OPERATIONS RESEARCH
Vol. 56, No. 1, January–February 2008, pp. 188–203
ISSN 0030-364X | EISSN 1526-5463 | 08 | 5601 | 0188

DOI 10.1287/opre.1070.0486

Regret in the Newsvendor Model with Partial Information

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Traditional stochastic inventory models assume full knowledge of the demand probability distribution. However, in practice, it is often difficult to completely characterize the demand distribution, especially in fast-changing markets. In this paper, we study the newsvendor problem with partial information about the demand distribution (e.g., mean, variance, symmetry, unimodality). In particular, we derive the order quantities that minimize the newsvendor's maximum regret of not acting optimally. Most of our solutions are tractable, which makes them attractive for practical application. Our analysis also generates insights into the choice of the demand distribution as an input to the newsvendor model. In particular, the distributions that maximize the entropy perform well under the regret criterion. Our approach can be extended to a variety of problems that require a robust but not conservative solution.

Subject classifications: distribution-free inventory policy; newsvendor model; robust optimization; entropy; value of information; semi-infinite linear optimization.
Area of review: Manufacturing, Service, and Supply Chain Operations.
History: Received November 2005; revisions received August 2006, March 2007, May 2007; accepted August 2007.

1. Introduction

Traditional stochastic inventory models assume full knowledge of the demand probability distribution. However, in reality, it is often difficult to completely characterize the demand, especially with little historical data or with subjective forecast methods (which are common practice; see Dalrymple 1987). If a decision maker wants to use a stochastic inventory model, she must select a demand probability distribution as an input to the model. But which distribution is she going to select: uniform, normal, gamma, or exponential? Each of these distributions gives rise to a different order quantity. Ideally, the decision must be robust, i.e., perform well under most demand scenarios. Naddor (1978) and Fortuin (1980) numerically observed that inventory decisions and costs are relatively insensitive to the choice of the distribution, when the mean and the variance are specified, giving rise to efficient distribution-free policies (Ehrhardt 1979). However, they considered only a few specific distributions. One might then wonder if their conclusions hold under more general demand scenarios, more general information levels, and with which performance guarantee.

In this paper, we study a single-period, single-item stochastic inventory problem, called the *newsvendor problem* (Porteus 1990), with limited information about the demand distribution (e.g., mean, mode, variance, symmetry). We derive the order quantities that minimize the newsvendor's maximum opportunity cost from choosing a particular demand distribution. We assume that past sales data are unavailable, thereby making the approaches based on Bayesian learning (Scarf 1959) or nonparametric learning (Godfrey and Powell 2001, Levi et al. 2007) unapplicable.

The problem of making an order decision with only partial information about the demand distribution has been studied since the origins of inventory theory. The traditional paradigm to decision making under uncertainty, called the *maximin approach*, consists in maximizing the worst-case profit. Scarf (1958) and Gallego and Moon (1993) derived the maximin order quantity when only the mean and the variance of the demand are specified. The maximin approach has also been applied to multiple-period inventory models, under continuous or periodic review (Gallego 1992, Moon and Gallego 1994, Gallego 1998), to the newsvendor model with customer balking (Moon and Choi 1995), and to finite-horizon models with discrete demand distributions, incompletely specified by selected moments and percentiles (Gallego et al. 2001). By definition, the maximin objective is conservative because it focuses on the worst-case demand scenario. Ben-Tal and Nemirovski (1999) and Bertsimas and Sim (2003) relaxed the level of conservativeness by imposing a "budget of uncertainty" within which the worst-case scenario is selected. However, their approach is essentially designed for problems with multiple random variables and applies only trivially to the newsvendor problem.

A less conservative approach is the minimax regret (Savage 1951), which we adopt in this paper. Using this approach, the firm minimizes its maximum opportunity cost from not making the optimal decision. The minimax regret is analogous to the "competitive ratio," popular in computer science (e.g., see Karp 1992). Chamberlain (2000) analyzed the regret when the type of the probability distribution is known, but there is uncertainty about the parameters of the distribution. Bergemann and Schlag (2005) and Lim and Shanthikumar (2007) considered a situation in which the probability distribution is known to lie within a neighborhood about a given distribution. Under interval uncertainty, the regret approach has been applied to solving combinatorial problems with uncertain cost parameters (e.g., see Kouvelis and Yu 1997). The regret has also been investigated in the newsvendor model by Morris (1959), Kasugai and Kasegai (1961), and Vairaktarakis (2000), when the support of the distribution is known, and by Yue et al. (2006) when the mean and the variance are known, without nonnegativity constraints. In this paper, we unify their results by developing a generic methodology, based on the moment problems, and consider more general information scenarios.

Our use of the regret is normative, i.e., we are interested in how people *should* behave. In contrast, the minimax regret has been used to describe how people *do* behave, as an alternative to the expected utility theory (Bell 1982, Loomes and Sugden 1982). Using the concept of regret, Schweitzer and Cachon (2000) and Brown and Tang (2006) explained an observed "bias" between practical order decisions and the newsvendor solution.

We will show that the minimax regret is related to entropy maximization. Entropy measures the amount of uncertainty associated with a probability distribution. Jaynes (1957, 2003) proposed to select as a prior the distribution that maximizes the entropy because it is the least informative. However, because the entropy criterion is independent of the inventory costs, there is no performance guarantee associated with this approach.

Identifying the maximum regret associated with a certain order decision can be formulated as a moment problem. Moment bound problems aim at maximizing a function over all possible random distributions that satisfy some moment constraints, and have had various applications in deriving Chebyshev-type inequalities, pricing options, and managing inventories (Smith 1995; Bertsimas and Popescu 2002, 2005; Popescu 2005).

In this paper, we not only derive robust order quantities, but also provide guidelines for selecting a demand distribution in the newsvendor model, with only partial information about the demand. The main contributions of our model are the following:

(1) We adopt an approach to solve a variety of problems that require a robust but not conservative solution.

(2) We characterize the robust order quantities in the presence of partial information, such as the moments (mean, variance) and the shape (range, symmetry, unimodality) of the demand distribution.

(3) We develop insights into which distribution must be selected as an input to the newsvendor problem. In particular, the distributions that maximize the entropy perform well under the minimax regret criterion.

(4) We measure the value of information of different assumptions about the demand distribution (moments and shape) and quantify the ability of the minimax regret approach to reduce the uncertainty inherent to the decision problem.

This paper is organized as follows. Section 2 introduces the newsvendor model and reviews different distribution-free approaches, as well as the concept of entropy. In §3, we characterize the problem of minimizing the maximum regret and formulate it as a moment problem. In §4, we derive the minimax regret order quantities with partial information about the demand distribution. In §5, we compare the value of information about the shape of the distribution, to the value of information about the moments of the distribution. Finally, we outline our conclusions and future directions for research in §6. Proofs appear in the appendix, in the online companion at http://or.pubs.informs.org/ecompanion.html.

2. Approaches for Decision Making Under Uncertainty

2.1. The Newsvendor Model

We first present the newsvendor model. Consider a make-to-stock firm that needs to determine its order quantity y before the selling season, without knowing the demand. The demand D is random and has a cumulative distribution function (c.d.f.) F. We assume linear costs and denote by r the unit selling price, s the unit salvage price, l the loss of goodwill cost per unit of unsatisfied demand, and c the unit cost. To avoid trivialities, we assume that $r > c > s$. The firm chooses its quantity to maximize its expected profit $\Pi_F(y)$:

$$\max_{y \geqslant 0} \Pi_F(y) \doteq rE_F[\min\{y, D\}] + sE_F[y-D]^+ - lE_F[D-y]^+ - cy. \tag{1}$$

The problem is a concave maximization problem. The optimal order quantity is the smallest y such that $F(y) \geqslant 1-\beta$, where $\beta \doteq (c-s)/(r+l-s)$. If the demand distribution is continuous, the optimality condition simplifies to $F(y) = 1-\beta$. In the sequel, we assume normalized costs such that $r+l-s=1$. The newsvendor's normalized expected profit can then be rewritten as $E_F[\min\{y, D\}] - \beta y - lE_F[D]$.

This model has been used as a building block for more complex inventory control problems (multiple periods, multiple stages, fixed-order costs, pricing, supply contracts); see Porteus (1990) for a review. However, it is often difficult to estimate the demand distribution, either because it is sensitive to factors that are beyond the firm's control (e.g., competitors' prices, availability of alternative products) or because the market conditions are changing rapidly. In what follows, we review different approaches for making decisions under uncertainty. Instead of considering a particular distribution F, we consider a class of distributions $\mathcal{D}$.

2.2. The Maximin Criterion

The traditional paradigm for robust optimization is the maximin approach: the firm chooses an order quantity to maximize its worst-case profit:

$$\max_{y\geqslant 0}\min_{F\in\mathcal{D}}\Pi_F(y). \tag{2}$$

By definition, the maximin approach is conservative because it focuses on the worst-case profit. In some situations, it recommends not ordering at all. For example, when only the mean is known and the lost sales cost is zero, the maximin order quantity is zero, even if the profit margin is high. Similar criteria for decision making under uncertainty are the maximax, often considered as too optimistic, and the Hurwicz criterion, which combines the maximin and the maximax criteria (Luce and Raiffa 1957).

2.3. The Minimax Regret Criterion

To overcome the conservativeness of the maximin approach, Savage (1951) introduced the concept of regret. Given a decision y and a probability distribution F, the regret measures the additional profit that could have been obtained with full information about the distribution, i.e., $\max_{z\geqslant 0}\{\Pi_F(z)\}-\Pi_F(y)$. The maximum regret $\rho(y)\doteq\max_{F\in\mathcal{D}}\max_{z\geqslant 0}\{\Pi_F(z)\}-\Pi_F(y)$ can be seen as the maximum price one would pay to know the exact demand distribution. The decision criterion consists of minimizing $\rho(y)$, i.e.,

$$\rho^*=\min_{y\geqslant 0}\rho(y)=\min_{y\geqslant 0}\max_{F\in\mathcal{D}}\max_{z\geqslant 0}\{\Pi_F(z)\}-\Pi_F(y). \tag{3}$$

In this paper, we consider $\mathcal{D}$ as the convex set of distributions with certain moments and shape (i.e., such that any convex combination, or continuum of convex combinations, of distributions from $\mathcal{D}$ belongs to $\mathcal{D}$), similar to Scarf (1958), Morris (1959), and Yue et al. (2006). The set of distributions $\mathcal{D}$ corresponds to the initial beliefs about the demand. For example, if demand is forecasted by a board of experts or executives, as is common in practice (Dalrymple 1987), different opinions must be reconciled into a common set $\mathcal{D}$. The regret can nevertheless be used with alternative characterizations of $\mathcal{D}$, such as the set of distributions belonging to the same family with uncertain parameters, or the neighborhood around a distribution of reference.

