HW #1 (CSE 4190.313)

Tuesday, March 31, 2020

Name: 정현진 ID No: 2013-11431

1. Suppose A is invertible and you exchange its i-th and j-th coulmns to reach B. Is the new matrix B invertible? Why? How would you find B^{-1} from A^{-1}?

P_{ij} 를 i번째 열과 j번째 열을 교환하는 치환 행렬이라고 하면

$AP_{ij} = B$ 로 나타낼 수 있다.

P는 invertible 이므로 AP_{ij} 또한 invertible 이고, 따라서 B도 invertible 이다.

이 때 $B^{-1} = P_{ij}^{-1} A^{-1} = P_{ij}^{T} A^{-1}$ 의 계산을 통해 구할 수 있다.

2. True or false (with a counterexample if false and a reason if true):

(a) A square matrix A with a column of zeros is not invertible.

(b) If A^T is invertible then A is invertible.

(a) A의 i번째 열이 0으로만 이루어져 있으면, A와 같은 크기 행렬 B가 있을 때

$$BA = B\begin{pmatrix} A_1 & A_2 & \cdots & \begin{smallmatrix}0\\0\\0\end{smallmatrix} & \cdots & A_n \end{pmatrix} = \begin{pmatrix} BA_1 & BA_2 & \cdots & 0 & \cdots & BA_n \end{pmatrix}$$

이 되므로, $BA = I = AB$ 를 만족하는 행렬 B가 존재하지 않는다.

따라서 A는 not invertible 이다.

(b) $A^T B = I$ 이고, 역행렬은 유일하므로 $(A^T B)^T = I^T$

$\Rightarrow B^T A = I$ 에서

A는 B^T를 역행렬로 가지게 된다.

따라서 A는 invertible이다.

3. Find the inverse of A

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

Gauss - Jordan 방법을 이용하면,

$$[A\ e_1\ e_2\ e_3] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$+\frac{1}{8} - \frac{4}{8}$

→ pivot

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{4} & 1 & 0 & 0 \\ 0 & \frac{1}{3} & 1 & 0 & -\frac{1}{3} & 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 1 \end{array}\right]$$

$-\frac{4}{12} + \frac{1}{12}$

$-\frac{1}{12}$

→ pivot

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{4} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{4} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & -\frac{3}{8} & -\frac{1}{2} & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|cccc} & 1 & 0 & 0 & 0 \\ I & -\frac{1}{4} & 1 & 0 & 0 \\ & -\frac{1}{4} & -\frac{1}{3} & 1 & 0 \\ & -\frac{1}{4} & -\frac{1}{3} & -\frac{1}{2} & 1 \end{array}\right] = [U\ \ L^{-1}]$$

이 때 $A^{-1} = U^{-1}L^{-1}$ 에서 $U^{-1} = I$ 이므로

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{3} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{3} & -\frac{1}{2} & 1 \end{bmatrix}$$

4. If A has column 1 + column 2 = column 3, show that A is not invertible:

(a) Find a nonzero solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{0}$.

(b) Explain why elimination keeps column 1 + column 2 = column 3.

(c) Explain why there is no third pivot.

x가 길이 n인 벡터일 때,

(a) x_i $(i=1,2,3,\cdots,n)$ 에서 $x_1 = x_2 = -x_3 = c$ $(c \neq 0)$

이고 $x_4 \sim x_n = 0$ 인 벡터 $x = \begin{pmatrix} c \\ c \\ -c \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ 는

임의의 행렬 A에 $Ax = 0$을 만족한다

(b) Elimination 과정에서 곱해지는 행렬을 E라 하면

$$EA = [\, EA_1 \mid EA_2 \mid EA_3 \mid \cdots \,]$$ 이고, $A_1 + A_2 = A_3$ 이므로,

$$= [\, EA_1 \mid EA_2 \mid E(A_1 + A_2) \mid \cdots \,]$$ 에서 분배법칙에 의해

$$= [\, EA_1 \mid EA_2 \mid EA_1 + EA_2 \mid \cdots \,]$$ 가 되므로

Column 1 + column 2 = column 3 이 보존된다.

(c) 2번째 pivot까지 소거를 진행하면,

$A_{13} = A_{23} = 0$ 이 된다.

(b)에 의해 소거 과정에서도 Column 1 + column 2 = column 3 이

유지되므로, $A_{33} = 0$이 되므로 세 번째 pivot이 존재하지 않는다.

