Chap 2. Vector Spaces

2.1 Vector Spaces and Subspaces

- We can add any two vectors, and we can multiply all vectors by scalars. ⇒ We can take linear combinations

6. Addition and scalar multiplication are required to satisfy these eight rules:

1. $x + y = y + x$.
2. $x + (y + z) = (x + y) + z$.
3. There is a unique "zero vector" such that $x + 0 = x$ for all x.
4. For each x there is a unique vector $-x$ such that $x + (-x) = 0$.
5. $1x = x$.
6. $(c_1 c_2)x = c_1(c_2 x)$.
7. $c(x + y) = cx + cy$.
8. $(c_1 + c_2)x = c_1 x + c_2 x$.

Ex0 : $\mathbb{R}^n$: all column vectors of n components

Ex1 : $\mathbb{R}^\infty$: vectors of infinitely many components.

$x = (1, 2, 1, 2, \cdots)$

Ex2 : The space of 3×2 matrices,

which is almost the same as $\mathbb{R}^6$

Ex3 : The space of functions $f(x)$, $0 \le x \le 1$,

$f(x) = x^2$, $g(x) = \sin x \Rightarrow (f+g)(x) = x^2 + \sin x$

The vectors are functions, and

the dimension is a larger infinity than for $\mathbb{R}^\infty$.

Other Examples

- Any plane through the origin in $\mathbb{R}^3$

  ↪ a subspace of $\mathbb{R}^3$

- Any line through the origin in $\mathbb{R}^3$

Def:

A subspace of a vector space is a nonempty subset that satisfies the requirements for a vector space: "Linear combination stay in the subspace".

① $\mathbf{x}, \mathbf{y}$ in the subspace $\Rightarrow \mathbf{x}+\mathbf{y}$ in the subspace

② $\left.\begin{array}{l}\mathbf{x} \text{ in the subspace} \\ c : \text{scalar}\end{array}\right\} \Rightarrow c\mathbf{x}$ in the subspace

- A subspace is a subset that is closed under addition and scalar multiplication.
- Zero vector will belong to every subspace. (Let $c=0$, then $\mathbf{0} = 0 \cdot \mathbf{x}$ is in the subspace)
- The smallest subspace $\mathbf{Z}$ contains only $\mathbf{0}$.
  ↳ zero-dimensional

Ex1

$\{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$: not a subspace.

(For $c = -1$, $(x,y) = (1,1)$,
$c(x,y) = (-1,-1)$ is not in this set.

$\{(x,y) \in \mathbb{R}^2 \mid (x \geq 0, y \geq 0) \text{ or } (x \leq 0, y \leq 0)\}$:
not a subspace
($\because$ $(1,2) + (-2,-1) = (-1,1)$ is not in this set)

Ex2: The vector space of $n \times n$ matrices.

The set of all lower triangular matrices.
The set of all symmetric matrices
↳ subspaces.

The Column Space of A

$m \times n$ matrix

· All linear combinations of the columns of A form a subspace of $\mathbb{R}^m$.

Ex
$$\begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$\boxed{2A}$ The system $A\mathbb{x} = \mathbb{b}$ is solvable if and only if the vector $\mathbb{b}$ is in the column space of A.

$$u \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} + v \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

o. $A\mathbb{x} = \mathbb{b}$ can be solved if and only if $\mathbb{b}$ lies in the plane that is spanned by the two column vectors of A.

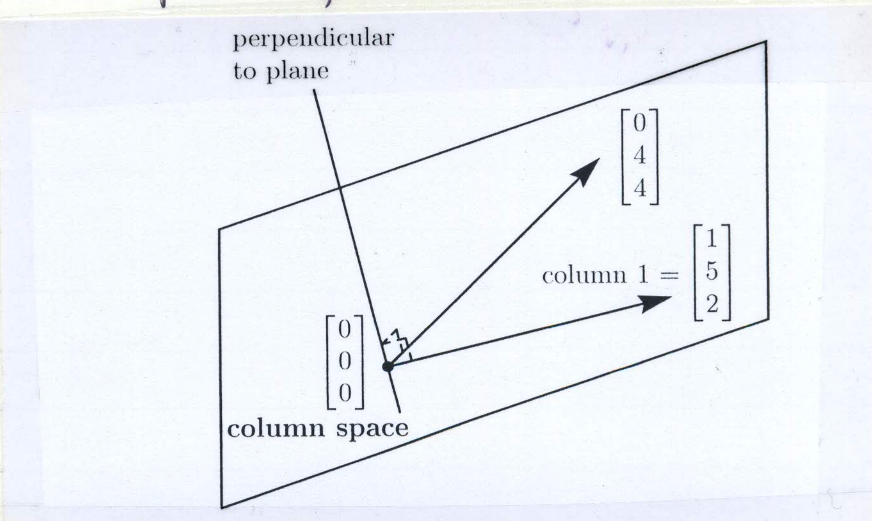


Figure 2.1 The column space $C(A)$, a plane in three-dimensional space.

