HW #3 (CSE 4190.313)

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1. Find a basis for the orthogonal complement of the row space of A:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}.$$

Split $\mathbf{x} = (3, 3, 3)^T$ into a row space component $\mathbf{x}_r$, and a nullspace component $\mathbf{x}_n$.

1. row space의 orthogonal complement는 A의 nullspace이다.
A의 nullspace를 구하기 위해서 $Ax=0$을 풀면,

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow \begin{matrix} x+2z=0 \\ x+y+4z=0 \end{matrix} \Rightarrow \begin{matrix} x=-2z \\ y=-2z \end{matrix}$$

따라서 $N(A) = z \cdot \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$ $(z \in R)$ 이고, basis는

$$\therefore \left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}$$

2. $x = x_r + x_n$ 으로 표현하기 위해 $N(A)$에서 $\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$을 가지는 x_n으로 하면,

$$\therefore x = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}}_{x_r} + \underbrace{\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}}_{x_n}.$$

$\because x_r \cdot x_n = 0$ 이므로 각각이 $C(A^T)$와 $N(A)$에 속함을 알 수 있다.

2. Suppose P is the projection matrix onto the subspace S and Q is the projection onto the orthogonal complement $S^{\perp}$. What are $P+Q$ and PQ? Show that $P-Q$ is its own inverse.

S와 $S^{\perp}$가 orthogonal complement이므로 $\mathbb{R}^n$은 S와 $S^{\perp}$로 나눌 수 있다.

임의의 벡터 $x = x_S + x_{S^{\perp}}$라 하면, $(x_S \in S,\ x_{S^{\perp}} \in S^{\perp})$

(1) $(P+Q)x = Px + Qx = P(x_S + x_{S^{\perp}}) + Q(x_S + x_{S^{\perp}}) = x_S + x_{S^{\perp}} = x$

$\therefore P+Q = I$

(2) $PQx = P(Qx) = P\cdot Q(x_S + x_{S^{\perp}}) = P\, x_{S^{\perp}} = P\cdot(0 + x_{S^{\perp}}) = 0$

$\therefore PQ = 0$

(3) $(P+Q)^2 - 4PQ = (P-Q)^2 = I$. 따라서 $(P-Q)^{-1} = P-Q$.

3. If $P_C = A(A^TA)^{-1}A^T$ is the projection onto the column space of A, what is the projection P_R onto the row space of A? Under what condition on A? Justify your answer.

(1) A의 row space는 A^T의 column space와 같으므로

P_R은 A^T의 column space에 대한 projection과 같다.

$\therefore P_R = A^T\cdot\left((A^T)^T A^T\right)^{-1}(A^T)^T = A^T\cdot(AA^T)^{-1}\cdot A$

(2) A의 row들이 서로 linearly independent 하여야 한다.

이때 P_C가 존재하므로 A의 column들도 linearly independent

하므로, A는 square matrix이고 invertible이다.

4. Prove that the trace of $P = \mathbf{a}\mathbf{a}^T / \mathbf{a}^T\mathbf{a}$ always equals 1.

trace는 행렬의 대각 성분의 합이다.

$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ 이라고 하면

$$P = \frac{1}{a_1^2 + a_2^2 + \cdots a_n^2} \begin{pmatrix} a_1^2 & & & \\ & a_2^2 & & \\ & & \ddots & \\ & & & a_n^2 \end{pmatrix}$$

$\therefore$ 따라서 trace는 항상 $\dfrac{\sum_{i=1}^{n} a_i^2}{\sum_{i=1}^{n} a_i^2} = 1$ 이 된다.

