

# Worksheet 8 Review

March 27, 2020

## Question 1

- a.  $\forall n \in \mathbb{N}, (0 \leq 1) \wedge (n \leq 2^n) \Rightarrow (n+1) \leq 2^{n+1}$

**Note:**

- **Induction:**  $\forall n \in \mathbb{N}, P(0) \wedge P(n) \Rightarrow P(n+1)$

- b. We will prove this statement by induction on  $n$ .

**Base Case:**

Let  $n = 0$ .

Then,

$$0 \leq 2^0 \tag{1}$$

$$0 \leq 1 \tag{2}$$

Since the above inequality is true, the base case holds.

**Inductive Case:**

Let  $n \in \mathbb{N}$ . Assume  $P(n)$ .

Then,

$$n \leq 2^n \tag{3}$$

$$n + 1 \leq 2^n + 1 \tag{4}$$

$$n + 1 \leq 2^n + 2^n \tag{5}$$

$$n + 1 \leq 2^{n+1} \tag{6}$$

by the fact  $2^k + 2^k = 2^{k+1}$ .

Then, it follows from proof by induction that the statement  $n \leq 2^n$  is true for all  $n$ .

**Correct Solution:**

We will prove this statement by induction on  $n$ .

**Base Case:**

Let  $n = 0$ .

Then,

$$0 \leq 2^0 \tag{7}$$

$$0 \leq 1 \tag{8}$$

Since the above inequality is true, the base case holds.

**Inductive Case:**

Let  $n \in \mathbb{N}$ . Assume  $P(n)$ .

**We want to show**  $n + 1 \leq 2^{n+1}$ .

Then,

$$n \leq 2^n \tag{9}$$

$$n + 1 \leq 2^n + 1 \tag{10}$$

$$n + 1 \leq 2^n + 2^n \tag{11}$$

$$n + 1 \leq 2^{n+1} \tag{12}$$

by the fact  $2^n + 2^n = 2^{n+1}$ .

Then, it follows from proof by induction that the statement  $n \leq 2^n$  is true for all  $n$ .

#### Notes:

- professor specifically states what we want to show in inductive case part of the proof. I thought it was obvious, and not necessary.
- When are the times 'we want to show x' in proof can be omitted?

## Question 2

- We will prove the statement by induction on natural number  $n$ .

#### Base Case:

Let  $n = 1$ .

Then,

$$\sum_{j=1}^1 T_j = 1 \cdot \frac{(1+1)(1+2)}{6} \tag{1}$$

$$= 1 \tag{2}$$

Since the data also shows value 1 at  $n = 1$ , the base case holds.

**Inductive Case:**

Let  $n \in \mathbb{N}$ . Assume  $\sum_{j=0}^n T_j = \frac{n \cdot (n+1)(n+2)}{6}$ .

We want to show  $\sum_{j=0}^{n+1} T_j = \frac{(n+1)(n+2)(n+3)}{6}$ .

It follows from the following table

n	1	2	3	4	5
$T_i = \frac{n \cdot (n+1)}{2}$	1	3	6	10	15
$\sum_{j=1}^n T_j$	1	4	10	20	35

that  $n + 1^{th}$  value of the summation is  $\frac{(n+1)(n+2)}{2}$  more than the  $n^{th}$  sum.

Then,

$$\sum_{j=0}^{n+1} T_j = \frac{n \cdot (n+1)(n+2)}{6} + \frac{(n+1)(n+2)}{2} \quad (3)$$

$$= \frac{n \cdot (n+1)(n+2)}{6} + \frac{3(n+1)(n+2)}{6} \quad (4)$$

$$= \frac{(n+1)(n+2)(n+3)}{6} \quad (5)$$

**Correct Solution:**

We will prove the statement by induction on natural number  $n$ .

**Base Case:**

**Let  $n = 0$ .**

Then,

$$\sum_{j=0}^1 T_j = \frac{0 \cdot (0+1)(0+2)}{6} \quad (1)$$

$$= 0 \quad (2)$$

Since

$$\sum_{j=0}^0 T_j = T_0 \quad (3)$$

and,

$$T_0 = \frac{0 \cdot (0+1)}{2} \quad (4)$$

$$= 0 \quad (5)$$

, the base case holds.

**Inductive Case:**

Let  $n \in \mathbb{N}$ . Assume  $\sum_{j=0}^n T_j = \frac{n \cdot (n+1)(n+2)}{6}$ .

We want to show  $\sum_{j=0}^{n+1} T_j = \frac{(n+1)(n+2)(n+3)}{6}$ .

It follows from the following table

n	1	2	3	4	5
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that  $n + 1^{th}$  value of the summation is  $\frac{(n+1)(n+2)}{2}$  more than the  $n^{th}$  sum.

Then,

$$\sum_{j=0}^{n+1} T_j = \frac{n \cdot (n+1)(n+2)}{6} + \frac{(n+1)(n+2)}{2} \quad (6)$$

$$= \frac{n \cdot (n+1)(n+2)}{6} + \frac{3(n+1)(n+2)}{6} \quad (7)$$

$$= \frac{(n+1)(n+2)(n+3)}{6} \quad (8)$$

**Notes:**

- I wasn't explicit about where the value 1 in data came from.

## Question 3

a. Let  $x \in \mathbb{R}$ .

**Correct Solution:**

Let  $x \in \mathbb{R}$ .

**We will prove the statement  $\forall n \in \mathbb{N}, (1+x)^n \geq 1+nx$  using induction on  $n$ .**

**Notes:**

- Professor separately introduced 'Let  $x \in \mathbb{R}$ ' from the rest of the statement.

- By using 'the standard proof structure to introduce  $x$ ', does it include the line up to 'we will prove the statement  $x$  by induction'?
- **Proof by Induction:**  $\forall k \in \mathbb{N}, P(k) \Rightarrow P(k + 1)$

b. **Base Case:**

Let  $n = 0$ .

Then,

$$1 = (1 + x)^0 \geq 1 + (0)x \quad (1)$$

$$\geq 1 \quad (2)$$

Because we know the inequality is true, we can conclude that the base case holds.

**Inductive Case:**

Let  $n \in \mathbb{N}$ . Assume  $(1 + x)^n \geq 1 + nx$ .

We want to show  $(1 + x)^{n+1} \geq 1 + (n + 1)x$ .

Because we know  $(1 + x)^{n+1} = (1 + x)^n(1 + x)$  and  $(1 + x)^n \geq 1 + nx$ , we can write

$$(1 + x)^{n+1} = (1 + x)^n(1 + x) \quad (3)$$

$$\geq (1 + nx)(1 + x) \quad (4)$$

$$\geq 1 + x + nx + nx^2 \quad (5)$$

Then,

$$1 + x + nx + nx^2 \geq 1 + x + nx \quad (6)$$

by the fact that  $nx^2 \geq 0$ .

Then,

$$1 + x + nx \geq 1 + x(n + 1) \tag{7}$$

Since  $(1 + x)^{n+1} \geq 1 + x(n + 1)$  is true, it follows from proof by induction that the statement  $(1 + x)^n \geq 1 + xn$  is true for all  $n$ .

**Notes:**

- $\mathbb{N} = \{0, 1, 2, \dots\}$
- $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$