CSC373 Worksheet 0 Solution

July 19, 2020

1. <u>Recurrence:</u> $T(n) = T(n-1) + n$

   <u>Guess:</u> $T(n) = \mathcal{O}(n^2)$.

   I need to show $T(n) \le c \cdot n^2$.

$$
\begin{align}
T(n) &\le c(n-1)^2 + n \tag{1} \\
&= c(n^2 - 2n + 1) + n \tag{2} \\
&= cn^2 - c2n + c + n \tag{3} \\
&\le cn^2 - c2n + cn + n \tag{4} \\
&= cn^2 - cn + n \tag{5} \\
&\le cn^2 - cn + cn \tag{6} \\
&= cn^2 \tag{7}
\end{align}
$$

<u>Notes:</u>

- Substitution method
    - Solves recurrences
        * Recurrence characters the running time of divide-and-conquer algorithm
    - How it works:
        1. Make a guess for the solution
        2. Use mathematical induction to prove the guess is correct or incorrect.

    <u>Example:</u>

    <u>Recurrence:</u> $T(n) = 2T(\lfloor n/2 \rfloor) + n$

    <u>Guess:</u> $T(n) = \mathcal{O}(n \log n)$,

    We need to show $T(n) \le cn \lg n$.

1. Assume the bound holds for all positive $m < n$, in particular $m = \lfloor n/2 \rfloor$
2. Find the upper bound of $T(m)$

$$T(\lfloor n/2 \rfloor) \le c \lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)$$

3. Show $T(n) = 2T(\lfloor n/2 \rfloor) + n$ leads to $T(n) \le cn \lg n$

$$
\begin{align}
T(n) &\le 2(c\lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)) + n \tag{8} \\
&\le cn \lg(n/2) + n \tag{9} \\
&= cn \lg(n) - cn \lg 2 + n \tag{10} \\
&= cn \lg(n) - cn + n \tag{11} \\
&\le cn \lg(n) - cn + cn \tag{12} \\
&\le cn \lg(n) \tag{13}
\end{align}
$$

4. Show that the boundary holds using mathematical induction

Doesn't have information in detail. Skipping this for now.

– Making good guess

  * Three suggestions

   1. Using recursion tree
   2. Through practice
   3. prove loose upper and lower bounds on the recurrence and then reduce the range of uncertainty

2. Recurrence: $T(n) = T(\lceil n/2 \rceil) + 1$

Guess: $T(n) = \mathcal{O}(\lg n)$.

I need to show $T(n) \le c \cdot \lg n$.

$$
\begin{align}
T(n) &\le c \lg(\lceil n/2 \rceil) + 1 \tag{1} \\
&\le c \lg(n/2) + 1 \tag{2} \\
&= c(\lg n - \lg 2) + 1 \tag{3} \\
&= c(\lg n - 1) + 1 \tag{4} \\
&= c \lg n - c + 1 \tag{5} \\
&\le c \lg n - c + c \tag{6}
\end{align}
$$

Correct Solution:

Recurrence: $T(n) = T(\lceil n/2 \rceil) + 1$

Guess: $T(n) = \mathcal{O}(\lg n)$.

I need to show $T(n) \leq c \cdot \lg n$.

$$
\begin{align}
T(n) &\leq c \lg(\lceil n/2 \rceil) + 1 \tag{1}\\
&\leq c \lg(n/2) + 1 \tag{2}\\
&= c(\lg n - \lg 2) + 1 \tag{3}\\
&= c(\lg n - 1) + 1 \tag{4}\\
&= c \lg n - c + 1 \tag{5}\\
&\leq c \lg n - c + c \tag{6}
\end{align}
$$

The solution holds for $c \geq 1$.

3. Recurrence: $T(n) = 2T(\lfloor n/2 \rfloor) + n$

   Guess (Upperbound): $T(n) = \mathcal{O}(n \lg n)$.

   I first need to show $T(n) \leq c \cdot n \lg n$.

$$
\begin{align}
T(n) &= 2T(\lfloor n/2 \rfloor) + n \tag{1}\\
&= 2c\lfloor n/2 \rfloor \lg \lfloor n/2 \rfloor + n \tag{2}\\
&\leq 2c \cdot (n/2) \lg(n/2) + n \tag{3}\\
&= c \cdot n(\lg n - 1) + n \tag{4}\\
&= cn \lg n - cn + n \tag{5}\\
&\leq cn \lg n - cn + cn \tag{6}\\
&\leq cn \lg n \tag{7}
\end{align}
$$

   The above inequality holds for $c \geq 1$.

