Linear and Nonlinear Computation Models

(CSE 4190.313)

Midterm Exam: May 2, 2011

(Solutions)

Problem	Score
1	
2	
3	
4	
5	
6	
7	
Total	

Name: ____________________

ID No: ____________________

Dept: ____________________

1. (15 points) True or false, with reason if true and counterexample if false:

   (a) (7 points) If $L_1U_1 = L_2U_2$ (upper triangular U's with nonzero diagonal, lower triangular L's with unit diagonal), then $L_1 = L_2$ and $U_1 = U_2$. The LU factorization is unique.

   (b) (4 points) If $A^2 + A = I$, then $A^{-1} = A + I$.

   (c) (4 points) If all diagonal entries of A are zero, then A is singular.

(a) True

$\because$ $L_1U_1 = L_2U_2 \Rightarrow L_2^{-1}L_1 = U_2 \cdot U_1^{-1}$

Lower triangular with 1's on the diagonal → Diagonal with 1's

upper triangular

$\Rightarrow L_2^{-1}L_1 = U_2 \cdot U_1^{-1} = I$

$\therefore L_1 = L_2$ and $U_1 = U_2$

(b) True

$\because$ $A(A+I) = I$ and $(A+I)A = I$

$\therefore (A+I) = A^{-1}$

(c) False

$\because$ Counterexample

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\det A = -1 \neq 0$

$\therefore A$ is nonsingular

2. (15 points) Solve $A\mathbf{x} = \mathbf{b}$ using the triangular systems $L\mathbf{c} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{c}$.

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

What part of A^{-1} have you found with this particular $\mathbf{b}$?

(a) $$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow c_1 = 0, \quad c_2 = 0, \quad c_3 = 1$

(b) $$\begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow x_3 = 1, \quad x_2 = -3, \quad 2x_1 = 6 - 4 = 2$

$\therefore x_1 = 1$

$$\therefore \mathbf{x} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

(c) The third column of A^{-1}

3. (20 points)

(a) (10 points) Reduce the following matrix equation $A\mathbf{x} = \mathbf{b}$ to $U\mathbf{x} = \mathbf{c}$ and then to $\mathbf{R}\mathbf{x} = \mathbf{d}$:

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix} = \mathbf{b}.$$

(b) (10 points) Find a particular solution $\mathbf{x}_p$ and all nullspace solutions $\mathbf{x}_n$.

(a)
$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 1 & 3 & 2 & 0 & 5 \\ 2 & 0 & 4 & 9 & 10 \end{array}\right] \rightarrow \left[\overbrace{\begin{array}{cccc} 1 & 0 & 2 & 3 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & 0 & 3 \end{array}}^{U} \middle| \overbrace{\begin{array}{c} 2 \\ 3 \\ 6 \end{array}}^{=\mathbf{c}}\right]$$

$$\rightarrow \left[\underbrace{\begin{array}{cccc} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}}_{R} \middle| \underbrace{\begin{array}{c} -4 \\ 3 \\ 2 \end{array}}_{=\mathbf{d}}\right]$$

(b)
$$\mathbf{x}_p = \begin{bmatrix} -4 \\ 3 \\ 0 \\ 2 \end{bmatrix}$$

$$\mathbf{x}_n = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

4. (10 points) Find bases for the four fundamental subspaces of

$$A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Basis of $C(A)$: $(1,1,1)$

Basis of $C(A^T)$: $(1,2,3)$

Basis of $N(A)$: $(-2,1,0), (-3,0,1)$

Basis of $N(A^T)$: $(1,-1,0), (1,0,-1)$

5. (20 points) Suppose L_1 is the line through the origin in the direction of $\mathbf{a}_1$ and L_2 is the line through $\mathbf{b}$ in the direction of $\mathbf{a}_2$.

(a) (10 points) To find the closest points $x_1\mathbf{a}_1$ and $\mathbf{b} + x_2\mathbf{a}_2$ on the two lines, write the two equations for the x_1 and x_2, that minimize $\|x_1\mathbf{a}_1 - x_2\mathbf{a}_2 - \mathbf{b}\|$.

(b) (10 points) Solve for x_1 and x_2 if $\mathbf{a}_1 = (1, 1, 0)$, $\mathbf{a}_2 = (0, 1, 0)$, $\mathbf{b} = (2, 1, 4)$.

(a) Solve $$\begin{bmatrix} a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_1^T a_1 & -a_1^T a_2 \\ -a_1^T a_2 & a_2^T a_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} a_1^T b \\ -a_2^T b \end{bmatrix}$$

(b) $$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

6. (10 points)

(a) (4 points) Find an orthonormal basis for the column space of A.

$$A = \begin{bmatrix} 1 & -6 \\ 3 & 6 \\ 4 & 8 \\ 5 & 0 \\ 7 & 8 \end{bmatrix}$$

(b) (3 points) Write A as QR, where Q has orthonormal columns and R is upper triangular.

(c) (3 points) Find the least-square solution to $A\mathbf{x} = \mathbf{b}$, if $\mathbf{b} = (-3, 7, 1, 0, 4)$.

(a) $$q_1 = \frac{1}{\|a\|} a = \frac{1}{10} \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix}$$

$$B = b - (q_1^T b) q_1 = \begin{bmatrix} -6 \\ 6 \\ 8 \\ 0 \\ 8 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{bmatrix}$$

$$q_2 = \frac{1}{\|B\|} B = \frac{1}{10} \begin{bmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{bmatrix}$$

(b) $$A = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} 10 & 10 \\ 0 & 10 \end{bmatrix}$$

(c) $$QRx = b \Rightarrow Rx = Q^T b$$

$$\begin{bmatrix} 10 & 10 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$\therefore x_1 = 0, \quad x_2 = 0.5$$

7. (10 points)

(a) The corners of a triangle are $(2,1)$, $(3,4)$, and $(0,5)$. What is the area?

(b) A new corner at $(-1,0)$ makes it four-sided. Find the area of the quadrangle.

(a) $\frac{1}{2}\begin{vmatrix} 1 & 3 \\ -2 & 4 \end{vmatrix} = \frac{1}{2} \times 10 = 5$

(b) $5 + \frac{1}{2}\begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix} = 5 + \frac{1}{2} \times 14 = 12$