Linear and Nonlinear Computation Models

(CSE 4190.313)

Final Exam: June 11, 2010

(Solutions)

Problem	Score	Problem	Score
1		6	
2		7	
3		8	
4		9	
5		Total	

Name: ____________________

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Dept: ____________________

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1. (10 points) Suppose A has eigenvalues $0, 3, 5$ with independent eigenvectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

   (a) (3 points) Give a basis for the nullspace and a basis for the column space.

   (b) (4 points) Find particular solution to $A\mathbf{x} = \mathbf{v} + \mathbf{w}$. Find all solutions.

   (c) (3 points) Show that $A\mathbf{x} = \mathbf{u}$ has no solution.

(a) $\mathbf{u}$: a basis for the nullspace of A ($\because A\mathbf{u} = \mathbf{0}$)

$\mathbf{v}, \mathbf{w}$: a basis for the column space of A

($\because A(\alpha\mathbf{v} + \beta\mathbf{w}) = 3\alpha\mathbf{v} + 5\beta\mathbf{w}$ for all $\alpha, \beta \in \mathbb{R}$)

(b) Let $\mathbf{x}_p = \frac{1}{3}\mathbf{v} + \frac{1}{5}\mathbf{w}$, then

$$A\mathbf{x}_p = \frac{1}{3}\cdot 3\mathbf{v} + \frac{1}{5}\cdot 5\mathbf{w} = \mathbf{v} + \mathbf{w}.$$

$\therefore$ $\mathbf{x}_p$ is a particular solution

Let $\mathbf{x} = \alpha\mathbf{u} + \frac{1}{3}\mathbf{v} + \frac{1}{5}\mathbf{w}$ for all $\alpha \in \mathbb{R}$,

then $A\mathbf{x} = \mathbf{0} + \frac{1}{3}\cdot 3\mathbf{v} + \frac{1}{5}\cdot 5\mathbf{w} = \mathbf{v} + \mathbf{w}$

$\therefore$ $\mathbf{x}$ is a general solution.

(c) If $\mathbf{u} = A\mathbf{x}$ for some $\mathbf{x}$, then

$\mathbf{u}\ (\neq \mathbf{0})$ is in the column space of A and

$\mathbf{u} = \alpha\mathbf{v} + \beta\mathbf{w}$ for some $\alpha, \beta \in \mathbb{R}$.

($\alpha \neq 0$ or $\beta \neq 0$)

$\mathbf{0} = A\mathbf{u} = 3\alpha\mathbf{v} + 5\beta\mathbf{w} \neq \mathbf{0}$ # → 두가지의 방법으로 모순을 보일수 있음!

This means $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are dependent #

Thus $A\mathbf{x} = \mathbf{u}$ has no solution

2. (10 points) True or false: If the n columns of S (eigenvectors of A) are independent, then

(a) (2 points) A is invertible,

(b) (3 points) A is diagonalizable,

(c) (2 points) S is invertible,

(d) (3 points) S is diagonalizable.

Justify your answer.

(a) False

「∵ A may have a zero eigenvalue」

(b) True

「∵ $AS = S\Lambda \Rightarrow S^{-1}AS = \Lambda$」

(c) True

「∵ S has rank n」

(d) False

「∵ Counterexample:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix}, \quad S = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

$$\Rightarrow S^{-1}AS = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

But, S is not diagonalizable

S has $\lambda = 3$ as a double eigenvalue

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_2 = 0$$

Thus, all eigenvectors of S are $\begin{bmatrix} \alpha \\ 0 \end{bmatrix}$, $\alpha \in \mathbb{R}$

∴ S has no two independent columns」

3. (10 points) If the transformation T is a reflection across the $45°$ line in the plane, find its matrix with respect to the standard basis $\mathbf{v}_1 = (1, 0), \mathbf{v}_2 = (0, 1)$, and also with respect to $\mathbf{V}_1 = (1, 1), \mathbf{V}_2 = (1, -1)$. Show that those matrices are similar.

$$[T]_{v \text{ to } v} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$[T]_{V \text{ to } V} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$M = [I]_{V \text{ to } v} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