   Guess (Lowerbound): $T(n) = \Omega(n \lg n)$.

I first need to show $d \cdot (n-2) \lg(n-2) \le T(n)$.

$$T(n) = 2T(\lfloor (n-2)/2 \rfloor) + n \tag{8}$$
$$\ge 2d \lfloor (n-2)/2 \rfloor \lg \lfloor (n-2)/2 \rfloor + n \tag{9}$$
$$\ge 2d \cdot ((n-2)/2) \lg((n-2)/2) + n \tag{10}$$
$$= d \cdot (n-2)(\lg(n-2) - 1) + n \tag{11}$$
$$= d \cdot (n-2) \lg(n-2) - d \cdot (n-2) + n \tag{12}$$
$$\ge d \cdot (n-2) \lg(n-2) - d \cdot (n-2) + (n-2) \tag{13}$$
$$\ge d \cdot (n-2) \lg(n-2) - d \cdot (n-2) + d \cdot (n-2) \tag{14}$$
$$= d \cdot (n-2) \lg(n-2) \tag{15}$$

The above inequality holds for $0 \le d < 1$.

Notes:

- Both upper bound and lower bound don't need to be the same

4.3-3

We saw that the solution of $T(n) = 2T(\lfloor n/2 \rfloor) + n$ is $O(n \lg n)$. Show that the solution of this recurrence is also $\Omega(n \lg n)$. Conclude that the solution is $\Theta(n \lg n)$.

First, we guess $T(n) \le cn \lg n$, ←————————————— upper bound

$$T(n) \le 2c \lfloor n/2 \rfloor \lg \lfloor n/2 \rfloor + n$$
$$\le cn \lg(n/2) + n$$
$$= cn \lg n - cn \lg 2 + n$$
$$= cn \lg n + (1-c)n$$
$$\le cn \lg n,$$

where the last step holds for $c \ge 1$. ←——— lower bound

Next, we guess $T(n) \ge c(n+2) \lg(n+2)$, ←

$$T(n) \ge 2c(\lfloor n/2 \rfloor + 2)(\lg(\lfloor n/2 \rfloor + 2) + n$$
$$\ge 2c(n/2 - 1 + 2)(\lg(n/2 - 1 + 2) + n$$
$$= 2c\frac{n+2}{2} \lg \frac{n+2}{2} + n$$
$$= c(n+2) \lg(n+2) - c(n+2) \lg 2 + n$$
$$= c(n+2) \lg(n+2) + (1-c)n - 2c$$
$$\ge c(n+2) \lg(n+2),$$

where the last step holds for $n \ge \frac{2c}{1-c}, 0 \le c < 1$.

4. Recurrence (Merge sort):

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \Theta(n) & \text{if } n > 1 \end{cases}$$

Guess (upper bound): $T(n) \leq c \cdot (n-2) \cdot \lg(n-2)$

$$
\begin{align}
T(n) &\leq c(\lceil n/2 \rceil - 2)\lg(\lceil n/2 \rceil - 2) + c(\lfloor n/2 \rfloor - 2)\lg(\lfloor n/2 \rfloor - 2) + dn \tag{1}\\
&= c(n/2 + 1 - 2)\lg(n/2 + 1 - 2) + c(n/2 + 1 - 2)\lg(n/2 + 1 - 2) + dn \tag{2}\\
&= c((n-2)/2)\lg((n-2)/2) + c((n-2)/2)\lg((n-2)/2) + dn \tag{3}\\
&= c(n-2)\lg((n-2)/2) + dn \tag{4}\\
&= c(n-2)\lg(n-2) - c(n-2) + dn \tag{5}\\
&= c(n-2)\lg(n-2) - (d-c)n + 2c \tag{6}\\
&= c(n-2)\lg(n-2) \tag{7}
\end{align}
$$

The bound holds as long as $c > d$.

