
- Written by: Hyun Jong Kim
- Created: 2/2/2024
- Last updated: 2/4/2024

These are notes for my fall 2022 GNTS presentation notes on Tuesday, 2/6/2023.

I would greatly appreciate comments and corrections to these notes; please send such suggestions to me at `hyunjong<dot>kim<at>math<dot>wisc<dot>edu` or to my latest email address.

> **Disclaimer**
>
> You might find it useful to use Obsidian.md if you are reading this as a Markdown file so that you can use the internal links in these notes. However, I may or may not make other notes that this note links to publicly available, so such links may be useless to you. Nevertheless, reading this as a Markdown file on Obsidian.md (alongside as a pdf file) could supplement your reading experience.

> **Title**
>
> A Zoo of L-functions

> **Abstract**
>
> I will talk about some different kinds of L-functions (and zeta functions) and maybe some problems surrounding them

---

0. Riemann zeta function

Reference: Bump et al, "Introduction to the Langlands Program", Chapter 1

$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$

Facts:

1. Converges absolutely for $\mathrm{Re}(s) > 1$
2. We have Euler product

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$$

3. We have functional equation: letting

$$\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s),$$

Λ is meromorphic (and therefore ζ is meromorphic) except for simple poles at $s = 0, 1$ and satisfies

$$\Lambda(1-s) = \Lambda(s)$$

1. Dirichlet L-functions

Reference: Bump et al, "Introduction to the Langlands Program", Chapter 1

- bump_et_al_ilp_3.1 Theorem
- Generalize ζ

Definition

Let $\chi : \mathbf{Z} \to \mathbf{C}$ be a Dirichlet character modulo m. The **Dirichlet L-series** is defined by

[

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p (1 - \chi(p)p^{-s})^{-1}$$

]

for $\operatorname{Re} s > 1$

Note that this generalizes ζ because $L(s, 1) = \zeta$.

Define $\Lambda(\chi, s)$ by

$$\Lambda(\chi, s) = \begin{cases} \pi^{-s/2}\Gamma(s/2)L(\chi, s) & \text{if } \chi(-1) = 1 \\ \pi^{-(s+1)/2}\Gamma((s+1)/2)L(\chi, s) & \text{if } \chi(-1) = -1 \end{cases}$$

We have the functional equation

$$\Lambda(\chi, s) = \varepsilon(\chi)q^{1/2-s}\Lambda(\overline{\chi}, 1-s)$$

(ε is definable with a "Gauss sum" and $\overline{\chi} : (\mathbf{Z}/m\mathbf{Z})^\times \to \mathbf{C}^\times$ is the Dirichlet character mod m "inducing" χ)

Therefore, $L(\chi, s)$ is extendable to a meromorphic function on the complex plane.

2. Dedekind zeta functions

Reference: Wikipedia

Definition

Let K be an algebraic number field. The **Dedekind zeta function** is defined by

$$\zeta_K(s) = \sum_{I \subseteq \mathcal{O}_K,\ \text{nonzero}} \frac{1}{(N_{K/\mathbb{Q}}(I))^s} = \prod_{\mathfrak{p} \subseteq \mathcal{O}_K \text{ prime}} \frac{1}{1 - N_{K/\mathbb{Q}}(\mathfrak{p})^{-s}}$$

for $\mathrm{Re}(s) > 1$.

$N_{K/\mathbb{Q}}(\mathfrak{p})$ equals the size of the residue field of $\mathfrak{p}$.

For $K = \mathbb{Q}$, $\zeta_K(s) = \zeta(s)$.

We have a functional equation and hence an analytic continuation to a meromorphic function.

2' Partial Dedekind zeta function

Reference: Drew Sutherland's MIT 18.785 Notes, Lecture 21

Definition

Let K be an algebraic number field, and let S be a set of primes of K. The **partial Dedekind zeta function associated to S** is defined by

$$\zeta_{K,S}(s) = \prod_{\mathfrak{p} \in S} \frac{1}{1 - N_{K/\mathbb{Q}}(\mathfrak{p})^{-s}}$$

for $\mathrm{Re}(s) > 1$.

If S is the set of all primes of K, then $\zeta_{K,S} = \zeta_K$.

