Worksheet 12 Review

March 31, 2020

Question 1

a. $g \in \mathcal{O}(1): \exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq c$, where $g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

Notes:

- $g \in \mathcal{O}(f) : \exists c, n_o \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq cf(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

b. **Predicate Logic** $\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq c$, where $g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

Proof. Let $n_0 = 1$, $c = 200$ and $g(n) = 100 + \frac{77}{n+1}$. Assume $n \geq n_0$.

We will prove the statement by showing

$$100 + \frac{77}{n+1} \leq c \tag{1}$$

It follows from the fact $n_0 \geq 1$ that we can write

$$100 + \frac{77}{n+1} \leq 100 + \frac{77}{1+1} \tag{2}$$
$$\leq 100 + \frac{77}{2} \tag{3}$$
$$\leq 100 + 77 \tag{4}$$
$$\leq 100 + 100 \tag{5}$$
$$\leq 200 \tag{6}$$

Then,

$$100 + \frac{77}{n+1} \leq c \tag{7}$$

by the fact that $c = 200$. $\square$

Question 2

- **Predicate Logic:** $\forall f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, $(\exists c_0, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \leq c_0 f(n)) \Rightarrow (\exists d_0, m_0 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq m_0 \Rightarrow f(n) \geq dg(n))$

  Proof. Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. Let $c = 2$, $n_0 = 1$ and $n \in \mathbb{N}$. Assume $n \geq m_0$. Let $d = \frac{1}{c}$ and $m_0 = n_0$. Assume $n \geq m_0$.

  We will prove that $d_0 g(n) \leq f(n)$ given $g(n) \leq c_0 f(n))$.

  It follows from the assumption $g(n) \leq f(n)$ that we can write

$$g(n) \leq cf(n) \tag{1}$$
$$\frac{1}{2} g(n) \leq f(n) \tag{2}$$
$$\frac{1}{2} g(n) \leq f(n) \tag{3}$$

Then since $d = \frac{1}{2}$,

$$d \cdot g(n) \le f(n) \tag{4}$$

$\square$

Question 3

- **Predicate Logic:** $\forall g : \mathbb{N} \to \mathbb{R}^{\ge 0}$, $\forall a \in \mathbb{R}^{\ge 0}$, $(\exists c, n_0 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \ge n_0 \Rightarrow g(n) \ge c) \Rightarrow (\exists c_1, c_2, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \ge n_1 \Rightarrow c_1 g(n) \le a + g(n) \le c_2 g(n))$

Proof. Let $g : \mathbb{N} \to \mathbb{R}^{\ge 0}$, and $a \in \mathbb{R}^{\ge 0}$. Assume $g \in \Omega(1)$, that is there exists $c, n_0 \in \mathbb{R}^+$, for every $n \in \mathbb{N}$ such that if $n \ge n_0$, $g(n) \ge c$. Let $c_1 = \frac{1}{2}$, $c_2 = \left(\frac{a}{c} + 1\right)$ and $n_1 = n_0$. Assume $n \ge n_1$.

We will prove $c_1 g(n) \le a + g(n) \le c_2 g(n)$ by diving into two parts, first by proving $c_1 g(n) \le a + g(n)$ is true, and then second by proving $a + g(n) \le c_2 g(n)$. Then, we will combine the two at the end to finish.

Part 1 ($c_1 g(n) \le a + g(n)$):

It follows from the fact $a \in \mathbb{R}^+$ that we can write

$$\begin{aligned} a + g(n) &\ge g(n) && (1) \\ &\ge \frac{1}{2} \cdot g(n) && (2) \end{aligned}$$

Then, because we know $c_1 = \frac{1}{2}$, we can conclude

$$a + g(n) \ge c_1 \cdot g(n) \tag{3}$$

Part 2 ($a + g(n) \leq c_2 g(n)$):

Using the value $c_2 = \left(\frac{a}{c} + 1\right)$, we can write

$$\begin{align} c_2 g(n) &= \left(\frac{a}{c} + 1\right) \cdot g(n) \tag{4} \\ &= \frac{a}{c} \cdot g(n) + g(n) \tag{5} \end{align}$$

Then,

$$c_2 g(n) \geq \frac{a}{c} \cdot c + g(n) \tag{6}$$

by the assumption that $g(n) \geq c$.

Then,

$$c_2 g(n) \geq a + g(n) \tag{7}$$

Since both $a + g(n) \leq c_2 g(n)$ and $c_1 g(n) \leq a + g(n)$ are true, we can conclude that the inequality $c_1 g(n) \leq a + g(n) \leq c_2 g(n)$ is true.

□

Notes:

- Noticed professor uses english phrase when expanding assumption.
- $g \in \Theta(f) :\ g \in \mathcal{O}(f) \land g \in \Omega(f)$
  or
  $g \in \Theta(f) :\ \exists c_1, c_2, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n)$, where $f, g :\ \mathbb{N} \to \mathbb{R}^{\geq 0}$
- $g \in \Omega(f) :\ \exists c, n_o \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \geq cf(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$
- $g \in \mathcal{O}(f) :\ \exists c, n_o \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \leq cf(n)$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

Question 4

a. $g \notin \mathcal{O}(f)$: $\exists g, f : \mathbb{N} \to \mathbb{R}^{\geq 0}, \forall c, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N}, (n \geq n_0) \wedge (g(n) > cf(n))$

b. **Predicate Logic:** $\forall a, b \in \mathbb{R}^+, a > b \Rightarrow n^a \notin \mathcal{O}(n^b)$

Expanded Predicate Logic: $\forall a, b \in \mathbb{R}^+, a > b \Rightarrow \forall c, n_0 \in \mathbb{R}^+,\ \exists n \in \mathbb{N},\ (n \geq n_0) \wedge (n^a > cn^b)$

Proof. Let $a, \in \mathbb{R}^+$. Assume $a > b$. Let $c, n_0 \in \mathbb{R}^+$ and $n = (n_0 + c)$.

We will prove the statement $(n \geq n_0) \wedge (n^a > cn^b)$ in two parts one for $(n \geq n_0)$ and the other for $(n^a > cn^b)$. Then, we will combine the two parts to finish.

Part 1 $(n \geq n_0)$:

Because we know $n_0, c \in \mathbb{R}^+$, we can conclude $n_0, c > 0$.

Using the fact $n_0, c > 0$, we can calculate

$$n_0 + c \geq n_0 \tag{1}$$

Then, because we know $n_0 + c = n$, we can conclude

$$n \geq n_0 \tag{2}$$

Part 2 $(n^a > cn^b)$:

Since $n = (n_0 + c)$, we can calculate that

$$cn^b = c(n_0 + c)^b \tag{3}$$

Then,

$$cn^b < (c + n_0)(n_0 + c)^b \tag{4}$$

by the fact $c, n_0 \in \mathbb{R}^+$ and $c + n_0 > c$.

Then,

$$cn^b < (c + n_0)^{a-b}(n_0 + c)^b \tag{5}$$

by the fact $a > b$.

Then,

$$cn^b < (c + n_0)^{a-b+b} \tag{6}$$
$$cn^b < (c + n_0)^a \tag{7}$$

Then, because we know $c + n_0 = n$, we can conclude

$$cn^b < n^a \tag{8}$$

Since part 1 and part 2 are both true, we can conclude $(n \geq n_0) \wedge (n^a > cn^b)$ is true.

$\square$

Notes:

- $\forall x, y \in \mathbb{R}^+, x > y \Leftrightarrow \log x > \log y$