Chap 6. Laplace Transforms

6.1 Laplace Transform

$f(t)$: function defined for $t \geq 0$

$\Rightarrow F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt$: Laplace Transform

$f(t) = \mathcal{L}^{-1}(F)$: inverse transform

Ex 1 (Laplace Transform)

$f(t) = 1$ for $t \geq 0$

$$\mathcal{L}(f) = \mathcal{L}(1) = \int_0^\infty e^{-st} \cdot 1\,dt = \left[-\frac{1}{s} e^{-st}\right]_{t=0}^{\infty} = \frac{1}{s} \quad (s>0)$$

Ex 2

$f(t) = e^{at}$ for $t \geq 0$ (a: constant)

$$\mathcal{L}(f) = \mathcal{L}(e^{at}) = \int_0^\infty e^{-st} \cdot e^{at}\,dt = \left[\frac{1}{a-s} e^{(a-s)t}\right]_{t=0}^{\infty} = \frac{1}{s-a}$$

(if $s > a$)

Th 1 (Linearity of the Laplace Transform)

$$\mathcal{L}(af(t) + bg(t)) = a\mathcal{L}(f(t)) + b\mathcal{L}(g(t))$$

Ex 3 ($\cosh at = \frac{1}{2}(e^{at} + e^{-at})$, $\sinh at = \frac{1}{2}(e^{at} - e^{-at})$)

$$\mathcal{L}(\cosh at) = \frac{1}{2}\mathcal{L}(e^{at}) + \frac{1}{2}\mathcal{L}(e^{-at})$$

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2-a^2}$$

$$\mathcal{L}(\sinh at) = \frac{1}{2}\mathcal{L}(e^{at}) - \frac{1}{2}\mathcal{L}(e^{-at}) = \frac{a}{s^2-a^2}$$

Ex 4 (Cosine and Sine)

$$\mathcal{L}(\cos\omega t) + i\mathcal{L}(\sin\omega t) = \mathcal{L}(\cos\omega t + i\sin\omega t)$$

$$= \mathcal{L}(e^{i\omega t}) = \frac{1}{s-i\omega} = \frac{s+i\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^2} + i\frac{\omega}{s^2+\omega^2}$$

$$\therefore \underline{\mathcal{L}(\cos\omega t) = \frac{s}{s^2+\omega^2}}, \quad \underline{\mathcal{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2}}$$

Th

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, \quad \text{for } n=0,1,2,\cdots$$

<proof>

$n=0$: $\mathcal{L}(t^0) = \mathcal{L}(1) = \frac{1}{s}$

By induction, $\mathcal{L}(t^{n+1}) = \int_0^\infty e^{-st} t^{n+1}\, dt$

$$= \underbrace{\left[-\frac{1}{s}e^{-st}t^{n+1}\right]_{t=0}^{\infty}}_{=0} + \frac{n+1}{s}\int e^{-st}t^n\, dt$$

$$= \frac{(n+1)}{s}\cdot\frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}}$$

Th

$$\mathcal{L}(t^a) = \frac{\Gamma(a+1)}{s^{a+1}}, \quad \text{for } a>0$$

<proof>

Let $st = x$

$$\mathcal{L}(t^a) = \int_0^\infty e^{-st} t^a\, dt = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^a \frac{dx}{s}$$

$$= \frac{1}{s^{a+1}}\int_0^\infty e^{-x}x^a\, dx = \frac{\Gamma(a+1)}{s^{a+1}}. \quad (s>0)$$

Th 2 (First Shifting Theorem, s-Shifting)

$F(s) = \mathcal{L}(f(t))$, where $s > k$

$\Rightarrow \mathcal{L}[e^{at} f(t)] = F(s-a)$ (where $s-a > k$)

$e^{at} f(t) = \mathcal{L}^{-1}[F(s-a)]$

Ex 5

$$\mathcal{L}[e^{at}\cos\omega t] = \frac{s-a}{(s-a)^2+\omega^2}, \quad \mathcal{L}[e^{at}\sin\omega t] = \frac{\omega}{(s-a)^2+\omega^2}$$

To find the inverse of $\mathcal{L}(f) = \dfrac{3s-137}{s^2+2s+401}$,

$$f = \mathcal{L}^{-1}\left[\frac{3(s+1)-140}{(s+1)^2+400}\right] = 3\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+20^2}\right] - 7\mathcal{L}^{-1}\left[\frac{20}{(s+1)^2+20^2}\right]$$

$$= e^{-t}(3\cos 20t - 7\sin 20t)$$

Th 3 (Existence Theorem for Laplace Transforms)

$f(t)$: piecewise continuous on $\forall [a,b]$ s.t $0 \le a < b$

$|f(t)| \le Me^{kt}$ for all $t \ge 0$ and for some k and M

$\Rightarrow \exists \mathcal{L}(f)$ for all $s > k$

<proof>

$$|\mathcal{L}(f)| = \left|\int_0^\infty e^{-st} f(t)\,dt\right| \le \int_0^\infty |f(t)| e^{-st}\,dt$$

$$\le \int_0^\infty Me^{kt}\cdot e^{-st}\,dt = \frac{M}{s-k} \text{ for } s > k.$$

Remark: The conditions are sufficient, but not necessary.

