

CSC236 Worksheet 6 Solution

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Question 1

- *Proof.* Assume that for all $k \in \mathbb{N}$, $R(3^k) = k3^k$.

I need to prove $R \in \mathcal{O}(n \lg n)$ and $R \in \Omega(n \lg n)$.

I will do so in parts.

Part 1 (Proving $R \in \mathcal{O}(n \lg n)$):

Define $n^* = 3^{\lceil \log_3 n \rceil}$. Then, we have,

$$\lceil \log_3 n \rceil - 1 < \log_3 n \leq \lceil \log_3 n \rceil \Rightarrow n^*/3 < n \leq n^* \quad (1)$$

I will also use the assumption (proved last week) that R is non-decreasing.

Let $d = 6$. Then $d \in \mathbb{R}^+$. Let $B = 3$. Then $B \in \mathbb{N}^+$. Let n be an arbitrary natural number no smaller than B . Then,

$$R(n) \leq R(n^*) \quad [\text{Since } n < n^*, \text{ and } R \text{ is non-decreasing}] \quad (2)$$

$$= n^* \log_3 n^* \quad [\text{By assumption, and replacing } n^* \text{ for } 3^k] \quad (3)$$

$$\leq 3n \log_3 3n \quad [\text{Since } n \leq n^* \Rightarrow 3n \leq 3n^*] \quad (4)$$

$$\leq 3n(\log_3 n + 1) \quad (5)$$

$$\leq 3n(\log_3 n + \log_3 n) \quad [\text{Since } n \geq 3 \Rightarrow \log_3 n \geq 1] \quad (6)$$

$$= 6n \log_3 n \quad (7)$$

$$\leq (6n \lg n) / \lg 3 \quad [\text{By change of basis to } \lg] \quad (8)$$

$$< 6n \lg n \quad (9)$$

$$= dn \lg n \quad [\text{Since } d = 6] \quad (10)$$

So $R \in \mathcal{O}(n \lg n)$, since $\log_3 n$ differs from $\lg n$ by a constant factor.

Part 2 (Proving $R \in \Omega(n \lg n)$):

Define $n^* = 3^{\lceil \log_3 n \rceil}$. Then, we have,

$$\lceil \log_3 n \rceil - 1 < \log_3 n \leq \lceil \log_3 n \rceil \Rightarrow n^*/3 < n \leq n^* \quad (11)$$

I will also use the assumption (proved last week) that R is non-decreasing.

Let $d = 1/(6 \lg 3)$. Then $d \in \mathbb{R}^+$. Let $B = 9$. Then $B \in \mathbb{N}^+$. Let n be an arbitrary natural number no smaller than B . Then,

$$R(n) \geq R(n^*/3) \quad [\text{Since } n^*/3 < n, \text{ and } R \text{ is non-decreasing}] \quad (12)$$

$$= (n^*/3) \cdot \log_3(n^*/3) \quad [\text{By assumption, and replacing } n^* \text{ for } 3^k] \quad (13)$$

$$\geq (n/3) \cdot \log_3(n/3) \quad [\text{Since } n^* \leq n \Rightarrow n^*/3 \leq n/3] \quad (14)$$

$$= (n/3) \cdot (\log_3 n - 1) \quad (15)$$

$$\geq (n/3) \cdot (\log_3 n - (\log_3 n)/2) \quad [\text{Since } n \geq 9 \Rightarrow (\log_3 n)/2 \geq 1] \quad (16)$$

$$= (n/6) \cdot \log_3 n \quad (17)$$

$$= (n/6) \cdot (\lg n / \lg 3) \quad (18)$$

$$= (n/(6 \lg 3)) \cdot \lg n \quad (19)$$

$$= dn \cdot \lg n \quad [\text{Since } d = 1/(6 \lg 3)] \quad (20)$$

So, $R \in \Omega(n \lg n)$.

□

Correct Solution:

Assume that for all $k \in \mathbb{N}$, $R(3^k) = k3^k$.

I need to prove $R \in \mathcal{O}(n \lg n)$ and $R \in \Omega(n \lg n)$.

I will do so in parts.

Part 1 (Proving $R \in \mathcal{O}(n \lg n)$):

Define $n^* = 3^{\lceil \log_3 n \rceil}$. Then, we have,

$$\lceil \log_3 n \rceil - 1 < \log_3 n \leq \lceil \log_3 n \rceil \Rightarrow n^*/3 < n \leq n^* \quad (21)$$

I will also use the assumption (proved last week) that R is non-decreasing.

