Chap 5. Eigenvalues and Eigenvectors

5.1 Introduction (all matrices are now square.)

$\underline{A\mathbf{x} = \lambda \mathbf{x} \text{ for some } \mathbf{x} \neq \mathbf{0}}$

Ex: (Initial Value Problem for ODE)

$$\begin{cases} \dfrac{dv}{dt} = 4v - 5w, & v = 8 \text{ at } t = 0 \\ \dfrac{dw}{dt} = 2v - 3w, & w = 5 \text{ at } t = 0 \end{cases}$$

$$u(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}, \quad u(0) = \begin{bmatrix} 8 \\ 5 \end{bmatrix}, \quad A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$$

Matrix Form: $\dfrac{du}{dt} = Au$ with $u = u(0)$ at $t = 0$

[Single equation: $\dfrac{du}{dt} = au$ with $u = u(0)$ at $t=0$.
Purely exponential solution: $u(t) = e^{at}u(0)$.]

$\Rightarrow \begin{cases} v(t) = e^{\lambda t} y \\ w(t) = e^{\lambda t} z \end{cases}$ or $\underline{u(t) = e^{\lambda t}\mathbf{x}}$ in vector notation

Look for pure exponential solutions:

$$\begin{cases} \lambda e^{\lambda t} y = 4e^{\lambda t} y - 5e^{\lambda t} z \\ \lambda e^{\lambda t} z = 2e^{\lambda t} y - 3e^{\lambda t} z \end{cases}$$

Cancelling $e^{\lambda t}$

Eigenvalue Problem: $\begin{cases} 4y - 5z = \lambda y \\ 2y - 3z = \lambda z \end{cases}$

Eigenvalue Equation: $\underline{A\mathbf{x} = \lambda\mathbf{x}}$ for some $\mathbf{x} \neq \mathbf{0}$

The Solutions of $A\mathbf{x} = \lambda\mathbf{x}$

$(A - \lambda I)\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$

$\boxed{5A}$ λ: eigenvalue of $A \Leftrightarrow A-\lambda I$ is singular

$\det(A-\lambda I)=0$: the characteristic equation

Each λ is associated with eigenvectors $\mathbf{x}$:

$(A-\lambda I)\mathbf{x}=\mathbf{0}$ or $A\mathbf{x}=\lambda\mathbf{x}$.

Ex: (Continued) $A-\lambda I=\begin{bmatrix}4-\lambda & -5\\ 2 & -3-\lambda\end{bmatrix}$

$|A-\lambda I| = \lambda^2-\lambda-2=(\lambda+1)(\lambda-2)=0$

$\therefore \lambda=-1$ or 2

① $\lambda_1=-1$: $(A-\lambda_1 I)\mathbf{x}=\begin{bmatrix}5 & -5\\ 2 & -2\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}\Rightarrow \mathbf{x}_1=\begin{bmatrix}1\\ 1\end{bmatrix}$

② $\lambda_2=2$: $(A-\lambda_2 I)\mathbf{x}=\begin{bmatrix}2 & -5\\ 2 & -5\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}\Rightarrow \mathbf{x}_2=\begin{bmatrix}5\\ 2\end{bmatrix}$

Pure exponential solutions to $d\mathbf{u}/dt=A\mathbf{u}$:

$$\left.\begin{aligned}\mathbf{u}(t)&=e^{\lambda_1 t}\mathbf{x}_1=e^{-t}\begin{bmatrix}1\\ 1\end{bmatrix}\\ \mathbf{u}(t)&=e^{\lambda_2 t}\mathbf{x}_2=e^{2t}\begin{bmatrix}5\\ 2\end{bmatrix}\end{aligned}\right\}\text{ special solutions}$$

Complete Solution : $\mathbf{u}(t)=c_1e^{\lambda_1 t}\mathbf{x}_1+c_2e^{\lambda_2 t}\mathbf{x}_2$

Initial Solution : $\mathbf{u}(0)=c_1\mathbf{x}_1+c_2\mathbf{x}_2$ or

$$\begin{bmatrix}1 & 5\\ 1 & 2\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}8\\ 5\end{bmatrix}\Rightarrow c_1=3,\ c_2=1$$

The solution to the original equation:

$$\mathbf{u}(t)=3e^{-t}\begin{bmatrix}1\\ 1\end{bmatrix}+e^{2t}\begin{bmatrix}5\\ 2\end{bmatrix}$$

Writing the two components separately,

Solution: $\begin{cases} v(t)=3e^{-t}+5e^{2t}, & v(0)=8\\ w(t)=3e^{-t}+2e^{2t}, & w(0)=5.\end{cases}$