(i) $\mathbb{b} = A\mathbb{x}$ and $\mathbb{b}' = A\mathbb{x}'$ for some $\mathbb{x}, \mathbb{x}'$

$\Rightarrow \mathbb{b} + \mathbb{b}' = A(\mathbb{x} + \mathbb{x}')$

(ii) $c\mathbb{b} = A(c\mathbb{x})$ for any scalar c

$\Rightarrow$ $\therefore C(A)$ is closed under (addition and scalar multiplication)

The zero matrix

- The smallest column space comes from $A = \underset{\uparrow}{O}$.
  The only combination of columns is $\mathbb{b} = \mathbb{0}$.
- For $A = I_{5\times 5}$, $C(A) = \mathbb{R}^5$.
- For any 5×5 non-singular matrix A, $C(A) = \mathbb{R}^5$. For such A, $A\mathbb{x} = \mathbb{b}$ is solvable for any $\mathbb{b} \in \mathbb{R}^5$.

The Nullspace of A

The dual to the column space of A.

Def

The null space of A, denoted by $N(A)$, is a subspace of $\mathbb{R}^n$ and consists of all $\mathbb{x} \in \mathbb{R}^n$ s.t $A\mathbb{x} = \mathbb{0}$.

(i) $A\mathbb{x} = \mathbb{0}$, $A\mathbb{x}' = \mathbb{0} \Rightarrow A(\mathbb{x} + \mathbb{x}') = \mathbb{0}$.

(ii) $A\mathbb{x} = \mathbb{0} \Rightarrow A(c\mathbb{x}) = \mathbb{0}$ for any scalar c.

$$\begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow N(A) = \{\mathbb{0}\}.$$

$\hookrightarrow$ independent columns.

Larger nullspace: $B = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix}$

Nullspace is a line: $\begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} c \\ c \\ -c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

2.2 Solving $A\mathbf{x}=\mathbf{0}$ and $A\mathbf{x}=\mathbf{b}$

When the nullspace contains more than the zero vector and/or the column space contains less than all vectors:

① Any vector $\mathbf{x}_n$ in the nullspace can be added to a particular solution $\mathbf{x}_p$. The solutions to all linear equations have this form: $\mathbf{x}=\mathbf{x}_p+\mathbf{x}_n$

Complete Solution: $\left.\begin{array}{l} A\mathbf{x}_p=\mathbf{b} \\ A\mathbf{x}_n=\mathbf{0} \end{array}\right]$ produce $A(\mathbf{x}_p+\mathbf{x}_n)=\mathbf{b}$.

② When the column space doesn't contain every $\mathbf{b}\in\mathbb{R}^m$, we need the conditions on $\mathbf{b}$ that makes $A\mathbf{x}=\mathbf{b}$ solvable.

Ex0: $0x=b$: 1×1 system

(i) $0x=b$ has no solution unless $b=0$.
The column space contains only $b=0$.

(ii) $0x=0$ has infinitely many solutions.
The null space contains all x. A particular solution is $x_p=0$, and the complete solution is $x=x_p+x_n$
$=0+(\text{any } x)$

Ex2

$$\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}b_1\\ b_2\end{bmatrix}$$: 2×2 system

(i) There is no solution unless $b_2=2b_1$.
The column space of A contains only the multiples of $(1,2)$.

(ii) When $b_2=2b_1$, there are infinitely many solutions.
For $(b_1,b_2)=(2,4)$, $\mathbf{x}_p=(1,1)$ and $\mathbf{x}_n=(-c,c)$

Complete Solution: $\mathbf{x}=\mathbf{x}_p+\mathbf{x}_n=\begin{bmatrix}1\\1\end{bmatrix}+c\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}1-c\\1+c\end{bmatrix}$

Echelon Form U and Row Reduced Form R

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix}$$

$$\to U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$: Echelon matrix

Lower triangular $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \Rightarrow A = LU.$

$\boxed{2B}$ For any $m \times n$ matrix A, there is a permutation P, a lower triangular L with unit diagonal, and an $m \times n$ echelon matrix U, such that $PA = LU$.

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

Reduced row echelon form

$Rx = 0$ has the same solutions as $Ux = 0$ and $Ax = 0$.

