$\mathbb{A}^1$-HOMOTOPY THEORY

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Contents

1. $\mathbb{A}^1$-homotopy categories

2. Simplicial sheaves in the Nisnevich topology as a category of spaces

S: A base scheme. This is assumed to be Noetherian (Definition D.0.4) of finite dimension (Definition D.0.9) when the $\mathbb{A}^1$-homotopy category $\mathcal{H}(S)$ is discussed.

Sch/S: The category of schemes over S.

Sm/S: The category of smooth schemes over S.

Sets: The category of sets.

Notation 2.0.1. Given a site T, let $\mathrm{Shv}(T) = \mathrm{Shv}(T, \mathbf{Sets})$ denote its category of sheaves of sets (Definition A.0.10). The category $\mathrm{Shv}(T)$ has all small limits and colimits and internal function objects. Moreover, $\mathrm{Shv}(T)$ has a final object (Lemma B.0.4), denoted by pt, $*$, or $\bullet$.

Definition 2.0.2. Let $\mathcal{C}$ be a (large) category (Definition A.0.1) with a final object (Definition A.0.4) $*$. By a *pointed object of $\mathcal{C}$*, we mean an object X of $\mathcal{C}$ equipped with a morphism $* \to X$. The *category of pointed objects of $\mathcal{C}$* refers to the under category/coslice category (Definition A.0.17) of $*$, i.e. the category whose

- objects are morphisms $* \to X$ in $\mathcal{C}$ and
- morphisms between objects $* \to X$ and $* \to Y$ are morphisms $X \to Y$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd} * \arrow[r] \arrow[dr] & X \arrow[d] \\ & Y \end{tikzcd}$$

The category of pointed objects of C may be denoted by notations such as $C_\bullet$ or C_*s.

Definition 2.0.3. Let T be a site (Definition A.0.9). The category of *pointed sheaves* (resp. of *simplicial pointed sheaves*) of sets on T is the category $\mathrm{Shv}(T)_\bullet = \mathrm{Shv}(T, \mathbf{Sets})_\bullet$ (resp. $\Delta^{\mathrm{op}} \mathrm{Shv}(T)_\bullet = \Delta^{\mathrm{op}} \mathrm{Shv}(T, \mathbf{Sets})_\bullet$) of pointed objects in (Definition 2.0.2) $\mathrm{Shv}(T) = \mathrm{Shv}(T, \mathbf{Sets})$ (resp. $\Delta^{\mathrm{op}} \mathrm{Shv}(T) = \Delta^{\mathrm{op}} \mathrm{Shv}(T, \mathbf{Sets})$); note that the category $\mathrm{Shv}(T)$ (resp. $\Delta^{\mathrm{op}} \mathrm{Shv}(T)$) has a final object (Lemma B.0.4) (♠ TODO: simplicial sheaves of sets has a final object).

It is also equivalent to refer to pointed sheaves of sets (resp. pointed simplicial sheaves of sets) as *sheaves of pointed sets* (resp. *simplicial sheaves of pointed sets*).

Definition 2.0.4. Let $\mathcal{C}$ be a (large) category (Definition A.0.1) with a final object (Definition A.0.4) $*$. By the *pointification of an object X of $\mathcal{C}$*, we mean the pointed object (Definition 2.0.2) coproduct (Definition A.0.14) $X_+ := X \coprod *$, if such an object exists, equipped with the natural morphism $* \to X_+$. If X_+ exists for all objects X of $\mathcal{C}$, then note that $X \mapsto X_+$ specifies a functor $\mathcal{C} \to \mathcal{C}_\bullet$ to the category of pointed objects of $\mathcal{C}$ (Definition 2.0.2).

In particular, we may talk about the pointification of an object in the following contexts:

- For a site (Definition A.0.9) T, we may talk about the pointification of a sheaf (Definition A.0.10) $X \in \mathrm{Shv}(T)$ of sets or of a simplicial (Definition 2.2.2) sheaf (Definition A.0.10) $X \in \Delta^{\mathrm{op}} \mathrm{Shv}(T)$ of sets; note that the categories $\mathrm{Shv}(T)$ and $\Delta^{\mathrm{op}} \mathrm{Shv}(T)$ both have final objects and (small) coproducts (♠ TODO:).
- We may talk about the pointification of topological spaces (Definition C.0.1); note that the category of topological spaces has a final object and (small) coproducts.

Moreover, in the context of discussing a pointification functor, the *forgetful functor* is the functor $\mathcal{C}_\bullet \to \mathcal{C}$ given by sending a pointed space $(X, * \to X)$ to X.

Lemma 2.0.5. Let $\mathcal{C}$ be a locally small (Definition A.0.6) category (Definition 2.2.6) with a final object (Definition A.0.4) $*$ and such that a pointification X_+ (Definition 2.0.4) of X exists for all objects X of $\mathcal{C}$, so that pointifcation is a functor $\mathcal{C} \to \mathcal{C}_\bullet$ to the category of pointed objects (Definition 2.0.2) of $\mathcal{C}$.

The pointification functor is left adjoint (Definition A.0.15) to the forgetful functor. In other words, for every object X of $\mathcal{C}$ and $(Y, y : * \to Y)$ of $\mathcal{C}_\bullet$, we have natural isomorphisms

$$\mathrm{Hom}_{\mathcal{C}_\bullet}(X_+, (Y, y)) \cong \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

Proof. A morphism $(X_+) = (X \coprod *, *) \to (Y, y)$ of pointed spaces consists of a morphism $(X \coprod *) \to Y$ such that the composition $* \to (X \coprod *) \to Y$ coincides with y. Included in this data is a morphism $X \to (X \coprod *) \to Y$, so there is a set map

$$\mathrm{Hom}_{\mathcal{C}_\bullet}(X_+, (Y, y)) \to \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

Conversely, given a morphism $X \to Y$, we can extend it uniquely to a morphism $X_+ \to (Y, y)$, yielding a set map

$$\mathrm{Hom}_{\mathcal{C}_\bullet}(X_+, (Y, y)) \leftarrow \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

These two set maps are in fact inverses. □

2.1. **Elementary distinguished squares and the Nisnevich topology.**

Definition 2.1.1. [See [Voe98, Definition 2.1], [MV99, Section 3 Definition 1.3]] Let S be a scheme (Definition D.0.2). An elementary distinguished square in the category Sm/S of smooth schemes (Definition D.0.5) over S is a square of the form

(A)
$$\begin{tikzcd} p^{-1}(U) \arrow[r] \arrow[d, "p"] & V \arrow[d, "p"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

(♠ TODO: open embedding, reduced subscheme, support) such that p is an étale morphism (Definition D.0.7), j is an open embedding, and $p^{-1}(X-U) \to X-U$ is an isomorphism (where $X-U$ is the maximal reduced subscheme with support in the closed subset $X-U$).

Definition 2.1.2. Let $f : X \to Y$ be a morphism of schemes (Definition D.0.3). We say that f is a *finite type morphism* if for every affine open $V = \mathrm{Spec}\, B \subseteq Y$ with $U = f^{-1}(V)$ affine, say $U = \mathrm{Spec}\, A$, the ring A is a finitely generated B-algebra.

When X is equipped with a finite type morphism $f : X \to Y$, we say that X is a *finite type scheme over Y* or a *finite type Y-scheme* or a *Y-scheme of finite type* (Definition D.0.8), etc.

Proposition 2.1.3 ([MV99, Section 3 Proposition 1.1])**.** Let S be a Noetherian (Definition D.0.4) scheme of finite dimension (Definition D.0.9). Let X be a scheme of finite type (Definition 2.1.2) over S and let $\{U_i \to X\}$ be a finite family of étale morphisms (Definition D.0.7) in Sch/S (Definition D.0.8). The following conditions are equivalent: (♠ TODO: residue field)

1. For any point x of X there is an i and a point u of U_i over x such that the corresponding morphism of residue fields is an isomorphism which maps to x with the same residue field.
2. For any point $x \in X$, the morphism
$$\coprod_i (U_i \times_X \mathrm{Spec}\, \mathscr{O}^h_{X,x}) \to \mathrm{Spec}\, \mathscr{O}^h_{X,x}$$
of S-schemes admits a section.

Moroever, the collection of families of étale morphisms $\{U_i \to X\}$ in Sm/S satisfying the equivalent conditions above forms a pretopology on Sm/S (Definition D.0.5).

Definition 2.1.4. [See [MV99, Section 3 Definition 1.2]] Let S be a Noetherian (Definition D.0.4) scheme of finite dimension (Definition D.0.9). The Grothendieck topology (Definition A.0.9) generated by the pretopology of Proposition 2.1.3 is called the *Nisnevich topology on* Sm/S. The site whose underlying category is Sm/S and whose Grothendieck topology is the Nisnevich topology is called the *(big) Nisnevich site of S* and is denoted by notations such as $(\mathrm{Sm}/S)_{\mathrm{Nis}}$, $(\mathbf{Sm}/S)_{\mathrm{Nis}}$, etc. A covering family in the Nisnevich site of S is called a *Nisnevich covering*.

(♠ TODO: establish that a nisnevich covering is a family of etale moprhisms such that there is an isomorphism of residue fields at every point)

Proposition 2.1.5 (See [Voe98, Definition 2.2], [MV99, Section 3 Proposition 1.4])**.** Let S be a Noetherian (Definition D.0.4) scheme of finite dimension (Definition D.0.9). A presheaf (Definition A.0.8) $F : \mathrm{Sm}/S \to \mathbf{Sets}$ is a sheaf in the Nisnevich topology (Definition 2.1.4) if and only if for any elementary distinguished square (Equation (A))

$$\begin{tikzcd} p^{-1}(U) \arrow[r] \arrow[d, "p"] & V \arrow[d, "p"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

the induced square

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(V) \arrow[d] \\ F(U) \arrow[r] & F(p^{-1}(U)) \end{tikzcd}$$

is Cartesian.

2.2. **Simplicial objects in a category.**

Definition 2.2.1. The *simplex category*, or *nonempty finite ordinal category*, denoted by Δ, is the locally small (Definition A.0.6) category (Definition A.0.1) whose

- objects are the finite nonempty totally ordered sets $[n] := \{0, 1, 2, \dots, n\}$ for each integer $n \geq 0$;
- morphisms are all order-preserving (non-decreasing) functions $\theta : [m] \to [n]$.

Composition in Δ is given by composition of functions.

Definition 2.2.2. Let $\mathcal{C}$ be a category (Definition A.0.1).

1. The *category of simplicial objects in $\mathcal{C}$*, commonly denoted by notations such as $\mathbf{s}\mathcal{C}$, $\mathrm{Simp}\,\mathcal{C}$, $\mathcal{C}_\Delta$, or $(\Delta^{\mathrm{op}})^{\mathcal{C}}$ (cf. Definition A.0.12), or $\Delta^{\mathrm{op}}C$, is the functor category (Definition A.0.12)
$$\mathbf{s}\mathcal{C} := \mathbf{Fun}(\Delta^{\mathrm{op}}, \mathcal{C}).$$
(Definition 2.2.1) In particular, a morphism between objects $X, Y : \Delta^{\mathrm{op}} \to \mathcal{C}$ in this category is a natural transform $X \Rightarrow Y$ from X to Y as functors.
An object X of $\mathbf{s}\mathcal{C}$ is called a *simplicial object of $\mathcal{C}$*, and, by definition (Definition A.0.2), consists of a family of objects $\{X_n\}_{n \geq 0}$ in $\mathcal{C}$ together with morphisms
$$X(\theta) : X_n \to X_m, \quad \text{for each } \theta : [m] \to [n] \text{ in } \Delta,$$
satisfying the functoriality conditions
$$X(\mathrm{id}_{[n]}) = \mathrm{id}_{X_n}, \qquad X(\theta \circ \psi) = X(\psi) \circ X(\theta).$$

2. Dually, the *category of cosimplicial objects in $\mathcal{C}$*, commonly denoted by notatoins such as $\mathbf{c}\mathcal{C}$, $\mathrm{Cosimp}\,\mathcal{C}$, or $\Delta^{\mathcal{C}}$ (cf. Definition A.0.12) is the functor category

$$\mathbf{c}\mathcal{C} := \mathbf{Fun}(\Delta, \mathcal{C}).$$

In particular, a morphism between objects $X, Y : \Delta \to \mathcal{C}$ in this category is a natural transform $X \Rightarrow Y$ from X to Y as functors.

An object Y of $\mathbf{c}\mathcal{C}$ is called a *cosimplicial object of $\mathcal{C}$*, and consists of a family of objects $\{Y^n\}_{n\geq 0}$ in $\mathcal{C}$ together with morphisms

$$Y(\theta) : Y^m \to Y^n, \quad \text{for each } \theta : [m] \to [n] \text{ in } \Delta,$$

satisfying the functoriality conditions

$$Y(\mathrm{id}_{[n]}) = \mathrm{id}_{Y^n}, \qquad Y(\theta \circ \psi) = Y(\theta) \circ Y(\psi).$$

For instance, a *(co)simplicial set, group, topological space, ring, etc.* refers to a (co)simplicial object in the category of sets, of groups, of topological spaces, of rings, etc. and such categories are denoted by notations such as $\mathbf{Sets}_\Delta$, $\mathbf{Grps}_\Delta$, $\mathbf{Top}_\Delta$, $\mathbf{Rings}_\Delta$, etc. Accordingly, a (co)simplicial map between such (co)simplicial objects refers to a morphism in the appropriate (co)simplicial category.

If $\mathcal{C}$ is locally small (Definition A.0.6), then both $\mathbf{s}\mathcal{C}$ and $\mathbf{c}\mathcal{C}$ are locally small as well.

Definition 2.2.3. Let $\mathcal{C}$ be a category (Definition A.0.1).

1. Let X be a simplicial object (Definition 2.2.2) in $\mathcal{C}$. An object $X_n := X([n])$ of $\mathcal{C}$ is called the *n-simplices of X*. In case that $\mathcal{C}$ is some kind of category of sets, an element of X_n is called an *n-simplex of X*, so X_n is the *set of n-simplices of X*. In this case, a *vertex of X* moreover refers to a 0-simplex of X and a *edge of X* refers to a 1-simplex of X.

   For each morphism $\theta : [m] \to [n]$ in Δ, the induced morphism

$$X(\theta) : X_n \to X_m$$

   in $\mathcal{C}$ is called a *simplicial morphism*.

   For each $0 \leq j \leq n$, the *jth face map of the n-simplicies* refers to the map

$$d_j = X(p_j) : X_n \to X_{n-1}, \quad p_j : [n-1] \to [n], \quad p_j(i) = \begin{cases} i & \text{if } i < j \\ i+1 & \text{if } i \geq j \end{cases}.$$

   d_j is also denoted by ∂_j.

   For each $0 \leq j \leq n$, the *jth degeneracy map of the n-simplicies* refers to the map

$$s_j = X(q_j) : X_n \to X_{n+1}, \quad q_j : [n+1] \to [n], \quad q_j(i) = \begin{cases} i & \text{if } i \leq j \\ i-1 & \text{if } i > j \end{cases}.$$

2. Let $Y : \Delta \to \mathcal{C}$ be a cosimplicial object (Definition 2.2.2) of $\mathcal{C}$. An object $Y^n := Y([n])$ of $\mathcal{C}$ is called the *n-cosimplicies of Y*. In case that $\mathcal{C}$ is some kind of category of sets, an element of Y_n is called an *n-cosimplex of Y*, so Y^n is the *set of n-cosimplices of Y*.

