

# CSC236 Worksheet 3

Hyungmo Gu

May 5, 2020

## Question 1

- Given the statement to prove

$P(x, y, z)$  : There are no positive integers  $x, y, z$  such that  $x^3 + 3y^3 = 9z^3$

*Proof.* We will prove  $P(x, y, z)$  using proof by contradiction.

Assume  $\exists x, y, z \in \mathbb{N}^+, x^3 + 3y^3 = 9z^3$ .

First, we need to show there is smallest element  $x_0 \in X$  with  $y_0, z_0 \in \mathbb{N}^+$  satisfying  $x^3 + 3y^3 = 9z^3$ , using well-ordering principle.

The header tells us there are elements  $x, y, z \in \mathbb{N}^+$ , satisfying  $x^3 + 3y^3 = 9z^3$ .

Then, we can write the set  $X = \{x \mid x \in \mathbb{N}^+, \exists y, z \in \mathbb{N}^+, x^3 + 3y^3 = 9z^3\}$  is not empty.

Then, using principle of well-ordering, we can write that there is smallest positive natural number  $x_0 \in X$  along with  $y_0, z_0 \in \mathbb{N}^+$  satisfying  $x^3 + 3y^3 = 9z^3$ .

Second, we need to show that  $x_1^3 = 9z_1^3 - 3y_1^3$  is satisfied, given  $x_0 > x_1$ .

We will do so in parts.

**Part 1 (Showing  $x_0 = 3 \cdot x_1$ ):**

We know that

$$x_0^3 + 3y_0^3 = 9z_0^3 \quad (1)$$

$$x_0^3 = 9z_0^3 - 3y_0^3 \quad (2)$$

Since  $3 \mid 9z_0^3 - 3y_0^3$ , we can write  $3 \mid x_0^3$ .

Then, since 3 is a prime number, by using the hint provided in question 3 of assignment 1, we can write there is  $x_1 \in \mathbb{Z}$ ,  $x_0 = 3 \cdot x_1$ .

Then, because we know  $x_0, 3 \in \mathbb{N}^+$ , we can conclude  $x_1 \in \mathbb{N}^+$ .

**Part 2 (Showing  $y_0 = 3 \cdot y_1$ ):**

We know that

$$x_0^3 + 3y_0^3 = 9z_0^3 \quad (3)$$

$$3y_0^3 = 9z_0^3 - x_0^3 \quad (4)$$

Then, using the fact  $x_0 = 3 \cdot x_1$  from part 1, we can calculate

$$3y_0^3 = 9z_0^3 - 3^3x_1^3 \quad (5)$$

$$y_0^3 = 3z_0^3 - 3^2x_1^3 \quad (6)$$

Since  $3 \mid 3z_0^3 - 3^2x_1^3$ , we can write that  $3 \mid y_0^3$ .

Then, since 3 is a prime number, by using the hint provided in question 3 of assignment 1, we can write there is  $y_1 \in \mathbb{Z}$ ,  $y_0 = 3 \cdot y_1$ .

Then, because we know  $y_0, 3 \in \mathbb{N}^+$ , we can conclude  $y_1 \in \mathbb{N}^+$ .

**Part 3 (Showing  $z_0 = 3 \cdot z_1$ ):**

We know that

$$9z_0^3 = x_0^3 + 3y_0^3 \quad (7)$$

Then, using the fact  $x_0 = 3 \cdot x_1$  from part 1, and  $y_0 = 3 \cdot y_1$  from part 2, we can calculate

$$9z_0^3 = 3^3x_1^3 + 3^4y_1^3 \quad (8)$$

$$z_0^3 = 3x_1^3 + 3^2y_1^3 \quad (9)$$

Since  $3 \mid 3x_1^3 + 3^2y_1^3$ , we can write that  $3 \mid z_0^3$ .

Then, since 3 is a prime number, by using the hint provided in question 3 of assignment 1, we can write there is  $z_1 \in \mathbb{Z}$ ,  $z_0 = 3 \cdot z_1$ .

Then, because we know  $z_0, 3 \in \mathbb{N}^+$ , we can conclude  $z_1 \in \mathbb{N}^+$ .

