Worksheet 8 Solution

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Question 1

a. $P(n) : \forall n \in \mathbb{N},\ n \leq 2^n$.
$\forall k \in \mathbb{N}, P(0) \wedge P(k) \Rightarrow P(k+1)$

Or, with P fully expanded,
$\forall k \in \mathbb{N},\ 0 \leq 2^0 \wedge k \leq 2^k \Rightarrow k+1 \leq 2^{k+1}$

b. **Base Case:**

Let $n = 0$.

Then,

$$(0) \leq 2^0 \tag{1}$$
$$0 \leq 1 \tag{2}$$

Since, $n \leq 2^n$ is true for $n = 0$, the base case holds.

Inductive Case:

Let $k \in \mathbb{N}$, and assume that $P(k)$ is true.

Then,

$$2^{k+1} = 2^k + 2^k \tag{1}$$
$$\geq k + k \tag{2}$$
$$\tag{3}$$

Then,

$$2^{k+1} \geq k + k \tag{4}$$
$$\geq k + 1 \tag{5}$$

by the fact that $k \in \mathbb{N}$ and $k \geq 1$.

Then, it follows from proof by induction that the statement $k \leq 2^k$ is true.

Question 2

- **Base Case:**

  Let $n = 0$.

  Then,

$$\sum_{j=0}^{0} T_j = \frac{(0)(0+1)(0+2)}{6} \tag{1}$$
$$= 0 \tag{2}$$

  Since $T_0 = 0$, the base case holds.

  Inductive Case:

  Let $k \in \mathbb{N}$, and assume that $\sum_{j=0}^{k} T_j = \frac{k(k+1)(k+2)}{6}$ is true.

Then,

$$\begin{aligned}
\sum_{j=0}^{k} T_j + T_{k+1} &= \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+2)}{2} && (1)\\
&= \frac{k(k+1)(k+2)}{6} + \frac{3(k+1)(k+2)}{6} && (2)\\
&= \frac{(k+1)(k+2)(k+3)}{6} && (3)
\end{aligned}$$

Then, it follows from proof by induction that the statement $\forall n \in \mathbb{N},\ \sum_{j=0}^{k} T_j = \frac{k(k+1)(k+2)}{6}$ is true.

Question 3

a. Let $x \in \mathbb{R}^+$, and let $n \in \mathbb{N}$. Assume $(1+x)^n \geq 1 + nx$.

Then,

$$\begin{aligned}
(1+x)^{n+1} &= (1+x)^n(1+x) && (1)\\
&\geq (1+nx)(1+x) && (2)
\end{aligned}$$

by the assumption $(1+x)^n \geq 1 + nx$.

Then,

$$\begin{aligned}
(1+x)^{n+1} &\geq (1+nx)(1+x) && (3)\\
&\geq 1 + x + nx + nx^2 && (4)\\
&\geq 1 + x(n+1) + nx^2 && (5)\\
&\geq 1 + x(n+1) && (6)
\end{aligned}$$

Then, it follows from proof by induction that the statement $\forall x \in \mathbb{R}^+, \forall n \in \mathbb{N},\ (1+x)^n \geq 1 + nx$ is true.

Question 4

a. $\forall n \in \mathbb{N},\ n \geq 8 \Rightarrow 30n \leq 2^n$.

b. **Base Case:**

Let $n = 8$.

Then,

$$30(8) \leq 2^{(8)} \tag{1}$$
$$240 \leq 256 \tag{2}$$

Then it follows from above that the base case holds.

Inductive Case:

Let $n \in \mathbb{N}$. Assume $30n \leq 2^n$, and $n \geq 8$.

Then,

$$2^{n+1} \geq 2^n + 2^n \tag{1}$$

by the fact $2^{n+1} = 2^n + 2^n$.

Then,

$$2^{n+1} \geq 30n + 30n \tag{2}$$

by the assumption $30n \leq 2^n$.

Then,

$$\begin{aligned} 2^{n+1} &\geq 30n + 30 && (3) \\ &\geq 30(n+1) && (4) \end{aligned}$$

Then, it follows from proof by induction that the statement $\forall n \in \mathbb{N}, n \geq 8 \Rightarrow 30n \leq 2^n$.