Summary and Examples

Ex1: (Diagonal Matrix)

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \lambda_1 = 3,\ \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix};\ \lambda_2 = 2,\ \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Ex2: (Projection Matrix) $\Rightarrow \lambda = 1$ or 0

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow \lambda_1 = 1,\ \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix};\ \lambda_2 = 0,\ \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We have $\lambda = 1$ when $\mathbf{x}$ projects to itself, and $\lambda = 0$ when $\mathbf{x}$ projects to the zero vector.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ has } \lambda = 1, 1, 0, 0$$

Ex3: (Triangular Matrix)

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 4 & 5 \\ 0 & \frac{3}{4}-\lambda & 6 \\ 0 & 0 & \frac{1}{2}-\lambda \end{vmatrix} = (1-\lambda)\left(\frac{3}{4}-\lambda\right)\left(\frac{1}{2}-\lambda\right)$$

- The Gaussian factorization $A = LU$ is not suited to the purpose of transforming A into a diagonal or triangular matrix without changing its eigenvalues.
- The eigenvalue problem is computationally more difficult than $A\mathbf{x} = \mathbf{b}$.
- Normally, the pivots, diagonal entries, and eigenvalues are completely different. But

$\boxed{5B}$ The sum of the n eigenvalues equals the sum of the n diagonal entries:

$$\text{Trace of } A = \lambda_1 + \cdots + \lambda_n = a_{11} + \cdots + a_{nn}.$$

The product of the n eigenvalues equals the $\det(A)$.

5.2 Diagonalization of a Matrix

The eigenvectors diagonalize a matrix.

$\boxed{5C}$ Suppose the $n \times n$ matrix A has n lin. indep. eigenvectors. If these eigenvectors are the columns of a matrix S, then $S^{-1}AS$ is a diagonal matrix Λ. The eigenvalues of A are on the diagonal of Λ:

$$\text{Diagonalization} \quad S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

the eigenvector matrix   the eigenvalue matrix

<proof>

$$AS = A\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} Ax_1 & \cdots & Ax_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$AS = S\Lambda, \quad S^{-1}AS = \Lambda, \quad A = S\Lambda S^{-1}.$$

(S is invertible since its columns are lin. independent) ∎

Remarks:

① If A has no repeated eigenvalue, then the n eigenvectors are automatically independent. (See 5D below.) Any matrix with distinct eigenvalues can be diagonalized.

② The diagonalizing matrix S is not unique.

③ Other matrices S will not produce a diagonal Λ. Suppose y is the first column of S. Then $\lambda_1 y$ is the first column of $S\Lambda$, which is Ay (the first column of AS). Then y must be an eigenvector: $Ay = \lambda_1 y$. The order of the eigenvectors of S and the eigenvalues in Λ is the same.

Remark 4: Not all matrices possess n lin. indep. eigenvectors, so <u>not all matrices are diagonalizable</u>. An example is

$$A=\begin{bmatrix}0&1\\0&0\end{bmatrix} \Rightarrow \det(A-\lambda I)=\begin{vmatrix}-\lambda&1\\0&-\lambda\end{vmatrix}=\lambda^2=0 \Rightarrow \lambda_1=\lambda_2=0.$$

All eigenvectors of A are multiples of $(1,0)$:

$$\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}y\\z\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \underset{\sim}{x}=\begin{bmatrix}c\\0\end{bmatrix}$$

$\lambda=0$ is a double eigenvalue – its algebraic multiplicity is 2. But the geometric multiplicity is only 1 – there is only one independent eigenvector. <u>We can't construct S.</u>

$\ulcorner$ ∵ Otherwise, since $\lambda_1=\lambda_2=0$, Λ should be the zero matrix. But, if $\Lambda=S^{-1}AS=0$, then $A=S\Lambda S^{-1}=0$ # $\lrcorner$

The failure of diagonalization came from $\lambda_1=\lambda_2$.

Ex: $A=\begin{bmatrix}3&1\\0&3\end{bmatrix} \Rightarrow \lambda_1=\lambda_2=3$; $A=\begin{bmatrix}2&-1\\1&0\end{bmatrix} \Rightarrow \lambda_1=\lambda_2=1$.

These matrices are not singular. But the problem is the shortage of eigenvectors – which are needed for S.

Note:

- Diagonalizability of A depends on enough eigenvectors.
- Invertibility of A depends on nonzero eigenvalues.

