Chap 21. Numerics for ODEs

21.1 Methods for First-Order ODEs

Initial Value Problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

We compute approximate numeric values of $y(x)$ at

$x_1 = x_0 + h, \quad x_2 = x_0 + 2h, \quad x_3 = x_0 + 3h, \cdots$

$$y(x+h) = y(x) + h y'(x) + \frac{1}{2}h^2 y''(x) + \cdots$$
$$\approx y(x) + h y'(x) = y(x) + h \cdot f(x, y)$$

$\Rightarrow$

$$y_1 = y_0 + h f(x_0, y_0), \quad y_2 = y_1 + h f(x_1, y_1), \cdots$$
$$\underline{y_{n+1} = y_n + h f(x_n, y_n)}, \quad (n = 0, 1, 2, \cdots)$$

↳ The Euler Method or the Euler–Cauchy Method

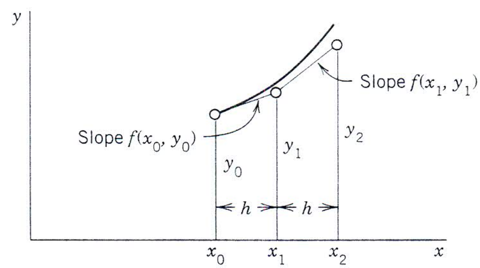


Fig. 448. Euler method

$$y(x+h) = y(x) + h y'(x) + \frac{1}{2}h^2 y''(\xi), \quad (x \leqq \xi \leqq x+h)$$

$\Rightarrow$ Local truncation error is $O(h^2)$

$\Rightarrow$ Global error is proportional to $h^2\left(\frac{1}{h}\right) = h$

since the number of steps is proportional to $\frac{1}{h}$.

Improved Euler Method (Heun's Method)

$$y_{n+1}^{*} = y_n + h\cdot f(x_n, y_n)$$

$$y_{n+1} = y_n + \frac{1}{2}h\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{*})\right]$$

Local Error

The local error of the Improved Euler Method is $O(h^3)$

<proof>

ⓐ $y(x_n+h) - y(x_n) = h\cdot \hat{f}_n + \frac{1}{2}h^2 \tilde{f}_n' + \frac{1}{6}h^3 \tilde{f}_n'' + \cdots$

where $\hat{f}_n = f(x_n, y(x_n))$

ⓑ $y_{n+1} - y_n \approx \frac{1}{2}h\cdot\left[\tilde{f}_n + \hat{f}_{n+1}\right]$

$$= \frac{1}{2}h\left[\tilde{f}_n + \left(\tilde{f}_n + h\cdot \tilde{f}_n' + \frac{1}{2}h^2 \tilde{f}_n'' + \cdots\right)\right]$$

$$= h\cdot \tilde{f}_n + \frac{1}{2}h^2 \tilde{f}_n' + \frac{1}{4}h^3 \hat{f}_n'' + \cdots$$

Hence, ⓑ−ⓐ is $\frac{h^3}{12}\tilde{f}_n'' + \cdots$, which is $O(h^3)$

Ex1 (Euler Method)

$$y' = f(x,y) = x+y, \quad y(0) = 0, \quad h = 0.2$$

$$\Rightarrow f(x_n, y_n) = x_n + y_n$$

$$y_{n+1} = y_n + 0.2(x_n + y_n)$$

Ex3 (Improved Euler Method)

$$y' = f(x,y) = x+y, \quad y(0) = 0, \quad h = 0.2$$

$$\Rightarrow k_1 = 0.2(x_n + y_n)$$

$$k_2 = 0.2 f(x_{n+1}, y_{n+1}^{*}) = 0.2(x_{n+1} + y_{n+1}^{*})$$

$$= 0.2(x_n + 0.2 + y_n + 0.2(x_n + y_n))$$

$$y_{n+1} = y_n + \frac{0.2}{2}(2.2x_n + 2.2y_n + 0.2)$$

$$= y_n + 0.22(x_n + y_n) + 0.02$$

Runge-Kutta Methods (RK Methods)