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For each morphism $\theta : [m] \to [n]$ in Δ, the induced morphism

$$Y(\theta) : Y^m \to Y^n$$

is called a *cosimplicial morphism*.

For each $0 \le j \le n$, the *jth coface map of the n-cosimplicies* refers to the map

$$d^j = Y(p_j) : Y^n \to Y^{n+1}, \quad p_j : [n] \to [n+1], \quad p_j(i) = \begin{cases} i & \text{if } i < j \\ i+1 & \text{if } i \ge j \end{cases}$$

d^j is also denoted by ∂^j.

For each $0 \le j \le n$, the *jth codegeneracy map of the n-cosimplicies* refers to the map

$$s^j = Y(q_j) : Y^n \to Y^{n-1}, \quad q_j : [n] \to [n-1], \quad q_j(i) = \begin{cases} i & \text{if } i \le j \\ i-1 & \text{if } i > j \end{cases}$$

A *(co)face/degeneracy of a the n-(co)simplicies of a (co)simplicial object* may also refer to the images of the (co)face/degeneracy maps.

Definition 2.2.4. Let $n \ge 0$ be an integer. The representable functor

$$\mathrm{Hom}_\Delta(-, [n]) : \Delta^{\mathrm{op}} \to \mathbf{Sets}$$

(Definition 2.2.1) is a simplicial set (Definition 2.2.2) often denoted by Δ^n, $\Delta^{[n]}$, or $\Delta[n]$, and is called the *standard n-simplex*. More generally, if J is a finite nonempty linearly ordered set, then we may speak of the simplicial set Δ^J given by the representable functor $\mathrm{Hom}_\Delta(-, J)$.

The *Standard simplicial n-simplex functor* refers to the functor

$$\Delta^\bullet : \Delta \to \mathbf{sSets}$$

given by $[n] \mapsto \Delta^n$. By construct, note that $\Delta^\bullet$ is a cosimplicial object (Definition 2.2.2) in the category of simplicial sets.

Dually, the functor

$$\mathrm{Hom}_\Delta([n], -) : \Delta \to \mathbf{Sets}$$

is a cosimplicial set (Definition 2.2.2) called the *standard n-cosimplex*.

Definition 2.2.5 (Model Category)**.** (♠ TODO: I don't like the axioms as stated here) A *model category*, or synonymously a *closed model category*, is a complete and cocomplete category $\mathcal{M}$ equipped with three distinguished classes of morphisms:

- *Weak equivalences* $\mathcal{W}$,
- *Fibrations* $\mathcal{F}$,
- *Cofibrations* $\mathcal{C}$,

1. **(Two-out-of-three)**: For any composable morphisms $f : X \to Y$, $g : Y \to Z$, if any two of f, g, or $g \circ f$ lie in $\mathcal{W}$, then so does the third.

2. **(Retracts)**: Each of the classes $\mathcal{W}, \mathcal{F}, \mathcal{C}$ is closed under retracts in the arrow category $\mathcal{M}^2$. That is, if f is a retract of g and g belongs to one of these classes, then f also belongs to that class.
3. **(Lifting)**: Given any commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

where $i \in \mathcal{C}$ and $p \in \mathcal{F}$, a diagonal filler (lift) exists making both triangles commute provided either
   - i is also a weak equivalence (called an acyclic cofibration), or
   - p is also a weak equivalence (called an acyclic fibration).

   Formally, acyclic cofibrations have the left lifting property with respect to all fibrations, and cofibrations have the left lifting property with respect to all acyclic fibrations.
4. **(Factorization)**: Every morphism $f : X \to Y$ in $\mathcal{M}$ admits two functorial factorizations:
   - $f = p \circ i$, where $i \in \mathcal{C}$ is a cofibration and $p \in \mathcal{F} \cap \mathcal{W}$ is an acyclic fibration.
   - $f = q \circ j$, where $j \in \mathcal{C} \cap \mathcal{W}$ is an acyclic cofibration and $q \in \mathcal{F}$ is a fibration.

Here, an *acyclic fibration* (or *trivial fibration*) is a morphism in $\mathcal{F} \cap \mathcal{W}$, and an *acyclic cofibration* (or *trivial cofibration*) is a morphism in $\mathcal{C} \cap \mathcal{W}$.

Definition 2.2.6 (Homotopy category of a model category)**.** Let $\mathcal{M}$ be a model category (Definition 2.2.5). The *homotopy category of* $\mathcal{M}$, denoted by notations such as $\mathrm{Ho}(\mathcal{M})$, $\mathrm{h}(\mathcal{M})$, etc. , is the category (Definition A.0.1) $\mathcal{M}[\mathcal{W}^{-1}]$ whose objects are those of $\mathcal{M}$, and whose morphisms are equivalence classes of morphisms in $\mathcal{M}$ under the relation of left and right homotopy, localized (Definition A.0.20) at the weak equivalences. Explicitly,

- The objects of $\mathrm{Ho}(\mathcal{M})$ are the same as those in $\mathcal{M}$.
- For objects X, Y in $\mathcal{M}$, the morphism set $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X, Y)$ consists of maps in $\mathcal{M}$ modulo homotopy, with weak equivalences formally inverted.

2.3. The category of spaces in $\mathbb{A}^1$-homotopy theory.

Definition 2.3.1. Let S be a Noetherian (Definition D.0.4) base scheme of finite dimension (Definition D.0.9). In the context of $\mathbb{A}^1$-homotopy theory, the category of *spaces* is the category $\Delta^{\mathrm{op}} \mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})$ of simplicial sheaves (Definition 2.2.2) for the Nisnevich site (Definition 2.1.4) on Sm/S. This category is also sometimes denoted by Spc. We may also call a space by names such as an *$\mathbb{A}^1$-homotopy (theoretic) space*, or a *motivic space*.

Remark 2.3.2. In [Voe98], Voevodsky developed the theory of the (unstable $\mathbb{A}^1$)-homotopy category by letting the category of spaces [Voe98, Definition 2.2] be merely the category $\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})$ of sheaves for the Nisnevich topology over S, instead of simplicial sheaves as in Definition 2.3.1. In [Voe98, Theorem 3.6], it is established that the (unstable) $\mathbb{A}^1$-homotopy category (Definition 3.3.7) (♠ TODO:) is equivalent to the homotopy category obtainable with sheaves, see Theorem 4.1.14 for a statement

Convention 2.3.3. Given a site T, there is a canonical functor, in fact a fully faithful embedding, $T \to \mathrm{Shv}(T)$ sending an object of T to the associated sheaf, i.e. the functor is given by

$$X \mapsto (Y \mapsto \mathrm{Hom}_T(Y, X));$$

more informally, this is to say that every object $X \in T$ embeds as a representable functor in $\mathrm{Shv}(T)$. As such, we often speak of objects of T as objects of $\mathrm{Shv}(T)$.

Moreover, given any category C, there is a "constant simplicial object functor" $C \to \Delta^{\mathrm{op}}C$ given by sending an object X of C to the simplicial object $\Delta^{\mathrm{op}} \to C$ given by $[n] \mapsto C$ for all $n \geq 0$ and $([n] \to [m]) \mapsto (C \xrightarrow{\mathrm{id}} C)$. Thus, given a sheaf $X \in \mathrm{Shv}(T)$, we can speak of X as though it were a simplicial sheaf $\Delta^{\mathrm{op}} \to \mathrm{Shv}(T)$.

(♠ TODO: Does this convention identify sipmlicial sets as simplicial sheaves?)

(♠ TODO: I might need to go fixing "Noetherian scheme" to "Noetherian scheme of finite dimension" to make sure the Nisnevich site is defined.)

Therefore, given a Noetherian (Definition D.0.4) base scheme S, one can consider S as a space (i.e. an object of $\Delta^{\mathrm{op}}\,\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})$). In fact, S is the final object of $\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})$ and of $\Delta^{\mathrm{op}}\,\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})$. As per **??**, this final object of $\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})$ may again be denoted by pt, $*$, or $\bullet$; note that pt also makes sense as a space by the above discussion.

See Definition B.0.7 for the definition of a point of a topos. For our purposes, we can alternatively define a point (of a site T) as a functor $x^* : \mathrm{Shv}(T) \to \mathbf{Sets}$ which commutes with finite limits and arbitrary colimits.

Lemma 2.3.4. Let T be a site. A point $x = (x_*, x^*, \varphi) : \mathbf{Sets} \to Shv(T)$, as a morphism of topoi, is equivalent to the data of a functor $\mathrm{Shv}(T) \to \mathbf{Sets}$ that commutes with finite limits and all colimits.

Proof. See [Sta25, Tag 00Y3] □

Convention 2.3.5. A point of a site T can be identified with a point of the topos $\mathrm{Shv}(T)$.

Definition 2.3.6. Let S be a Noetherian (Definition D.0.4) base scheme. In the context of $\mathbb{A}^1$-homotopy theory, a *point* of a space $X \in \Delta^{\mathrm{op}}\,\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})$ is a morphism $\mathrm{pt} \to X$. A *pointed space* is a space equipped with a point — as such, the category of pointed spaces is the category $\Delta^{\mathrm{op}}\,\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})_\bullet$ of pointed simplicial sheaves in the Nisnevich topology on Sm/S.

Remark 2.3.7. Note that we have two notions of points — one is the notion of a point of a topos (Definition B.0.7), say of the topos $\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})$. Another is the notion of a point of a space (Definition 2.0.3). It may be fitting to think of the former type of point as a "point of the base scheme S".

3. How to define the $\mathbb{A}^1$-homotopy category

3.1. The simplicial model structure on the category of simplicial sheaves on a site.

For a general small site T, there is a model category structure on $\Delta^{\mathrm{op}}\,\mathrm{Shv}(T)$ called the simplicial model category structure. Defining this for $T = (\mathrm{Sm}/S)_{\mathrm{Nis}}$ is a preliminary step towards defining the $\mathbb{A}^1$-homotopy categories $\mathcal{H}(S)$ and $\mathcal{H}_\bullet(S)$.

Definition 3.1.1 (Standard model category structure on simplicial sets)**.** Let sSet denote the category of simplicial sets (Definition 2.2.2). The *standard model category structure on* sSet, also known as the *Kan model structure* or the *Quillen model structure* or the *Kan-Quillen model structure*, is the model category (Definition 2.2.5) given by specifying three classes of morphisms:

- **Weak equivalences:** A map $f : X \to Y$ in sSet is a weak equivalence, also called a *weak homotopy equivalence between the simplicial sets X and Y*, if the induced map of geometric realizations $|f| : |X| \to |Y|$ is a weak homotopy equivalence of topological spaces (i.e., induces isomorphisms on all homotopy groups for all choices of basepoint).
- **Fibrations:** A map $f : X \to Y$ is a *(Kan) fibration* if it has the right lifting property with respect to the horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ for all $n \geq 1$ and $0 \leq k \leq n$.
- **Cofibrations:** A map $f : X \to Y$ is a cofibration if it is a monomorphism (Definition A.0.16) (i.e., injective at each level).

For this model category structure, all objects are cofibrant, and the fibrant objects are exactly the Kan complexes (Definition 4.1.9).

Definition 3.1.2 (See e.g. [MV99, Section 2 Definition 1.2])**.** Let $f : X \to Y$ be a morphism of simplicial sheaves on some site T.

1. f is called a *(simplicial) weak eqiuvalence* if for any point (Definition B.0.7) x of the site T, the morphism $x^*(f) : x^*(Y) \to x^*(Y)$ is a weak equivalence (Definition 3.1.1) of simplicial sets.
2. f is called a *(simplicial) cofibration* if it is a monomorphism (Definition A.0.16). (♠ TODO: right lifting property)
3. f is called a *(simplicial) fibration* if it has the right lifting property with respect to any cofibration that is also a weak equivalence.

A morphism $f : X \to Y$ of pointed (Definition 2.0.2) simplicial (Definition 2.2.2) sheaves (Definition A.0.10) of sets (on some site) is called a *simplicial weak equivalence* (resp. *simplicial cofibration*, resp. *simplicial fibration*) if it is a simplicial weak equivalence (resp. simplicial cofibration, resp. simplicial fibration) as a morphism of simplicial sheaves.

Theorem 3.1.3 (See e.g. [MV99, Section 2 Theorem 1.4])**.** For any small (Definition A.0.6) site (Definition A.0.9) T, the classes of simplicial weak equivalences, simplicial cofibrations, and simplicial fibrations (Definition 3.1.2) give $\Delta^{\mathrm{op}}\,\mathrm{Shv}(T)$ the structure of a model category (Definition 2.2.5).

There is a similar model category structure on $\Delta^{\mathrm{op}}\,\mathrm{Shv}(T)_\bullet$.

Definition 3.1.4. Let T be a small (Definition A.0.6) site (Definition A.0.9). The model category structure on $\Delta^{\mathrm{op}}\,\mathrm{Shv}(T)$ discussed in Theorem 3.1.3 is referred to as the *simplicial model category structure on* $\Delta^{\mathrm{op}}\,\mathrm{Shv}(T)$. We call the associated homotopy category (Definition 2.2.6) the *simplicial homotopy category of T (or of* $\mathrm{Shv}(T)$ *or of* $\Delta^{\mathrm{op}}\,\mathrm{Shv}(T)$*)* and denote it by $\mathcal{H}_s(T)$.

Similarly, the model category structure on $\Delta^{\mathrm{op}}\,\mathrm{Shv}(T)_\bullet$ is referred to as the *(pointed) simplicial model category structure on* $\Delta^{\mathrm{op}}\,\mathrm{Shv}(T)_\bullet$. We call the associated homotopy category the *pointed simplicial homotopy category of T (or of* $\mathrm{Shv}(T)$ *or of* $\Delta^{\mathrm{op}}\,\mathrm{Shv}(T)$*)* and denote it by $\mathcal{H}_s(T)_\bullet$ or $\mathcal{H}_{s\bullet}(T)$.

3.2. **General model structures on categories of simplicial sheaves.** There are two more general model category structures on $\Delta^{\mathrm{op}}(\mathrm{Shv}(T))$ whose theories [MV99] relies upon to define the $\mathbb{A}^1$-model category structure. One (Definition 3.2.1) is defined with respect to a set A of morphisms in $\mathcal{H}_s(T)$ (Definition 3.1.4) and the other (Definition 3.2.4) is defined with respect to a site with interval. The $\mathbb{A}^1$-simplicial model structure is a special case of the latter, which in turn is a special case of the former.

In defining these model category structures, there is a general pattern — we define "local" objects in $\mathcal{H}_s(T)$, then define what it means for a morphism in $\Delta^{\mathrm{op}}(\mathrm{Shv}(T))$ to be a "weak-equivalence" or to be a "fibration".

Definition 3.2.1 (See e.g. [MV99, Section 2 Definition 2.1, Definition 2.2])**.** Let T be a small (Definition A.0.6) site (Definition A.0.9) and let A be a set of morphisms in $\mathcal{H}_s(T)$ (Definition 3.1.4).