Counterexample: $1/\sqrt{t} \to \infty$ as $t \to 0$. But $\mathcal{L}(1/\sqrt{t}) = \sqrt{\pi/s}$.

6.2 Transforms of Derivatives and Integrals

Th1 (Laplace Transform of Derivative)

$f(t)$: continuous and $|f(t)| \leqq Me^{kt}$, for all $t \geqq 0$.

$f'(t)$: piecewise continuous on $\forall [a, b]$ s.t $0 \leqq a < b$.

$\Rightarrow \mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ for $s > k$

<proof>

$$\mathcal{L}(f') = \int_0^\infty e^{-st} f'(t)\,dt = \left[e^{-st} f(t)\right]_{t=0}^{\infty} + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= s\mathcal{L}(f) - f(0) \quad \text{for } s > k \;//$$

Remark

$$\mathcal{L}(f'') = s\mathcal{L}(f') - f'(0) = s\left[s\mathcal{L}(f) - f(0)\right] - f'(0)$$

$$= s^2\mathcal{L}(f) - sf(0) - f'(0)$$

Th2 (Laplace Transform of Derivatives of Any Order)

$f(t), f'(t), \cdots, f^{(n-1)}(t)$: continuous for all $t \geqq 0$

$|f^{(i)}(t)| \leqq M \cdot e^{kt}$ for all $t \geqq 0$, for $i = 0, \cdots, n-1$

$f^{(n)}(t)$: piecewise continuous on $\forall [a, b]$ s.t $0 \leqq a < b$

$\Rightarrow \mathcal{L}(f^{(n)}) = s^n \mathcal{L}(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$.

Th3 (Laplace Transform of Integral)

f: piecewise continuous and $|f(t)| \leqq Me^{kt}$ for all $t \geqq 0$.

$\Rightarrow \mathcal{L}\left[\int_0^t f(\tau)\,d\tau\right] = \frac{1}{s}F(s)$ and $\int_0^t f(\tau)\,d\tau = \mathcal{L}^{-1}\left[\frac{1}{s}F(s)\right]$

for $s > 0$, $s > k$, $t > 0$

<proof> Let $g(t) = \int_0^t f(\tau)\,d\tau$,

$$|g(t)| \leqq \int_0^t |f(\tau)|\,d\tau \leqq M\int_0^t e^{k\tau}\,d\tau = \frac{M}{k}\left[e^{k\tau}\right]_{\tau=0}^{t} = \frac{M}{k}(e^{kt} - 1)$$

$$\leqq \frac{M}{k} e^{kt} \quad (k > 0)$$

$g'(t) = f(t)$

$\mathcal{L}[f(t)] = \mathcal{L}[g'(t)] = s\mathcal{L}[g(t)] - \overset{0}{g(0)}$

$\therefore \mathcal{L}\left[\int_0^t f(\tau)\,d\tau\right] = \frac{1}{s}F(s)$.

$\underline{\text{Ex 1}}$ $f(t) = t\sin\omega t$, $f(0) = 0$

$f'(t) = \sin\omega t + \omega t\cos\omega t$, $f'(0) = 0$

$f''(t) = 2\omega\cos\omega t - \omega^2 f(t)$

$\Rightarrow \mathcal{L}(f'') = 2\omega\mathcal{L}(\cos\omega t) - \omega^2\mathcal{L}(f)$

$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0) = s^2\mathcal{L}(f)$

$\therefore \mathcal{L}(f) = \dfrac{2\omega}{s^2+\omega^2}\mathcal{L}(\cos\omega t) = \dfrac{2\omega s}{(s^2+\omega^2)^2}$

$\underline{\text{Ex 2}}$

o. $f(t) = \cos\omega t \Rightarrow f''(t) = -\omega^2\cos\omega t$, $f(0) = 1$, $f'(0) = 0$

