Chap 2. Second-Order Linear ODEs

2.1 Homogeneous Linear ODEs of Second Order

$$y'' + p(x)\,y' + q(x)\,y = 0$$

Ex1 (Superposition of Solutions)

$y'' + y = 0$

$y = \cos x$ and $y = \sin x$ are solutions

$y = c_1 \cos x + c_2 \sin x$ is also a solution

Th1 (Fundamental Theorem of the Homogeneous Linear ODE)

$y_1(x)$, $y_2(x)$: solutions

$\Rightarrow c_1 y_1(x) + c_2 y_2(x)$: solution (c_1 and c_2 : constants)

<proof>

$$\begin{aligned} y'' + py' + qy &= (c_1 y_1 + c_2 y_2)'' + p(c_1 y_1 + c_2 y_2)' + q(c_1 y_1 + c_2 y_2) \\ &= c_1 y_1'' + c_2 y_2'' + p(c_1 y_1' + c_2 y_2') + q(c_1 y_1 + c_2 y_2) \\ &= c_1(y_1'' + py_1' + qy_1) + c_2(y_2'' + py_2' + qy_2) = 0 \end{aligned}$$

Caution!

This theorem holds for homogeneous linear ODEs only, but does not hold for nonhomogeneous linear or nonlinear ODEs.

Ex2 (a Nonhomogeneous Linear ODE)

$y'' + y = 1$,

$y_1 = 1 + \cos x$ and $y_2 = 1 + \sin x$ are solutions.

But, $y_1 + y_2$ is not a solution.

Ex3 (a Nonlinear ODE)

$y'' y - x y' = 0 \Rightarrow y_1 = x^2$ and $y_2 = 1$ are solutions

But, $y_1 + y_2 = x^2 + 1$ is not a solution

Neither is $-y_1 = -x^2$!

Initial Value Problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = K_0, \quad y'(x_0) = K_1$$

Ex4 (Initial Value Problem)

$y'' + y = 0, \quad y(0) = 3.0, \quad y'(0) = -0.5$

<solution>

$y = c_1 \cos x + c_2 \sin x$: a general solution

$y' = -c_1 \sin x + c_2 \cos x$

$\Rightarrow y(0) = c_1 = 3.0$ and $y'(0) = c_2 = -0.5$

$\therefore y(x) = 3.0 \cos x - 0.5 \sin x$

Def (General Solution)

$c_1 y_1 + c_2 y_2$, where y_1 and y_2 : lin. indep. solutions

$$\left[\begin{array}{l} k_1 y_1(x) + k_2 y_2(x) = 0 \\ \Leftrightarrow k_1 = k_2 = 0 \end{array} \right]$$

$\{y_1, y_2\}$: basis

Ex 5&6

$y'' + y = 0 \Rightarrow \{\cos x, \sin x\}$: basis

$y'' - y = 0 \Rightarrow \{e^x, e^{-x}\}$: basis

Ex 7 (Reduction of Order)

$x^2 y'' - xy' + y = 0, \quad x > 0$

$y_1 = x$ is a solution

Let $y_2 = ux \Rightarrow y_2' = u'x + u, \; y_2'' = u''x + 2u'$

$x^2(u''x + 2u') - x(u'x + u) + ux = 0$

$x^3 u'' + x^2 u' = 0 \Rightarrow \underline{xv' + v = 0} \quad (v = u')$

First-Order Linear ODE

Find a Basis if One Solution is Known (Reduction of Order)

$$y'' + p(x)\,y' + q(x)\,y = 0$$

$y_1(x)$: a known solution

Let $y_2(x) = u(x)\,y_1(x)$ for some $u(x)$

$$y_2' = u' y_1 + u y_1'$$

$$y_2'' = u'' y_1 + 2u' y_1' + u y_1''$$

$$\Rightarrow (u'' y_1 + 2u' y_1' + u y_1'') + p(u' y_1 + u y_1') + q(u y_1) = 0$$

$$u'' y_1 + u'(2y_1' + p y_1) + u\underbrace{(y_1'' + p y_1' + q y_1)}_{=0} = 0$$

$$u'' + u' \cdot \frac{2y_1' + p y_1}{y_1} = 0$$

$$U' + \left(\frac{2y_1'}{y_1} + p\right) \cdot U = 0, \quad \text{where } U = u'$$

$$\frac{dU}{U} = -\left(\frac{2y_1'}{y_1} + p\right) dx$$

$$\ln|U| = -2\ln|y_1| - \int p\,dx$$

$$U = \frac{1}{y_1^2} \cdot e^{-\int p\,dx}$$

$$y_2 = u y_1 = y_1 \cdot \int U\,dx$$

2.2 Homogeneous Linear ODEs with Constant Coefficients

$$y'' + ay' + by = 0, \quad \text{where } a, b : \text{constants}$$

Let $y = e^{\lambda x} \Rightarrow \underline{(\lambda^2 + a\lambda + b)}\, e^{\lambda x} = 0$