1. An object X of $\mathcal{H}_s(T)$ is called *A-local* if for any $Y \in \mathcal{H}_s(T)$ and any $f : Z_1 \to Z_2$ in A, the map
$$\mathrm{Hom}_{\mathcal{H}_s(T)}(Y \times Z_2, X) \to \mathrm{Hom}_{\mathcal{H}_s(T)}(Y \times Z_1, X)$$
is a bijection. Write $\mathcal{H}_{s,A}(T)$ for the full subcategory of $\mathcal{H}_s(T)$ of A-local objects.
2. A morphism $f : X_1 \to X_2$ in $\Delta^{\mathrm{op}}(\mathrm{Shv}(T))$ is called an *A-weak equivalence* if for any A-local object Y the map
$$\mathrm{Hom}_{\mathcal{H}_s(T)}(X_2, Y) \to \mathrm{Hom}_{\mathcal{H}_s(T)}(X_1, Y)$$
induced by f is a bijection.
3. A morphism $f : X_1 \to X_2$ in $\Delta^{\mathrm{op}}(\mathrm{Shv}(T))$ is called an *A-fibration* if it has the right lifting property with respect to monomorphisms (i.e. simplicial cofibrations) that are also A-weak equivalences.

A morphism $f : X \to Y$ in $\mathcal{H}_s(T)_\bullet$ is called an *A-weak equivalence* (resp. *A-fibration*) if it is an A-weak equivalence (resp. A-fibration) as an unpointed morphism.

Theorem 3.2.2 ([MV99, Section 2 Theorem 2.5])**.** Let T be a small (Definition A.0.6) site (Definition A.0.9) and let A be a set of morphisms in $\mathcal{H}_s(T)$ (Definition 3.1.4). The classes of A-weak equivalences, monomorphisms, and A-fibrations (Definition 3.2.1) give $\Delta^{\mathrm{op}}(\mathrm{Shv}(T))$ (Definition 2.2.2) (Definition A.0.10) the structure of a model category (Definition 2.2.5).

The inclusion functor $\mathcal{H}_{s,A}(T) \hookrightarrow \mathcal{H}_s(T)$ (Definition 3.2.1) has a left adjoint functor (Definition A.0.15)
$$L_A : \mathcal{H}_s(T) \to \mathcal{H}_{s,A}(T),$$
which we call the *A-localization functor*, which identifies $\mathcal{H}_{s,A}(T)$ with the localization of $\mathcal{H}_s(T)$ with respect to A-weak equivalences. (♠ TODO: localizatoin)

Definition 3.2.3 (See [MV99, Section 2.3])**.** Let T be a site (Definition A.0.9) with enough points (Definition B.0.8). An *interval in T* is a sheaf (Definition A.0.10) I of sets on T together with sheaf morphisms (Definition A.0.10)

$$\mu : I \times I \to I$$
$$i_0, i_1 : \mathrm{pt} \to I$$

(Definition A.0.14)

satisfying the following:

- Writing p be the canonical morphism $I \to \mathrm{pt}$,
$$\mu(i_0 \times \mathrm{id}) = \mu(\mathrm{id} \times i_0) = i_0 p$$
$$\mu(i_1 \times \mathrm{id}) = \mu(\mathrm{id} \times i_1) = \mathrm{id}$$
- The morphism $i_0 \coprod i_1 : \mathrm{pt} \coprod \mathrm{pt} \to I$ (Definition A.0.14) is a monomorphism.

A site T with enough points equipped with an interval is called a *site with interval*.

Definition 3.2.4 (See [MV99, Section 2 Definition 3.1])**.** Let (T, I) be a site with interval (Definition 3.2.3).

1. A simplicial (Definition 2.2.2) sheaf (Definition A.0.10) X (of sets) is called *I-local* if for any simplicial sheaf Y the map
$$\mathrm{Hom}_{\mathcal{H}_s(T)}(Y \times I, X) \to \mathrm{Hom}_{\mathcal{H}_s(T)}(Y, X)$$
induced by $i_0 : \mathrm{pt} \to I$ is a bijection. Write $\mathcal{H}_{s,I}(T)$ for the full subcategory of $\mathcal{H}_s(T)$ (Definition 3.1.4) of I-local objects.
2. A morphism $f : X_1 \to Y_2$ in $\Delta^{\mathrm{op}}(\mathrm{Shv}(T))$ (Definition 2.2.2) is called an *I-weak equivalence* if for any I-local object Y the map
$$\mathrm{Hom}_{\mathcal{H}_s(T)}(X_2, Y) \to \mathrm{Hom}_{\mathcal{H}_s(T)}(X_1, Y)$$
is a bijection.
3. A morphism $f : X_1 \to Y_2$ in $\Delta^{\mathrm{op}}(\mathrm{Shv}(T))$ is called an *I-weak fibration* if it has the right lifting property with respect to monomorphisms (i.e. simplicial cofibrations) that are also I-weak equivalences.
(♠ TODO: What is an I-fibration and how does it differ from an I-weak fibration?)

A morphism $f : X \to Y$ in $\Delta^{\mathrm{op}}(\mathrm{Shv}(T))_\bullet$ is called an *I-weak equivalence* (resp. *I-fibration*) if it is an I-weak equivalence (resp. I-fibration) as an unpointed morphism.

Lemma 3.2.5. Let (T, I) be a small (Definition A.0.6) site (Definition A.0.9) with interval (Definition 3.2.3). Let $A = \{i_0\}$.

- A simplicial (Definition 2.2.2) sheaf (Definition A.0.10) (of sets) $X \in \Delta^{\mathrm{op}}(\mathrm{Shv}(T))$ is I-local (Definition 3.2.4) if and only if it is A-local (Definition 3.2.1).
- A morphism $f : X_1 \to X_2$ in $\Delta^{\mathrm{op}}(\mathrm{Shv}(T))$ is an I-weak equivalence (Definition 3.2.4) if and only if it is an A-weak equivalence (Definition 3.2.1).
(♠ TODO: What is an I-fibration and how does it differ from an I-weak fibration?)

- A morphism $f : X_1 \to X_2$ in $\Delta^{\mathrm{op}}(\mathrm{Shv}(T))$ is an I-fibration (Definition 3.2.4) if and only if it is an A-weak fibration (Definition 3.2.1).

Proof. These are all clear by the definitions. $\square$

Theorem 3.2.6 ([MV99, Section 2 Theorem 3.2])**.** Let (T, I) be a small (Definition A.0.6) site with interval (Definition 3.2.3). The classes of I-weak equivalences, monomorphisms, and I-fibrations (Definition 3.2.4) give $\Delta^{\mathrm{op}}(\mathrm{Shv}(T))$ the structure of a model category (Definition 2.2.5). We may refer to this model category as the *I-model category on* $\Delta^{\mathrm{op}}\,\mathrm{Shv}(T)$.

The inclusion functor $\mathcal{H}_{s,I}(T) \hookrightarrow \mathcal{H}_s(T)$ (Definition 3.2.4, Definition 3.1.4) has a left adjoint (Definition A.0.15) functor $L_I : \mathcal{H}_s(T) \to \mathcal{H}_{s,I}(T)$, which we call the *$I$-localization functor*, which identifies $\mathcal{H}_{s,I}(T)$ with the homotopy category (Definition 2.2.6) of $\mathcal{H}_s(T)$, i.e. the localization (Definition A.0.20) of $\mathcal{H}_s(T)$ with respect to I-weak equivalences.

Remark 3.2.7. [MV99, Section 2 Remark 3.3] The I-model category structure (Theorem 3.2.6) on $\Delta^{\mathrm{op}}\,\mathrm{Shv}(T)$ only depends on the object I and not on the morphism i_0.

3.3. **The $\mathbb{A}^1$-simplicial model structure on the category of spaces and the $\mathbb{A}^1$-homotopy category.**

Convention 3.3.1. Let T be a site (Definition A.0.9). Recall that the standard simplicial n-simplex functor $\Delta^\bullet$ (Definition 2.2.4) is the functor

$$\Delta^\bullet : \Delta \to \mathbf{sSets} = \Delta^{\mathrm{op}}\mathbf{Sets}$$

(Definition 2.2.1) (Definition 2.2.2) given by $[n] \mapsto \Delta^n = \mathrm{Hom}_\Delta(-, [n])$. Identifying objects of $\Delta^{\mathrm{op}}\mathbf{Sets}$ as objects of $\Delta^{\mathrm{op}}\,\mathrm{Shv}(T)$ (Convention 2.3.3), we identify $\Delta^\bullet$ as a functor

$$\Delta^\bullet : \Delta \to \Delta^{\mathrm{op}}\,\mathrm{Shv}(T).$$

Definition 3.3.2 ([MV99, Section 3 Definition 2.1])**.** Let S be a Noetherian (Definition D.0.4) scheme of finite dimension (Definition D.0.9).

1. A simplicial (Definition 2.2.2) sheaf (Definition A.0.10) X on $(\mathrm{Sm}/S)_{\mathrm{Nis}}$ (Definition 2.1.4) is called *$\mathbb{A}^1$-local* if for any simplicial sheaf Y the map

$$\mathrm{Hom}_{\mathcal{H}_s((\mathrm{Sm}/S)_{\mathrm{Nis}})}(Y, X) \to \mathrm{Hom}_{\mathcal{H}_s((\mathrm{Sm}/S)_{\mathrm{Nis}})}(Y \times \mathbb{A}^1, X)$$

   (Definition 3.2.1, Definition 3.1.4, Definition 2.1.4) induced by the projection $Y \times \mathbb{A}^1 \to Y$ is a bijection. Write

$$\mathcal{H}_{s,\mathbb{A}^1}(S) = \mathcal{H}_{s,\mathbb{A}^1}(\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}}))$$

   for the full subcategory of $\mathcal{H}_s(\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}}))$ (Definition 3.1.4) of $\mathbb{A}^1$-local objects. A posteriori, it turns out (Corollary 3.3.6) that $\mathcal{H}_{s,\mathbb{A}^1}(S)$ is equivalent to the unstable $\mathbb{A}^1$-homotopy category $\mathcal{H}_s^{\mathbb{A}^1}(S)$ (Definition 3.3.7).
2. A morphism $f : X_1 \to X_2$ in $\Delta^{\mathrm{op}}\,\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})$ is called an *$\mathbb{A}^1$-weak equivalence* if for any $\mathbb{A}^1$-local, simplicially fibrant (i.e. the morphism $Z \to \mathrm{pt}$ is a simplicial fibration) sheaf Z, the map

$$\mathrm{Hom}_{\Delta^{\mathrm{op}}\,\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})}(X_2 \times \Delta^\bullet, Z) \to \mathrm{Hom}_{\Delta^{\mathrm{op}}\,\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})}(X_1 \times \Delta^\bullet, Z)$$

(Convention 3.3.1, Definition 2.2.2) induced by f is a weak equvialence of simplicial sets.

3. A morphism $f : X \to Y$ in $\Delta^{\mathrm{op}} \operatorname{Shv}((\operatorname{Sm}/S)_{\mathrm{Nis}})$ is called an *$\mathbb{A}^1$-fibration* if it has the right lifting property with respect to monomorphisms that are also $\mathbb{A}^1$-weak equivalences.

A morphism $f : X \to Y$ in $\Delta^{\mathrm{op}}(\operatorname{Shv}(T))_\bullet$ is called an *$\mathbb{A}^1$-weak equivalence* (resp. *$\mathbb{A}^1$-fibration*) if it is an $\mathbb{A}^1$-weak equivalence (resp. $\mathbb{A}^1$-fibration) as an unpointed morphism.

The following lemma shows that $\mathbb{A}^1$-homotopy theory is an instance of homotopy theory for sites with interval where the interval is given by the affine line $\mathbb{A}^1$.

Lemma 3.3.3. Let S be a Noetherian (Definition D.0.4) scheme of finite dimension (Definition D.0.9). The following specifies an interval (Definition 3.2.3) I in $(\operatorname{Sm}/S)_{\mathrm{Nis}}$: (♠ TODO: Why does the nisnevich site have enough points)

- The sheaf I of sets is the sheaf represented by $\mathbb{A}^1_S$.
- $\mu : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ is given by $(x, y) \mapsto (xy)$.
- $i_0, i_1 : \mathrm{pt} = S \to \mathbb{A}^1_S$ are given by embedding the point into $t = 0$ and $t = 1$ respectively where t is the coordinate of $\mathbb{A}^1$.

For this interval I,

1. a simplicial sheaf X on $(\operatorname{Sm}/S)_{\mathrm{Nis}}$ is $\mathbb{A}^1$-local (Definition 3.3.2) if and only if it is I-local (Definition 3.2.4),
2. a morphism $f : X_1 \to X_2$ in $\Delta^{\mathrm{op}} \operatorname{Shv}((\operatorname{Sm}/S)_{\mathrm{Nis}})$ is an $\mathbb{A}^1$-weak equivalence (Definition 4.1.12) if and only if it is an I-weak equivalence (Definition 3.2.4), and
3. a morphism $f : X_1 \to X_2$ in $\Delta^{\mathrm{op}} \operatorname{Shv}((\operatorname{Sm}/S)_{\mathrm{Nis}})$ is an $\mathbb{A}^1$-fibration (Definition 4.1.12) if and only if it is an I-weak fibration (Definition 3.2.4).

Proof. It is not difficult to see that the proposed interval is indeed an interval. The notions of I-locality and $\mathbb{A}^1$-locality coincide by definition. See [Cho08, Lemma 2.5.5] for an argument that shows that I-weak equivalence and $\mathbb{A}^1$-weak equivalence are equivalent. The notions of I-fibrations and of $\mathbb{A}^1$-fibrations then coincide by definition. □

Definition 3.3.4 (Simplicial Model Structure)**.** Let $\mathcal{M}$ be a category equipped with a model structure (Definition 2.2.5). Suppose $\mathcal{M}$ is also enriched over (Definition A.0.18) the category of simplicial sets **sSet** (Definition 2.2.2): that is, there is a bifunctor

$$\operatorname{Map}(-,-) : \mathcal{M}^{op} \times \mathcal{M} \to \mathbf{sSet}$$

(♠ TODO: tensor, cotensor) satisfying the standard enrichment axioms. Denote by $- \otimes K$ the tensor (or simplicial action) of an object of $\mathcal{M}$ by a simplicial set $K \in \mathbf{sSet}$ and by $(-)^K$ the cotensor.

Then *simplicial model structure on $\mathcal{M}$* consists of the following data and axioms:

- The model category structure on $\mathcal{M}$,
- The simplicial enrichment $\operatorname{Map}(-,-)$,

- The tensoring $- \otimes K : \mathcal{M} \times \mathbf{sSet} \to \mathcal{M}$ and cotensoring $(-)^K : \mathcal{M} \times \mathbf{sSet}^{op} \to \mathcal{M}$,

such that the following axioms hold:

1. (Compatibility) For all cofibrations (Definition 3.1.1) $i : A \to B$ in $\mathcal{M}$ and all cofibrations $j : K \to L$ in $\mathbf{sSet}$, the induced map
$$i \square j : (B \otimes K) \coprod_{A \otimes K} (A \otimes L) \to B \otimes L$$
is a cofibration in $\mathcal{M}$ which is a weak equivalence (Definition 3.1.1) if either i or j is.
2. (Simplicial hom) For all cofibrations $i : A \to B$ in $\mathcal{M}$ and all fibrations $p : X \to Y$ in $\mathcal{M}$, the induced map of simplicial sets
$$\mathrm{Map}(B, X) \to \mathrm{Map}(A, X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B, Y)$$
is a fibration (Definition 3.1.1) of simplicial sets which is a weak equivalence if either i or p is.