$\mathcal{L}(f'') = -\omega^2\mathcal{L}(f)$

$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - \overset{0}{\cancel{f'(0)}}$ $\Rightarrow \mathcal{L}(f) = \dfrac{s}{s^2+\omega^2} = \mathcal{L}(\cos\omega t)$

o. $g(t) = \sin\omega t \Rightarrow g'(t) = \omega\cos\omega t$, $g(0) = 0$

$\left.\begin{array}{l}\mathcal{L}(g') = \omega\mathcal{L}(\cos\omega t) \\ \mathcal{L}(g') = s\mathcal{L}(g) - \cancel{g(0)}\end{array}\right] \Rightarrow \mathcal{L}(\sin\omega t) = \dfrac{\omega}{s}\mathcal{L}(\cos\omega t) = \dfrac{\omega}{s^2+\omega^2}$

$\underline{\text{Ex 3}}$ $\mathcal{L}(f) = \dfrac{1}{s(s^2+\omega^2)}$, $\mathcal{L}(g) = \dfrac{1}{s^2(s^2+\omega^2)} \Rightarrow f(t), g(t)?$

<solution>

$\mathcal{L}(\sin\omega t) = \dfrac{\omega}{s^2+\omega^2} \Rightarrow \mathcal{L}^{-1}\left[\dfrac{1}{s^2+\omega^2}\right] = \dfrac{1}{\omega}\sin\omega t$

$f(t) = \mathcal{L}^{-1}\left[\dfrac{1}{s(s^2+\omega^2)}\right] = \displaystyle\int_0^t \dfrac{1}{\omega}\sin\omega\tau\, d\tau = \dfrac{1}{\omega^2}(1-\cos\omega t)$

$$g(t) = \mathcal{L}^{-1}\left[\dfrac{1}{s^2(s^2+\omega^2)}\right] = \int_0^t \frac{1}{\omega^2}(1-\cos\omega\tau)\,d\tau$$

$$= \frac{1}{\omega^2}\left[\tau - \frac{1}{\omega}\sin\omega\tau\right]_{\tau=0}^{\tau=t}$$

$$= \frac{1}{\omega^2}\left(t - \frac{1}{\omega}\sin t\right) = \frac{t}{\omega^2} - \frac{\sin t}{\omega^3}$$

Differential Equations

$y'' + ay' + by = r(t)$, $\quad y(0) = K_0$, $\quad y'(0) = K_1$

$$\Rightarrow [s^2 Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = \underset{\mathcal{L}(r)}{R(s)}$$

$$(s^2 + as + b)Y = (s+a)y(0) + y'(0) + R(s)$$

$$Y(s) = [(s+a)y(0) + y'(0)]Q(s) + R(s)Q(s), \text{ where } Q(s) = \frac{1}{s^2+as+b}$$

[If $y(0) = y'(0) = 0$, $\quad Y(s) = R(s)Q(s)$

$$Q(s) = \frac{Y(s)}{R(s)} = \frac{\mathcal{L}(y(t))}{\mathcal{L}(r(t))} = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})}$$]

$$\Rightarrow y(t) = \mathcal{L}^{-1}(Y)$$

Ex 4 $\quad y'' - y = t$, $\quad y(0) = 1$, $\quad y'(0) = 1$

$$s^2 Y - sy(0) - y'(0) - Y = \frac{1}{s^2}$$

$$(s^2 - 1)Y = s + 1 + \frac{1}{s^2} \Rightarrow Y = \frac{1}{s-1} + \frac{1}{s^2-1} - \frac{1}{s^2}$$

$$\therefore y(t) = \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2-1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2}\right)$$

$$= e^t + \sinh t - t$$

Ex 6 (Shifted Data Problems)

$y'' + y = 2t$, $\quad y(\tfrac{1}{4}\pi) = \tfrac{1}{2}\pi$, $\quad y'(\tfrac{1}{4}\pi) = 2 - \sqrt{2}$

Let $\tilde{t} = t - \frac{\pi}{4}$, then $\tilde{y}'' + \tilde{y} = 2(\tilde{t} + \frac{\pi}{4})$, $\quad \tilde{y}(0) = \frac{1}{2}\pi$, $\quad \tilde{y}'(0) = 2 - \sqrt{2}$

$$s^2 \tilde{Y} - s \cdot \tfrac{1}{2}\pi - (2 - \sqrt{2}) + \tilde{Y} = \frac{2}{s^2} + \frac{\pi/2}{s}$$

$$\tilde{Y} = \frac{2}{(s^2+1)s^2} + \frac{\frac{1}{2}\pi}{(s^2+1)s} + \frac{\frac{1}{2}\pi s}{s^2+1} + \frac{2-\sqrt{2}}{s^2+1}$$

$$\tilde{y} = 2\tilde{t} + \tfrac{1}{2}\pi - \sqrt{2}\sin\tilde{t}$$

$$y = 2t - \sin t + \cos t$$

6.3 Unit Step Function

$$u(t-a) = \begin{cases} 0 & \text{if } t<a \\ 1 & \text{if } t>a \end{cases}$$

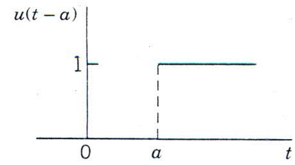


Fig. 118. Unit step function $u(t-a)$

Time Shifting

Th1 (Second Shifting Theorem; Time Shifting)

$$\tilde{f}(t) = f(t-a)u(t-a) = \begin{cases} 0 & \text{if } t<a \\ f(t-a) & \text{if } t>a \end{cases}$$

$$\mathcal{L}[f(t-a)u(t-a)] = e^{-as}F(s), \text{ where } F(s)=\mathcal{L}[f(t)]$$

