Worksheet 17 Solution

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Question 1

a. We need to determine $|\mathcal{I}_n|$.

The problem tells that the values in inputs are either 1 or 0, and we know $\mathcal{I}_n$ represents all possible inputs of size n containing binary values.

After watching lecture videos, and reading notes, I do not yet understand the details of how to evaluate the $\mathcal{I}_n$, but from the pattern below

$[0], [1], [1,0], [0,1], [1,1], [0,0], [0,0,0], [0,0,1], [0,1,0], [1,0,0], [1,1,0], [1,0,1], [0,1,1], [1,1,1]$

we can see the inputs of size 1 have 2 different inputs, the inputs of size 2 have 4 different inputs, and the inputs of size 3 have 8 different inputs.

Using this pattern, I can make an educated guess that $|\mathcal{I}_n| = 2^n$.

Notes:

- The idea of average-case analysis is that some data structures and algorithms have poor worst-case performance but perform well in vast majority of others.
- Average-case analysis looks at running time on sets of inputs
- Average case: $AVG_{func}(n) = avg\{\text{runtime of func(x)} \mid x \in \mathcal{I}_n\}$
- Worst case: $WC_{func}(n) = max\{\text{runtime of func(x)} \mid x \in \mathcal{I}_n\}$

b.

n	i	Sets	$\lvert S_{n,i}\rvert$
2	0	$\{[0]\}$	1
2	0	$\{[0,1],[0,0]\}$	2
2	1	$\{[1,0]\}$	1
3	0	$\{[0,1,1],[0,0,1],[0,0,0]\}$	3
3	1	$\{[1,0,1],[1,0,0]\}$	2
3	2	$\{[1,1,0]\}$	1

By the pattern outlined above, we can deduce that $|S_{n,i}| = n - i$.

Correct Solution:

n	i	Sets	$\lvert S_{n,i}\rvert$
1	0	$\{[0]\}$	1
2	0	$\{[0,1],[0,0]\}$	2
2	1	$\{[1,0]\}$	1
3	0	$\{[0,1,1],[0,0,1],[0,1,0],[0,0,0]\}$	4
3	1	$\{[1,0,1],[1,0,0]\}$	2
3	2	$\{[1,1,0]\}$	1

By the pattern outlined above, we can deduce that $|S_{n,i}| = 2^{n-i-1}$.

c. Because we know there is only one list in a set S_n containing all 1s, we can conclude $|S_{n,n}| = 1$.

d. We will prove the statement informally using proof by cases.

Case 1 (when list doesn't have 0s):

The definition of $S_{n,i}$ tells us $0 \leq i \leq n$, $S_{n,i}$ contains all lists with 0 starting at ith position.

Using the fact, we can conclude $S_{n,n}$ is a set of lists containing 0 at n^{th} position.

Since i in a list starts at $i = 0$ and ends at $i = n - 1$, there are no 0 in the list of a set $S_{n,n}$.

Then, we can conclude $S_{n,n}$ is one and the only that contains a list of only 1s.

Case 2 (when list has one or more 0s):

Since this list has 0 starting at i^{th} position, we can conclude this list exists in the set $S_{n,i}$.

Attempt 2:

We will prove the statement informally using proof by cases.

Case 1 (when list doesn't have 0s):

The definition of $S_{n,i}$ tells us $0 \leq i \leq n$, $S_{n,i}$ contains all lists with 0 starting at ith position.

Using the facts, we can conclude $S_{n,n}$ is a set of lists containing 0 at n^{th} position.

Since i in a list starts at $i = 0$ and ends at $i = n-1$, there are no 0 in the list of a set $S_{n,n}$.

Then, we can conclude $S_{n,n}$ is one and the only that contains a list of only 1s.

Case 2 (when list has one or more 0s):

The definition of $S_{n,i}$ tells us, the set $S_{n,i}$ contains all lists with 0 starting at i^{th} position.

Because we know this list has 0 starting at i^{th} position, using the fact, we can conclude this list exists in the set $S_{n,i}$.

e. The definition of exact expression for average-case running time is

$$AVG_{\text{has\_even}}(n) = \frac{1}{|\mathcal{I}_n|} \cdot \sum_{lst \in \mathcal{I}_n} \text{Runtime of has\_even(lst)} \tag{1}$$

Then,

$$AVG_{\text{has\_even}}(n) = \frac{1}{2^n} \cdot \sum_{lst \in \mathcal{I}_n} \text{Runtime of has\_even(lst)} \tag{2}$$

by the fact $|\mathcal{I}_n| = 2^n$ from the solution of question 1.a.

Then,

$$AVG_{\text{has\_even}}(n) = \frac{1}{2^n} \cdot \sum_{i=0}^{n-1} \sum_{\substack{lst \in S_{n,i} \\ lst[i]=0}} \text{Runtime of has\_even(lst)} \tag{3}$$

by the fact $\sum\limits_{lst \in \mathcal{I}_n}$ can be re-expressed as $\sum\limits_{i=0}^{n-1} \sum\limits_{\substack{lst \in S_{n,i} \\ lst[i]=0}}$.

