

CSC236 Worksheet 6 Review

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Question 1

- Rough Work:

Assume that $\forall k \in \mathbb{N}, R(3^k) = k3^k$.

I need to prove $R \in \Theta(n \lg n)$. That is, $R \in \mathcal{O}(n \lg n)$ and $R \in \Omega(n \lg n)$.

I will do so in parts.

1. Part 1 (Proving $R \in \mathcal{O}(n \lg n)$)

Let $n \in \mathbb{N}$. Define $n^* = 3^{\lceil \log_3 n \rceil}$. Then, we have

$$\lceil \log_3 n \rceil - 1 < \log_3 n \leq \lceil \log_3 n \rceil \quad (1)$$

I will also use the fact proved in week 7 tutorial exercise that R is non-decreasing.

Let $d = 6$. Then, $d \in \mathbb{R}^+$. Let $B = 3$. Then, $B \in \mathbb{R}^+$. Assume $n \geq B$.

I need to show $R(n) \leq dn \lg n$.

Starting from $R(n)$, we have

$$R(n) \leq R(n^*) \quad [\text{Since } n \leq n^* \text{ and } R \text{ is non-decreasing}] \quad (2)$$

$$= n^* \log_3 n^* \quad [\text{By replacing } 3^k \text{ for } n^*] \quad (3)$$

$$\leq 3n \log_3 3n \quad [\text{Since } n \leq n^* \Rightarrow 3n \leq 3n^*] \quad (4)$$

$$= 3n(\log_3 n + 1) \quad (5)$$

$$= 3n(\log_3 n + \log_3 n) \quad [\text{Since } n \leq B = 3 \Rightarrow \log_3 n \leq 1] \quad (6)$$

$$\leq 6n \log_3 n \quad (7)$$

$$\leq dn \log_3 n \quad [\text{Since } d = 6] \quad (8)$$

$$\leq dn \lg n \quad (9)$$

2. Part 2 (Proving $R \in \Omega(n \lg n)$)

Let $n \in \mathbb{N}$. Define $n^* = 3^{\lceil \log_3 n \rceil}$. Then, we have

$$\lceil \log_3 n \rceil - 1 < \log_3 n \leq \lceil \log_3 n \rceil \quad (10)$$

I will also use the fact proved in week 7 tutorial exercise that R is non-decreasing.

Let $d = 1/(6 \lg 3)$. Then, $d \in \mathbb{R}^+$. Let $B = 9$. Then, $B \in \mathbb{R}^+$. Assume $n \geq B$.

I need to show $R(n) \geq dn \lg n$.

Starting from $R(n)$, we have

$$R(n) \geq R(n/3) \quad [\text{Since } n^*/3 < n \text{ and } R \text{ is non-decreasing}] \quad (11)$$

$$= (n^*/3) \log_3(n^*/3) \quad [\text{By replacing } 3^k \text{ for } n^*] \quad (12)$$

$$= (n/3) \log_3(n/3) \quad [\text{Since } n < n^* \Rightarrow (n/3) \leq (n^*/3)] \quad (13)$$

$$= (n/3)(\log_3 n - 1) \quad (14)$$

$$\geq (\log_3 n - (\log_3 n)/2) \quad [\text{Since } n \geq B = 9 \Rightarrow (\log_3 n)/2 \geq 1] \quad (15)$$

$$= (n \log_3 n)/6 \quad (16)$$

$$= (n \lg n)/(6 \lg 3) \quad (17)$$

$$= dn \lg n \quad (18)$$