

CSC236 Term Test 1 Version 2 Review

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May 12, 2020

Question 1

- *Proof.* Define $P(n) : f(n) = 3^n$.

I will use complete induction to prove that $\forall n \in \mathbb{N}, P(n)$.

Base Case ($n = 0$):

Let $n = 0$.

Then,

$$f(n) = 1 \quad \text{[By def.]} \quad (1)$$

$$\leq 3^0 \quad (2)$$

$$= 3^n \quad (3)$$

Thus, $P(n)$ follows in this step.

Base Case ($n = 1$):

Let $n = 1$.

Then,

$$f(n) = 1 \quad \text{[By def., since } n = 1\text{]} \quad (4)$$

$$\leq 3^1 \quad (5)$$

$$= 3^n \quad (6)$$

Thus, $P(n)$ follows in this step.

Base Case ($n = 2$):

Let $n = 2$.

Then,

$$f(n) = 9 \quad [\text{By def., since } n = 2] \quad (7)$$

$$\leq 3^2 \quad (8)$$

$$= 3^n \quad (9)$$

Thus, $P(n)$ follows in this step.

Base Case ($n = 3$):

Let $n = 3$.

Then,

$$f(n) = f(n-1) + 3f(n-2) + 9f(n-3) \quad [\text{By def., since } n = 2] \quad (10)$$

$$= 9 + 3 \cdot 3 + 9 \cdot 1 \quad [\text{By def., since } n-1 = 2, n-2 = 1, n-3 = 0] \quad (11)$$

$$= 3^2 + 3^2 + 3^2 \quad (12)$$

$$= 3^3 \quad (13)$$

$$= 3^n \quad (14)$$

$$\leq 3^n \quad (15)$$

Thus, $P(n)$ follows in this step.

Case ($n > 3$):

Let $n > 3$.

Then, since $0 \leq n-3 < n-2 < n-1 < n$, $P(n-3)$, $P(n-2)$, $P(n-1)$ holds by induction hypothesis. That is, $P(n-3) \leq 3^{n-3}$, $P(n-2) \leq 3^{n-2}$, $P(n-1) \leq 3^{n-1}$.

Thus,

$$f(n) = f(n-1) + 3f(n-2) + 9f(n-3) \quad [\text{By def., since } n > 2] \quad (16)$$

$$\leq 3^{n-1} + 3 \cdot 3^{n-2} + 9 \cdot 3^{n-3} \quad [\text{By header}] \quad (17)$$

$$= 3^{n-1} + 3^{n-1} + 3^{n-1} \quad (18)$$

$$= 3^n \quad (19)$$

So, $P(n)$ follows from $H(n)$ in this step.

□

Question 2

- *Proof.* Define for convenience

$P(x, y, z, w)$: There are no positive integers x, y, z, w such that $x^4 + 3y^4 + 9z^4 = 27w^4$.

I will prove $P(x, y, z, w)$ by contradiction.

Assume $\neg P(x, y, z, w)$. That is, $\exists x, y, z, w \in \mathbb{N}^+, x^4 + 3y^4 + 9z^4 = 27w^4$.

Then, $X = \{x \in \mathbb{N}^+ \mid \exists y, z, w \in \mathbb{N}^+, x^4 + 3y^4 + 9z^4 = 27w^4\}$ is not empty.

Then, by the principle of well-ordering, X has smallest element.

Let $x_0 \in X$ be its smallest element, and let $y_0, z_0, w_0 \in \mathbb{N}^+, x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4$.

Then,

$$x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \Rightarrow x_0 = 27w_0^4 - 3y_0^4 + 9z_0^4 \quad (20)$$

$$\Rightarrow x \mid x_0^4 \quad (21)$$

$$\Rightarrow 3 \mid x_0 \quad \begin{array}{l} \text{[By hint, since 3 is prime]} \\ (22) \end{array}$$

$$\text{Let } x_1 \in \mathbb{N}^+, x_0 = 3x_1 \Rightarrow 3^4x_1^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \quad (23)$$

$$\Rightarrow 3y_0^4 = 27w_0^4 - 9z_0^4 - 3^4x_1^4 \quad (24)$$

$$\Rightarrow y_0^4 = 9w_0^4 - 3z_0^4 - 3^3x_1^4 \quad (25)$$

$$\Rightarrow 3 \mid y_0^4 \quad (26)$$

$$\Rightarrow 3 \mid y_0 \quad \begin{array}{l} \text{[By hint, since 3 is prime]} \\ (27) \end{array}$$

$$\text{Let } y_1 \in \mathbb{N}^+, y_0 = 3y_1 \Rightarrow 3^4x_1^4 + 3^5y_1^4 + 9z_0^4 = 27w_0^4 \quad (28)$$

$$\Rightarrow 9z_0^4 = 27w_0^4 - 3^5y_1^4 - 3^4x_1^4 \quad (29)$$

$$\Rightarrow z_0^4 = 3w_0^4 - 3^3y_1^4 - 3^2x_1^4 \quad (30)$$

$$\Rightarrow 3 \mid z_0^4 \quad (31)$$

$$\Rightarrow 3 \mid z_0 \quad \begin{array}{l} \text{[By hint, since 3 is prime]} \\ (32) \end{array}$$

$$\text{Let } z_1 \in \mathbb{N}^+, z_0 = 3z_1 \Rightarrow 3^4x_1^4 + 3^5y_1^4 + 3^6z_1^4 = 27w_0^4 \quad (33)$$

$$\Rightarrow 3x_1^4 + 3^2y_1^4 + 3^3z_1^4 = w_0^4 \quad (34)$$

$$\Rightarrow 3 \mid w_0^4 \quad (35)$$

$$\Rightarrow 3 \mid w_0 \quad \begin{array}{l} \text{[By hint, since 3 is prime]} \\ (36) \end{array}$$

$$\text{Let } w_1 \in \mathbb{N}^+, w_0 = 3w_1 \Rightarrow 3^4x_1^4 + 3^5y_1^4 + 3^6z_1^4 = 3^7w_1^4 \quad (37)$$

$$\Rightarrow x_1^4 + 3y_1^4 + 3^2z_1^4 = 3^3w_1^4 \quad (38)$$

$$\Rightarrow x_1 \in X \quad (39)$$

Then, since $x_1 < x_0$ and $x_1 \in X$, but x_0 is the smallest element in X , this leads to contradiction.

Thus, the assumption is false, and $P(x, y, z, w)$ holds. \square