

# CSC373 Worksheet 1 Solution

July 23, 2020

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1. Strassen_Algorithm(A,B):
2     n = A.rows
3     let C be a new n x n matrix
4
5     if n == 1
6         C_11 = A_11 * B_11
7
8     else partition as in step 3 of strassen's algorithm
9
10        p1 = Strassen_Algorithm(A_11, B_12) -
11             Strassen_Algorithm(A_11, B_22)
12
13        p2 = Strassen_Algorithm(A_11, B_22) +
14             Strassen_Algorithm(A_12, B_22)
15
16        p3 = Strassen_Algorithm(A_21, B_11) +
17             Strassen_Algorithm(A_22, B_11)
18
19        p4 = Strassen_Algorithm(A_22, B_21) -
20             Strassen_Algorithm(A_22, B_11)
21
22        p5 = Strassen_Algorithm(A_11, B_11) +
23             Strassen_Algorithm(A_11, B_22) +
24             Strassen_Algorithm(A_22, B_11) +
25             Strassen_Algorithm(A_22, B_22)
26
27        p6 = Strassen_Algorithm(A_12, B_21) +
28             Strassen_Algorithm(A_12, B_22) -
29             Strassen_Algorithm(A_22, B_21) -
30             Strassen_Algorithm(A_22, B_22)
31
32        p7 = Strassen_Algorithm(A_11, B_11) +
33             Strassen_Algorithm(A_11, B_12) -
34             Strassen_Algorithm(A_21, B_11) -
35             Strassen_Algorithm(A_21, B_12)
36
37        C_11 = p5 + p4 - p2 + p6
38        C_12 = p1 + p2
39        C_21 = p3 + p4
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40         C_22 = p5 + p1 - p3 - p7
41
42     return C
43

```

**Notes:**

- Strassen's method for matrix multiplication
  - Reduces the time complexity of matrix multiplication from  $O(n^3)$  to  $O(n^{\log_2 7}) = O(n^{2.81})$
  - Has four steps

- 1) Divide the input matrices  $A$  and  $B$  and output matrix  $C$  into  $n/2 \times n/2$  submatrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$

- 2) Create 10 matrices,  $S_1, S_2, \dots, S_{10}$  each of which is  $n/2 \times n/2$  and is the sum or difference of two matrices created in step 1

$$\begin{aligned}
 S_1 &= B_{12} - B_{22} \\
 S_2 &= A_{11} + A_{12} \\
 S_3 &= A_{21} + A_{22} \\
 S_4 &= B_{21} - B_{11} \\
 S_5 &= A_{11} + A_{22} \\
 S_6 &= B_{11} + B_{22} \\
 S_7 &= A_{12} - A_{22} \\
 S_8 &= B_{21} + B_{22} \\
 S_9 &= A_{11} - A_{21} \\
 S_{10} &= B_{11} + B_{12}
 \end{aligned}$$

- 3) Recursively multiply  $n/2 \times n/2$  matrices seven times to compute the following  $n/2 \times n/2$  matrices

$$\begin{aligned}
 P_1 &= A_{11} \cdot S_1 = A_{11} \cdot B_{12} - A_{11} \cdot B_{22} \\
 P_2 &= S_2 \cdot B_{22} = A_{11} \cdot B_{22} + A_{12} \cdot B_{22} \\
 P_3 &= S_3 \cdot B_{11} = A_{21} \cdot B_{11} + A_{22} \cdot B_{11} \\
 P_4 &= A_{22} \cdot S_4 = A_{21} \cdot B_{11} + A_{22} \cdot B_{11} \\