Proposition 3.3.5 (See [MV99, Section 3 Definition 2.1, Section 3 after Proposition 2.12])**.** Let S be a Noetherian (Definition D.0.4) base scheme of finite dimension (Definition D.0.9). The classes of $\mathbb{A}^1$-weak equivalences (Definition 3.3.2), monomorphisms (Definition A.0.16), and $\mathbb{A}^1$-fibrations (Definition 3.3.2) form a proper (Definition 4.1.13) simplicial (Definition 3.3.4) model structure (Definition 2.2.5) on the category $\Delta^{\mathrm{op}} \mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})$ (Definition 2.2.2) (Definition 2.1.4).

Similarly, the classes of $\mathbb{A}^1$-weak equivalences, monomorphisms, and $\mathbb{A}^1$-fibrations form a proper simplicial model structure on the category $\Delta^{\mathrm{op}} \mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})_\bullet$.

The inclusion functor $\mathcal{H}_{s,\mathbb{A}^1}(S) \hookrightarrow \mathcal{H}_s(\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}}))$ (Definition 3.3.2, Definition 3.1.4) has a left adjoint (Definition A.0.15)
$$L_{\mathbb{A}^1} : \mathcal{H}_s(\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})) \to \mathcal{H}_{s,\mathbb{A}^1}(S),$$
which we call the $\mathbb{A}^1$*-localization functor of spaces*, which identifies $\mathcal{H}_{s,\mathbb{A}^1}(S)$ with the localization of $\mathcal{H}_s(\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}}))$ with respect to $\mathbb{A}^1$-weak equivalences.

Similarly, there is an $\mathbb{A}^1$*-localization functor of pointed spaces* which we also denote by $L_{\mathbb{A}^1}$.

Proof. This follows by Theorem 3.2.6 and Lemma 3.3.3. □

Corollary 3.3.6. Let S be a Noetherian (Definition D.0.4) base scheme of finite dimension (Definition D.0.9). Letting I be the interval (Definition 3.2.3) given in Lemma 3.3.3, the I-model category on $\Delta^{\mathrm{op}} \mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})$ (Theorem 3.2.6) coincides with the $\mathbb{A}^1$-model structure on $\Delta^{\mathrm{op}} \mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})$ (Definition 3.3.7). Moreover, the category $\mathcal{H}_{s,I}((\mathrm{Sm}/S)_{\mathrm{Nis}})$ (Definition 3.2.4) is equivalent to the category $\mathcal{H}_{s,\mathbb{A}^1}(S)$ (Definition 3.3.2) and the I-localization (Theorem 3.2.6) and $\mathbb{A}^1$-localization functors (Proposition 3.3.5)
$$L_I : \mathcal{H}_s(\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})) \to \mathcal{H}_{s,I}(\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}}))$$
$$L_{\mathbb{A}^1} : \mathcal{H}_s(\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})) \to \mathcal{H}_{s,\mathbb{A}^1}(S),$$
are equivalent. Moreover, $\mathcal{H}_{s,\mathbb{A}^1}(S)$ is identified with the homotopy category of $\mathcal{H}_s(\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}}))$.

Proof. Lemma 3.3.3 implies that $\mathcal{H}_{s,I}(\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}}))$ and $\mathcal{H}_{s,\mathbb{A}^1}(S)$ are eiquvalent by definition. Moreover, since L_I and $L_{\mathbb{A}^1}$ are each left adjoint to their corresponding right adjoint inclusion functors, Theorem 3.2.6 and Proposition 3.3.5 imply that L_I and $L_{\mathbb{A}^1}$ are equivalent. Since $\mathcal{H}_{s,I}(\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}}))$ is identified with the homotopy category of $\mathcal{H}_s(T)$, $\mathcal{H}_{s,\mathbb{A}^1}(S)$ is identified with the homotopy category of $\mathcal{H}_{s,\mathbb{A}^1}(S)$. ∎

Definition 3.3.7 (Unstable motivic homotopy category over a scheme, See [MV99, Section 3 Definition 2.1, Section 3 after Proposition 2.12])**.** Let S be a Noetherian (Definition D.0.4) base scheme of finite dimension (Definition D.0.9).

1. The simplicial (Definition 3.3.4) model (Definition 2.2.5) structure on $\Delta^{\mathrm{op}}\,\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})$ discussed in Proposition 3.3.5 is called the $\mathbb{A}^1$*-model structure on* $\Delta^{\mathrm{op}}\,\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})$.
2. The associated homotopy category (Definition 2.2.6) is called the *(unstable)* $\mathbb{A}^1$*-homotopy category (of smooth schemes over* S*)* or the *(unstable) motivic category over* S and may be denoted by notations such as $\mathcal{H}(S)$ or $\mathcal{H}^{\mathbb{A}^1}(S)$.

   Note that $\mathcal{H}^{\mathbb{A}^1}(S)$ is equivalent to the category $\mathcal{H}_{s,\mathbb{A}^1}(S)$ (Definition 3.3.2) by Corollary 3.3.6
3. Similarly, the simplicial model structure on $\Delta^{\mathrm{op}}\,\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})_\bullet$ discussed in Proposition 3.3.5 is called the *(pointed)* $\mathbb{A}^1$*-model structure on* $\Delta^{\mathrm{op}}\,\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})_\bullet$.
4. The associated homotopy category (Definition 2.2.6) is called the *(unstable) pointed* $\mathbb{A}^1$*-homotopy category (of smooth scehemes over* S*)* or the *(unstable) motivic homotopy category over* S and may be denoted by notations such as $\mathcal{H}_\bullet(S)$ $\mathcal{H}_\bullet^{\mathbb{A}^1}(S)$, $\mathcal{H}(S)_\bullet$, or $\mathcal{H}^{\mathbb{A}^1}(S)_\bullet$.

Given two objects $X, Y \in \mathcal{H}_\bullet(S)$, denote by $[X,Y]_\bullet$ the (pointed) set $\mathrm{Hom}_{\mathcal{H}_\bullet(S)}(X,Y)$ of morphisms. Given two objects $X, Y \in \mathcal{H}(S)$, denote by $[X,Y]$ the (pointed) set $\mathrm{Hom}_{\mathcal{H}(S)}(X,Y)$ of morphisms. (♠ TODO: Why are the homotopy groups actual sheaves of groups?)

Lemma 3.3.8. Let S be a Noetherian (Definition D.0.4) base scheme of finite dimension (Definition D.0.9). The pointification and forgetful functors (Definition 2.0.4)

$$(X \mapsto X_+) : \Delta^{\mathrm{op}}\,\mathrm{Shv} \leftrightarrows \Delta^{\mathrm{op}}\,\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})_\bullet : ((Y,y) \mapsto Y)$$

(Definition 2.2.2) (Definition 2.0.3) Both functors preserve simplicial weak equivalences (Definition 3.1.2) and $\mathbb{A}^1$-weak equivalences (Definition 3.3.2) and thus induce adjoint pairs (Definition A.0.15) of functors

$$(X \mapsto X_+) : \mathcal{H}_s((\mathrm{Sm}/S)_{\mathrm{Nis}}) \leftrightarrows \mathcal{H}_s((\mathrm{Sm}/S)_{\mathrm{Nis}})_\bullet : ((Y,y) \mapsto Y)$$

(Definition 3.1.4) and

$$(X \mapsto X_+) : \mathcal{H}^{\mathbb{A}^1}(S) \leftrightarrows \mathcal{H}^{\mathbb{A}^1}(S)_\bullet : ((Y,y) \mapsto Y)$$

(Definition 3.3.7)

Proof. (♠ TODO: establish that the fucntors preserve weak equivalences) The adjunctions follow from that of Lemma 2.0.5. ∎

See [MV99, After Section 2 Lemma 2.30].

Definition 4.0.1. Let T be a site (Definition A.0.9). For $(X,x),(Y,y) \in \Delta^{\mathrm{op}}(\mathrm{Shv}(T))_\bullet$ (Definition 2.2.2) (Definition 2.0.2) (Definition A.0.10), define their *wedge product* $(X,x)\vee(Y,y)$ and their *smash product* $(X,x)\wedge(Y,y)$ as follows:

$$(X,x)\vee(Y,y) := \left(X \coprod_{\mathrm{pt}} Y, x=y\right)$$
$$(X,x)\wedge(Y,y) := \left(X\times Y/\left((X,x)\coprod(Y,y)\right), x\times y\right).$$

(♠ TODO: TODO: prove this)

Claim 4.0.2. Let T be a site (Definition A.0.9) and fix $(Y,y)\in\Delta^{\mathrm{op}}(\mathrm{Shv}(T))$. The functor

$$\Delta^{\mathrm{op}}(\mathrm{Shv}(T))_\bullet \to \Delta^{\mathrm{op}}(\mathrm{Shv}(T))_\bullet, \quad (X,x)\mapsto (X,x)\wedge(Y,y)$$

has a right adjoint (Definition A.0.15), denoted by

$$(Z,z)\mapsto \underline{\mathrm{Hom}}_\bullet((Y,y),(Z,z)).$$

Definition 4.0.3. Let T be a site (Definition A.0.9). (♠ TODO: $\Delta^n, \partial\Delta^n$)

1. Let S^1_s the quotient sheaf $\Delta^1/\partial\Delta^1$, pointed by the image of $\partial\Delta^1$. It is called the *simplicial circle (on T)*. As per Convention 2.3.3, it may also denote the constant pointed simplicial sheaf induced by S^1_s.
2. Define the *simplicial suspension functor*

$$\Sigma_s : \Delta^{\mathrm{op}}(\mathrm{Shv}(T))_\bullet \to \Delta^{\mathrm{op}}(\mathrm{Shv}(T))_\bullet, \quad (X,x)\mapsto S^1_s\wedge(X,x).$$

   (Definition 4.0.1)
3. Define the *simplicial loop functor* $\Omega^1_s(-) := \underline{\mathrm{Hom}}_\bullet(S^1_s,-)$, which is right adjoint to $\Sigma_s(-)$.
4. We write by $\Sigma^n = \Sigma^n_s$ to be the n-time composition of Σ. We write by $\Omega^n = \Omega^n_s$ to be the n-time composition of Ω^1_s.

Definition 4.0.4 (Multiplicative group scheme)**.** Let S be a scheme. The *multiplicative group scheme over S*, denoted $\mathbb{G}_{m,S}$, is the group scheme over S defined as the open subscheme

$$\mathbb{G}_{m,S} := \mathbb{A}^1_S \setminus \{0\}_S$$

of the affine line $\mathbb{A}^1_S$, equipped with the group law given by multiplication of functions:

$$m : \mathbb{G}_{m,S}\times_S \mathbb{G}_{m,S} \to \mathbb{G}_{m,S}, \quad (x,y)\mapsto xy.$$

The identity section is the morphism

$$e : S \to \mathbb{G}_{m,S}, \quad s\mapsto 1,$$

and the inversion morphism is given by

$$i : \mathbb{G}_{m,S}\to\mathbb{G}_{m,S}, \quad x\mapsto x^{-1}.$$

Definition 4.0.5. (♠ TODO: figure out other formulations of T, e.g. it might be isomorphic to $\mathbb{A}^1/(\mathbb{A}^1 - \{0\}) \cong \mathbb{P}^1$) Let S be a Noetherian scheme (Definition D.0.4) of finite dimension (Definition D.0.9).

The *Tate sphere over S*, or *motivic sphere over S*, is the smash product

$$T := S^1_s \wedge \mathbb{G}_m$$

where S^1_s is the simplicial circle (Definition 4.0.3) on $(\mathrm{Sm}/S)_{\mathrm{Nis}}$ (Definition 2.1.4), and $\mathbb{G}_m/S$ is the multiplicative group (Definition 4.0.4) over S regarded as a (simplicial) sheaf on the $(\mathrm{Sm}/S)_{\mathrm{Nis}}$.

In the homotopy category, T is $\mathbb{A}^1$-weakly equivalent to the pointed projective line $(\mathbb{P}^1, \infty)$. (♠ TODO: verify that this is true and also which homotopy category? Stable or unstable?)

4.1. The $\mathbb{A}^1$-homotopy sheaves of a pointed space.

Definition 4.1.1 (See e.g. [Mor12, Before Definition 1.7], cf. [MV99, After Section 3 Lemma 2.13])**.** Let S be a Noetherian (Definition D.0.4) base scheme of finite dimension (Definition D.0.9). Given an object $X \in \mathcal{H}_\bullet(S)$ (Definition 3.3.7), define the *nth $\mathbb{A}^1$-homotopy sheaf* $\pi_n^{\mathbb{A}^1}(X)$ as the Nisnevich sheaf of sets associated (Definition A.0.11) to the presheaf $U \mapsto [\Sigma^n(U_+), X]_\bullet$ (Definition 3.3.7, Definition 4.0.3, Definition 2.0.4). It is a sheaf of groups of $n = 1$ and a sheaf of abelian groups for $n \geq 2$.

We can also define the sheaf of connected components for an unpointed space X:

Definition 4.1.2. Let S be a Noetherian (Definition D.0.4) base scheme of finite dimension (Definition D.0.9). Given an object $X \in \mathcal{H}(S)$, define $\pi_0^{\mathbb{A}^1}(X)$ as the Nisnevich (Definition 2.1.4) sheaf (Definition A.0.10) of sets associated (Definition A.0.11) to the presheaf $U \mapsto [U, X]$ (Definition 3.3.7). It is called the *sheaf of $\mathbb{A}^1$-connected components of X*.

Lemma 4.1.3. Let S be a Noetherian (Definition D.0.4) base scheme of finite dimension (Definition D.0.9). Given a pointed space X, the sheaf $\pi_0^{\mathbb{A}^1}(X)$ in the sense of Definition 4.1.1 is isomorphic to that of the underlying space of X in the sense of Definition 4.1.2.

Proof. It suffices to show that the presheaves

$$(\mathrm{Sm}/S)_{\mathrm{Nis}} \to \mathbf{Sets}, \quad U \mapsto [U_+, X]_\bullet$$

and

$$(\mathrm{Sm}/S)_{\mathrm{Nis}} \to \mathbf{Sets}, \quad U \mapsto [U, X]$$

are naturally isomorphic. This is true by the adjunction bewteen the functor $+$ and the forgetful functor stated in Claim 2.0.5. □

Remark 4.1.4. (♠ TODO: The morl voevodsky notion of A1-homotopy group) The definitions of the $\mathbb{A}^1$-homotopy group of a pointed space as given in [MV99] and in [Mor12, Section 6.3 after Conjecture 6.34] are different from that of Definition 4.1.1. The former definition uses the theory of resolution functors. The latter ostensibly defines $\pi_n^{\mathbb{A}^1}(X)$ as the sheafification of the presheaf $U \mapsto \pi_n(L_{\mathbb{A}^1}(X)(U))$ where π_n is the nth homotopy of a simplicial set. We nevertheless accept, without proof, that all three definitions are equivalent in this current writing for convenience.