**Part 4 (Showing  $x_1^3 = 9z_1^3 - 3y_1^3$ ):**

We know that

$$9z_0^3 = x_0^3 + 3y_0^3 \quad (10)$$

Then, using the fact  $x_0 = 3 \cdot x_1$  from part 1,  $y_0 = 3 \cdot y_1$  from part 2, and  $z_0 = 3 \cdot z_1$  we can calculate

$$3^5z_1^3 = 3^3x_1^3 + 3^4y_1^3 \quad (11)$$

$$3^2z_1^3 = x_1^3 + 3y_1^3 \quad (12)$$

$$9z_1^3 = x_1^3 + 3y_1^3 \quad (13)$$

Finally, the part 4 tells us

$$9z_1^3 = x_1^3 + 3y_1^3 \quad (14)$$

where  $x_1 < x_0$ .

Then, because we know  $x_0$  is the smallest number satisfying  $x^3 + 3y^3 = 9z^3$ , we can conclude above leads to contradiction.

Then, we can conclude the the assumption is false.

□

## Notes:

- **Proof By Contradiction:**  $\neg P \Rightarrow \neg Q \wedge Q$  (Assuming we are proving  $P \Rightarrow Q$ )
- **Principle of Well-Ordering:** Any nonempty subset  $A$  of  $\mathbb{N}$  contains a minimum element; i.e. for any  $A \subseteq \mathbb{N}$  such that  $A \neq \emptyset$ , there is some  $a \in A$  such that for all  $a' \in A$ ,  $a \leq a'$ .
- examples of well-ordered sets
  1.  $\mathbb{N} \cup \{0\}$
  2.  $\mathbb{N} \cup \{1, 2\}$
  3.  $\{n \in \mathbb{N} : n > 5\}$
- examples of non-well-ordered sets
  1.  $\mathbb{R}$  and the open interval  $(0, 2)$
  2.  $\mathbb{Z}$
- Learned the  $P$  in  $\neg P \Rightarrow \neg Q \wedge Q$  is the statement

$P(x, y, z)$  : There are no positive integers  $x, y, z$  such that  $x^3 + 3y^3 = 9z^3$

And learned the  $Q$  is the principle of well-ordering on  $P$ .

- Learned the goal of contradiction is to show the assumption violates principle of well-ordering. That is, there is  $x_1 \in X$  less than  $x_0$  satisfying  $x^3 + 3y^3 = 9z^3$
- Noticed professor reduced wordiness of work using short notations.

Then

$$\begin{aligned} x_0^3 + 3y_0^3 = 9z_0^3 &\Rightarrow x_0^3 = 9z_0^3 - 3y_0^3 \Rightarrow 3 \mid x_0^3 \Rightarrow 3 \mid x_0 && \# \text{ by clue for A1 Q3} \\ \text{let } x_1 \in \mathbb{N}^+, 3x_1 = x_0 &\Rightarrow 3^3 x_1^3 = 9z_0^3 - 3y_0^3 \Rightarrow 3^2 x_1^3 = 3z_0^3 - y_0^3 && \# \text{ divide through by 3} \\ &\Rightarrow y_0^3 = 3z_0^3 - 3^2 x_1^3 \Rightarrow 3 \mid y_0^3 \Rightarrow 3 \mid y_0 \\ \text{let } y_1 \in \mathbb{N}^+, 3y_1 = y_0 &\Rightarrow 3^3 y_1^3 = 3z_0^3 - 3^2 x_1^3 \Rightarrow 3^2 y_1^3 = z_0^3 - 3x_1^3 && \# \text{ divide through by 3} \\ &\Rightarrow 3x_1^3 + 3^2 y_1^3 = z_0^3 \Rightarrow 3 \mid z_0^3 \Rightarrow 3 \mid z_0 \\ \text{let } z_1 \in \mathbb{N}^+, 3z_1 = z_0 &\Rightarrow 3x_1^3 + 3^2 y_1^3 = 3^3 z_1^3 \Rightarrow x_1^3 + 3y_1^3 = 9z_1^3 && \# \text{ divide through by 3} \\ &\Rightarrow x_1 \in X \end{aligned}$$

## Question 2

- *Proof.* **Basis:**

We need to show that the property is true for the simplest members  $x, y, z$ .

