CSC373 Worksheet 6 Solution

August 14, 2020

1. 1. Multiply objective function by - 1

Maximize

$$-2x_1 - 7x_2 - x_3$$

Subject to

$$\begin{aligned} x_1 - x_3 &= 7 \\ 3x_1 + x_2 &\geq 7 \\ x_2 &\geq 0 \\ x_3 &\leq 0 \end{aligned}$$

2. Replace non-nonnegative constraints x_1

Maximize

$$-2x_1' + 2x_1'' - 7x_2 - x_3$$

Subject to

$$\begin{aligned} x_1' - x_1'' - x_3 &= 7 \\ 3x_1' - 3x_1'' + x_2 &\geq 7 \\ x_1', x_1'', x_2 &\geq 0 \\ x_3 &\leq 0 \end{aligned}$$

3. Replace non-nonnegative constraints x_3

Maximize

$$-2x_1' + 2x_1'' - 7x_2 - x_3' + x_3''$$

Subject to

$$\begin{aligned} x_1' - x_1'' - x_3' + x_3'' &= 7 \\ 3x_1' - 3x_1'' + x_2 &\geq 7 \\ x_1', x_1'', x_2, x_3', x_3'' &\geq 0 \end{aligned}$$

4. Replace equality constraints with $\geq$ and $\leq$

Maximize

$$-2x_1' + 2x_1'' - 7x_2 - x_3' + x_3''$$

Subject to

$$\begin{aligned} x_1' - x_1'' - x_3' + x_3'' &\leq 7 \\ x_1' - x_1'' - x_3' + x_3'' &\geq 7 \\ 3x_1' - 3x_1'' + x_2 &\geq 7 \\ x_1', x_1'', x_2, x_3', x_3'' &\geq 0 \end{aligned}$$

5. Correct greater-than-or-equal-to inequality constraints

Maximize

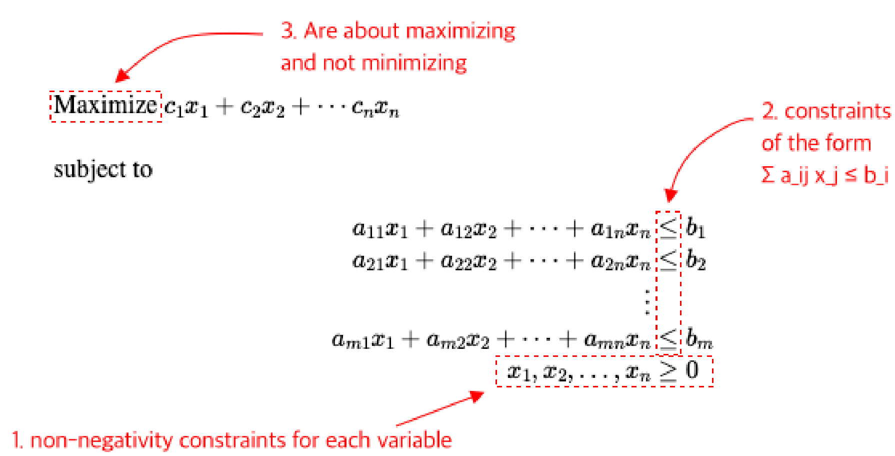
$$-2x_1' + 2x_1'' - 7x_2 - x_3' + x_3''$$

Subject to

$$
\begin{aligned}
x_1' - x_1'' - x_3' + x_3'' &\leq 7 \\
-x_1' + x_1'' + x_3' - x_3'' &\leq -7 \\
-3x_1' + 3x_1'' - x_2 &\leq 7 \\
x_1', x_1'', x_2, x_3', x_3'' &\geq 0
\end{aligned}
$$

Notes:

- **Linear Programming**
  - Is a method to achieve the best outcome (such as maximum profit or lowest cost) in a mathematical model whose requirements are represented by linear relationships. [1]
  - Is named to make it sound cool for government funding
    - Like dynamic programming
  - Applications
    - Microeconomics (maximize profits, minimize costs)
    - Company management
- **Standard Form**
  - Is a form of linear programming
  - Are about maximizing, not minimizing [2]
  - All have a positivity constraint for each variable [2]
  - All other constraints are all of the form "linear combination of variables $\leq$ constant". [2]