Case I : $(a^2 - 4b > 0)$: $\underline{y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$,
where $\lambda_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b})$, $\lambda_2 = \frac{1}{2}(-a - \sqrt{a^2-4b})$

Case II : $(a^2 - 4b = 0)$: $\underline{y = (c_1 + c_2 x)\, e^{-\frac{ax}{2}}}$

Case III : $(a^2 - 4b < 0)$: $\underline{y = e^{-\frac{ax}{2}}(A\cos\omega x + B\sin\omega x)}$,
where $\lambda_1 = -\frac{a}{2} + i\omega$, $\lambda_2 = -\frac{a}{2} - i\omega$

$\omega = \frac{\sqrt{4b - a^2}}{2}$

pf) Let λ_1, λ_2 : the roots of $\lambda^2 + a\lambda + b = 0$

Consider $c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} = 0$

$$c_1 \lambda_1 e^{\lambda_1 x} + c_2 \lambda_2 e^{\lambda_2 x} = 0$$

$$\begin{bmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(\lambda_2 - \lambda_1) e^{\lambda_1 x} \cdot e^{\lambda_2 x} \neq 0 \quad \text{if } \lambda_1 \neq \lambda_2$$

$$\therefore \underline{c_1 = c_2 = 0} \quad \text{if } \lambda_1 \neq \lambda_2.$$

If $\lambda_1 = \lambda_2$, consider $c_1 e^{-\frac{a}{2}x} + c_2 x e^{-\frac{a}{2}x} = 0$
(Case II)

$$c_1 + c_2 x = 0$$

$$\therefore \underline{c_1 = c_2 = 0} \quad \lrcorner$$

Remark (If $a^2 - 4b < 0$)

$$e^{\lambda_1 x} = e^{-\frac{a}{2}x} \cdot e^{i\omega x} = e^{-\frac{a}{2}x}(\cos\omega x + i\sin\omega x)$$

$$e^{\lambda_2 x} = e^{-\frac{a}{2}x} \cdot e^{-i\omega x} = e^{-\frac{a}{2}x}(\cos\omega x - i\sin\omega x)$$

$$\Rightarrow \frac{1}{2}(e^{\lambda_1 x} + e^{\lambda_2 x}) = e^{-\frac{a}{2}x} \cdot \cos\omega x$$

$$\frac{1}{2i}(e^{\lambda_1 x} - e^{\lambda_2 x}) = e^{-\frac{a}{2}x} \cdot \sin\omega x$$

Ex2 (Distinct Real Roots)

$y'' + y' - 2y = 0, \quad y(0) = 4, \quad y'(0) = -5$

$\lambda^2 + \lambda - 2 = 0, \quad (\lambda - 1)(\lambda + 2) = 0$

$$y = c_1 e^{x} + c_2 e^{-2x}$$
$$y' = c_1 e^{x} - 2c_2 e^{-2x} \Rightarrow \begin{cases} c_1 + c_2 = 4 \\ c_1 - 2c_2 = -5 \end{cases} \Rightarrow c_1 = 1, \ c_2 = 3$$

$$\therefore \ y = e^{x} + 3e^{-2x}$$

Ex4 (Double Root)

$y'' + y' + 0.25y = 0, \quad y(0) = 3.0, \quad y'(0) = -3.5$

$\lambda^2 + \lambda + 0.25 = 0, \quad (\lambda + 0.5)^2 = 0$

$$y = (c_1 + c_2 x)e^{-0.5x}$$
$$y' = c_2 e^{-0.5x} - 0.5(c_1 + c_2 x)e^{-0.5x} \Rightarrow c_1 = 3.0, \ c_2 = -2$$

$$\therefore \ y = (3.0 - 2.0x)e^{-0.5x}$$

Ex5 (Complex Roots)

$y'' + 0.4y' + 9.04y = 0, \quad y(0) = 0, \quad y'(0) = 3$

$\lambda^2 + 0.4\lambda + 9.04 = 0, \quad \lambda = -0.2 \pm 3i, \quad \omega = 3$

