HW #3 (CSE 4190.313)

Saturday, May 9, 2020

Name: ______________________ **ID No:** ______________________

1. Find a basis for the orthogonal complement of the row space of A:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}.$$

Split $\mathbf{x} = (3, 3, 3)^T$ into a row space component $\mathbf{x}_r$, and a nullspace component $\mathbf{x}_n$.

2. Suppose P is the projection matrix onto the subspace S and Q is the projection onto the orthogonal complement $S^{\perp}$. What are $P+Q$ and PQ? Show that $P-Q$ is its own inverse.

3. If $P_C = A(A^TA)^{-1}A^T$ is the projection onto the column space of A, what is the projection P_R onto the row space of A? Under what condition on A? Justify your answer.

4. Prove that the trace of $P = \mathbf{a}\mathbf{a}^T/\mathbf{a}^T\mathbf{a}$ always equals 1.

5. Find the projection of $\mathbf{b}$ onto the column space of A:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}.$$

6. If you know the average $\hat{x}_N$ of N numbers $b_1, \cdots, b_N$, how can you quickly find the average $\hat{x}_{N+1}$ with one more number b_{N+1}? The idea of recursive least squares is to avoid adding $N+1$ numbers.

7. Consider the closest cubic $b = C + Dt + Et^2 + Ft^3$ to the four points with $b = 0, 8, 8, 20$ at $t = 0, 1, 3, 4$. Write the four equations $A\mathbf{x} = \mathbf{b}$. Solve them by elimination. This cubic now goes exactly through the four points. What are $\mathbf{p}$ and $\mathbf{e}$?

8. The average of the four times t_i, $(i = 1, 2, 3, 4)$, is $\bar{t} = \frac{1}{4}(0 + 1 + 3 + 4) = 2$. Moreover, the average of the four b_i, $(i = 1, 2, 3, 4)$, is $\bar{b} = \frac{1}{4}(0 + 8 + 8 + 20) = 9$.

   (a) Show that the best line goes through the center point $(\bar{t}, \bar{b}) = (2, 9)$.

   (b) Explain why $C + D\bar{t} = \bar{b}$ comes from the first equation in $A^T A\, \hat{\mathbf{x}} = A^T \mathbf{b}$.

9. (a) In the following Gram-Schmidt formula, show that C is orthogonal to $\mathbf{q}_1$ and $\mathbf{q}_2$:

$$C = \mathbf{c} - (\mathbf{q}_1^T \mathbf{c})\mathbf{q}_1 - (\mathbf{q}_2^T \mathbf{c})\mathbf{q}_2.$$

(b) In the following modified Gram-Schmidt steps, show that the vector $\bar{C}$ is the same as the vector C in the above equation:

$$C^* = \mathbf{c} - (\mathbf{q}_1^T \mathbf{c})\mathbf{q}_1, \quad \bar{C} = C^* - (\mathbf{q}_2^T C^*)\mathbf{q}_2.$$

10. (a) If the columns of A are orthogonal to each other, what can you say about the form of $A^T A$? If the columns are orthonormal, what can you say then?

    (b) Under what conditions on the columns of A (which may be rectangular) is $A^T A$ invertible? Justify your answer.

11. Find the fourth Legendre polynomial, which is orthogonal to 1, x, and $x^2 - \frac{1}{3}$, over the interval $-1 \leq x \leq 1$.

12. Apply the Gram-Schmidt process to

$$\mathbf{a} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

and write the result in the form $A = QR$.

13. Apply the Gram-Schmidt process to

$$\mathbf{a} = \sin t, \quad \mathbf{b} = \cos t, \quad \mathbf{c} = 1,$$

under the inner product $\langle f, g \rangle = \int_0^\pi f(t)g(t)dt$.

14. What is the closest function $a \cos x + b \sin x$ to the function $f(x) = \sin 2x$ on the interval from $-\pi$ to π? What is the closest straight line $c + dx$?