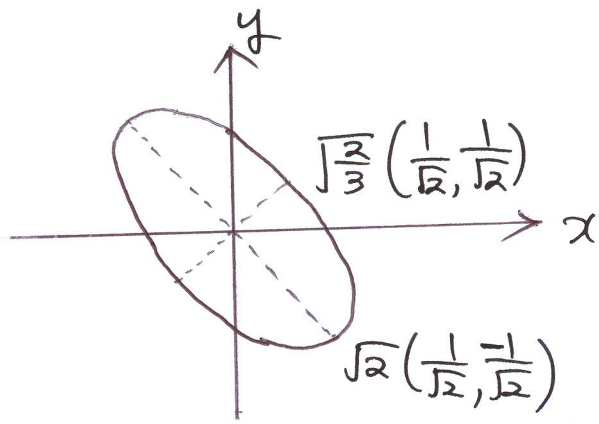
$$M^{-1} = [I]_{v \text{ to } V} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$[T]_{V \text{ to } V} = M^{-1} [T]_{v \text{ to } v} M$$

$$\left(\begin{aligned} \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned} \right)$$

$\therefore$ The two matrices are similar.

4. (10 points) Draw the tilted ellipse $x^2 + xy + y^2 = 1$ and find the half-lengths of its axes from the eigenvalues of the corresponding matrix A.



eigenvalues / unit eigenvectors

$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$x^2 + xy + y^2 = \frac{3}{2}\left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right)^2 + \frac{1}{2}\left(\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}\right)^2$$

Half-leng of the major axis : $\sqrt{2}$

— 〃 — the minor axis : $\sqrt{\frac{2}{3}}$

5. (10 points)

(a) (5 points) Find $U\Sigma V^T$ if A has orthogonal columns $\mathbf{w}_1, \ldots, \mathbf{w}_n$ of lengths $\sigma_1, \ldots, \sigma_n$.

(b) (5 points) If an $m \times n$ matrix Q has orthogonal columns, what is Q^+?

(a)

$$A = \overbrace{\begin{bmatrix} \mathbf{w}_1/\sigma_1 & \cdots & \mathbf{w}_n/\sigma_n & \underbrace{\begin{matrix} \; & \cdots & \; \end{matrix}}_{\substack{(m-n) \\ \text{orthonormal} \\ \text{vectors}}} \end{bmatrix}}^{m} \overbrace{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}}^{n} \overbrace{\begin{bmatrix} I_{n\times n} \end{bmatrix}}^{n}$$

(b)

$$Q = \overbrace{\begin{bmatrix} \mathbf{q}_1/\|\mathbf{q}_1\| & \cdots & \mathbf{q}_n/\|\mathbf{q}_n\| \end{bmatrix}}^{n} \begin{bmatrix} \|\mathbf{q}_1\| & & \\ & \ddots & \\ & & \|\mathbf{q}_n\| \end{bmatrix}$$

$$Q^+ = \begin{bmatrix} 1/\|\mathbf{q}_1\| & & \\ & \ddots & \\ & & 1/\|\mathbf{q}_n\| \end{bmatrix} \begin{bmatrix} \mathbf{q}_1^T/\|\mathbf{q}_1\| \\ \vdots \\ \mathbf{q}_n^T/\|\mathbf{q}_n\| \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{q}_1^T/\mathbf{q}_1^T\mathbf{q}_1 \\ \vdots \\ \mathbf{q}_n^T/\mathbf{q}_n^T\mathbf{q}_n \end{bmatrix}$$

6. (10 points) Find the minimum values of

$$R_1(\mathbf{x}) = \frac{x_1^2 - x_1x_2 + x_2^2}{x_1^2 + x_2^2}, \quad \text{and} \quad R_2(\mathbf{x}) = \frac{x_1^2 - x_1x_2 + x_2^2}{4x_1^2 + x_2^2},$$

(a) $A_1 = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$ $\quad (\lambda-1)^2 - \frac{1}{4} = 0$

$\lambda_{min} = \frac{1}{2}, \ \lambda_{max} = \frac{3}{2}$

$\therefore \min R_1(\mathbf{x}) = \lambda_{min} = \frac{1}{2}$

(b) $R_2(\mathbf{x}) = \dfrac{\frac{1}{4}(2x_1)^2 - \frac{1}{2}(2x_1)x_2 + x_2^2}{(2x_1)^2 + x_2^2}$ $\quad \begin{pmatrix} y_1 = 2x_1 \\ y_2 = x_2 \end{pmatrix}$

$= \dfrac{\frac{1}{4}y_1^2 - \frac{1}{2}y_1y_2 + y_2^2}{y_1^2 + y_2^2}$

$A_2 = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & 1 \end{bmatrix} \Rightarrow (\lambda - \frac{1}{4})(\lambda - 1) - \frac{1}{4^2}$