$y = e^{-0.2x}(A\cos 3x + B\sin 3x) \Rightarrow A = 0$ since $y(0) = 0$

$y = Be^{-0.2x}\sin 3x$

$y' = B(-0.2e^{-0.2x}\sin 3x + 3e^{-0.2x}\cos 3x)$

$y'(0) = 3B = 3 \quad \Rightarrow B = 1$

$$\therefore \ y = e^{-0.2x}\sin 3x$$

Ex6 (Complex Roots)

a general solution of the ODE

$$y'' + \omega^2 y = 0 \qquad (\omega : \text{non-zero constant})$$

is $y = A\cos\omega x + B\sin\omega x$

2.3 Differential Operators

$Dy = y'$, $D^2y = D(Dy) = y''$

$D(x^2) = 2x$, $D(\sin x) = \cos x$, $D^2(\sin x) = -\sin x$

$L = P(D) = D^2 + aD + bI$: 2nd order diff. operator

$L[y] = P(D)[y] = (D^2 + aD + bI)[y] = y'' + ay' + by$

L is a linear operator : $L[\alpha y + \beta w] = \alpha L[y] + \beta L[w]$

α, β : constants

$y'' + ay' + by = 0 \Rightarrow L[y] = P(D)[y] = 0$

$D[e^{\lambda x}] = \lambda e^{\lambda x}$

$D^2[e^{\lambda x}] = \lambda^2 e^{\lambda x}$

$P(D)[e^{\lambda x}] = (\lambda^2 + a\lambda + b)e^{\lambda x} = P(\lambda)e^{\lambda x} = 0$

$e^{\lambda x}$ is a solution of $P(D)[y] = 0$

$\Leftrightarrow$ λ is a solution of $P(\lambda) = 0$

If $P(\lambda)$ has a double root, consider

$$P(D)[e^{\lambda x}] = P(\lambda)e^{\lambda x}$$

Differentiate by λ,

$$P(D)[xe^{\lambda x}] = (P'(\lambda) + xP(\lambda))e^{\lambda x} = 0$$

$(\because P(\lambda) = P'(\lambda) = 0)$

$\therefore$ $xe^{\lambda x}$ is also a solution of $P(D)[y] = 0$.

2.5 Euler-Cauchy Equations

$$x^2 y'' + ax y' + by = 0$$

Let $y = x^m \Rightarrow x^2 m(m-1) x^{m-2} + ax \cdot m x^{m-1} + b x^m = 0$

$$\underline{m^2 + (a-1)m + b = 0}$$

Case I : (Different Real Roots : m_1 and m_2)

$$\underline{y = c_1 x^{m_1} + c_2 x^{m_2}}$$

Case II : (Double Root : $m = \frac{1}{2}(1-a)$)

$$\underline{y = (c_1 + c_2 \ln x) x^m}$$

Case III : (Complex Conjugate Roots : $m_{1,2} = \mu \pm i\nu$)

$$x^{m_1} = x^{\mu} \cdot x^{i\nu} = x^{\mu} \cdot e^{i\nu \ln x}$$
$$= x^{\mu} \cdot [\cos(\nu \ln x) + i \sin(\nu \ln x)]$$
$$x^{m_2} = x^{\mu} \cdot x^{-i\nu} = x^{\mu} \cdot e^{-i\nu \ln x}$$
$$= x^{\mu} \cdot [\cos(\nu \ln x) - i \sin(\nu \ln x)]$$
$$\underline{y = x^{\mu} \cdot [A \cos(\nu \ln x) + B \sin(\nu \ln x)]}$$

Case II :

$$P(D) = x^2 D^2 + axD + bI$$
$$P(D)[x^m] = (m^2 + (a-1)m + b)(x^m)$$

Differentiate by m,

$$P(D)[\ln x \cdot x^m] = (\underbrace{2m + (a-1)}_{=0})(x^m) + (\underbrace{m^2 + (a-1)m + b}_{=0})(\ln x \cdot x^m)$$

if $m = \frac{1-a}{2}$

$$P(D)[\ln x \cdot x^m] = 0$$

$\therefore$ $\ln x \cdot x^m$ is also a solution of $P(D)[y] = 0$

Ex 1: (Different Real Roots)

$$x^2 y'' + 1.5 x y' - 0.5 y = 0$$

$$m^2 + 0.5 m - 0.5 = 0$$

$$(m - 0.5)(m + 1) = 0$$

$$y = c_1 x^{0.5} + c_2 x^{-1} = c_1 \sqrt{x} + \frac{c_2}{x}, \quad x > 0$$

Ex 2: (a Double Root)

