

CSC373 Worksheet 6 Solution

August 13, 2020

1. Multiply objective function by - 1

Maximize

$$-2x_1 - 7x_2 - x_3$$

Subject to

$$x_1 - x_3 = 7$$

$$3x_1 + x_2 \geq 7$$

$$x_2 \geq 0$$

$$x_3 \leq 0$$

2. Replace non-nonnegative constraints x_1

Maximize

$$-2x'_1 + 2x''_1 - 7x_2 - x_3$$

Subject to

$$x'_1 - x''_1 - x_3 = 7$$

$$3x'_1 - 3x''_1 + x_2 \geq 7$$

$$x'_1, x''_1, x_2 \geq 0$$

$$x_3 \leq 0$$

3. Replace non-nonnegative constraints x_3

Maximize

$$-2x'_1 + 2x''_1 - 7x_2 - x'_3 + x''_3$$

Subject to

$$x'_1 - x''_1 - x'_3 + x''_3 = 7$$

$$3x'_1 - 3x''_1 + x_2 \geq 7$$

$$x'_1, x''_1, x_2, x'_3, x''_3 \geq 0$$

4. Replace equality constraints with \geq and \leq

Maximize

$$-2x'_1 + 2x''_1 - 7x_2 - x'_3 + x''_3$$

Subject to

$$x'_1 - x''_1 - x'_3 + x''_3 \leq 7$$

$$x'_1 - x''_1 - x'_3 + x''_3 \geq 7$$

$$3x'_1 - 3x''_1 + x_2 \geq 7$$

$$x'_1, x''_1, x_2, x'_3, x''_3 \geq 0$$

5. Correct greater-than-or-equal-to inequality constraints

Maximize

$$-2x'_1 + 2x''_1 - 7x_2 - x'_3 + x''_3$$

Subject to

$$\begin{aligned}
 x'_1 - x''_1 - x'_3 + x''_3 &\leq 7 \\
 -x'_1 + x''_1 + x'_3 - x''_3 &\leq -7 \\
 -3x'_1 + 3x''_1 - x_2 &\leq 7 \\
 x'_1, x''_1, x_2, x'_3, x''_3 &\geq 0
 \end{aligned}$$

Notes:

• Linear Programming

- Is a method to achieve the best outcome (such as maximum profit or lowest cost) in a mathematical model whose requirements are represented by linear relationships. ^[1]
- Is named to make it sound cool for government funding
 - * Like dynamic programming
- Applications
 - * Microeconomics (maximize profits, minimize costs)
 - * Company management

• Standard Form

- Is a form of linear programming
- Are about maximizing, not minimizing ^[2]
- All have a positivity constraint for each variable ^[2]
- All other constraints are all of the form “linear combination of variables \leq constant”. ^[2]



• Converting Linear Programming to Standard Form

- 1) The objective function might be a minimization rather than a maximization
- Negate coefficients of the objective function

multiply by -1

$\begin{array}{ll} \text{minimize} & -2x_1 + 3x_2 \\ \text{subject to} & \end{array}$ $\begin{array}{rcl} x_1 + x_2 & = & 7 \\ x_1 - 2x_2 & \leq & 4 \\ x_1 & \geq & 0 \end{array}$	$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2 \\ \text{subject to} & \end{array}$ $\begin{array}{rcl} x_1 + x_2 & = & 7 \\ x_1 - 2x_2 & \leq & 4 \\ x_1 & \geq & 0 \end{array}$
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- 2) There might be variables without nonnegativity constraints
- Replace each non-nonnegative variable x_i with x'_i and x''_i
 - Modify linear program