Pivot Variables and Free Variables

Nullspace of R (Pivot columns in boldface) : $Rx = \begin{bmatrix} \mathbf{1} & 3 & \mathbf{0} & -1 \\ \mathbf{0} & 0 & \mathbf{1} & 1 \\ \mathbf{0} & 0 & \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

pivot variables

columns with pivots

columns without pivots

free variables

To find the most general solution to $R\mathbf{x}=\mathbf{0}$:

$$R\mathbf{x}=\mathbf{0} \Rightarrow \begin{cases} u+3v-y=0 \\ w+y=0 \end{cases} \Rightarrow \begin{cases} u=-3v+y \\ w=\quad -y \end{cases}$$

There is a double infinity of solutions, with v and y free and independent. The complete solution is a combination of two special solutions :

special solutions

Nullspace contains all combinations of special solutions

$$\mathbf{x}=\begin{bmatrix} -3v+y \\ v \\ -y \\ y \end{bmatrix} = v\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$v=1,\ y=0$ $\qquad$ $v=0,\ y=1$

all solutions are linear combinations of these two special solutions

① After reaching $R\mathbf{x}=\mathbf{0}$, identify the pivot variables and free variables.

② Give one free variable the value 1, set the other free variables to 0, and solve $R\mathbf{x}=\mathbf{0}$ for the pivot variables. This $\mathbf{x}$ is a special solution.

③ Every free variable produces its own "special solution". The combinations of special solutions form the nullspace.

Nullspace matrix (Columns are special solutions)

$$N=\begin{bmatrix} -3 & 1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{array}{l} \text{not free} \\ \text{free} \\ \text{not free} \\ \text{free} \end{array}$$

Solving $A\mathbf{x}=\mathbf{b}$, $U\mathbf{x}=\mathbf{c}$ and $R\mathbf{x}=\mathbf{d}$

$A\mathbf{x}=\mathbf{b}=(b_1,b_2,b_3)$

$$U\mathbf{x}=\mathbf{c}: \quad \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2-2b_1 \\ b_3-2b_2+5b_1 \end{bmatrix}$$

$$\Uparrow$$
$$L^{-1}\mathbf{b}=\mathbf{c}$$

- The equation is inconsistent unless $b_3-2b_2+5b_1=0$.
- $A\mathbf{x}=\mathbf{b}$ can be solved $\Leftrightarrow$ $\mathbf{b}$ lies in the column space of A.

Columns of A span the column space
$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$$

the columns without pivots

- Their combinations fill out a plane in 3D space.

$$\left[\begin{array}{l} \text{Column } 2 = 3 \times \text{column } 1 \\ \text{Column } 4 = \text{column } 3 - \text{column } 1 \end{array}\right] \text{ dependent columns}$$

- The column space $C(A)$ is generated by columns 1 and 3. Equivalently, $C(A)$ is the plane of all vectors $\mathbf{b}$ with $b_3-2b_2+5b_1=0$. Geometrically, $(5,-2,1)$ is perpendicular to each column.

- If $\mathbf{b}$ belongs to $C(A)$, the solutions of $A\mathbf{x}=\mathbf{b}$ are easy to find. The last equation of $U\mathbf{x}=\mathbf{c}$ is $0=0$. To the free variables v and y, we may assign any values. The pivot values u and w are still determined by back-substitution.

Ex: $b_3 - 2b_2 + 5b_1 = 0$, with $\mathbb{b} = (1, 5, 5)$

$$A\mathbf{x} = \mathbb{b}: \quad \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$$

Forward elimination produces

$$U\mathbf{x} = \mathbb{c}: \quad \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

Back-substitution gives

$$\begin{cases} 3w + 3y = 3 \\ u + 3v + 3w + 2y = 1 \end{cases} \Rightarrow \begin{cases} w = 1 - y \\ u = -2 - 3v + y. \end{cases}$$

free

There is a double infinity of solutions:

Complete solution

$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$

$$\mathbf{x} = \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

a particular solution to $A\mathbf{x} = \mathbb{b}$.

- Every solution to $A\mathbf{x} = \mathbb{b}$ is the sum of one particular solution and a solution to $A\mathbf{x} = \mathbb{0}$.

  $\mathbf{x}_{complete} = \mathbf{x}_{particular} + \mathbf{x}_{nullspace}$.
- The particular solution comes from solving the equation with all free variables set to zero.
- Geometrically, the solutions again fill a 2D surface, but it is not a subspace. It does not contain $\mathbf{x} = \mathbb{0}$. It is parallel to the nullspace, shifted by $\mathbf{x}_p$.

① Reduce $A\mathbf{x}=\mathbf{b}$ to $U\mathbf{x}=\mathbf{c}$

② With free variables $=0$, find a particular solution to $A\mathbf{x}_p=\mathbf{b}$ and $U\mathbf{x}_p=\mathbf{c}$.

