Worksheet 16 Review

April 2, 2020

Question 1

a. Let $k \in \mathbb{N}$.

Here, the minimum possible change occurs for the loop variable in a single iteration when $i = i + 1$.

The maximum possible change occurs for the loop variable in a single iteration when $i = i + 6$.

The exact upper bound of the variable after k iteration is

$$i_k \leq 6k \tag{1}$$

The exact lower bound of the variable after k iteration is

$$k \leq i_k \tag{2}$$

Using the fact that the termination occurs when $i_k = n$, we can calculate that for the upper bound, the loop terminates when

$$6k \geq n \tag{3}$$
$$k \geq \frac{n}{6} \tag{4}$$

Because we know $\frac{n}{6}$ may be a decimal, we can conclude the closest value at which the loop terminates is when

$$k = \left\lceil \frac{n}{6} \right\rceil \tag{5}$$

Using the same fact, we can calculate that for the lower bound, the loop terminates when

$$k \geq n \tag{6}$$

It follows from above that for the lower bound, the smallest value of k at which the loop termination occurs is when

$$k = n \tag{7}$$

Then, we can conclude the function has asymptotic lower bound of $\Omega(n)$, and asymptotic upper bound of $\mathcal{O}(n)$.

Then, since both Ω and $\mathcal{O}$ have the same value, $\Theta(n)$ is also true.

Correct Solution:

Here, the minimum possible change occurs for the loop variable in a single iteration when $i = i + 1$.

The maximum possible change occurs for the loop variable in a single iteration when $i = i + 6$.

The exact upper bound of the variable after k iteration is

$$i_k \leq 6k \tag{8}$$

The exact lower bound of the variable after k iteration is

$$k \leq i_k \tag{9}$$

Using the fact that the termination occurs when $i_k = n$, we can calculate that for the upper bound, the loop terminates when

$$6k \geq n \tag{10}$$
$$k \geq \frac{n}{6} \tag{11}$$

Because we know $\frac{n}{6}$ may be a decimal, we can conclude the closest value at which the loop terminates is when

$$k = \left\lceil \frac{n}{6} \right\rceil + 1 \tag{12}$$

Using the same fact, we can calculate that for the lower bound, the loop terminates when

$$k \geq n \tag{13}$$

It follows from above that for the lower bound, the smallest value of k at which the loop termination occurs is when

$$k = n + 1 \tag{14}$$

Then, we can conclude the function has asymptotic lower bound of $\Omega(n)$, and asymptotic upper bound of $\mathcal{O}(n)$.

Since both Ω and $\mathcal{O}$ have the same value, $\Theta(n)$ is also true.

Notes:

- Realized $\mathbf{+\ 1}$ required after thinking there are n iterations between $i = 0$ and $i = n - 1$ when $i = i + 1$.
- Realized $\mathbf{+\ 1}$ in $\frac{n}{6}$ is required after playing with the following example [0,1,2,3,4,5] and [0,1,2,3,4,5,6].
- Realized $\lceil \frac{5}{6} \rceil = 1$ and $\lceil \frac{6}{6} \rceil = 1$, and indeed $\mathbf{+\ 1}$ is required to reach loop termination.
- Perhaps the choice of ceiling and floor can be determined by playing with examples.

b. Let $k \in \mathbb{N}$.

The minimum possible change occurs when $i = i \cdot 2$.

The maximum possible change occurs when $i = i \cdot 3$.

Next, we will determine the exact lower bound of loop iteration.

Because we know $i = i \cdot 2$ increases smallest, we can conclude the smallest possible change in i will result when $i = i \cdot 2$ happens entirely.

Then, it follows from above that at k^{th} iteration, the smallest value of i is 2^k.

For

Given this, the value of i at k^{th} iteration is 3^k.

Using the above fact, the exact upper bound and the lower bound of the loop iteration is

$$2^k \leq i_k \leq 3^k \tag{1}$$

Now, we will determine the asymptotic running time of the upper bound $i_k \leq 3^k$.

We want

Question 2

Question 3