$\begin{matrix} 1 & 3 \\ 0 & 2 \end{matrix} \to$ C

5. If A and B have nonzeros in the positions marked by $*$, which zeros are still zero in their factors L and U?

$$A = \begin{bmatrix} * & * & * & * \\ * & * & * & 0 \\ 0 & * & * & * \\ 0 & 0 & * & * \end{bmatrix}, \qquad B = \begin{bmatrix} * & * & * & 0 \\ * & * & 0 & * \\ * & 0 & * & * \\ 0 & * & * & * \end{bmatrix}$$

$$A = \overset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & \boxed{0} \\ \boxed{0} & l_{32} & 1 & 0 \\ \boxed{0} & \boxed{0} & l_{43} & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} * & * & * & * \\ 0 & * & * & - \\ \boxed{0} & 0 & * & * \\ \boxed{0} & \boxed{0} & 0 & * \end{bmatrix}} \to 0 - l_{21} \cdot A_{14} \neq 0$$

$$B = \overset{L}{\begin{bmatrix} 1 & 0 & 0 & \boxed{0} \\ l_{21} & 1 & \boxed{0} & 0 \\ l_{31} & \boxed{0} & 1 & 0 \\ \boxed{0} & l_{42} & l_{43} & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} * & * & * & \boxed{0} \\ 0 & * & - & * \\ 0 & \boxed{0} & * & * \\ \boxed{0} & 0 & 0 & * \end{bmatrix}} \to 0 - l_{21} \cdot A_{13} \neq 0$$

ㅁ 표시를 한 곳들의 0이 각각 L과 U에 보존된다.

6. The less familiar form $A = LPU$ exchanges rows only at the end:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \rightarrow L^{-1}A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix} = PU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

What is L in this case?

$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ 라 하면 $LP = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 0 & 1 \\ l_{31} & 1 & l_{32} \end{bmatrix}$

$A = LPU = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 0 & 1 \\ l_{31} & 1 & l_{32} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix}$ 을 계산하면, $l_{21} = 1$, $l_{31} = 2$, $l_{32} = 0$

따라서 $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

7. Find the inverse of A

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

Gauss - Jordan method 를 이용하면,

$$\begin{bmatrix} A & I \end{bmatrix} = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 3 & 4 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

pivot

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 3 & 4 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$$

pivot

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 4 & -3 & 1 & 0 \\ 0 & 0 & 4 & 1 & 4 & -3 & 0 & 1 \end{array}\right]$$

pivot

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 4 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 & -12 & 9 & -4 & 1 \end{array}\right] = \begin{bmatrix} U & L^{-1} \end{bmatrix}$$

이 때 A는 lower triangular matrix 이므로 $A = LU = LI$

$A^{-1} = I^{-1}L^{-1} = L^{-1}$

따라서 $A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 4 & -3 & 1 & 0 \\ -12 & 9 & -4 & 1 \end{bmatrix}$

8. (a) If P is any permutation matrix, find a nonzero vector $\mathbf{x}$ so that $(I - P)\mathbf{x} = \mathbf{0}$.

(b) If P has 1s on the anitidiagonal from $(1, n)$ to $(n, 1)$, describe PAP.

(a) x의, 모든 원소가 같은 값을 가지는 벡터

$x_i = c$ $(i=1,2,3, \cdots n)$, $c \neq 0$ 는

$(I-P)x = 0$ 을 만족한다.

$$\therefore x = \begin{pmatrix} c \\ c \\ c \\ c \\ \vdots \\ c \end{pmatrix} \quad (c \neq 0)$$

(b) PA는 행렬을 위 아래로 뒤집고,

AP는 행렬을 좌우로 뒤집으므로,

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$ 일 때

$$PAP = \begin{bmatrix} a_{nn} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{11} \end{bmatrix}$$ 로 나타낼 수 있다.

9. (a) What matrix E has the same effect as these three steps? Subtract row 1 from row 2, subtract row 1 from row 3, then substract row 2 from row 3.

(b) What single matrix L has the same effect as these three reverse steps? Add row 2 to row 3, add row 1 to row 3, then add row 1 to row 2.

(a)
$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

(b)
$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

0. (a) Explain why the inner product $\mathbf{x}^T\mathbf{y}$ of $\mathbf{x}$ and $\mathbf{y}$ equals the inner product of $P\mathbf{x}$ and $P\mathbf{y}$, where P is a permutation matrix.

(b) With $\mathbf{x}^T = (1, 2, 3)$ and $\mathbf{y}^T = (1, 4, 2)$, choose a 3×3 permutation matrix P to show that $(P\mathbf{x})^T\mathbf{y}$ is not always equal to $\mathbf{x}^T(P\mathbf{y})$.

(a) $x^T y = x_1 y_1 + x_2 \cdot y_2 + x_3 y_3 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i$

P가 벡터의 임의의 i번째 원소와 j번째 원소를 교환할 때, $x_i = Px_j$, $y_i = Py_j$ 이다.