③ Find the special solutions to $A\mathbf{x}=\mathbf{0}$ (or $U\mathbf{x}=\mathbf{0}$ or $R\mathbf{x}=\mathbf{0}$)
Each free variable, in turn, is 1. Then
$\mathbf{x}=\mathbf{x}_p+$ (any combination $\mathbf{x}_n$ of special solutions)

How does the reduced form R make this solution even clearer?

Reduced equation $R\mathbf{x}=\mathbf{d}$

$$\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix}=\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Our particular solution $\mathbf{x}_p$ has free variables $v=y=0$. Then $u=-2$, $w=1$
The entries of $\mathbf{d}$ go directly into $\mathbf{x}_p$.

$\boxed{2D}$

- Suppose elimination reduces $A\mathbf{x}=\mathbf{b}$ to $U\mathbf{x}=\mathbf{c}$ and $R\mathbf{x}=\mathbf{d}$, with r pivot rows and r pivot columns. The rank is r. The last $(m-r)$ rows of U and R are zero, so there is a solution only if the last $(m-r)$ entries of $\mathbf{c}$ and $\mathbf{d}$ are zero.
- The complete solution is $\mathbf{x}=\mathbf{x}_p+\mathbf{x}_n$. One particular solution $\mathbf{x}_p$ has all free variables zero. Its pivot variables are the first r entries of $\mathbf{d}$, so $R\mathbf{x}_p=\mathbf{d}$.
- The nullspace solution $\mathbf{x}_n$ are combinations of $(n-r)$ special solutions, with one free variable equal to 1. The pivot variables in that special solution can be found in the corresponding column of R (with sign reversed).

2.3 Linear Independence, Basis, and Dimension

$\boxed{2E}$

Suppose $c_1 v_1 + \cdots + c_k v_k = 0$ only happes when $c_1 = \cdots = c_k = 0$. Then the vectors $v_1, \cdots, v_k$ are linearly independent. If any c's are nonzero, the v's are linearly dependent. One vector is a combination of the others.

Ex3:

No zeros on the diagonal : $A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow$ The columns are linearly independent

[The columns of A are independent exactly when $N(A) = \{\text{zero vector}\}$]

∘ The nonzero rows of any echelon matrix U must be independent. The columns that contain the pivots are linearly independent.

Ex

Two independent rows

Two independent columns

$$U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

independent

independent

$\boxed{2F}$

The r nonzero rows of an echelon matrix U and a reduced matrix R are linearly independent. So are the r columns that contain pivots.

Ex4

the nxn identity matrix

The columns of $I_{n\times n}$ are independent.

To check any set of vectors $v_1, \cdots, v_n$ for independence, let $A = [v_1 \cdots v_n]$ and solve $Ac = 0$.
The vectors are dependent if there is a solution $c \neq 0$.
With no free variables (rank n), they are independent.
If the rank is less than n, the columns are dependent.

$\boxed{2G}$
A set of n vectors in $\mathbb{R}^m$ must be linearly dependent if $n > m$.

Pf) $A = [v_1 \cdots v_n]$: $m \times n$ matrix.
There cannot be n pivots $\Rightarrow$ the rank $< n$.
$\therefore Ac = 0$ has nonzero solutions $c \neq 0$ ⌟

Spanning a Subspace

$\boxed{2H}$ If a vector space V consists of all linear combinations of $w_1, \cdots, w_\ell$, then these vectors span the space.
Every vector v in V is some combination of the w's:
$\underline{v = c_1 w_1 + \cdots + c_\ell w_\ell}$, for some coefficients c_i.

∘. a different combination of w's could give the same v.
The c's need not be unique. The spanning set might be exessively large. It could include 0 or even all vectors.

Ex6
$w_1 = (1,0,0)$, $w_2 = (0,1,0)$, $w_3 = (-2,0,0)$ span $\underline{\text{a plane}}$.
w_1 and w_2 also span this plane. the xy-plane

Ex7
The column (row) space of A is exactly the space that is spanned by its columns (rows).

Basis for a Vector Space

- To decide if $\mathbb{b}$ is a combination of the columns, we try to solve $A\mathbf{x}=\mathbb{b}$.
  (Spanning involves the column space.)
- To decide if the columns are independent, we solve $A\mathbf{x}=\mathbb{0}$.
  (Independence involves the nullspace.)

$\boxed{2I}$

A basis for V is a sequence of vectors s.t.

① The vectors are linearly independent (not too many)

② They span the space V (not too few).

