CSC236 Worksheet 5 Solution

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Question 1

a. *Proof.* For convenience, define $H(k) : R(3^k) = 3^k k$. Notice that when $n = 3^k$, $3^k k$ is the same as $n \log_3 n$. I will use simple induction to prove $\forall k \in \mathbb{N}$, $H(k)$.

Base Case $(k = 0)$:

Let $k = 0$.

Then,

$$
\begin{align}
3^k \cdot k &= 3^0 \cdot 0 \tag{1} \\
&= 0 \tag{2} \\
&= R(n) \qquad \text{[By def.]} \tag{3}
\end{align}
$$

Thus, $H(0)$ is verified.

Inductive Step:

Let $k \in \mathbb{N}$. Assume $H(k)$. That is $R(3^k) = 3^k k$.

I will show that $H(k+1)$ follows. That is, $R(3^{k+1}) = (k+1)3^{k+1}$.

The definition of $R(3^{k+1})$ tells us,

$$
\begin{align}
R(3^{k+1}) &= 3^{k+1} + 3R(\lceil 3^{k+1}/3 \rceil) \tag{4} \\
&= 3^{k+1} + 3R(3^k) \tag{5}
\end{align}
$$

Then, using this fact, we can conclude

$$R(3^{k+1}) = 3^{k+1} + 3 \cdot 3^k k \qquad\qquad \text{[By I.H]} \qquad\qquad (6)$$
$$= 3^{k+1} + 3^{k+1} k \qquad\qquad\qquad\qquad (7)$$
$$= 3^{k+1}(k + 1) \qquad\qquad\qquad\qquad (8)$$
$$\square$$

Correct Solution:

For convenience, define $H(k) : R(3^k) = 3^k k$. Notice that when $n = 3^k$, $3^k k$ is the same as $n \log_3 n$. I will use simple induction to prove $\forall k \in \mathbb{N}$, $H(k)$.

Base Case ($k = 0$):

Let $k = 0$.

Then,

$$3^k \cdot k = 3^0 \cdot 0 \qquad\qquad\qquad\qquad (9)$$
$$= 0 \qquad\qquad\qquad\qquad (10)$$
$$= R(n) \qquad\qquad \text{[By def.]} \qquad\qquad (11)$$

Thus, $H(0)$ is verified.

Inductive Step:

Let $k \in \mathbb{N}$. Assume $H(k)$. That is $R(3^k) = 3^k k$.

I will show that $H(k + 1)$ follows. That is, $R(3^{k+1}) = (k + 1)3^{k+1}$.

Since $k + 1 > 0$, $3^{k+1} > 1$.

So the definition of $R(3^{k+1})$ tells us,

$$R(3^{k+1}) = 3^{k+1} + 3R(\lceil 3^{k+1}/3 \rceil) \qquad\qquad (12)$$
$$= 3^{k+1} + 3R(3^k) \qquad\qquad\qquad (13)$$

Then, using this fact, we can conclude

$$R(3^{k+1}) = 3^{k+1} + 3 \cdot 3^k k \qquad \text{[By I.H]} \qquad (14)$$
$$= 3^{k+1} + 3^{k+1}k \qquad (15)$$
$$= 3^{k+1}(k+1) \qquad (16)$$

<u>Notes:</u>

- Noticed that professor used the phrase 'Notice that when $n = 3^k$, $3^k k$ is the same as $n \log_3 n$.' to express n interms of 3^k.
- I feel I should review this problem to make sure I understood.

b. **<u>Rough Works:</u>**

Define $P(k)$ as

$$P(k) : \text{for every postive integer } m, \text{ if } m < k, \text{ then } R(3^m) \leq R(3^k)$$

I will use complete induction to prove $\forall k \in \mathbb{N}, P(k)$.