 P_5 &= S_5 \cdot S_6 = A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} \cdot B_{11} + A_{22} \cdot B_{22} \\
 P_6 &= S_7 \cdot S_8 = A_{12} \cdot B_{21} + A_{12} \cdot B_{22} - A_{22} \cdot B_{21} - A_{22} \cdot B_{22} \\
 P_7 &= S_9 \cdot S_{10} = A_{11} \cdot B_{11} + A_{11} \cdot B_{12} - A_{21} \cdot B_{11} - A_{21} \cdot B_{12}
 \end{aligned}$$

- 4) Construct the four  $n/2 \times n/2$  submatrices of the product  $C$

$$\begin{aligned}
 C_{11} &= P_5 + P_4 - P_2 + P_6 = A_{11} \cdot B_{11} + A_{12} \cdot B_{12} \\
 C_{12} &= P_1 + P_2 = A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\
 C_{21} &= P_3 + P_4 = A_{21} \cdot B_{11} + A_{22} \cdot B_{21} \\
 C_{22} &= P_5 + P_1 - P_3 - P_7 = A_{22} \cdot B_{22} + A_{21} \cdot B_{12}
 \end{aligned}$$

**Example:** Use Strassen's algorithm to compute the matrix product

$$\begin{pmatrix} 1 & 3 \\ 7 & 5 \end{pmatrix} \begin{pmatrix} 6 & 8 \\ 4 & 2 \end{pmatrix}$$

\* **STEP 1**

$$A_{11} = 1, A_{12} = 3, A_{21} = 7, A_{22} = 5$$

$$B_{11} = 6, B_{12} = 8, B_{21} = 4, B_{22} = 2$$

\* **STEP 2**

$$S_1 = B_{12} - B_{22} = 4 - 2 = 2$$

$$S_2 = A_{11} + A_{12} = 1 + 3 = 4$$

$$S_3 = A_{21} + A_{22} = 7 + 5 = 12$$

$$S_4 = B_{21} - B_{11} = 4 - 6 = -2$$

$$S_5 = A_{11} + A_{22} = 1 + 5 = 6$$

$$S_6 = B_{11} + B_{22} = 6 + 2 = 8$$

$$S_7 = A_{12} - A_{22} = 3 - 5 = -2$$

$$S_8 = B_{21} + B_{22} = 4 + 2 = 6$$

$$S_9 = A_{11} - A_{21} = 1 - 7 = -6$$

$$S_{10} = B_{11} + B_{12} = 6 + 4 = 10$$

\* **STEP 3**

$$P_1 = A_{11} \cdot S_1 = A_{11} \cdot B_{12} - A_{11} \cdot B_{22} = 1 \cdot 4 - 1 \cdot 2 = 2$$

$$P_2 = S_2 \cdot B_{22} = A_{11} \cdot B_{22} + A_{12} \cdot B_{22} = 1 \cdot 2 + 3 \cdot 2 = 8$$

$$P_3 = S_3 \cdot B_{11} = A_{21} \cdot B_{11} + A_{22} \cdot B_{11} = 7 \cdot 6 + 5 \cdot 6 = 72$$

$$P_4 = A_{22} \cdot S_4 = A_{22} \cdot B_{21} - A_{22} \cdot B_{11} = 5 \cdot 4 - 5 \cdot 6 = -10$$

$$P_5 = S_5 \cdot S_6 = A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} \cdot B_{11} + A_{22} \cdot B_{22} = 48$$

$$P_6 = S_7 \cdot S_8 = A_{12} \cdot B_{21} + A_{12} \cdot B_{22} - A_{22} \cdot B_{21} - A_{22} \cdot B_{22} = -20$$

$$P_7 = S_9 \cdot S_{10} = A_{11} \cdot B_{11} + A_{11} \cdot B_{12} - A_{21} \cdot B_{11} - A_{21} \cdot B_{12} = -20$$

\* **STEP 4**

$$C_{11} = P_5 + P_4 - P_2 + P_6 = 48 - 10 - 8 - 20 = 10$$

$$C_{12} = P_1 + P_2 = 10$$

$$C_{21} = P_3 + P_4 = 62$$

$$C_{22} = P_5 + P_1 - P_3 - P_7 = 48 + 2 - 72 + 20 = -2$$

– Is not preferred in practical purposes

- 1) The constants used in Strassen's method are high and for a typical application Naive method works better.
- 2) For Sparse matrices, there are better methods especially designed for them.
- 3) The submatrices in recursion take extra space.