The regret is a pure intellectual construct. In practice, there does not exist a "true" probability distribution, especially in a single-period setting, and the actual opportunity cost is never measured. Instead, the motivation behind the minimax regret is to make a decision that would perform well under most demand scenarios. Among the distributions in $\mathcal{D}$, some lead to more extreme decisions than others. In general, extreme order quantities (such as the maximin solutions) are obtained with the distributions at the boundary of $\mathcal{D}$. By comparing the distribution-free decision y to the optimal newsvendor solution, the minimax regret objective tries to get away from the boundaries of $\mathcal{D}$, and to lead to a decision that performs well under most probability distributions. An alternative approach to escape from the boundaries of $\mathcal{D}$ is the entropy maximization.

2.4. Entropy Maximization

Entropy maximization is not a criterion for decision making under uncertainty. Instead, it is a criterion for selecting a probability distribution as an input to a stochastic model. Because the selection of the demand distribution is distinct from the optimization of the order quantity, the entropy approach for choosing an order quantity does not have a performance guarantee, unlike the minimax regret.

The principle of insufficient reason, proposed by Laplace, states that, with no information available, all possible outcomes should be considered as equally likely (Luce and Raiffa 1957). Jaynes (1957, 2003) generalized this principle by proposing to consider the distribution that maximizes the entropy over the set of distributions $\mathcal{D}$. The entropy of a probability distribution represents the amount of uncertainty associated with the distribution. The distribution that maximizes the entropy is thus a good prior distribution because it is the "maximally noncommittal with regard to missing information" (Jaynes 1957). In fact, the entropy is similar to a barrier function (Boyd and Vandenberghe 2004) and can therefore be seen as a way to approach the analytical center of $\mathcal{D}$.

Classical examples of entropy-maximizing distributions are the following: the uniform distribution when only the range of the distribution is known (consistent with the principle of insufficient reason); the exponential distribution when the distribution is known to be nonnegative and have a certain mean; and the normal distribution when the distribution has known mean and variance (but not necessarily nonnegative).

3. Methodology

In this section, we propose a methodology for solving the minimax regret problem (3). In particular, we formulate the problem of identifying the worst-case demand scenario as a moment problem. Problem (3) can be reformulated as follows, by inverting the order of maximization:

$$\begin{aligned}\min_{y\geqslant 0}\rho(y)&=\min_{y\geqslant 0}\max_{z\geqslant 0}\left\{\max_{F\in\mathcal{D}}\Pi_F(z)-\Pi_F(y)\right\}\\&=\min_{y\geqslant 0}\max_{z\geqslant 0}\left\{\max_{F\in\mathcal{D}}\int_0^\infty(\min\{x,z\}\right.\\&\qquad\left.-\min\{x,y\})\,dF(x)\right\}+\beta(y-z).\end{aligned}$$

Let $\Omega\subset\mathcal{R}$ be the known support of the distribution, and $\mathcal{B}$ be a σ-algebra of measurable subsets of Ω. We assume that the decision maker knows the n first moments of the distribution, $q_1, \ldots, q_n$. For convenience, we explicitly enforce the normalization requirement by taking $q_0 = 1 = \int_\Omega dF(x)$. Hence, any distribution $F \in \mathscr{D}$ is defined on $(\Omega, \mathscr{B})$ and must satisfy $\int_\Omega x^i dF(x) = q_i$ for all $i = 0, \ldots, n$. We also assume that certain Slater conditions hold on the moment constraints (specifically, that the moment vector is interior to the set of feasible moments). The inner problem, consisting of finding the distribution that maximizes the regret for given y and z, can then be formulated as follows:

$$\begin{aligned} &\max_{F \in \mathscr{D}} \int_\Omega (\min\{x, z\} - \min\{x, y\})\, dF(x) \\ &\text{s.t. } \int_\Omega x^i dF(x) = q_i \quad \forall i = 0, \ldots, n. \end{aligned} \tag{4}$$

Because the normalization and moment constraints are explicitly enforced, problem (4), defined over the convex set $\mathscr{D}$ of probability measures, can be relaxed over the corresponding cone of $\mathscr{C}$ (i.e., the set of nonnegative measures defined on $(\Omega, \mathscr{B})$ satisfying the same "shape" properties as the probability distributions in $\mathscr{D}$). By strong duality (under Slater's conditions), problem (4) is equivalent to the following dual problem (Popescu 2005):

$$\begin{aligned} &\min_{\alpha_0, \ldots, \alpha_n} \sum_{i=0}^{n} \alpha_i q_i \\ &\text{s.t. } \sum_{i=0}^{n} \alpha_i x^i - (\min\{x, z\} - \min\{x, y\}) \in \mathscr{C}^*, \end{aligned} \tag{5}$$

where $\mathscr{C}^*$ is the polar of $\mathscr{C}$. In particular, when $\mathscr{D}$ is the set of nonnegative distributions with support Ω, the dual problem (5) simplifies to a semi-infinite linear optimization problem. When the problem has an optimal solution, there exists an optimal solution that is a convex combination of $n+1$ "basic" probability measures. Examples of such "basic" probability measures include Diracs when $\mathscr{D}$ has only moment constraints (Smith 1995; Bertsimas and Popescu 2002, 2005); pairs of symmetric Diracs when $\mathscr{D}$ is restricted to symmetric probability distributions; and uniform distributions, with one of the boundaries at M, when $\mathscr{D}$ is restricted to unimodal distributions with mode at M (see Popescu 2005 for details).

Bertsimas and Popescu (2002, 2005) showed that problem (4), when $\mathscr{D}$ has only moment constraints, can be formulated as a semidefinite optimization problem and be therefore efficiently solved. Similar results hold for the general conic dual problem under certain conditions (Popescu 2005). However, in this paper, we solve these problems in closed form. The next proposition characterizes the optimal solution to problem (3).

Proposition 1. (a) *The function*

$$G(z; y) = \max_{F \in \mathscr{D}} \{\Pi_F(z) - \Pi_F(y)\}$$

is concave on the interval $z \in [0, y]$ *and on the semi-interval* $z \in [y, \infty)$, *but not necessarily on* $[0, \infty)$.

(b) *The function* $\rho(y)$ *is convex. Moreover,*

$$y^* \doteq \arg\min \rho(y)$$

is such that

$$\max_{z \in [0, y^*]} G(z; y^*) = \max_{z \in [y^*, \infty)} G(z; y^*).$$

Part (a) states that the optimization problem over z is not a concave problem, unless we distinguish the two cases $y \leqslant z$ and $y \geqslant z$. According to part (b), the quantity y^* that minimizes the maximum regret is such that the regret of ordering too little is equal to the regret of ordering too much.

4. Derivation of Minimax Regret Order Quantities

In this section, we derive the minimax regret order quantities, with limited demand information.

Range. We first assume that only the range (or support) of the demand distribution is known. The support of the demand distribution can be viewed as a "budget of uncertainty" (Bertsimas and Sim 2003). By restricting the size of the interval, the decision maker adjusts the amount of variability she wants to cover with her decision. The minimax regret quantity was first derived by Morris (1959), but we provide a new proof of this result.

Theorem 1 (Morris 1959, Kasugai and Kasegai 1961, Vairaktarakis 2000). *If the demand distribution is nonnegative, with support* $[A, B]$, *the minimax regret order quantity is equal to*

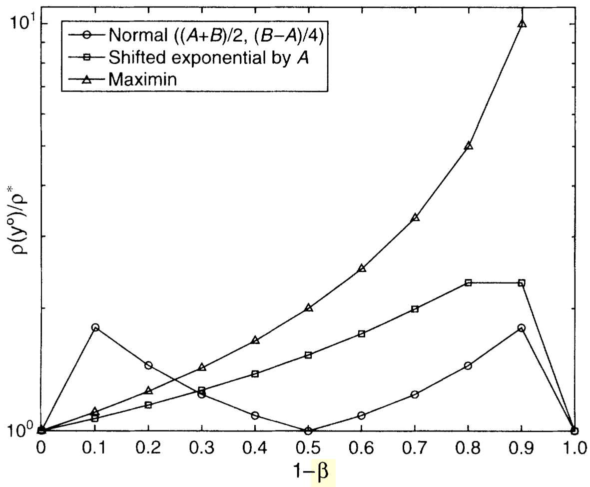
$$y^* = \beta A + (1-\beta)B \tag{6}$$

and $\rho^* = \beta(1-\beta)(B-A)$.

The maximum regret is attained with two Diracs, at A and at B. In contrast, only the low-demand scenario is relevant to the maximin decision (when the lost sales cost l is zero).

In reality, one needs to specify a demand distribution to use the newsvendor model (1). With a uniform distribution, the newsvendor solution coincides with (6), and the maximum regret equals ρ^*, consistent with the principle of insufficient reason (or the maximum entropy) that suggests a uniform prior when only the range is known. Figure 1 compares the minimax regret ρ^* to the maximum regret $\rho(y^\circ) = \max\{(1-\beta)(B-y^\circ), \beta(y^\circ - A)\}$, where y° is the maximin order quantity (equal to A when $l = 0$), or the newsvendor solution when the demand distribution is normal with $[A, B] = [\mu - 2\sigma, \mu + 2\sigma]$ or exponential, shifted by A and with B corresponding to the 95th percentile. It turns out that the ratios $\rho(y^\circ)/\rho^*$ are independent of the specific values of A and B. The maximum ratios are equal to one for the uniform distribution, 1.872 for the normal distribution, and 2.456 for the shifted exponential distribution. Interestingly, the newsvendor solution with a normal distribution is less robust, from a regret perspective, than the maximin solution, for small profit margins.

Figure 1. Comparison (on a logarithmic scale) between the minimax regret ρ^* and the maximum regret $\rho(y^\circ)$ when the range is known, where y° is the maximin quantity, or the newsvendor solution derived with a normal or an exponential distribution.



Mean. Suppose that the newsvendor knows only the mean of the demand distribution. Indeed, one often needs to make order decisions based on single-point forecasts. The next theorem derives the associated minimax regret order quantity. The next result can also be extended to the case when the newsvendor knows both the mean and the support of the distribution (see Zhu et al. 2006 and Roels 2006).

