CSC373 Worksheet 6 Solution

August 13, 2020

1. 1. Multiply objective function by - 1

   Maximize

   $$-2x_1 - 7x_2 - x_3$$

   Subject to

   $$\begin{aligned} x_1 - x_3 &= 7 \\ 3x_1 + x_2 &\geq 7 \\ x_2 &\geq 0 \\ x_3 &\leq 0 \end{aligned}$$

   2. Replace non-nonnegative constraints x_1

   Maximize

   $$-2x_1' + 2x_1'' - 7x_2 - x_3$$

   Subject to

   $$\begin{aligned} x_1' - x_1'' - x_3 &= 7 \\ 3x_1' - 3x_1'' + x_2 &\geq 7 \\ x_1', x_1'', x_2 &\geq 0 \\ x_3 &\leq 0 \end{aligned}$$

   3. Replace non-nonnegative constraints x_3

Maximize

$$-2x_1' + 2x_1'' - 7x_2 - x_3' + x_3''$$

Subject to

$$\begin{aligned} x_1' - x_1'' - x_3' + x_3'' &= 7 \\ 3x_1' - 3x_1'' + x_2 &\geq 7 \\ x_1', x_1'', x_2, x_3', x_3'' &\geq 0 \end{aligned}$$

4. Replace equality constraints with $\geq$ and $\leq$

Maximize

$$-2x_1' + 2x_1'' - 7x_2 - x_3' + x_3''$$

Subject to

$$\begin{aligned} x_1' - x_1'' - x_3' + x_3'' &\leq 7 \\ x_1' - x_1'' - x_3' + x_3'' &\geq 7 \\ 3x_1' - 3x_1'' + x_2 &\geq 7 \\ x_1', x_1'', x_2, x_3', x_3'' &\geq 0 \end{aligned}$$

5. Correct greater-than-or-equal-to inequality constraints

Maximize

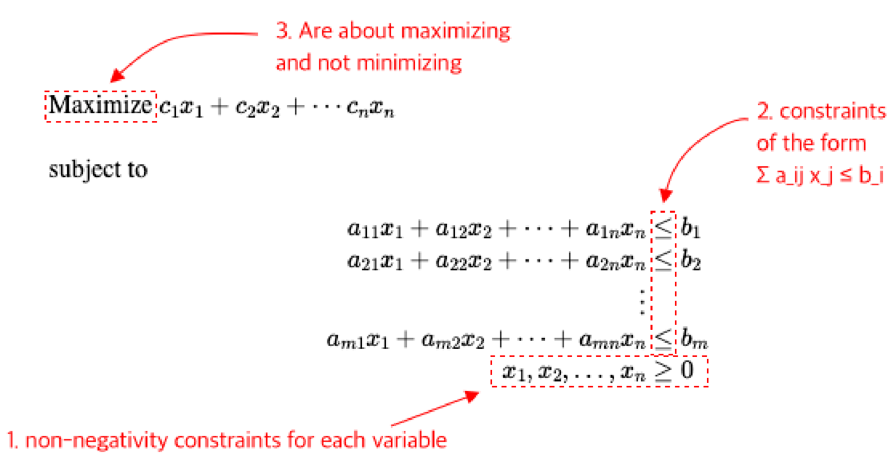
$$-2x_1' + 2x_1'' - 7x_2 - x_3' + x_3''$$

Subject to

$$
\begin{aligned}
x_1' - x_1'' - x_3' + x_3'' &\le 7 \\
-x_1' + x_1'' + x_3' - x_3'' &\le -7 \\
-3x_1' + 3x_1'' - x_2 &\le 7 \\
x_1', x_1'', x_2, x_3', x_3'' &\ge 0
\end{aligned}
$$

Notes:

- **Linear Programming**
  - Is a method to achieve the best outcome (such as maximum profit or lowest cost) in a mathematical model whose requirements are represented by linear relationships. [1]
  - Is named to make it sound cool for government funding
    - Like dynamic programming
  - Applications
    - Microeconomics (maximize profits, minimize costs)
    - Company management
- **Standard Form**
  - Is a form of linear programming
  - Are about maximizing, not minimizing [2]
  - All have a positivity constraint for each variable [2]
  - All other constraints are all of the form "linear combination of variables $\le$ constant". [2]



- **Converting Linear Programming to Standard Form**

1) The objective function might be a minimization rather than a maximization
   - Negate coefficients of the objective function

multiply by -1

$$\begin{array}{llllll} \boxed{\text{minimize} & -2x_1 & + & 3x_2} \\ \text{subject to} \\ & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \le & 4 \\ & x_1 & & & \ge & 0 \end{array} \qquad \begin{array}{llllll} \boxed{\text{maximize} & 2x_1 & - & 3x_2} \\ \text{subject to} \\ & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \le & 4 \\ & x_1 & & & \ge & 0 \end{array}$$

2) There might be variables without nonnegativity constraints
   - Replace each non-nonnegative variable x_i with x_i' and x_i''
   - Modify linear program

Replace x_i with x_i' and x_i''

$$\begin{array}{llllll} \text{maximize} & 2x_1 & - & 3x_2 \\ \text{subject to} \\ & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \le & 4 \\ & x_1 & & & \ge & 0 \end{array} \qquad \begin{array}{llllllll} \text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' \\ \text{subject to} \\ & x_1 & + & x_2' & - & x_2'' & = & 7 \\ & x_1 & - & 2x_2' & + & 2x_2'' & \le & 4 \\ & x_1, x_2', x_2'' & & & & & \ge & 0 \end{array}$$

x_2 is not nonnegative :(They are now nonnegative :)! Yayy!!