Every vector in the space is a combination of the basis vectors, and the combination is unique.

because they span

「∵ If $v = a_1 v_1 + \cdots + a_k v_k = b_1 v_1 + \cdots + b_k v_k$,

$\sum (a_i - b_i) v_i = \mathbb{0} \Rightarrow a_i - b_i = 0$ and $a_i = b_i$ 」

Ex9:

Echelon matrix: $U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ → Four columns span the column space of U; but they are not independent

The columns for pivots are a basis for the column space

- But, the vector space has infinitely many different bases.
  (ex. the first and last columns are also a basis for $C(U)$)
- $C(U)$ is not the same as the column space $C(A)$ before elimination — but, the number of independent columns didn't change.

Dimension of a Vector Space

$\boxed{2J}$ Any two bases for a vector space V contain the same number of vectors, the dimension of V.

↓proof

$\boxed{2K}$ If $v_1, \cdots, v_m$ and $w_1, \cdots, w_n$ are both bases for the same vector space, then $m = n$.

「pf Suppose $n > m$, then we will arrive at a contradiction.

$\Rightarrow w_1 = a_{11} v_1 + \cdots + a_{m1} v_m, \cdots$

$$W = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_m \end{bmatrix} \begin{bmatrix} a_{11} & & \\ \vdots & \cdots & \\ a_{m1} & & \end{bmatrix} = V\underbrace{A}_{m \times n}$$

There is a nonzero solution to $Ax = 0$ for $x \neq 0$.

$\Rightarrow VAx = 0$ and $Wx = 0$ for $x \neq 0$.

This is impossible since $w_1, \cdots w_n$ are independent.

Similarly, we can show that $n < m$ is impossible」

$\boxed{2L}$ ◦ Any linearly independent set in V can be extended to a basis, by adding more vectors if necessary.

◦ Any spanning set in V can be reduced to a basis, by discarding vectors if necessary.

◦ A basis is a <u>maximal independent</u> set. It cannot be made larger without loosing independence.

◦ A basis is also a <u>minimal spanning</u> set. It cannot be made smaller and still span the space.

<u>Note</u>: Never use, ↓ these terms are meaningless

"basis of ~~a~~ matrix", "rank of ~~a~~ space", "~~dimension~~ of a basis"

2.4 The Four Fundamental Subspaces

Fundamental Theorem of Linear Algebra, Part I

1. $C(A)$ = column space of A; dimension r.
2. $N(A)$ = nullspace of A; dimension $n-r$.
3. $C(A^T)$ = row space of A; dimension r.
4. $N(A^T)$ = left nullspace of A; dimension $m-r$.

Basic Example : $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \to U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$N(A), C(A^T)$: subspaces of R^n

$N(A^T), C(A)$: subspaces of R^m

$\boxed{2M}$ The row space $C(A^T)$ of A has the same dimension r as the row space of U, and it has the same bases, because the row spaces of A and U (and R) are the same.

$\boxed{2N}$ The nullspace $N(A)$ of A has dimension $(n-r)$. The special solutions are a basis of $N(A)$. ($N(A)$ is the same as the nullspace of U and R.)

$\boxed{2O}$ The dimension of $C(A)$ equals the rank r. A basis for $C(A)$ is formed by the r columns of A that corresponds, in U, to the columns containing pivots.

$\ulcorner \because$ $A\mathbf{x} = 0$ exactly when $U\mathbf{x} = 0$.

The two systems are equivalent and have the same solutions.

If a set of columns of A is independent, then so are the corresponding columns of U, and vice versa. $\lrcorner$

$\boxed{2P}$ The left nullspace $N(A^T)$ has dimension $m-r$

- The rows of A combine to produce the $(m-r)$ zero rows of U. Start from $PA = LU$, or $L^{-1}PA = U$. The last $(m-r)$ rows of the invertible matrix $L^{-1}P$ must be a basis of y's in the left nullspace.
- In our 3×4 matrix example, $b_3 - 2b_2 + 5b_1 = 0$. Thus the components of y are $5, -2, 1$. The vector y is a basis for the left nullspace of dimension $m-r=1$. It is the last row of $L^{-1}P$, and produces the zero row in U.

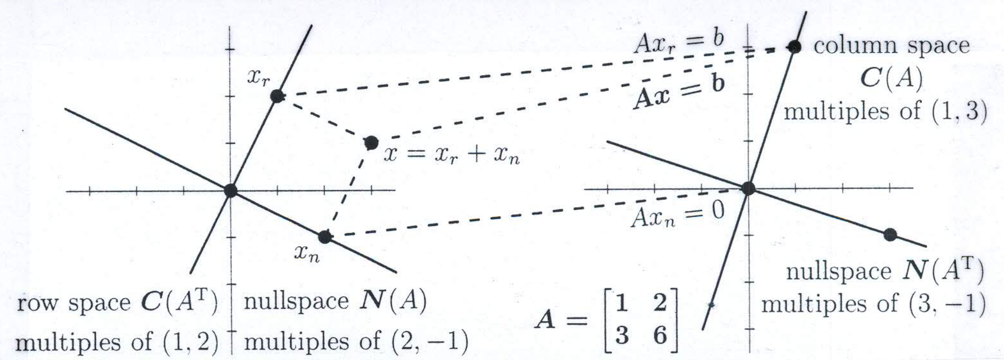


Figure 2.5 The four fundamental subspaces (lines) for the singular matrix A.