Classical Runge-Kutta Method of Fourth Order

$$k_1 = h f(x_n, y_n)$$
$$k_2 = h f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1)$$
$$k_3 = h f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_2)$$
$$k_4 = h f(x_n + h, y_n + k_3)$$
$$\Rightarrow \begin{cases} x_{n+1} = x_n + h \\ y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{cases}$$

Ex3 (Classical Runge-Kutta Method)

$$y' = f(x, y) = x + y, \quad h = 0.2$$
$$\Rightarrow k_1 = 0.2(x_n + y_n)$$
$$k_2 = 0.2(x_n + 0.1 + y_n + 0.5k_1) = 0.22(x_n + y_n) + 0.02$$
$$k_3 = 0.2(x_n + 0.1 + y_n + 0.5k_2) = 0.222(x_n + y_n) + 0.022$$
$$k_4 = 0.2(x_n + 0.2 + y_n + k_3) = 0.2444(x_n + y_n) + 0.0444$$

$$\Rightarrow y_{n+1} = y_n + 0.2214(x_n + y_n) + 0.0214$$

Fehlberg's fifth-order RK Method

$$y_{n+1} = y_n + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6$$

Fehlberg's fourth-order RK Method

$$y_{n+1}^{*} = y_n + \frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4104}k_4 - \frac{1}{5}k_5$$

where $k_1 = h f(x_n, y_n)$, $k_2 = h f(x_n + \frac{1}{4}h, y_n + \frac{1}{4}k_1)$

$k_3 = h f(x_n + \frac{3}{8}h, y_n + \frac{3}{32}k_1 + \frac{9}{32}k_2)$,

....

21.3 Methods for Systems and Higher Order ODEs

Euler Method

$y'' + 2y' + 0.75y = 0, \quad y(0) = 3, \; y'(0) = -2.5, \; h = 0.2$

$\Rightarrow \mathbf{y}_{n+1} = \mathbf{y}_n + h \cdot \mathbf{f}(x_n, \mathbf{y}_n)$

$$\begin{cases} y_{1,n+1} = y_{1,n} + h f_1(x_n, y_{1,n}, y_{2,n}) \\ y_{2,n+1} = y_{2,n} + h f_2(x_n, y_{1,n}, y_{2,n}) \end{cases}$$

$$\begin{bmatrix} f_1(x, y_1, y_2) \\ f_2(x, y_1, y_2) \end{bmatrix} = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_2 \\ -2y_2 - 0.75y_1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} y_{1,n+1} = y_{1,n} + 0.2\, y_{2,n} \\ y_{2,n+1} = y_{2,n} + 0.2(-2y_{2,n} - 0.75\, y_{1,n}) \end{cases}$$

See Table 21.10

The results are not accurate enough for practical purposes.

Runge-Kutta Methods

$\mathbf{y}(x_0) = \mathbf{y}_0$

$$\begin{cases} \mathbf{k}_1 = h\mathbf{f}(x_n, \mathbf{y}_n) \\ \mathbf{k}_2 = h\mathbf{f}(x_n + \frac{1}{2}h, \mathbf{y}_n + \frac{1}{2}\mathbf{k}_1) \\ \mathbf{k}_3 = h\mathbf{f}(x_n + \frac{1}{2}h, \mathbf{y}_n + \frac{1}{2}\mathbf{k}_2) \\ \mathbf{k}_4 = h\mathbf{f}(x_n + h, \mathbf{y}_n + \mathbf{k}_3) \end{cases}$$

$\Rightarrow \mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4)$

21.4 Methods for Elliptic PDEs

Quasilinear : $\underline{a u_{xx} + 2b u_{xy} + c u_{yy}} = F(x, y, u, u_x, u_y)$

Elliptic : $ac - b^2 > 0$ (Ex : Laplace Eq.)
Parabolic : $ac - b^2 = 0$ (Ex : Heat Eq.)
Hyperbolic : $ac - b^2 < 0$ (Ex : Wave Eq.)

