

CSC236 Term Test 1 Version 2 Solution

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May 6, 2020

Question 1

- *Proof.* Define $P(n) : f(n) = 3^n$.

I will use complete induction to prove that $\forall n \in \mathbb{N}, n > 2 \Rightarrow P(n)$.

Base Case ($n = 0$):

Let $n = 0$.

Then, the definition of $f(n)$ tells us $f(n) = 1$.

Then, we have

$$\begin{aligned} f(n) &= 3^0 & (1) \\ &= 3^n & (2) \end{aligned}$$

Thus, $P(n)$ follows.

Base Case ($n = 1$):

Let $n = 1$.

Then, the definition of $f(n)$ tells us $f(n) = 3$.

Then, we have

$$\begin{aligned} f(n) &= 3^1 & (3) \\ &= 3^n & (4) \end{aligned}$$

Thus, $P(n)$ follows.

Base Case ($n = 2$):

Let $n = 2$.

Then, the definition of $f(n)$ tells us $f(n) = 9$.

Then, we have

$$f(n) = 3^2 \tag{5}$$

$$= 3^n \tag{6}$$

Thus, $P(n)$ follows.

Case ($n > 2$):

Assume $n > 2$.

Then, since $0 \leq n - 1 < n$, $0 \leq n - 2 < n$, and $0 \leq n - 3 < n$, the complete induction tells us $P(n - 1)$, $P(n - 2)$, and $P(n - 3)$, i.e. $f(n - 1) = 3^{n-1}$, $f(n - 2) = 3^{n-2}$, and $f(n - 3) = 3^{n-3}$, respectively.

Then, using these facts, we can write

$$f(n) = f(n - 1) + 3f(n - 2) + 9f(n - 3) \tag{7}$$

$$= 3^{n-1} + 3 \cdot 3^{n-2} + 3^2 \cdot 3^{n-3} \tag{8}$$

$$= 3^{n-1} + 3^{n-1} + 3^{n-1} \tag{9}$$

$$= 3^{n-1}(1 + 1 + 1) \tag{10}$$

$$= 3^{n-1}3 \tag{11}$$

$$= 3^n \tag{12}$$

Thus, $P(n)$ follows. □

Correct Solution:

Define $P(n) : f(n) = 3^n$.

I will use complete induction to prove that $\forall n \in \mathbb{N}, P(n)$.

Inductive Step:

Let $n \in \mathbb{N}$. Assume $H(n) : \bigwedge_{i=0}^{n-1} P(i)$. I will prove $P(n)$ follows. That is, $f(n) = 3^n$.

Base Case ($n = 0$):

Let $n = 0$.

Then, the definition of $f(n)$ tells us $f(n) = 1$.

Then, we have

$$f(n) = 3^0 \tag{13}$$

$$= 3^n \tag{14}$$

Thus, $P(n)$ follows.

Base Case ($n = 1$):

Let $n = 1$.

Then, the definition of $f(n)$ tells us $f(n) = 3$.

Then, we have

$$f(n) = 3^1 \tag{15}$$

$$= 3^n \tag{16}$$

Thus, $P(n)$ follows.

Base Case ($n = 2$):

Let $n = 2$.

Then, the definition of $f(n)$ tells us $f(n) = 9$.

Then, we have

$$f(n) = 3^2 \tag{17}$$

$$= 3^n \tag{18}$$

Thus, $P(n)$ follows.

Case ($n > 2$):

Assume $n > 2$.

Then, since $0 \leq n-1 < n$, $0 \leq n-2 < n$, and $0 \leq n-3 < n$, the complete induction tells us $P(n-1)$, $P(n-2)$, and $P(n-3)$, i.e. $f(n-1) = 3^{n-1}$, $f(n-2) = 3^{n-2}$, and $f(n-3) = 3^{n-3}$, respectively.

Then, using these facts, we can write

$$f(n) = f(n-1) + 3f(n-2) + 9f(n-3) \quad [\text{By definition, since } n > 2] \quad (19)$$

$$= 3^{n-1} + 3 \cdot 3^{n-2} + 3^2 \cdot 3^{n-3} \quad (20)$$

$$= 3^{n-1} + 3^{n-1} + 3^{n-1} \quad (21)$$

$$= 3^{n-1}(1 + 1 + 1) \quad (22)$$

$$= 3^{n-1}3 \quad (23)$$

$$= 3^n \quad (24)$$

Thus, $P(n)$ follows.