Replace x_i with x'_i and x''_i

$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2 \\ \text{subject to} & \end{array}$ $\begin{array}{rcl} x_1 + x_2 & = & 7 \\ x_1 - 2x_2 & \leq & 4 \\ x_1 & \geq & 0 \end{array}$ <p style="text-align: center; color: orange;">x_2 is not nonnegative :(</p>	$\begin{array}{ll} \text{maximize} & 2x_1 - 3x'_2 + 3x''_2 \\ \text{subject to} & \end{array}$ $\begin{array}{rcl} x_1 + x'_2 - x''_2 & = & 7 \\ x_1 - 2x'_2 + 2x''_2 & \leq & 4 \\ x_1, x'_2, x''_2 & \geq & 0 \end{array}$ <p style="text-align: center; color: orange;">They are now nonnegative :) Yayy!!</p>
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- 3) There might be **equality constraints**, which have an equal sign rather than a less-than-or-equal-to sign
- Replace equality constraint $f(x_1, x_2, \dots, x_n) = b$ with $f(x_1, x_2, \dots, x_n) \leq b$ and $f(x_1, x_2, \dots, x_n) \geq b$

Multiply incorrect constraints by -1

$\begin{array}{ll} \text{maximize} & 2x_1 - 3x'_2 + 3x''_2 \\ \text{subject to} & \end{array}$ $\begin{array}{rcl} x_1 + x'_2 - x''_2 & \leq & 7 \\ x_1 + x'_2 - x''_2 & \geq & 7 \\ x_1 - 2x'_2 + 2x''_2 & \leq & 4 \\ x_1, x'_2, x''_2 & \geq & 0 \end{array}$	$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2 + 3x_3 \\ \text{subject to} & \end{array}$ $\begin{array}{rcl} x_1 + x_2 - x_3 & \leq & 7 \\ -x_1 - x_2 + x_3 & \leq & -7 \\ x_1 - 2x_2 + 2x_3 & \leq & 4 \\ x_1, x_2, x_3 & \geq & 0 \end{array}$
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- 4) There might be **inequality constraints**, but instead of having a less-than-or-equal-to-sign
- Multiply incorrect inequality constraints by -1

Replace = with \leq and \geq

<p>maximize $2x_1 - 3x'_2 + 3x''_2$</p> <p>subject to</p> $\begin{array}{rcl} x_1 + x'_2 - x''_2 & = & 7 \\ x_1 - 2x'_2 + 2x''_2 & \leq & 4 \\ x_1, x'_2, x''_2 & \geq & 0 \end{array}$	<p>maximize $2x_1 - 3x'_2 + 3x''_2$</p> <p>subject to</p> $\begin{array}{rcl} x_1 + x'_2 - x''_2 & \leq & 7 \\ x_1 + x'_2 - x''_2 & \geq & 7 \\ x_1 - 2x'_2 + 2x''_2 & \leq & 4 \\ x_1, x'_2, x''_2 & \geq & 0 \end{array}$
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References:

- 1) Wikipedia, Linear Programming, [link](#)
 - 2) Instituto de Mathematicas, Standard form for Linear Programs, [link](#)
- 2.

$$\begin{aligned} z &= 2x_1 - 6x_3 \\ x_4 &= 7 - x_1 - x_2 + x_3 \\ x_5 &= -8 + 3x_1 - x_2 \\ x_6 &= -x_1 + 2x_2 + 2x_3 \end{aligned}$$

The basic variables are variables on the lhs, and the non-basic variables are the variables on the rhs of the expressions.

The basic variables	$\begin{aligned} z &= 2x_1 - 6x_3 \\ x_4 &= 7 - x_1 - x_2 + x_3 \\ x_5 &= -8 + 3x_1 - x_2 \\ x_6 &= -x_1 + 2x_2 + 2x_3 \end{aligned}$	The non-basic variables
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Notes:

- Slack Form

- Is a form of linear programming
- Is for efficient solving of linear programming problem using simplex algorithm

Slack variables

The value of objective function

$$\begin{array}{rcl}
 z & = & 2x_1 - 3x_2 + 3x_3 \\
 x_4 & = & 7 - x_1 - x_2 + x_3 \\
 x_5 & = & -7 + x_1 + x_2 - x_3 \\
 x_6 & = & 4 - x_1 + 2x_2 - 2x_3
 \end{array}$$