<u>Ex1</u> A has $m=n=2$ and rank $r=1$

1. The column space $C(A)$ contains all multiples of $\begin{bmatrix}1\\3\end{bmatrix}$.
2. The nullspace $N(A)$ contains all multiples of $\begin{bmatrix}-2\\1\end{bmatrix}$.
3. The row space $C(A^T)$ contains all multiples of $\begin{bmatrix}1\\2\end{bmatrix}$.
4. The left nullspace $N(A^T)$ contains all multiples of $y = \begin{bmatrix}-3\\1\end{bmatrix}$.

- When the last entry of A is changed from 6 to 7, the column space and row space have dimension $r=2$. The matrix is invertible.

Existence of Inverses

20 **EXISTENCE: Full row rank $r = m$.** $Ax = b$ has ***at least*** one solution x for every b if and only if the columns span $\mathbf{R}^m$. Then A has a **right-inverse** C such that $AC = I_m$ (m by m). This is possible only if $m \le n$.

UNIQUENESS: Full column rank $r = n$. $Ax = b$ has ***at most*** one solution x for every b if and only if the columns are linearly independent. Then A has an n by m **left-inverse** B such that $BA = I_n$. This is possible only if $m \ge n$.

- In the existence case, one possible solution is $\mathbf{x} = C\mathbf{b}$, since then $A\mathbf{x} = AC\mathbf{b} = \mathbf{b}$. But there will be other solutions if there are other right-inverses. When the columns span R^m, the number of solutions is 1 or ∞.
- In the uniqueness case, if there is a solution to $A\mathbf{x} = \mathbf{b}$, it has to be $\mathbf{x} = BA\mathbf{x} = B\mathbf{b}$. But there may be no solution. The number of solutions is 0 or 1.
- Simple formulas for the best left and right inverses, if they exist: $B = (A^TA)^{-1}A^T$ and $C = A^T(AA^T)^{-1}$.

Certainly, $BA = I$ and $AC = I$.
What is not so certain is that A^TA and AA^T are invertible.

$$\begin{cases} A^TA \text{ has an inverse if the rank is } n. \\ AA^T \text{ has an inverse if the rank is } m. \end{cases}$$

Ex2: $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \Rightarrow r = m = 2$

$$AC = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{5} \\ c_{31} & c_{32} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Best right-inverse $\quad \underbrace{A^T(AA^T)^{-1}}_{\text{the pseudoinverse}} = \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{25} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{5} \\ 0 & 0 \end{bmatrix} = C$

A^T is an example of infinitely many left-inverses:

$$BA^T = \begin{bmatrix} \frac{1}{4} & 0 & b_{13} \\ 0 & \frac{1}{5} & b_{23} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The best left inverse $(A^TA)^{-1}A^T$ (also the pseudoinverse) has $b_{13} = b_{23} = 0$. This is a uniqueness case when $r = n$. There is no free variable, since $n - r = 0$. the rank
If there is a solution, it will be the only one.

<u>The condition for invertibility is full rank: $r = m = n$.</u>

1. The columns span $\mathbb{R}^n$, so $A\mathbf{x} = \mathbf{b}$ has at least one solution.
2. The columns are independent, so $A\mathbf{x} = \mathbf{0}$ has only $\mathbf{x} = \mathbf{0}$ as the solution.
3. The rows of A span $\mathbb{R}^n$.
4. The rows are independent.
5. Elimination can be completed: $PA = LDU$, with all n pivots.
6. The determinant of A is not zero.
7. Zero is not an eigenvalue of A.
8. A^TA is positive definite

<u>Matrices of Rank 1</u>

Every matrix of rank 1 has the simplest form:
$A = u \cdot v^T$ = column times row.

