Worksheet 9 Solution

March 18, 2020

Question 1

a. Every set S of size 0 has $\frac{0(0-1)}{2} = 0$ subsets of size 2

b. Let $n = 0$, and S be an arbitrary set. Assume S has size 0.

   Then, S only has empty subsets by the fact that S has size 0.

   Since empty subset has size 0, there are 0 subsets with size 2.

c. **Section 1:**

   Every set of size k has $\frac{k(k-1)}{2}$ subsets of size 2.

   Section 2:

   Every set of sie $k+1$ has $\frac{(k+1)k}{2}$ subsets of size 2.

   Section 3.1:

   Because we know

Index	Set	# of subsets of size 2 containing last element
2	$\{s_1, s_2\}$	has 1 subset containing s_2
3	$\{s_1, s_2, s_3\}$	has 2 subsets containing s_3
4	$\{s_1, s_2, s_3, s_4\}$	has 3 subsets containing s_4

, we can deduce from above that the number of subsets of size 2 containing s_{k+1} is k.

Section 3.2:

P(n): $\forall n \in \mathbb{N}$, every set of size n has $\frac{n(n-1)}{2}$ subsets of size 2

Let $k \in \mathbb{N}$, and assume P(k).

Then, the number of subsets of S of size 2 that don't contain s_{k+1} is $\frac{k(k-1)}{2}$.

Section 3.3:

$$
\begin{align}
\frac{k(k-1)}{2} + k &= \frac{k(k-1)}{2} + \frac{2k}{2} \tag{1}\\
&= \frac{k}{2}\left[(k-1)+2\right] \tag{2}\\
&= \frac{k(k+1)}{2} \tag{3}
\end{align}
$$

Then, it follows from the proof of induction that the statement $\forall n \in \mathbb{N}$, every set of size n has $\frac{n(n-1)}{2}$ is true.

Question 2

a. P(n): Every finite set S of size n has exactly $\frac{n(n-1)(n-2)}{6}$ subsets of size 3.
Base Case ($n = 0$):

Let the size of S be 0.

Then, S only contains empty subsets.

Since an empty subset has size 0, S has 0 subsets of size 3.

Inductive Step:

Let $n \in \mathbb{N}$.

By the table below

Index	Set	# of subsets of size 3 containing last element
0	$\{\}$	0
1	$\{s_1\}$	0
2	$\{s_1, s_2\}$	0
3	$\{s_1, s_2, s_3\}$	1
4	$\{s_1, s_2, s_3, s_4\}$	3
5	$\{s_1, s_2, s_3, s_4, s_5\}$	6

,we can deduce that the number of subsets of size 3 containing s_{k+1} is $\frac{k(k-1)}{2}$.

Since the number of subsets of S of size 3 that doesn't contain s_{k+1} is $\frac{(k)(k-1)(k-2)}{6}$,

$$
\begin{aligned}
\frac{k(k-1)(k-2)}{6} + \frac{k(k-1)}{2} &= \frac{k(k-1)(k-2)}{6} + \frac{3k(k-2)}{6} && (1) \\
&= \frac{k(k-1)}{6}(k-2+3) && (2) \\
&= \frac{(k+1)k(k-1)}{6} && (3)
\end{aligned}
$$

Then, it follows from the proof of induction that the statement every finite set S of size n has exactly $\frac{n(n-1)(n-2)}{6}$ subsets of size 3 is true.

Question 3

a. **Part 1 (The subset of S that contain the element 3):**

$\{1, 3\}, \{2, 3\}, \{3\}, \{1, 2, 3\}$

Part 2 (The subset of S that do not contain the element 3):

$\{\}, \{1\}, \{2\}, \{1, 2\}$