Diagonalization can fail only if there are repeated eigenvalues. Even then, it does not always fail. $A=I$ has repeated eigenvalues $1,1,\cdots,1$, but it is already diagonal!

For an eigenvalue that is repeated p times, we need to check whether there are p independent eigenvectors — i.e., whether $A-\lambda I$ has rank $n-p$. To complete that circle of ideas, we have to show that distinct eigenvalues present no problem.

$\boxed{5D}$

If eigenvectors $\mathbf{x}_1, \cdots, \mathbf{x}_k$ correspond to different eigenvalues $\lambda_1, \cdots, \lambda_k$, then those eigenvectors are linearly independent.

Proof

Suppose $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = \mathbf{0}$, $\Rightarrow A(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) = c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 = \mathbf{0}$.

$\Rightarrow c_1(\lambda_1-\lambda_2)\mathbf{x}_1 = c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 - \lambda_2(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) = \mathbf{0}$

Since $\lambda_1 \neq \lambda_2$ and $\mathbf{x}_1 \neq \mathbf{0}$, we have $c_1 = 0$

and similarly $c_2 = 0$.

By induction, we can extend this argument to any number of eigenvectors. $\lrcorner$

Examples of Diagonalization

Ex1: $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, AS = S\Lambda = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

Ex2: $K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$: $90°$ rotation $\Rightarrow \det(K-\lambda I) = \lambda^2+1 = 0 \Rightarrow \lambda = \pm i$

$\lambda_1 = i$: $(K-iI)\mathbf{x}_1 = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$\lambda_2 = -i$: $(K+iI)\mathbf{x}_2 = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$$S = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \text{ and } S^{-1}KS = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

Powers and Products : A^k and AB

The eigenvalues of A^2 are exactly $\lambda_1^2, \cdots, \lambda_n^2$, and every eigenvector of A is also an eigenvector of A^2.
By squaring $S^{-1}AS$, we have $S^{-1}A^2S = S^{-1}ASS^{-1}AS = \Lambda^2$.
The matrix A^2 is diagonalized by the same S, so the eigenvectors are unchanged. The eigenvalues are squared.

$\boxed{5E}$

The eigenvalues of A^k are $\lambda_1^k, \cdots, \lambda_n^k$, and each eigenvector of A is still an eigenvector of A^k. When S diagonalizes A, it diagonalizes A^k:

$$\Lambda^k = (S^{-1}AS)(S^{-1}AS)\cdots(S^{-1}AS) = S^{-1}A^kS.$$

If A is invertible, this rule also applies to its inverse ($k=-1$)
The eigenvalues of A^{-1} are $1/\lambda_i$.

Pf: If $Ax = \lambda x$, then $x = \lambda A^{-1}x$ and $\frac{1}{\lambda}x = A^{-1}x$.

Ex3: $K = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{90° \text{ rotation}} \Rightarrow K^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $K^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\lambda_1 = i,\ \lambda_2 = -i\ ;\ \lambda_1^2 = -1,\ \lambda_2^2 = -1\ ;\ \frac{1}{\lambda_1} = -i,\ \frac{1}{\lambda_2} = i.$

$K^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\Lambda^4 = \begin{bmatrix} (i)^4 & 0 \\ 0 & (-i)^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- For a product of two matrices, the eigenvalues of AB have no good answer. The eigenvalues of AB and $A+B$ have nothing to do with $\mu\lambda$ and $\lambda+\mu$, where λ and μ are eigenvalues of A and B, respectively

$$\left[\begin{array}{l} ABx = A\mu x = \mu Ax \neq \mu\lambda x \\ (A+B)x \neq (\lambda+\mu)x \end{array}\right.$$ since A and B may not share the same eigenvector.

Counter Example: $AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \lambda_1 = 1, \lambda_2 = 0$

But, both A and B have zero eigenvalue.

$\boxed{5H}$ Diagonalizable matrices share the same eigenvector matrix S $\Leftrightarrow$ $AB=BA$

<proof>

$\Rightarrow$) If the same S diagonalizes both $A=S\Lambda_1 S^{-1}$ and $B=S\Lambda_2 S^{-1}$

$AB = S\Lambda_1 S^{-1} S\Lambda_2 S^{-1} = S\Lambda_1\Lambda_2 S^{-1}$ and

$BA = S\Lambda_2 S^{-1} S\Lambda_1 S^{-1} = S\Lambda_2\Lambda_1 S^{-1}$.

Since $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$ (diagonal matrices always commute), we have $AB=BA$.