<u>Ex:</u>

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 8 & 4 & 4 \\ -2 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}$$

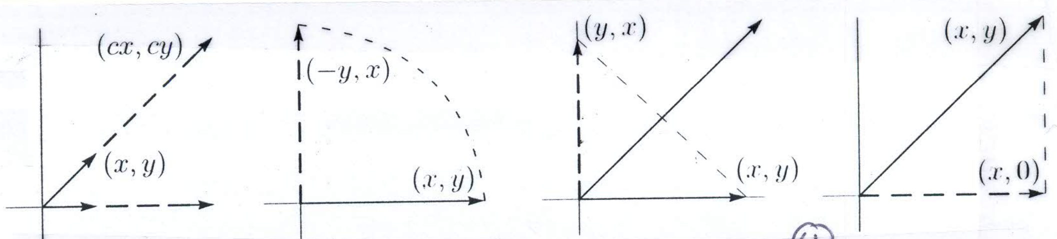
↳ rank 1

2.6 Linear Transformations

$\boxed{2T}$ For all numbers c and d and all vectors $\mathbf{x}$ and $\mathbf{y}$, matrix multiplication satisfies the rule of linearity:

$$A(c\mathbf{x}+d\mathbf{y}) = c(A\mathbf{x})+d(A\mathbf{y}).$$

Every transformation $T(\mathbf{x})$ that meets this requirement is a linear transformation.



① stretching ② 90° rotation ③ reflection (45° mirror) ④ projection on axis

① $\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$ ② $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ ③ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ④ $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Ex1: $A = \frac{d}{dt}$: Differentiation is linear

$$Ap(t) = \frac{d}{dt}(a_0 + \cdots + a_n t^n) = a_1 + \cdots + n a_n t^{n-1}$$

$N(A)$ = the one-dim. space of constants.

$C(A) = P_{n-1}$, the n-dim. space of polynomials of degree $(n-1)$.

Ex2

Integration from 0 to t:

$$Ap(t) = \int_0^t (a_0 + \cdots + a_n t^n)\,dt = a_0 t + \cdots + \frac{a_n}{n+1} t^{n+1}$$

$N(A) = \{0\}$. $C(A) = P_{n+1} \setminus \{\text{constants}\}$

Note:

$Ap = p^2$, $Ap = p+1 \Rightarrow$ Not linear!

Transformations Represented by Matrices

Linearity: If $x = c_1 x_1 + \cdots + c_n x_n$, then

$$Ax = c_1(Ax_1) + \cdots + c_n(Ax_n).$$

∘. If we know Ax for each vector in a basis, then we know Ax for each vector in the entire space.

Ex4: $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow Ax_1 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$; $x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow Ax_2 = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}$

$$\Rightarrow \quad A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \\ 4 & 8 \end{bmatrix}$$

Starting with a different basis $(1,1)$ and $(2,-1)$, the same A is also the only linear transformation with

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 12 \end{bmatrix} \text{ and } A\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Differentiation

Basis for P_3: $p_1 = 1$, $p_2 = t$, $p_3 = t^2$, $p_4 = t^3$

Action of $\frac{d}{dt}$: $Ap_1 = 0$, $Ap_2 = p_1$, $Ap_3 = 2p_2$, $Ap_4 = 3p_3$

$$\text{Differentiation matrix}: A_{diff} = \begin{array}{cccc} & & & \\ \end{array}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$\begin{array}{cccc} Ap_1 & Ap_2 & Ap_3 & Ap_4 \end{array}$$

p_1: basis of $N(A_{diff})$

p_1, p_2, p_3: basis of $C(A_{diff})$

For $p = 2 + t - t^2 - t^3$,

$$\frac{dp}{dt} = Ap \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -3 \\ 0 \end{bmatrix} \rightarrow 1 - 2t - 3t^2$$

$\boxed{24}$ $x_1, \cdots, x_n$: a basis for the space V

$y_1, \cdots, y_m$: —— " —— W.

Each linear transformation T from V to W is represented by a matrix A. The jth column of A is found by $T(x_j) = Ax_j = a_{1j}y_1 + \cdots + a_{mj}y_m$.

Integration

$V = P_3$, $W = P_4$.

$y_1 = 1$, $y_2 = t$, $y_3 = t^2$, $y_4 = t^3$, $y_5 = t^4$: a basis for W

$\int_0^t 1\,dt = t$ or $Ax_1 = y_2$, $\cdots$, $\int_0^t t^3 dt = \frac{1}{4}t^4$ or $Ax_4 = \frac{1}{4}y_5$

Integration Matrix : $A_{int} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$

Integration followed by differentiation brings back the original function.