- 4) Because of the limited precision of computer arithmetic on noninteger values, larger errors accumulate in Strassen's algorithm than in Naive Method

**References:**

- 1) GeeksForGeeks, Divide and Conquer — Set 5 (Strassen's Matrix Multiplication), [link](#)
- Regular matrix multiplication
    -
  - The master method for solving recurrences
    - provides 'cookbook' method for solving recurrences of the form

$$T(n) = aT(n/b) + f(n)$$

– depends on the following theorem

- \* Let  $a \leq 1$  and  $b > 1$  be constants, let  $f(n)$  be a function and let  $T(n)$  be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n)$$

where we interpret  $n/b$  to mean either  $\lfloor n/b \rfloor$  or  $\lceil n/b \rceil$ . Then  $T(n)$  has the following asymptotic bounds:

1. If  $f(n) = O(n^{\log_b a - \epsilon})$  for some constant  $\epsilon > 0$ , then  $T(n) = \Theta(n^{\log_b a})$
2. If  $f(n) = \Theta(n^{\log_b a})$ , then  $T(n) = \Theta(n^{\log_b a} \lg n)$
3. If  $f(n) = \Omega(n^{\log_b a + \epsilon})$  for some constant  $\epsilon > 0$ , and if  $af(n/b) \leq cf(n)$  for some constant  $c < 1$ , and all sufficiently large  $n$ , then  $T(n) = \Theta(f(n))$ .

**Example:**

$$T(n) = 9T(n/3) + n$$

Here,  $a = 9$ ,  $b = 3$ , and  $f(n) = n = O(n^{\log_3 9-1})$  where  $\epsilon = 1$ .

Thus,  $T(n) = \Theta(n^{\log_3 9})$  or  $T(n) = \Theta(n^2)$

**Example 2:**

$$T(n) = T(2n/3) + 1$$

Here,  $a = 1$ ,  $b = 3/2$ ,  $f(n) = 1 = \Theta(n^{\log_{3/2} 1})$ .

Thus,  $T(n) = \theta(\lg n)$

**Example 3:**

$$T(n) = T(n/4) + n \lg n$$

Here  $a = 1$ ,  $b = 4$ , and  $f(n) = n \lg n$  has asymptotic lowerbound of  $f(n) = \Omega(n^{\log_4 3+\epsilon}) = \Omega(n)$  where  $\epsilon \approx 0.2$

Furthermore,

$$\begin{aligned} af(n/b) &= (3n/4) \lg n/4 \\ &= (3/4)n \lg n/4 \\ &= (3/4)n \lg n/4 \\ &= 3/4n \lg n - \lg 4 \\ &< 3/4n \lg n \\ &= cf(n) \end{aligned}$$

where  $c = 3/4$ .

Thus,  $T(n) = \Theta(n \lg n)$

**Example 4:**

$$T(n) = 2T(n/2) + n \lg n$$

Here,  $a = 2$ ,  $b = 2$ ,  $f(n) = n \lg n$ .

2. Let  $n = 3^m$  where  $m$  is an element of  $\mathbb{Z}^+ \cup \{0\}$

Then we know the time it takes to multiply  $n \times n$  matrices in  $3 \times 3$  matrices is  $T(n) = kT(\frac{n}{3}) + \Theta(n^2)$ .

Now, I need to look for the upper bound of  $k$  in  $T(n) = \Theta(n^{\log_3 k})$  satisfying  $O(n^{\lg 7}) \approx O(n^{2.81})$ .

And using master's theorem, we can write that the upper limit of  $k$  is 21.