5. Find the projection of $\mathbf{b}$ onto the column space of A:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}.$$

$$\hat{P} = A(A^TA)^{-1}A^T \cdot b$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{pmatrix} \left(\begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \right)^{-1} \cdot \begin{pmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 6 & -8 \\ -8 & 16 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$$

$$= \quad // \quad \cdot \frac{1}{44} \begin{pmatrix} 18 & 8 \\ 8 & 6 \end{pmatrix} \cdot \quad //$$

$$= \frac{1}{44} \cdot \begin{pmatrix} 26 & 14 \\ 10 & 2 \\ -4 & 8 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$$

$$= \frac{1}{44} \cdot \begin{pmatrix} 40 & 12 & 4 \\ 12 & 8 & -12 \\ 4 & -12 & 40 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$$

$$= \frac{1}{44} \cdot \begin{pmatrix} 92 \\ -56 \\ 260 \end{pmatrix}$$

$\therefore$ 따라서 b를 A에 projection하면 $\frac{1}{44} \cdot \begin{pmatrix} 92 \\ -56 \\ 260 \end{pmatrix}$ 이다.

6. If you know the average $\hat{x}_N$ of N numbers $b_1, \cdots, b_N$, how can you quickly find the average $\hat{x}_{N+1}$ with one more number b_{N+1}? The idea of recursive least squares is to avoid adding $N+1$ numbers.

$\hat{x}_N = \frac{b_1 + \cdots + b_N}{N}$ 이고, $\hat{x}_{N+1} = \frac{b_1 + \cdots + b_{N+1}}{N+1}$ 이므로

$\hat{x}_N$과 b_{N+1}을 알고 있으면

$\therefore$ $\hat{x}_{N+1} = \frac{N \cdot \hat{x}_N + b_{N+1}}{N+1}$ 의 식으로 구할 수 있다.

7. Consider the closest cubic $b = C + Dt + Et^2 + Ft^3$ to the four points with $b = 0, 8, 8, 20$ at $t = 0, 1, 3, 4$. Write the four equations $A\mathbf{x} = \mathbf{b}$. Solve them by elimination. This cubic now goes exactly through the four points. What are $\mathbf{p}$ and $\mathbf{e}$?

1. $C = 0$

$C + D + E + F = 8$

$C + 3D + 9E + 27F = 8$

$C + 4D + 16E + 64F = 20$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 3 & 9 & 27 & 8 \\ 0 & 4 & 16 & 64 & 20 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 0 & 6 & 24 & -16 \\ 0 & 0 & 12 & 60 & -12 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 0 & 6 & 24 & -16 \\ 0 & 0 & 0 & 12 & 20 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 0 & 1 & 4 & -8/3 \\ 0 & 0 & 0 & 1 & 5/3 \end{array}\right)$$

2. 이제 풀어보면, 우선

$C = 0$, $F = \frac{5}{3}$,

$E + 4F = -\frac{8}{3}$

$D + E + F = 8$

$\Rightarrow$ $E = -\frac{8}{3} - 4F = -\frac{28}{3}$

$D = 8 - E - F = 8 + \frac{28}{3} - \frac{5}{3} = \frac{47}{3}$

$\therefore$ $(C, D, E, F) = \left(0, \frac{47}{3}, -\frac{28}{3}, \frac{5}{3}\right)$ 이다.

3. 4개의 점을 모두 지나므로, 프로젝션을 하면 자기 자신이 되고, 따라서 에러는 없다.

$\therefore$ $p = b$, $e = 0$

8. The average of the four times t_i, $(i = 1, 2, 3, 4)$, is $\bar{t} = \frac{1}{4}(0 + 1 + 3 + 4) = 2$. Moreover, the average of the four b_i, $(i = 1, 2, 3, 4)$, is $\bar{b} = \frac{1}{4}(0 + 8 + 8 + 20) = 9$.

(a) Show that the best line goes through the center point $(\bar{t}, \bar{b}) = (2, 9)$.

(b) Explain why $C + D\bar{t} = \bar{b}$ comes from the first equation in $A^TA\,\hat{\mathbf{x}} = A^T\mathbf{b}$.

