Problem Set 2 Solution

March 17, 2020

Question 1

a.

b. **Predicate Logic:** $\forall k, n \in \mathbb{Z}^+$, $\forall p \in \mathbb{N}$, $Prime(p) \land p^k < n < p^k + p \Rightarrow gcd(p^k, n) = 1$

Let $k, n \in \mathbb{Z}^+$, and $p \in \mathbb{N}$. Assume $Prime(p)$, and $p^k < n < p^k + p$.

Then, p^k can either be divided by 1 or p by fact 3.

Since, $p^k < n < p^k + p$, n cannot be written in multiples of p.

Then, it follows from the definition of divisibility that $p \nmid n$.

Since $p \nmid n$, but $1 \mid p^k$ and $1 \mid n$, $gcd(p^k, n) = 1$.

c. **Predicate Logic:** $\forall m \in \mathbb{Z}$, $\forall n_0 \in \mathbb{N}$, $\exists n \in \mathbb{N}\ n > n_0 \land gcd(n, n+m) = 1$

Since there are infinitely many primes by fact 4, let $Prime(n)$ and $n > m$.

Since $Prime(n)$, by fact 3, n can either be divided by 1 or n.

Since $n \mid n$, but $n \nmid m$, $n \nmid (n+m)$, and n can't be chosen as the greatest common divisor of n and $n+m$.

Since $gcd(n, n+m) \neq n$ but $1 \mid n$ and $1 \mid (n+m)$, $gcd(n, n+m) = 1$.

Then, it follows from above that the statement $\forall m \in \mathbb{Z}$, $\forall n_0 \in \mathbb{N}$, $\exists n \in \mathbb{N}$ $n > n_0 \wedge gcd(n, n+m) = 1$ is true.

d. **Definition of Primary Gap:** Let $a \in \mathbb{N}$. We say that a is a prime gap when there exists a prime p such that $p + a$ is also prime, and none of the numbers between p and $p + a$ (exclusive) are prime.

Case 1 ($a > 2$):

Let $a, p \in \mathbb{Z}^+$. Assume $PrimaryGap(a)$, $Primary(p)$, and $a > 2$.

Then, $2 \nmid p$ and $2 \nmid p + a$.

Then,

$$2 \mid (p+a) - a \tag{1}$$
$$2 \mid a \tag{2}$$

by fact 1.

Then it follows from above that in case $a > 2$, primary gap is divisible by 2.

Case 2 ($a \leq 2$):

Let $a, p \in \mathbb{Z}^+$. Assume $PrimaryGap(a)$, $Primary(p)$, and $a \leq 2$.

Then, only two primary numbers in $\mathbb{Z}^+$ exist - 1 and 2.

Then,

$$a = 2 - 1 \tag{1}$$
$$a = 1 \tag{2}$$

Then, it follows from above that in case $a \leq 2$, the value of primary gap is 1.

Question 2

a. Let $n \in \mathbb{N}$, and $x \in \mathbb{R}$.

Because we know $\forall x \in \mathbb{R},\ 0 \leq x - \lfloor x \rfloor < 1$ from fact 1, we can conclude $\lfloor x \rfloor \leq x < 1 + \lfloor x \rfloor$.

Then,

$$\begin{align}
\lfloor nx \rfloor - n\lfloor x \rfloor &\leq nx - \lfloor x \rfloor \tag{1}\\
&\leq n(x - \lfloor x \rfloor) \tag{2}
\end{align}$$

by using the above.

Then,

$$\begin{align}
\lfloor nx \rfloor - n\lfloor x \rfloor &\leq n(x - \lfloor x \rfloor) \tag{3}\\
&\leq k \tag{4}
\end{align}$$

by choosing $k = n(x - \lfloor x \rfloor)$.

Then, it follows from the above that the statement the statement $\forall n \in \mathbb{N},\ \exists k \in \mathbb{N},\ x \in \mathbb{R},\ \lfloor nx \rfloor - n\lfloor x \rfloor \leq k$ is true.

Question 3