Notes:

1. Learned that $n > i$ in $\forall n \in \mathbb{N}, n > i \Rightarrow P(n)$ is used when $P(n)$ is true starting $i + 1$.

If $P(n)$ is true for all natural numbers, then $\forall n \in \mathbb{N}, P(n)$ is used.

2. Learned that ‘Assume $n > 2$ ’ in ‘Let $n \in \mathbb{N}$. Assume $n > 2$. Assume $H(n) : \bigwedge_{i=0}^{n-1} P(i)$ ’. is used when $P(i)$ is true starting $n = 3$.

If $P(i)$ is true for all natural numbers, then, ‘Let $n \in \mathbb{N}$. Assume $H(n) : \bigwedge_{i=0}^{n-1} P(i)$ ’ is used.

Question 2

- Given the statement to prove

$P(x, y, z, w) : \text{There are no positive integers } x, y, z, w \text{ such that } x^4 + 3y^4 + 9z^4 = 27w^4.$

Proof. I will prove $P(x, y, z, w)$ using proof by contradiction.

Assume $\exists x, y, z, w \in \mathbb{N}^+, x^4 + 3y^4 + 9z^4 = 27w^4$.

Then, the set $X = \{x \in \mathbb{N}^+ \mid \exists y, z, w \in \mathbb{N}^+, x^4 + 3y^4 + 9z^4 = 27w^4\}$ is not empty.

Then, by principle of well-ordering, there is smallest positive integer $x_0 \in X$, and positive integers $y_0, z_0, w_0 \in \mathbb{N}^+$ such that $x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4$.

Then,

$$\begin{aligned} x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 &\Rightarrow x_0^4 = 27w_0^4 - 3y_0^4 - 9z_0^4 \\ &\Rightarrow 3 \mid x_0^4 \Rightarrow 3 \mid x_0 \end{aligned} \quad \text{[By hint]} \quad (1)$$

$$\begin{aligned} \text{Let } \exists x_1 \in \mathbb{N}^+, x_0 = 3x_1 &\Rightarrow x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \\ &\Rightarrow 3y_0^4 = 27w_0^4 - 9z_0^4 - x_0^4 \\ &\Rightarrow 3y_0^4 = 27w_0^4 - 9z_0^4 - 3^4x_1^4 \\ &\Rightarrow y_0^4 = 9w_0^4 - 3z_0^4 - 3^3x_1^4 \\ &\Rightarrow 3 \mid y_0^4 \Rightarrow 3 \mid y_0 \end{aligned} \quad \text{[By hint]} \quad (2)$$

$$\begin{aligned} \text{Let } \exists y_1 \in \mathbb{N}^+, y_0 = 3y_1 &\Rightarrow x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \\ &\Rightarrow 9z_0^4 = 27w_0^4 - 3y_0^4 - x_0^4 \\ &\Rightarrow 9z_0^4 = 27w_0^4 - 3^5y_1^4 - 3^4x_1^4 \\ &\Rightarrow z_0^4 = 3w_0^4 - 3^3y_1^4 - 3^2x_1^4 \\ &\Rightarrow 3 \mid z_0^4 \Rightarrow 3 \mid z_0 \end{aligned} \quad \text{[By hint]} \quad (3)$$

$$\begin{aligned} \text{Let } \exists w_1 \in \mathbb{N}^+, w_0 = 3w_1 &\Rightarrow x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \\ &\Rightarrow 3^4x_1^4 + 3^5y_1^4 + 3^6z_1^4 = 3^7w_1^4 \\ &\Rightarrow x_1^4 + 3y_1^4 + 3^2z_1^4 = 3^3w_1^4 \\ &\Rightarrow x_1^4 + 3y_1^4 + 9z_1^4 = 27w_1^4 \\ &\Rightarrow x_1 \in X \end{aligned} \quad (4)$$

Then, this leads to contradiction, because we know $x_1 < x_0$, $x_1 \in X$, but x_0 is the smallest number in X .

Thus, we can conclude the assumption is false.

□

Correct Solution:

Given the statement to prove

$P(x, y, z, w)$: There are no positive integers x, y, z, w such that $x^4 + 3y^4 + 9z^4 = 27w^4$.

Proof. I will prove $P(x, y, z, w)$ using proof by contradiction.