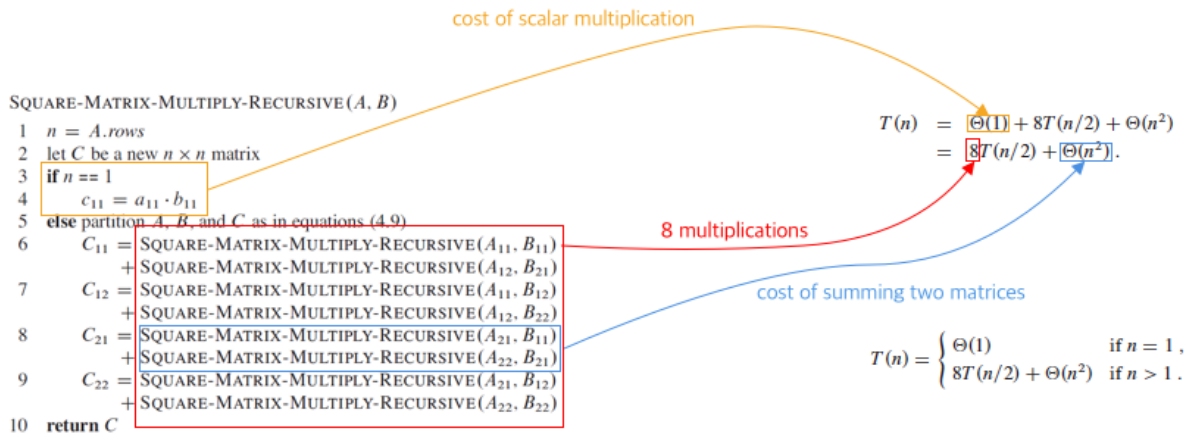
**Improved Solution:**

Let  $n = 3^m$  where  $m$  is an element of  $\mathbb{Z}^+ \cup \{0\}$

Then we know the time it takes to multiply  $n \times n$  matrices in  $3 \times 3$  matrices is  $T(n) = kT(\frac{n}{3}) + \Theta(n^2)$ .

Now, I need to look for the upper bound of  $k$  in  $T(n) = \Theta(n^{\log_3 k})$  satisfying  $O(n^{\lg 7}) \approx O(n^{2.81})$ .

And using master's theorem, we can write that the upper limit of  $k$  is 21 (Following the first condition  $f(n) = O(n^{\log_3 k - \epsilon})$  where  $\epsilon \approx 0.81$ ).

**Notes:**

- $T(n)$  represents the time it takes to multiply two  $n \times n$  matrices.
  - At base case scalar multiplication is performed. So,  $T(1) = \Theta(1)$ .
  - 8 represents the number of recursive calls on the function SQUARE-MATRIX-MULTIPLY-RECURSIVE
  - $\Theta(n^2)$  represents the addition of two  $\frac{n}{2} \times \frac{n}{2}$  matrices
3. •  $68 \times 68$  matrices using 132,464 multiplications:
- Has recurrence of form  $T(n) = 132,464T(\frac{n}{68}) + \Theta(n^2)$
  - Has  $a = 132,464, b = 68, f(n) = \Theta(n^2)$
  - Since  $f(n) = \Theta(n^{\log_b a - \epsilon})$  where  $\epsilon \approx 0.80$ , case 1 of master's theorem applies and  $T(n) = \Theta(n^{\log_b a}) \approx \Theta(n^{2.80})$
- $70 \times 70$  matrices using 143,640 multiplications:

- Has recurrence of form  $T(n) = 143,640T(\frac{n}{70}) + \Theta(n^2)$
- Has  $a = 143,640, b = 70, f(n) = \Theta(n^2)$
- Since  $f(n) = \Theta(n^{\log_b a - \epsilon})$  where  $\epsilon \approx 0.80$ , case 1 of master's theorem applies and  $T(n) = \Theta(n^{\log_b a}) \approx \Theta(n^{2.80})$
- $72 \times 72$  matrices using 155,424 multiplications:
  - Has recurrence of form  $T(n) = 155,424(\frac{n}{72}) + \Theta(n^2)$
  - Has  $a = 155,424, b = 72, f(n) = \Theta(n^2)$
  - Since  $f(n) = \Theta(n^{\log_b a - \epsilon})$  where  $\epsilon \approx 0.80$ , case 1 of master's theorem applies and  $T(n) = \Theta(n^{\log_b a}) \approx \Theta(n^{2.80})$

They all have the same asymptotic running time ( $\Theta(n^{2.80})$ ).

In comparison to Strassen method (which has  $\Theta(n^{\lg 7}) \approx \Theta(n^{2.81})$ ), the above three divide and conquer algorithms are a bit faster.

### Correct Solution:

We need to find the divide and conquer method that yields the best asymptotic running time.