<proof>

$$e^{-as}F(s) = e^{-as}\int_0^\infty e^{-s\tau}f(\tau)\,d\tau = \int_0^\infty e^{-s(a+\tau)}f(\tau)\,d\tau$$

$$= \int_a^\infty e^{-st}f(t-a)\,dt \quad (t=a+\tau)$$

$$= \int_0^\infty e^{-st}f(t-a)u(t-a)\,dt = \mathcal{L}[f(t-a)u(t-a)]$$

Ex 0

$$\mathcal{L}[u(t-a)] = e^{-as}\mathcal{L}[1] = \frac{e^{-as}}{s}$$

Ex 1

$$f(t) = 2u(t) - 2u(t-\pi) + u(t-2\pi)\sin t = \begin{cases} 2 & \text{if } 0<t<\pi \\ 0 & \text{if } \pi<t<2\pi \\ \sin t & \text{if } t>2\pi \end{cases}$$

$\sin t = \sin(t-2\pi)$

$$\mathcal{L}[f] = \frac{2}{s} - \frac{2e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s^2+1}$$

Ex 2

$$F(s) = \frac{2}{s^2} - \frac{2e^{-2s}}{s^2} - \frac{4e^{-2s}}{s} + \frac{se^{-\pi s}}{s^2+1}$$

$$\mathcal{L}^{-1}[F] = 2t - 2(t-2)u(t-2) - 4u(t-2) + \cos(t-\pi)u(t-\pi)$$

$$= 2t - 2t\,u(t-2) - \cos t\; u(t-\pi)$$

Ex 3 (Response of an RC-Circuit to a Single Rectangular Wave)

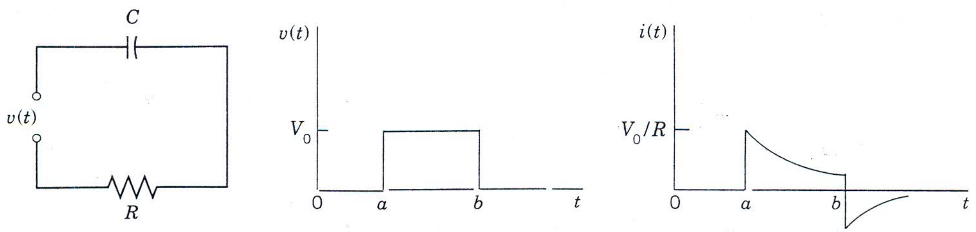


Fig. 123. *RC*-circuit, electromotive force *v*(*t*), and current in Example 3

$$R\,i(t) + \frac{1}{C}\int_0^t i(\tau)\,d\tau = v(t) = V_0\left[u(t-a) - u(t-b)\right]$$

$$R\,I(s) + \frac{I(s)}{sC} = \frac{V_0}{s}\left[e^{-as} - e^{-bs}\right]$$

$$I(s) = \frac{V_0/R}{s + 1/(RC)}\left[e^{-as} - e^{-bs}\right] = F(s)\left[e^{-as} - e^{-bs}\right]$$

$$\mathcal{L}^{-1}\left[F(s)\right] = \mathcal{L}^{-1}\left[\frac{V_0/R}{s - \left(-\frac{1}{RC}\right)}\right] = \frac{V_0}{R}e^{-\frac{t}{RC}} \cdot 1 = f(t)$$

$$\therefore \mathcal{L}^{-1}\left[I(s)\right] = f(t-a)\,u(t-a) - f(t-b)\,u(t-b)$$

$$= \frac{V_0}{R}\left[e^{-\frac{t-a}{RC}} \cdot u(t-a) - e^{-\frac{t-b}{RC}} \cdot u(t-b)\right]$$

$$= \begin{cases} \dfrac{V_0}{R}\, e^{\frac{a}{RC}} \cdot e^{-\frac{t}{RC}} & \text{if } a < t < b \\ \dfrac{V_0}{R}\left(e^{\frac{a}{RC}} - e^{\frac{b}{RC}}\right) \cdot e^{-\frac{t}{RC}} & \text{if } t > b. \end{cases}$$