If S has all but finitely many primes, then $\zeta_{K,S}$ and ζ_K differ by the product of finitely many factors, so $\zeta_{K,S}$ extends to a meromorphic function

3. Hecke L-function of a Hecke character of a number field

Reference: Bump, Chapter 1, Section 5

Definition

Let K be a number field and let $\chi : I_{\mathfrak{m}} \to \mathbf{C}^\times$[1] be a Hecke character of weight ξ_∞ for the modulus $\mathfrak{m}$ of K, where $\xi_\infty : P_{\mathfrak{m}} \to \mathbf{C}^\times$ is a (unitary) character[2].
The Hecke L-function of χ is

$$L(\chi, s) = \sum_{\mathfrak{a}} \chi(\mathfrak{a})(N\mathfrak{a})^{-s} = \prod_{\mathfrak{p}} \left(1 - \chi(\mathfrak{p})(N\mathfrak{p})^{-s}\right)^{-1}$$

For the trivial character $\chi = \chi_0$, the Hecke L-function is the Dedekind zeta function, i.e. $L(\chi_0, s) = \zeta_K(s)$.

Hecke characters have a functional equation and an analytic continuation as entire functions.

4. L-function of an extension of number fields

Reference: Bump et al, Chapter 1, Section 5
bump_et_al_ilp_page_12

Let E/K be an abelian extension of number fields

Definition

Let E/K be an abelian extension of number fields with Galois group G. For $\rho \in \hat{G}$ (i.e. $\rho : G \to \mathbb{C}$) Define the L-function

$$L(\rho, s) = \prod_{\mathfrak{p} \text{ prime of } K} (1 - \rho(\sigma_{\mathfrak{p}})(N_{\mathfrak{p}})^{-s})^{-1}$$

where $\rho(\sigma_{\mathfrak{p}})$ is the image of the Frobenius element by the Galois representation induced by ρ in $\mathbf{C}^{I_{\mathfrak{p}}}$.

bump_et_al_ilp_5.9 Theorem

(Artin)

Let K be a number field, E/K a finite abelian extension with Galois group G, let $\rho : G \to \mathrm{C}^{\times}$ be a Galois character and $L(\rho, s)$ the associated L-function.

Then there exists a unique primitive Hecke character χ of K, of modulus m such that

$$L(\rho, s) = L(\chi, s).$$

When $K = \mathbb{Q}$, we have a theorem that yields the Kronecker-Weber theorem (which states that all abelian extensions of K are subfields of cyclotomic fields).

5. Artin L-function of a global Galois representation (of a number field)

Reference: Gabor Wiese's notes "Galois Representations"

wiese_gr_1.4.1 Definition

Definition

Let K be a number field, and let $\rho : G_K \to \mathrm{GL}_n(\mathbb{C})$ be a Galois representation (i.e. an Artin representation). Define the L-function

$$L(\rho, s) = \prod_{\mathfrak{p}} \frac{1}{\mathrm{char\,poly}(\mathrm{Frob}_{\mathfrak{p}})(\rho)(N(\mathfrak{p})^{-s})}$$

One can also more generally define a "partial L-function" for representations over topological fields, assuming that the characteristic polynomial of Frobenius are in $\overline{\mathbb{Q}}[X]$.

This is a generalization of 4, which only discussed Galois characters.

Conjecture

(Artin)
If ρ is a non-trivial Artin representation, then $L(\rho, s)$ has a holomorphic continuation to the whole complex plane.

We know that $L(\rho, s)$ has a functional equation and hence a meromorphic continuation.

See also Silverman's "Advanced Topics in the Arithmetic of Elliptic curves", Chapter II, Section 10 for a discussion on the special case of elliptic curves

6. Zeta function of a nice variety over a finite field

Reference: Poonen's notes "Lectures on Rational points on Curves",

Definition

Given a "nice" variety over a finite field $\mathbb{F}_q$, define

$$Z_X(T) = \prod_{\text{closed points } P \in X} (1 - T^{\deg P})^{-1}$$

$$\zeta_X(s) = Z_X(q^{-s})$$

By the Weil-conjectures, $Z_X(T)$, which is a priori in $\mathbb{Q}[[T]]$, is in fact in $\mathbb{Q}(T)$. Moreover, Z_X has a functional equation.

More generally, we can define a zeta function for an arbitrary scheme of finite type over $\mathbb{Z}$.

For a nice genus g-curve X, we have

$$Z_X(T) = \frac{P_1(T)}{(1-T)(1-qT)}$$

for some integer polynomial $P_1(T)$ of the form
$1 + a_1T + a_2T^2 + \cdots + a_gT^g + qa_{g-1}T^{g+1} + q^2a_{g-2}T^{g+2} + \cdots + q^gT^{2g}$.