$\Leftarrow$) Suppose $AB=BA$. Starting from $A\mathbf{x}=\lambda\mathbf{x}$, we have

$AB\mathbf{x} = BA\mathbf{x} = B\lambda\mathbf{x} = \lambda B\mathbf{x}$.

Thus $\mathbf{x}$ and $B\mathbf{x}$ are both eigenvectors of A, sharing the same λ (or else $B\mathbf{x}=\mathbf{0}$).

If we assume for convenience that the eigenvalues of A are distinct – the eigenspaces are all one-dimensional – then $B\mathbf{x}$ must be a multiple of $\mathbf{x}$. In other words, $\mathbf{x}$ is an eigenvector of B as well as A.

The proof with repeated eigenvalues is a little longer. ⌟

<u>(Spectral Theorem)</u>

Every real symmetric matrix A can be diagonalized by an orthogonal matrix Q (i.e. $Q^TQ=I$, $Q^T=Q^{-1}$):

$$Q^{-1}AQ = \Lambda \quad \text{or} \quad A = Q\Lambda Q^T$$

The columns of Q contain orthonormal eigenvectors of A.

5.3 Difference Equations and Powers A^k

You invest \$1000 at 6% interest. Compounded once a year, the principal P is multiplied by 1.06. This is a difference equation $P_{k+1} = AP_k = 1.06\,P_k$ with a time step of one year. After 5 years, the original $P_0 = 1000$ has been multiplied 5 times.

$$\text{Yearly: } P_5 = (1.06)^5 P_0 \text{ which is } (1.06)^5 1000 = \$1338.$$

Now the time step is reduced to a month. The new difference equation is $p_{k+1} = (1 + 0.06/12)\,p_k$. After 5 years, or 60 months, you have \$11 more

$$\text{Monthly: } p_{60} = \left(1 + \frac{0.06}{12}\right)^{60} p_0 \text{ which is } (1.005)^{60} 1000 = \$1349.$$

The next step is to compound every day, on 5×365 days.

$$\text{Daily: } \left(1 + \frac{0.06}{365}\right)^{5\times365} 1000 = \$1349.83$$

The interest is added at every instant. The bank could compound the interest N times a year, so $\Delta t = 1/N$.

$$\text{Continuously: } \left(1 + \frac{0.06}{N}\right)^{5N} 1000 \to e^{0.30}\, 1000 = \$1349.87$$

The bank can switch to a differential equation – the limit of the difference equation $p_{k+1} = (1 + 0.06\,\Delta t)\,p_k$.

$$\text{Discrete to Continuous: } \frac{p_{k+1} - p_k}{\Delta t} = 0.06\,p_k \text{ approaches } \frac{dp}{dt} = 0.06\,p.$$

The solution is $p(t) = e^{0.06t} p_0$. After $t = 5$ years, this amounts to \$1349.87. The improvement over compounding every day is only 4 cents.

Fibonacci Numbers

Fibonacci numbers: 0, 1, 1, 2, 3, 5, 8, 13, ⋯

Fibonacci equation: $F_{k+2} = F_k + F_{k+1}$

How can we find F_{1000} without starting at $F_0 = 0$ and $F_1 = 1$, and working all the way out to F_{1000}?

$$\begin{cases} F_{k+2} = F_k + F_{k+1} \\ F_{k+1} = F_{k+1} \end{cases} \text{ becomes } u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = A u_k.$$

The solution to a difference equation $u_{k+1} = A u_k$ is $u_k = A^k u_0$.
The real problem is to find some quick way to compute A^k.

$\boxed{5G}$ If A can be diagonalized, $A = S \Lambda S^{-1}$, then $A^k = S \Lambda^k S^{-1}$,

$$u_k = A^k u_0 = S \Lambda^k S^{-1} u_0.$$

The columns of S are eigenvectors of A. Writing $S^{-1} u_0 = c$,

$$u_k = S \Lambda^k c = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c_1 \lambda_1^k x_1 + \cdots + c_n \lambda_n^k x_n$$

After k steps, u_k is a combination of the n pure solutions $\lambda^k x$.

In a specific example, the first step is to find the eigenvalues.