Definition 4.1.5. Let V be a real vector space of finite dimension. A *k-simplex in topology* (or a *geometric k-simplex*) is the convex hull of $k+1$ affinely independent points $v_0, v_1, \ldots, v_k \in V$, and is denoted by

$$[v_0, v_1, \ldots, v_k] := \left\{ \sum_{i=0}^{k} t_i v_i \;\middle|\; t_i \geq 0, \; \sum_{i=0}^{k} t_i = 1 \right\}.$$

It is also standard to talk of the *standard topological n-simplex* — the topological space $|\Delta^n|$ defined as the subset of Euclidean space $\mathbb{R}^{n+1}$ given by

$$|\Delta^n| = \left\{ (t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} t_i = 1, \text{ and } t_i \geq 0 \text{ for all } i \right\}$$

equipped with the induced topology from the usual Euclidean topology on $\mathbb{R}^{n+1}$.

(♠ TODO: comment on how $|\Delta^n|$ makes sense via a geometric realization)

Definition 4.1.6. The *standard topological/geometric simplicial (topological) space* refers to the simplicial (Definition 2.2.2) topological space (Definition C.0.1), denoted by notations such as $\Delta^\bullet$ or $|\Delta^\bullet|$ whose n-simplices, for each $n \geq 0$, is the standard topological n-simplex $|\Delta^n|$ (Definition 4.1.5) and whose face and degeneracy maps are given in the "obvious" way (♠ TODO:).

Definition 4.1.7. Let S be a scheme (Definition D.0.2). By the *"standard" simplicial scheme over S*, we will mean the simplicial (Definition 2.2.2) scheme $\Delta^\bullet_S$ whose n-simplices (Definition 2.2.3), for each $n \geq 0$, is the closed subschemes $\Delta^n_S \subset \mathbb{A}^{n+1}_S$ given by the equation $\sum_{i=0}^n x_i = 1$ and whose face and degeneracy maps are given in the "obvious" way (♠ TODO:) (♠ TODO: this might actually be a cosimplicial scheme)

We will call $\Delta^\bullet_S$ the "standard" simplicial scheme over S becaues it is analogous to the standard simplicial topological space $|\Delta^\bullet|$ (Definition 4.1.6).

Definition 4.1.8. Let S be a noetherian (Definition D.0.4) scheme (Definition D.0.2) of finite dimension (Definition D.0.9). Let $X \in \mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})$ be a sheaf of sets (Definition A.0.10) on the Nisnevich site over S (Definition 2.1.4), and let U/S be a smooth scheme (Definition D.0.5).

(♠ TODO: figure out if Δ^n_S is a simplicial scheme or a cosimplicial scheme.) Let $\mathrm{Sing}_n(X)(U) = \mathrm{Hom}_{\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})}(U \times \Delta^n_S, X)$ where Δ^n_S is the sheaf represented by the scheme Δ^n_S (Definition 4.1.7). The $\mathrm{Sing}_n(X)(U)$ form a simplicial set $\mathrm{Sing}_\bullet(X)(U)$. We may refer to this as the *aigebraic singular simplicial set (of morphisms from U to X)*.

Definition 4.1.9. A *Kan complex* or a *Kan simplicial set* is a simplicial set (Definition 2.2.2) K satisfying the Kan extension condition for the horns $\Lambda^n_i \subset \Delta^n$ for all n and all $0 \leq i < n$. More explicitly, for any $0 \leq i \leq n$, any map $f_0 : \Lambda^n_i \to K$ admits an extension $f : \Delta^n \to K$ (Definition 2.2.4).

Kan-complexes are equivalent to ∞-groupoids (**??**).

Definition 4.1.10 (Homotopy Groups of a Kan Simplicial Set)**.** Let X be a Kan simplicial set (Definition 4.1.9), and fix a basepoint $x \in X_0$.

For each integer $n \geq 1$, define the *n-th homotopy group of X at x*, denoted $\pi_n(X, x)$, as the set of homotopy classes of maps (♠ TODO: homotopy, boundary of Δ^n)

$$f : \partial\Delta^{n+1} \to X$$

that restrict to the constant map at x on the basepoint simplex, modulo homotopies relative to the boundary.

Equivalently, $\pi_n(X, x)$ can be described as the set of equivalence classes of n-simplices (Definition 2.2.2) whose faces are degenerate at x, with composition induced by the combinatorial structure of simplices.

These sets carry natural group structures for $n \geq 1$, with $\pi_1(X, x)$ being the fundamental group and $\pi_n(X, x)$ for $n \geq 2$ being abelian groups.

Definition 4.1.11. Let S be a noetherian (Definition D.0.4) scheme (Definition D.0.2) of finite dimension (Definition D.0.9). Let $X \in \mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})$ be a sheaf of sets on the Nisnevich site, let U/S be a smooth scheme (Definition D.0.5), and let $x \in \mathrm{Hom}(U, X)$. Define *homotopy groups* $\pi^{\mathbb{A}^1}_{i,U}(X, x)$ to be the homotopy groups (Definition 4.1.10) of the Kan simplicial set (Definition 4.1.9) $\mathrm{Sing}_\bullet(Ex^\infty(X))(U)$ with respect to the base point x. (♠ TODO: Ex^∞)

Definition 4.1.12 ([Voe98, Definition 3.4])**.** Let S be a noetherian (Definition D.0.4) scheme (Definition D.0.2) of finite dimension (Definition D.0.9). A morphism $f : X \to Y$ in the category $\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})$ of sheaves (Definition A.0.10) of sets on $(\mathrm{Sm}/S)_{\mathrm{Nis}}$ (Definition 2.1.4) is called an *$\mathbb{A}^1$-weak equivalence* or a *weak equivalence* if for any smooth scheme U/S, any $x \in \mathrm{Hom}_{\Delta^{\mathrm{op}}((\mathrm{Sm}/S)_{\mathrm{Nis}})}(U, X)$, and any $i \geq 0$, the corresponding map

$$\pi^{\mathbb{A}^1}_{i,U}(X, x) \to \pi^{\mathbb{A}}_{i,U}(Y, f(x))$$

(Definition 4.1.11) is a bijection.

The unstable $\mathbb{A}^1$-homotopy category can alternatively be obtained as a homotopy category not of simplicial sheaves, but of ordinary sheaves on $(\mathrm{Sm}/S)_{\mathrm{Nis}}$.

Definition 4.1.13 (Proper Model Category)**.** Let $\mathcal{M}$ be a model category (Definition 2.2.5) with classes of weak equivalences $\mathcal{W}$, cofibrations $\mathcal{C}of$, and fibrations $\mathcal{F}ib$. (♠ TODO: pushout, pullback)

1. Then $\mathcal{M}$ is called *left proper* if weak equivalences are preserved under pushouts along cofibrations, i.e., if for every pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow^{f \in \mathcal{W}} & & \downarrow \\ C & \to & D \end{array}$$

with $i \in \mathcal{C}of$, the induced map $C \to D$ lies in $\mathcal{W}$.

2. Dually, $\mathcal{M}$ is called *right proper* if weak equivalences are preserved under pullbacks along fibrations, i.e., if for every pullback diagram

$$\begin{array}{ccc} P & \to & X \\ \downarrow^{g} & & \downarrow^{p\in\mathcal{F}ib} \\ Y & \xrightarrow{h\in\mathcal{W}} & Z \end{array}$$

the induced map $P \to Y$ is in $\mathcal{W}$.
3. A model category $\mathcal{M}$ is *proper* if it is both left proper and right proper.

Theorem 4.1.14 ([MV99, Theorems 3.6, 3.7])**.** Let S be a Noetherian (Definition D.0.4) base scheme of finite dimension (Definition D.0.9). The following specifies a proper (Definition 4.1.13) model category structure (Definition 2.2.6) on the category $\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})$ of sheaves of sets (Definition A.0.10) on $(\mathrm{Sm}/S)_{\mathrm{Nis}}$ (Definition 2.1.4):

1. The weak equivalences are the $\mathbb{A}^1$-weak equivalence (Definition 4.1.12)
2. The cofibrations are the monomorphisms (Definition A.0.16).
3. The fibrations are the class of morphisms having the right lifting property with respect to morphisms that are simultaneously weak equivalences and cofibrations.

Moreover, the homotopy category (Definition 2.2.6) of this model category structure is equivalent (Definition A.0.5) to the unstable $\mathbb{A}^1$-homotopy category $\mathcal{H}(S)$ (Definition 4.1.1).

Proposition 4.1.15 ([MV99, Section 3 Proposition 2.14])**.** Let S be a Noetherian (Definition D.0.4) base scheme of finite dimension (Definition D.0.9). A morphism $f : X \to Y$ in $\Delta^{\mathrm{op}}\,\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})_{\bullet}$ is an $\mathbb{A}^1$-weak equivalence if and only if the natural morphisms

$$\pi_n^{\mathbb{A}^1}(X) \to \pi_n^{\mathbb{A}^1}(Y)$$

of the $\mathbb{A}^1$-homotopy sheaves (Definition 4.1.1) are isomorphisms for all $n \geq 0$.

5. $\mathbb{A}^1$-invariance of Nisnevich sheaves

(♠ TODO: TODO: discuss the merit of A1-invariance)

Notation 5.0.1. Whenever S is a Noetherian (Definition D.0.4) base scheme of finite dimension (Definition D.0.9) let H^i_{Nis} denote the ith sheaf cohomology functor for $(\mathrm{Sm}/S)_{\mathrm{Nis}}$. More explicitly, given a sheaf G of groups on $(\mathrm{Sm}/S)_{\mathrm{Nis}}$ and a $X \in \mathrm{Sm}/S$, we may speak of the group $H^0_{\mathrm{Nis}}(X;G)$ and the set $H^1_{\mathrm{Nis}}(X;G)$ (Definition A.0.27) as the zeroth and first nonabelian sheaf cohomology sets on the site $((\mathrm{Sm}/S)_{\mathrm{Nis}})_{/X}$ over X (Definition A.0.28) and if G is further a sheaf of abelian groups, then we may speak of the groups $H^i_{\mathrm{Nis}}(X;G)$ as the abelian sheaf cohomology groups (Definition A.0.23) of G on $((\mathrm{Sm}/S)_{\mathrm{Nis}})_{/X}$. Note that $H^0_{\mathrm{Nis}}(X;G)$ and $H^1_{\mathrm{Nis}}(X;G)$ are unambiguous when G is a sheaf of abelian groups by Proposition A.0.29

Definition 5.0.2 (cf. [Mor12, Definition 1.7], [Mor06, Definition 3.1])**.** Let S be a base scheme.

1. A presheaf (Definition A.0.8) P of sets on Sm/S is said to be *$\mathbb{A}^1$-invariant* if for any $X \in \mathrm{Sm}/S$, the map
$$P(X) \to P(\mathbb{A}^1_S \times_S X)$$
induced by the projection $\mathbb{A}^1_S \times_S X \to X$ is a bijection.
2. Assume that S is Noetherian (Definition D.0.4) of finite dimension (Definition D.0.9) (so that the Nisnevich topology is defined). A sheaf (Definition A.0.10) G of groups on $(\mathrm{Sm}/S)_{\mathrm{Nis}}$ (Definition 2.1.4) is said to be *strongly $\mathbb{A}^1$-invariant* if for any $X \in \mathrm{Sm}/S$, the map
$$H^i_{\mathrm{Nis}}(X; G) \to H^i_{\mathrm{Nis}}(\mathbb{A}^1_S \times_S X; G)$$
(Notation 5.0.1) induced by the projection $\mathbb{A}^1_S \times_S X \to X$ is a bijection for $i \in \{0, 1\}$.
3. Assume that S is Noetherian (Definition D.0.4) of finite dimension (Definition D.0.9). A sheaf G of abelian groups on $(\mathrm{Sm}/S)_{\mathrm{Nis}}$ is said to be *strictly $\mathbb{A}^1$-invariant* if for any $X \in \mathrm{Sm}/S$, the map
$$H^i_{\mathrm{Nis}}(X; G) \to H^i_{\mathrm{Nis}}(\mathbb{A}^1_S \times_S X; G)$$
induced by the projection $\mathbb{A}^1_S \times_S X \to X$ is a bijection for any $i \geq 0$.

Here are some facts about these notions of $\mathbb{A}^1$-invariance:

Example 5.0.3. Let k be a field.

1. [Mor06, After Theorem 3.2] The constant sheaf (Definition A.0.26) $\mathbb{Z}$ on $(\mathrm{Sm}/k)_{\mathrm{Nis}}$ (Definition 2.1.4) is strictly $\mathbb{A}^1$-invariant (Definition 5.0.2). In fact, $H^i_{\mathrm{Nis}}(X; \mathbb{Z}) = 0$ (Definition A.0.23) for $i > 0$.
2. [Mor06, After Theorem 3.2] The sheaf on $(\mathrm{Sm}/k)_{\mathrm{Nis}}$ represnted by an abelian variety over k is strictly $\mathbb{A}^1$-invariant. In fact, $H^i_{\mathrm{Nis}}(X; \mathbb{Z}) = 0$ for $i > 0$.
3. [Mor06, After Theorem 3.2] The sheaf on $(\mathrm{Sm}/k)_{\mathrm{Nis}}$ represented by the multiplicative group $\mathbb{G}_m = \mathbb{A}^1 - \{0\}$ is strictly $\mathbb{A}^1$-invariant.

Lemma 5.0.4 (cf. [Mor12, Remark 1.8])**.** Let S be a Noetherian (Definition D.0.4) base scheme of finite dimension (Definition D.0.9). A sheaf X of sets on $(\mathrm{Sm}/S)_{\mathrm{Nis}}$ is $\mathbb{A}^1$-invariant (Definition 5.0.2) if and only if it is $\mathbb{A}^1$-local (Definition 3.3.2) as a (Convention 2.3.3) space (Definition 2.3.1).

By definition, a strictly $\mathbb{A}^1$-invariant sheaf of groups is clearly a strongly $\mathbb{A}^1$-invariant sheaf. If the sheaf is of abelian groups, then the converse is true:

Theorem 5.0.5 (See [Mor12, Theorem 1.16, Theorem 5.46], [Mor06, Theorem 3.2])**.** (♠ TODO: perfect field) Let k be a perfect field. A sheaf G of abelian groups on $(\mathrm{Sm}/k)_{\mathrm{Nis}}$ is strongly $\mathbb{A}^1$-invariant (Definition 5.0.2) if and only if it is strictly $\mathbb{A}^1$-invariant (Definition 5.0.2).

Remark 5.0.6. Morel's text [Mor12] assumes that the base field k is perfect; the only reason for this assumption throughout the text, as explained in [Mor12, Remark 1.17], is to ensure that Theorem 5.0.5 holds. As such, it may be the case that some statements taken from loc. cit. do not require the perfectness hypothesis.

Moreover, it is known over perfect fields that $\mathbb{A}^1$-homotopy group sheaves are strongly $\mathbb{A}^1$-invariant.

Theorem 5.0.7 ([Mor12, Theorem 1.9])**.** Let k be a perfect field, and let X be a pointed space. The sheaf $\pi_1^{\mathbb{A}^1}(X)$ (Definition 4.1.1) of groups is strongly $\mathbb{A}^1$-invariant (Definition 5.0.2) and for any $n \geq 2$ the sheaf $\pi_n^{\mathbb{A}^1}(X)$ is strictly $\mathbb{A}^1$-invariant (Definition 5.0.2).

