CSC236 Midterm 2 Version 1 Solution

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Question 1

- Let $n, q \in \mathbb{N}$. Let $r \in \{0, 1\}$

  Assume $n > 2$, and $n = 2q + r$.

  I need to find a closed form for $T(2q + r)$, using repeated subtitution.

  Starting from $T(n)$, we have

$$
\begin{aligned}
T(n) &= n + T(n-2) && \text{[By def. since } n > 2] & (1)\\
T(2q + r) &= 2q + r + T(2q + r - 2) && \text{[By replacing } n \text{ for } 2q + r] & (2)\\
&= 2q + r + T(2(q-1) + r) && & (3)\\
&\;\;\vdots && & (4)\\
&= \sum_{i=0}^{i=q-1} (2(q-i) + r) + T(r) && \text{[After } q - 1 \text{ repeatitions]} & (5)\\
&= 2\sum_{i=0}^{i=q-1} (q-i) + \sum_{i=0}^{i=q-1} r + T(r) && & (6)\\
&= 2\sum_{i=0}^{i=q-1} (q-i) + \sum_{i=0}^{i=q-1} r && \text{[Since } T(r) = 0] & (7)\\
&= 2\sum_{i'=1}^{i=q} i' + \sum_{i=0}^{i=q-1} r && & (8)\\
&= 2\sum_{i'=1}^{i'=q} i' + \sum_{i=0}^{i=q-1} r && & (9)\\
&= 2\sum_{i'=1}^{i'=q} i' + \sum_{i=0}^{i=q-1} r && & (10)\\
&= 2(q(q+1))/2 + \sum_{i=0}^{i=q-1} r && \text{[By using } \sum_{i=1}^{i=n} i = (n(n+1))/2] & (11)\\
&= q(q+1) + rq && & (12)\\
&= q(q+1+r) && & (13)
\end{aligned}
$$

- *Proof.* For convenience, define $H(q) : q(q + r + 1) = T(2q + r)$.

  I will use simple induction to prove $\forall q \in \mathbb{N}, H(q)$.

  <u>**Base Case $(q = 0)$:**</u>

  Let $q = 0$.

  Then,

$$
\begin{aligned}
q(q + r + 1) &= 0 && & (1)\\
&= T(2 \cdot 0 + r) && \text{[By def.]} & (2)\\
&= T(2q + r) && & (3)
\end{aligned}
$$

Thus, $T(2q + r)$ verifies in this step.

Inductive Step:

Let $q \in \mathbb{N}$. Assume $H(q)$.

I need to show $H(q + 1)$ follows. That is, $(q + 1)\Big[(q + 1) + r + 1\Big] = T(2(q + 1) + r)$.

Starting with $(q + 1)\Big[(q + 1) + r + 1\Big]$, we have

$$
\begin{aligned}
(q + 1)\Big[(q + 1) + r + 1\Big] &= (q + 1)(q + 1) + (q + 1)r + (q + 1) & (4)\\
&= q^2 + 2q + 1 + (qr + r) + (q + 1) & (5)\\
&= (q^2 + qr + q) + (2q + r + 2) & (6)\\
&= q(q + r + 1) + (2(q + 1) + r) & (7)\\
&= T(2q + r) + 2(q + 1) + r & \text{[By I.H]} \quad (8)\\
&= T(2(q + 1) + r) & \text{[By def.]} \quad (9)
\end{aligned}
$$

Thus, $H(q + 1)$ follows from $H(q)$ in this step.

$\square$

- *Proof.* Define for convenience

$$
H(n) : \bigwedge_{i=0}^{n-1} T(n) - T(i) \geq 0 \tag{1}
$$

I will use complete induction to prove that $\forall n \in \mathbb{N}, H(n)$.

Inductive Step:

Let $n \in \mathbb{N}$. Assume $\bigwedge_{i=0}^{n-1} H(i)$. I will show $H(n)$ follows.

Base Case($n < 2$):

Assume $n < 2$.