- **Converting Linear Programming to Standard Form**

1) The objective function might be a minimization rather than a maximization
   - Negate coefficients of the objective function

multiply by -1

$$\begin{array}{llllll} \boxed{\text{minimize} & -2x_1 & + & 3x_2} & & \\ \text{subject to} & & & & & \\ & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \leq & 4 \\ & x_1 & & & \geq & 0 \end{array} \qquad \begin{array}{llllll} \text{maximize} & 2x_1 & - & 3x_2 & & \\ \text{subject to} & & & & & \\ & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \leq & 4 \\ & x_1 & & & \geq & 0 \end{array}$$

2) There might be variables without nonnegativity constraints
   - Replace each non-nonnegative variable x_i with x_i' and x_i''
   - Modify linear program

Replace x_i with x_i' and x_i''

$$\begin{array}{llllll} \text{maximize} & 2x_1 & - & 3x_2 & & \\ \text{subject to} & & & & & \\ & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \leq & 4 \\ & x_1 & & & \geq & 0 \end{array} \qquad \begin{array}{llllllll} \text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\ \text{subject to} & & & & & & & \\ & x_1 & + & x_2' & - & x_2'' & = & 7 \\ & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\ & x_1, x_2', x_2'' & & & & & \geq & 0 \end{array}$$

x_2 is not nonnegative :(

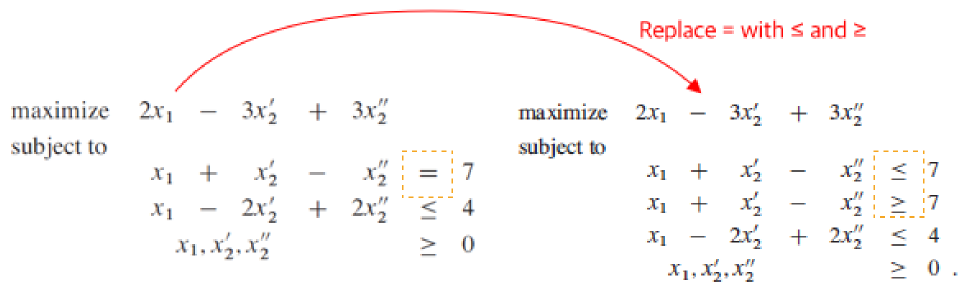
They are now nonnegative :)! Yayy!!

3) There might be **equality constraints**, which have an equal sign rather than a less-than-or-equal-to sign
   - Replace equality constraint $f(x_1, x_2, ..., x_n) = b$ with $f(x_1, x_2, ..., x_n) \leq b$ and $f(x_1, x_2, ..., x_n) \geq b$

Multiply incorrect constraints by -1

$$\begin{array}{llllllll} \text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\ \text{subject to} & & & & & & & \\ & x_1 & + & x_2' & - & x_2'' & \leq & 7 \\ & x_1 & + & x_2' & - & x_2'' & \geq & 7 \\ & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\ & x_1, x_2', x_2'' & & & & & \geq & 0\ . \end{array} \qquad \begin{array}{llllllll} \text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\ \text{subject to} & & & & & & & \\ & x_1 & + & x_2 & - & x_3 & \leq & 7 \\ & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\ & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\ & x_1, x_2, x_3 & & & & & \geq & 0 \end{array}$$

4) There might be **inequality constraints**, but instead of having a less-than-or-equal-to-sign
   - Multiply incorrect inequality constraints by -1



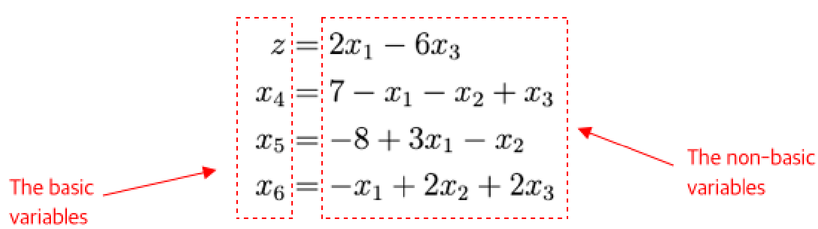
References:

1) Wikipedia, Linear Programming, link
2) Instituto de Mathematicas, Standard form for Linear Programs, link

2.

$$
\begin{aligned}
z &= 2x_1 - 6x_3 \\
x_4 &= 7 - x_1 - x_2 + x_3 \\
x_5 &= -8 + 3x_1 - x_2 \\
x_6 &= -x_1 + 2x_2 + 2x_3
\end{aligned}
$$

The basic variables are variables on the lhs (i.e $B = 4, 5, 6$), and the non-basic variables are the variables on the rhs of the expressions (i.e. $N = 1, 2, 3$).