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \text{ has } \det(A - \lambda I) = \lambda^2 - \lambda - 1$$

Two eigenvalues $\quad \lambda_1 = \frac{1}{2}(1+\sqrt{5})$ and $\lambda_2 = \frac{1}{2}(1-\sqrt{5})$

The eigenvectors $\quad x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

$$u_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } S^{-1} u_0 = c$$

$$S^{-1} u_0 = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ gives } c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Fibonacci Numbers: $F_k = c_1 \lambda_1^k + c_2 \lambda_2^k = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^k - \left(\frac{1-\sqrt{5}}{2} \right)^k \right].$

Markov Matrices

Each year $\frac{1}{10}$ of the people outside California move in, and $\frac{2}{10}$ of the people inside California move out. We start with y_0 people outside and z_0 inside. At the end of the first year the numbers outside and inside are y_1 and z_1:

Difference equation: $\begin{aligned} y_1 &= 0.9y_0 + 0.2z_0 \\ z_1 &= 0.1y_0 + 0.8z_0 \end{aligned}$ or $\begin{bmatrix} y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}\begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$

Two essential properties of a Markov process:

1. The total number of people stay fixed.
2. The numbers outside and inside can never be negative.

We solve this Markov difference equation using $u_k = S\Lambda^k S^{-1} u_0$

$$A - \lambda I = \begin{bmatrix} 0.9-\lambda & 0.2 \\ 0.1 & 0.8-\lambda \end{bmatrix} \text{ has } \det(A-\lambda I) = \lambda^2 - 1.7\lambda + 0.7$$

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 0.7 \end{aligned} \Rightarrow A = S\Lambda S^{-1} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix}\begin{bmatrix} 1 & \\ & 0.7 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} y_k \\ z_k \end{bmatrix} = A^k \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix}\begin{bmatrix} 1^k & \\ & 0.7^k \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

$$= (y_0 + z_0)\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} + (y_0 - 2z_0)(0.7)^k \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix}$$

In the long run, the factor $(0.7)^k$ becomes extremely small. The solution approaches a limiting state $u_\infty = (y_\infty, z_\infty)$

Steady State: $\begin{bmatrix} y_\infty \\ z_\infty \end{bmatrix} = (y_0 + z_0)\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$

$$\begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} \quad \text{or} \quad Au_\infty = u_\infty$$

The steady state is the eigenvector of A corresponding to $\lambda = 1$.

$\boxed{5K}$ A Markov matrix A has all $a_{ij} \geq 0$, with each column adding to 1.

(a) $\lambda_1 = 1$ is an eigenvalue of A.

(b) Its eigenvector x_1 is nonnegative and $Ax_1 = x_1$ (steady state).

(c) The other eigenvalues satisfy $|\lambda_i| \leq 1$.

(d) If A or any power of A has all positive entries, these other $|\lambda_i|$ are below 1. The solution $A^k u_0$ approaches a multiple of $x_1 \Rightarrow$ the steady state u_∞.

Remark

- Why is $\lambda = 1$ always an eigenvalue? Each column of $A - I$ adds up to $1 - 1 = 0$. The rows of $A-I$ add up to the zero row. They are linearly dependent and $\det(A - I) = 0$.
- In the formula $u_k = c_1 \lambda_1^k x_1 + \cdots + c_n \lambda_n^k x_n$, no eigenvalue can be larger than 1. (Otherwise, u_k would blow up.) If all other eigenvalues are strictly smaller than $\lambda_1 = 1$, then the first term in the formula will be dominant. The other λ_i^k go to zero, and $u_k \to c_1 x_1 = u_\infty =$ steady state.

Stability of $u_{k+1} = A u_k$

We want to study the behavior of $u_{k+1} = A u_k$ as $k \to \infty$. Assuming that A can be diagonalized, u_k will be a combination of pure solutions: $u_k = S \Lambda^k S^{-1} u_0 = c_1 \lambda_1^k x_1 + \cdots + c_n \lambda_n^k x_n$. The growth of u_k is governed by the λ_i^k. Stability depends on λ_i

$\boxed{5J}$ The difference equation $u_{k+1} = A u_k$ is

stable if all eigenvalues $|\lambda_i| < 1$;

neutrally stable if some $|\lambda_i| = 1$ and all the other $|\lambda_i| < 1$;

unstable if at least one eigenvalue $|\lambda_i| > 1$.

- In the stable case, A^k approaches zero and so does $u_k = A^k u_0$.

5.4 Differential Equations and e^{At}

Differential Equation : $\frac{du}{dt} = Au = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} u$

The first step is to find the eigenvalues (-1 and -3) and the eigenvectors: $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-3)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Solution: $u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{-t} & \\ & e^{-3t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

Initial Condition : $u(0) = c_1 x_1 + c_2 x_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = Sc$

$\Rightarrow u(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = S\begin{bmatrix} e^{-t} & \\ & e^{-3t} \end{bmatrix} S^{-1} u_0$.