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Then, all $T(n)$ and $T(n-1)$ are 0 by definition.

So, $T(n) - T(n-1) \geq 0$.

Thus, $C(n)$ follows in this step.

Case $(n \geq 2)$:

Assume $n \geq 2$.

Then,

$$
\begin{aligned}
T(n) - T(n-1) &= n + T(n-2) \\
&\quad - \Big[(n-1) + T(n-3)\Big] && \text{[By def.]} && (2) \\
&= 1 + T(n-2) - T(n-3) && && (3) \\
&\geq 1 && \text{[By I.H, since } 0 \leq n-2 < n] && (4) \\
&> 0 && && (5)
\end{aligned}
$$

Thus, $C(n)$ follows from $H(n)$ in this step. $\qquad\square$

Correct Solution:

Proof. Define for convenience

$$
H(n) : \bigwedge_{i=1}^{n-1} T(n) - T(i) \geq 0 \tag{1}
$$

I will use complete induction to prove that $\forall n \in \mathbb{N}, H(n)$.

Inductive Step:

Let $n \in \mathbb{N}$. Assume $\bigwedge_{i=1}^{n-1} H(i)$. I will show $H(n)$ follows.

Base Case$(n < 2)$:

Assume $n < 2$.

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Then, all $T(n)$ and $T(n-1)$ are 0 by definition.

So, $T(n) - T(n-1) \geq 0$.

Thus, $C(n)$ follows in this step.

Case ($n = 2$):

Let $n = 2$.

Then,

$$
\begin{aligned}
T(n) - T(n-1) &= (2 + T(1)) - (1 + T(0)) && \text{[By def.]} && (2) \\
&= 1 && \text{[Since T(1) = T(0) = 0 by def.]} && (3) \\
&> 0 && && (4)
\end{aligned}
$$

Thus, $C(n)$ follows in this step.

Case ($n > 2$):

Assume $n > 2$.

Then,

$$
\begin{aligned}
T(n) - T(n-1) &= n + T(n-2) && \text{[By def.]} \\
&\quad - \Big[(n-1) + T(n-3)\Big] && \\
& && (5) \\
&= 1 + T(n-2) - T(n-3) && (6) \\
&\geq 1 && \text{[By I.H, since } 0 \leq n-3 < n-2 < n] \\
& && (7) \\
&> 0 && (8)
\end{aligned}
$$

Thus, $C(n)$ follows from $H(n)$ in this step. $\qquad\square$

Question 2

Let $n \in \mathbb{N}$.

Define $P(n)$: If *a_list* is a python list, then the function *reversi(a_list)* terminates, and returns *a_list[::-1]*, which is *a_list[:]* in reverse order.

I will use complete induction to prove $\forall n \in \mathbb{N}, P(n)$.

1. Inductive Step

   > Let $n \in \mathbb{N}$. Assume $H(n) : \bigwedge_{i=0}^{i=n-1} P(i)$. I will prove $P(n)$ follows. That is, the function *reversi(a_list)* terminates and returns *a_list* of size n in reverse order.

2. Base Case ($n < 1$)

   > Let $n < 1$.
   >
   > Then, the if part of the *reversi(a_list)* activates.
   >
   > Then, by code, *a_list[:]* $= []$ is returned, which is *a_list[::-1]*
   >
   > Thus, $P(n)$ follows in this step.

3. Case ($n \geq 1$)

   > Assume $n \geq 1$.
   >
   > Then, the else part of the *reversi(a_list)* activates.
   >
   > Then, since *reversi(a_list[1:])* has length $n - 1$ and $0 \leq n - 1 < n$, *reversi(a_list[1:])* is a list with elements in reverse order.
   >
   > Then, since *reversi(a_list[1:])* $+ [$*a_list[0]*$]$ adds element *a_list[0]* at the the end of list, *a_list[::-1]* follows.
   >
   > Thus, $P(n)$ follows from $H(n)$ in this step.