Notes:

- **Slack Form**

- Is a form of linear programming
- Is for efficient solving of liner programming problem using simplex algorithm

The value of objective function

Slack variables

$$
\begin{array}{rcrcrcrcr}
\boxed{z} & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 .
\end{array}
$$

- **Converting Linear Programs into Slack Form**
  1) Start from the standard form of linear programming
  2) Shift objective functions to right

$$
\begin{array}{ll}
\text{maximize} & \boxed{2x_1 - 3x_2 + 3x_3} \\
\text{subject to} & \\
& \begin{array}{rcrcrcr}
x_1 & + & x_2 & - & x_3 & \le & 7 \\
-x_1 & - & x_2 & + & x_3 & \le & -7 \\
x_1 & - & 2x_2 & + & 2x_3 & \le & 4 \\
 & & x_1, x_2, x_3 & & & \ge & 0
\end{array}
\end{array}
$$

Introduce slack variables

Shift objective functions to right

$$
\begin{array}{ll}
\text{maximize} & \boxed{\qquad\qquad 2x_1 - 3x_2 + 3x_3} \\
\text{subject to} & \\
& \begin{array}{rcrcrcrcr}
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
 & & x_1, x_2, x_3, x_4, x_5, x_6 & & & \ge & 0
\end{array}
\end{array}
$$

  3) Introduce slack variable x_i to lhs and move expressions $\sum_{j=1}^{n} a_{ij}x_j$ to rhs

$$
\begin{array}{ll}
\text{maximize} & 2x_1 - 3x_2 + 3x_3 \\
\text{subject to} & \\
& \begin{array}{rcl}
\boxed{\begin{array}{rcrcr}
x_1 & + & x_2 & - & x_3 \\
-x_1 & - & x_2 & + & x_3 \\
x_1 & - & 2x_2 & + & 2x_3
\end{array}} & \begin{array}{c} \le \\ \le \\ \le \end{array} & \begin{array}{r} 7 \\ -7 \\ 4 \end{array} \\
x_1, x_2, x_3 & \ge & 0
\end{array}
\end{array}
$$

Move expressions to rhs

Introduce slack variables

$$
\begin{array}{ll}
\text{maximize} & 2x_1 - 3x_2 + 3x_3 \\
\text{subject to} & \\
& \boxed{\begin{array}{c} x_4 \\ x_5 \\ x_6 \end{array}} \begin{array}{c} = \\ = \\ = \end{array} \boxed{\begin{array}{rcrcrcr}
7 & - & x_1 & - & x_2 & + & x_3 \\
-7 & + & x_1 & + & x_2 & - & x_3 \\
4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}} \\
& x_1, x_2, x_3, x_4, x_5, x_6 \ge 0
\end{array}
$$

Introduce slack variables

4) Change inequalities in linear programming to equality

$$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2 + 3x_3 \\ \text{subject to} & \\ & x_1 + x_2 - x_3 \le 7 \\ & -x_1 - x_2 + x_3 \le -7 \\ & x_1 - 2x_2 + 2x_3 \le 4 \\ & x_1, x_2, x_3 \ge 0 \end{array}$$

Change inequality sign to equality

Introduce slack variables

$$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2 + 3x_3 \\ \text{subject to} & \\ & x_4 = 7 - x_1 - x_2 + x_3 \\ & x_5 = -7 + x_1 + x_2 - x_3 \\ & x_6 = 4 - x_1 + 2x_2 - 2x_3 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \ge 0 \end{array}$$

5) Use Variable z to denote objective function

$$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2 + 3x_3 \\ \text{subject to} & \\ & x_4 = 7 - x_1 - x_2 + x_3 \\ & x_5 = -7 + x_1 + x_2 - x_3 \\ & x_6 = 4 - x_1 + 2x_2 - 2x_3 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \ge 0 \end{array}$$

Use variable z to denote objective function and omit the nonnegativity constraints.