Using master's method, we have:

- $T(n) = 132,464T(\frac{n}{68}) + \Theta(n^2) \rightarrow T(n) \approx \Theta(n^{2.7951284873613815})$
- $T(n) = 143,640T(\frac{n}{70}) + \Theta(n^2) \rightarrow T(n) \approx \Theta(n^{2.795122689748337})$
- $T(n) = 155,424T(\frac{n}{72}) + \Theta(n^2) \rightarrow T(n) \approx \Theta(n^{2.795147391093449})$

Based on the above, the second method  $T(n) = 143,640T(\frac{n}{70})$  has the best asymptotic running time.

In comparison to Strassen method (which has  $\Theta(n^{\lg 7}) \approx \Theta(n^{2.81})$ ), the above three divide and conquer algorithm is a bit faster.

#### 4. Consider three multiplications

1.  $P_1 = ac - ad$
2.  $P_2 = ac - ad$
3.  $P_3 = bd + bc$

Then, we have

$$P_1 + P_2 = (ac - ad) + (ad - bd) = ac - bd$$

$$P_2 + P_3 = (ad - bd) + (bd + bc) = ad + bc$$

Notes:

- I arrived to solution by working backward, by laying the solution, the known pieces, and then finding the missing ones.

The image shows handwritten mathematical work on a dark background. It consists of two parts. The top part shows the subtraction of  $P_2$  from  $P_1$  to find  $ac - bd$ . The bottom part shows the addition of  $P_3$  to  $P_2$  to find  $ad + bc$ .

$$\begin{array}{r} P_1 = ac - ad \\ P_2 = ad - bd \\ \hline ac - bd \end{array}$$

$$\begin{array}{r} P_2 = ad - bd \\ P_3 = bd + bc \\ \hline ad + bc \end{array}$$

5. a) Here,  $a = 2, b = 2, f(n) = 4$ .

Since  $f(n) = n^{\log_2 2+3} = n^{\log_b a + \epsilon}$  where  $\epsilon = 3$ , and  $af(\frac{n}{b}) = 2\left(\frac{n^4}{16}\right) = \frac{n^4}{8} \leq cn^4$  where  $c = \frac{1}{8}$  for sufficiently large  $n$ , the case 3 of master's theorem applies.

Thus,  $T(n)$  has upper bound of  $\mathcal{O}(n^4)$  and lower bounds of  $\Omega(n^4)$ , or  $\Theta(n^4)$ .

b) Here  $a = 1, b = \frac{10}{7}, f(n) = n$ .

Since  $f(n) = 1 = n^{0+1} = n^{\log_{10/7}(1)+1} = n^{\log_b(a)+\epsilon}$ , where  $\epsilon = 1$ , and  $af(\frac{n}{b}) = \frac{7n}{10} \leq cn^4$  where  $c = \frac{7}{10}$  for sufficiently large  $n$ , the case 3 of master's theorem applies.

Thus,  $T(n)$  has upper bound of  $\mathcal{O}(n)$  and lower bounds of  $\Omega(n^4)$ , or  $\Theta(n)$ .



c) Here we have  $a = 16, b = 4, f(n) = n^2$ .

Since  $f(n) = n^2 = n^{\log_4 16} = n^{\log_b a}$ , case 2 of master's theorem applies.

Thus,  $T(n)$  has upper bound of  $\mathcal{O}(n^2 \lg n)$  and lower bounds of  $\Omega(n^2 \lg n)$ .

d) Here we have  $a = 7, b = 3, f(n) = n^2$ .

Since  $n^2 = n^{\log_3 7 + \epsilon} = n^{\log_b a + \epsilon}$  where  $\epsilon \approx 0.23$ , and  $af(\frac{n}{b}) \leq cn^2$  where  $c = \frac{7}{9}$ , the case 3 of master's theorem applies.

Thus,  $T(n)$  has upper bound of  $\mathcal{O}(n^2)$  and lower bounds of  $\Omega(n^2)$ , or  $\Theta(n^2)$ .

e) Here we have  $a = 7, b = 2, f(n) = n^2$ .