THEOREM 2. *If the demand distribution is nonnegative with mean μ, the minimax regret order quantity is equal to*

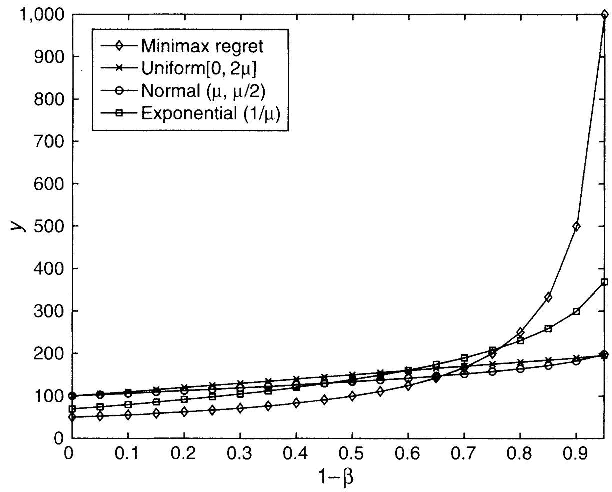
$$y^* = \begin{cases} \mu(1-\beta) & \text{if } \frac{1}{2} \leqslant \beta, \\ \dfrac{\mu}{4\beta} & \text{if } \frac{1}{2} \geqslant \beta, \end{cases} \tag{7}$$

and the minimax regret equals

$$\rho^* = \begin{cases} \beta(1-\beta)\mu & \text{if } \frac{1}{2} \leqslant \beta, \\ \dfrac{1}{4}\mu & \text{if } \frac{1}{2} \geqslant \beta. \end{cases}$$

Similar to the case where only the range is known, the maximum value of the regret is attained with two extreme distributions. The low-demand scenario is a two-point distribution with probability $1-\epsilon$ at zero, and probability ϵ at μ/ϵ, where $\epsilon \to 0$. This distribution also fully characterizes the worst-case profit of the maximin criterion, when $l = 0$, leading to a maximin order of zero. The high-demand scenario depends on the value of β: when $\beta \geqslant 1/2$, it is a unit impulse at μ (i.e., a deterministic demand); when $\beta \leqslant 1/2$, it is a two-point distribution, with probability $1-2\beta$ at zero, and probability 2β at $\mu/(2\beta)$.

Figure 2. Comparison between the minimax regret quantity and the newsvendor solutions obtained with a uniform, a normal, and an exponential distribution.

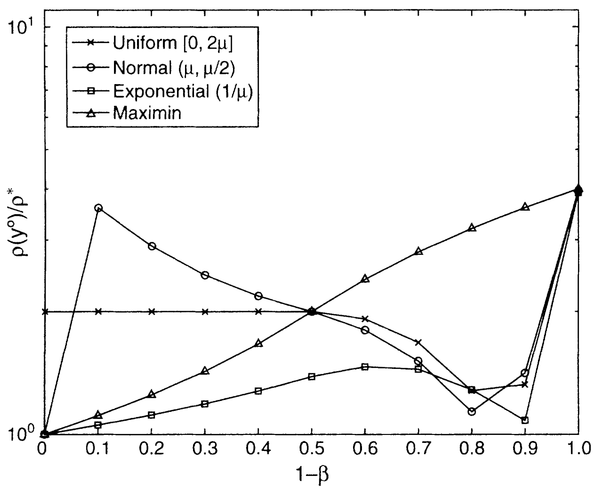


It is common to use a uniform, normal, or exponential distribution in the newsvendor model (1). In fact, the exponential distribution is expected to be the most robust, given that it is entropy-maximizing over the class of nonnegative distributions with known mean. Figure 2 compares the minimax regret order quantity y^* to the newsvendor solutions with a uniform distribution over $[0, 2\mu]$, a normal distribution with a coefficient of variation of 1/2, and an exponential distribution. The minimax regret quantity is typically smaller than the newsvendor solutions when $1-\beta \leqslant 0.7$ and grows to infinity when $1-\beta \to 1$. In fact, the three distributions have finite variance, in contrast to the distribution that attains the maximum regret, and therefore give rise to smaller order quantities when the profit margin gets large. Evidently, tighter results can be obtained when the variance of the demand distribution is known or bounded from above (see Theorem 8).

Figure 3 compares the minimax regret ρ^* to the maximum regret $\rho(y^\circ)$ associated with the maximin solution, and the newsvendor solutions with uniform, normal, and exponential distributions. Consistent with the entropy principle, the exponential distribution is very robust when only the mean is known, at least when $1-\beta \leqslant 0.9$: in this range of profit margins, the maximum ratios $\rho(y^\circ)/\rho^*$ equal 1.471 with the exponential distribution, 2 with the uniform distribution, and 3.744 with the normal distribution.

The minimax regret order quantity is guaranteed to not perform too badly in the worst case. But how does it perform on average? Table 1 displays the mean loss of profit incurred with (7) when the decision maker knows only the mean demand, equal to 100, for three levels of profit margins $1-\beta$: 0.2, 0.4, and 0.6. We assumed that the underlying demand distribution, unknown to the decision maker, was beta with shape parameters a and b and its probability density function (p.d.f) had one of the following

Figure 3. Comparison (on a logarithmic scale) between the minimax regret ρ^* and the maximum regret $\rho(y^\circ)$ when the mean is known, where y° is the newsvendor solution with a uniform, a normal, or an exponential distribution.



shapes: U-shaped ($a \in (0,1)$, $b \in (0,1)$), strictly decreasing ($a = 1$, $b \in (1,3]$), bell-shaped ($a \in (1,3]$, $b \in (1,3]$), and strictly increasing ($a \in (1,3]$, $b = 1$). For each profit margin and shape of p.d.f., we randomly picked 300 values for a and b and computed the associated optimal order quantity y°, the maximum expected profit $\Pi(y^\circ)$, and the expected profit with (7), $\Pi(y^*)$. All quantities were scaled by $100(a+b)/a$ to make the mean demand equal to 100. We then computed the 95% confidence intervals for the newsvendor solutions y° (averaged over all demand distribution samples) and the profit loss from ordering the minimax regret quantity (7) instead of y°, that is, $\Pi(y^\circ) - \Pi(y^*)$ (averaged over all demand distribution samples). The bottom line of the table shows the values of the minimax regret quantity (7) and the minimax regret ρ^*.

The minimax regret approach performs best when the profit margins are close to 0.5. When the profit margins increase, both the worst-case bound and the average profit losses increase. Indeed, the minimax regret quantity is typically smaller than the newsvendor solutions (see Figure 2), making it unable to reap large profits when demand is high. The worse performance of the minimax regret approach when the profit margins are small is more interesting because it contrasts with the worst-case bound behavior. In fact, there is a higher level of uncertainty about the order quantity (both within a particular p.d.f. class and among different classes) when the profit margins are small. The profit losses incurred with (7) are accordingly larger and also more variable in this uncertain environment.

Interestingly, the minimax regret exhibits the worst performance with a bell-shaped distribution, which is arguably the most common type of distribution. Nevertheless, the average profit loss is still significantly smaller than the worst-case theoretical bound.

Mean and Median. We now assume that the median m of the demand distribution is known, in addition to the mean μ. By definition, there is a 50% chance that the demand realization falls below m and a 50% chance that it is above m. For many continuous unimodal distributions (e.g., the Pearson family), the mean is right of the median under right skew, and left of the median under left skew; this property is, however, not satisfied with more general distributions, including multimodal and discrete distributions (see von Hippel 2005).

THEOREM 3. *If the distribution is nonnegative with mean μ and median m, the minimax regret order quantity is equal to the following if $\beta \geqslant 1/2$:*

$$y^* = \begin{cases} 2m(1-\beta) & \text{if } \mu \geqslant m, \\ 2(1-\beta)(2\mu - m) & \text{if } \mu \leqslant m \text{ and } \beta \geqslant \frac{3}{4}, \\ \dfrac{2\mu - m}{4(2\beta - 1)} & \text{if } \mu \leqslant m \text{ and } \dfrac{3}{4} \geqslant \beta \geqslant \dfrac{1}{4} + \dfrac{\mu}{2m}, \\ 2m\dfrac{\mu - \beta m}{2\mu - m} & \text{if } \mu \leqslant m \text{ and } \beta \leqslant \dfrac{1}{4} + \dfrac{\mu}{2m}, \end{cases}$$

and $\rho^* = y^*(\beta - 1/2)$.

When $\beta \leqslant 1/2$, the minimax regret order quantity is equal to the following:

$$y^* = \begin{cases} 2\mu + 2\beta(m - 2\mu) & \text{if } \beta \geqslant \frac{1}{4}, \\ m + \dfrac{2\mu - m}{8\beta} & \text{if } \dfrac{1}{4} \geqslant \beta, \end{cases}$$

and $\rho^* = \beta(y^* - m)$.

The maximum regret is attained with two extreme demand distributions. When $\beta \geqslant 1/2$, the low-demand demand

Table 1. Mean optimal order quantities y° and mean profit loss from ordering (7) instead of y° under different demand scenarios, when only the mean is known ($\mu = 100$).

$1-\beta$	0.2		0.4		0.6	
Demand p.d.f.	y°	$\Pi(y^\circ) - \Pi(y^*)$	y°	$\Pi(y^\circ) - \Pi(y^*)$	y°	$\Pi(y^\circ) - \Pi(y^*)$
U-shaped	30 ± 4	3.8 ± 0.5	32 ± 0	0.5 ± 0.0	56 ± 1	3.6 ± 0.2
Decreasing	63 ± 5	6.9 ± 0.8	69 ± 1	2.2 ± 0.1	85 ± 1	7.3 ± 0.3
Bell-shaped	99 ± 5	9.2 ± 0.8	111 ± 0	5.5 ± 0.1	113 ± 0	9.3 ± 0.3
Increasing	161 ± 6	5.3 ± 0.4	165 ± 0	2.7 ± 0.0	143 ± 1	1.1 ± 0.1
y^*, ρ^*	20	16	40	24	62	25

Table 2. Comparison between the minimax regret quantity y^* and the newsvendor solution y°, with a gamma, log-normal, or negative binomial demand distributions, in terms of absolute difference in order quantities and ratios of regrets when the mean and the median are known ($\mu = 100$).

