

Worksheet 8 Solution

March 17, 2020

Question 1

a. $P(n) : \forall n \in \mathbb{N}, n \leq 2^n.$

$$\forall k \in \mathbb{N}, P(0) \wedge P(k) \Rightarrow P(k+1)$$

Or, with P fully expanded,

$$\forall k \in \mathbb{N}, 0 \leq 2^0 \wedge k \leq 2^k \Rightarrow k+1 \leq 2^{k+1}$$

b. **Base Case:**

Let $n = 0$.

Then,

$$(0) \leq 2^0 \tag{1}$$

$$0 \leq 1 \tag{2}$$

Since, $n \leq 2^n$ is true for $n = 0$, the base case holds.

Inductive Case:

Let $k \in \mathbb{N}$, and assume that $P(k)$ is true.

Then,

$$2^{k+1} = 2^k + 2^k \tag{1}$$

$$\geq k + k \tag{2}$$

$$\tag{3}$$

Then,

$$2^{k+1} \geq k + k \tag{4}$$

$$\geq k + 1 \tag{5}$$

by the fact that $k \in \mathbb{N}$ and $k \geq 1$.

Then, it follows from proof by induction that the statement $k \leq 2^k$ is true.

Question 2

- **Base Case:**

Let $n = 0$.

Then,

$$\sum_{j=0}^0 T_j = \frac{(0)(0+1)(0+2)}{6} \tag{1}$$

$$= 0 \tag{2}$$

Since $T_0 = 0$, the base case holds.

Inductive Case:

Let $k \in \mathbb{N}$, and assume that $\sum_{j=0}^k T_j = \frac{k(k+1)(k+2)}{6}$ is true.

Then,

$$\sum_{j=0}^k T_j + T_{k+1} = \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+2)}{2} \quad (1)$$

$$= \frac{k(k+1)(k+2)}{6} + \frac{3(k+1)(k+2)}{6} \quad (2)$$

$$= \frac{(k+1)(k+2)(k+3)}{6} \quad (3)$$

Then, it follows from proof by induction that the statement $\forall n \in \mathbb{N}, \sum_{j=0}^k T_j = \frac{k(k+1)(k+2)}{6}$ is true.

Question 3

a. Let $x \in \mathbb{R}^+$, and let $n \in \mathbb{N}$. Assume $(1+x)^n \geq 1+nx$.

Then,

$$(1+x)^{n+1} = (1+x)^n(1+x) \quad (1)$$

$$\geq (1+nx)(1+x) \quad (2)$$

by the assumption $(1+x)^n \geq 1+nx$.

Then,

$$(1+x)^{n+1} \geq (1+nx)(1+x) \quad (3)$$

$$\geq 1+x+nx+nx^2 \quad (4)$$

$$\geq 1+x(n+1)+nx^2 \quad (5)$$

$$\geq 1+x(n+1) \quad (6)$$

Then, it follows from proof by induction that the statement $\forall x \in \mathbb{R}^+, \forall n \in \mathbb{N}, (1+x)^n \geq 1+nx$ is true.

Question 4

a. $\forall n \in \mathbb{N}, n \geq 8 \Rightarrow 30n \leq 2^n$.