c_1 \cdot g(n)) \leq \log(f(n)) \tag{2}$$

Then, using the fact $b > 1$, we can calculate

$$\frac{\log(c_1 \cdot g(n))}{\log b} \leq \frac{\log(f(n))}{\log b} \tag{3}$$
$$\frac{\log(c_1) + \log(g(n))}{\log b} \leq \frac{\log(f(n))}{\log b} \tag{4}$$

Then,

$$\frac{\log(g(n))}{\log b} \leq \frac{\log(f(n))}{\log b} \tag{5}$$

by the fact $c_1 = 1$ and $\log c_1 = 0$.

Then, since $\frac{\log f(x)}{\log b} = \log_b f(x)$,

$$\log_b(g(n)) \leq \log_b(f(n)) \tag{6}$$

Then, because we know $d_1 = 1$, we can conclude

$$\log_b(g(n)) \leq d_1 \cdot \log_b(f(n)) \tag{7}$$

Part 2 ($\log_b f(n) \leq d_2 \cdot \log_b g(n)$):

The assumption tells us

$$f(n) \leq c_2 \cdot g(n) \tag{8}$$

Then, it follows from the fact $\forall x, y \in \mathbb{R}^+, x \geq y \Leftrightarrow \log x \geq \log y$

$$\log(f(n)) \leq \log(c_2 \cdot g(n)) \tag{9}$$

Then, using the fact $b > 1$, we can calculate

$$\frac{\log(f(n))}{\log b} \leq \frac{\log(c_2 \cdot g(n))}{\log b} \tag{10}$$
$$\frac{\log(f(n))}{\log b} \leq \frac{\log(c_2) + \log(g(n))}{\log b} \tag{11}$$

Then, since $c_2 = b$,

$$\frac{\log(f(n))}{\log b} \leq \frac{\log(b) + \log(g(n))}{\log b} \tag{12}$$

Then, using the fact $g(n)$ is eventually $\geq b$, we can write

$$\frac{\log(f(n))}{\log b} \leq \frac{\log(g(n)) + \log(g(n))}{\log b} \tag{13}$$
$$\frac{\log(f(n))}{\log b} \leq \frac{2 \cdot \log(g(n))}{\log b} \tag{14}$$

Then, since $\frac{\log f(x)}{\log b} = \log_b f(x)$,

$$\log_b(f(n)) \leq 2 \cdot \log_b(g(n)) \tag{15}$$

Then, because we know $d_2 = 2$, we can conclude

$$\log_b(f(n)) \leq d_2 \cdot \log_b(g(n)) \tag{16}$$

$\square$

Notes:

- $\forall x, y \in \mathbb{R}^+, x \geq y \Leftrightarrow \log x \geq \log y$
- $\exists c_1, c_2, n_0 \in \mathbb{R}^+,\ \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow c_1 \cdot g(n) \leq f(n) \leq c2 \cdot g(n)$
- **Definition of Eventually:** $\exists n_0 \in \mathbb{N}, n \geq n_0 \Rightarrow P$, where $P : \mathbb{N} \to \{\text{True}, \text{False}\}$

Question 2

Question 3

Question 4