		Gamma			Log-normal			Negative binomial		
$1-\beta$	m	$\lvert y^\circ - y^*\rvert$	p.v. (%)	$\rho(y^\circ)/\rho^*$	$\lvert y^\circ - y^*\rvert$	p.v. (%)	$\rho(y^\circ)/\rho^*$	$\lvert y^\circ - y^*\rvert$	p.v. (%)	$\rho(y^\circ)/\rho^*$
[0, 0.95]	10	12.1	0	1.21	23.8	0	1.47	11.8	0	1.21
	50	10.3	1	1.22	7.3	6	1.13	10.2	1	1.21
	80	6.6	83	1.11	15.5	0	1.34	6.7	83	1.11
[0.95, 1]	10	2,022	0	1.40	1,628	0	1.43	1,606	0	1.41
	50	1,516	0	1.72	1,410	0	1.64	1,520	0	1.73
	80	2,589	0	2.11	2,584	0	2.12	2,591	0	2.11

scenario is a two-point distribution, with equal probabilities at zero and at 2μ. The high-demand scenario depends on the value of β and on the relative order between μ and m: if $\mu \geqslant m$, it is a two-point distribution, with probability $1-\epsilon$ at m, and probability ϵ at $m + (\mu - m)/\epsilon$, when $\epsilon \to 0$; if $m \geqslant \mu$ and $\beta \geqslant 3/4$, it is a two-point distribution, with equal probabilities at $2\mu - m$ and at m; if $m \geqslant \mu$ and if $1/4 + \mu/(2m) \leqslant \beta \leqslant 3/4$, it is a three-point distribution, with probability $3/2 - 2\beta$ at zero, probability $2\beta - 1$ at $(2\mu - m)/(4\beta - 2)$, and probability $1/2$ at m; finally, if $m \geqslant \mu$ and $1/2 \leqslant \beta \leqslant 1/4 + \mu/(2m)$, it is a two-point distribution, with probability $1 - \mu/m$ at zero, and μ/m at m. The worst-case distributions when $\beta \leqslant 1/2$ are similar.

The demand distribution for slow-moving items is often asymmetric, with the median less than the mean, because demand occurs sporadically. Accordingly, the gamma, log-normal, and negative binomial distribution are often used to model right-skewed demand distributions (Fortuin 1980). Table 2 evaluates the robustness of the newsvendor solutions obtained with these distributions, when only the mean and the median are known. In this numerical study, we fixed the mean μ to 100, and considered three values for the median m: 10, 50, and 80. For each case, we randomly chose 300 values for the profit margin $1-\beta$, uniformly distributed over [0, 0.95] or over [0.95, 1]. Table 2 reports the mean absolute difference in order quantities, $|y^\circ - y^*|$, as well as the p-value (p.v.) of a two-sample Kolmogorov-Smirnov (KS) test. (The p-value represents the confidence level with which the two distributions of order quantities can be considered as equal.) The robustness of a particular distribution is measured by the ratio of the maximum regret $\rho(y^\circ)$ to the minimax regret ρ^*, when both quantities are averaged over the 300 random values of β.

As shown in Table 2, both the gamma and the negative binomial distribution are robust, in terms of regret, when $\beta \geqslant 0.05$. The order quantities under these distributions are typically within a few units from the minimax regret quantities (especially when $m = 80$, where the K-S hypothesis cannot be rejected), and there is only a 10%–20% increase in the mean regret. However, when $\beta \leqslant 0.05$, there is a significant difference in order quantities; yet, the increase in regret remains moderate. In fact, the gamma, log-normal, and negative binomial distributions have finite variance, unlike the distribution that attains the minimax regret, giving rise to smaller order quantities when $\beta \leqslant 0.05$ (similar to Figure 2); nevertheless, profits are not significantly affected by the poor choice of order quantity because β is small.

The next corollary derives the minimax regret order quantity when the mean equals the median, as for the normal and uniform distributions.

Corollary 1. *If the distribution is nonnegative with its mean μ equal to the median, the minimax regret order quantity is equal to*

$$y^* = \begin{cases} 2(1-\beta)\mu & \text{if } \beta \geqslant \frac{1}{4}, \\ \mu + \mu\dfrac{1}{8\beta} & \text{if } \beta \leqslant \dfrac{1}{4}. \end{cases}$$

The additional requirement that the mean equals the median makes the order quantity larger than (7) if $\beta \geqslant 1/8$. In particular, it is now optimal to order more than the mean demand when $1-\beta \geqslant 0.5$, instead of when $1-\beta \geqslant 0.25$.

In practice, the normal, uniform, and Poisson distributions are often selected as priors in the newsvendor model when the mean and the median are known to be equal. Figure 4 measures the robustness of the following distributions: normal with mean 100 and standard deviation 100/3, uniform between zero and 200, Poisson with mean 100, and a U-shaped symmetric beta with parameters $a = b = 0.5$, multiplied by 200. The robustness of a distribution is measured by the ratio of the maximum regret associated with the newsvendor solutions $\rho(y^\circ)$ to the minimax regret ρ^*.

When the mean is known to be equal to the median, the uniform outperforms the other distributions and exhibits a good degree of robustness when $1-\beta \leqslant 0.9$. Somewhat surprisingly, the U-shaped beta distribution also exhibits a good level of robustness, for small profit margins. In general, there is a smaller loss of robustness around the median. Effectively, when $1-\beta = 0.5$, one knows with certainty that one must order the mean/median, independently of the shape of demand distribution. Similarly to

Figure 4. Comparison (on a logarithmic scale) between the minimax regret ρ^* and the maximum regret $\rho(y^\circ)$ when the mean is known to be equal to the median, where y° is the newsvendor solution with a uniform, a normal, a Poisson, or a U-shaped beta distribution.

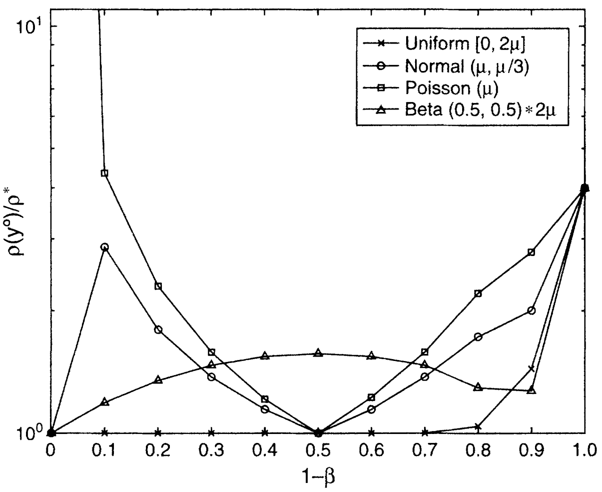


Figure 3, the regret with the newsvendor solutions is four times larger than the minimax regret when the profit margins are high. Note also that the Poisson distribution leads to larger regrets than the normal distribution because its coefficient of variation is smaller.

Mean and Symmetry. We restrict further the class of distributions considered by assuming symmetry. A symmetric demand distribution gives the same probability to $\mu - x$ and $\mu + x$ for all $0 \leqslant x \leqslant \mu$. As shown in the next theorem, the robust order quantity is the same as in Corollary 1, when $\beta \geqslant 1/4$.

THEOREM 4. *When the demand distribution is known to be nonnegative, symmetric, with mean μ, the minimax regret order quantity is equal to*

$$y^* = 2\mu(1-\beta),$$

and the minimax regret amounts to

$$\rho^* = \begin{cases} \mu(2\beta-1)(1-\beta) & \text{if } 1/2 \leqslant \beta, \\ \mu(1-2\beta)\beta & \text{if } 1/2 \geqslant \beta. \end{cases}$$

From the nonnegativity of demand, a symmetric demand distribution with mean μ is bounded from above by 2μ. Interestingly, the minimax regret order quantity with a symmetric demand is the same as the newsvendor solution with a uniform demand distribution. Intuitively, the uniform distribution is "in the middle" of all symmetric distributions. At one extreme, we have the two-point distribution, giving positive probability to both 0 and 2μ, which also attains the maximin profit; the other extreme is the deterministic case, with all probability mass at μ, which also characterizes the maximax profit.

The order quantity derived in Theorem 4 is the same as the one derived in Theorem 1, when the demand distribution is known to have its support on $[0, 2\mu]$. However, the value of the minimax regret is lower in the case of symmetry.

Unimodality, Mode, and Range. We now assume that the demand distribution is unimodal, with its mode equal to M. A unimodal c.d.f. with mode M is convex to the left of M and concave to the right of M. If the distribution is continuous, the p.d.f. is increasing to the left of the mean and decreasing to the right.

THEOREM 5. *If the demand distribution has its support on $[A, B]$, and is unimodal with mode M, the minimax regret order quantity is*

$$y^* = \begin{cases} A + \sqrt{(M-A)(1-\beta)(B(1-\beta) - A(1+\beta) + 2\beta M)} \\ \quad \text{if } M(1-2\beta(1-\beta)) \geqslant \beta^2 A + (1-\beta)^2 B, \\ B - \sqrt{\beta(B-M)(B(2-\beta) - \beta A - 2M(1-\beta))} \\ \quad \text{if } M(1-2\beta(1-\beta)) \leqslant \beta^2 A + (1-\beta)^2 B, \end{cases} \tag{8}$$

and the minimax regret equals

$$\rho^* = \begin{cases} (1-\beta)\left(\dfrac{B+M}{2} - \beta\dfrac{B-M}{2} - y^*\right) \\ \quad \text{if } M(1-2\beta(1-\beta)) \geqslant \beta^2 A + (1-\beta)^2 B, \\ \beta\left(y^* - M + \beta\dfrac{M-A}{2}\right) \\ \quad \text{if } M(1-2\beta(1-\beta)) \leqslant \beta^2 A + (1-\beta)^2 B. \end{cases}$$

The worst-case distributions that maximize the regret are uniform. For example, when $y \leqslant M$, the low-demand and high-demand scenarios are uniform distributions, over $[A, M]$ and $[M, B]$, respectively. Therefore, when $B \to \infty$, the regret becomes infinite, and the minimax regret quantity is not well defined.

Table 3 displays the performance of the minimax regret approach under various types of beta p.d.f.s, with mode $M = 100$ and parameters a and b: strictly decreasing ($a = 1, b \in (1, 3]$), bell-shaped ($a = 2, b \in (1, 3]$), and strictly increasing ($a \in (1, 3], b = 1$). The range was taken as $[0, 300]$. For each profit margin and shape of p.d.f., we randomly picked 300 values for a and b and computed the associated optimal order quantity, the maximum expected profit, and the expected profit with (8). All distributions were scaled or shifted to make the mode equal to 100. We then computed the 95% confidence intervals for the newsvendor solution y° (averaged over all sampled distributions), and the profit loss resulting from ordering the minimax regret quantity (8) instead of the optimal newsvendor solution y°, that is, $\Pi(y^\circ) - \Pi(y^*)$ (averaged over all sampled distributions). The bottom line shows the minimax regret quantities y^* and the values of the minimax regret ρ^*.

