

Midterm 2 Version 3 Solution

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April 5, 2020

Question 1

a.

$$165 \div 2 = 82, \text{ remainders } \mathbf{1}$$

$$82 \div 2 = 41, \text{ remainders } \mathbf{0}$$

$$41 \div 2 = 20, \text{ remainders } \mathbf{1}$$

$$20 \div 2 = 10, \text{ remainders } \mathbf{0}$$

$$10 \div 2 = 5, \text{ remainders } \mathbf{0}$$

$$5 \div 2 = 2, \text{ remainders } \mathbf{1}$$

$$2 \div 2 = 1, \text{ remainders } \mathbf{0}$$

$$1 \div 2 = 0, \text{ remainders } \mathbf{1}$$

From the above, we can conclude the binary representation of the decimal number 165 is $(10100101)_2$

b. The largest number that can be expressed by an n -digit balanced ternary representation is

$$\sum_{i=0}^{n-1} 3^i = \frac{1}{2} \cdot (3^n - 1) \quad (1)$$

Notes:

- Geometric Series

$$\sum_{i=0}^n r^i = \frac{1 - r^{n+1}}{1 - r}, \text{ where } |r| > 1$$

c.

$f(n) \in \mathcal{O}(n)$	True	$g(n) \in \Omega(n)$	True	$f(n) \in \Omega(g(n))$	True
$f(n) \in \Theta(g(n))$	False	$g(n) \in \Theta(n)$	False	$f(n) + g(n) \in \Theta(g(n))$	False

Correct Solution:

$f(n) \in \mathcal{O}(n)$	True	$g(n) \in \Omega(n)$	True	$f(n) \in \Omega(g(n))$	True
$f(n) \in \Theta(g(n))$	False	$g(n) \in \Theta(n)$	False	$f(n) + g(n) \in \Theta(g(n))$	True

Notes:

- Note that for $f(n) + g(n) \in \Theta(g(n))$, large values of n causes $g(n) = n^{\log_2 n}$ to dominate $f(n) = \frac{3n}{\log_2 n + 8}$. This causes the inequality to be simplified to

$$c_1 \cdot n^{\log_2 n} \leq n^{\log_2 n} \leq c_2 \cdot n^{\log_2 n} \quad (1)$$

It follows from above the answer is True.

d.

k	0	1	2
$i * i$	$3 = 3^{2^0}$	$9 = 3^{2^1}$	$81 = 3^{2^4}$

From the rough work, we can deduce the value of i after k iterations is

$$3^{2^k} \quad (1)$$

- e. Loop termination occurs when $i_k \geq n^3$.

We need to find the smallest value of k , and the value is

$$\lceil \log_2 3 \log_3 n \rceil \quad (1)$$

Question 2

- **Statement:** $\forall n \in \mathbb{N}, n \geq 1 \Rightarrow \sum_{i=1}^n \frac{1}{\sqrt{i}} > \sqrt{n} - 1$

Proof. Let $n \in \mathbb{N}$. Assume $n \geq 1$.

We will prove the statement using induction on n .

Base Case ($n = 1$):

Let $n = 1$.

We want to show $\sum_{i=1}^1 \frac{1}{\sqrt{i}} > \sqrt{1} - 1$

Starting from the left hand side of the inequality, we can calculate

$$\sum_{i=1}^1 \frac{1}{\sqrt{i}} = \frac{1}{\sqrt{1}} \quad (1)$$

$$= 1 \quad (2)$$

$$> 0 \quad (3)$$

$$= \sqrt{1} - 1 \quad (4)$$

Inductive Case:

Let $n \in \mathbb{N}$. Assume $\sum_{i=1}^n \frac{1}{\sqrt{i}} > \sqrt{n} - 1$.

We want to show $\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} > \sqrt{n+1} - 1$.