Guess (lower bound): $c \cdot (n-2) \cdot \lg(n-2) \leq T(n)$

$$
\begin{align}
T(n) &\leq c(\lceil n/2 \rceil + 1)\lg(\lceil n/2 \rceil + 1) + c(\lfloor n/2 \rfloor + 1)\lg(\lfloor n/2 \rfloor + 1) + dn \tag{8}\\
&\leq c(n/2 - 1 + 1)\lg(n/2 - 1 + 1) + c(n/2 - 1 + 1)\lg(n/2 - 1 + 1) + dn \tag{9}\\
&= c(n/2)\lg(n/2) + c(n/2)\lg(n/2) + dn \tag{10}\\
&= cn\lg(n/2) + dn \tag{11}\\
&= cn\lg(n) - cn + dn \tag{12}\\
&= cn\lg(n) + (d-c)n \tag{13}\\
&\leq c(n-1)\lg(n-1) \tag{14}
\end{align}
$$

The bound holds as long as $d > c$, and $0 \leq c < 1$

Notes:

- the n here is asymptotically large

5. Recurrence: $T(n) = 2T(\lfloor n/2 \rfloor + 17) + n$

Guess (upper bound): $cn \lg n$

$$T(n) \le 2c(\lfloor n/2 \rfloor + 17) \lg(\lfloor n/2 \rfloor + 17) + n \qquad (15)$$
$$\le 2c((n/2) + 17) \lg((n/2) + 17) + n \qquad (16)$$
$$= 2c(n/2) \lg(n/2) + n \qquad (17)$$
$$= cn(\lg(n) - 1) + n \qquad (18)$$
$$= cn \lg(n) - cn + n \qquad (19)$$
$$\le cn \lg(n) - cn + cn \qquad (20)$$
$$= cn \lg(n) \qquad (21)$$

6.

$$T(n) = 4T(n/3) + n \qquad (1)$$
$$\le 4c(n/3)^{\log_3 4} + n \qquad (2)$$
$$\le 4c(1/3)^{\log_3 4} n^{\log_3 4} + n \qquad (3)$$
$$\le (4/4)cn^{\log_3 4} + n \qquad (4)$$
$$\le cn^{\log_3 4} + n \qquad (5)$$

We cannot advance further since n in $cn^{\log_3 4} + n$ cannot be eliminated.

With the new guess $T(n) \le cn^{\log_3 4} - dn$, we have

$$T(n) = 4T(n/3) + n \qquad (6)$$
$$\le 4c(n/3)^{\log_3 4} - d(n/3) + n \qquad (7)$$
$$= 4c(n/3)^{\log_3 4} - d(n/3) + n \qquad (8)$$
$$= (4/3^{\log_3 4})cn^{\log_3 4} - d(n/3) + n \qquad (9)$$
$$= (4/4)cn^{\log_3 4} - d(n/3) + n \qquad (10)$$
$$= cn^{\log_3 4} - d(n/3) + n \qquad (11)$$
$$\le cn^{\log_3 4} - d(n/3) + n \qquad (12)$$
$$\le cn^{\log_3 4} \qquad (13)$$

The bound holds as long as $d \ge 3$ and $c \ge 1$.

Correct Solution:

Recurrence: $T(n) = 2T(\lfloor n/2 \rfloor + 17) + n$

Guess (upper bound): $cn \lg n$

$$T(n) \leq 2c(\lfloor n/2 \rfloor + 17) \lg(\lfloor n/2 \rfloor + 17) + n \tag{14}$$
$$\leq 2c((n/2) + 17) \lg((n/2) + 17) + n \tag{15}$$
$$= 2c(n/2) \lg(n/2) + n \tag{16}$$
$$= cn(\lg(n) - 1) + n \tag{17}$$
$$= cn \lg(n) - cn + n \tag{18}$$
$$\leq cn \lg(n) - cn + cn \tag{19}$$
$$= cn \lg(n) \tag{20}$$

$$T(n) = 4T(n/3) + n \tag{1}$$
$$\leq 4c(n/3)^{\log_3 4} + n \tag{2}$$
$$\leq 4c(1/3)^{\log_3 4} n^{\log_3 4} + n \tag{3}$$
$$\leq (4/4)cn^{\log_3 4} + n \tag{4}$$
$$\leq cn^{\log_3 4} + n \tag{5}$$

We cannot advance further since n in $cn^{\log_3 4} + n$ cannot be eliminated.