6.4 Short Impulses. Dirac's Delta Function

$$f_k(t-a) = \begin{cases} 1/k & \text{if } a \leq t \leq a+k \\ 0 & \text{otherwise} \end{cases}$$

Impulse of f_k

$$I_k = \int_0^{\infty} f_k(t-a)\,dt = \int_a^{a+k} \frac{1}{k}\,dt = 1$$

$$f_k(t-a) = \frac{1}{k}\left[u(t-a) - u(t-(a+k))\right]$$

$$\mathcal{L}\left[f_k(t-a)\right] = \frac{1}{ks}\left[e^{-as} - e^{-(a+k)s}\right] = e^{-as} \cdot \frac{1-e^{-ks}}{ks}$$

Dirac Delta Function

$$\delta(t-a) = \lim_{k\to 0} f_k(t-a) = \begin{cases} \infty & \text{if } t=a \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{L}\left[\delta(t-a)\right] = \lim_{k\to 0} e^{-as} \cdot \frac{1-e^{-ks}}{ks} = e^{-as}$$

Ex2 (Hammerblow Response of a Mass-Spring System)

$$y'' + 3y' + 2y = \delta(t-1), \quad y(0)=0,\ y'(0)=0$$

$$s^2Y + 3sY + 2Y = e^{-s}$$

$$Y = e^{-s}\left(\frac{1}{s+1} - \frac{1}{s+2}\right) \qquad F(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

$$y(t) = \mathcal{L}^{-1}\left[e^{-s}F(s)\right] \qquad f(t) = \mathcal{L}^{-1}\left[F(s)\right] = e^{-t} - e^{-2t}$$

$$= f(t-1)\,u(t-1) = \begin{cases} 0 & \text{if } 0 \leq t < 1 \\ e^{-(t-1)} - e^{-2(t-1)} & \text{if } t > 1 \end{cases}$$

More on Partial Fractions

$$Y(s) = \frac{F(s)}{G(s)}, \text{ where } G(s) = (s-a), (s-a)^2, \cdots, \text{ or } (s-a)(s-\bar{a}), [(s-a)(s-\bar{a})]^2, \cdots$$

- Repeated real factors $(s-a)^2, (s-a)^3$, etc, require partial fractions

$$\frac{A_2}{(s-a)^2} + \frac{A_1}{s-a} \Rightarrow (A_2 t + A_1)e^{at}$$

Inverse transforms

$$\frac{A_3}{(s-a)^3} + \frac{A_2}{(s-a)^2} + \frac{A_1}{s-a} \Rightarrow \left(\frac{1}{2}A_3 t^2 + A_2 t + A_1\right)e^{at}$$

- Unrepeated complex factors $(s-a)(s-\bar{a})$, $a = \alpha + i\beta$, $\bar{a} = \alpha - i\beta$. require a partial fraction

$$\frac{As + B}{(s-\alpha)^2 + \beta^2}$$

Ex $\hat{4}$

$$y'' - 3y' + 2y = 4t, \quad y(0) = 1, \ y'(0) = -1$$

$$s^2 Y - s + 1 - 3sY + 3 + 2Y = \frac{4}{s^2}$$

$$(s^2 - 3s + 2)Y = \frac{s^3 - 4s^2 + 4}{s^2}$$

$$Y(s) = \frac{s^3 - 4s^2 + 4}{s^2(s-2)(s-1)} = \frac{2}{s^2} + \frac{3}{s} - \frac{1}{s-2} - \frac{1}{s-1}$$

$$\mathcal{L}^{-1}[Y] = 2t + 3 - e^{2t} - e^{t}$$

6.5 Convolution

$$h(t) = (f*g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau : \text{ convolution of } f \text{ and } g$$

Th1 (Convolution Theorem)

$$\mathcal{L}(f*g) = \mathcal{L}(f)\cdot\mathcal{L}(g)$$

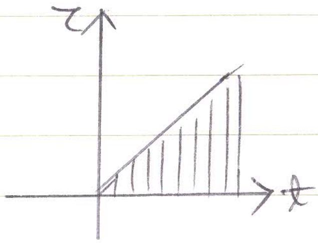
[But, in general $\mathcal{L}(f\cdot g) \neq \mathcal{L}(f)\cdot\mathcal{L}(g)$ ← the product of f and g]

<proof>

$$e^{-s\tau}G(s) = \mathcal{L}[g(t-\tau)u(t-\tau)]$$

$$= \int_0^\infty e^{-st} g(t-\tau)u(t-\tau)\,dt$$

$$= \int_\tau^\infty e^{-st} g(t-\tau)\,dt$$



$$F(s)G(s) = \int_0^\infty e^{-s\tau} f(\tau) G(s)\, d\tau$$

$$= \int_0^\infty f(\tau) \int_\tau^\infty e^{-st} g(t-\tau)\,dt\,d\tau$$

$$= \int_0^\infty e^{-st} \int_0^t f(\tau)\, g(t-\tau)\,d\tau\,dt$$

$$= \int_0^\infty e^{-st} (f*g)(t)\,dt$$

Properties of Convolution

- $f*g = g*f$
- $f*(g_1+g_2) = f*g_1 + f*g_2$
- $(f*g)*h = f*(g*h)$
- $f*0 = 0*f = 0$

