Midterm 2 Version 3 Solution

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Question 1

a.

$$
\begin{aligned}
165 \div 2 &= 82, \text{remainders } \mathbf{1} \\
82 \div 2 &= 41, \text{remainders } \mathbf{0} \\
41 \div 2 &= 20, \text{remainders } \mathbf{1} \\
20 \div 2 &= 10, \text{remainders } \mathbf{0} \\
10 \div 2 &= 5, \text{remainders } \mathbf{0} \\
5 \div 2 &= 2, \text{remainders } \mathbf{1} \\
2 \div 2 &= 1, \text{remainders } \mathbf{0} \\
1 \div 2 &= 0, \text{remainders } \mathbf{1}
\end{aligned}
$$

From the above, we can conclude the binary representation of the decimal number 165 is $(10100101)_2$

b. The largest number that can be expressed by an n-digit balanced ternary representation is

$$
\sum_{i=0}^{n-1} 3^i = \frac{1}{2} \cdot (3^n - 1) \tag{1}
$$

Notes:

- Geometric Series

$$
\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}, \text{ where } |r| > 1
$$

c.

$f(n) \in \mathcal{O}(n)$	True	$g(n) \in \Omega(n)$	True	$f(n) \in \Omega(g(n))$	True
$f(n) \in \Theta(g(n))$	False	$g(n) \in \Theta(n)$	False	$f(n) + g(n) \in \Theta(g(n))$	False

Correct Solution:

$f(n) \in \mathcal{O}(n)$	True	$g(n) \in \Omega(n)$	True	$f(n) \in \Omega(g(n))$	True
$f(n) \in \Theta(g(n))$	False	$g(n) \in \Theta(n)$	False	$f(n) + g(n) \in \Theta(g(n))$	True

Notes:

- Note that for $f(n) + g(n) \in \Theta(g(n))$, large values of n causes $g(n) = n^{\log_2 n}$ to dominate $f(n) = \frac{3n}{\log_2 n + 8}$. This causes the inequality to be simplified to

$$c_1 \cdot n^{\log_2 n} \leq n^{\log_2 n} \leq c_2 \cdot n^{\log_2 n} \tag{1}$$

It follows from above the answer is True.

d.

k	0	1	2
$i * i$	$3 = 3^{2^0}$	$9 = 3^{2^1}$	$81 = 3^{2^4}$

From the rough work, we can deduce the value of i after k iterations is

$$3^{2^k} \tag{1}$$

e. Loop termination occurs when $i_k \geq n^3$.

We need to find the smallest value of k, and the value is

$$\lceil \log_2 3 \log_3 n \rceil \tag{1}$$

Question 2

- **Statement:** $\forall n \in \mathbb{N},\ n \geq 1 \Rightarrow \sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n} - 1$

Proof. Let $n \in \mathbb{N}$. Assume $n \geq 1$.

We will prove the statement using induction on n.

Base Case ($n = 1$)**:**

Let $n = 1$.

We want to show $\sum_{i=1}^{1} \frac{1}{\sqrt{i}} > \sqrt{1} - 1$

Starting from the left hand side of the inequality, we can calculate

$$\begin{aligned} \sum_{i=1}^{1} \frac{1}{\sqrt{i}} &= \frac{1}{\sqrt{1}} && (1) \\ &= 1 && (2) \\ &> 0 && (3) \\ &= \sqrt{1} - 1 && (4) \end{aligned}$$

Inductive Case:

Let $n \in \mathbb{N}$. Assume $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n} - 1$.

We want to show $\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} > \sqrt{n+1} - 1$.