Remark 5.0.8. Ayoub showed that [Ayo23] $\pi_0^{\mathbb{A}^1}(X)$ is generally not $\mathbb{A}^1$-invariant. On the other hand [Mor12, Remark 1.13], $\pi_0^{\mathbb{A}^1}(X)$ is $\mathbb{A}^1$-invariant for smooth k-schemes of dimension ≤ 1.

Proposition 5.0.9 ([Bac24, Lemma 1.18])**.** Let k be a field (Definition D.0.10). Let K/k be a separable field extension (Definition D.0.17) and write $\eta = \operatorname{Spec} K$ (Definition D.0.15). Note that there is a base change functor (♠ TODO:) $(\mathrm{Sm}/k)_{\mathrm{Nis}} \to (\mathrm{Sm}/K)_{\mathrm{Nis}} : F \mapsto F|_\eta$.

1. Let $\{X_\lambda\}$ be a cofiltered system (Definition D.0.12) of smooth (Definition D.0.6) k-varieties (Definition D.0.13) such that $\varprojlim_\lambda X_\lambda \in \mathrm{Sm}_\eta$ (Definition D.0.12). For any sheaf F of sets on $(\mathrm{Sm}/k)_{\mathrm{Nis}}$ we have
$$F|_\eta(\varprojlim_\lambda X_\lambda) \cong \varinjlim_\lambda F(X_\lambda).$$
2. The base change functor
   (a) preserves $\mathbb{A}^1$-invariant (Definition 5.0.2) sheaves (of sets),
   (b) commutes with taking homotopy sheaves (♠ TODO: what does this mean)
   (c) preserves strongly $\mathbb{A}^1$-invariant sheaves (Definition 5.0.2) of groups and preserves strictly $\mathbb{A}^1$-invariant sheaves (Definition 5.0.2) of abelian groups

(♠ TODO: My actual goal is something along the following lines: Let R be a Dedekind ring. If F is strictly A1-invariant over R, then it is strictly A!-invariant over R_p for all p.

I might need a statement for topoi of the following form: Given a Grothendieck topos $\mathcal{E}$, given a filtered category I, a filtered diagram $F_\bullet : I \to \mathrm{Ab}(\mathcal{E})$ of abelian sheaves wtih colimit $F = \operatorname{colim}_{i\in I} F_i$, we have
$$H^q(\mathcal{E}, F) \cong \operatorname{colim}_{i\in I} H^q(\mathcal{E}, F_i)$$
SGA 4 Expose VII, Theorem 5.7 does this for the etale site.)

(♠ TODO:

Proof of Preservation of Strong $\mathbb{A}^1$-invariance under Localization

Lemma 5.0.10. Let R be a Dedekind ring and let A be a strongly $\mathbb{A}^1$-invariant Nisnevich sheaf on Sm/R. Let $\mathfrak{p} \subset R$ be a prime ideal and let $S = \operatorname{Spec}(R_\mathfrak{p})$. Then the restriction $A|_S$ is a strongly $\mathbb{A}^1$-invariant sheaf on Sm/S.

Proof. The localization $S = \operatorname{Spec}(R_\mathfrak{p})$ can be written as the cofiltered limit of open affine neighborhoods of $\mathfrak{p}$ in $X = \operatorname{Spec}(R)$:
$$S \cong \varprojlim_{U \ni \mathfrak{p}} U.$$

This limit is a limit of schemes with affine transition maps (open immersions). A standard result in the theory of topoi (SGA 4, Exposé VII, Prop. 5.7) states that the topos of sheaves on such a limit is equivalent to the limit of the topoi. Specifically, for Nisnevich cohomology, we have the continuity property:

$$H^i_{\mathrm{Nis}}(S, \mathcal{F}|_S) \cong \varinjlim_{U \ni \mathfrak{p}} H^i_{\mathrm{Nis}}(U, \mathcal{F}|_U)$$

for any sheaf $\mathcal{F}$ on Sm/R.

To prove that $A|_S$ is strongly $\mathbb{A}^1$-invariant, we must show that for any smooth scheme $Y \in \mathrm{Sm}/S$ and $i \in \{0, 1\}$, the projection $p : Y \times \mathbb{A}^1_S \to Y$ induces an isomorphism:

$$p^* : H^i_{\mathrm{Nis}}(Y, A|_Y) \xrightarrow{\sim} H^i_{\mathrm{Nis}}(Y \times \mathbb{A}^1_S, A|_{Y \times \mathbb{A}^1_S}).$$

Since Y is of finite presentation over S, it descends to a smooth scheme Y_U over some open neighborhood U of $\mathfrak{p}$. By shrinking U if necessary, we may assume $Y = Y_U \times_U S$.

We compute the cohomology of Y as the limit of the cohomology of its models Y_V over open sets $V \subseteq U$:

$$H^i_{\mathrm{Nis}}(Y, A|_Y) \cong \varinjlim_{V \subseteq U} H^i_{\mathrm{Nis}}(Y_V, A|_{Y_V}).$$

Since A is strongly $\mathbb{A}^1$-invariant on Sm/R, its restriction to any open $V \subset \mathrm{Spec}(R)$ is also strongly $\mathbb{A}^1$-invariant. Therefore, for every V, we have an isomorphism:

$$H^i_{\mathrm{Nis}}(Y_V, A|_{Y_V}) \cong H^i_{\mathrm{Nis}}(Y_V \times \mathbb{A}^1_V, A|_{Y_V \times \mathbb{A}^1_V}).$$

Taking the filtered colimit on both sides, and using the continuity of cohomology for $Y \times \mathbb{A}^1_S \cong \varprojlim (Y_V \times \mathbb{A}^1_V)$, we obtain:

$$\varinjlim_{V \subseteq U} H^i_{\mathrm{Nis}}(Y_V, A|_{Y_V}) \cong \varinjlim_{V \subseteq U} H^i_{\mathrm{Nis}}(Y_V \times \mathbb{A}^1_V, A|_{Y_V}) \cong H^i_{\mathrm{Nis}}(Y \times \mathbb{A}^1_S, A|_Y).$$

Thus, the isomorphism holds for $A|_S$, proving it is strongly $\mathbb{A}^1$-invariant. □

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(♠ TODO: And then an argument for strict $\mathbb{A}^1$-invariance is probably needed separately, because for strict invariance, we are using nonabelian sheaf cohomology:

Preservation for Sheaves of Groups (Non-Abelian)

Lemma 5.0.11. Let G be a strongly $\mathbb{A}^1$-invariant sheaf of groups on Sm/R. Let $S = \mathrm{Spec}(R_{\mathfrak{p}}) \cong \varprojlim U$. Then the restriction $G|_S$ is strongly $\mathbb{A}^1$-invariant.

Proof. Strong $\mathbb{A}^1$-invariance is defined by the bijectivity of the maps $H^i_{\mathrm{Nis}}(Y, G) \to H^i_{\mathrm{Nis}}(Y \times \mathbb{A}^1, G)$ for $i = 0, 1$.

Case $i = 0$ (Sections): By the definition of the inverse image sheaf on a pro-object, the sections on Y are defined as the colimit of sections on the models Y_U:

$$H^0(Y, G|_S) = G(Y) \cong \varinjlim_U G(Y_U) = \varinjlim_U H^0(Y_U, G).$$

The colimit of bijections is a bijection, so the property holds for $i = 0$.

Case $i = 1$ (Torsors): The set $H^1_{\mathrm{Nis}}(S, G)$ classifies G-torsors. Since G-torsors are schemes of finite presentation over S, Grothendieck's limit theorems (EGA IV_3, 8.8.2) apply:

1. Every G-torsor $P \to Y$ over the limit descends to a G-torsor $P_U \to Y_U$ for some U (essential surjectivity).
2. Any isomorphism between torsors over Y descends to an isomorphism over some Y_V (fullness).

This implies that the functor $H^1_{\mathrm{Nis}}(-, G)$ commutes with filtered limits of schemes:

$$H^1_{\mathrm{Nis}}(Y, G|_S) \cong \varinjlim_U H^1_{\mathrm{Nis}}(Y_U, G).$$

Since the map $H^1(Y_U) \to H^1(Y_U \times \mathbb{A}^1)$ is a bijection for all U (by the strong invariance of G on Sm/R), the limit map is also a bijection. $\square$

)

6. $\mathbb{A}^1$-derived category

$\mathbb{A}^1$-homotopy theory yields an $\mathbb{A}^1$-derived category of complexes of Nisnevich sheaves of abelian groups obtained by inverting all the $\mathbb{A}^1$-quasi isomorphisms. Spaces can be considered as objects of this derived category and their homology objects are referred to as $\mathbb{A}^1$-homology sheaves.

Notation 6.0.1. Given a Noetherian (Definition D.0.4) base scheme S of finite dimension (Definition D.0.9), let $\mathcal{A}b(S)$ denote the abelian category of abelian groups for the Nisnevich topology on Sm/S. Let $C_*(\mathcal{A}b(S))$ denote the category of chain complexes of objects in $\mathcal{A}b(S)$. Let $D(\mathcal{A}b(S))$ denote the derived category of $\mathcal{A}b(S)$ in the standard homological algebra sense, i.e. it is the class obtained from $C_*(\mathcal{A}b(S))$ by inverting the class of quasi-isomorphisms between chain complexes.

In general, given a site T and a simplicial sheaf $X \in \Delta^{\mathrm{op}} \mathrm{Shv}(T)$, denote by $\mathbb{Z}(X)$ the free abelian sheaf generated by X . More precisely, $\mathbb{Z}(X) \in \Delta^{\mathrm{op}} \mathrm{Shv}(T)$ is the simplicial sheaf of abelian groups associated to the presheaf $U \mapsto \mathbb{Z}[X(U)]$ where $\mathbb{Z}[X(U)]$ is the free abelian simplicial group generated by the simplicial set $X(U)$ (Definition D.0.1).

(♠ TODO: TODO: figure out what the chain complex $C_*(X)$ associated to a space means)

In particular, given a space $X \in \Delta^{\mathrm{op}} \mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})$, we can speak of the Nisnevich simplicial sheaf $\mathbb{Z}(X)$.

Definition 6.0.2 (cf. [MV99, Definition 6.17])**.** Let S be a Noetherian (Definition D.0.4) base scheme of finite dimension (Definition D.0.9).

1. A chain complex $D_* \in C_*(\mathrm{Ab}(S))$ is called *$\mathbb{A}^1$-local* if and only if for any $C_* \in C_*(\mathrm{Ab}(S))$, the projection
$$C_* \otimes \mathbb{Z}(\mathbb{A}^1) \to C_*$$
induces a bijection
$$\mathrm{Hom}_{D(\mathrm{Ab}(S))}(C_*, D_*) \to \mathrm{Hom}_{D(\mathrm{Ab}(S))}(C_* \otimes \mathbb{Z}(\mathbb{A}^1), D_*).$$
Denote by $D_{\mathbb{A}^1\text{-loc}}(\mathrm{Ab}(S))$ the full subcategory of $D(\mathrm{Ab}(S))$ consisting of $\mathbb{A}^1$-local complexes.
2. A morphism $f : C_* \to D_*$ in $C_*(\mathrm{Ab}(S))$ is called an *$\mathbb{A}^1$-quasi isomorphism* if and only if for every $\mathbb{A}^1$-local chain complex $E_* \in C_*(\mathrm{Ab}(S))$, the morphism
$$\mathrm{Hom}_{D(\mathrm{Ab}(S))}(D_*, E_*) \to \mathrm{Hom}_{D(\mathrm{Ab}(S))}(C_*, E_*)$$
is bijective. Denote by $\mathbb{A}^1$-Qis the class of $\mathbb{A}^1$-quasi isomorphisms.
3. The *$\mathbb{A}^1$-derived category* $D_{\mathbb{A}^1}(\mathrm{Ab}(S))$ is the category obtained from $C_*(\mathrm{Ab}(S))$ by inverting $\mathbb{A}^1$-Qis.

We have the following analogue of Proposition 3.3.5

(♠ TODO: TODO: talk about A1-localization of chain complexes)

Theorem 6.0.3 (cf. [Mor12, Lemma 6.18, Corollary 6.19])**.** Let k be a field. There exists a functor $L^{\mathrm{ab}}_{\mathbb{A}^1} : C_*(\mathrm{Ab}(k)) \to C_*(\mathrm{Ab}(k))$ called the *(abelian) $\mathbb{A}^1$-localization functor* together with a natural transformation
$$\theta : \mathrm{id} \to L^{\mathrm{ab}}_{\mathbb{A}^1}$$
such that for any chain complex $C_* \in C_*(\mathrm{Ab}(k))$, the morphism $\theta_{C_*} : C_* \to L^{\mathrm{ab}}_{\mathbb{A}^1}(C_*)$ is an $\mathbb{A}^1$-quasi isomorphism whose target is an $\mathbb{A}^1$-local fibrant chain complex.

Furthermore, $L^{\mathrm{ab}}_{\mathbb{A}^1}$ induces a functor
$$D(\mathrm{Ab}(k)) \to D_{\mathbb{A}^1\text{-loc}}(\mathrm{Ab}(k))$$
which is left adjoint to the inclusion $D_{\mathbb{A}^1\text{-loc}}(\mathrm{Ab}(k)) \subset D(\mathrm{Ab}(k))$, and which induces an equivalence
$$D_{\mathbb{A}^1}(\mathrm{Ab}(k)) \to D_{\mathbb{A}^1\text{-loc}}(\mathrm{Ab}(k))$$
of categories.

Definition 6.0.4. Let k be a field. Denote by $C^{\mathbb{A}^1}_*$ the functor $L^{\mathrm{ab}}_{\mathbb{A}^1} \circ C_* : \mathcal{H}(k) \to D_{\mathbb{A}^1}(\mathrm{Ab}(k))$. Given a space $X \in \mathcal{H}(k)$, we call $C^{\mathbb{A}^1}_*(X)$ the *$\mathbb{A}^1$-chain complex of X*. (♠ TODO: TODO: notate C_*) Given a space $X \in \mathcal{H}(k)$, denote by $H^{\mathbb{A}^1}_n(X)$ the *nth $\mathbb{A}^1$-homology sheaf of X*, defined to be the nth homology object of the $C^{\mathbb{A}^1}_*(X)$. Denote by $\widetilde{H}^{\mathbb{A}^1}_n(X)$ the *nth $\mathbb{A}^1$-reduced homology sheaf of X*, defined by
$$\widetilde{H}^{\mathbb{A}^1}_n(X) = \ker\left(H^{\mathbb{A}^1}_n(X) \to H^{\mathbb{A}^1}_n(\mathrm{pt})\right).$$

Remark 6.0.5. Just as for classical topology, we have that $H_n^{\mathbb{A}^1}(\mathrm{pt}) = 0$ for $n \neq 0$ and $\mathbb{Z}$ for $n = 0$ and hence we have an isomorphism

$$H_*^{\mathbb{A}^1}(X) = \mathbb{Z} \oplus \widetilde{H}_*^{\mathbb{A}^1}(X)$$

of graded abelian sheaves where $\mathbb{Z}$ here is concentrated in degree 1.

