

# CSC373 Worksheet 5 Solution

August 8, 2020

1. *Proof.* Assume that a flow network  $G = (V, E)$  violates the assumption that the network contains a path  $s \rightsquigarrow v \rightsquigarrow t$  for all vertices  $v \in V$ . Let  $u$  be a vertex for which there is no path  $s \rightsquigarrow u \rightsquigarrow t$ .

I must show such that there is no flow at vertex  $u$ . That is, there exists a maximum flow  $f$  in  $G$  such that  $f(u, v) = f(v, u) = 0$  for all vertices  $v \in V$ .

Assume for the sake of contradiction that there is some vertex  $u$  with flow  $f$ . That is, there exists some vertices  $v \in V$  such that  $f(u, v) > 0$  or  $f(v, u) > 0$ .

I see that three cases follows, and I will prove each separately.

1. **Cases 1:**  $f(u, v) = 0$  and  $f(v, u) > 0$

Here, assume that  $f(u, v) = 0$  for all  $v \in V$  and  $f(v, u) > 0$  for some  $v \in V$ .

Then, we can write  $\sum_{v \in V} f(u, v) = 0$  and  $\sum_{v \in V} f(v, u) > 0$

But this violates the flow conservation property (i.e  $\sum_{v \in V} f(u, v) = \sum_{v \in V} f(v, u)$ )

Thus, by proof by contradiction,  $f(u, v) = 0$  and  $f(v, u) = 0$  for all  $v \in V$  and all  $u \in V$  with no path  $s \rightsquigarrow u \rightsquigarrow t$ .

2. **Cases 2:**  $f(u, v) > 0$  and  $f(v, u) = 0$

Here, assume that  $f(u, v) > 0$  for some  $v \in V$  and  $f(v, u) = 0$  for all  $v \in V$ .

Then, by similar work as case 1, the same result follows.

### 3. Cases 3: $f(u, v) > 0$ and $f(v, u) > 0$

Here, assume that  $f(u, v) > 0$  and  $f(v, u) > 0$  for some  $v \in V$ .

Since  $s \rightsquigarrow v \rightsquigarrow t$  and  $u$  is connected by some vertices  $v$ , we can write  $s \rightsquigarrow u \rightsquigarrow t$ .

Then, this violates the fact in header that the vertex  $u$  has no path  $s \rightsquigarrow u \rightsquigarrow t$ .

Thus, by proof by contradiction,  $f(u, v) = 0$  and  $f(v, u) = 0$  for all  $v \in V$  and all  $u \in V$  with no path  $s \rightsquigarrow u \rightsquigarrow t$ .

□

## Notes

### • Maximum Flow:

- Finds a flow of maximum value <sup>[1]</sup>

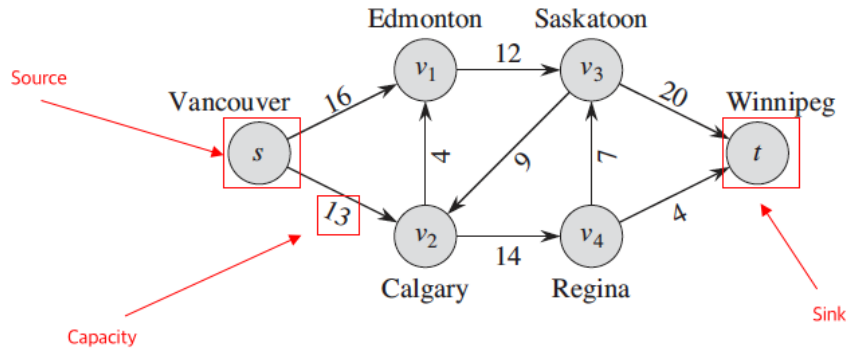
### Example



Here, the maximum flow is  $10 + 5 + 13 = 28$

### • Flow Network:

- $G = (V, E)$  is a directed graph in which each edge  $(u, v) \in E$  has a nonnegative capacity  $c(u, v) \geq 0$ .
- Two vertices must exist: **source**  $s$  and **sink**  $t$
- **path** from source  $s$  to vertex  $v$  to sink  $t$  is represented by  $s \rightsquigarrow v \rightsquigarrow t$



- **Capacity:**

- Is a non-negative function  $f : V \times V \rightarrow \mathbb{R}_{\geq 0}$
- Has **capacity constraint** where for all  $u, v \in V$   $0 \leq f(u, v) \leq c(u, v)$ 
  - \* Means flow cannot be above capacity constraint

- **Flow:**

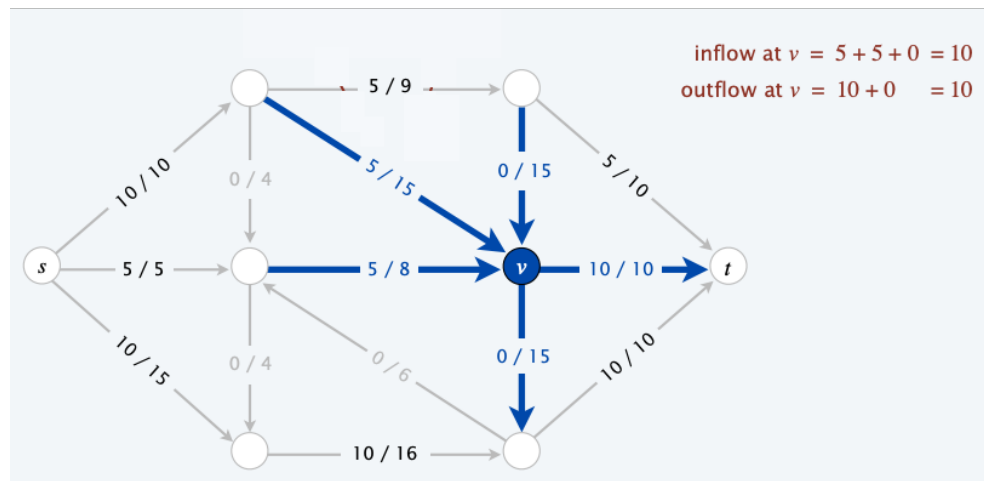
- Is a real valued function  $f : V \times V \rightarrow \mathbb{R}$  in  $G$
- Satisfies **capacity constraint** (i.e for all  $u, v \in V$ ,  $0 \leq f(u, v) \leq c(u, v)$ )
- Satisfies **flow conservation**

For all  $u \in V - \{s, t\}$ , we require

$$\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v) \quad (1)$$

Means flow into vertex  $u$  is the same as flow going out of vertex  $u$ . <sup>[1]</sup>

**Example:**



## References

1) Princeton University, Network Flow 1, link

## 2. Rough Works:

I need to show whether both of professor Adam's two children can go to the same school as maximum-flow problem.

1.