Starting from the left hand side of the inequality, we can conclude

$$\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} = \sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}} \tag{5}$$

Then, it follows from induction hypothesis (i.e. $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n} - 1$) that

$$\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} > \sqrt{n+1} - 1 + \frac{1}{\sqrt{n+1}} \tag{6}$$

The hint tells us the following

$$\forall n \in \mathbb{Z}^{+},\ \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}}, \text{ and } \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n+1}} \tag{7}$$

Using the hint, we can calculate

$$\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} > (\sqrt{n} - 1) + \frac{1}{\sqrt{n+1} + \sqrt{n}} \tag{8}$$

$$= (\sqrt{(}n) - 1) + (\sqrt{n+1} - \sqrt{n}) \tag{9}$$

$$= \sqrt{n+1} - 1 \tag{10}$$

□

Question 3

- **Statement:** $\forall a \in \mathbb{R}^+$, $a > 1 \Rightarrow a^n + 3 \in \Theta(2^n)$

  Negation Statement: $\exists a \in \mathbb{R}^+$, $(a > 1) \wedge \Big[\forall c_1, c_2, n_0 \in \mathbb{R}^+$, $\exists n \in \mathbb{N}$, $(n \geq n_0) \wedge \big((c_1 \cdot (a^n + 3) > 2^n) \vee (2^n > c_2 \cdot (a^n + 3))\big)\Big]$

  Proof. Let $a = \frac{3}{2}$. Let $c_1, c_2, n_0 \in \mathbb{R}^+$, and $n = \left\lceil max(n_0, \frac{\log(2c_1)}{\log\frac{4}{3}}, \frac{\log(6c_1)}{2}) \right\rceil + 1$.

  We need to show $a > 1$, $n > n_0$ and $a^n + 3 < \frac{1}{c_1} \cdot 2^n$.

  We will do so in parts.

  Part 1 (showing $a > 1$):

  We need to show $a > 1$.

  Because we know $a = \frac{3}{2}$ from header, we can conclude

$$a > 1 \tag{1}$$

  Part 2 (showing $n > n_0$):

  We need to show $n \geq n_0$.

  Using the fact $\left\lceil max(n_0, \frac{\log(2c_1)}{\log\frac{4}{3}}, \frac{\log(6c_1)}{2}) \right\rceil \geq n_0$, we can conclude

$$n = \left\lceil max(n_0, \frac{\log(2c_1)}{\log\frac{4}{3}}, \frac{\log(6c_1)}{2}) \right\rceil + 1 > n_0 \tag{2}$$

Part 3 (Showing $a^n + 3 < \frac{1}{c_1} \cdot 2^n$):

We need to show $a^n + 3 < \frac{1}{c_1} \cdot 2^n$.

We will do so by showing $a^n < \frac{2^n}{2c_1}$, $3 < \frac{2^n}{2c_1}$, and then combining the two.

For the inequality $a^n < \frac{2^n}{2c_1}$, using the following fact

$$\frac{\log(2c_1)}{\log\frac{4}{3}} < \left\lceil max(n_0, \frac{\log(2c_1)}{\log\frac{4}{3}}, \frac{\log(6c_1)}{2}) \right\rceil + 1 = n \tag{3}$$

we can calculate

$$\log(2c_1) < n \log\frac{4}{3} \tag{4}$$

$$2c_1 < \left(\frac{4}{3}\right)^n \tag{5}$$

$$2c_1 < \frac{2^n}{\left(\frac{3}{2}\right)^n} \tag{6}$$

$$\left(\frac{3}{2}\right)^n < \frac{2^n}{2c_1} \tag{7}$$

$$\tag{8}$$

Then, since $a = \frac{3}{2}$, we can conclude

$$a^n < \frac{2^n}{2c_1} \tag{9}$$

For the inequality $3 < \frac{2^n}{2c_1}$, using the following fact

$$\frac{\log(6c_1)}{2} < \left\lceil max(n_0, \frac{\log(2c_1)}{\log\frac{4}{3}}, \frac{\log(6c_1)}{2}) \right\rceil + 1 = n \tag{10}$$

we can conclude

$$6c_1 < 2^n \tag{11}$$

$$3 < \frac{2^n}{2c_1} \tag{12}$$

Finally, by combining the two, we can conclude

$$a^n + 3 < \frac{2^n}{2c_1} + \frac{2^n}{2c_1} \tag{13}$$

$$a^n + 3 < \frac{2^n}{c_1} \tag{14}$$

□

Question 4