Starting from the left hand side of the inequality, we can conclude

$$\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} = \sum_{i=1}^n \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}} \quad (5)$$

Then, it follows from induction hypothesis (i.e. $\sum_{i=1}^n \frac{1}{\sqrt{i}} > \sqrt{n} - 1$) that

$$\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} > \sqrt{n+1} - 1 + \frac{1}{\sqrt{n+1}} \quad (6)$$

The hint tells us the following

$$\forall n \in \mathbb{Z}^+, \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}}, \text{ and } \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n+1}} \quad (7)$$

Using the hint, we can calculate

$$\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} > (\sqrt{n} - 1) + \frac{1}{\sqrt{n+1} + \sqrt{n}} \quad (8)$$

$$= (\sqrt{n} - 1) + (\sqrt{n+1} - \sqrt{n}) \quad (9)$$

$$= \sqrt{n+1} - 1 \quad (10)$$

□

Question 3

- **Statement:** $\forall a \in \mathbb{R}^+, a > 1 \Rightarrow a^n + 3 \in \Theta(2^n)$

Negation Statement: $\exists a \in \mathbb{R}^+, (a > 1) \wedge \left[\forall c_1, c_2, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N}, (n \geq n_0) \wedge ((c_1 \cdot (a^n + 3) > 2^n) \vee (2^n > c_2 \cdot (a^n + 3))) \right]$

Proof. Let $a = \frac{3}{2}$. Let $c_1, c_2, n_0 \in \mathbb{R}^+$, and $n = \left\lceil \max(n_0, \frac{\log(2c_1)}{\log \frac{4}{3}}, \frac{\log(6c_1)}{2}) \right\rceil + 1$.

We need to show $a > 1$, $n > n_0$ and $a^n + 3 < \frac{1}{c_1} \cdot 2^n$.

We will do so in parts.

Part 1 (showing $a > 1$):

We need to show $a > 1$.

Because we know $a = \frac{3}{2}$ from header, we can conclude

$$a > 1 \quad (1)$$

Part 2 (showing $n > n_0$):

We need to show $n \geq n_0$.

Using the fact $\left\lceil \max(n_0, \frac{\log(2c_1)}{\log \frac{4}{3}}, \frac{\log(6c_1)}{2}) \right\rceil \geq n_0$, we can conclude

$$n = \left\lceil \max(n_0, \frac{\log(2c_1)}{\log \frac{4}{3}}, \frac{\log(6c_1)}{2}) \right\rceil + 1 > n_0 \quad (2)$$

Part 3 (Showing $a^n + 3 < \frac{1}{c_1} \cdot 2^n$):

We need to show $a^n + 3 < \frac{1}{c_1} \cdot 2^n$.

We will do so by showing $a^n < \frac{2^n}{2c_1}$, $3 < \frac{2^n}{2c_1}$, and then combining the two.

For the inequality $a^n < \frac{2^n}{2c_1}$, using the following fact

$$\frac{\log(2c_1)}{\log \frac{4}{3}} < \left\lceil \max(n_0, \frac{\log(2c_1)}{\log \frac{4}{3}}, \frac{\log(6c_1)}{2}) \right\rceil + 1 = n \quad (3)$$

we can calculate

$$\log(2c_1) < n \log \frac{4}{3} \quad (4)$$

$$2c_1 < \left(\frac{4}{3}\right)^n \quad (5)$$

$$2c_1 < \frac{2^n}{\left(\frac{3}{2}\right)^n} \quad (6)$$

$$\left(\frac{3}{2}\right)^n < \frac{2^n}{2c_1} \quad (7)$$

$$(8)$$

Then, since $a = \frac{3}{2}$, we can conclude

$$a^n < \frac{2^n}{2c_1} \quad (9)$$

For the inequality $3 < \frac{2^n}{2c_1}$, using the following fact

$$\frac{\log(6c_1)}{2} < \left\lceil \max(n_0, \frac{\log(2c_1)}{\log \frac{4}{3}}, \frac{\log(6c_1)}{2}) \right\rceil + 1 = n \quad (10)$$

we can conclude

$$6c_1 < 2^n \quad (11)$$

$$3 < \frac{2^n}{2c_1} \quad (12)$$

Finally, by combining the two, we can conclude

$$a^n + 3 < \frac{2^n}{2c_1} + \frac{2^n}{2c_1} \quad (13)$$

$$a^n + 3 < \frac{2^n}{c_1} \quad (14)$$

□

Question 4

a. *Proof.* Let $n \in \mathbb{N}$.