3) There might be **equality constraints**, which have an equal sign rather than a less-than-or-equal-to sign
   - Replace equality constraint $f(x_1, x_2, ..., x_n) = b$ with $f(x_1, x_2, ..., x_n) \le b$ and $f(x_1, x_2, ..., x_n) \ge b$

Multiply incorrect constraints by -1

$$\begin{array}{llllllll} \text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' \\ \text{subject to} \\ & x_1 & + & x_2' & - & x_2'' & \le & 7 \\ & x_1 & + & x_2' & - & x_2'' & \ge & 7 \\ & x_1 & - & 2x_2' & + & 2x_2'' & \le & 4 \\ & x_1, x_2', x_2'' & & & & & \ge & 0\ . \end{array} \qquad \begin{array}{llllllll} \text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 \\ \text{subject to} \\ & x_1 & + & x_2 & - & x_3 & \le & 7 \\ & -x_1 & - & x_2 & + & x_3 & \le & -7 \\ & x_1 & - & 2x_2 & + & 2x_3 & \le & 4 \\ & x_1, x_2, x_3 & & & & & \ge & 0 \end{array}$$

4) There might be **inequality constraints**, but instead of having a less-than-or-equal-to-sign
   - Multiply incorrect inequality constraints by -1

Replace = with ≤ and ≥

$$
\begin{array}{llrrrrrrrr}
\text{maximize} & & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & & \\
& & x_1 & + & x_2' & - & x_2'' & \boxed{=} & 7 \\
& & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& & & & x_1, x_2', x_2'' & & & \geq & 0
\end{array}
\qquad
\begin{array}{llrrrrrrrr}
\text{maximize} & & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & & \\
& & x_1 & + & x_2' & - & x_2'' & \boxed{\leq} & 7 \\
& & x_1 & + & x_2' & - & x_2'' & \boxed{\geq} & 7 \\
& & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& & & & x_1, x_2', x_2'' & & & \geq & 0 .
\end{array}
$$

References:

1) Wikipedia, Linear Programming, link
2) Instituto de Mathematicas, Standard form for Linear Programs, link

2. **Notes:**

- **Slack Form**
  - Is a form of linear programming
  - Is for efficient solving of liner programming problem using simplex algorithm

The value of objective function

Slack variables

$$
\begin{array}{rcrrrrrrr}
\boxed{z} & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
\boxed{x_4} & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
\boxed{x_5} & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
\boxed{x_6} & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 .
\end{array}
$$

- **Converting Linear Programs into Slack Form**
  1) Start from the standard form of linear programming
  2) Shift objective functions to right

$$
\begin{array}{ll}
\text{maximize} & \boxed{2x_1 - 3x_2 + 3x_3} \\
\text{subject to} & \\
& \begin{array}{rcrcrcl}
x_1 & + & x_2 & - & x_3 & \leq & 7 \\
-x_1 & - & x_2 & + & x_3 & \leq & -7 \\
x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
 & & x_1, x_2, x_3 & & & \geq & 0
\end{array}
\end{array}
$$

Introduce slack variables

$$
\begin{array}{ll}
\text{maximize} & \boxed{2x_1 - 3x_2 + 3x_3} \\
\text{subject to} & \\
& \begin{array}{rcrcrcrcr}
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
\multicolumn{5}{c}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & &
\end{array}
\end{array}
$$

3) Introduce slack variable x_i to lhs and move expressions $\sum_{j=1}^{n} a_{ij}x_j$ to rhs

4) Change inequalities in linear programming to equality

$$
\begin{array}{ll}
\text{maximize} & 2x_1 - 3x_2 + 3x_3 \\
\text{subject to} & \\
& \begin{array}{rcrcrcl}
x_1 & + & x_2 & - & x_3 & \boxed{\leq} & 7 \\
-x_1 & - & x_2 & + & x_3 & \boxed{\leq} & -7 \\
x_1 & - & 2x_2 & + & 2x_3 & \boxed{\leq} & 4 \\
 & & x_1, x_2, x_3 & & & \boxed{\geq} & 0
\end{array}
\end{array}
$$

Introduce slack variables

$$
\begin{array}{ll}
\text{maximize} & 2x_1 - 3x_2 + 3x_3 \\
\text{subject to} & \\
& \begin{array}{rcrcrcrcr}
x_4 & \boxed{=} & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & \boxed{=} & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & \boxed{=} & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
\multicolumn{5}{c}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & &
\end{array}
\end{array}
$$

5) Use Variable z to denote objective function

6) Omit the nonnegativivty constraints

$$
\begin{array}{ll}
\text{maximize} & 2x_1 - 3x_2 + 3x_3 \\
\text{subject to} & \\
& \begin{array}{rcrcrcrcr}
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
\multicolumn{5}{c}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & &
\end{array}
\end{array}
$$

Use variable z to denote objective function and omit the nonnegativity constraints.

$$
\begin{array}{rcrcrcrcr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}
$$

References:

1) Cambridge University, Linear Programming, link