For an elliptic curve $E/\mathbb{F}_q$, we have $Z_E(E,T) = (1 - aT + pT^2)$, where a is the "trace of Frobenius" satisfying $a = q + 1 - \#E(\mathbb{F}_q)$.

Hasse-Weil zeta function for nice varieties over number fields (5 mixed with 6ish for elliptic curves)

Reference: Silverman's "Advanced Topics in the Arithmetic of Elliptic curves", Chapter II, Section 10

Now given $K/\mathbb{Q}$ a number field and let E/L be an elliptic curve. For each prime $\mathfrak{p}$ of K, define the **local L-series of E at $\mathfrak{p}$** by

$$L_{\mathfrak{p}}(E/K,T) = \begin{cases} Z_{E/\mathbb{F}_{\mathfrak{p}}}(T) & \text{if } E \text{ has good reduction at } \mathfrak{p} \\ 1-T & \text{if } E \text{ has split multiplicative reduction at } \mathfrak{p} \\ 1+T & \text{if } E \text{ has nonsplit multiplicative reduction at } \mathfrak{p} \\ 1 & \text{if } E \text{ has additive reduction at } \mathfrak{p} \end{cases}$$

where $\mathbb{F}_{\mathfrak{p}}$ is the residue field of $\mathfrak{p}$ and define the **global L-series of E** by

$$L(E/K,s) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(E/K, \#\mathbb{F}_{\mathfrak{p}}^{-s})^{-1}$$

This is an example of Artin L-function of a global Galois representation. Also, these ideas can be generalized to algebraic varieties V over number fields K.

Whether $L(E/K,s)$ has a functional equation[4] is generally not known. However, by the modularity theorem, which shows that $L(E/\mathbb{Q},s) = L(f,s)$ for some newform f, we can conclude that $L(E/\mathbb{Q},s)$ has a functional equation because $L(f,s)$ does. Deuring also showed that a CM elliptic curve E/K has an L-function that is expressible either as the L-function of a Grössencharacter or the product of two such characters. A Grössencharacter has a functional equation, so $L(E/K,s)$ has a functional equation if E has CM.

1. whether the L-function of a nice variety over a number field has a functional equation
2. whether the zeta function of a nice variety over a finite field has a functional equation

So the following type of question is still quite open:

Conjecture

$L(E/K,s)$ has a functional equation when K is a general number field (and E/K is not a CM elliptic curve)

Moreover, Artin's holomorphy conjecture, which is stated above, states that $L(E/K,s)$ has a holomorphic continuation to the whole complex plane.

6'. $\mathbb{A}^1$-enriched logarithmic zeta function

Reference: Bilu, Ho, Srinivasan, Vogt, and Wickelgren's paper "Quadratic enrichment of the logarithmic derivative of the zeta function"

> Let X be a smooth, proper variety over a field k. Let $\varphi : X \to X$ be an endomorphism. The **$\mathbb{A}^1$-logarithmic zeta function of (X,φ)** is defined by
>
> $$\operatorname{dlog} \zeta_{X,\varphi}^{\mathbb{A}^1} := \sum \operatorname{Tr}(\varphi^m) t^{m-1} \in \operatorname{GW}(k)[[t]]$$

The motivation for this definition[5] is to generalize the classical zeta function $\zeta_X(T)$ towards $\mathbb{A}^1$-enumerative geometry: when $k = \mathbb{F}_q$ and φ is the (geometric) Frobenius endomorphism on X, we have

$$\zeta_X(T) = \exp\left(\sum_{m\geq 1} \frac{|X(\mathbb{F}_{q^m})|}{m} T^m\right),$$

so

$$\operatorname{dlog}\zeta_X(T) := \frac{d}{dT}\log\zeta_X(T) = \sum_{m\geq 1} |X(\mathbb{F}_{q^m})| T^{m-1}.$$

Moreover, when φ is the Frobenius endomorphism, $\operatorname{Tr}(\varphi^m)$ is an "$\mathbb{A}^1$-enriched version of $|X(\mathbb{F}_{q^m})|$", where counting is done not with integers, but with elements of the Grothndieck-Witt ring $\operatorname{GW}(k)$ whose elements of generated by nondegenerate symmetric bilinear forms on finite dimension k-vector spaces[6].