But,

- $f*1 \neq f$
- $(f*f)(t) \ngeq 0$

Ex 1

$$H(s) = \frac{1}{(s-a)s}, \quad \left(\begin{array}{l} \mathcal{L}[e^{at}] = \frac{1}{s-a} \\ \mathcal{L}[1] = \frac{1}{s} \end{array} \right)$$

$$\Rightarrow h(t) = e^{at} * 1 = \int_0^t e^{a\tau} \cdot 1 \, d\tau = \frac{1}{a}(e^{at} - 1)$$

Ex 2

$$H(s) = \frac{1}{(s^2+w^2)^2}$$

$$\Rightarrow h(t) = \frac{\sin wt}{w} * \frac{\sin wt}{w}$$

$$= \frac{1}{w^2} \int_0^t \sin w\tau \cdot \sin w(t-\tau) \, d\tau$$

$$= \frac{1}{2w^2} \int_0^t \left[-\cos wt + \cos w\tau \right] d\tau$$

$$= \frac{1}{2w^2} \left[-t \cos wt + \frac{\sin wt}{w} \right]$$

Ex 4

$$y'' + w_0^2 y = K \sin w_0 t, \quad y(0) = 0, \; y'(0) = 0$$

$$s^2 Y + w_0^2 Y = \frac{K w_0}{s^2 + w_0^2}$$

$$Y = \frac{K w_0}{(s^2 + w_0^2)^2}$$

$$y(t) = \frac{K w_0}{2 w_0^2} \left(-t \cos w_0 t + \frac{\sin w_0 t}{w_0} \right)$$

$$= \frac{K}{2 w_0^2} \left(- w_0 t \cos w_0 t + \sin w_0 t \right)$$

Nonhomogeneous Linear ODEs

$$y'' + ay' + by = r(t), \quad y(0) = y'(0) = 0$$

$$s^2Y + asY + bY = R$$

$$Y(s) = R(s)Q(s), \quad \text{where } Q(s) = \frac{1}{s^2+as+b}$$

$$\Rightarrow y(t) = \int_0^t q(t-\tau)\, r(\tau)\, d\tau$$

Ex 5

$$y'' + 3y' + 2y = r(t) = \begin{cases} 1 & \text{if } 1 < t < 2 \\ 0 & \text{otherwise} \end{cases}, \quad \text{where } \begin{matrix} y(0)=0 \\ y'(0)=0 \end{matrix}$$

$$Q(s) = \frac{1}{s^2+3s+2} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$q(t) = e^{-t} - e^{-2t}$$

$$\int q(t-\tau)\,d\tau = \int \left[e^{-(t-\tau)} - e^{-2(t-\tau)}\right] d\tau$$

$$= e^{-(t-\tau)} - \frac{1}{2} e^{-2(t-\tau)}$$

Case 1: If $t < 1$, $\int_0^t q(t-\tau)\, r(\tau)\, d\tau = 0$

Case 2: If $1 < t < 2$,

$$\int_0^t q(t-\tau)\, r(\tau)\, d\tau = \int_1^t q(t-\tau)\, d\tau$$

$$= \frac{1}{2} - e^{-(t-1)} + \frac{1}{2} e^{-2(t-1)}$$

Case 3: If $t > 2$,

$$\int_0^t q(t-\tau)\, r(\tau)\, d\tau = \int_1^2 q(t-\tau)\, d\tau$$

$$= e^{-(t-2)} - e^{-(t-1)} - \frac{1}{2}\left[e^{-2(t-2)} - e^{-2(t-1)}\right]$$

Integral Equations

Ex 6

$$y(t) - \int_0^t y(\tau)\sin(t-\tau)\,d\tau = t$$

$$Y(s) - Y(s)\cdot\frac{1}{s^2+1} = Y(s)\frac{s^2}{s^2+1} = \frac{1}{s^2}$$

$$Y(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}$$

$$\Rightarrow y(t) = t + \frac{t^3}{6} \qquad \left(\because \mathcal{L}(t^n) = \frac{n!}{s^{n+1}}\right)$$