Use variable
z to denote
objective function

$$\begin{array}{l} z = 2x_1 - 3x_2 + 3x_3 \\ x_4 = 7 - x_1 - x_2 + x_3 \\ x_5 = -7 + x_1 + x_2 - x_3 \\ x_6 = 4 - x_1 + 2x_2 - 2x_3 \end{array}$$

6) Omit the nonnegativivty constraints

$$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2 + 3x_3 \\ \text{subject to} & \\ & x_4 = 7 - x_1 - x_2 + x_3 \\ & x_5 = -7 + x_1 + x_2 - x_3 \\ & x_6 = 4 - x_1 + 2x_2 - 2x_3 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \ge 0 \end{array}$$

Use variable z to denote objective function and omit the nonnegativity constraints.

remove nonnegative
constraints

$$\begin{array}{l} z = 2x_1 - 3x_2 + 3x_3 \\ x_4 = 7 - x_1 - x_2 + x_3 \\ x_5 = -7 + x_1 + x_2 - x_3 \\ x_6 = 4 - x_1 + 2x_2 - 2x_3 \end{array}$$

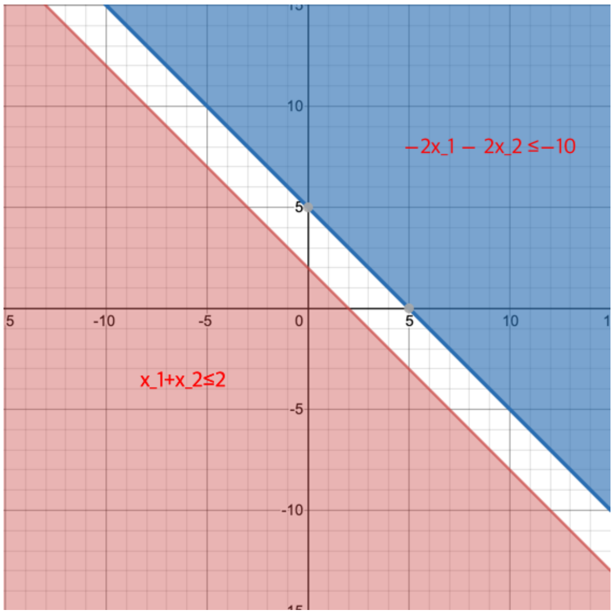
References:

1) Cambridge University, Linear Programming, link

3. Multiplying the first expression (under subject to) by 2, and summing the inequality constraints, we have

$$0 \leq -6 \tag{1}$$

which is impossible.



Notes

- I noticed that infeasible solution has non-overlapping region
- **Infeasible**
  - A Linear Program is infeasible if there is no solution that satisfies all of the constraints

4. Let $x_1 = 3r$ and $x_2 = r$ where $r \geq 0$. Then the inequality constraints become

$$\begin{aligned} -2(3r) + (r) = -5r &\leq -1 \\ -(3r) - 2(r) = -5r &\leq -2 \\ 3r, r &\geq 0 \end{aligned}$$

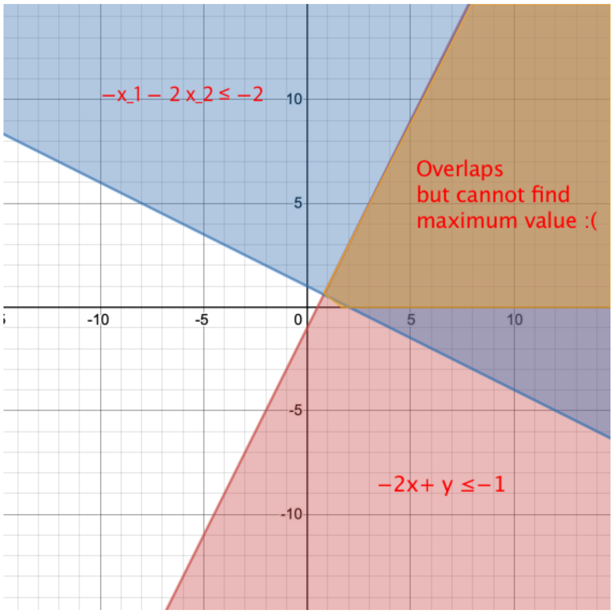
and are valid.

Now, looking at the objective functions, with $x_1 = 3r$ and $x_2 = r$, it becomes

$$3r - r = 2r$$

which increases without bound.

Thus, there is no maximum, and the linear program is unboudned.



Notes

- I learned that to show an LP is unbounded, I first have to subtitute x_i with a common variable r (e.g. $x_1 = 3r$, $x_2 = r$), check inequality constraints, and then look at objective functions and see if I can get max/min.
- **Unbounded**
  - A Linear Program is unbounded if it has some feasible solutions but does not have a finite optimal objective value

References:

1) CLRS Solutions, 29.1 Standard and slack forms, link

5. At worst case, the upper bound of variables and constraints in conversion to standard form is $2n$ and $2m$.

   Proof. Suppose a linear program has n variables and m constraints.