One can define appropriately a notion of "dlog-rationality" for the power series $\operatorname{dlog}\zeta^{\mathbb{A}^1}_{X,\varphi}$. It turns out that there is a class of schemes (smooth projective schemes over k with "cellular structure") for which $\operatorname{dlog}\zeta^{\mathbb{A}^1}_{X,\varphi}$ is dlog-rational. There are also schemes, such as some elliptic curves, for which $\operatorname{dlog}\zeta^{\mathbb{A}^1}_{X,\varphi}$ is not dlog-rational.

The functional equation has been verified for $\operatorname{dlog}^{\mathbb{A}^1}_{\mathbb{P}^n}(T)$ with the Frobenius endomorphism.

7. L-function of a modular form

Reference: Diamond, Shurman, Chapter 5 Section 9

Definition

Given a weight k-modular form f for $\Gamma_1(N)$, i.e. $f \in \mathcal{M}_k(\Gamma_1(N))$, write its fourier expansion by

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n$$

where $q = e^{2\pi i\tau}$.
Its $\boxed{L\textbf{-function}}$ is defined by the Dirichlet series

$$L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

Whenever $f \in \mathcal{S}_k(\Gamma_1(N))^{\pm}$, there is a functional equation for $L(s, f)$ and hence $L(s, f)$ has an analytic continuation to the complex plane.

The following fact addresses whether $L(s, f)$ has an Euler product expansion

Theorem

The following are equivalent for $f \in \mathcal{M}_k(\Gamma_1(N))$:

- f is a normalized eigenform, i.e. it is a eigenvector for all Hecke operators T_n and $\langle n\rangle$ such that $a_1 = 1$.
- $L(s, f)$ has an Euler product expansion

$$L(s,f)=\prod_p \left(1-a_p p^{-s}+\chi(p)p^{k-1-2s}\right)^{-1}$$

Also, given $f\in\mathcal{M}_k(N,\chi)$[7] for a Dirichlet character χ modulo N, f is a normalized eigenform if and only if we have the recursive relations:

1. $a_1(f)=1$,
2. $a_{p^r}(f)=a_p(f)a_{p^{r-1}}(f)-\chi(p)p^{k-1}a_{p^{r-2}}(f)$ for all p prime and $r\geq 2$,
3. $a_{mn}(f)=a_m(f)a_n(f)$ when $(m,n)=1$.

For elliptic curves over $E/\mathbb{Q}$, we have the very same recursive relations for the traces a_{p^r} of Frobenius for $E/\mathbb{F}_{p^r}$ when p is a prime of good reduction.

In fact, one can say more:

Theorem

(Modularity theorem, "version L")
Let $E/\mathbb{Q}$ be an elliptic curve with conductor N_E. Then for some newform[8] $f\in\mathcal{S}_2(\Gamma_0(N_E))$,
$L(s,f)=L(s,E)$[9]

As mentioned above, this modularity theorem allows us to conclude that $L(s,E)$ has a functional equation and thus has an analytic continuation.

7'. L-function of a newform by getting an Artin L-function from an abelian variety constructed from the newform

Given a newform $f\in\mathcal{S}_2(\Gamma_1(M_f))$ at a level M_f, one can construct an abelian variety

$$A_f=J_1(M_f)/I_fJ_1(M_f)$$

Here,

- $J_1(M_f)$ is the Jacobian of $X_1(M_f)$,
- there is an action of the (level M_f) "Hecke algebra" $\mathbb{T}_\mathbb{Z}$[11] on $J_1(M_f)$
- I_f denotes the kernel of the homomorphism $\mathbb{T}_\mathbb{Z}\to\mathbb{C}$ given by sending $T\in\mathbb{T}_\mathbb{Z}$ to the eigenvalue $\lambda_f(T)$ of T on f, i.e.

$$T_f=\lambda_f(T)f$$

and

$$I_f=\{T\in\mathbb{T}_\mathbb{Z}:\lambda_f(T)=0\}.$$

Fact

(See Diamond, Shurman Lemma 9.5.3)
Let χ be a Dirichlet character modulo N. Let $f \in \mathcal{S}_2(N, \chi)$ be a normalized eigenform. Then $V_\ell(A_f) := T_\ell(A_f) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is a free module of rank 2 over $\mathbb{K}_f \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ where $\mathbb{K}_f$ is the number field generated over $\mathbb{Q}$ by the Fourier coefficients a_n of f[12].