Fundamental formula $Se^{\Lambda t}S^{-1}u(0)$ solves the differential eq:

$u(t) = Se^{\Lambda t}S^{-1}u(0)$ with $\Lambda = \begin{bmatrix} -1 & \\ & -3 \end{bmatrix}$ and $e^{\Lambda t} = \begin{bmatrix} e^{-t} & \\ & e^{-3t} \end{bmatrix}$

Matrix Exponential : $e^{At} = I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \cdots$

The series always converges, and its sum e^{At} has the properties

① $(e^{As})(e^{At}) = e^{A(s+t)}$ and $(e^{At})(e^{-At}) = I$.

② $\frac{d}{dt}(e^{At}) = Ae^{At} \Rightarrow u(t) = e^{At}u(0)$ solves the ODE.

③ This solution must be the same as the form $Se^{\Lambda t}S^{-1}u(0)$.

$$\begin{aligned} e^{At} &= I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \cdots \\ &= I + S\Lambda S^{-1}t + \frac{1}{2!}S\Lambda^2 S^{-1}t^2 + \frac{1}{3!}S\Lambda^3 S^{-1}t^3 + \cdots \\ &= S\left(I + \Lambda t + \frac{1}{2!}\Lambda^2 t^2 + \frac{1}{3!}\Lambda^3 t^3 + \cdots\right)S^{-1} \\ &= Se^{\Lambda t}S^{-1} \end{aligned}$$

Ex1: $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \Rightarrow \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$

$$e^{At} = S e^{\Lambda t} S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & \\ & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

$$= \frac{1}{2} \begin{bmatrix} e^{-t} + e^{-3t} & e^{-t} - e^{-3t} \\ e^{-t} - e^{-3t} & e^{-t} + e^{-3t} \end{bmatrix}$$

at $t=0$, we get $e^0 = I$. The infinite series e^{At} gives the result for all t, but a series can be hard to compute. The form $Se^{\Lambda t}S^{-1}$ gives the same answer when A can be diagonalized; it requires n independent eigenvectors in S. This simpler form leads to a combination of n exponentials $e^{\lambda t}x$ — which is the best solution of all.

$\boxed{5L}$

If A can be diagonalized, $A = S\Lambda S^{-1}$, then $du/dt = Au$ has the solution: $u(t) = e^{At}u(0) = Se^{\Lambda t}S^{-1}u(0)$.

The columns of S are the eigenvectors $x_1, \cdots, x_n$ of A.

$$u(t) = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} S^{-1} u(0)$$

$$= c_1 e^{\lambda_1 t} x_1 + \cdots + c_n e^{\lambda_n t} x_n = \text{combination of } e^{\lambda t}x.$$

The constants that match the initial conditions $u(0)$ are $c = S^{-1}u(0)$.

- e^{At} is never singular ($\because \det e^{At} = e^{\lambda_1 t} \cdots e^{\lambda_n t} = e^{\text{trace}\,At}$)

  Quick proof: Just recognize e^{-At} as its inverse.

- $[e^{At}v_1 \cdots e^{At}v_n] = e^{At}[v_1 \cdots v_n]$

  $\Rightarrow$ If n solutions are linearly independent at $t=0$, they remain linearly independent forever.

5.6 Similarity Transformations

The matrices A and $M^{-1}AM$ are similar. Two questions:

① What do the similar matrices $M^{-1}AM$ have in common?

② With a special choice of M, what special form can be achieved by $M^{-1}AM$? ⇒ The Jordan form.

∘ The combinations $M^{-1}AM$ arise in a differential or difference equation, when a change of variables $u = Mv$ introduces v.

$$\frac{du}{dt} = Au \text{ becomes } M\frac{dv}{dt} = AMv \text{ or } \frac{dv}{dt} = M^{-1}AMv.$$

$$u_{n+1} = Au_n \text{ becomes } Mv_{n+1} = AMv_n \text{ or } v_{n+1} = M^{-1}AMv_n.$$

In the special case $M = S$, the system is uncoupled because $\Lambda = S^{-1}AS$ is diagonal. Other M's are also useful.

$\boxed{5P}$

Suppose that $B = M^{-1}AM$. Then A and B have the same eigenvalues. Every eigenvector x of A corresponds to an eigenvector $M^{-1}x$ of B.

∵ $Ax = \lambda x \Rightarrow A = MBM^{-1}$ and $MBM^{-1}x = \lambda x \Rightarrow BM^{-1}x = \lambda(M^{-1}x)$.