Laplace Eq : $\nabla^2 u = u_{xx} + u_{yy} = 0$
Poisson Eq : $\nabla^2 u = u_{xx} + u_{yy} = f(x,y)$

$$\begin{cases} u_x(x,y) \approx \frac{1}{2h}\left[u(x+h,y) - u(x-h,y)\right] \\ u_y(x,y) \approx \frac{1}{2k}\left[u(x,y+k) - u(x,y-k)\right] \end{cases}$$

$$u_{xx}(x,y) \approx \frac{1}{h^2}\left[u(x+h,y) - 2u(x,y) + u(x-h,y)\right]$$
$$u_{yy}(x,y) \approx \frac{1}{k^2}\left[u(x,y+k) - 2u(x,y) + u(x,y-k)\right]$$
$$u_{xy}(x,y) \approx \frac{1}{4hk}\left[u(x+h,y+k) - u(x-h,y+k) - u(x+h,y-k) + u(x-h,y-k)\right]$$

Let $h = k$ and
substitute into Poisson and Laplace Eqs

$\Rightarrow$ <u>Difference Equation for the Poisson Eq.</u>

$$u(x+h,y) + u(x,y+h) + u(x-h,y) + u(x,y-h) - 4u(x,y) = h^2 f(x,y)$$
or $u(E) + u(N) + u(W) + u(S) - 4u(x,y) = h^2 f(x,y)$

<u>Difference Equation for the Laplace Eq.</u>

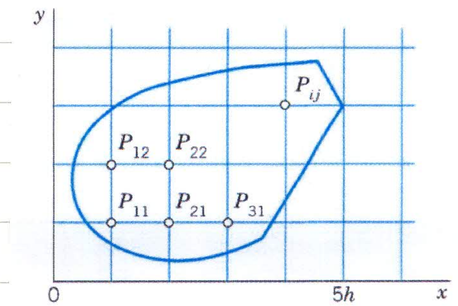
$$u(x+h,y) + u(x,y+h) + u(x-h,y) + u(x,y-h) - 4u(x,y) = 0$$

Dirichlet Problem

(u is prescribed on the boundary curve C of a region R)

Let $P_{ij} = (ih, jh)$ $\Rightarrow$ $\underline{u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{ij} = 0}$

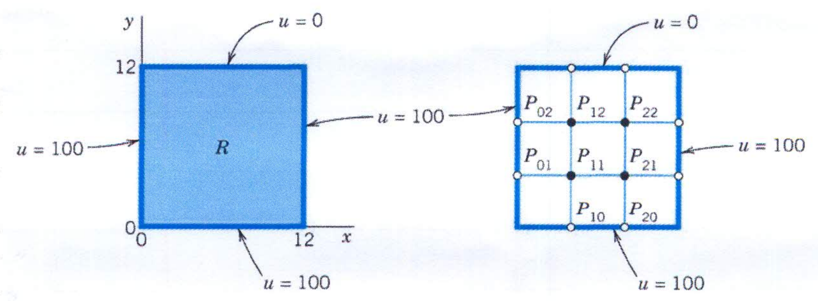
$u_{ij} = u(ih, jh)$



Ex1:

$u_{xx} + u_{yy} = 0$

$$\begin{cases} -4u_{11} + u_{21} + u_{12} = -200 \\ u_{11} - 4u_{21} + u_{22} = -200 \\ u_{11} - 4u_{12} + u_{22} = -100 \\ u_{21} + u_{12} - 4u_{22} = -100 \end{cases}$$



(a) Given problem (b) Grid and mesh points

$$\begin{cases} u_{11} = 0.25u_{21} + 0.25u_{12} + 50 \\ u_{21} = 0.25u_{11} + 0.25u_{22} + 50 \\ u_{12} = 0.25u_{11} + 0.25u_{22} + 25 \\ u_{22} = 0.25u_{21} + 0.25u_{12} + 25 \end{cases}$$