Let $d = 6$. Then $d \in \mathbb{R}^+$. Let $B = 3$. Then $B \in \mathbb{N}^+$. Let n be an arbitrary natural number no smaller than B . Then,

$$R(n) \leq R(n^*) \quad [\text{Since } n < n^*, \text{ and } R \text{ is non-decreasing}] \quad (22)$$

$$= n^* \log_3 n^* \quad [\text{By assumption, and replacing } n^* \text{ for } 3^k] \quad (23)$$

$$\leq 3n \log_3 3n \quad [\text{Since } n \leq n^* \Rightarrow 3n \leq 3n^*] \quad (24)$$

$$= 3n(\log_3 n + 1) \quad (25)$$

$$\leq 3n(\log_3 n + \log_3 n) \quad [\text{Since } n \geq 3 \Rightarrow \log_3 n \geq 1] \quad (26)$$

$$= 6n \log_3 n \quad (27)$$

$$= (6n \lg n) / \lg 3 \quad [\text{By change of basis to } \lg] \quad (28)$$

$$< 6n \lg n \quad (29)$$

$$= dn \lg n \quad [\text{Since } d = 6] \quad (30)$$

So $R \in \mathcal{O}(n \lg n)$, since $\log_3 n$ differs from $\lg n$ by a constant factor.

Part 2 (Proving $R \in \Omega(n \lg n)$):

Define $n^* = 3^{\lceil \log_3 n \rceil}$. Then, we have,

$$\lceil \log_3 n \rceil - 1 < \log_3 n \leq \lceil \log_3 n \rceil \Rightarrow n^*/3 < n \leq n^* \quad (31)$$

I will also use the assumption (proved last week) that R is non-decreasing.

Let $d = 1/(6 \lg 3)$. Then $d \in \mathbb{R}^+$. Let $B = 9$. Then $B \in \mathbb{N}^+$. Let n be an arbitrary natural number no smaller than B . Then,

$$R(n) \geq R(n^*/3) \quad [\text{Since } n^*/3 < n, \text{ and } R \text{ is non-decreasing}] \quad (32)$$

$$= (n^*/3) \cdot \log_3(n^*/3) \quad [\text{By assumption, and replacing } n^* \text{ for } 3^k] \quad (33)$$

$$\geq (n/3) \cdot \log_3(n/3) \quad [\text{Since } n^* \leq n \Rightarrow n^*/3 \leq n/3] \quad (34)$$

$$= (n/3) \cdot (\log_3 n - 1) \quad (35)$$

$$\geq (n/3) \cdot (\log_3 n - (\log_3 n)/2) \quad [\text{Since } n \geq 9 \Rightarrow (\log_3 n)/2 \geq 1] \quad (36)$$

$$= (n/6) \cdot \log_3 n \quad (37)$$

$$= (n/6) \cdot (\lg n / \lg 3) \quad (38)$$

$$= (n/(6 \lg 3)) \cdot \lg n \quad (39)$$

$$= dn \cdot \lg n \quad [\text{Since } d = 1/(6 \lg 3)] \quad (40)$$

So, $R \in \Omega(n \lg n)$, since $\log_3 n$ differs from $\lg n$ by a constant factor.

Notes:

- Learned that if there is trouble going from $\log_3 n - 1$ to $dn \lg n$, a good approach is to increase the value of B.
- Noticed that professor used 'Let $d = \underline{\hspace{1cm}}$. Then $d \in \mathbb{R}^+$ ' to define variable's value as well as its type.

- $g \in \Theta(f) : g \in \mathcal{O}(f) \wedge g \in \Omega(f)$

or

$$g \in \Theta(f) : \exists c_1, c_2, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n), \text{ where } f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$$

- $g \in \Omega(f) : \exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \geq cf(n), \text{ where } f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$

- $g \in \mathcal{O}(f) : \exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq cf(n), \text{ where } f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$

Question 2

• **Rough Work:**

Let $n \in \mathbb{N}$ and bits $b_0, \dots, b_k \in \{0, 1\}$ be such that $n = \sum_{i=0}^{i=k} 2^i b_i$. I will use identities:

$$\lfloor n/2 \rfloor = \sum_{i=1}^{i=k} 2^{i-1} b_i \quad (41)$$

$$n \equiv b_0 \pmod{2} \quad (42)$$

Define $P(n)$: “If n is a natural number, then *decimal_to_binary*(n)” terminates and returns binary string representing n with no leading zeros, except if n is 0.

I will use complete induction to prove $\forall n \in \mathbb{N}, P(n)$.

1. Inductive Step

Let $n \in \mathbb{N}$. Assume $\bigwedge_{i=0}^{i=n-1} P(i)$. I will show $P(n)$ follows.

2. Base Case ($n < 2$)

Let $n < 2$.

Then, the if part of *decimal_to_binary*(n) executes, and the program terminates by outputting the input value.

Since, binary rep of 0 is 0, and binary rep of 1 is 1, $P(n)$ follows in this step.

3. Case ($n \geq 2$)

Notes:

- **Correct:** A program is correct if it produces output on every acceptable input
- **Precondition** and **Postcondition** are assertions involving some of the variables of the program
 - **Precondition** states what must be true *before* program starts execution

- **Postcondition** states what must be true when the program *ends*