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

(a) best estimate는 $\hat{x} = (A^TA)^{-1}A^Tb$ 로 구해지므로, 구해보면

$$\hat{x} = \left(\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 8 \\ 8 & 26 \end{pmatrix}^{-1} \quad //$$

$$= \frac{1}{40} \cdot \begin{pmatrix} 26 & -8 \\ -8 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}$$

$$= \frac{1}{40} \cdot \begin{pmatrix} 26 & 18 & 2 & -6 \\ -8 & -4 & 4 & 8 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}$$

$$= \frac{1}{40} \cdot \begin{pmatrix} 40 \\ 160 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

∴ 따라서 $\hat{x} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ 이고 best line은 $b = 1 + 4t$. $t = 2$이면 $b = 9$ 이다.

b) $A^TA\hat{x} = A^Tb$ 를 정리하면 다음과 같은 식이 된다.

$$A^TA \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = A^Tb \Rightarrow \begin{bmatrix} 4 & \sum_{i=1}^{4} t_i \\ \sum_{i=1}^{4} t_i & \sum_{i=1}^{4} t_i^2 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{4} b_i \\ \sum_{i=1}^{4} t_i b_i \end{bmatrix}$$

첫번째 식을 전개하면

$$4 \cdot \hat{C} + \sum_{i=1}^{4} t_i \cdot \hat{D} = \sum_{i=1}^{4} b_i \Rightarrow \hat{C} + \frac{\sum_{i=1}^{4} t_i}{4} \cdot \hat{D} = \frac{\sum_{i=1}^{4} b_i}{4}$$

여기서 $\frac{\sum_{i=1}^{4} t_i}{4} = \bar{t}$, $\frac{\sum_{i=1}^{4} b_i}{4} = \bar{b}$ 이므로 위식은 $\hat{C} + \bar{t} \cdot \hat{D} = \bar{b}$ 가 된다.

∴ A^TA의 계산 과정에서 t_i의 갯수, t_i의 합, b_i의 합이 나오므로 $\bar{t}$과 $\bar{b}$이 도출되게 된다.

9. (a) In the following Gram-Schmidt formula, show that C is orthogonal to $\mathbf{q}_1$ and $\mathbf{q}_2$:

$$C = \mathbf{c} - (\mathbf{q}_1^T\mathbf{c})\mathbf{q}_1 - (\mathbf{q}_2^T\mathbf{c})\mathbf{q}_2.$$

(b) In the following modified Gram-Schmidt steps, show that the vector $\bar{C}$ is the same as the vector C in the above equation:

$$C^* = \mathbf{c} - (\mathbf{q}_1^T\mathbf{c})\mathbf{q}_1, \quad \bar{C} = C^* - (\mathbf{q}_2^T C^*)\mathbf{q}_2.$$

q_1, q_2, q_3 가 서로 수직이므로,

(a) (i) $q_1^T C = q_1^T \cdot c - q_1^T \cdot (q_1^T \cdot c) q_1 - q_1^T (q_2^T c) \cdot q_2$

$= q_1^T \cdot c - (q_1^T \cdot c) \cdot \underbrace{(q_1^T \cdot q_1)}_{=1} - (q_2^T c) \cdot \underbrace{(q_1^T \cdot q_2)}_{=0}$

$= q_1^T \cdot c - q_1^T \cdot c - 0$

$= 0.$

(ii) $q_2^T C = q_2^T \cdot c - q_2^T \cdot (q_1^T \cdot c) \cdot q_1 - q_2^T \cdot (q_2^T \cdot c) \cdot q_2$

$= q_2^T \cdot c - (q_1^T \cdot c) \cdot \underbrace{(q_2^T \cdot q_1)}_{=0} - (q_2^T \cdot c)\underbrace{(q_2^T \cdot q_2)}_{=1}$

$= q_2^T \cdot c - 0 - q_2^T \cdot c$

$= 0$

∴ (i)과 (ii)에 의해 $C \perp q_1$, $C \perp q_2$ 이다.

(b) $\bar{C} = (c - (q_1^T c) \cdot q_1) - (q_2^T \cdot (c - (q_1^T \cdot c) \cdot q_1)) q_2$

$= c - (q_1^T \cdot c) \cdot q_1 - (q_2^T c) q_2 + \underbrace{q_2^T \cdot (q_1^T \cdot c) \cdot q_1 \cdot q_2}_{\text{x라고}}$

여기서 X를 보면 $X = q_2^T \cdot (q_1^T \cdot c) \cdot q_1 \cdot q_2$

$= (q_1^T \cdot c) \cdot \underbrace{(q_2^T \cdot q_1)}_{=0} \cdot q_2$

$= 0.$

∴ 따라서 $\bar{C} = C$,

10. (a) If the columns of A are orthogonal to each other, what can you say about the form of A^TA? If the columns are orthonormal, what can you say then?