7. $\mathbb{A}^1$-Brouwer degrees

7.1. **The classical Brouwer degree.** Let S^n be the (topological) n-sphere. Recall the following classical theorem in topology:

Theorem 7.1.1. Let $n > 0$ be an integer. Let S^n be the n-sphere and write π_i for the ith fundamental group of a (connected) space. For an integer $i > 0$,

$$\pi_i(S^n) = \begin{cases} 0 & \text{if } i < n \\ \mathbb{Z} & \text{if } i = n. \end{cases}$$

A consequence of this theorem is that one can speak of the *Brouwer degree of a continuous map $f : S^n \to S^n$* — it is defined as the unique integer d such that the induced group homomorphism $\pi_n(f) : \pi_n(S^n) \to \pi_n(S^n)$, which is identifiable with a group homomorphism $\mathbb{Z} \to \mathbb{Z}$ is given by $a \mapsto da$.

Morel's $\mathbb{A}^1$-Brouwer degree generalizes this idea in such a way that $(\mathbb{P}^1)^{\wedge n}$ (over a field k) takes the place of S^n. These "enriched" degrees take values in the Grothendieck-Witt ring $\mathrm{GW}(k)$ (♠ TODO: TODO: ref), which can be regarded as the degree 0 part of the Milnor-Witt K-theory ring K_*^{MW} (Lemma 7.2.5).

7.2. **Milnor-Witt K-theory and the Grothendieck-Witt ring of a field.**

Definition 7.2.1. [See [MV99, Definition 1.21]] Let F be a (commutative) field. The *Milnor-Witt K-theory of F* is the graded associated ring $K_*^{MW}(F)$ generated by the degree 1 symbols $[u]$ for each $u \in F^\times$ along with one degree -1 symbol η subject to the following relations:

1. (Steinberg relation) $[a] \cdot [1 - a] = 0$ for each $a \in F^\times - \{1\}$.
2. $[ab] = [a] + [b] + \eta \cdot [a] \cdot [b]$ for each $(a, b) \in (F^\times)^2$.
3. $[u] \cdot \eta = \eta \cdot [u]$ for each $u \in F^\times$.
4. $\eta \cdot h = 0$ where $h := \eta \cdot [-1] + 2$.

Definition 7.2.2. Let F be a (commutative) field. The *Milnor K-theory of F* is the graded associated ring $K_*^M(F)$ generated by the degree 1 symbols $[u]$ for each $u \in F^\times$ subject to the Steinberg relations: $[a] \cdot [1 - a] = 0$ for each $a \in F^\times - \{1\}$.

Remark 7.2.3. Let F be a field. Note that $K_*^M(F)$ is the quotient of $K_*^{MW}(F)$ by η.

Definition 7.2.4. Let F be a (commutative) field. The *Grothendieck-Witt ring of F* is the commutative ring $\mathrm{GW}(F)$ whose underlying group is the group completion of the commutative monoid of isomorphism classes of non-degenerate symmetric bilinear forms under the direct sum, and whose multiplication structure is determined by tensor products of symmetric bilinear forms.

Equivalently (see e.g. [HM73, Chapter IV Lemma 1.1], cf. [Mor12, Lemma 3.9]), $\mathrm{GW}(F)$ is generated by elements $\langle u\rangle$ for $u \in F^\times$ subject to the following relations:

1. $\langle u(v^2)\rangle = \langle u\rangle$
2. $\langle u\rangle + \langle -u\rangle = \langle 1\rangle + \langle -1\rangle$
3. $\langle u\rangle + \langle v\rangle = \langle u+v\rangle + \langle (u+v)uv\rangle$ if $u+v \neq 0$.

Lemma 7.2.5 ([Mor12, Lemma 3.10, cf. Definition 1.21])**.** Let F be a field. There is a ring isomorphism

$$\mathrm{GW}(F) \to K_0^{MW}(F)$$

given by $\langle u\rangle \mapsto \eta[u] + 1$.

The following relates the Milnor-Witt K-theory of a field F with that of its function field $F(T)$:

Theorem 7.2.6. (See e.g. [Mor12, Theorem 3.24]) For any field F the following is a (split) short exact sequence of $K_*^{MW}(F)$-modules:

$$0 \to K_n^{MW}(F) \to K_n^{MW}(F(T)) \xrightarrow{\Sigma\partial^P_{(P)}} \bigoplus_P K_{n-1}^{MW}(F[T]/P) \to 0$$

7.3. **The $\mathbb{Z}$-graded sheaf of unramified Milnor-Witt K-theory of a field.** Let k be a perfect field. There is a sheaf $\underline{\mathbf{K}}_*^{MW}$ on $(\mathrm{Sm}/k)_{\mathrm{Nis}}$ referred to as the *$\mathbb{Z}$-graded sheaf of unramified Milnor Witt K-theory*. See e.g. [Mor12, The discussions surrounding Lemma 3.32, Theorem 2.46] for an establishment of this sheaf. For any field extension F of k of finite transcendence degree, $\underline{\mathbf{K}}_*^{MW}(F) = K_*^{MW}(F)$. Moreover, the following is a consequence of [Mor12, Theorem 3.22]:

Theorem 7.3.1. Let k be a perfect field. Let F be a field extension of k of finite transcendence degree. Let v be a discrete valuation on F. The ring $\underline{\mathbf{K}}_*^{MW}(\mathcal{O}_v)$ is the subring of $K_*^{MW}(F)$ generated by the elements η and $[u] \in K_1^{MW}(F)$ for $u \in \mathcal{O}_v^\times$ where $\mathcal{O}_v$ is the ring of integers of F with respect to v.

7.4. Analogously to classical topology, we have Hurewicz theorems. To state them, we first need the definition of $\mathbb{A}^1$-connectedness:

Definition 7.4.1. Let S be a Noetherian (Definition D.0.4) base scheme of finite dimension (Definition D.0.9). A pointed space $X \in \Delta^{\mathrm{op}}\,\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})_\bullet$ is said to be *n-$\mathbb{A}^1$-connected or $\mathbb{A}^1$-n-connected* for $n \geq 0$ if the sheaves $\pi_i^{\mathbb{A}^1}(X)$ are isomorphic to pt for $i \leq n$. It is said to be *$\mathbb{A}^1$-connected* if it is 0-$\mathbb{A}^1$-connected. An unpointed space $X \in \Delta^{\mathrm{op}}\,\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})$ is said to be $\mathbb{A}^1$-connected if $\pi_0^{\mathbb{A}^1}(X)$ is isomorphic to pt.

Lemma 7.4.2. Let S be a Noetherian (Definition D.0.4) base scheme of finite dimension (Definition D.0.9). A pointed space $X \in \Delta^{\mathrm{op}}\,\mathrm{Shv}((\mathrm{Sm}/S)_{\mathrm{Nis}})_\bullet$ is $\mathbb{A}^1$-connected if and only if its underlying space is $\mathbb{A}^1$-connected.

Proof. This follows by Lemma 4.1.3 □

Theorem 7.4.3 (Hurewciz theorems for $\mathbb{A}^1$-homotopy theory)**.** (♠ TODO: TODO; state the Hurewicz theorems) Let k be a perfect field. Let $X \in \mathcal{H}_\bullet(k)$ be a pointed $\mathbb{A}^1$-connected space.

1. [Mor12, Theorem 6.35] The Hurewicz morphism
$$\pi_1^{\mathbb{A}^1}(X) \to H_1^{\mathbb{A}^1}(X)$$
is the initial morphism from $\pi_1^{\mathbb{A}^1}(X)$ to a strictly $\mathbb{A}^1$-invariant sheaf of groups. More precisely, any morphism
$$\pi_1^{\mathbb{A}^1}(X) \to M$$
to a strictly $\mathbb{A}^1$-invariant sheaf M of groups factors uniquely through $H_1^{\mathbb{A}^1}(X)$.
2. [Mor12, Theorem 6.37] Let $n \geq 2$ be an integer and let $X \in \mathcal{H}_\bullet(k)$ be a pointed $(n-1)$-$\mathbb{A}^1$-connected space. For each $i \in \{0, \dots, n-1\}$, we have
$$\widetilde{H}_i^{\mathbb{A}^1}(X) = 0$$
and the Hurewicz morphism
$$\pi_n^{\mathbb{A}^1}(X) \to H_n^{\mathbb{A}^1}(X)$$
is an isomorphism between strictly $\mathbb{A}^1$-invariant sheaves.

Theorem 7.4.4 ([Mor12, Theorem 1.22])**.** Let k be a perfect field. For $n \geq 1$, there is a morphism
$$(\mathbb{G}_m)^{\wedge n} \to \underline{\mathbf{K}}_n^{MW}, \quad (U_1, \dots, U_n) \mapsto [U_1, \dots, U_n]$$
of sheaves on $(\mathrm{Sm}/k)_{\mathrm{Nis}}$ and this morphism is the universal one to a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups. More precisely, any morphism $(\mathbb{G}_m)^{\wedge n} \to M$ where M is a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups factors uniquely through $\underline{\mathbf{K}}_n^{MW}$.

The following is a consequence:

(♠ TODO: TODO: find the relationship between the n-fold wedge of multiplicative group and punctured affine spcae, and n-fold wedge of projective line)

Theorem 7.4.5 ([Mor12, Theorem 1.23])**.** Let k be a perfect field. For $n \geq 2$, there is an isomorphism
$$\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}) \cong \pi_n^{\mathbb{A}^1}((\mathbb{P}^1)^{\wedge n}) \cong \underline{\mathbf{K}}_n^{MW}$$
of sheaves on $(\mathrm{Sm}/k)_{\mathrm{Nis}}$.

Theorem 7.4.6 ([Mor12, Corollary 1.24])**.** Let k be a perfect field. The canonical morphism
$$[\mathbb{A}^n - \{0\}, \mathbb{A}^n - \{0\}]_\bullet \cong [(\mathbb{P}^1)^{\wedge n}, (\mathbb{P}^1)^{\wedge n}] \to K_0^{MW}(k) = \mathrm{GW}(k)$$
is an epimorphism for $n = 1$ and an isomorphism for $n \geq 2$.

8. $\mathbb{A}^1$-homotopy groups of spaces

9. Stable $\mathbb{A}^1$-categories

(♠ TODO:)

Definition 9.0.1. The *unstable $\mathbb{A}^1$-homotopy category*, denoted $\mathcal{H}(S)$, is the homotopy category of $\mathrm{Spc}(S)$ with respect to the *$\mathbb{A}^1$-model structure*. This model structure is the left Bousfield localization of the Nisnevich-local model structure on $\mathrm{Spc}(S)$ with respect to the set of projection maps $\{X \times \mathbb{A}^1 \to X \mid X \in \mathrm{Sm}/S\}$. The pointed homotopy category $\mathcal{H}_\bullet(S)$ is constructed similarly from $\mathrm{Spc}_\bullet(S)$.

Definition 9.0.2. Let S be a Noetherian scheme (Definition D.0.4) of finite dimension (Definition D.0.9) and let T be the Tate sphere (Definition 4.0.5).

1. A *motivic spectrum* (or *T-spectrum*) E consists of a sequence $(E_n)_{n\geq 0}$ of pointed motivic spaces $E_n \in \mathrm{Spc}_\bullet(S)$ together with structure morphisms σ_n:
$$\sigma_n : T \wedge E_n \to E_{n+1}$$
(Definition 4.0.1)
2. A *morphism of motivic spectra* $f : E \to F$ is a sequence of maps $f_n : E_n \to F_n$ in $\mathrm{Spc}_\bullet(S)$ compatible with the structure morphisms, i.e., $f_{n+1} \circ \sigma_n^E = \sigma_n^F \circ (\mathrm{id}_T \wedge f_n)$.
3. We denote the category of motivic spectra over S by $\mathrm{Spt}_T(S)$.

(♠ TODO: distingiush between the "fully stable category" = T-spectra category = $\mathbb{P}^1$-spectra category and the S^1-spectra category, stabilized only with respect to the simplicial circle S^1)

Definition 9.0.3. (♠ TODO: separate out the stable model structure) The *stable $\mathbb{A}^1$-homotopy category*, denoted by notations such as $\mathcal{SH}(S)$, or $\mathbf{SH}(S)$, is the homotopy category of the category of motivic spectra $\mathrm{Spt}_T(S)$ with respect to the *stable model structure*. This model structure is the stabilization of the pointed $\mathbb{A}^1$-model structure (Definition 3.3.7) on $\mathrm{Spc}_\bullet(S)$ with respect to the endofunctor $-\wedge T$, meaning the weak equivalences are the stable $\mathbb{A}^1$-weak equivalences (morphisms inducing isomorphisms on all stable homotopy sheaves).

10. $\mathbb{A}^1$-connectivity theorems

(♠ TODO: state unstable version) (♠ TODO: state stable version of Schmidt-Strunk over Dedekind schemes) (♠ TODO: state Gabber's geometric presentation lemma over fields)

(♠ TODO: State Gabber's geometric presentation lemma by Schmidt-Strunk (over dedekind schemes with infinite residue fields), then by Deshmukh, Hogadi, Kulkarni, and Yadav (DHKY) (over Noetherian domains with infinite residue fields) and Kulkarni-Hogadi's over finite fields) (♠ TODO: state Gabber's geometric presentation lemma by Druzhinin for henselian local essentially smooth schemes, see Remark 3 of loc.cit.)

Theorem 10.0.1 (Stable $\mathbb{A}^1$ connectivity theorm [Dru22, Theorems 2, 7])**.** (♠ TODO: essentially smooth) Let S be a scheme of Krull dimension (Definition D.0.9) d. Let U be an essentially smooth local henselian scheme over a base scheme S. Let $F \in \mathbf{SH}_{S^1}(S)$ be a 0-connective spectrum. Then

$$[U, F \wedge S^i]_{\mathbf{SH}_{S^1}(S)} = 0, \quad [U, \Sigma^\infty_{\mathbb{G}_m} F \wedge S^i]_{\mathbf{SH}_{S^1}(S)} = 0 \text{ for all } i > d.$$

(♠ TODO: henselian scheme) (♠ TODO: stable homotopy group) (♠ TODO: connective spectrum) (♠ TODO: $\mathbf{SH}_{S^1}(S)$, S^1, $[,]$, $\Sigma^\infty_{\mathbb{G}_m}$)

Appendix A. Category Theory

Definition A.0.1 (Category). A *category* *category* $\mathcal{C}$ consists of the following data:

- A class of *objects* denoted $\mathrm{Ob}(\mathcal{C})$.
- For each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a class

  $$\mathrm{Hom}_{\mathcal{C}}(X, Y)$$

  of *morphisms* (also called *arrows* or *homs*). If the category $\mathcal{C}$ is clear, then this *hom-class* is also denoted by $\mathrm{Hom}(X, Y)$. It may also be denoted by $\mathrm{hom}_{\mathcal{C}}(X, Y)$ or $\mathrm{hom}(X, Y)$, especially to distinguish from other types of hom's (e.g. internal hom's)
- For each triple of objects X, Y, Z, a composition law

  $$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z),$$

  denoted $(g, f) \mapsto g \circ f$.
- For each object X, an *identity morphism*

  $$\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X).$$

These data satisfy the following axioms:

- (Associativity) For all morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, and $h \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$,

  $$h \circ (g \circ f) = (h \circ g) \circ f.$$
- (Identity) For all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$,

  $$\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X.$$

One often writes $X \in \mathcal{C}$ synonymously with $X \in \mathrm{Ob}(\mathcal{C})$, i.e. to denote that X is an object of of $\mathcal{C}$.