Then,

$$AVG_{\text{has\_even}}(n) = \frac{1}{2^n} \cdot \sum_{i=0}^{n-1} \sum_{\substack{lst \in S_{n,i} \\ lst[i]=0}} (i+1) \tag{4}$$

by the fact the loop starts at 0 and ends at i.

Then,

$$AVG_{\text{has\_even}}(n) = \frac{1}{2^n} \cdot \sum_{i=0}^{n-1} 2^{n-i-1}(i+1) \tag{5}$$

$$= \sum_{i=0}^{n-1} \frac{i+1}{2^{i+1}} \tag{6}$$

by the fact there are total of 2^{n-i-1} many lists in each $S_{n,i}$ from the solution of question 1.b.

Correct Solution:

The definition of exact expression for average-case running time is

$$AVG_{\text{has\_even}}(n) = \frac{1}{|\mathcal{I}_n|} \cdot \sum_{lst \in \mathcal{I}_n} \text{Runtime of has\_even(lst)} \tag{1}$$

Then,

$$AVG_{\text{has\_even}}(n) = \frac{1}{2^n} \cdot \sum_{lst \in \mathcal{I}_n} \text{Runtime of has\_even(lst)} \tag{2}$$

by the fact $|\mathcal{I}_n| = 2^n$ from the solution of question 1.a.

Then,

$$AVG_{\text{has\_even}}(n) = \frac{1}{2^n} \cdot \sum_{i=0}^{n} \sum_{\substack{lst \in S_{n,i} \\ lst[i]=0}} \text{Runtime of has\_even(lst)} \tag{3}$$

by the fact $\sum\limits_{lst \in \mathcal{I}_n}$ can be re-expressed as $\sum\limits_{i=0}^{n-1} \sum\limits_{\substack{lst \in S_{n,i} \\ lst[i]=0}}$.

Then,

$$AVG_{\text{has\_even}}(n) = \frac{1}{2^n} \cdot \sum_{i=0}^{n} \sum_{\substack{lst \in S_{n,i} \\ lst[i]=0}} (i+1) \tag{4}$$

by the fact there are total of $i+1$ many iterations from 0 to i.

Then,

$$\begin{aligned} AVG_{\text{has\_even}}(n) &= \frac{1}{2^n} \cdot \left[\sum_{i=0}^{n-1} \sum_{\substack{lst \in S_{n,i} \\ lst[i]=0}} (i+1) + \sum_{\substack{lst \in S_{n,n} \\ lst[n]=0}} (n+1) \right] && (5) \\ &= \frac{1}{2^n} \cdot \left[\left(\sum_{i=0}^{n-1} 2^{n-i-1}(i+1) \right) + (n+1) \right] && (6) \end{aligned}$$

by the fact $|S_{n,i}| = 2^{n-i-1}$ for $0 \le i < n$, and $|S_{n,n}| = 1$ by the solution to question 1.b and 1.c.

Then,

$$\begin{aligned} AVG_{\text{has\_even}}(n) &= \frac{1}{2^n} \cdot \left[\sum_{i=0}^{n-1} \sum_{\substack{lst \in S_{n,i} \\ lst[i]=0}} (i+1) + \sum_{\substack{lst \in S_{n,n} \\ lst[n]=0}} (n+1) \right] && (7) \\ &= \frac{1}{2^n} \cdot \left[\left(\sum_{i'=1}^{n} 2^{n-i'} i' \right) + 1 \cdot (n+1) \right] && (8) \end{aligned}$$

by replacing $i+1$ with i'.

f. The solution to question 1.e tells us

$$\begin{aligned} AVG_{\text{has\_even}}(n) &= \frac{1}{2^n} \cdot \left[\left(\sum_{i'=1}^{n} 2^{n-i'} i' \right) + 1 \cdot (n+1) \right] && (1) \\ &= \left(\sum_{i'=1}^{n} \left(\frac{1}{2}\right)^{i'} i' \right) + \frac{(n+1)}{2^n} && (2) \end{aligned}$$

Because we know n is asymptotically large and 2^n dominates $n+1$, the expression can be simplified to

$$AVG_{\text{has\_even}}(n) = \sum_{i'=1}^{\infty} \left(\frac{1}{2}\right)^{i'} i' \tag{3}$$

Then,

$$\begin{aligned} AVG_{has\_even}(n) &= \frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^2} && (4) \\ &= \frac{1}{\frac{1}{2}} && (5) \\ &= 2 && (6) \end{aligned}$$

by the hint $\forall x \in \mathbb{R},\ |x| < 1 \Rightarrow \sum\limits_{i=1}^{\infty} i x^i = \frac{x}{(1-x)^2}$

Then, it follows from above the upper bound of average-case running time is $\mathcal{O}(1)$.