• Converting Linear Programs into Slack Form

- 1) Start from the standard form of linear programming
- 2) Shift objective functions to right

maximize
subject to

$$\begin{array}{rcl}
 2x_1 & - & 3x_2 + 3x_3 \\
 x_1 + x_2 - x_3 & \leq & 7 \\
 -x_1 - x_2 + x_3 & \leq & -7 \\
 x_1 - 2x_2 + 2x_3 & \leq & 4 \\
 x_1, x_2, x_3 & \geq & 0
 \end{array}$$

Introduce slack variables

maximize
subject to

$$\begin{array}{rcl}
 2x_1 & - & 3x_2 + 3x_3 \\
 x_4 & = & 7 - x_1 - x_2 + x_3 \\
 x_5 & = & -7 + x_1 + x_2 - x_3 \\
 x_6 & = & 4 - x_1 + 2x_2 - 2x_3 \\
 x_1, x_2, x_3, x_4, x_5, x_6 & \geq & 0
 \end{array}$$

Shift objective functions to right

- 3) Introduce slack variable x_i to lhs and move expressions $\sum_{j=1}^n a_{ij}x_j$ to rhs

maximize
subject to

$$\begin{array}{rcl}
 2x_1 & - & 3x_2 + 3x_3 \\
 x_1 + x_2 - x_3 & \leq & 7 \\
 -x_1 - x_2 + x_3 & \leq & -7 \\
 x_1 - 2x_2 + 2x_3 & \leq & 4 \\
 x_1, x_2, x_3 & \geq & 0
 \end{array}$$

Move expressions to rhs

Introduce slack variables

maximize
subject to

$$\begin{array}{rcl}
 2x_1 & - & 3x_2 + 3x_3 \\
 x_4 & = & 7 - x_1 - x_2 + x_3 \\
 x_5 & = & -7 + x_1 + x_2 - x_3 \\
 x_6 & = & 4 - x_1 + 2x_2 - 2x_3 \\
 x_1, x_2, x_3, x_4, x_5, x_6 & \geq & 0
 \end{array}$$

4) Change inequalities in linear programming to equality

maximize $2x_1 - 3x_2 + 3x_3$
 subject to

$$\begin{array}{rclclcl} x_1 & + & x_2 & - & x_3 & & 7 \\ -x_1 & - & x_2 & + & x_3 & & -7 \\ x_1 & - & 2x_2 & + & 2x_3 & & 4 \\ x_1, x_2, x_3 & & & & & & \geq 0 \end{array}$$

Change inequality sign to equality

Introduce slack variables

maximize $2x_1 - 3x_2 + 3x_3$
 subject to

$$\begin{array}{rclclclcl} x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\ x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\ x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\ x_1, x_2, x_3, x_4, x_5, x_6 & & & & & & \geq & 0 \end{array}$$

5) Use Variable z to denote objective function

maximize $2x_1 - 3x_2 + 3x_3$
 subject to

$$\begin{array}{rclclclcl} x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\ x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\ x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\ x_1, x_2, x_3, x_4, x_5, x_6 & & & & & & \geq & 0 \end{array}$$

Use variable z to denote objective function

Use variable z to denote objective function and omit the nonnegativity constraints.

$$\begin{array}{rclclclcl} z & = & 2x_1 & - & 3x_2 & + & 3x_3 \\ x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\ x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\ x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \end{array}$$

6) Omit the nonnegativity constraints

maximize $2x_1 - 3x_2 + 3x_3$
 subject to

$$\begin{array}{rclclclcl} x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\ x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\ x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\ x_1, x_2, x_3, x_4, x_5, x_6 & & & & & & \geq & 0 \end{array}$$

remove nonnegative constraints

Use variable z to denote objective function and omit the nonnegativity constraints.

$$\begin{array}{rclclclcl} z & = & 2x_1 & - & 3x_2 & + & 3x_3 \\ x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\ x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\ x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \end{array}$$

References:

- 1) Cambridge University, Linear Programming, [link](#)