Since  $f(n) = n^2 = n^{\log_2 7 - \epsilon}$ , where  $\epsilon \approx 0.81$ , case 1 of master theorem applies.

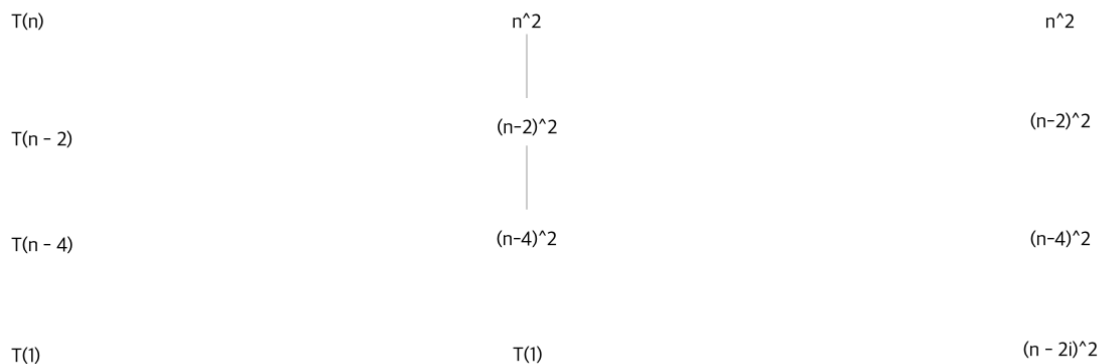
Thus,  $T(n)$  has upper bound of  $\mathcal{O}(n^{\log_2 7})$  and lower bound of  $\Omega(n^{\log_2 7})$ , or  $\Theta(n^{\log_2 7})$ .

f) Here we have  $a = 2, b = 4, f(n) = \sqrt{n}$ .

Since  $f(n) = \sqrt{n} = n^{\log_4 2} = n^{\log_b a}$ , case 2 of master's theorem applies.

Thus  $T(n)$  has upper bound of  $\mathcal{O}(\sqrt{n} \lg n)$ , and lower bound of  $\Omega(\sqrt{n} \lg n)$ , or  $\Theta(\sqrt{n} \lg n)$ .

g) **Solution:**



Using recurrence tree method, we can see that the tree has depth of  $\frac{n}{2}$ , level cost of  $(n-2i)^2$  where  $i = 0, 1, \dots, n-1$ , and bottom level cost of  $T(1)$  or  $\Theta(1)$ .

So, the total cost of  $T(n)$  is:

$$T(n) = \sum_{i=0}^{\frac{n}{2}-1} (n - 2i) + \Theta(1) \quad (1)$$

$$= \frac{n^2}{2} - 2 \sum_{i=0}^{\frac{n}{2}-1} i + \Theta(1) \quad (2)$$

$$= \frac{n^2}{2} - \left(\frac{n}{2}\right)\left(\frac{n}{2} - 1\right) + \Theta(1) \quad (3)$$

$$= \left(\frac{n^2}{2}\right) - \left(\frac{n^2}{4} - \frac{n}{2}\right) + \Theta(1) \quad (4)$$

$$= \frac{n^2}{4} + \frac{n}{2} + \Theta(1) \quad (5)$$

$$= \Theta(n^2) \quad (6)$$

And to verify  $T(n)$ , I will use substitution method.

Let the guess be  $T(n) \leq cn^3$ .

Then,

$$T(n) = T(n-2) + n^2 \quad (7)$$

$$\leq c(n-2)^3 + n^2 \quad (8)$$

$$= c(n^3 - 6n^2 + 12n - 8) + n^2 \quad (9)$$

$$= c(n^3 - 5n^2 + 12n - 8) - n^2(c-1) \quad (10)$$

$$\leq c(n^3 + 12n - 8) - n^2(c-1) \quad (11)$$

$$= cn^3 - n^2(c-1) \quad \text{[Since } n^3 \text{ dominates } n] \quad (12)$$

$$\leq cn^3 \quad (13)$$

and the boundary holds as long as  $c \geq 1$ .