Table 3. Mean newsvendor solution y° and mean profit loss from ordering (8) instead of y° under different demand scenarios, when only the mode and the range are known ($M = 100$, $A = 0$, $B = 300$).

$1-\beta$	0.2		0.4		0.6	
Demand p.d.f.	y°	$\Pi(y^\circ)-\Pi(y^*)$	y°	$\Pi(y^\circ)-\Pi(y^*)$	y°	$\Pi(y^\circ)-\Pi(y^*)$
Decreasing	122 ± 1	8.9 ± 0.1	149 ± 2	10.2 ± 0.3	176 ± 2	7.7 ± 0.5
Bell-shaped	56 ± 1	0.5 ± 0.1	84 ± 1	1.6 ± 0.3	110 ± 2	3.6 ± 0.5
Increasing	43 ± 1	3.6 ± 0.3	62 ± 1	9.6 ± 0.3	76 ± 1	16.8 ± 0.2
y^*, ρ^*	66	11	98	17	130	21

As shown in Table 3, the profit losses from ordering (8) instead of the newsvendor solution (had we had full information about the demand distribution) are the largest with decreasing p.d.f.s when the profit margins are small, and increasing p.d.f.s when the profit margins are large, consistent with the distributions characterizing the worst case. Interestingly, the regret performs best under bell-shaped distributions, in contrast to the case where only the mean is known (see Table 1).

Unimodality, Mode, and Median. The next theorem derives the minimax regret order quantity for unimodal distributions with known mode and median. By having the mode left (respectively, right) to the median, one can expect the distribution to be right (respectively, left) skewed, although there are exceptions to the rule (e.g., the Weibull).

THEOREM 6. *If the demand distribution is unimodal with mode M and has its median at m, with $2m \geqslant M$, the minimax regret order quantity is*

$$y^* = \begin{cases} m+(1-2\beta)\sqrt{m(M-m)} \\ \quad \text{if } m \leqslant M,\ 1-M/(2m) \leqslant \beta \leqslant 1/2, \\ 2m\sqrt{\beta(1-\beta)} \quad \text{if } m \leqslant M,\ \beta \geqslant 1/2, \\ 2\sqrt{(1-\beta)M(2\beta M-\beta m+m-M)} \\ \quad \text{if } M \leqslant m \leqslant M\dfrac{8\beta^2-12\beta+5}{4(\beta-1)^2},\ \beta \geqslant 1/2, \\ m-\sqrt{(m-M)(2\beta-1)(4\beta M-2\beta m-4M+3m)} \\ \quad \text{if } m \geqslant M\dfrac{8\beta^2-12\beta+5}{4(\beta-1)^2},\ \beta \geqslant 1/2, \\ \text{not defined}, \quad \text{otherwise}, \end{cases}$$

and the minimax regret amounts to

$$\rho^* = \begin{cases} (1-2\beta)^2(M-2\sqrt{m(M-m)})\frac{1}{4} \\ \quad \text{if } m \leqslant M,\ 1-M/(2m) \leqslant \beta \leqslant 1/2, \\ m(1-2\sqrt{\beta(1-\beta)})(1-\beta) \quad \text{if } m \leqslant M,\ \beta \geqslant 1/2, \\ (m-\beta(m-M)-y^*)(1-\beta) \\ \quad \text{if } M \leqslant m \leqslant M\dfrac{8\beta^2-12\beta+5}{4(\beta-1)^2},\ \beta \geqslant 1/2, \\ (2y^*-3M+2\beta M)(2\beta-1)\frac{1}{4} \\ \quad \text{if } m \geqslant M\dfrac{8\beta^2-12\beta+5}{4(\beta-1)^2},\ \beta \geqslant 1/2, \\ \text{not defined}, \quad \text{otherwise}. \end{cases}$$

The worst-case distributions that maximize the regret are mixtures of uniform distributions with boundaries at zero, m, and M. When the median is greater than the mode and $\beta \leqslant 1/2$, or when the median is less than the mode and $\beta \leqslant 1 - M/(2m)$, the minimax regret is not defined because the regret is infinite, independently of the order quantity.

The gamma, log-normal, and negative binomial distributions are often used to model skewed demand distributions. Table 4 reports the mean absolute difference in order quantities, between the minimax regret and the gamma, lognormal, and negative binomial distributions, as well as the p-value of the KS test. We assumed that $m = 100$ and considered three values for the mode M: 10, 50, and 80. For each case, we randomly chose 300 different profit margins in the interval [0, 0.5]. (The minimax regret order quantity is not defined for larger profit margins.) Table 4 also displays the increase in regret by comparing the minimax regret ρ^* to the maximum regret $\rho(y^\circ)$ associated with the newsvendor solutions.

Among those three distributions, the gamma and the lognormal are the most robust, especially when the mode M is close to the median m. Otherwise, they approximate poorly the minimax regret order quantity because almost all KS tests are rejected, and the maximum regret associated with these distributions is about twice as large as ρ^*. Therefore, when the demand distribution is known to be skewed, the minimax regret approach can lead to significant profit improvements over the newsvendor solutions.

When the mode equals the median, the expression for the minimax regret order quantity greatly simplifies, as shown in the next corollary.

COROLLARY 2. *If the demand distribution is unimodal with mode and median m, the minimax regret order quantity is*

$$y^* = \begin{cases} 2m\sqrt{\beta(1-\beta)} & \text{if } \beta \geqslant 1/2, \\ \text{not defined}, & \text{otherwise}. \end{cases}$$

Mean, Unimodality, and Symmetry. When the mode is equal to the median, the minimax regret quantity is not defined for high-margin products (see Corollary 2). In contrast, when the distribution is symmetric, the regret is always finite. Although the joint requirement for symmetry and unimodality might seem restrictive, many distributions satisfy these conditions: uniform, truncated normal,

Table 4. Comparison between the minimax regret quantity y^* and the newsvendor solution y°, with a gamma, log-normal, or negative binomial demand distributions, when the median and the mode are known ($m = 100$).

		Gamma			Log-normal			Negative binomial		
$1-\beta$	M	$\lvert y^\circ - y^*\rvert$	p.v. (%)	$\rho(y^\circ)/\rho^*$	$\lvert y^\circ - y^*\rvert$	p.v. (%)	$\rho(y^\circ)/\rho^*$	$\lvert y^\circ - y^*\rvert$	p.v. (%)	$\rho(y^\circ)/\rho^*$
[0, 0.5]	10	19.1	0	2.45	14.0	0	2.02	19.8	0	3.21
	50	6.7	0	1.33	7.4	0	1.48	13.7	0	2.7
	80	2.9	38	1.19	4.2	3	1.29	13.1	0	2.60

and Poisson if μ is large. Moreover, no assumption is made about the variance.

THEOREM 7. *If the distribution is known to be nonnegative, symmetric, unimodal, with mean μ, the minimax regret order quantity is equal to*

$$y^* = \begin{cases} 2\mu\sqrt{\beta(1-\beta)} & \text{if } \frac{1}{2} \leqslant \beta, \\ 2\mu(1-\sqrt{\beta(1-\beta)}) & \text{if } \frac{1}{2} \geqslant \beta, \end{cases} \tag{9}$$

and the minimax regret equals

$$\rho^* = \begin{cases} (1-\beta)\mu(1-2\sqrt{\beta(1-\beta)}) & \text{if } \frac{1}{2} \leqslant \beta, \\ \beta\mu(1-2\sqrt{\beta(1-\beta)}) & \text{if } \frac{1}{2} \geqslant \beta. \end{cases}$$

The maximum regret is attained with two distributions: a uniform distribution over $[0, 2\mu]$ (instead of the two-point distribution in Theorem 4), and a unit impulse at μ (i.e., deterministic demand). Incidentally, these two distributions also characterize the maximin and the maximax solutions, respectively.

The minimax regret quantity is identical to the newsvendor solution with the following unimodal symmetric distribution:

$$F(y) = \begin{cases} \dfrac{1}{2} - \dfrac{1}{2}\sqrt{1-\left(\dfrac{y}{\mu}\right)^2}, & 0 \leqslant y \leqslant \mu, \\ \dfrac{1}{2} + \dfrac{1}{2}\sqrt{1-4\left(1-\dfrac{y}{2\mu}\right)^2}, & \mu \leqslant y \leqslant 2\mu, \end{cases} \tag{10}$$

obtained by equating $1-\beta$ with $\bar{F}(y)$ in Theorem 7, in the same way the minimax regret quantity is identical to the newsvendor solution with a uniform distribution when only the range is known (see Theorem 1). Hence, even though the worst-case distributions are uniform or unit impulses, the probability distribution associated with the minimax regret order quantity is more reminiscent of the normal or the Laplace distribution. One can check that the distribution (10) has mean μ and standard deviation $\mu\sqrt{5/3 - \pi/2} \approx 0.3096\mu$.

The normal and the uniform distributions are often considered when the demand distribution is known to be symmetric and unimodal. Figure 5 compares the minimax regret ρ^* to the maximum regret $\rho(y^\circ)$ obtained with the newsvendor model using a uniform distribution over $[0, 2\mu]$ or a normal distribution with a coefficient of variation of 1/3, as well as that obtained when ordering the mean demand (i.e., deterministic demand). Similar to Figures 1 and 3, the ratios are scale-free, i.e., do not depend on μ. Although the newsvendor solutions are close to the minimax regret quantity, their regret is significantly larger than ρ^*, especially when $1-\beta$ tends to 0.5 (despite the fact that it is optimal to order the mean demand, independently of the shape of the distribution, when $\beta = 0.5$). In fact, the chosen uniform and normal distributions have a larger coefficient of variation than the distribution associated with the minimax regret solution (10), thereby making the regret tend to zero slower than ρ^*. Therefore, considering a coefficient of variation larger than 0.31 for symmetric unimodal distributions is overly conservative, and will distort the

Figure 5. Comparison (on a logarithmic scale) between the minimax regret ρ^* and the maximum regret $\rho(y^\circ)$ when the distribution is known to be unimodal and symmetric, where y° is the newsvendor solution derived with a uniform, a normal, or a deterministic distribution.