$= \lambda^2 - \frac{5}{4}\lambda + \frac{3}{4^2}$

$= (\lambda - \frac{5}{8})^2 - \frac{13}{8^2}$

$\therefore \lambda_{min} = \frac{5-\sqrt{13}}{8}$

$\therefore \min R_2(\mathbf{x}) = \lambda_{min}$

$= \frac{5-\sqrt{13}}{8}$

7. (10 points) Show that for any two different vectors of the same length, $\|\mathbf{x}\| = \|\mathbf{y}\|$, the Householder transformation with $\mathbf{v} = \mathbf{x} - \mathbf{y}$ gives $H\mathbf{x} = \mathbf{y}$ and $H\mathbf{y} = \mathbf{x}$.

$$Hx = \left[I - 2 \cdot \frac{(x-y)(x-y)^T}{(x-y)^T(x-y)} \right] x$$

$$= x - 2 \cdot \frac{(x-y)\cdot(x^Tx - y^Tx)}{x^Tx - y^Tx - x^Ty + y^Ty}$$

$$= x - (x-y) \cdot \frac{2(x^Tx - x^Ty)}{2x^Tx - 2x^Ty} \qquad \left(\begin{array}{l} \because x^Tx = y^Ty \\ x^Ty = y^Tx \end{array} \right)$$

$$= x - x + y$$

$$= y$$

$$Hy = y - 2 \cdot \frac{(x-y)(x-y)^T}{(x-y)^T(x-y)} \cdot y$$

$$= y - (x-y) \cdot \frac{2(x^Ty - y^Ty)}{x^Tx - y^Tx - x^Ty + y^Ty}$$

$$= y + (x-y) \cdot \frac{2(y^Ty - x^Ty)}{2y^Ty - 2x^Ty}$$

$$= y + x - y$$

$$= x$$

8. (10 points) Find the tridiagonal matrix HAH^{-1} that is similar to

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 1 & 0 \\ 4 & 0 & 0 \end{bmatrix}.$$

$$v = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix}$$

$$H = I - 2 \cdot \frac{v \cdot v^T}{v^T \cdot v}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{80} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 64 & 32 \\ 0 & 32 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.6 & -0.8 \\ 0 & -0.8 & 0.6 \end{bmatrix}$$

$$HAH^{-1} = HAH$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.6 & -0.8 \\ 0 & -0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 \\ 3 & 1 & 0 \\ 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.6 & -0.8 \\ 0 & -0.8 & 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 4 \\ -5 & -0.6 & 0 \\ 0 & -0.8 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.6 & -0.8 \\ 0 & -0.8 & 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -5 & 0 \\ -5 & 0.36 & 0.48 \\ 0 & 0.48 & 0.64 \end{bmatrix}$$

9. (20 points) In solving the following matrix equation

$$\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

(a) (5 points) using the Jacobi method,

(b) (5 points) the Gauss-Seidel method, and

(c) (10 points) the SOR,

compute the maximum absolute eigenvalues of their respective crucial matrices $S^{-1}T$:

(a) $$S^{-1}T = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{bmatrix}$$

$$\therefore \lambda = \pm \frac{1}{3}, \quad |\lambda|_{max} = \frac{1}{3} = \mu_{max}$$

(b) $$S^{-1}T = \begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} \\ 0 & \frac{1}{9} \end{bmatrix}$$

$$\lambda_1 = 0, \ \lambda_2 = \frac{1}{9}, \quad |\lambda|_{max} = \frac{1}{9} = \left(\frac{1}{3}\right)^2$$

(c) $$S^{-1}T = \begin{bmatrix} 3 & 0 \\ -\omega & 3 \end{bmatrix}^{-1} \begin{bmatrix} 3(1-\omega) & \omega \\ 0 & 3(1-\omega) \end{bmatrix}$$

$$= \begin{bmatrix} 1-\omega & \frac{1}{3}\omega \\ \frac{1}{3}\omega & 1-\omega+\frac{1}{9}\omega^2 \end{bmatrix}$$

$$\omega_{opt} = \frac{2\left(1-\sqrt{1-\mu_{max}^2}\right)}{\mu_{max}^2} = 18\left(1-\sqrt{\frac{8}{9}}\right)$$

$$|\lambda|_{max} = \omega_{opt} - 1 = \underline{17 - 12\sqrt{2}}$$