CSC236 Worksheet 5 Review

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Question 1

a. *Proof.* Define $P(k) : R(3^k) = k3^k$. Note that when $n = 3^k$, this is equivalent to $R(n) = n \log_3 n$. I will use simple induction to prove $P(k)$.

Base Case ($k = 0$):

Let $k = 0$.

Then,

$$\begin{aligned} R(3^k) &= 0 && \text{[By def., since } n = 3^0 = 1\text{]} && (1) \\ &= 0 \cdot 3^0 && && (2) \\ &= k \cdot 3^k && && (3) \end{aligned}$$

Thus, $P(k)$ is verified in this step.

Inductive Step:

Let $k \in \mathbb{N}$. Assume $P(k)$. That is, $R(3^k) = k \cdot 3^k$. I need to prove $P(k+1)$ follows. That is, $R(3^{k+1}) = (k+1) \cdot 3^{k+1}$

Starting from $R(3^{k+1})$, we have

$$\begin{aligned} R^(3^{k+1}) &= 3^{k+1} + 3R(\lceil 3^{k+1}/3 \rceil) && \text{[By def., since } 0 < k+1 \text{, and } 1 < 3^{k+1}\text{]} && (4) \\ &= 3^{k+1} + 3R(\lceil 3^k \rceil) && && (5) \\ &= 3^{k+1} + 3R(3^k) && \text{[Since } \lceil 3^k \rceil = 3^k\text{]} && (6) \\ &= 3^{k+1} + 3(k \cdot 3^k) && \text{[By I.H]} && (7) \\ &= 3^{k+1} + (k \cdot 3^{k+1}) && && (8) \\ &= (k+1) \cdot 3^{k+1} && && (9) \end{aligned}$$

$\square$

b. For convenience, define $P(n) : \bigwedge_{i=1}^{i=n} R(i) \leq R(n)$.

I will use complete induction to prove that $\forall n \in \mathbb{N}, 0 < n \Rightarrow P(n)$.

<u>**Inductive Step:**</u>

Let $n \in \mathbb{N} \setminus \{0\}$. Assume $\bigwedge_{i=1}^{i=n-1} P(i)$. I will show that $P(n)$ follows.

<u>**Base Case ($n = 1$):**</u>

Let $n = 1$.

Then, $\bigwedge_{i=1}^{i=n} R(i) = R(n)$.

Thus, $P(n)$ follows in this step.

<u>**Base Case ($n = 2$):**</u>

Let $n = 2$.

In this step, I need to prove $P(n)$ follows. That is, $R(1) \leq R(2)$ and $R(2) \leq R(2)$.

I will do so in parts.

Part 1 (Proving $R(1) \leq R(2)$):

The definition tells us $R(1) = 1$ and $R(2) = 2 + 3R(\lceil 2/3 \rceil) = 2 + 3R(1) = 2$.

Since $R(1) = 1 < R(2) = 2$, we can conclude $R(1) \leq R(2)$ holds.

Part 2 (Proving $R(2) \leq R(2)$):

Since $R(2) = R(2)$, $R(2) \leq R(2)$ holds.

<u>**Case ($n > 2$):**</u>

Since $n > 2$, $1 \leq n - 1 < n$. So, by induction hypothesis, $P(n-1)$ holds. Then, by transitivity of $\leq$, it is suffice to prove $P(n)$ by showing $R(n-1) \leq R(n)$.

Starting with $R(n-1)$, we have

$$
\begin{aligned}
R(n-1) &= n - 1 + 3R(\lceil (n-1)/3 \rceil) && \text{[By def., since } n > 2 \text{ and } n-1 > 1\text{]} \quad (10)\\
&\leq n + 3R(\lceil (n-1)/3 \rceil) && (11)\\
&\leq n + 3R(\lceil n/3 \rceil) && \text{[By I.H, since } 1 \leq \lceil (n-1)/3 \rceil < \lceil n/3 \rceil < n\text{]} \quad (12)\\
&= R(n) && (13)
\end{aligned}
$$

Thus, $P(n)$ follows from $\bigwedge_{i=1}^{i=n-1} P(i)$ in this step.