   When converting to standard form, the areas that affect the number of constraints and variables are:

   1. Variables without nonnegativity constraints
   2. Existence of equality constraints

   So, in worst case, all of the variables are not nonnegative, and all of the expressions have equality constraints.

   Since addressing each of non-nonnegative constraints adds an additional variable, we can write there would be total of $2n$ variables at the end.

   And since addressing each equality constraints result in 1 additional constraint, we can conclude there would be total of $2m$ constraints.

   □

   References:

   1) CLRS Solutions, 29.1 Standard and slack forms, link

6. The following is an example of a linear program for which the feasible region is not bounded, but the optimal objective value is finite.

   Minimum

$$x_1 - x_2$$

   Subject to

$$\begin{aligned} -2x_1 + x_2 &\leq -1 \\ -x_1 + 2x_2 &\leq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Proof. Let $x_1 = 2r$ and $x_2 = r$ where $r \geq 0$. Then, the inequality constraints become

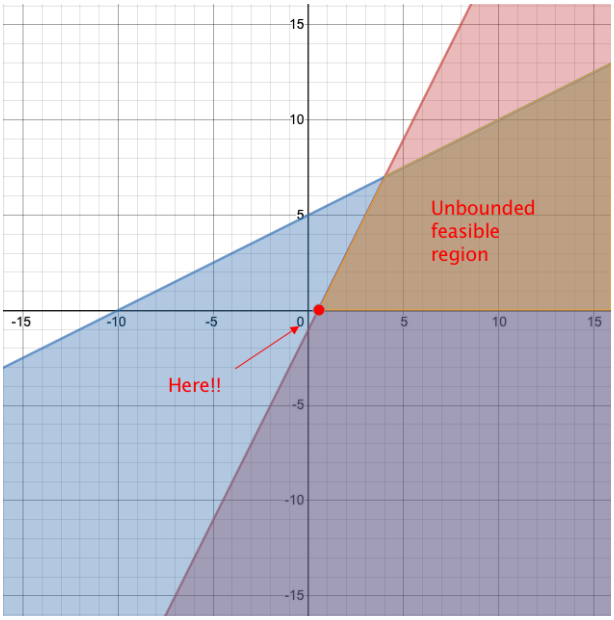
$$-2(2r) + (r) = -3r \leq -1 \quad (2)$$
$$-(2r) + 2(r) = 0 \leq 10 \quad (3)$$
$$2r, r \geq 0 \quad (4)$$

which is valid.

Now, looking at the objective function, we have $2r - r = r$.

Since we are looking for the minimum value and $r \geq 0$, we can write $min(r) = 0$ (if we are looking for the maximum, then r is ever-increasing and the linear program is unbounded).

Thus, the optimal objective value in this example is finite.



□

<u>Notes:</u>

- **Feasible Region**
  - Is the set of all possible points (sets of values of the choice variables) of an optimization problem that satisfy the problem's constraints. [1]

References:

1) Wikipedia, Feasible region, link

7.

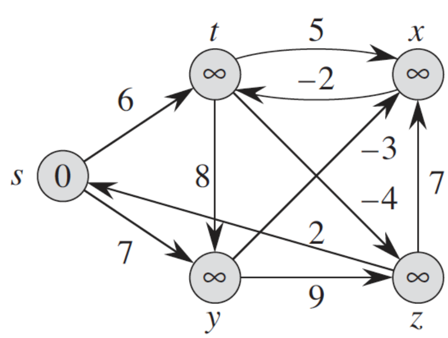
$$
\begin{array}{ll}
\text{Maximize} & \displaystyle\sum_{v \in V - \{s\}} d_v \\
\text{Subject To} & d_v - d_u \leq w(u,v) \text{ for each edge } (u,v) \in E \\
& d_s = 0
\end{array}
$$

Notes:

- I chose positive sign. But is there a symstematic approaches to determining signs in objective functions? What does positive x_i means? what about negative x_i?
- **Formulating Problems as Linear Program (Shortest Path)**
  - Single-source shortest-path problem can be formulated as a linear program.
  - Goal is computing the weight of a shortest path from s to t.

