

# CSC236 Worksheet 6 Solution

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## Question 1

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### Rough Work:

Assume that for all  $k \in \mathbb{N}$ ,  $R(3^k) = k3^k$ .

I need to prove  $R \in \mathcal{O}(n \lg n)$  and  $R \in \Omega(n \lg n)$ .

I will do so in parts.

1. Prove that  $R \in \mathcal{O}(n \lg n)$ .

Define  $n^* = 3^{\lceil \log_3 n \rceil}$ . Then, we have,

$$\lceil \log_3 n \rceil - 1 < \log_3 n \leq \lceil \log_3 n \rceil \Rightarrow n^*/3 < n \leq n^* \quad (1)$$

I will also use the assumption (proved last week) that  $R$  is non-decreasing.

Let  $d = 6$ . Then  $d \in \mathbb{R}^+$ . Let  $B = 3$ . Then  $B \in \mathbb{N}^+$ . Let  $n$  be an arbitrary natural number no smaller than  $B$ . Then,

$$R(n) \leq R(n^*) \quad [\text{Since } n < n^*, \text{ and } R \text{ is non-decreasing}] \quad (2)$$

$$= n^* \log_3 n^* \quad [\text{By assumption, and replacing } n^* \text{ for } 3^k] \quad (3)$$

$$\leq 3n \log_3 3n \quad [\text{Since } n \leq n^* \Rightarrow 3n \leq 3n^*] \quad (4)$$

$$\leq 3n(\log_3 n + 1) \quad (5)$$

$$\leq 3n(\log_3 n + \log_3 n) \quad [\text{Since } n \geq 3 \Rightarrow \log_3 n \geq 1] \quad (6)$$

$$= 6n \log_3 n \quad (7)$$

$$\leq (6n \lg n) / \lg 3 \quad [\text{By change of basis to } \lg] \quad (8)$$

$$< 6n \lg n \quad (9)$$

$$= dn \lg n \quad [\text{Since } d = 6] \quad (10)$$

So  $R \in \mathcal{O}(n \lg n)$ , since  $\log_3 n$  differs from  $\lg n$  by a constant factor.

2. Prove that  $R \in \Omega(n \log n)$

Define  $n^* = 3^{\lceil \log_3 n \rceil}$ . Then, we have,

$$\lceil \log_3 n \rceil - 1 < \log_3 n \leq \lceil \log_3 n \rceil \Rightarrow n^*/3 < n \leq n^* \quad (11)$$

I will also use the assumption (proved last week) that  $R$  is non-decreasing.

Let  $d = 1/6$ . Then  $d \in \mathbb{R}^+$ . Let  $B = 9$ . Then  $B \in \mathbb{N}^+$ . Let  $n$  be an arbitrary natural number no smaller than  $B$ . Then,

$$R(n) \geq R(n^*/3) \quad [\text{Since } n^*/3 < n, \text{ and } R \text{ is non-decreasing}] \quad (12)$$

$$= (n^*/3) \cdot \log_3(n^*/3) \quad [\text{By assumption, and replacing } n^* \text{ for } 3^k] \quad (13)$$

$$\geq (n/3) \cdot \log_3(n/3) \quad [\text{Since } n^* \leq n \Rightarrow n^*/3 \leq n/3] \quad (14)$$

$$\geq (n/3) \cdot (\log_3 n - 1) \quad (15)$$

$$\geq (n/3) \cdot (\log_3 n - (\log_3 n)/2) \quad [\text{Since } n \geq 9 \Rightarrow (\log_3 n)/2 \geq 1] \quad (16)$$

$$\geq (n/6) \cdot \log_3 n \quad (17)$$

$$\geq dn \cdot \log_3 n \quad [\text{Since } d = 1/6] \quad (18)$$

**Notes:**

- $g \in \Theta(f) : g \in \mathcal{O}(f) \wedge g \in \Omega(f)$

or

$g \in \Theta(f) : \exists c_1, c_2, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n)$ , where  $f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$

- $g \in \Omega(f) : \exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \geq cf(n)$ , where  $f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$
- $g \in \mathcal{O}(f) : \exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq cf(n)$ , where  $f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$