Assume $\exists x, y, z, w \in \mathbb{N}^+, x^4 + 3y^4 + 9z^4 = 27w^4$.

Then, the set $X = \{x \in \mathbb{N}^+ \mid \exists y, z, w \in \mathbb{N}^+, x^4 + 3y^4 + 9z^4 = 27w^4\}$ is not empty.

Then, **since X is subset of \mathbb{N}** , by principle of well-ordering, there is smallest positive integer $x_0 \in X$. **Furthermore**, there are positive integers $y_0, z_0, w_0 \in \mathbb{N}^+$ such that $x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4$.

Then,

$$\begin{aligned} x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 &\Rightarrow x_0^4 = 27w_0^4 - 3y_0^4 - 9z_0^4 \\ &\Rightarrow 3 \mid x_0^4 \Rightarrow 3 \mid x_0 \end{aligned} \quad \begin{array}{l} \text{[By hint]} \\ (1) \end{array}$$

$$\begin{aligned} \text{Let } \exists x_1 \in \mathbb{N}^+, x_0 = 3x_1 &\Rightarrow x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \\ &\Rightarrow 3y_0^4 = 27w_0^4 - 9z_0^4 - x_0^4 \\ &\Rightarrow 3y_0^4 = 27w_0^4 - 9z_0^4 - 3^4x_1^4 \\ &\Rightarrow y_0^4 = 9w_0^4 - 3z_0^4 - 3^3x_1^4 \\ &\Rightarrow 3 \mid y_0^4 \Rightarrow 3 \mid y_0 \end{aligned} \quad \begin{array}{l} \text{[By hint]} \\ (2) \end{array}$$

$$\begin{aligned} \text{Let } \exists y_1 \in \mathbb{N}^+, y_0 = 3y_1 &\Rightarrow x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \\ &\Rightarrow 9z_0^4 = 27w_0^4 - 3y_0^4 - x_0^4 \\ &\Rightarrow 9z_0^4 = 27w_0^4 - 3^5y_1^4 - 3^4x_1^4 \\ &\Rightarrow z_0^4 = 3w_0^4 - 3^3y_1^4 - 3^2x_1^4 \\ &\Rightarrow 3 \mid z_0^4 \Rightarrow 3 \mid z_0 \end{aligned} \quad \begin{array}{l} \text{[By hint]} \\ (3) \end{array}$$

$$\begin{aligned} \text{Let } \exists w_1 \in \mathbb{N}^+, w_0 = 3w_1 &\Rightarrow x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 \\ &\Rightarrow 3^4x_1^4 + 3^5y_1^4 + 3^6z_1^4 = 3^7w_1^4 \\ &\Rightarrow x_1^4 + 3y_1^4 + 3^2z_1^4 = 3^3w_1^4 \\ &\Rightarrow x_1^4 + 3y_1^4 + 9z_1^4 = 27w_1^4 \\ &\Rightarrow x_1 \in X \end{aligned} \quad (4)$$

Then, this leads to contradiction, because we know $x_1 < x_0$, $x_1 \in X$, but x_0 is the smallest number in X .

Thus, we can conclude the assumption is false.

□

Note:

- Noticed professor wrote ‘Divide by 3’ as a reasoning in calculation.

$$\begin{aligned} \text{Let } z_1 \in \mathbb{N}^+, 3z_1 = z_0 &\Rightarrow 81z_1^4 = 3w_0^4 - (9x_1^4 + 27y_1^4) \\ &\Rightarrow 27z_1^4 = w_0^4 - (3x_1^4 + 9y_1^4) \Rightarrow 3x_1^4 + 9y_1^4 + 27z_1^4 = w_0^4 \quad \# \text{ divide by 3} \\ &\Rightarrow 3 \mid w_0^4 \Rightarrow 3 \mid w_0 \quad \# \text{ since 3 divides LHS and allowed assumption} \end{aligned}$$

Question 3

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Rough Work:

Define \mathcal{T} as the smallest set such that

- $() \in \mathcal{T}$
- If $t_1, t_2 \in \mathcal{T}$, $(t_1, t_2) \in \mathcal{T}$

I need to prove $\forall t \in \mathcal{T}, P(t)$. That is, left(t) is odd.

- Basis

Let $() \in \mathcal{T}$.

Then, since there is only 1 left parenthesis and since this is odd, $P(t)$ holds.

- Inductive Step