$$x^2 y'' - 5 x y' + 9 y = 0$$

$$m^2 - 6m + 9 = 0$$

$$(m - 3)^2 = 0$$

$$y = (c_1 + c_2 \ln x) x^3$$

Ex 3: (Complex Conjugate Roots)

$$x^2 y'' + 0.6 x y' + 16.04 y = 0$$

$$m^2 - 0.4 m + 16.04 = 0$$

$$(m - 0.2)^2 + 4^2 = 0$$

$$m_1 = 0.2 + 4i, \quad m_2 = 0.2 - 4i$$

$$y = x^{0.2} \left[A \cos(4 \ln x) + B \sin(4 \ln x) \right]$$

Ex 4: (Electric Potential Field between Two Concentric Spheres)

$$r \cdot v'' + 2 v' = 0, \quad v' = \frac{dv}{dr}$$

$$r^2 \cdot v'' + 2 r v' = 0$$

$$m^2 + m = 0$$

$$m = 0, -1$$

$$v(r) = c_1 + c_2 / r$$

2.6 Existence and Uniqueness of Solutions. Wronskian

$y'' + p(x)y' + q(x)y = 0$

$y = c_1 y_1 + c_2 y_2$: general solution

$y(x_0) = K_0$, $y'(x_0) = K_1$: initial conditions

open interval

Th1: $p(x), q(x)$: continuous on I s.t $x_0 \in I$

$\Rightarrow$ I.V.P. has a unique solution $y(x)$ on I

Th2: $y_1(x), y_2(x)$: linearly dependent on I

$$\Leftrightarrow W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' = 0$$

for some $x_0 \in I$

Th2*

If $W(y_1, y_2) = 0$ for some $x_0 \in I \Rightarrow W(y_1, y_2) \equiv 0$ for all $x \in I$

(That is, if $W(y_1, y_2) \neq 0$ for some $x_1 \in I \Rightarrow y_1, y_2$: lin. indep.)

Ex 1

$y_1 = \cos\omega x$, $y_2 = \sin\omega x$: solutions of $y'' + \omega^2 y = 0$

$$W(\cos\omega x, \sin\omega x) = \begin{vmatrix} \cos\omega x & \sin\omega x \\ -\omega\sin\omega x & \omega\cos\omega x \end{vmatrix} = \omega$$

$\cos\omega x, \sin\omega x$: lin. indep $\Leftrightarrow \omega \neq 0$

Ex 2

$y'' - 2y' + y = 0 \Rightarrow y = (c_1 + c_2 x)e^x$: general solution

$$W(e^x, xe^x) = \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix} = e^{2x} \neq 0$$

$\therefore e^x, xe^x$: lin. indep

2.7 Nonhomogeneous ODEs

$$y'' + p(x)y' + q(x)y = r(x), \qquad r(x) \neq 0$$

$\Rightarrow y(x) = y_h(x) + y_p(x)$

↳ a particular solution of the above

↳ a general solution of $y''+py'+qy=0$

Method of Undetermined Coefficients

$$y'' + ay' + by = r(x)$$

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
kx^n $(n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

Ex1 (Basic Rule)

$$y'' + y = 0.001x^2, \quad y(0) = 0, \quad y'(0) = 1.5$$

<solution>

Step 1 : $\underbrace{y'' + y = 0}_{\text{Homogeneous ODE}} \Rightarrow y_h = A\cos x + B\sin x$

Step 2 : $y_p = K_2x^2 + K_1x + K_0, \quad y_p'' = 2K_2$

$y_p'' + y_p = K_2x^2 + K_1x + (K_0 + 2K_2) = 0.001x^2$

$K_2 = 0.001, \quad K_1 = 0, \quad K_0 = -0.002$

$y_p = 0.001x^2 - 0.002$

$y = A\cos x + B\sin x + 0.001x^2 - 0.002$

Step 3 : $y(0) = A - 0.002 = 0 \Rightarrow A = 0.002$

$y' = y_h' + y_p' = -A\sin x + B\cos x + 0.002x \Rightarrow y'(0) = B = 1.5$

$\therefore y = 0.002\cos x + 1.5\sin x + 0.001x^2 - 0.002$

Ex 2 (Modification Rule)

$$y'' + 2y' + y = (D+1)^2[y] = e^{-x}, \quad y(0) = -1, \; y'(0) = 1$$

$$y_h = (c_1 + c_2 x)e^{-x}$$

$$y_p = cx^2 e^{-x} \Rightarrow y_p' = c(2x - x^2)e^{-x}, \; y_p'' = c(2 - 4x + x^2)e^{-x}$$