Now take a look at its $\mathbb{Q}_\ell$-Galois representation of $G_{\mathbb{Q}}$ acting on $V_\ell(A_f) \cong (K_f \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^2$. Note that $K_f \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ decomposes in the form $\prod_{\lambda \mid \ell} \mathbb{K}_{f,\lambda}$ where the product is over primes λ of $\mathbb{K}_f$ lying over ℓ, so by projective, for each λ we get a 2-dimensional representation

$$\rho_{f,\lambda} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2\left(\mathbb{K}_{f,\lambda}\right)$$

such a representation is called a **modular representation** and it has its own Artin L-function

Many forms of the modularity theorem

Above, we say the following modularity theorem:

Theorem

(Modularity theorem, "version L")
Let $E/\mathbb{Q}$ be an elliptic curve with conductor N_E. Then for some newform7 $f \in \mathcal{S}_2(\Gamma_0(N_E))$, $L(s, f) = L(s, E)$9

There are also many other formulations of the modularity theorem, all taken from Diamond and Shurman:

Ther following is a formulation that says that the ℓ-adic Galois representation of an elliptic curve over $\mathbb{Q}$ equals a representation of some newform:

Theorem

(Modularity theorem, "version R")
Let E be an elliptic curve over $\mathbb{Q}$. Then $\rho_{E,\ell}$ is a modular representation for some ℓ, i.e. there is some newform $f \in \mathcal{S}_2(\Gamma_0(M_f))$ such that $\mathbb{K}_{f,\lambda} = \mathbb{Q}_\ell$ for some maximal ideal λ of $\mathcal{O}_{\mathbb{K}_f}$ lying over ℓ such that $\rho_{f,\lambda}$ is equivalent to $\rho_{E,\ell}$.

Here is a stronger formulation of the above:

Theorem

(Modularity theorem, "strong version R")
Let E be an elliptic curve over $\mathbb{Q}$ with conductor N. Then for some newform $f \in \mathcal{S}_2\left(\Gamma_0(N)\right)$ with number field $\mathbb{K}_f = \mathbb{Q}$, $\rho_{f,\lambda} \sim \rho_{E,\ell}$ for all ℓ

Here is a statement concerning how the traces of Frobenii/Hasse invariants of an elliptic curve over $\mathbb{Q}$ are equal to the Fourier coefficients of some newform:

Theorem

(Modularity Theorem, Version a_p). Let E be an elliptic curve over $\mathbb{Q}$ with conductor N_E. Then for some newform $f \in \mathcal{S}_2\left(\Gamma_0\left(N_E\right)\right)$,

$$a_p(f)=a_p(E) \quad \text{for all primes } p$$

Here are some modularity theorem statements for elliptic curves over $\mathbb{Q}$ about how elliptic curves are covered by modular curves.

Theorem

(Modularity theorem, "version $X_{\mathbb{Q}}$")
Let E be an elliptic curve over $\mathbb{Q}$. Then for some positive integer N there exists a surjective morphism over $\mathbb{Q}$ of curves over $\mathbb{Q}$ from the modular curve $X_0(N)_{\text{alg}}$ to the elliptic curve E,

$$X_0(N)_{\text{alg}} \longrightarrow E$$

(Modularity Theorem, "version $A_{\mathbb{Q}}$") Let E be an elliptic curve over $\mathbb{Q}$. Then for some positive integer N and some newform $f \in \mathcal{S}_2\left(\Gamma_0(N)\right)$ there exists a surjective morphism over $\mathbb{Q}$ of varieties over $\mathbb{Q}$

$$A_{f, \text{alg}}^{\prime} \longrightarrow E$$

(Modularity Theorem, Version $J_{\mathbb{Q}}$). Let E be an elliptic curve over $\mathbb{Q}$. Then for some positive integer N there exists a surjective morphism over $\mathbb{Q}$ of varieties over $\mathbb{Q}$

$$\mathrm{J}_0(N)_{\text{alg}} \longrightarrow E$$

And here are some modularity statements for complex elliptic curves with j-invariants over $\mathbb{Q}$:

Theorem

(Modularity Theorem, Version $X_{\mathbb{C}}$)
Let E be a complex elliptic curve with $j(E) \in \mathbb{Q}$. Then for some positive integer N there exists a surjective holomorphic homomorphism of compact Riemann surfaces from the modular curve $X_0(N)$ to the elliptic curve E,

$$X_0(N) \longrightarrow E$$

(Modularity Theorem, Version $A_{\mathbb{C}}$)
Let E be a complex elliptic curve with $j(E) \in \mathbb{Q}$.
Then for some positive integer N and some newform $f \in \mathcal{S}_2\left(\Gamma_0(N)\right)$ there exists a surjective holomorphic homomorphism of complex tori

$$A_f^{\prime} \longrightarrow E$$

(Modularity Theorem, Version $J_{\mathbb{C}}$) Let E be a complex elliptic curve with $j(E) \in \mathbb{Q}$. Let $J_0(N)$ denotes the Jacobian of $X_0(N)$. Then for some positive integer N there exists a surjective holomorphic homomorphism of complex tori

$$J_0(N) \longrightarrow E$$

See Also

- https://en.wikipedia.org/wiki/Category:Zeta_and_L-functions - Lists many Wikipedia articles that have to do with Zeta and L-functions.

Meta

References

Citations and Footnotes

---

1. $I_{\mathfrak{m}}$ denotes the subgroup of the group I of fractional ideals in K defined by

   $$I_{\mathfrak{m}} = \{a \in I \mid (a, m) = 1\}$$

   where $\mathfrak{m}$ is a nonzero integral ideal of the number field K.
   ↩
2. Of more technically speaking, induced from a unitary character $K^\times/\mathbf{Q}^\times \to \mathbf{C}^\times$ such that $U_{\mathfrak{m}} \subset \ker \xi_\infty$[3]↩
3. $U_{\mathfrak{m}}$ denotes the group of units in $P_{\mathfrak{m}}$ where $\mathfrak{m}$ is a modulus of the number field K.
   - $P_{\mathfrak{m}}$

   ↩
4. Do not confuse the following questions:↩
5. The logarithmic derivative of the zeta function is used instead of the zeta function itself because the question of rationality for the $\mathbb{A}^1$-logarithmic zeta function becomes difficult to ask because the Grothendieck-Witt ring in general has torsion elements.↩
6. Moreover, taking the ranks of these vector spaces should in principle return any "classical" count, i.e. counts with familiar numbers.↩
7. $\mathcal{M}_k(N, \chi)$ denotes the χ -eigenspace of $\mathcal{M}_k(\Gamma_1(N))$ where χ is a Dirichlet character modulo N.

   In other words, it is the space of $f \in \mathcal{M}_k(\Gamma_1(N))$ such that

   $$f[\gamma]_k = \chi(d_\gamma) f$$

   for all $\gamma \in \Gamma_1(N)$.

   #_meta/TODO/notation d_γ
   - $\mathcal{M}_k(\Gamma)$
   - $\Gamma_1(N)$
   - $[\gamma_k]$

   ; d_γ is denotes the lower right entry of $\gamma \in \Gamma_0(N)$.↩

8. A newform is a normalized eigenform in $\mathcal{S}_k(\Gamma_1(N))^{\text{new}}$[10]↩
9. Here $L(s,E)$ is the global L-series $L(E/\mathbb{Q},s)$ as defined in 5 mixed with 6ish for elliptic curves↩
10. $\mathcal{S}_k(\Gamma_1(N))^{\text{new}}$ denotes the space of newforms at level N of weight k.

    It is the orthogonal complement of $\mathcal{S}_k(\Gamma_1(N))^{\text{old}}$ with respect to the Petersson inner product, i.e.

    [

    $$\mathcal{S}_k(\Gamma_1(N))^{\text{new}} = (\mathcal{S}_k(\Gamma_1(N))^{\text{old}})^{\perp}.$$

    ]

    ↩
11. $\mathbb{T}_{\mathbb{Z}}$ denotes the Hecke algebra (of level N; N is omitted from the notation). It is the algebra of endomorphisms of $\mathcal{S}_2(\Gamma_1(N))$ generated over $\mathbb{Z}$ by Hecke operators

    $$\mathbb{T}_{\mathbb{Z}} = \mathbb{Z}\left[\left\{T_n, \langle n \rangle : n \in \mathbb{Z}^+\right\}\right]$$

    - $\mathcal{S}_2$
    - $\Gamma_1(N)$
    - T_n
    - $\langle n \rangle$

    ↩
12. There turns out to be an isomorphism $\mathbb{T}_{\mathbb{Z}}/I_f \xrightarrow{\sim} \mathcal{O}_f = \mathbb{Z}[\{a_n : n \in \mathbb{Z}^+\}]$. Moreover, as noted above, $\mathbb{T}_{\mathbb{Z}}/I_f$ acts on A_f, and hence $\mathcal{O}_f$ acts on $T_\ell(A_f)$ as well.↩