Worksheet 5 Review

March 23, 2020

Question 1

- **Predicate Logic:** $\forall x, y \in \mathbb{Z}, Odd(x) \wedge Odd(y) \Rightarrow Odd(xy)$

  Let $x, y \in \mathbb{Z}$. Assume $Odd(x)$ and $Odd(y)$.

  Then, $\exists k, m \in \mathbb{Z}$,

$$x = 2k - 1 \tag{1}$$
$$y = 2m - 1 \tag{2}$$

  Then,

$$xy = (2k-1)(2m-1) \tag{3}$$
$$xy = (4km - 2k - 2m + 2) - 1 \tag{4}$$
$$xy = 2(2km - k - m + 1) - 1 \tag{5}$$
$$xy = 2o - 1 \tag{6}$$

  by setting $o = 2km - k - m + 1$.

  Since, $o \in \mathbb{Z}$, it follows from the definition of odd that the statement $\forall x, y \in \mathbb{Z}, Odd(x) \wedge Odd(y) \Rightarrow Odd(xy)$ is true.

Question 2

a. $\forall n, m \in \mathbb{Z},\ Even(n) \wedge Odd(m) \Rightarrow m^2 - n^2 = m + n$

b. The flaw is that the value k in $n = 2k$ and $m = 2k + 1$ cannot be the same.

Question 3

a. $Dom(f, g) : \forall n \in \mathbb{Z},\ g(n) \leq f(n)$, where $f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$

b. Let $f(n) = 3n$, $g(n) = n$, and $n \in \mathbb{N}$.

   Then,

$$\begin{align} g(n) = n &\leq n + n + n \tag{1} \\ &\leq 3n \tag{2} \\ &\leq f(n) \tag{3} \end{align}$$

   Then, it follows from the definition of '**is dominated by**' that g is dominated by f.

c. **Negation:** $\neg Dom(f, g) : \exists n \in \mathbb{Z},\ g(n) > f(n)$, where $f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$

   Let $n = 1$, $f(n) = 3n$, and $g(n) = n$.

   Then,

$$\begin{align} n + 165 &= (1) + 165 \tag{1} \\ &= 166 \tag{2} \\ &> 1 \tag{3} \\ &> (1)^2 \tag{4} \\ &> n^2 \tag{5} \end{align}$$

   Then it follows from the negation of $Dom(f, g)$ that g is not dominated by f.

d. **Predicate Logic:** $\forall f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$, $f(n) = n^2 \wedge g(n) = n + 165 \Rightarrow (\exists m \in \mathbb{N}, g(m) > f(m))$

Let $f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$ and $n = 1$. Assume $f(n) = n^2$, and $g(n) = n + 165$.

Then,

$$
\begin{align}
g(1) = (1) + 165 &= 166 \tag{1}\\
&> 1 \tag{2}\\
&> (1)^2 \tag{3}\\
&> f(1) \tag{4}
\end{align}
$$

Then, it follows from above statement that g is not dominated by f.

Question 4

- Let $x \in \mathbb{R}^{\geq 0}$, and $\epsilon = x - \lfloor x \rfloor$. Assume $x \geq 4$.

  Then,

$$
\begin{align}
(\lfloor x \rfloor)^2 &= (x - \epsilon)^2 \tag{1}\\
&= x^2 - 2x\epsilon + \epsilon^2 \tag{2}
\end{align}
$$

  Since

$$
\begin{align}
x &\geq 4 \tag{3}\\
x^2 &\geq 4x \tag{4}\\
\frac{1}{2}x^2 &\geq 2x \tag{5}\\
& \tag{6}
\end{align}
$$

  ,and

$$\frac{1}{2}x^2 \geq 2x\epsilon \tag{7}$$

by using the fact $\forall x \in \mathbb{R},\ 0 \leq x - \lfloor x \rfloor < 1$,

$$\begin{aligned}
(\lfloor x \rfloor)^2 &= x^2 - 2x\epsilon + \epsilon^2 && (8)\\
&\geq \frac{1}{2}x^2 + \epsilon^2 && (9)\\
&\geq \frac{1}{2}x^2 && (10)
\end{aligned}$$

Then it follows from above that the statement $\forall x \in \mathbb{R}^{\geq 0},\ x \geq 4 \Rightarrow (\lfloor x \rfloor)^2 \geq \frac{1}{2}x^2$ is true.