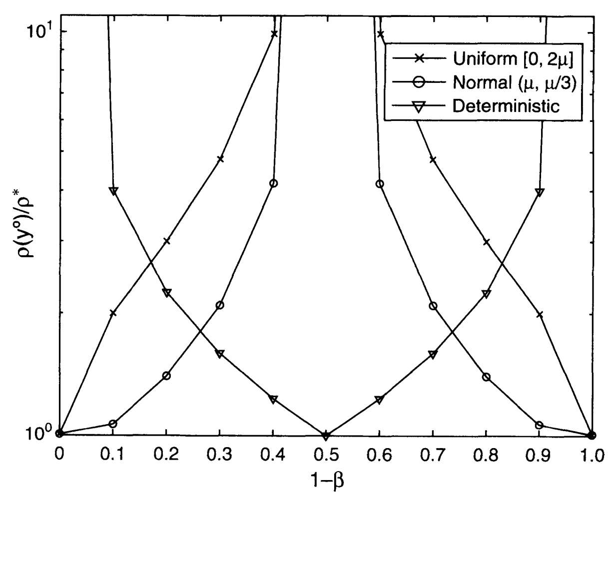
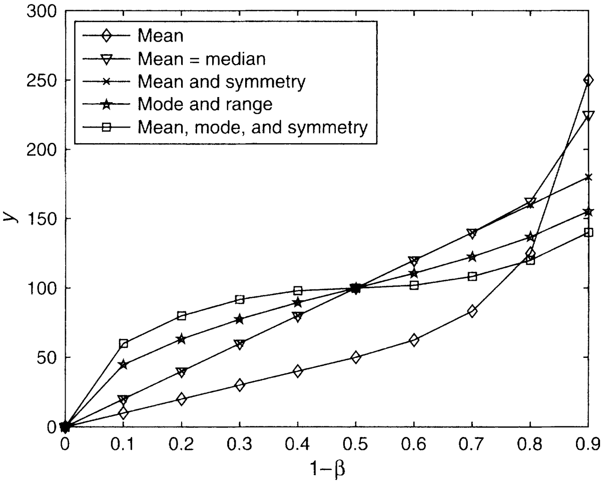


Figure 6. Minimax regret order quantities when the mean is known, in addition to the shape of the distribution ($\mu = 100$).



order quantities upwards when $\beta \leqslant 1/2$, and downwards when $\beta \geqslant 1/2$.

Figure 6 displays the evolution of the minimax regret order quantity as additional information is available about the shape of the distribution. As the distribution is constrained to be more "regular," the order quantity is shifted about the mean. Intuitively, the conditions of symmetry and unimodality tend to accumulate more probability mass around the mean, thereby making the optimal order quantity closer to the mean demand. Similarly, for the normal distribution, the order quantity becomes closer to the mean demand as the coefficient of variation falls. This suggests that joint efforts could be devoted to both reducing forecast error variability and better characterizing the shape of the demand distribution.

Mean and Variance. The next theorem characterizes the minimax regret when the mean and variance are known.

THEOREM 8. *If the demand distribution is known to be nonnegative, with mean μ and variance σ^2, the minimax regret order quantity y^* is the solution to the following equality:*

$$\max\left\{\max_{\max\{\mu,\,y\}\leqslant x\leqslant(\sigma^2+\mu^2)/\mu}\left(\frac{\mu}{x}-\beta\right)(x-y),\right.$$
$$\left.\max_{\max\{y,\,(\sigma^2+\mu^2)/\mu\}\leqslant x\leqslant y+\sqrt{\sigma^2+(y-\mu)^2}}\left(\frac{\sigma^2}{\sigma^2+(x-\mu)^2}-\beta\right)(x-y)\right\}$$
$$=\left\{\max_{\min\{\mu,y\}\geqslant x\geqslant\max\{0,\,y-\sqrt{\sigma^2+(\mu-y)^2}\}}\left(\frac{(x-\mu)^2}{\sigma^2+(x-\mu)^2}-\beta\right)(x-y)\right\}. \tag{11}$$

The robust order quantity can be efficiently found with a gradient search because, by Proposition 1, the inner maximization problems are concave. It is easy to show that when the variance grows to infinity, the minimax regret order quantity (11) tends to (7), as if only the mean was known. In particular, if $\beta \leqslant 1/2$, $y^* \to \mu/(4\beta)$, which is one quarter of the upper bound derived by Gallego et al. (2007). At the other extreme, when the variance tends to zero, it becomes optimal to order exactly the mean demand. In Roels (2006), we derive the following approximation to the minimax regret order quantity for $\sigma/\mu \leqslant \sqrt{1-\beta}$: $y^* \approx \max\{0, \mu + (2/5)\sigma(1-2\beta)/\sqrt{\beta(1-\beta)}\}$. This approximation is the sum of the mean demand and some positive (respectively, negative) safety stock whenever $\beta \leqslant 1/2$ (respectively, $\beta \geqslant 1/2$), proportional to the demand's standard deviation.

When the mean and the variance are known, the following three distribution-free approaches are available: the maximin proposed by Scarf (1958), the minimax regret without nonnegativity constraints, proposed by Yue et al. (2006), and the minimax regret with nonnegativity requirements (11). Table 5 evaluates the robustness of the different approaches, with the mean absolute difference in order quantities, the p-value of the KS test, and the relative increase in mean regret, over 300 values

Table 5. Comparison between the minimax regret, the maximin, and the minimax regret without nonnegativity constraints, when the mean and the variance are known ($\mu = 100$).

		Maximin			Minimax regret w/o nonnegativity		
$1-\beta$	σ/μ	$\lvert y^\circ - y^*\rvert$	p.v. (%)	$\rho(y^\circ)/\rho^*$	$\lvert y^\circ - y^*\rvert$	p.v. (%)	$\rho(y^\circ)/\rho^*$
[0.01, 0.1]	0.3	39.6	0	1.93	23.5	0	2.14
	0.6	21.2	0	1.35	26.7	0	2.39
	2	6.70	0	1.09	6.70	0	1.09
[0.1, 0.9]	0.3	3.2	9	1.14	4.7	0	1.36
	0.6	15.7	0	1.39	10.2	0	1.40
	2	67.7	0	2.48	57.3	0	2.03
[0.9, 0.99]	0.3	15.0	0	1.29	0.12	100	1.00
	0.6	31.2	0	1.29	0.22	100	1.00
	2	101.7	0	1.30	1.45	83	1.01

of the profit margin, randomly chosen over the following intervals: [0.01, 0.1], [0.1, 0.9], and [0.9, 0.99], and for three levels of coefficients of variation: 0.3, 0.6, and 2. The minimax regret quantity without nonnegativity constraints was truncated to zero whenever it was negative.

Despite its worst-case focus, the maximin approach performs well in terms of regret, at least when the coefficient of variation is less than one. Scarf (1958) pointed out the similarity between the maximin quantity and the newsvendor solution under a normal demand distribution. In fact, the good performance of the maximin objective applies to *any* demand distribution with the specified mean and variance. In contrast, ignoring the nonnegativity constraints in the minimax regret approach can significantly distort the recommended order quantities when the profit margins are low, i.e., when $1-\beta \leqslant 0.1$, but becomes negligible when the profit margins increase.

In the newsvendor model, one often assumes a normal distribution for the demand, mostly because of the central limit theorem. The entropy principle provides another justification: among all distributions with given mean and variance, the normal distribution is entropy-maximizing, i.e., is the most "random." In addition, Scarf (1958) numerically showed that the normality assumption was robust from the maximin point of view. However, the actual demand, unlike the normal distribution, is always nonnegative. Ignoring the nonnegativity constraints in the newsvendor model can impact the order quantity in the following two ways. First, the newsvendor solution becomes negative, for small profit margins and large variances. Second, the order quantity becomes infinite when a high service level is maintained while the variance grows to infinity, whereas it is never optimal to order more than μ/β (Gallego et al. 2007).

To circumvent this nonnegativity issue, one can truncate to zero the newsvendor solution whenever it is negative. A more involved approach consists in truncating the whole distribution, by readjusting the mean and the variance, before solving the newsvendor model (e.g., see Fisher and Raman 1996). Alternatively, one can use the gamma distribution whenever the coefficient of variation is larger than 0.3. Table 6 evaluates the robustness of the three suggested approaches with the mean difference in order quantities, the p-value of the KS test, and the relative increase in mean regret. The minimax regret and maximin order quantities are plotted as a function of the profit margin in Figure 7, when $\mu=\sigma=100$, as well as the newsvendor solutions under a gamma and a truncated normal distribution.

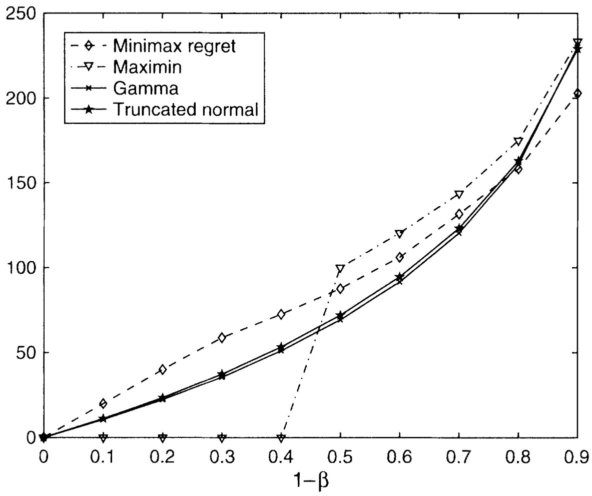
Not surprisingly, the normal distribution is robust for low coefficients of variation and low profit margins. Indeed, Naddor (1978) already observed that the optimal stock levels and profits are relatively insensitive to the choice of the demand distribution when the first two moments are known. Hence, the normality assumption is robust, in an inventory control setting, from both a maximin and a minimax regret standpoint. For large coefficients of variation and small profit margins, a more robust solution can be obtained by truncating the normal distribution. (The p-values of the KS two-sample tests are almost always significant in these cases.) For larger profit margins however, the truncated normal distribution underestimates the minimax regret quantity, and it is more robust to use the gamma distribution (or the normal distribution if the coefficient of variation is not too large) instead. Ultimately, the choice between the newsvendor or the minimax regret approach is governed by trading off the percentage reduction in regret against the increase in complexity.

The minimax regret hedges profits against the worst case; but how does it perform under the most common demand distributions? Table 7 measures the 95% confidence intervals for the average profit loss, over 300 random values of $1-\beta$, chosen in the intervals $[0.01, 0.1]$, $[0.1, 0.9]$, and $[0.9, 0.99]$, from using (11) instead of the optimal newsvendor solution, when the demand distribution is truncated normal, gamma, log-normal, or binomial. Compared to the upper bound ρ^*, the minimax regret approach performs well across all demand scenarios, except for the gamma and the negative binomial when the coefficient of variation is large.

Table 6. Comparison between the minimax regret and the newsvendor model, with a normal, left-truncated normal, or gamma demand distributions, when the mean and the variance are known ($\mu=100$).