There are three cases:  $e = x$ ,  $e = y$  and  $e = z$ . In each of the cases  $s_2(e) = 1$  and  $s_1(e) = 0$ .

Using this fact, starting from the left hand side, we can conclude

$$s_1(e) = 0 = 3 \cdot 0 \tag{1}$$

$$= 3 \cdot (1 - 1) \tag{2}$$

$$= 3 \cdot (s_2(e) - 1) \tag{3}$$

### **Inductive Step:**

Let  $e_1$  and  $e_2$  be arbitrary elements of  $\varepsilon$ . Assume  $H(e_1, e_2) : P(e_1)$  and  $P(e_2)$ . That is,  $e_1$  and  $e_2$  have the property  $s_1(e_1) = 3 \cdot (s_2(e_1) - 1)$  and  $s_1(e_2) = 3 \cdot (s_2(e_2) - 1)$ .

We need to show all possible combinations of  $e_1$  and  $e_2$  have the property. That is,  $P((e_1 + e_2))$ ,  $P((e_1 \times e_2))$ .

There are two cases, depending on how  $e$  is constructed from  $e_1$  and  $e_2$ :  $e = (e_1 + e_2)$ ,  $e = (e_1 \times e_2)$ . In each case we have

$$s_1(e) = s_1(e_1) + s_1(e_2) + 3 \tag{4}$$

$$s_2(e) = s_2(e_1) + s_2(e_2) \tag{5}$$

Then, using above fact, we can conclude

$$s_1(e) = s_1(e_1) + s_1(e_2) + 3 \tag{6} \quad \text{[by 4]}$$

$$= 3 \cdot (s_2(e_1) - 1) + 3 \cdot (s_2(e_2) - 1) + 3 \tag{7} \quad \text{[by induction hypothesis]}$$

$$= 3 \cdot s_2(e_1) - 3 + 3 \cdot s_2(e_2) - 3 + 3 \tag{8}$$

$$= 3 \cdot s_2(e_1) + 3 \cdot s_2(e_2) - 6 + 3 \tag{9}$$

$$= 3 \cdot (s_2(e_1) + s_2(e_2)) - 3 \tag{10}$$

$$= 3 \cdot s_2(e) - 3 \tag{11} \quad \text{[by 5]}$$

□

## Notes:

### • Structural Induction

- is a proof method used in mathematical logic, computer science, graph theory.
- is a generalization of mathematical induction over natural numbers.
- is a recursion method
- **Example:**

Define  $\varepsilon$ : The smallest set such that

\*  $x, y, z \in \varepsilon$  **# variables**

\*  $e_1, e_2 \in \varepsilon \Rightarrow (e_1 + e_2), (e_1 - e_2), (e_1 \times e_2), (e_1 \div e_2) \in \varepsilon$  **# operators**

(steps omitted). Prove  $P(e) : \mathbf{vr}(e) = \mathbf{op}(e) + 1$  **#  $\mathbf{vr}$  means number of variable,  $\mathbf{op}$  means number of operators**

to prove above using structural induction:

1. **Verify Base Case(s):** Show that the property is true for the simplest members,  $x, y, z$ . That is show  $P(x)$ ,  $P(y)$ , and  $P(z)$ .

There are three cases:  $e = x$ ,  $e = y$ , and  $e = z$ . In each case  $\mathbf{vr}(e) = 1$  and  $\mathbf{op}(e) = 0$ , so  $P(e)$  holds for the basis.

2. **Inductive Step:** Let  $e_1$  and  $e_2$  be arbitrary elements of  $\varepsilon$ . Assume  $H(e_1, e_2) : P(e_1)$  and  $P(e_2)$ . That is,  $e_1$  and  $e_2$  have the property  $\mathbf{vr}(e_1) = \mathbf{op}(e_1) + 1$  and  $\mathbf{vr}(e_2) = \mathbf{op}(e_2) + 1$ .

We need to show all possible combinations of  $e_1$  and  $e_2$  have the property. That is,  $P((e_1 + e_2))$ ,  $P((e_1 - e_2))$ ,  $P((e_1 \times e_2))$ , and  $P((e_1 \div e_2))$ .