(b) Under what conditions on the columns of A (which may be rectangular) is A^TA invertible? Justify your answer.

$A = (a_1\ a_2\ a_3 \cdots a_n)$ 이라고 하면

(a) $A^TA = \begin{pmatrix} a_1^2 & & \cdots & \\ & a_2^2 & a_i a_j & \\ & a_j a_i & \ddots & \\ & & & a_n^2 \end{pmatrix}$ 에서, A의 column들이 서로 orthogonal 하므로

대각선 성분을 제외한 나머지의 값이 모두 0이 된다.

만약 서로 orthonormal 하면 대각선 성분의 값이 1이 되므로

$A^TA = I$가 된다.

(b) A^TA의 null space $N(A^TA) = \{0\}$ 이어야 A^TA는 invertible 이다.

이때 $N(A^TA) = N(A)$ 이므로, $N(A) = \{0\}$ 이어야 하고, 이는 A의 column들이 모두 linearly independent 해야 한다.

$\therefore$ A의 column 들이 서로 linearly independent 해야 한다.

11. Find the fourth Legendre polynomial, which is orthogonal to 1, x, and $x^2 - \frac{1}{3}$, over the interval $-1 \le x \le 1$.

$v_1 = 1,\quad v_2 = x,\quad v_3 = x^2 - \frac{1}{3}.$

$-\ (1, x^3) = \int_{-1}^{1} x^3 = 0$

$-\ (x, x^3) = \int_{-1}^{1} x^4 \neq 0$

$-\ (x^2 - \frac{1}{3}, x^3) = \int_{-1}^{1} x^5 - \frac{1}{3}\int_{-1}^{1} x^3 = 0$

따라서 x와 x^3 사이의 각도만 수직으로 변환 하게 된다.

$$v_4 = x^3 - \underbrace{\frac{(1, x^3)}{(1,1)} \cdot 1}_{=0} - \frac{(x, x^3)}{(x,x)} \cdot x - \underbrace{\frac{(x^2 - \frac{1}{3}, x^3)}{(x^2 - \frac{1}{3}, x^2 - \frac{1}{3})} \cdot (x^2 - \frac{1}{3})}_{=0}$$

$$= x^3 - \frac{(x, x^3)}{(x,x)} \cdot x$$

$$= x^3 - \frac{\int_{-1}^{1} x^4}{\int_{-1}^{1} x^2} \cdot x = x^3 - \frac{3}{5}x$$

$\therefore v_4 = x^3 - \frac{3}{5}x$

12. Apply the Gram-Schmidt process to

$$\mathbf{a} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

and write the result in the form $A = QR$.

우선 $A = (a\ b\ c)$ 에 대해 Q를 구하면,

(i) $q_1 = \frac{a}{|a|} = \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$

(ii) $B = b - (q_1^T \cdot b) \cdot q_1$

$= \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} - 0 \cdot q_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

$q_2 = \frac{B}{|B|} = \frac{1}{3}\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

(iii) $C = c - (q_1^T \cdot c) \cdot q_1 - (q_2^T \cdot c) \cdot q_2$

$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{5\sqrt{2}}{2} \cdot \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} - 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{5}{2} \\ \frac{5}{2} \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

$\frac{\sqrt{2}}{2}$

$1 \quad 2 \, ?$

$-\frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2}$

$\therefore q_3 = \frac{C}{|C|} = \sqrt{2} \cdot \begin{pmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$

$\therefore A = QR = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix} \begin{pmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & \frac{5\sqrt{2}}{2} \\ 0 & 3 & 1 \\ 0 & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}$

13. Apply the Gram-Schmidt process to

$$\mathbf{a} = \sin t, \quad \mathbf{b} = \cos t, \quad \mathbf{c} = 1,$$

under the inner product $\langle f, g\rangle = \int_0^\pi f(t)g(t)dt$.