		Normal			Truncated normal			Gamma		
$1-\beta$	σ/μ	$\lvert y^\circ-y^*\rvert$	p.v. (%)	$\rho(y^\circ)/\rho^*$	$\lvert y^\circ-y^*\rvert$	p.v. (%)	$\rho(y^\circ)/\rho^*$	$\lvert y^\circ-y^*\rvert$	p.v. (%)	$\rho(y^\circ)/\rho^*$
$[0.01, 0.1]$	0.3	6.9	0	1.21	6.9	0	1.21	9.1	0	1.28
	0.6	13.4	0	1.19	6.5	0	1.11	5.5	0	1.29
	2	6.9	0	1.09	0.6	20	1.01 +	6.9	0	1.09
$[0.1, 0.9]$	0.3	4.9	1	1.24	4.9	1	1.25	5.1	0	1.27
	0.6	8.9	5	1.23	11.8	0	1.32	10.0	0	1.29
	2	48.6	0	1.63	8.7	1	1.12	34.0	0	1.67
$[0.9, 0.99]$	0.3	7.9	0	1.19	7.9	0	1.19	7.74	0	1.20
	0.6	15.0	0	1.18	16.4	0	1.21	16.7	0	1.21
	2	51.5	0	1.19	114.5	0	1.62	59.5	0	1.14

Figure 7. Order quantities with the minimax regret, the maximin, and the newsvendor model, using a gamma or a truncated normal distribution, when $\mu = \sigma = 100$.



5. Robust Value of Additional Information

Using the mean and the variance to guide inventory decisions is a popular practice, especially because it gives rise to efficient strategies to aggregate inventories and pool demand risk. On the other hand, estimating the variance is sometimes more an art than a science, especially when forecasting is done subjectively (Dalrymple 1987). One might then wonder if it is necessary to estimate the variance to make a good inventory decision.

In this section, we compare the minimax regret ρ^* when the mean and the variance have been estimated to that when only the mean and the shape of the demand distribution have been characterized. In fact, the minimax regret quantifies the maximum price to obtain full information about the demand distribution. Comparing the values of regrets will then reveal which type of information is the most valuable. The purpose of this comparison is not to discriminate between the two forecasting methods (ideally, one would like to characterize both the shape and the moments of the distribution), but to generate insights into the value of information about the shape of the distribution, sometimes overlooked in practice.

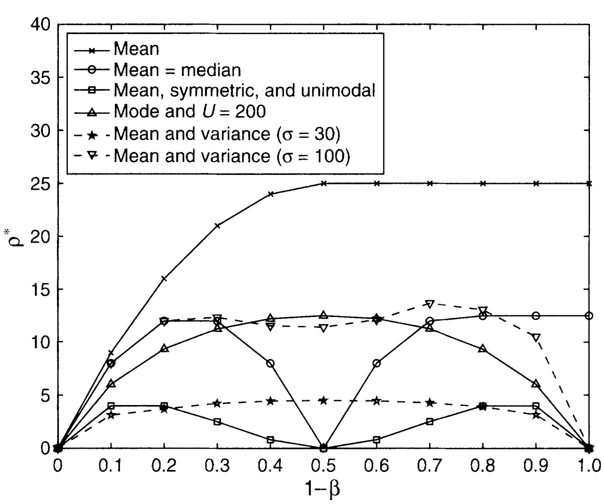
We first consider "regular" demand distributions. Figure 8 compares the minimax regrets under the following levels of information about the demand distribution: mean; mean and median; unimodal with mode equal to 100 and upper bound equal to 200; unimodal and symmetric; mean and coefficient of variation of 0.3; mean and coefficient of variation of 1. As more information is available about the shape of the demand distribution, the value of the regret decreases. The regret is generally largest when $1-\beta = 0.5$, except when the median is known to be equal to the mean, in which case the regret is equal to zero at $\beta = 1/2$ (when it is optimal to order the median, independently of the shape of the distribution).

When the profit margin is between 0.2 and 0.8, the regret with a symmetric and unimodal distribution is smaller than that with a coefficient of variation greater than 0.3. That is, one would be ready to pay a higher price to completely characterize the demand distribution when the variance is known than when the demand is known to be symmetric and unimodal. Indeed, we observed from Theorem 7 that

Table 7. Profit loss from ordering the minimax regret quantity (11) instead of the optimal newsvendor solution, under different demand scenarios, when the mean and the variance are known ($\mu = 100$).

σ/μ	Demand p.d.f.	[0.01, 0.1]	[0.1, 0.9]	[0.9, 0.99]
0.3	Truncated normal	0.07 ± 0.01	0.14 ± 0.01	0.10 ± 0.01
	Gamma	0.11 ± 0.01	0.16 ± 0.01	0.09 ± 0.01
	Log-normal	0.16 ± 0.02	0.17 ± 0.01	0.09 ± 0.01
	Negative binomial	0.11 ± 0.01	0.16 ± 0.01	0.09 ± 0.01
	ρ^*	2.37 ± 0.07	4.12 ± 0.05	2.32 ± 0.07
0.6	Truncated normal	0.12 ± 0.01	0.40 ± 0.04	0.25 ± 0.02
	Gamma	0.06 ± 0.01	0.37 ± 0.03	0.22 ± 0.02
	Log-normal	0.33 ± 0.01	0.38 ± 0.03	0.13 ± 0.01
	Negative binomial	0.05 ± 0.00	0.37 ± 0.03	0.22 ± 0.02
	ρ^*	4.07 ± 0.17	7.92 ± 0.09	4.79 ± 0.13
2	Truncated normal	0.00 ± 0.00	0.20 ± 0.02	1.87 ± 0.06
	Gamma	1.81 ± 0.10	3.89 ± 0.28	0.33 ± 0.03
	Log-normal	0.02 ± 0.00	1.02 ± 0.03	0.49 ± 0.05
	Negative binomial	1.85 ± 0.10	3.90 ± 0.28	0.33 ± 0.03
	ρ^*	5.04 ± 0.25	17.44 ± 0.27	15.64 ± 0.42

Figure 8. Minimax regret for regular distributions ($\mu = 100$, $r + l - s = 1$).



assuming a coefficient of variation greater than 0.3 is conservative when the distribution is symmetric unimodal.

There are two possible reasons for which the minimax regret is so low when the demand distribution is assumed to be symmetric and unimodal. First, the assumption of symmetry and unimodality might be so restrictive that, no matter what decision is made, the value of the regret will always be low. Second, the minimax regret approach might be more efficient at reducing the level of uncertainty under these shape assumptions.

In fact, the second reason prevails. We measure the level of uncertainty of the decision problem with the difference between the maximax and the maximin profits, i.e., $\Delta = \max_y \max_{F\in\mathcal{D}} \Pi_F(y) - \max_y \min_{F\in\mathcal{D}} \Pi_F(y)$, which is the range of possible profits if a "good" decision is made. (The actual range of profits, for any decision, would be the difference between the maximax and the minimin profits.) Accordingly, the ratio ρ^*/Δ quantifies the percentage reduction in profit uncertainty due to the minimax regret approach. Figure 9 displays this ratio, as a function of the profit margin, under three levels of information about the demand. This ratio can be computed explicitly for all cases considered in this paper (except for the mean and variance) and is independent of the scale of the demand. When the mean is known, the maximax profit equals $\mu(1-\beta)$ (deterministic demand) and the maximin profit equals zero (unit impulse at zero). In this case, the relative regret ρ^*/Δ is decreasing; thus, the regret approach is more efficient with high profit margins. A similar decreasing shape for the relative regret was also observed when the range is known ($\rho^*/\Delta = \beta$) and when the mode is known. When the mean and variance are known, the maximax profit equals $\mu(1-\beta)$ (see Yue et al. 2006) and the maximin profit equals $\max\{0, \mu(1-\beta) - \sigma\sqrt{\beta(1-\beta)}\}$ (see Scarf 1958). The relative regret is first decreasing when the profit margins are small, and then remains constant at about 30%. The consistence of the good performance of the minimax regret makes the mean-variance approach appealing in practice, especially when the same inventory control system is used for products with different profit margins. When the coefficient of variation grows, we observed that the curve becomes closer to that obtained when only the mean is known, as the variance becomes less informative. Finally, when the demand distribution is symmetric and unimodal, the maximax profit equals $\mu(1-\beta)$ (deterministic demand) and the maximin profit equals $\mu(1-\beta)^2$ (uniform between zero and 2μ). Under this assumption (or when the mean is equal to the median), the relative regret has a bowl shape. Thus, the minimax regret approach is most efficient when the profit margins have intermediate values. In particular, when the profit margin is between 0.2 and 0.8, the minimax regret is more efficient under the assumption of symmetry and unimodality than under the assumption of mean and variance.

Figure 9. Ratio of the minimax regret to the difference between the maximax and the maximin profits, when the following assumptions are made about the demand: mean; symmetry and unimodality; mean and coefficient of variation of 0.3.

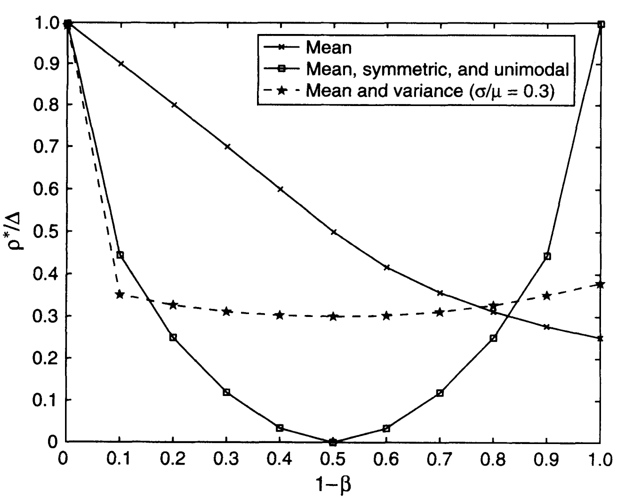
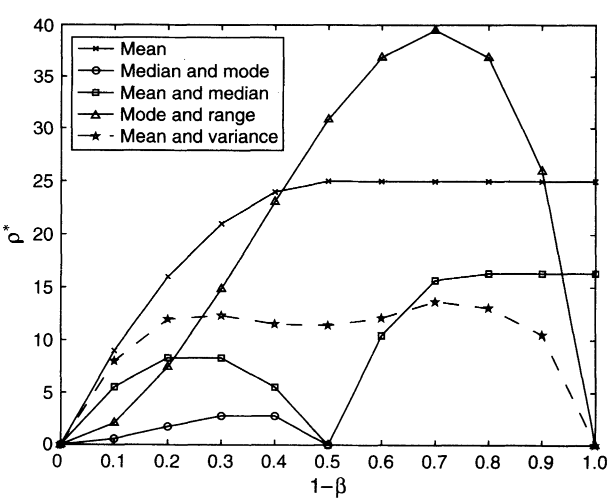


Figure 10 illustrates the value of the regret when the distribution is skewed. We consider the following levels of information about the distribution: mean only; median and mode; mode and median; mode and range; and mean and variance. We chose the parameters consistent with an exponential distribution with mean 100 (mode 0, median 69, and the upper bound equal to 460, corresponding to the 99th percentile of the distribution). When $1-\beta \leqslant 1/2$, knowing the median in addition to the mean or the mode leads to a lower regret than knowing the mean and the variance. However, for larger profit margins, the variance describes more accurately the demand distribution than the characteristics about the shape.

