CSC236 Term Test 1 Version 2 Solution

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Question 1

- *Proof.* Define $P(n) : f(n) = 3^n$.

  I will use complete induction to prove that $\forall n \in \mathbb{N}, n > 2 \Rightarrow\ P(n)$.

  Base Case ($n = 0$):

  Let $n = 0$.

  Then, the definition of $f(n)$ tells us $f(n) = 1$.

  Then, we have

$$\begin{aligned} f(n) &= 3^0 && (1)\\ &= 3^n && (2) \end{aligned}$$

  Thus, $P(n)$ follows.

  Base Case ($n = 1$):

  Let $n = 1$.

  Then, the definition of $f(n)$ tells us $f(n) = 3$.

  Then, we have

$$\begin{aligned} f(n) &= 3^1 && (3)\\ &= 3^n && (4) \end{aligned}$$

Thus, $P(n)$ follows.

Base Case ($n = 2$):

Let $n = 2$.

Then, the definition of $f(n)$ tells us $f(n) = 9$.

Then, we have

$$f(n) = 3^2 \tag{5}$$
$$= 3^n \tag{6}$$

Thus, $P(n)$ follows.

Case ($n > 2$):

Assume $n > 2$.

Then, since $0 \leq n-1 < n$, $0 \leq n-2 < n$, and $0 \leq n-3 < n$, the complete induction tells us $P(n-1)$, $P(n-2)$, and $P(n-3)$, i.e. $f(n-1) = 3^{n-1}$, $f(n-2) = 3^{n-2}$, and $f(n-3) = 3^{n-3}$, respectively.

Then, using these facts, we can write

$$f(n) = f(n-1) + 3f(n-2) + 9f(n-3) \tag{7}$$
$$= 3^{n-1} + 3 \cdot 3^{n-2} + 3^2 \cdot 3^{n-3} \tag{8}$$
$$= 3^{n-1} + 3^{n-1} + 3^{n-1} \tag{9}$$
$$= 3^{n-1}(1 + 1 + 1) \tag{10}$$
$$= 3^{n-1}3 \tag{11}$$
$$= 3^n \tag{12}$$

Thus, $P(n)$ follows. $\square$

Correct Solution:

Define $P(n) : f(n) = 3^n$.

I will use complete induction to prove that $\forall n \in \mathbb{N},\ P(n)$.

Inductive Step:

Let $n \in \mathbb{N}$. Assume $n > 2$. Assume $H(n) : \bigwedge_{i=0}^{n-1} P(i)$. I will prove $P(n)$ follows. That is, $f(n) = 3^n$.

Base Case ($n = 0$):

Let $n = 0$.

Then, the definition of $f(n)$ tells us $f(n) = 1$.

Then, we have

$$\begin{aligned} f(n) &= 3^0 && (13) \\ &= 3^n && (14) \end{aligned}$$

Thus, $P(n)$ follows.

Base Case ($n = 1$):

Let $n = 1$.

Then, the definition of $f(n)$ tells us $f(n) = 3$.

Then, we have

$$\begin{aligned} f(n) &= 3^1 && (15) \\ &= 3^n && (16) \end{aligned}$$

Thus, $P(n)$ follows.

Base Case ($n = 2$):

Let $n = 2$.

Then, the definition of $f(n)$ tells us $f(n) = 9$.

Then, we have

$$\begin{aligned} f(n) &= 3^2 && (17) \\ &= 3^n && (18) \end{aligned}$$

Thus, $P(n)$ follows.

<u>**Case ($n > 2$):**</u>

Assume $n > 2$.

Then, since $0 \leq n-1 < n$, $0 \leq n-2 < n$, and $0 \leq n-3 < n$, the complete induction tells us $P(n-1)$, $P(n-2)$, and $P(n-3)$, i.e. $f(n-1) = 3^{n-1}$, $f(n-2) = 3^{n-2}$, and $f(n-3) = 3^{n-3}$, respectively.

Then, using these facts, we can write

$$
\begin{aligned}
f(n) &= f(n-1) + 3f(n-2) + 9f(n-3) && \text{[By definition, since } n > 2\text{]} \quad (19)\\
&= 3^{n-1} + 3 \cdot 3^{n-2} + 3^2 \cdot 3^{n-3} && (20)\\
&= 3^{n-1} + 3^{n-1} + 3^{n-1} && (21)\\
&= 3^{n-1}(1+1+1) && (22)\\
&= 3^{n-1}3 && (23)\\
&= 3^n && (24)
\end{aligned}
$$

Thus, $P(n)$ follows.

Question 2

Question 3