따라서 $x_i y_i = Px_j \cdot Py_j \quad (1 \le i, j \le n)$

이 되므로

$$x^T y = \sum_{i=1}^{n} x_i y_i = \sum_{j=1}^{n} Px_j Py_j = Px \cdot Py.$$

$$\therefore x^T y = Px \cdot Py$$

(b) $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ 일 때,

$$(Px)^T y = (2\ 3\ 1)\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = 14$$

$$x^T(Py) = (1\ 2\ 3)\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = 11$$

으로, $(Px)^T y$ 와 $x^T(Py)$가 같지 않다.

11. (a) Suppose you solve $A\mathbf{x} = \mathbf{b}$ for three special right-hand sides $\mathbf{b}$:

$$A\mathbf{x}_1 = \mathbf{e}_1, \quad A\mathbf{x}_2 = \mathbf{e}_2, \quad A\mathbf{x}_3 = \mathbf{e}_3.$$

If the solutions $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are the columns of a matrix X, what is AX?

(b) Find the inverses of

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 8 \end{bmatrix}.$$

(a) $AX = A \cdot [x_1\ x_2\ x_3] = [Ax_1\ Ax_2\ Ax_3]$

$= [e_1\ e_2\ e_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore AX = I$

(b) $A_1 = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{pmatrix} \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$

$\therefore A_1^{-1} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \frac{1}{2} & & \\ & & \frac{1}{3} & \\ & & & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

A_2는 Gauss-Jordan을 통해

$[U\ L^{-1}] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 6 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 & -24 & 12 & -4 & 0 \end{array}\right]$ 이 되고, A_2가 lower triangular matrix 이므로,

$\therefore A_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 6 & -3 & 1 & 0 \\ -24 & 12 & -4 & 0 \end{bmatrix}$

A_3는 Gauss-Jordan을 하면, $[U\ L^{-1}] = \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 5 & 6 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{2}{5} & 0 & 0 & -\frac{7}{5} & 1 \end{array}\right]$

$\Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -4 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{7}{2} & -\frac{5}{2} \end{array}\right] = [I\ A^{-1}]$

$\therefore A_3^{-1} = \begin{bmatrix} -2 & 1 & 0 & 0 \\ \frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -4 & 3 \\ 0 & 0 & \frac{7}{2} & -\frac{5}{2} \end{bmatrix}$

12. Write down the 5×5 finite-difference matrix equation ($h = \frac{1}{6}$) for

$$-\frac{d^2u}{dx^2} = f(x), \qquad \frac{du}{dx}(0) = \frac{du}{dx}(1) = 0.$$

$-u''(x) = f(x)$, $u'(0) = u'(1) = 0$,

$u'(x) = \frac{u(x+h)-u(x)}{h}$, $u'(x-h) = \frac{u(x)-u(x-h)}{h}$

$u''(x) = \frac{1}{h^2}(u(x+h)-2u(x)+u(x-h)) = -f(x)$

즉 $h^2 f(x) = -u(x+h)+2u(x)-u(x-h)$ 가 되고

$u'(0) = \frac{u(h)-u(0)}{h} = 0 \rightarrow u(0) = u(h)$ 이고, 마찬가지로 $u(1) = u(h)$

$u_i = u(i \cdot h)$ ($i = 0, 1, 2, \dots, 6$, $h = \frac{1}{6}$) 로 나타내면

difference equation은 $-u_{j+1} + 2u_j - u_{j-1} = h^2 \cdot f(jh)$ ($j = 1, 2, 3, \dots, n$)이 된다.

$j=1 \Rightarrow -U_2 + 2U_1 - U_0 = h^2 \cdot f(h)$

$\vdots$

$j=5 \Rightarrow -U_6 + 2U_5 - U_4 = h^2 \cdot f(5h)$

이 때 $U_0 = U(0) = u(h)$ 이고, $u(h) = U(1 \cdot h) = U_1$ 이 된다.

마찬가지로 $U_6 = U_1$ 이 된다.

따라서 $j=1$ 일때와 $j=5$ 일때 식은 각각

$j=1 \Rightarrow -U_2 + U_1 = h^2 \cdot f(h)$

$j=5 \Rightarrow -U_1 + 2U_5 - U_4 = h^2 \cdot f(5h)$

가 되고, 이를 matrix 형식으로 나타내면

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix} = h^2 \cdot \begin{bmatrix} f(h) \\ f(2h) \\ f(3h) \\ f(4h) \\ f(5h) \end{bmatrix} \quad \text{where } h = \frac{1}{6}$$