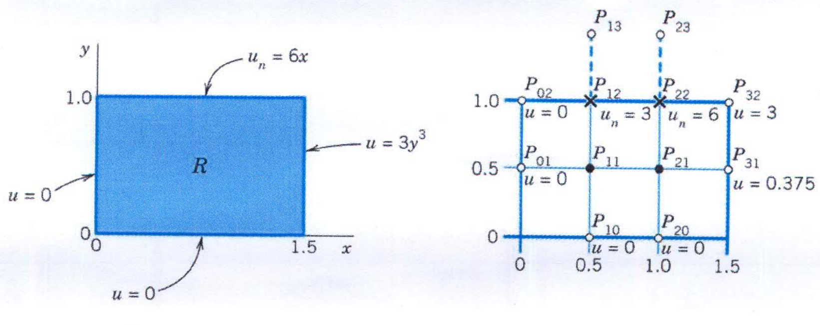
$\Rightarrow$ Apply the Gauss-Seidel Iteration

21.5 Neumann and Mixed Problems

Neumann Problem : $u_n = \frac{\partial u}{\partial n}$ is prescribed on the boundary C

Mixed Problem : u is prescribed on a part of C

u_n —— 〃 —— on the remaining part



(a) Region R and boundary values

(b) Grid ($h = 0.5$)

Ex 1 :

$$\nabla^2 u = u_{xx} + u_{yy} = f(x,y) = 12xy, \quad h = 0.5,$$

$$h^2 f(x,y) = 3xy$$

$$\begin{cases} -4u_{11} + u_{21} + u_{12} = 12(0.5)(0.5)\cdot\frac{1}{4} - 0 = 0.75 \\ u_{11} - 4u_{21} + u_{22} = 12(1)(0.5)\cdot\frac{1}{4} - 0.375 = 1.125 \end{cases}$$

$$\begin{cases} u_{11} - 4u_{12} + u_{22} + u_{13} = 12(0.5)(1)\cdot\frac{1}{4} - 0 = 1.5 \\ u_{21} + u_{12} - 4u_{22} + u_{23} = 12(1)(1)\cdot\frac{1}{4} - 3 = 0 \end{cases}$$

$$u_{13} = u_{11} + 3 \quad \left(\because\ 3 = \frac{\partial u_{12}}{\partial y} \approx \frac{u_{13} - u_{11}}{2h} = u_{13} - u_{11}\right)$$

$$u_{23} = u_{21} + 6 \quad \left(\because\ 6 = \frac{\partial u_{22}}{\partial y} \approx \frac{u_{23} - u_{21}}{2h} = u_{23} - u_{21}\right)$$

$$\begin{cases} 2u_{11} - 4u_{12} + u_{22} = 1.5 - 3 = -1.5 \\ 2u_{21} + u_{12} - 4u_{22} = 0 - 6 = -6 \end{cases}$$

$$\therefore \begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 2 & 0 & -4 & 1 \\ 0 & 2 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0.75 \\ 1.125 \\ -1.5 \\ -6 \end{bmatrix}$$

21.6 Methods for Parabolic PDEs

Heat Equation

$$\begin{cases} u_t = u_{xx}, & 0 \leq x \leq 1, \quad t \geq 0 \\ u(x,0) = f(x) \\ u(0,t) = u(1,t) = 0 \end{cases}$$

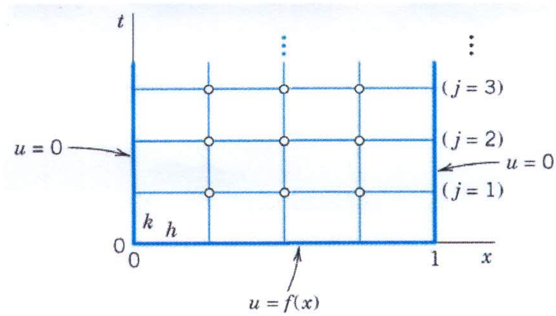
$$(4) \quad \frac{1}{k}\left(u_{i,j+1} - u_{ij}\right) = \frac{1}{h^2}\left(u_{i+1,j} - 2u_{ij} + u_{i-1,j}\right)$$