With the new guess $T(n) \leq cn^{\log_3 4} - dn$, we have

$$T(n) = 4T(n/3) + n \tag{6}$$
$$\leq 4c(n/3)^{\log_3 4} - 4d(n/3)4d(n/3) + n \tag{7}$$
$$= 4d(n/3) = 4c(n/3)^{\log_3 4} - 4d(n/3) + n \tag{8}$$
$$= 4d(n/3) = (4/3^{\log_3 4})cn^{\log_3 4} - 4d(n/3) + n \tag{9}$$
$$= (4/4)cn^{\log_3 4} - 4d(n/3) + n \tag{10}$$
$$= cn^{\log_3 4} - 4d(n/3) + n \tag{11}$$
$$\leq cn^{\log_3 4} - 4d(n/3) + n \tag{12}$$
$$\leq cn^{\log_3 4} - 4d(n/2) + n \tag{13}$$
$$\leq cn^{\log_3 4} - 2dn + n \tag{14}$$
$$\leq cn^{\log_3 4} - 2dn + dn \tag{15}$$
$$\leq cn^{\log_3 4} - dn \tag{16}$$

7. I need to show $T(n) \leq cn^2$

$$T(n) = 4T(n/2) + n \tag{17}$$
$$\leq 4c(n/2)^2 + n \tag{18}$$
$$= (4/4)cn^2 + n \tag{19}$$
$$= cn^2 + n \tag{20}$$

We cannot advance further since n in $cn^2 + n$ cannot be eliminated.

But with the new guess $T(n) \leq cn^2 - dn$, we have

$$T(n) = 4T(n/2) + n \tag{21}$$
$$\leq 4c(n/2)^2 - 4d(n/2) + n \tag{22}$$
$$= (4/4)cn^2 - 2dn + n \tag{23}$$
$$\leq cn^2 - 2dn + dn \tag{24}$$
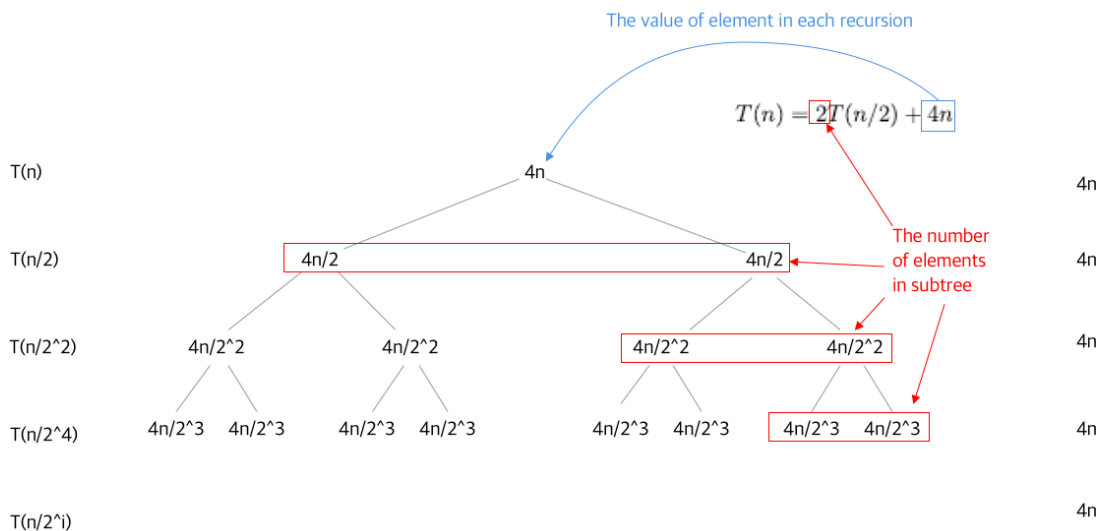$$= cn^2 - dn \tag{25}$$

The bound holds as long as $d \geq 1$ and $c \geq 1$.

8. **Notes:**

- Recursion Tree
  - Provides a straightforward way to provide a good guess.
  - Is then verified using subtitution method

  Example:

  <u>Recurrence:</u> $T(n) = 2T(n/2) + 4n, T(1) = 4$

1. Finding number of levels in recursion tree

$$1 = n/2^i \tag{1}$$
$$2^i = n \tag{2}$$
$$i = \log_2 n \tag{3}$$

2. Finding the value of guess

$$\sum_{i=0}^{\log_2 n} 4n = 4n \cdot \sum_{i=0}^{\log_2 n} 1 \tag{4}$$
$$= 4n(\log_2 n + 1) \tag{5}$$