There are four cases, depending on how  $e$  is constructed from  $e_1$  and  $e_2$ :  $e = (e_1 + e_2)$ ,  $e = (e_1 - e_2)$ ,  $e = (e_1 \times e_2)$  and  $e = (e_1 \div e_2)$ . In each case we have

$$\mathbf{vr}(e) = \mathbf{vr}(e_1) + \mathbf{vr}(e_2) \quad (12)$$

$$\mathbf{op}(e) = \mathbf{op}(e_1) + \mathbf{op}(e_2) + 1 \text{ **# +1 is from + in } e_1 + e_2 \text{** } \quad (13)$$

Thus,

$$\mathbf{vr}(e) = \mathbf{vr}(e_1) + \mathbf{vr}(e_2) \quad [\text{by (4.1)}] \quad (14)$$

$$= (\mathbf{op}(e_1) + 1) + (\mathbf{op}(e_2) + 1) \quad [\text{by induction hypothesis}] \quad (15)$$

$$= (\mathbf{op}(e_1) + \mathbf{op}(e_2)) + 2 \quad (16)$$

$$= (\mathbf{op}(e) - 1) + 2 \quad [\text{by (4.2)}] \quad (17)$$

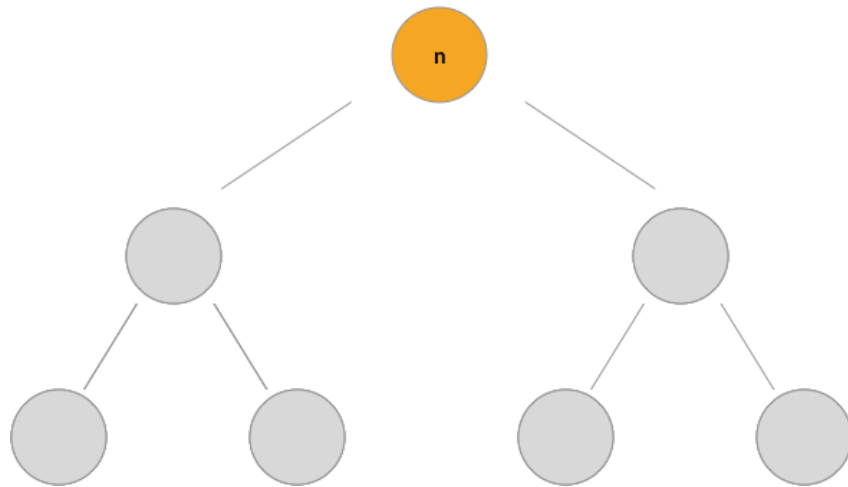
$$= \mathbf{op}(e) + 1 \quad (18)$$

### Question 3

#### • Rough Work:

Define the set of non-empty full binary trees,  $\mathcal{T}$ , as the smallest set such that:

- Any single node is an element of  $\mathcal{T}$
- If  $t_1, t_2 \in \mathcal{T}$ ,  $n$  is a node that belongs to neither  $t_1$  nor  $t_2$ , and  $t_1, t_2$  have no nodes in common, then  $n$  together with edges to the **root nodes**  $t_1$  and  $t_2$  is also an element of  $\mathcal{T}$ .



Prove  $P(t) : \mathbf{leaf\_node}(t) = \mathbf{internal\_node}(t) + 1$

#### 1. Basis

There is one case, where  $t$  is the binary tree with one node. In this case, the node is leaf node. So,  $\mathbf{leaf\_node}(t) = 1$ . So,  $P(t)$  holds for the case.

#### 2. Inductive Step

Let  $t_1$  and  $t_2$  be arbitrary element of  $\mathcal{T}$ . Assume  $H(t_1, t_2) : P(t_1)$  and  $P(t_2)$ . That is,  $t_1$  and  $t_2$  have the property **leaf\_node**( $t_1$ ) = **internal\_node**( $t_1$ ) + 1 and **leaf\_node**( $t_2$ ) = **internal\_node**( $t_2$ ) + 1. Let  $n$  be a node that belongs to neither  $t_1$  nor  $t_2$ . Assume  $t_1$  and  $t_2$  have no nodes in common.

We need to show  $P(t)$  follows. That is, **leaf\_node**( $t$ ) = **internal\_node**( $t$ ) + 1.

1. a