Midterm 2 Version 2 Solution

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Question 1

a.

$$\begin{aligned}
100 \div 3 &= 33, \text{Remainder } \mathbf{1} \\
33 \div 3 &= 11, \text{Remainder } \mathbf{0} \\
11 \div 3 &= 3, \text{Remainder } \mathbf{2} \\
3 \div 3 &= 1, \text{Remainder } \mathbf{0} \\
1 \div 3 &= 0, \text{Remainder } \mathbf{1}
\end{aligned}$$

It follows from above that the ternary representation of 100 is $(10201)_3$.

Attempt 2:

$$\begin{aligned}
100 + (-1 \cdot 3^4) &= 100 - 81 = 19 \\
19 + (-1 \cdot 3^3) &= 19 - 27 = -8 \\
-8 + (+1 \cdot 3^2) &= -8 + 9 = 1 \\
1 + (0 \cdot 3^1) &= 1 + 0 = 1 \\
1 + (-1 \cdot 3^0) &= 1 - 1 = 0
\end{aligned}$$

So by flipping the signs, and reading from top to bottom, we can conclude the balanced ternary representation of 100 is $(11T101)_{bt}$

Notes:

- Balanced ternary representation expresses a decimal using 1, 0 and -1
- **T** represents negative sign in balanced ternary representation.
- Is my way of calculating balanced ternary representation correct? My approach was 'which sign should be used given 3^n so the calculation stops at 3^0?'

b. The largest number expressible by an n-digit binary representation is

$$\sum_{i=0}^{n-1} 2^i \tag{1}$$

Correct Solution:

$$\sum_{i=0}^{n-1} 2^i = \frac{1-2^{n-1+1}}{1-2} = 2^n - 1 \tag{1}$$

Notes:

- Noticed professor simplified solution using geometric series
- Geometric series with finite sum

$$\sum_{i=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}, \text{ where } |r| > 1 \tag{2}$$

c.

$f(n) \in \mathcal{O}(n)$	True	$g(n) \in \Omega(n)$	False	$f(n) \in \Omega(g(n))$	True
$f(n) \in \Theta(g(n))$	False	$g(n) \in \Theta(\log_3 n)$	False	$f(n) + g(n) \in \Theta(f(n))$	True

Notes:

- Learned $\sqrt{n}$ rises faster than $\log n$.
- Learned if $g(n) \in \Theta(f(n))$ is true then $f(n) + g(n) \in \Theta(f(n))$ is true.

d.

k	0	1	2	3
$i \cdot i \cdot i$	$2 = 2^{3^0}$	$2^3 = 2^{3^1}$	$2^9 = 2^{3^2}$	$2^27 = 2^{3^3}$

We can deduce from above that $i_k = 2^{3^k}$

e. $\lceil \log_3(\log_2(n) - 1) \rceil$

Correct Solution:

We want to find the smallest value of k satisfying $2 \cdot i_k \geq n$, and the value is

$$\lceil \log_3(\log_2(n) - 1) \rceil$$

Question 2

- **Statement:** $\forall n \in \mathbb{N}, n \geq 2 \Rightarrow \prod_{i=1}^{n} \frac{2^i-1}{2^i} \geq \frac{1}{2n}$

Proof. Let $n \in \mathbb{N}$. Assume $n \geq 2$.

We will prove the statement using induction on n.

Base Case (n = 2):

Let $n = 2$.

We want to show $\prod_{i=1}^{2} \frac{2^i-1}{2^i} \geq \frac{1}{2\cdot(2)}$

Starting from $\prod_{i=1}^{2} \frac{2^i-1}{2^i}$, we can conclude

$$\begin{aligned} \prod_{i=1}^{2} \frac{2^i - 1}{2^i} = \left(\frac{1}{2}\right) \cdot \left(\frac{3}{4}\right) &= \frac{3}{8} && (1) \\ &\geq \frac{2}{8} && (2) \\ &\geq \frac{1}{4} && (3) \end{aligned}$$

Inductive Case:

Let $n \in \mathbb{N}$. Assume $\prod_{i=1}^{n} \frac{2^i-1}{2^i} \geq \frac{1}{2n}$.

We want to show $\prod_{i=1}^{n+1} \frac{2^i-1}{2^i} \geq \frac{1}{2(n+1)}$.

