Problem Set 2 Solution

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Question 1

a.

b. **Predicate Logic:** $\forall k, n \in \mathbb{Z}^+$, $\forall p \in \mathbb{N}$, $Prime(p) \wedge p^k < n < p^k + p \Rightarrow gcd(p^k, n) = 1$

Let $k, n \in \mathbb{Z}^+$, and $p \in \mathbb{N}$. Assume $Prime(p)$, and $p^k < n < p^k + p$.

Then, p^k can either be divided by 1 or p by fact 3.

Since, $p^k < n < p^k + p$, n cannot be written in multiples of p.

Then, it follows from the definition of divisibility that $p \nmid n$.

Since $p \nmid n$, but $1 \mid p^k$ and $1 \mid n$, $gcd(p^k, n) = 1$.

c. **Predicate Logic:** $\forall m \in \mathbb{Z}$, $\forall n_0 \in \mathbb{N}$, $\exists n \in \mathbb{N}\ n > n_0 \wedge gcd(n, n+m) = 1$

Since there are infinitely many primes by fact 4, let $Prime(n)$ and $n > m$.

Since $Prime(n)$, by fact 3, n can either be divided by 1 or n.

Since $n \mid n$, but $n \nmid m$, $n \nmid (n+m)$, and n can't be chosen as the greatest common divisor of n and $n+m$.

Since $gcd(n, n+m) \neq n$ but $1 \mid n$ and $1 \mid (n+m)$, $gcd(n, n+m) = 1$.

Then, it follows from above that the statement $\forall m \in \mathbb{Z}, \forall n_0 \in \mathbb{N}, \exists n \in \mathbb{N}$ $n > n_0 \wedge gcd(n, n+m) = 1$ is true.

d. **Definition of Primary Gap:** Let $a \in \mathbb{N}$. We say that a is a prime gap when there exists a prime p such that $p + a$ is also prime, and none of the numbers between p and $p + a$ (exclusive) are prime.

Case 1 ($a > 2$):

Let $a, p \in \mathbb{Z}^+$. Assume $PrimaryGap(a)$, $Primary(p)$, and $a > 2$.

Then, $2 \nmid p$ and $2 \nmid p + a$.

Then,

$$2 \mid (p+a) - a \quad (1)$$
$$2 \mid a \quad (2)$$

by fact 1.

Then it follows from above that in case $a > 2$, primary gap is divisible by 2.

Case 2 ($a \leq 2$):

Let $a, p \in \mathbb{Z}^+$. Assume $PrimaryGap(a)$, $Primary(p)$, and $a \leq 2$.

Then, only two primary numbers in $\mathbb{Z}^+$ exist - 1 and 2.

Then,

$$a = 2 - 1 \quad (1)$$
$$a = 1 \quad (2)$$

Then, it follows from above that in case $a \leq 2$, the value of primary gap is 1.

Question 2

a. Let $n \in \mathbb{N}$, and $x \in \mathbb{R}$.

Because we know $\forall x \in \mathbb{R},\ 0 \leq x - \lfloor x \rfloor < 1$ from fact 1, we can conclude $\lfloor x \rfloor \leq x < 1 + \lfloor x \rfloor$.

Then,

$$\begin{align} \lfloor nx \rfloor - n\lfloor x \rfloor &\leq nx - n\lfloor x \rfloor \tag{1} \\ &\leq n(x - \lfloor x \rfloor) \tag{2} \end{align}$$

by using the above.

Then,

$$\begin{align} \lfloor nx \rfloor - n\lfloor x \rfloor &\leq n(x - \lfloor x \rfloor) \tag{3} \\ &< n \tag{4} \\ &< k \tag{5} \end{align}$$

by using fact 1 and choosing $k = n$.

Then, it follows that the statement the statement $\forall n \in \mathbb{N},\ \exists k \in \mathbb{N},\ x \in \mathbb{R},\ \lfloor nx \rfloor - n\lfloor x \rfloor \leq k$ is true.

b. **Negation of statement:** $\forall k \in \mathbb{N},\ \exists m \in \mathbb{N},\ \exists x \in \mathbb{R},\ \lfloor nx \rfloor - n\lfloor x \rfloor > k$

Let $x = 0.5$ and $n = 2(k + 1)$.

Then,

$$\begin{align} \lfloor nx \rfloor - n\lfloor x \rfloor &= \lfloor \frac{2(k+1)}{2} \rfloor - n\lfloor 0.5 \rfloor \tag{1} \\ &= k + 1 - 0 \tag{2} \\ &= k + 1 \tag{3} \\ &> k \tag{4} \end{align}$$

Then it follows that the statement $\exists k \in \mathbb{N}, \forall n \in \mathbb{N}, \forall x \in \mathbb{R}, \lfloor nx \rfloor - n\lfloor x \rfloor \leq k$ is false.

Question 3

a. **Predicate Logic**: $\forall f : \mathbb{R} \to \mathbb{R}$, $f(x) = f(-x) \wedge -f(-x) = f(x) \leftrightarrow f = 0$

Part 1: Proving in $\Rightarrow$ direction

Let $f : \mathbb{R} \to \mathbb{R}$. Assume $f(x) = f(-x) \wedge -f(-x) = f(x)$.

Then,

$$f(-x) - f(-x) = 2f(x) \tag{1}$$
$$0 = 2f(x) \tag{2}$$

by adding $f(x) = f(-x)$ and$-f(-x) = f(x)$ together.

Then,

$$0 = f(x) \tag{3}$$

Then it follows that the statement $\forall f : \mathbb{R} \to \mathbb{R}$, $f(x) = f(-x) \wedge -f(-x) = f(x) \Rightarrow f = 0$ is true.

Part 2: Proving in $\Leftarrow$ direction

Let $f : \mathbb{R} \to \mathbb{R}$. Assume $f(x) = 0$.

Then,

$$-f(-x) = -(-0) \tag{1}$$
$$= 0 \tag{2}$$
$$= f(x) \tag{3}$$

It follows from above that $f(x) = 0$ is an odd function.

Also,

$$
\begin{aligned}
f(-x) &= (-0) && (4)\\
&= 0 && (5)\\
&= f(x) && (6)
\end{aligned}
$$

It follows from above that $f(x) = 0$ is an odd function.

Because we know $f(x) = 0$ is both even and odd, we can conclude that the statement $\forall f : \mathbb{R} \to \mathbb{R},\ f = 0 \Rightarrow f(x) = f(-x) \wedge -f(-x) = f(x)$ is true.

b. **Predicate Logic:** $\forall f : \mathbb{R} \to \mathbb{R}, \exists f_1, f_2 : \mathbb{R} \to \mathbb{R}, -f_1(x) = f_1(x) \wedge f_2(-x) = f_2(fx) \wedge f(x) = f_1(x) + f_2(x)$

Negation: $\exists f : \mathbb{R} \to \mathbb{R},\ \forall f_1, f_2 : \mathbb{R} \to \mathbb{R},\ -f_1(-x) \neq f_1(x) \vee f_2(x) \neq f_2(x) \vee f(x) \neq f_1(x) + f_2(x)$

Let f be an even function.

Then,

$$
\begin{aligned}
f(-x) &= (f_1(-x) + f_2(-x)) && (1)\\
&= f_1(x) - f_2(x) && (2)\\
&\neq f(x) && (3)
\end{aligned}
$$

Then, it follows from negation of the statement that every function cannot be written as a sum of an even function and an odd function.