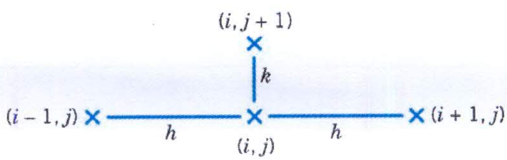
$$(5) \quad \underline{u_{i,j+1} = (1-2r)u_{ij} + r\left(u_{i+1,j} + u_{i-1,j}\right)}, \quad r = \frac{k}{h^2}$$

Crucial to the convergence is the condition

$$r = \frac{k}{h^2} \leq \frac{1}{2}$$

Intuitively, this means that we should not move too fast in the t-direction



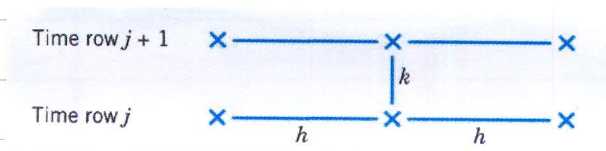


Crank-Nicolson Method

(7) $$\frac{1}{k}\left(u_{i,j+1}-u_{ij}\right)=\frac{1}{2h^2}\left(u_{i+1,j}-2u_{ij}+u_{i-1,j}\right)+\frac{1}{2h^2}\left(u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1}\right)$$

Let $r=\frac{k}{h^2}$, then we have

(8) $$(2+2r)\,u_{i,j+1}-r\left(u_{i+1,j+1}+u_{i-1,j+1}\right)=(2-2r)\,u_{ij}+r\left(u_{i+1,j}+u_{i-1,j}\right)$$



When we choose $r=1$,

$$4u_{i,j+1}-u_{i+1,j+1}-u_{i-1,j+1}=u_{i+1,j}+u_{i-1,j}$$

Ex1:

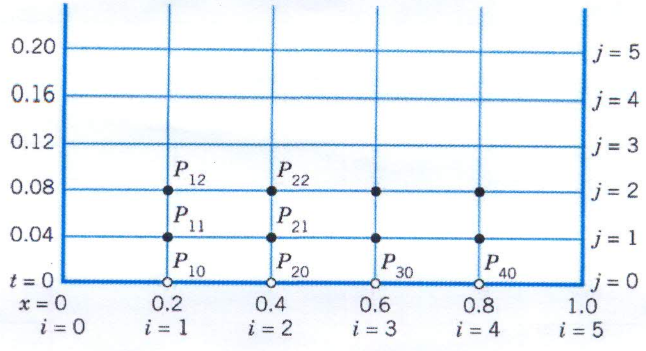
$$\begin{cases} u_t=u_{xx}, & 0\leq x\leq 1,\ t\geq 0 \\ u(x,0)=f(x)=\sin\pi x \\ u(0,t)=u(1,t)=0 \end{cases}$$

Let $h=0.2$ and $r=1$, find $u(x,t)$ for $0\leq t\leq 0.2$

$\Rightarrow$ $r=\frac{k}{h^2}=1$, $k=0.04$, $\frac{0.2}{0.04}=5$ steps

$$\begin{cases} u_{10}=\sin 0.2\pi=0.587785 \\ u_{20}=\sin 0.4\pi=0.951057 \\ u_{30}=u_{20} \\ u_{40}=u_{10} \end{cases}$$

($\because f(x)=\sin\pi x$ is symmetric w.r.t $x=0.5$)



$u_{10} = \sin 0.2\pi = 0.587785, \qquad u_{20} = \sin 0.4\pi = 0.951057.$

(9) $\underline{4u_{i,j+1} - u_{i+1,j+1} - u_{i-1,j+1} = u_{i+1,j} + u_{i-1,j}}$

For $j=0$,

$(i=1):\ 4u_{11} - u_{21} - \cancel{u_{01}}^{\,0} = u_{20} + \cancel{u_{00}}^{\,0} = 0.951057$

$(i=2):\ -u_{11} + 4u_{21} - \underset{\underset{u_{31}}{\parallel}}{u_{21}} = \underset{\underset{u_{30}}{\parallel}}{u_{20}} + u_{10} = 1.538842$

