Linear and Nonlinear Computation Models

(CSE 4190.313)

Midterm Exam: April 25, 2012

(Solutions)

Problem	Score
1	
2	
3	
4	
5	
6	
7	
8	
Total	

Name: ______________________

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1. (15 points) True or false, with reason if true or counterexample if false:

   (a) (5 points) If A is invertible and its rows are in reverse order in B, then B is invertible.

   (b) (5 points) If A and B are symmetric, then AB is symmetric.

   (c) (5 points) Every nonsingular matrix can be factored into the product $A = LU$ of a lower triangular L and an upper triangular U.

(a) True

$\because$ Otherwise, B is singular.

$$\Rightarrow \exists\, x = (x_1, \cdots, x_n)^T \neq 0 \text{ s.t } Bx = 0$$

Let $y = (x_n, \cdots, x_1)^T \neq 0$. then

$$Ay = 0 \text{ with } y \neq 0 \quad \#$$

$\Downarrow$

(This means that A is non-invertible)

(b) False

Counterexample:

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}}_{\text{symmetric}} = \begin{bmatrix} 7 & 7 \\ 8 & 10 \end{bmatrix} : \text{asymmetric}$$

(c) False

Counterexample:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} ad & ae \\ bd & be+cf \end{bmatrix}$$

$ad = 0 \Rightarrow a = 0$ or $d = 0$

(i) $a = 0 \Rightarrow ae = 0 \neq 1$ #

(ii) $d = 0 \Rightarrow bd = 0 \neq 1$ #

$\therefore \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \neq LU$

2. (10 points) Construct a matrix whose nullspace consists of all combinations of $(2, 2, 1, 0)$ and $(3, 1, 0, 1)$.

$$A = \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & -1 \end{bmatrix}$$

$(2,2,1,0)$ and $(3,1,0,1)$ are special solutions of the above reduced row echelon form.

3. (10 points) Suppose A is 5 by 4 with rank 4. Show that $A\mathbf{x} = \mathbf{b}$ has no solution when the 5 by 5 matrix $[A\ \mathbf{b}]$ is invertible. Show that $A\mathbf{x} = \mathbf{b}$ is solvable when $[A\ \mathbf{b}]$ is singular.

(a) When the square matrix $[A\ \mathbf{b}]$ is invertible, $\mathbf{b}$ cannot be represented as a linear combination of the columns of A. Thus, $\mathbf{b}$ is not in the column space of A $\Rightarrow$ $\underline{A\mathbf{x} = \mathbf{b} \text{ has no solution}}$.

(b) When $[A\ \mathbf{b}]$ is singular, $[A\ \mathbf{b}]$ has rank 4.

$\exists c_i \neq 0$ s.t $c_1 a_1 + \cdots + c_4 a_4 + c_5 \mathbf{b} = \mathbf{0}$,

where $A = [a_1 \cdots a_4]$.

$\Rightarrow c_5 \neq 0$ [$\because$ otherwise, $c_1 = \cdots = c_4 = 0$ as $a_1, \cdots, a_4$ are lin. indep.]

$$\mathbf{b} = -\frac{c_1}{c_5} a_1 - \cdots - \frac{c_4}{c_5} a_4 \in C(A)$$

$\therefore A\mathbf{x} = \mathbf{b}$ is solvable

4. (15 points) Add the extra column $\mathbf{b}$ and reduce A to echelon form:

$$[A\ \mathbf{b}] = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{bmatrix}.$$

A combination of the rows of A has produced the zero row. What combination is it? Which vectors are in the nullspace of A^T and which are in the nullspace of A?

(a) (row 1) $- 2\times$ (row 2) $+$ (row 3) $= 0$

(b) $\alpha(1,-2,1) \in N(A^T)$, $\alpha \in \mathbb{R}$

(c) $U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = R$

$\beta(1,-2,1) \in N(A)$, $\beta \in \mathbb{R}$

5. (10 points) Why is each of these statements false?

(a) (4 points) $(1,1,1)$ is perpendicular to $(1,1,-2)$, so the planes $x+y+z=0$ and $x+y-2z=0$ are orthogonal subspaces.

(b) (3 points) The subspace spanned by $(1,1,0,0,0)$ and $(0,0,0,1,1)$ is the orthogonal complement of the subspace spanned by $(1,-1,0,0,0)$ and $(2,-2,3,4,-4)$.

(c) (3 points) Two subspaces that meet only in the zero vector are orthogonal.

(a) The two planes share nonzero vectors on the line: $x+y=0$, $z=0$

$\Rightarrow$ They cannot be orthogonal subspaces.

(b) The orthogonal complement should be 3-dimensional.

(c) Two lines $x+y=0$ and $2x+y=0$ meet only in the zero vector.

But, they are not orthogonal subspaces.