$$c(2 - 4x + x^2) + 2c(2x - x^2) + cx^2 = 1 \Rightarrow c = \frac{1}{2}$$

$$y = (c_1 + c_2 x)e^{-x} + \frac{1}{2}x^2 e^{-x}$$

$$y(0) = c_1 = -1, \quad y'(0) = c_2 + 1 = 1 \Rightarrow c_2 = 0$$

$$\therefore y = \left(\frac{1}{2}x^2 - 1\right)e^{-x}$$

Ex 3 (Sum Rule)

$$y'' + 2y' + 5y = e^{0.5x} + 40\cos 10x - 190\sin 10x$$

$$y(0) = 0.16, \quad y'(0) = 40.08$$

<solution>

Step1: $\lambda^2 + 2\lambda + 5 = (\lambda+1)^2 + 2^2 = 0, \quad \lambda = -1 \pm 2i$

$$y_h = e^{-x}(A\cos 2x + B\sin 2x)$$

Step2: $y_p = y_{p_1} + y_{p_2}, \begin{cases} y_{p_1} = Ce^{0.5x} \\ y_{p_2} = K\cos 10x + M\sin 10x \end{cases}$

$$y_{p_1}' = 0.5Ce^{0.5x}, \quad y_{p_1}'' = 0.25Ce^{0.5x}$$

$$y_{p_2}' = -10K\sin 10x + 10M\cos 10x$$

$$y_{p_2}'' = -100K\cos 10x - 100M\sin 10x$$

$$y_{p_1}'' + 2y_{p_1}' + 5y_{p_1} = 6.25Ce^{0.5x} = e^{0.5x} \Rightarrow C = \frac{1}{6.25}$$

$$y_{p_2}'' + 2y_{p_2}' + 5y_{p_2} = (-95K + 20M)\cos 10x + (-20K - 95M)\sin 10x$$

$$= 40\cos 10x - 190\sin 10x \Rightarrow K = 0, \; M = 2$$

$$\therefore y = y_h + y_{p_1} + y_{p_2} = e^{-x}(A\cos 2x + B\sin 2x) + 0.16e^{0.5x} + 2\sin 10x$$

Step3: $y(0) = A + 0.16 = 0.16, \quad y'(0) = -A + 2B + 0.08 + 20 = 40.08$

$$\Rightarrow A = 0, \; B = 10$$

$$\therefore y = 10e^{-x}\sin 2x + 0.16e^{0.5x} + 2\sin 10x$$

2.9 Modeling: Electric Circuits

$$LI' + RI + \frac{1}{C}\int I\,dt = E(t) = E_0 \sin \omega t$$

$$LI'' + RI' + \frac{1}{C} I = E_0 \omega \cos \omega t$$

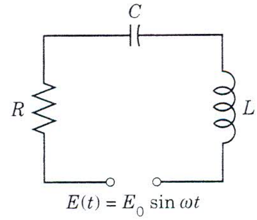

Fig. 60. *RLC*-circuit

Let

$$I_p = a\cos\omega t + b\sin\omega t = I_0 \sin(\omega t - \theta)$$

$$I_p' = \omega(-a\sin\omega t + b\cos\omega t)$$

$$I_p'' = \omega^2(-a\cos\omega t - b\sin\omega t)$$

$$\Rightarrow a = \frac{-E_0 S}{R^2+S^2}, \quad b = \frac{E_0 R}{R^2+S^2} \Rightarrow I_0 = \sqrt{a^2+b^2} = \frac{E_0}{\sqrt{R^2+S^2}}$$

Impedance

where $S = \omega L - \frac{1}{\omega C}$: reactance

$I_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$, where λ_1, λ_2 : solution of $\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$

2.10 Solution by Variation of Parameters

$$y'' + p(x)y' + q(x)y = r(x)$$

$$\Rightarrow y_p(x) = -y_1 \int \frac{y_2 r}{W}\,dx + y_2 \int \frac{y_1 r}{W}\,dx,$$

where $\{y_1, y_2\}$: basis of the sols of $y'' + p(x)y' + q(x)y = 0$

Ex1

$$y'' + y = \sec x = \frac{1}{\cos x}$$

$$\Rightarrow y_1 = \cos x, \quad y_2 = \sin x, \quad W(y_1, y_2) = 1$$

$$y_p = -\cos x \int \sin x \cdot \sec x\,dx + \sin x \int \cos x \cdot \sec x\,dx$$

$$= \cos x \ln|\cos x| + x \sin x$$

$$\therefore y = y_h + y_p = (c_1 + \ln|\cos x|)\cos x + (c_2 + x)\sin x$$