Although no general conclusion can be drawn from Figure 10, because the minimax regrets are a function of the

Figure 10. Minimax regret for a skewed distribution ($\mu = 100$, $m = 69$, $B = 460$, $\sigma = 100$).



values of the mean, mode, median, and variance, these two examples illustrate the potential advantage of alternative, and somewhat more qualitative, descriptions of the demand distribution. In fact, both the moments and the shape of the distribution are useful to predict demand. The power of the minimax regret approach relies on its ability to combine both a qualitative and a quantitative description of the demand distribution to lead to nonconservative decisions.

6. Conclusions

In this paper, we propose a robust approach to inventory management with partial demand information. In particular, we derive order quantities that minimize the newsvendor's maximum regret. The minimax regret objective balances the risks of ordering too little against the risk of ordering too much and is consequently less conservative than the maximin approach. The minimax regret approach is able to combine both a qualitative and a quantitative description of the demand distribution. Therefore, in practice, one should not underemphasize the qualitative behavior of the demand, especially when the variance is difficult to estimate.

Most of the derived order quantities are simple functions, which makes them attractive for practical application. The minimax regret approach also generates insight into "robust" probability distributions to be used in the newsvendor model (1). In general, the robust distributions are also entropy-maximizing: exponential when only the mean is known; uniform when only the range is known, or when the distribution is known to be symmetric; and normal when only the mean and variance are known, but the coefficient of variation is small. For larger coefficients of variation, it is recommended (from a minimax regret perspective) to truncate the normal distribution, and for larger profit margins ($1-\beta \geqslant 0.9$), to use a gamma distribution instead. For skewed distributions, the gamma distribution performs well, as long as there is moderate spread between the median and the mean and/or mode, and the profit margins are not too high.

In the presence of capacity constraints, affecting both the actual decision y and the benchmark decision z, the minimax regret ρ^* is always finite (unlike the unconstrained case when the mode and the median are known). In Roels (2006), we characterize the minimax regret decisions with multiple products subject to capacity constraints.

The minimax regret is a generic approach. It can therefore be applied to other cases than those explored in the paper, e.g., when one knows the mean and the mode, or when the mode is known to lie within some interval (Popescu 2005). These situations might not be solvable analytically, but the moment problem (4) can be solved using semidefinite programming (Bertsimas and Popescu 2002, 2005; Popescu 2005), and the minimax regret problem can be solved efficiently (Proposition 1). The minimax regret approach also applies to situations where the decision maker knows the distribution but is uncertain about the values of the parameters of the distribution.

7. Electronic Companion

An electronic companion to this paper is available as part of the online version that can be found at http://or.pubs.informs.org/ecompanion.html.

Acknowledgments

The authors thank the associate editor, the anonymous referees, Steve Graves, David Simchi-Levi, Jonathan Kluberg, and the participants of the OM Seminar at the Sloan School of Management for their insightful comments, which have significantly improved this paper. Preparation of this paper was supported, in part, by the PECASE Award DMI-9984339 from the National Science Foundation and the Singapore-MIT Alliance Program.

References

Bell, D. E. 1982. Regret in decision making under uncertainty. *Oper. Res.* **30**(5) 961–981.

Ben-Tal, A., A. Nemirovski. 1999. Robust solutions of uncertain linear programs. *Oper. Res. Lett.* **25**(1) 1–13.

Bergemann, D., K. Schlag. 2005. Robust monopoly pricing: The case of regret. Cowles Foundation discussion paper, Cowles Foundation, New Haven, CT.

Bertsimas, D., I. Popescu. 2002. On the relation between option and stock prices: A convex optimization approach. *Oper. Res.* **50**(2) 358–374.

Bertsimas, D., I. Popescu. 2005. Optimal inequalities in probability theory: A convex optimization approach. *SIAM J. Optim.* **15**(3) 780–804.

Bertsimas, D., M. Sim. 2003. The price of robustness. *Oper. Res.* **52**(1) 35–53.

Boyd, S., L. Vandenberghe. 2004. *Convex Optimization*. Cambridge University Press, Cambridge, UK.

Brown, A. O., C. S. Tang. 2006. The impact of alternative measures on single-period inventory policy. *J. Indust. Management Optim.* **2**(3) 297–318.

Chamberlain, G. 2000. Econometrics and decision theory. *J. Econometrics* **95**(2) 255–283.

Dalrymple, D. J. 1987. Sales forecasting practices. Results from a United States survey. *Internat. J. Forecasting* **3** 379–391.

Ehrhardt, R. 1979. The power approximation for computing (S, s) inventory policies. *Management Sci.* **25**(8) 777–786.

Fisher, M., A. Raman. 1996. Reducing the cost of uncertainty through accurate response to early sales. *Oper. Res.* **44**(1) 87–99.

Fortuin, L. 1980. Five popular probability density functions: A comparison in the field of stock-control models. *J. Oper. Res. Soc.* **31**(10) 937–942.

Gallego, G. 1992. A minimax distribution-free procedure for the (Q, R) inventory model. *Oper. Res. Lett.* **11** 55–60.

Gallego, G. 1998. New bounds and heuristics for (Q, R) policies. *Management Sci.* **44**(2) 219–233.

Gallego, G., I. Moon. 1993. The distribution free newsboy problem: Review and extensions. *J. Oper. Res. Soc.* **44**(8) 825–834.

Gallego, G., K. Katircioglu, B. Ramachandran. 2007. Inventory management under highly uncertain demand. *Oper. Res. Lett.* **35**(3) 281–289.

Gallego, G., J. K. Ryan, D. Simchi-Levi. 2001. Minimax analysis for finite-horizon inventory models. *IIE Trans.* **33** 861–874.

Godfrey, G., W. B. Powell. 2001. An adaptive, distribution-free algorithm for the newsvendor problem with censored demands, with application to inventory and distribution problems. *Management Sci.* **47** 1101–1112.

Jaynes, E. T. 1957. Information theory and statistical mechanics. *Physical Rev.* **106** 620–630.

Jaynes, E. T. 2003. *Probability Theory: The Logic of Science*. Cambridge University Press, Cambridge, UK.

Karp, R. M. 1992. On-line algorithms vs. off-line algorithms: How much is it worth knowing the future? J. van Leeuwen, ed. *Proc. World Comput. Congress*, Vol. 1. Elsevier Science Publishers, Amsterdam, 416–429.

Kasugai, H., T. Kasegai. 1961. Note on minimax regret ordering policy—Static and dynamic solutions and a comparison to maximin policy. *J. Oper. Res. Soc. Japan* **3** 155–169.

Kouvelis, P., G. Yu. 1997. *Robust Discrete Optimization and Applications*. Kluwer Academic Publishers, Boston.

Levi, R., R. Roundy, D. B. Shmoys. 2007. Provably near-optimal sampling-based policies for stochastic inventory control models. *Math. Oper. Res.* **32**(4) 821–839.

Lim, A. E. B., J. G. Shanthikumar. 2007. Relative entropy, exponential utility, and robust dynamic pricing. *Oper. Res.* **55**(2) 198–214.

Loomes, G., R. Sugden. 1982. Regret theory: An alternative theory of rational choice under uncertainty. *Econom. J.* **92** 805–824.

Luce, R. F., H. Raiffa. 1957. *Games and Decisions*. John Wiley and Sons, New York.

Moon, I., S. Choi. 1995. The distribution free newsboy problem with balking. *J. Oper. Res. Soc.* **46**(4) 537–542.

Moon, I., G. Gallego. 1994. Distribution free procedures for some inventory models. *J. Oper. Res. Soc.* **45**(6) 651–658.

Morris, W. T. 1959. Inventorying for unknown demand. *J. Indust. Engrg.* **10**(4) 299–302.

Naddor, E. 1978. Sensitivity to distributions in inventory systems. *Management Sci.* **24**(16) 1769–1772.

Popescu, I. 2005. A semidefinite approach to optimal-moment bounds for convex class of distributions. *Math. Oper. Res.* **30**(3) 632–657.

Porteus, E. L. 1990. Stochastic inventory theory. D. Heyman, M. Sobel, eds. *Handbooks in Operations Research and Management Science*, Vol. 2. Elsevier, North-Holland, Amsterdam, 605–653.

Roels, G. 2006. Information and decentralization in inventory, supply chain, and transportation systems. Ph.D. thesis, Massachusetts Institute of Technology, Cambridge, MA.

Savage, L. J. 1951. The theory of statistical decisions. *J. Amer. Statist. Assoc.* **46**(253) 55–67.

Scarf, H. E. 1958. A min-max solution to an inventory problem. K. J. Arrow, S. Karlin, H. E. Scarf, eds. *Studies in Mathematical Theory of Inventory and Production*. Stanford University Press, Stanford, CA, 201–209.

Scarf, H. E. 1959. Bayes solutions of the statistical inventory problem. *Ann. Math. Statist.* **30**(2) 490–508.

Schweitzer, M. A., G. P. Cachon. 2000. Decision bias in the newsvendor problem with a known demand distribution: Experimental evidence. *Management Sci.* **46**(3) 404–420.

Smith, J. E. 1995. Generalized Chebyshev inequalities: Theory and applications in decision analysis. *Oper. Res.* **43**(5) 807–825.

Vairaktarakis, G. L. 2000. Robust multi-item newsboy models with a budget constraint. *Internat. J. Production Econom.* **66** 213–226.

von Hippel, P. T. 2005. Mean, median, and skew: Correcting a textbook rule. *J. Statist. Ed.* **13**(2). www.amstat.org/publications/jse/v13n2/vonhippel.html.

Yue, J., B. Chen, M.-C. Wang. 2006. Expected value of distribution information for the newsvendor problem. *Oper. Res.* **54**(6) 1128–1136.

Zhu, Z., J. Zhang, Y. Ye. 2006. Newsvendor optimization with limited distribution information. Working paper, Stanford University, Stanford, CA.