$\Rightarrow u_{11} = 0.399274,\ u_{21} = 0.646039$

For $j=1$,

$(i=1):\ 4u_{12} - u_{22} = u_{01} + u_{21} = 0.646039$

$(i=2):\ -u_{12} + 3u_{22} = u_{11} + u_{21} = 1.045313$

$\Rightarrow u_{12} = 0.271221,\ u_{22} = 0.438844$

t	$x = 0$	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$	$x = 1$
0.00	0	0.588	0.951	0.951	0.588	0
0.04	0	0.399	0.646	0.646	0.399	0
0.08	0	0.271	0.439	0.439	0.271	0
0.12	0	0.184	0.298	0.298	0.184	0
0.16	0	0.125	0.202	0.202	0.125	0
0.20	0	0.085	0.138	0.138	0.085	0

21.7 Methods for Hyperbolic PDEs

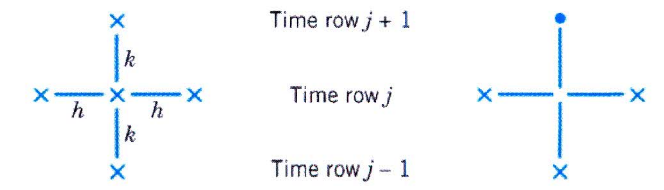
Wave Equation

$$\begin{cases} u_{tt} = u_{xx} \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \\ u(0,t) = u(1,t) = 0 \end{cases}$$

$$(5) \quad \frac{1}{k^2}\left(u_{i,j+1} - 2u_{ij} + u_{i,j-1}\right) = \frac{1}{h^2}\left(u_{i+1,j} - 2u_{ij} + u_{i-1,j}\right)$$

When we choose $r^* = \frac{k^2}{h^2} = 1$, we have

$$(6) \quad \underline{u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}}$$



(a) Formula (5) (b) Formula (6)

From $u_t(x,0) = g(x)$,

$$\frac{1}{2k}\left(u_{i,1} - u_{i,-1}\right) = g_i \Rightarrow u_{i,-1} = u_{i,1} - 2k g_i$$

$$\Rightarrow u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,-1}$$

$$= u_{i-1,0} + u_{i+1,0} - u_{i,1} + 2k \cdot g_i$$

$$\therefore u_{i,1} = \frac{1}{2}\left(u_{i-1,0} + u_{i+1,0}\right) + k \cdot g_i$$

Ex 1:

$$\begin{cases} u_{tt} = u_{xx}, & 0 \leq x \leq 1,\ t \geq 0 \\ u(x,0) = f(x) = \sin \pi x \\ u_t(x,0) = g(x) = 0 \\ u(0,t) = u(1,t) = 0, & h = k = 0.2 \end{cases}$$

$u_{00} = u_{50} = 0$

$u_{10} = u_{40} = \sin 0.2\pi = 0.587785$

$u_{20} = u_{30} = \sin 0.4\pi = 0.951057$

$$\underline{u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0})}$$

$(i=1)$: $u_{11} = \frac{1}{2}(u_{00} + u_{20}) = 0.475528$

$(i=2)$: $u_{21} = \frac{1}{2}(u_{10} + u_{30}) = 0.769421$

$u_{31} = u_{21}$, $u_{41} = u_{11}$ by symmetry

(6) $\underline{u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}}$ with $j = 1$

$(i=1)$: $u_{12} = \underbrace{u_{01}}_{0} + u_{21} - u_{10} = 0.181636$

$(i=2)$: $u_{22} = u_{11} + u_{31} - u_{20} = 0.293892$

$u_{32} = u_{22}$, $u_{42} = u_{12}$ by symmetry

t	$x = 0$	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$	$x = 1$
0.0	0	0.588	0.951	0.951	0.588	0
0.2	0	0.476	0.769	0.769	0.476	0
0.4	0	0.182	0.294	0.294	0.182	0
0.6	0	−0.182	−0.294	−0.294	−0.182	0
0.8	0	−0.476	−0.769	−0.769	−0.476	0
1.0	0	−0.588	−0.951	−0.951	−0.588	0

These values are exact, the exact solution being

$u(x,t) = \sin \pi x \cos \pi t$