$A_{diff} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$ and $A_{diff}A_{int} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$

o. Differentiation is a left-inverse of integration. Rectangular matrices cannot have two-sided inverses! Thus $A_{int}A_{diff} = I_{5\times5}$ cannot be true. ($\because$ only of rank 4)

In fact, $A_{int}A_{diff} = \begin{bmatrix} 0 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \rightarrow$ rank 4

Rotations Q, Projections P, and Reflections H

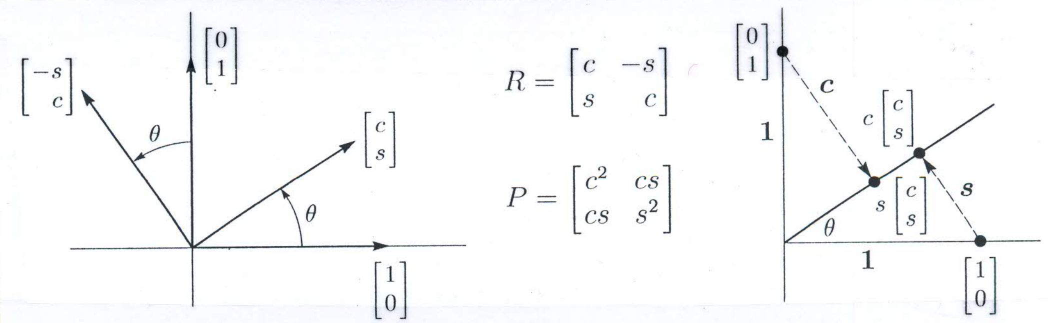


Figure 2.10 Rotation through θ (left). Projection onto the θ-line (right).

① Rotation:

$$Q_\theta Q_\varphi = \begin{bmatrix} C_\theta & -S_\theta \\ S_\theta & C_\theta \end{bmatrix}\begin{bmatrix} C_\varphi & -S_\varphi \\ S_\varphi & C_\varphi \end{bmatrix} = \begin{bmatrix} C_\theta C_\varphi - S_\theta S_\varphi & -C_\theta S_\varphi - S_\theta C_\varphi \\ S_\theta C_\varphi + C_\theta S_\varphi & -S_\theta S_\varphi + C_\theta C_\varphi \end{bmatrix}$$

$$= \begin{bmatrix} C_{\theta+\varphi} & -S_{\theta+\varphi} \\ S_{\theta+\varphi} & C_{\theta+\varphi} \end{bmatrix} = Q_{\theta+\varphi}$$

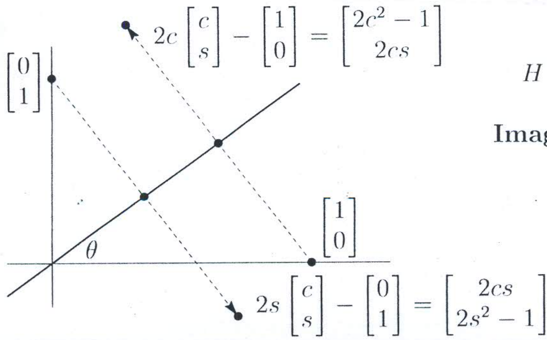
$$Q_\theta \cdot \underbrace{Q_{-\theta}}_{\text{inverse of } Q_\theta} = Q_0 = I, \qquad Q_\theta^n = Q_{n\theta}.$$

② Projection:

The projection of $(1,0)$ onto the θ-line has length $c = \cos\theta$ in the direction of (c,s); thus it falls at (c^2, cs). Similarly, the projection of $(0,1)$ has length s and falls at $s(c,s) = (cs, s^2)$.

Projection onto θ-line: $P = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} \Rightarrow P^2 = P$.

- The matrix P has no inverse.
- $N(P)$ is the perpendicular line to the θ-line.
- Points on the θ-line are projected to themselves!



$$H = 2P - I = \begin{bmatrix} 2c^2-1 & 2cs \\ 2cs & 2s^2-1 \end{bmatrix}$$

Image + original = 2 × projection

$$Hx + x = 2Px$$

Figure 2.11 Reflection through the θ-line: the geometry and the matrix.

③ Reflection:

Reflection matrix : $H = \begin{bmatrix} 2c^2-1 & 2cs \\ 2cs & 2s^2-1 \end{bmatrix} \Rightarrow H^2 = I$.

- Two reflections bring back the original
- a reflection is its own inverse : $H = H^{-1}$.
- Relationship to projections : $H = 2P - I$.
  $H\mathbf{x} + \mathbf{x} = 2P\mathbf{x}$: the image plus the original equals twice the projection.
- $H = 2P - I$ also confirms $H^2 = I$:
  $H^2 = (2P-I)^2 = 4P^2 - 4P + I = I$ since $P^2 = P$.

Dependence of Matrix Representation on the Basis

Suppose $(\cos\theta, \sin\theta)$, $(-\sin\theta, \cos\theta)$ form a basis,

$\Rightarrow$ (i) $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and (ii) $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow H = 2P - I$

But, (iii) $Q = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ is not changed.

⊛ In general, the matrix A is altered to $S^{-1}AS$ under a change of basis accounted for by S.

$\Rightarrow$ We come back to it in Chap 5.