6. (15 points)

(a) (7 points) Suppose you guess your professor's age, making errors $e = -2, -1, 5$ with probabilities $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$. Check that the expected error $E(e)$ is zero and find the variance $E(e^2)$.

(b) (8 points) If the professor guesses too (or tries to remember), making errors $-1, 0, 1$ with probabilities $\frac{1}{8}, \frac{6}{8}, \frac{1}{8}$, what weights w_1 and w_2 give the reliability of your guess and the professor's guess.

(a) $E(e_1) = (-2)\times\frac{1}{2} + (-1)\times\frac{1}{4} + 5\times\frac{1}{4} = 0$ +3

$E(e_1^2) = (-2)^2\times\frac{1}{2} + (-1)^2\times\frac{1}{4} + 5^2\times\frac{1}{4} = \frac{17}{2}$ +4

(b) $E(e_2) = (-1)\times\frac{1}{8} + 0\times\frac{6}{8} + 1\times\frac{1}{8} = 0$ +4

$E(e_2^2) = (-1)^2\times\frac{1}{8} + 1^2\times\frac{1}{8} = \frac{1}{4}$

$w_1 = \frac{\sqrt{34}}{17}$, $w_2 = 2$ +4

7. (10 points) Compute the Fourier coefficients a_0, a_1, b_1:

$$a_0 = \frac{(y, 1)}{(1, 1)}, \quad a_1 = \frac{(y, \cos x)}{(\cos x, \cos x)}, \quad b_1 = \frac{(y, \sin x)}{(\sin x, \sin x)}$$

of the following step function $y(x)$, for $0 \le x \le 2\pi$:

$$y(x) = \begin{cases} 1 & \text{for } 0 \le x \le \pi, \\ 0 & \text{for } \pi < x \le 2\pi. \end{cases}$$

$(1,1) = \int_0^{2\pi} dx = 2\pi$, $(y,1) = \int_0^{\pi} dx = \pi$

$\therefore a_0 = \frac{1}{2}$ +3

$(y, \cos x) = \int_0^{\pi} \cos x \, dx = [\sin x]_0^{\pi} = 0 \Rightarrow a_1 = 0$ +3

$(y, \sin x) = \int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi} = 2$

$(\sin x, \sin x) = \int_0^{2\pi} \sin^2 x \, dx = \frac{1}{2}\int_0^{2\pi} (1 - \cos 2x)\, dx = \pi$

$\therefore b_1 = \frac{2}{\pi}$ +4

8. (15 points)

(a) (5 points) Find the determinant of an $n \times n$ matrix M_j which is obtained when a vector $\mathbf{x} = (x_1, \cdots, x_n)^T$ replaces the j-th column of the identity matrix $I_{n \times n}$.

(b) (5 points) If $A\mathbf{x} = \mathbf{b}$, show that AM_j is the matrix B_j which is obtained when the vector $\mathbf{b} = (b_1, \cdots, b_n)^T$ replaces the j-th column of the $n \times n$ matrix A.

(c) (5 points) By taking determinants in $AM_j = B_j$, show that $x_j = \frac{\det B_j}{\det A}$.

(a)

$$M_j = [\mathbf{e}_1 \cdots \mathbf{e}_{j-1} \ \mathbf{x} \ \mathbf{e}_{j+1} \cdots \mathbf{e}_n] = \begin{bmatrix} 1 & 0 & & x_1 & & 0 \\ 0 & 1 & & \vdots & & \vdots \\ \vdots & 0 & \ddots & x_j & & \vdots \\ \vdots & \vdots & & \vdots & & 0 \\ 0 & 0 & & x_n & & 1 \end{bmatrix}$$

$$\det M_j = \det \begin{bmatrix} 1 & & 0 & & 0 \\ & \ddots & \vdots & & \vdots \\ & & 1 & & \vdots \\ & & x_j & & \vdots \\ & & 0 & 1 & \\ & & \vdots & & \ddots \\ & & 0 & & 1 \end{bmatrix} = x_j$$

(b) $AM_j = A[\mathbf{e}_1 \cdots \mathbf{e}_{j-1} \ \mathbf{x} \ \mathbf{e}_{j+1} \cdots \mathbf{e}_n]$, where $A = [\mathbf{a}_1 \cdots \mathbf{a}_n]$

$$= [A\mathbf{e}_1 \cdots A\mathbf{e}_{j-1} \ A\mathbf{x} \ A\mathbf{e}_{j+1} \cdots A\mathbf{e}_n]$$

$$= [\mathbf{a}_1 \cdots \mathbf{a}_{j-1} \ \mathbf{b} \ \mathbf{a}_{j+1} \cdots \mathbf{a}_n]$$

$$= B_j$$

(c) $\det(AM_j) = \det(B_j)$

$\det(A) \cdot \det(M_j) = \det(B_j)$

$\underline{x_j = \det(M_j) = (\det B_j) / (\det A)}$