Ex 7

$$y(t) - \int_0^t (1+\tau)\,y(t-\tau)\,d\tau = 1 - \sinh t$$

$$y - (1+t) * y = 1 - \sinh t$$

$$Y(s)\left[1 - \left(\frac{1}{s} + \frac{1}{s^2}\right)\right] = \frac{1}{s} - \frac{1}{s^2-1}$$

$$Y(s)\cdot\frac{s^2-s-1}{s^2} = \frac{s^2-1-s}{s(s^2-1)}$$

$$Y(s) = \frac{s}{s^2-1}$$

$$\Rightarrow y(t) = \cosh t$$

6.6 Differentiation and Integration of Transforms

$$F(s)=\int_0^\infty e^{-st}f(t)\,dt\ ,\quad F'(s)=-\int_0^\infty e^{-st}\,t\,f(t)\,dt$$

$$\therefore\ \mathcal{L}[t f(t)]=-F'(s)\ \text{and}\ \mathcal{L}^{-1}[F'(s)]=-t f(t)$$

Ex 1

$$\mathcal{L}[\sin\beta t]=\frac{\beta}{s^2+\beta^2}\ ,\quad \mathcal{L}[\cos\beta t]=\frac{s}{s^2+\beta^2}$$

$$\mathcal{L}[t\sin\beta t]=-\frac{d}{ds}\left(\frac{\beta}{s^2+\beta^2}\right)=\frac{2\beta s}{(s^2+\beta^2)^2}$$

$$\mathcal{L}[t\cos\beta t]=-\frac{d}{ds}\left(\frac{s}{s^2+\beta^2}\right)=\frac{s^2-\beta^2}{(s^2+\beta^2)^2}$$

$$\mathcal{L}\left[t\cos\beta t\pm\frac{1}{\beta}\sin\beta t\right]=\frac{s^2-\beta^2\pm(s^2+\beta^2)}{(s^2+\beta^2)^2}$$

Integration of Transforms

$|f(t)|\leqq Me^{kt}$ for all $t\geqq 0$, and $\exists \lim_{t\to 0}\frac{f(t)}{t}$

$$\Rightarrow \mathcal{L}\left[\frac{f(t)}{t}\right]=\int_s^\infty F(\tilde{s})\,d\tilde{s}$$

Γ_0^∞

$$\int_s^\infty F(\tilde{s})\,d\tilde{s}=\int_s^\infty\int_0^\infty e^{-\tilde{s}t}f(t)\,dt\,d\tilde{s}$$

$$=\int_0^\infty\int_s^\infty e^{-\tilde{s}t}f(t)\,d\tilde{s}\,dt$$

$$\cdots$$

$$=\int_0^\infty e^{-st}\frac{f(t)}{t}\,dt=\mathcal{L}\left[\frac{f(t)}{t}\right]$$

Ex2 (Better solution than the textbook)

$$\mathcal{L}^{-1}\left[\ln\left(1+\frac{\omega^2}{s^2}\right)\right]=f(t)$$

$$\frac{d}{ds}\left[\ln\left(1+\frac{\omega^2}{s^2}\right)\right]=\frac{-2\omega^2}{s(s^2+\omega^2)}=-\frac{2}{s}+\frac{2s}{s^2+\omega^2}$$

$$-tf(t)=-2+2\cos\omega t$$

$$\therefore \mathcal{L}^{-1}\left[\ln\left(1+\frac{\omega^2}{s^2}\right)\right]=f(t)=\frac{2}{t}(1-\cos\omega t)$$

Special Linear ODEs with Variable Coefficients

$$\mathcal{L}[tf(t)]=-F'(s)$$

$$\mathcal{L}[ty']=-\frac{d}{ds}[sY-y(0)]=-Y-sY'(s)$$

$$\mathcal{L}[ty'']=-\frac{d}{ds}[s^2Y-sy(0)-y'(0)]=-2sY-s^2Y'(s)+y(0)$$

Ex3 (Laguerre's Equation)

$ty''+(1-t)y'+ny=0$: Laguerre's ODE

$$[-2sY-s^2Y'(s)+y(0)]+sY-y(0)+[Y+sY'(s)]+nY=0$$

$$(s-s^2)Y'(s)+(n+1-s)Y=0$$

$$\frac{dY}{Y}=\frac{n+1-s}{s^2-s}ds=\left(\frac{n}{s-1}-\frac{n+1}{s}\right)ds$$

$$Y=\frac{(s-1)^n}{s^{n+1}},$$ Let $l_n=\mathcal{L}^{-1}(Y)$

$$\Rightarrow\ l_0=1,\ l_n(t)=\frac{e^t}{n!}\frac{d^n}{dt^n}\left(t^ne^{-t}\right),\ n=1,2,3,\cdots$$