First, we need to analyze the number of iterations of loop 2 per iteration of loop 1

The code tells us loop 2 starts at $j = 0$, ends at $j = i - 1$, and j increases by 2 per iteration.

Using the fact, we can conclude that the number of iterations of loop 2 is

$$\left\lceil \frac{i - 1 - 0 + 1}{2} \right\rceil = \left\lceil \frac{i}{2} \right\rceil \quad (1)$$

or

$$\frac{i}{2} \quad (2)$$

since we are ignoring floor and ceiling symbols.

Next, we need to add this number over all iterations of loop 1.

The code tell us us it starts at $i = 0$ and ends at $i = \sqrt{n-1}$ with i increasing by 1 per iteration.

Using this fact, the total number of iterations of loop 2 is

$$\frac{0}{2} + \frac{1}{2} + \dots + \frac{\sqrt{n-1}}{2} = \sum_{i=0}^{\sqrt{n-1}} \frac{i}{2} \quad (3)$$

$$= \frac{1}{2} \cdot \sum_{i=0}^{\sqrt{n-1}} i \quad (4)$$

Then, it follows from the fact $\sum_{i=0}^n i = \frac{n(n+1)}{2}$ that,

$$\frac{1}{2} \cdot \sum_{i=0}^{\sqrt{n-1}} i = \frac{1}{2} \cdot \frac{(\sqrt{n-1})(\sqrt{n-1}+1)}{2} \quad (5)$$

$$= \frac{(\sqrt{n-1})(\sqrt{n-1}+1)}{4} \quad (6)$$

□

Correct Solution:

Let $n \in \mathbb{N}$.

First, we need to analyze the number of iterations of loop 2 per iteration of loop 1

The code tells us loop 2 starts at $j = 0$, ends at $j = i - 1$, and j increases by 2 per iteration.

Using the fact, we can conclude that the number of iterations of loop 2 is

$$\left\lceil \frac{i-1-0+1}{2} \right\rceil = \left\lceil \frac{i}{2} \right\rceil \quad (1)$$

or

$$\frac{i}{2} \quad (2)$$

since we are ignoring floor and ceiling symbols.

Next, we need to add this number over all iterations of loop 1.

The code tell us us it starts at $i = 0$ with i increasing by 1 per iteration until the terminating condition of $i^2 \geq n$.

And from the inequality, we know the loop finishes at $i = \sqrt{n} - 1$.

Using these facts, the total number of iterations of loop 2 is

$$\frac{0}{2} + \frac{1}{2} + \dots + \frac{\sqrt{n}-1}{2} = \sum_{i=0}^{\sqrt{n}-1} \frac{i}{2} \quad (3)$$

$$= \frac{1}{2} \cdot \sum_{i=0}^{\sqrt{n}-1} i \quad (4)$$

Then, it follows from the fact $\sum_{i=0}^n i = \frac{n(n+1)}{2}$ that,

$$\frac{1}{2} \cdot \sum_{i=0}^{\sqrt{n}-1} i = \frac{1}{2} \cdot \frac{(\sqrt{n}-1)(\sqrt{n}-1+1)}{2} \quad (5)$$

$$= \frac{(\sqrt{n}-1)\sqrt{n}}{4} \quad (6)$$

Notes:

- How can I construct a proof in a situation where the slight change in a problem causes the flow of proof to deviate from the main?

One example of how I mean by above is the following

The code tell us us it starts at $i = 0$ with i increasing by 1 per iteration until the terminating condition of $i^2 \geq n$.

And from the inequality, we know the loop finishes at $i = \sqrt{n} - 1$.

- I get the same feeling when solving question 4.b