$V_1 = \sin t$라고 하면

$$V_2 = \cos t - \frac{\overset{\to 0}{\langle \cos t, \sin t\rangle}}{\langle \sin t, \sin t\rangle} \cdot \sin t$$

$$= \cos t$$

$$V_3 = 1 - \frac{\langle 1, \sin t\rangle}{\langle \sin t, \sin t\rangle} \cdot \sin t - \frac{\overset{\to 0}{\langle 1, \cos t\rangle}}{\langle \cos t, \cos t\rangle} \cdot \cos t$$

$$= 1 - \frac{\overbrace{\int_0^\pi \sin t\, dt}^{\boxed{b}}}{\underbrace{\int_0^\pi \sin^2 t\, dt}_{\boxed{a}}} \cdot \sin t$$

$$a = \int_0^\pi \frac{1-\cos 2t}{2} dt = \left[\frac{1}{2} t\right]_0^\pi - \left[\frac{1}{4}\sin 2t\right]_0^\pi$$

$$= \frac{\pi}{2}$$

$$b = \int_0^\pi \sin t\, dt = [-\cos t]_0^\pi = 2.$$

$$\Rightarrow V_3 = 1 - \frac{\frac{2}{1}}{\frac{\pi}{2}} \cdot \sin t$$

$$= 1 - \frac{4}{\pi} \cdot \sin t.$$

$\therefore$ 따라서 $(V_1, V_2, V_3) = (\sin t, \cos t, 1 - \frac{4}{\pi} \sin t)$

가 orthogonal basis 이다.

14. What is the closest function $a\cos x + b\sin x$ to the function $f(x) = \sin 2x$ on the interval from $-\pi$ to π? What is the closest straight line $c + dx$?

(1) $|E|^2 = |a\cos x + b\sin x - \sin 2x|^2$ 이 최소가 되는 a, b를 찾는다.

$$|E|^2 = \int_{-\pi}^{\pi} a^2\cos^2 x + b^2\sin^2 x + \sin^2 2x + 2ab\cdot\cos x\cdot\sin x - 2a\cdot\cos x\cdot\sin 2x - 2b\cdot\sin x\cdot\sin 2x \, dx$$

를 계산하면,

$$|E|^2 = a^2\cdot\int_{-\pi}^{\pi}\frac{1+\cos 2x}{2}dx + b^2\cdot\int_{-\pi}^{\pi}\frac{1-\cos 2x}{2}dx + \int_{-\pi}^{\pi}\frac{1-\cos 4x}{2}dx + 2ab\cdot\int_{-\pi}^{\pi}\frac{\sin 2x}{2}dx$$

$$-4a\cdot\int_{-\pi}^{\pi}\cos^2 x\cdot\sin x\,dx - 4b\cdot\int_{-\pi}^{\pi}\cos x\cdot\sin^2 x\,dx.$$

$$= a^2\cdot\pi + b^2\cdot\pi + \pi + 0 - 0 - 0$$

$$= (a^2+b^2+1)\cdot\pi$$

따라서 $a=0$, $b=0$일 때 E^2은 최소가 되므로

∴ closest function 은 $y=0$.

(2) $A^TA\hat{x} = A^Tb$ 를 구하면

$$\begin{bmatrix} (1,1) & (1,x) \\ (x,1) & (x,x) \end{bmatrix}\begin{bmatrix} \hat{c} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} (1, \sin 2x) \\ (x, \sin 2x) \end{bmatrix}$$

위의 식을 계산하면

$$\begin{bmatrix} 2\pi & 0 \\ 0 & \frac{2}{3}\pi^3 \end{bmatrix}\begin{bmatrix} \hat{c} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 0 \\ -\pi \end{bmatrix}$$

따라서 $\hat{c} = 0$, $\hat{D} = -\frac{3}{2\pi^2}$ 이고,

∴ closest straight line은 $y = -\frac{3}{2\pi^2}$ 이다.