∘ In fact, $A - \lambda I$ and $B - \lambda I$ have the same determinant.

Product of matrices $B - \lambda I = M^{-1}AM - \lambda I = M^{-1}(A - \lambda I)M$.

Product rule $\det(B - \lambda I) = \det M^{-1} \det(A - \lambda I) \det M = \det(A - \lambda I)$

The polynomials $\det(A - \lambda I)$ and $\det(B - \lambda I)$ are equal.

Their roots — the eigenvalues of A and B — are the same.

Ex1:

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has $\lambda = 1$ and 0. Each B is $M^{-1}AM$.

If $M = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$, then $B = \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}$: triangular with $\lambda = 1$ and 0.

If $M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, then $B = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$: projection with $\lambda = 1$ and 0.

If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $B =$ arbitrary matrix with $\lambda = 1$ and 0.

Change of Basis = Similarity Transformation

Every linear transformation is represented by a matrix. The matrix depends on the choice of basis! If we change the basis by M, we change the matrix A to a similar matrix B.

Suppose we have a basis $v_1, \cdots, v_n$. The jth column of A comes from $\underline{Tv_j = a_{1j}v_1 + \cdots + a_{nj}v_n}$: combination of v's.

For a new basis $V_1, \cdots, V_n$, the new matrix B is constructed in the same way : $\underline{TV_j = b_{1j}V_1 + \cdots + b_{nj}V_n}$: combination of V's

Each V must be a combination of v's : $\underline{V_j = \sum m_{ij} v_i}$

o. The matrix is really representing the identity transformation when the only thing happening is the change of basis (T is I). The inverse matrix M^{-1} also represents the identity transform when the basis is changed from the v's back to the V's

$\boxed{SQ}$

The matrices A and B that represent the same transformation T with respect to two different bases are similar :

$$[T]_{V \text{ to } V} = [I]_{v \text{ to } V} \; [T]_{v \text{ to } v} \; [I]_{V \text{ to } v}$$

$$B = M^{-1} \quad A \quad M$$

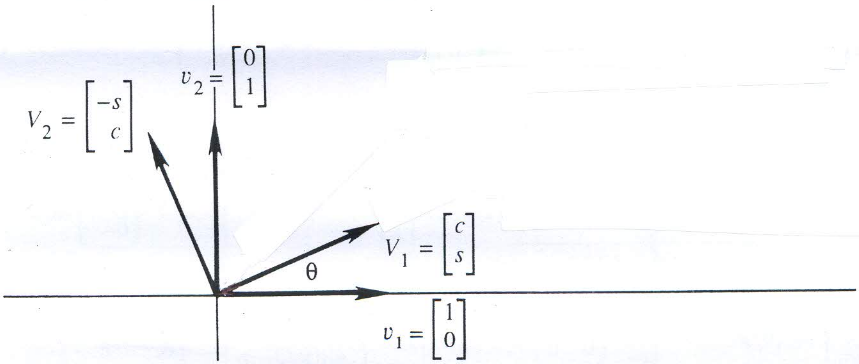

Fig. 5.5. Change of basis to make the projection matrix diagonal.

$TV_1 = V_1$ and $TV_2 = 0$. The matrix is diagonal

Eigenvector basis $\quad B = [T]_{V \text{ to } V} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$V_1 = (\cos\theta)v_1 + (\sin\theta)v_2$ and $V_2 = (-\sin\theta)v_1 + (\cos\theta)v_2$.

Change of basis $\quad M = [I]_{V \text{ to } v} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$

The inverse matrix M^{-1} goes from v to V.

Standard basis $\quad A = MBM^{-1} = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$

- The way to diagonalize A is to find its eigenvectors. They go into the columns of M (or S) and $M^{-1}AM$ is diagonal.
- In the language of linear transformation: Choose a basis consisting of eigenvectors. The standard basis led to A, which was not simple. The right basis led to B: diagonal.
- Note: $M^{-1}AM$ does not arise in solving $A\mathbf{x} = \mathbf{b}$. There the basic operation was to multiply A (on the left) by a matrix that preserves the nullspace and row space of A. It normally changes the eigenvalues.
- Eigenvalues are actually calculated by a sequence of simple similarities. The matrix goes gradually toward a triangular form, and the eigenvalues gradually appear on the main diagonal (see more details in Chap 7). This is much better than trying to compute $\det(A - \lambda I) = 0$. For a large matrix, it is numerically impossible to concentrate all that information into the polynomial and then get it out again.