Proof $\mathcal{L}(t^ne^{-t})=\dfrac{n!}{(s+1)^{n+1}},\quad \mathcal{L}\left[\dfrac{d^n}{dt^n}(t^ne^{-t})\right]=\dfrac{n!\,s^n}{(s+1)^{n+1}}$

$$\mathcal{L}(l_n)=\frac{(s-1)^n}{s^{n+1}}=Y\quad \lrcorner$$

morning glory

6.7 Systems of ODEs

$$\begin{cases} y_1' = a_{11} y_1 + a_{12} y_2 + g_1(t) \\ y_2' = a_{21} y_1 + a_{22} y_2 + g_2(t) \end{cases}$$

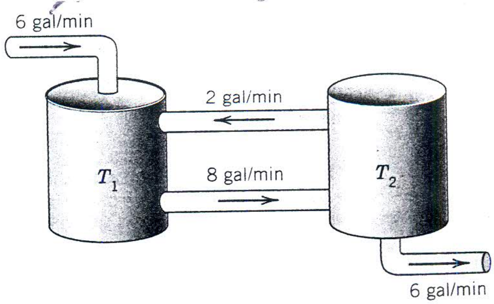
$$\Rightarrow \begin{cases} sY_1 - y_1(0) = a_{11} Y_1 + a_{12} Y_2 + G_1(s) \\ sY_2 - y_2(0) = a_{21} Y_1 + a_{22} Y_2 + G_2(s) \end{cases}$$

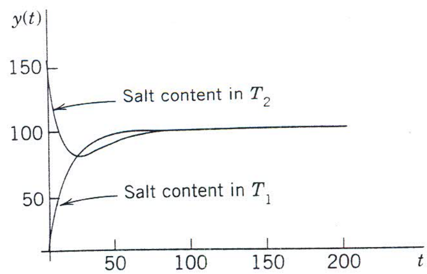
$$\text{or } \begin{cases} (a_{11} - s) Y_1 + a_{12} Y_2 = -y_1(0) - G_1(s) \\ a_{21} Y_1 + (a_{22} - s) Y_2 = -y_2(0) - G_2(s) \end{cases}$$

$\Rightarrow$ Solve for $Y_1(s)$ and $Y_2(s)$ algebraically.

Ex 1

containing 6 lb of salt





$$\begin{cases} y_1' = -0.08 y_1 + 0.02 y_2 + 6, & y_1(0) = 0 \\ y_2' = 0.08 y_1 - 0.08 y_2, & y_2(0) = 150 \end{cases}$$

$$\begin{cases} sY_1 - y_1(0) = -0.08 Y_1 + 0.02 Y_2 + \frac{6}{s} \\ sY_2 - y_2(0) = 0.08 Y_1 - 0.08 Y_2 \end{cases}$$

$$\begin{cases} (-0.08 - s) Y_1 + 0.02 Y_2 = -\frac{6}{s} \\ 0.08 Y_1 - (0.08 + s) Y_2 = -150 \end{cases}$$

$$\begin{cases} Y_1 = \frac{100}{s} - \frac{62.5}{s+0.12} - \frac{37.5}{s+0.04} \\ Y_2 = \frac{100}{s} + \frac{125}{s+0.12} - \frac{75}{s+0.04} \end{cases}$$

$$\begin{cases} y_1 = 100 - 62.5 e^{-0.12t} - 37.5 e^{-0.04t} \\ y_2 = 100 + 125 e^{-0.12t} - 75 e^{-0.04t} \end{cases}$$

Laplace Transformation

$F(s) = \int_0^\infty e^{-st} f(t)dt = \mathcal{L}(f)$

$\mathcal{L}(1) = \frac{1}{s}$, $\mathcal{L}(e^{at}) = \frac{1}{s-a}$, $\mathcal{L}\left[e^{at} f(t)\right] = F(s-a)$,
$\mathcal{L}[\cos\omega t] = \frac{s}{s^2+\omega^2}$, $\mathcal{L}[\sin\omega t] = \frac{\omega}{s^2+\omega^2}$
$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$, for $n = 1, 2, 3, \ldots$

$\mathcal{L}[f^{(n)}] = s^n \mathcal{L}(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots \; f^{(n-1)}(0)$
$\mathcal{L}[\int_0^t f(\tau)d\tau] = \frac{1}{s} F(s)$

$F'(s) = -\mathcal{L}[t f(t)]$, $\int_s^\infty F(\tilde{s})d\tilde{s} = \mathcal{L}\left[\frac{f(t)}{t}\right]$

$\mathcal{L}[f(t-a)u(t-a)] = e^{-as} F(s)$: t-shifting
$\mathcal{L}[\delta(t-a)] = e^{-as}$

$F(s) = \mathcal{L}[f]$, $G(s) = \mathcal{L}[g] \Rightarrow F(s)G(s) = \mathcal{L}[h]$,
where $h(t) = (f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$