

CSC373 Worksheet 0 Solution

July 19, 2020

1. Recurrence: $T(n) = T(n-1) + n$

Guess: $T(n) = \mathcal{O}(n^2)$.

I need to show $T(n) \leq c \cdot n^2$.

$$T(n) \leq c(n-1)^2 + n \tag{1}$$

$$= c(n^2 - 2n + 1) + n \tag{2}$$

$$= cn^2 - c2n + c + n \tag{3}$$

$$\leq cn^2 - c2n + cn + n \tag{4}$$

$$= cn^2 - cn + n \tag{5}$$

$$\leq cn^2 - cn + cn \tag{6}$$

$$= cn^2 \tag{7}$$

Notes:

- Substitution method
 - Solves recurrences
 - * Recurrence characterizes the running time of divide-and-conquer algorithm
 - How it works:
 1. Make a guess for the solution
 2. Use mathematical induction to prove the guess is correct or incorrect.

Example:

Recurrence: $T(n) = 2T(\lfloor n/2 \rfloor) + n$

Guess: $T(n) = \mathcal{O}(n \log n)$,

We need to show $T(n) \leq cn \lg n$.

1. Assume the bound holds for all positive $m < n$, in particular $m = \lfloor n/2 \rfloor$
2. Find the upper bound of $T(m)$

$$T(\lfloor n/2 \rfloor) \leq c \lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)$$

3. Show $T(n) = 2T(\lfloor n/2 \rfloor) + n$ leads to $T(n) \leq cn \lg n$

$$T(n) \leq 2(c \lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)) + n \quad (8)$$

$$\leq cn \lg(n/2) + n \quad (9)$$

$$= cn \lg(n) - cn \lg 2 + n \quad (10)$$

$$= cn \lg(n) - cn + n \quad (11)$$

$$\leq cn \lg(n) - cn + cn \quad (12)$$

$$\leq cn \lg(n) \quad (13)$$

4. Show that the boundary holds using mathematical induction

Doesn't have information in detail. Skipping this for now.

– Making good guess

* Three suggestions

1. Using recursion tree
2. Through practice
3. prove loose upper and lower bounds on the recurrence and then reduce the range of uncertainty

2. Recurrence: $T(n) = T(\lceil n/2 \rceil) + 1$

Guess: $T(n) = \mathcal{O}(\lg n)$.

I need to show $T(n) \leq c \cdot \lg n$.

$$T(n) \leq c \lg(\lceil n/2 \rceil) + 1 \quad (1)$$

$$\leq c \lg(n/2) + 1 \quad (2)$$

$$= c(\lg n - \lg 2) + 1 \quad (3)$$

$$= c(\lg n - 1) + 1 \quad (4)$$

$$= c \lg n - c + 1 \quad (5)$$

$$\leq c \lg n - c + c \quad (6)$$

Correct Solution:

Recurrence: $T(n) = T(\lceil n/2 \rceil) + 1$

Guess: $T(n) = \mathcal{O}(\lg n)$.

I need to show $T(n) \leq c \cdot \lg n$.

$$T(n) \leq c \lg(\lceil n/2 \rceil) + 1 \quad (1)$$

$$\leq c \lg(n/2) + 1 \quad (2)$$

$$= c(\lg n - \lg 2) + 1 \quad (3)$$

$$= c(\lg n - 1) + 1 \quad (4)$$

$$= c \lg n - c + 1 \quad (5)$$

$$\leq c \lg n - c + c \quad (6)$$

The solution holds for $c \geq 1$.

3. Recurrence: $T(n) = T(\lceil n/2 \rceil) + 1$

Guess (Upperbound): $T(n) = \mathcal{O}(\lg n)$.

I first need to show $T(n) \leq c \cdot n \lg n$.

$$T(n) = 2T(\lfloor n/2 \rfloor) + n \quad (1)$$

$$= 2c \lfloor n/2 \rfloor \lg \lfloor n/2 \rfloor + n \quad (2)$$

$$\leq 2c \cdot (n/2) \lg(n/2) + n \quad (3)$$

$$= c \cdot n(\lg n - 1) + n \quad (4)$$

$$= cn \lg n - cn + n \quad (5)$$

$$\leq cn \lg n - cn + cn \quad (6)$$

$$\leq cn \lg n \quad (7)$$

The solution holds for $c \geq 1$.