Starting from $\frac{1}{2(n+1)}$, because we know $n \geq 1$, we can conclude

$$\frac{1}{2(n+1)} \leq \frac{1}{2 \cdot (n+n)} \tag{4}$$
$$= \frac{1}{2 \cdot 2n} \tag{5}$$

Then, using inductive hypothesis $\prod_{i=1}^{n} \frac{2^i-1}{2^i} \geq \frac{1}{2n}$, we can conclude that

$$\frac{1}{2 \cdot (n+1)} \leq \prod_{i=1}^{n} \frac{2^i - 1}{2^i} \cdot \frac{1}{2} \tag{6}$$
$$= \prod_{i=1}^{n} \frac{2^i - 1}{2^i} \cdot \left(1 - \frac{1}{2}\right) \tag{7}$$
$$< \prod_{i=1}^{n} \frac{2^i - 1}{2^i} \cdot \left(1 - \frac{1}{2^{n+1}}\right) \tag{8}$$
$$= \prod_{i=1}^{n} \frac{2^i - 1}{2^i} \cdot \left(\frac{2^{n+1}}{2^{n+1}} - \frac{1}{2^{n+1}}\right) \tag{9}$$
$$= \prod_{i=1}^{n} \frac{2^i - 1}{2^i} \cdot \left(\frac{2^{n+1} - 1}{2^{n+1}}\right) \tag{10}$$
$$= \prod_{i=1}^{n+1} \frac{2^i - 1}{2^i} \tag{11}$$

□

Notes:

- Solved the inductive part of the problem by starting from the left hand side and calculating to the right as far as I can, and then repeating the same procedure from the right to the left until there were few missing steps left in between.

Question 3

- **Statement:** $\exists a \in \mathbb{R}^+, an^2 + 1 \in \Theta(n^4)$

  Negation of Statement: $\forall a, c_1, c_2, n_0 \in \mathbb{R}^+,\ \exists n \in \mathbb{N},\ (n \geq n_0) \wedge \Big((c_1 n^4 > an^2 + 1) \vee (an^2 + 1 > c_2 n^4)\Big)$

Proof. Let $a, c_1, n_0 \in \mathbb{R}^+$, and $n = \left\lceil max(n_0, \sqrt[4]{\frac{2}{c_1}}, \sqrt{\frac{2a}{c_1}}) \right\rceil + 1$.

We will disprove the statement by showing $(n \geq n_0)$ and $(c_1 n^4 > an^2 + 1)$.

Part 1 ($n \geq n_0$):

Because we know $\left\lceil max(n_0, \sqrt[4]{\frac{2}{c_1}}, \sqrt{\frac{2a}{c_1}}) \right\rceil \geq n_0$, we can conclude

$$n = \left\lceil max(n_0, \sqrt[4]{\frac{2}{c_1}}, \sqrt{\frac{2a}{c_1}}) \right\rceil + 1 > n_0 \tag{1}$$

Part 2 (Showing $c_1 n^4 > an^2 + 1$):

We need to show $c_1 n^4 > an^2 + 1$.

We will do so by showing $\frac{c_1 n^4}{2} > an^2$ and $\frac{c_1 n^4}{2} > 1$, and then combining them.

For the first inequality, using the fact that $\sqrt{\frac{2a}{c_1}} < \left\lceil max(n_0, \sqrt[4]{\frac{2}{c_1}}, \sqrt{\frac{2a}{c_1}}) \right\rceil + 1 = n$, we can calculate

$$\begin{align} \sqrt{\frac{2a}{c_1}} &< n \tag{2} \\ \frac{2a}{c_1} &< n^2 \tag{3} \\ 2a &< c_1 n^2 \tag{4} \\ a &< \frac{c_1 n^2}{2} \tag{5} \\ an &< \frac{c_1 n^3}{2} \tag{6} \end{align}$$

For the second inequality, using the fact that $\sqrt[4]{\frac{2}{c_1}} < \left\lceil max(n_0, \sqrt[4]{\frac{2}{c_1}}, \sqrt{\frac{2a}{c_1}}) \right\rceil + 1 = n$, we can calculate

$$\begin{align} \sqrt[4]{\frac{2}{c_1}} &< n \tag{7} \\ \frac{2}{c_1} &< n^4 \tag{8} \\ 1 &< \frac{c_1 \cdot n^4}{2} \tag{9} \end{align}$$

Then, by combining the two results together,

$$1 + an^2 < \frac{c_1 n^4}{2} + \frac{c_1 n^4}{2} \tag{10}$$
$$< c_1 n^4 \tag{11}$$

$\square$

Question 4