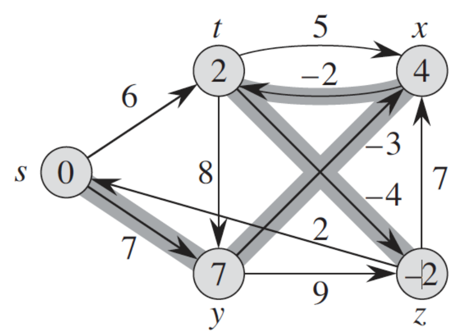
$$
\begin{array}{lrcll}
\text{maximize} & d_t & & & \\
\text{subject to} & & & & \\
& d_v & \leq & d_u + w(u,v) & \text{for each edge } (u,v) \in E \\
& d_s & = & 0\,. &
\end{array}
$$

Example:



$$
\begin{array}{ll}
\text{Maximize} & d_t \\
\text{Subject To} & d_t - d_s \le 6 \\
& d_t - d_x \le -2 \\
& d_x - d_t \le 5 \\
& d_x - d_y \le -3 \\
& d_x - d_z \le 7 \\
& d_y - d_t \le 8 \\
& d_y - d_s \le 7 \\
& d_z - d_t \le -4 \\
& d_z - d_y \le 9 \\
& d_s - d_z \le 2 \\
& d_s = 0
\end{array}
$$

On continued calculation, the values of each variables should be $d_s = 0$, $d_y = 7$, $d_x = 4$, $d_t = 2$, and $d_z = -2$.



References:

1) University of Missouri St. Louis, Linear Programming, link

8. **Rough Works:**

Goal: Find the feasible flow of minimum cost using linear program

$$
\begin{array}{ll}
\text{Minimize} & \displaystyle\sum_{u,v \in V} a(u,v) f_{uv} \\
\text{Subject To} &
\end{array}
$$

Notes:

- **Minimum-cost Flow Problem**
  - **Goal** $\rightarrow$ Finding the cheapest possible way of sending a certain amount of flow through a flow network [1]

$$
\begin{array}{llll}
\text{minimize} & \boxed{\displaystyle\sum_{(u,v)\in E} a(u,v)\, f_{uv}} & & \text{Total Cost}\\
\text{subject to} & & & \text{Capacity Constraint 1}\\
& \boxed{f_{uv} \le c(u,v)} & \text{for each } u,v\in V\ , & \\
& \boxed{\displaystyle\sum_{v\in V} f_{vu} - \sum_{v\in V} f_{uv} = 0} & \text{for each } u \in V-\{s,t\} & \text{Flow Conservation}\\
& \boxed{\displaystyle\sum_{v\in V} f_{sv} - \sum_{v\in V} f_{vs} = d}\ , & & \text{The flow value } |f|\\
& \boxed{f_{uv}\ge 0} & \text{for each } u,v\in V\ . & \\
& & & \text{Capacity constraint 2}
\end{array}
$$

  - Applications
    1) Finding delivery route from factory to a warehouse [1]

- **Multicommodity Flow Problem**
  - **Goal** $\rightarrow$ Finding whether a flow exists

$$
\begin{array}{lrcll}
\text{minimize} & & 0 & & \\
\text{subject to} & & & & \\
& \displaystyle\sum_{i=1}^{k} f_{iuv} & \le & c(u,v) & \text{for each } u,v\in V\ ,\\
& \displaystyle\sum_{v\in V} f_{iuv} - \sum_{v\in V} f_{ivu} & = & 0 & \text{for each } i = 1,2,\ldots,k \text{ and}\\
& & & & \text{for each } u\in V-\{s_i,t_i\}\ ,\\
& \displaystyle\sum_{v\in V} f_{i,s_i,v} - \sum_{v\in V} f_{i,v,s_i} & = & d_i & \text{for each } i = 1,2,\ldots,k\ ,\\
& f_{iuv} & \ge & 0 & \text{for each } u,v\in V \text{ and}\\
& & & & \text{for each } i = 1,2,\ldots,k\ .
\end{array}
$$

  - Is a network flow problem with multiple commodities (flow demands) between different source and sink nodes [2].
  - Applications [3]
    1) **Telephone Network** $\rightarrow$ Each call is a commodity with its own source and sink
    2) **Transportation Network** $\rightarrow$ Each trip is its own commodity
    3) **Supply Chain** $\rightarrow$ Different goods move through the network

<u>**Example:**</u>

- **Aggregate Flow**
  - Is the sum of various commodity flows

References:

1) Wikipedia, Minimum-cost flow problem, link
2) Wikipedia, Multicommodity flow problem, link
3) Rensselaer Polytechnique Institute, Math Models of OR Multicommodity Network Flow Problems, link