# HW #2 (CSE 4190.313)

**Tuesday, April 21, 2020**

**Name:** ______________________ **ID No:** ______________________

1. Suppose $A$ is the sum of two matrices of rank one: $A = \mathbf{u}\mathbf{v}^T + \mathbf{w}\mathbf{z}^T$.

   (a) Which vectors ________________ span the column space of $A$?

   Which vectors ________________ span the row space of $A$?

   (b) The rank is less than 2 if ____________________or if ____________________.

2. If the rows of an $m \times n$ matrix $A$ are linearly independent, then the rank is _______,

   the column space is ____________________, and the left nullspace is ____________________.

3. $A$ is an $m \times n$ matrix of rank $r$. Suppose there are right-hand sides $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has no solution.

   (a) What inequalityes ($<$ or $\leq$) must be true between $m, n, r$? Explain why.

   (b) How do you know that $A^T\mathbf{y} = \mathbf{0}$ has a nonzero solution?

4. Reduce the following matrix $A$ to a reduced echelon form $R$:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 7 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix}.$$

Find a special solution for each free variable and describe every soluition to $A\mathbf{x} = \mathbf{0}$.

5. Under what condition on $b_1, b_2, b_3$ is the following system solvable? Find all solutions when that condition holds.

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

6. Using the fact that the total number of $5 \times 5$ permutation matrices is $5!$, answer the following yes/no questions.

   (a) Are they linearly independent? Explain why.

   (b) Do they span the space of all $5 \times 5$ matrices? Explain why.

7. On the vector space $\mathbf{P}_3$ of cubic polynomials, what matrix represents $\frac{d^2}{dt^2}$ ? Construct the $4 \times 4$ matrix $A$ from the standard basis $1, t, t^2, t^3$. Find its nullspace and column space. What do they mean in terms of polynomials?

8. Find all vectors that are perpendicular to $(1, 4, 4, 1)$ and $(2, 9, 8, 2)$.

9. Given an $m \times n$ matrix $A$ with rank $r$, if you know a particular solution $\mathbf{x}_p$ (free variables $= 0$) and all special solutions $\mathbf{x}_1, \cdots, \mathbf{x}_{n-r}$, for

$$A\mathbf{x} = \mathbf{b},$$

(a) Find a solution $\begin{bmatrix} \mathbf{y}_p \\ \mathbf{Y}_p \end{bmatrix}$ and all special solutions $\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{Y}_1 \end{bmatrix}, \cdots, \begin{bmatrix} \mathbf{y}_{2n-r} \\ \mathbf{Y}_{2n-r} \end{bmatrix}$, for

$$\begin{bmatrix} A & 2A \\ 3A & 6A \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{Y} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ 3\mathbf{b} \end{bmatrix}.$$

(b) Find a solution $\begin{bmatrix} \mathbf{y}_p \\ \mathcal{Y}_p \\ \mathbf{Y}_p \end{bmatrix}$ and all special solutions $\begin{bmatrix} \mathbf{y}_1 \\ \mathcal{Y}_1 \\ \mathbf{Y}_1 \end{bmatrix}, \cdots, \begin{bmatrix} \mathbf{y}_{3n-r} \\ \mathcal{Y}_{3n-r} \\ \mathbf{Y}_{3n-r} \end{bmatrix}$, for

$$\begin{bmatrix} A & A & A \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathcal{Y} \\ \mathbf{Y} \end{bmatrix} = \mathbf{b}.$$

10. (a) $A\mathbf{x} = \mathbf{b}$ has a solution under what conditions on $\mathbf{b}$, for the following $A$ and $\mathbf{b}$?

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 6 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

(b) Find a basis for the nullspace of $A$?

(c) Find the general solution to $A\mathbf{x} = \mathbf{b}$, when a solution exists.

(d) (2 points) What is a basis for the column space of $A$?

11. True or false: If we know $T(\mathbf{v}_i)$ for $n$ different nonzero vectors $\mathbf{v}_i$ in $\mathbf{R}^n$, $(i = 1, \cdots, n)$, then we know $T(\mathbf{v})$ for all vectors $\mathbf{v}$ in $\mathbf{R}^n$. Explain the reason why.

12. (a) What matrix $M$ transforms $(1, 0)$ and $(0, 1)$ to $(r, t)$ and $(s, u)$?

    (b) What matrix $M$ transforms $(a, c)$ and $(b, d)$ to $(1, 0)$ and $(0, 1)$?

    (c) What conditions on $a, b, c, d$ will make part (b) impossible? Explain why.

13. Suppose $A$ is a symmetric matrix ($A^T = A$).

   (a) Why is its column space perpendicular to its nullspace?

   (b) If $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{z} = 5\mathbf{z}$, which subspaces contain these eigenvectors $\mathbf{x}$ and $\mathbf{z}$? Explain why.

14. What matrix $P$ projects every point in $\mathbf{R}^3$ onto the line of intersection of the planes $x + y + t = 0$ and $x - t = 0$?