The Jordan Form

If A has a full set of eigenvectors, we take $M=S$ and arrive at $J=S^{-1}AS=\Lambda$. Then the Jordan form coincides with the diagonal Λ. This is impossible for a defective (nondiagonalizable) matrix. For every missing eigenvector, the Jordan form will have a 1 just above its main diagonal. The eigenvalues appear on the diagonal because J is triangular. And distinct eigenvalues can always be decoupled. It is only a repeated λ that may (or may not) require an off-diagonal 1 in J.

$\boxed{5U}$

If A has s independent eigenvectors, it is similar to a matrix with s blocks:

Jordan form: $$J = M^{-1}AM = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix}$$

Each Jordan block J_i is a triangular matrix that has only a single eigenvalue λ_i and only one eigenvector:

Jordan block $$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

The same λ_i will appear in several blocks, if it has several independent eigenvectors. Two matrices are similar if and only if they share the same Jordan form J.

- Not all matrices are diagonalizable, and the Jordan form is the most general case. Its construction is both technically and extremely unstable. A slight change in A can put back all the missing eigenvectors, and remove the off-diagonal 1s.

Ex4:

all lead to

$$T=\begin{bmatrix}1&2\\0&1\end{bmatrix},\ A=\begin{bmatrix}2&-1\\1&0\end{bmatrix},\ B=\begin{bmatrix}1&0\\1&1\end{bmatrix}\Rightarrow J=\begin{bmatrix}1&1\\0&1\end{bmatrix}$$

These four matrices have eigenvalues 1 and 1 with only one eigenvector – so J consists of one block. The determinants all equal 1. The traces are 2. The eigenvalues satisfy $1\cdot1=1$ and $1+1=2$. The eigenvalues are on the diagonal. These matrices are similar – they belong to the same family.

$$(T)\ M^{-1}TM=\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}1&2\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&\frac{1}{2}\end{bmatrix}=\begin{bmatrix}1&1\\0&1\end{bmatrix}=J$$

$$(B)\ P^{-1}BP=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1&0\\1&1\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}=\begin{bmatrix}1&1\\0&1\end{bmatrix}=J$$

$$(A)\ U^{-1}AU=\begin{bmatrix}1/\sqrt{2}&1/\sqrt{2}\\1/\sqrt{2}&-1/\sqrt{2}\end{bmatrix}\begin{bmatrix}2&-1\\1&0\end{bmatrix}\begin{bmatrix}1/\sqrt{2}&1/\sqrt{2}\\1/\sqrt{2}&-1/\sqrt{2}\end{bmatrix}=\begin{bmatrix}1&1\\0&1\end{bmatrix}=T$$

and then $M^{-1}TM=J$.

Ex5: $A=\begin{bmatrix}0&1&2\\0&0&1\\0&0&0\end{bmatrix}$ and $B=\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}\Rightarrow\lambda_1=\lambda_2=\lambda_3=0.$

There can be a single 3x3 block, or a 2x2 block and a 1x1 block, or three 1x1 blocks. Three possible Jordan forms are

$$J_1=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix},\ J_2=\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix},\ J_3=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

The only eigenvector of A is (1,0,0). Its Jordan form has only one block, and A must be similar to J_1. The matrix B has the additional eigenvector (0,1,0), and its Jordan form is J_2 with two blocks. As for $J_3=0$, it is in a family by itself; the only matrix similar to J_3 is $M^{-1}OM=0$.

Ex 6: Application to difference and differential equations (powers and exponentials). If A can be diagonalized, the powers of $A = S\Lambda S^{-1}$ are easy: $A^k = S\Lambda^k S^{-1}$.
In every case we have Jordan's similarity: $A = MJM^{-1}$, so now we need the powers of J: $A^k = MJ^kM^{-1}$.
J is block-diagonal, the powers $(J_i)^k$ can be taken separately

$$(J_i)^k = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{1}{2}k(k-1)\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}.$$

This block J_i will enter when λ is a triple eigenvalue with a single eigenvector. Its exponential is in the solution to the corresponding differential equation:

$$\text{Exponential} \quad e^{J_i t} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2}t^2e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$$

The third column of this exponential comes directly from solving

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \text{ starting from } u_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

This can be solved by back-substitution.

These powers and exponentials of J are a part of the solutions u_k and $u(t)$. The other part is the M that connects A to J:

if $u_{k+1} = Au_k$, then $u_k = A^k u_0 = MJ^kM^{-1}u_0$

if $du/dt = Au$, then $u(t) = e^{At}u(0) = Me^{Jt}M^{-1}u(0)$.