We may call a category as above an *ordinary category* to distinguish this notion from the notions of *categories enriched in monoidal categories* or higher/n-categories. (♠ TODO: TODO: define n-categories)

A category as defined above may be called called a *large category* or a *class category* to emphasize that the hom-classes may be proper classes rather than sets (note, however, that the possibility that hom-classes are sets is not excluded for large categories). Accordingly, a *category* may often refer to a locally small category (Definition A.0.6), which is a category whose hom-classes are all sets.

Definition A.0.2. Let $\mathcal{C}$ and $\mathcal{D}$ be (large) categories (Definition A.0.1).

1. A *functor $F : \mathcal{C} \to \mathcal{D}$ (from $\mathcal{C}$ to $\mathcal{D}$)* consists of :
   - For each object X in $\mathcal{C}$, an object $F(X)$ in $\mathcal{D}$.
   - For each morphism $f : X \to Y$ in $\mathcal{C}$, a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$,

   such that:

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \text{for all objects } X \text{ in } \mathcal{C},$$
$$F(g \circ f) = F(g) \circ F(f) \quad \text{for all } X, Y, Z \in \mathrm{Ob}(\mathcal{C}) \text{ and all } f : X \to Y, g : Y \to Z \text{ in } \mathcal{C}.$$

   Functors as defined above are also referred to as *covariant functors* to distinguish them from contravariant functors
2. A *contravariant functor from $\mathcal{C}$ to $\mathcal{D}$* refers to a covariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. Equivalently, such a functor consists of
   - For each object X in $\mathcal{C}$, an object $F(X)$ in $\mathcal{D}$.
   - For each morphism $f : X \to Y$ in $\mathcal{C}$, a morphism $F(f) : F(Y) \to F(X)$ in $\mathcal{D}$,

   such that:

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \text{for all objects } X \text{ in } \mathcal{C},$$
$$F(g \circ f) = F(f) \circ F(g) \quad \text{for all } X, Y, Z \in \mathrm{Ob}(\mathcal{C}) \text{ and all } f : X \to Y, g : Y \to Z \text{ in } \mathcal{C}.$$

   A synonym for a "contravariant functor from $\mathcal{C}$ to $\mathcal{D}$" is a "presheaf on $\mathcal{C}$ with values in $\mathcal{D}$ (Definition A.0.8)".

Note that declarations such as "Let $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ be a contravariant functor" can be common; such declarations usually mean "Let F be a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$" as opposed to "Let F be a contravariant functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$". further note that a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is equivalent to a covariant functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$.

Definition A.0.3. Let $\mathcal{C}$ and $\mathcal{D}$ be (large) categories (Definition A.0.1). Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors (Definition A.0.2).

A *natural transformation η between F and G* is a family of morphisms $\eta_X : F(X) \to G(X)$ in $\mathcal{D}$, one for each object X in $\mathcal{C}$, such that for every morphism $f : X \to Y$ in $\mathcal{C}$,

$$G(f) \circ \eta_X = \eta_Y \circ F(f)$$

in $\mathcal{D}$. In other words, the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

We write such a natural transformation by *$\eta : F \Rightarrow G$*.

If η_X is an isomorphism for all objects X of $\mathcal{C}$, then η is said to be a *natural isomorphism*.

Definition A.0.4. Let $\mathcal{C}$ be a category (Definition A.0.1).

1. An object $I \in \mathcal{C}$ is called an *initial object* if for every object $X \in \mathcal{C}$ there exists a unique morphism
$$I \to X.$$
Equivalently, an initial object is a limit (Definition A.0.13) of the empty diagram (Definition A.0.12), if such a limit exists.
2. An object $F \in \mathcal{C}$ is called a *final object* (or *terminal object*) if for every object $X \in \mathcal{C}$ there exists a unique morphism
$$X \to F.$$
Equivalently, a final object is a colimit (Definition A.0.13) of the empty diagram (Definition A.0.12), if such a colimit exists.
3. An object $Z \in \mathcal{C}$ is called a *zero object* if Z is both initial and final in $\mathcal{C}$. In particular, for every object $X \in \mathcal{C}$ there exist unique morphisms
$$Z \to X \quad \text{and} \quad X \to Z.$$

Definition A.0.5. An *equivalence of categories* between two (large) categories (Definition A.0.1) $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of functors (Definition A.0.2)
$$F : \mathcal{C} \to \mathcal{D} \quad \text{and} \quad G : \mathcal{D} \to \mathcal{C}$$
together with natural isomorphisms (Definition A.0.3)
$$\eta : \mathrm{Id}_{\mathcal{C}} \xrightarrow{\sim} G \circ F \quad \text{and} \quad \epsilon : F \circ G \xrightarrow{\sim} \mathrm{Id}_{\mathcal{D}}.$$
Such functors F and G may be called *(natural) inverses of each other*.

When $\mathcal{C}$ and $\mathcal{D}$ are locally small categories (Definition A.0.6), F is an equivalence of categories if and only if F is fully faithful and essentially surjective

Definition A.0.6 (Locally small category)**.** A (large) category (Definition A.0.1) $\mathcal{C}$ is called a *locally small category* if for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the collection $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms between them is a (small) *set* (as opposed to a proper class). In other words, each hom-class is a set and may even be called a *hom-set*.

In some contexts, a locally small category may simply be called a *category*, especially when genuinely large categories are not considered.

A category $\mathcal{C}$ is called a *small category* if it is a locally small category and the class $\mathrm{Ob}(\mathcal{C})$ of objects is a set.

Remark A.0.7. Many "concrete" categories considered in "classical mathematics" or outside of more "abstract" category theory tend to be locally small. For example, the categories of sets, groups, R-modules, vector spaces, topological spaces, schemes, manifolds, sheaves on "small enough" sites are all locally small.

Definition A.0.8 (Presheaf on a category)**.** Let C and $\mathcal{A}$ be (large) categories (Definition A.0.1).

1. A *presheaf $\mathcal{F}$ on C with values in $\mathcal{A}$* is a functor
$$\mathcal{F} : C^{\mathrm{op}} \to \mathcal{A}.$$

In other words, a presheaf $\mathcal{F}$ on C with values in $\mathcal{A}$ is simply a contravariant functor (Definition A.0.2) from C to $\mathcal{A}$. Explicitly, for every object U in C, one has an object $\mathcal{F}(U)$ in $\mathcal{A}$ (called the *U-valued sections/sections evaluated at U of $\mathcal{F}$*), and for every morphism $f : V \to U$ in C, one has a morphism (called the restriction map)

$$\mathcal{F}(f) : \mathcal{F}(U) \to \mathcal{F}(V)$$

in $\mathcal{A}$, such that for all composable morphisms $W \xrightarrow{g} V \xrightarrow{f} U$ in C, the following diagram in $\mathcal{A}$ commutes:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\mathcal{F}(f)"] \arrow[rr, bend left, "\mathcal{F}(f\circ g)"] & \mathcal{F}(V) \arrow[r, "\mathcal{F}(g)"] & \mathcal{F}(W) \end{tikzcd}$$

That is,

$$\mathcal{F}(g) \circ \mathcal{F}(f) = \mathcal{F}(f \circ g),$$

and for every object U in C, $\mathcal{F}(\mathrm{id}_U) = \mathrm{id}_{\mathcal{F}(U)}$.

2. Let $\mathcal{F}, \mathcal{G} : C^{\mathrm{op}} \to \mathcal{A}$ be two presheaves on C with values in $\mathcal{A}$. A *morphism of presheaves*

$$\varphi : \mathcal{F} \to \mathcal{G}$$

is a natural transformation of functors (Definition A.0.3): for each object U of C, one has a morphism

$$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

in $\mathcal{A}$, such that for every morphism $f : V \to U$ in C, the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\varphi_U"'] & \mathcal{F}(V) \arrow[d, "\varphi_V"] \\ \mathcal{G}(U) \arrow[r, "\mathcal{G}(f)"'] & \mathcal{G}(V) \end{tikzcd}$$

commutes, i.e.,

$$\varphi_V \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \varphi_U$$

for all objects and morphisms in C.

3. Given a universe U, a *U-presheaf on $\mathcal{C}$* typically refers to a presheaf of U-sets on C.
4. The *presheaf category/category of $\mathcal{A}$-valued presheaves on $\mathcal{C}$* is the (large) category whose objects are the presheaves on C with values in $\mathcal{A}$ and whose morphisms are the presheaf morphisms. Common notations for the presheaf category include, but are not limited to: $\mathcal{A}^{\mathcal{C}^{\mathrm{op}}}$, $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$, $[\mathcal{C}^{\mathrm{op}}, \mathcal{A}]$. If the value category $\mathcal{A}$ is clear from context, then notations such as $\mathrm{PreShv}(\mathcal{C})$ are also common. Note that the presheaf category $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ is equivalent to the category of functors (Definition A.0.12) $\mathcal{C}^{\mathrm{op}} \to \mathcal{A}$ and hence notations for the functor categories are applicable as notations for presheaf categories.

Definition A.0.9 (Grothendieck topology). (♠ TODO: check definitions using sites to make sure that the categories are locally small.) Let $\mathscr{U}$ be a universe.

1. Let C be a locally small category (Definition A.0.6). A *Grothendieck topology on C* assigns to each object U of C a collection of families of morphisms $\{U_i \to U\}_{i \in I}$, called *coverings of U*, satisfying:
   - (Isomorphism) If $f : V \to U$ is an isomorphism in C, then $\{f : V \to U\}$ is a covering of U.
   - (Stability under base change) If $\{U_i \to U\}_{i \in I}$ is a covering of U and $V \to U$ is any morphism, then the family $\{U_i \times_U V \to V\}_{i \in I}$ is a covering of V.
   - (Transitivity) If $\{U_i \to U\}_{i \in I}$ is a covering of U and for each i, $\{V_{ij} \to U_i\}_{j \in J_i}$ is a covering of U_i, then the family $\{V_{ij} \to U\}_{i \in I, j \in J_i}$ is a covering of U.

   Alternatively, a Grothendieck topology J on a category C is an assignment of a collection $J(U)$ of sieves on each object $U \in \mathrm{Ob}(C)$ such that:
   (a) the maximal sieve $\{f : V \to U \mid f \in \mathrm{Mor}(C)\}$ belongs to $J(U)$,
   (b) if $S \in J(U)$ and $f : V \to U$, then the pullback sieve f^*S on V belongs to $J(V)$,
   (c) if S is a sieve on U, and if there exists $R \in J(U)$ such that for all $f : V \to U$ in R the pullback sieve f^*S is in $J(V)$, then $S \in J(U)$.

   Some will refer to a Grothendieck topology as simply a *topology*, not to be confused with the related, but less general, notion of a topology on a set (Definition C.0.1).
2. A *site* is a locally small category C equipped with a Grothendieck topology.
3. Let $(\mathcal{C}, J)$ be a site (Definition A.0.9), where J is a Grothendieck topology on $\mathcal{C}$. A family of objects $\{U_i\}_{i \in I}$ in $\mathcal{C}$ is called a *topologically generating family* if for every object $X \in \mathcal{C}$ and every covering sieve $S \in J(X)$, the sieve S is generated by pullbacks of covering families from the family $\{U_i\}$.

   More precisely, this means that for any $S \in J(X)$, the sieve S contains a covering family $\{V_j \to X\}$ such that each morphism $V_j \to X$ factors through some U_i, and the covering families of the U_i generate the topology J. Equivalently, the Grothendieck topology J is the smallest Grothendieck topology containing all coverings of the U_i.
4. A *$\mathscr{U}$-site* is a site whose underlying category C is $\mathscr{U}$-locally small (Definition A.0.6) and which has a $\mathscr{U}$-small topologically generating family. A $\mathscr{U}$-site is called *$\mathscr{U}$-small* if its underlying category is $\mathscr{U}$-small. Similarly, A *small site* is a site whose underlyign category is a set.

Definition A.0.10 (Sheaf on a site). (♠ TODO: There might be some need to say that $\mathcal{A}$ is a category for which sheaves on the site "can be defined") (♠ TODO: go through statements using the notion of sheaves and make sure that the value categories have small products and that the categories have small generating families.) Let $(\mathcal{C}, J)$ be a site (Definition A.0.9) with a small topological generating family (Definition A.0.9) (or a U-small topologically generating family if a universe U is available) and let $\mathcal{A}$ be a (large) category (Definition A.0.1) that has all small (Definition A.0.6) products (Definition A.0.14) (Some common examples of categories that have small products and thus play the role of $\mathcal{A}$ here include $\mathcal{A} = \mathrm{Set}$, Ab, $R-\mathbf{mod}$ for a fixed ring R, rings).

1. A presheaf (Definition A.0.8) $\mathcal{F} : C^{\mathrm{op}} \to \mathcal{A}$ is a *sheaf on the site $(\mathcal{C}, J)$ valued in $\mathcal{A}$* if, for every object U of $\mathcal{C}$ and every covering $\{U_i \to U\}_{i \in I}$ in J, the following sequence is an equalizer in $\mathcal{A}$:

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$$

where the first map sends s to $(\mathcal{F}(U_i \to U)(s))_i$ and the arrows to $(\mathcal{F}(U_i \times_U U_j \to U_i)(s_i))_{i,j}$ and $(\mathcal{F}(U_i \times_U U_j \to U_j)(s_j))_{i,j}$, respectively.

2. A *morphism of sheaves* $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$ is a morphism as presheaves (Definition A.0.8).
3. Let U be a universe. A *U-sheaf* typically refers to a U-presheaf that is a sheaf for a U-site. In other words, a U-sheaf is a sheaf on a site whose underlying category is U-locally small (Definition A.0.6) and which has a U-small topologically generating family such that the sheaf is valued in U-sets.
4. The *sheaf category/category of $\mathcal{A}$-valued sheaves on $\mathcal{C}$* is the (large) category defined as the full subcategory of $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ whose objects are the sheaves on C with values in $\mathcal{A}$. Common notations for the sheaf category include $\mathrm{Shv}(\mathcal{C}, \mathcal{A})$, $\mathrm{Shv}(\mathcal{C}, J, \mathcal{A})$, $\mathrm{Sh}(\mathcal{C}, \mathcal{A})$, $\mathrm{Sh}(\mathcal{C}, J, \mathcal{A})$. If the value category $\mathcal{A}$ is clear from context, then notations such as $\mathrm{Shv}(\mathcal{C})$, $\mathrm{Shv}(\mathcal{C}, J)$, $\mathrm{Sh}(\mathcal{C})$, $\mathrm{Sh}(\mathcal{C}, J)$ are also common.

Definition A.0.11. Let $\mathcal{C}$ be a site (Definition A.0.9) and let $\mathcal{A}$ be a (large) category (Definition A.0.1) A *sheafification functor* refers to a functor

$$a : \mathrm{PreShv}(\mathcal{C}, \mathcal{A}) \to \mathrm{Shv}(\mathcal{C}, \mathcal{A})$$

that is left adjoint (Definition A.0.15) to the inclusion functor

$$i : \mathrm{Shv}(\mathcal{C}, \mathcal{A}) \hookrightarrow \mathrm{PreShv}(\mathcal{C}, \mathcal{A}).$$

If such a sheafification functor exists, then it is unique up to unique natural isomorphism. Given a presheaf P, the sheafification $a(P)$ is also sometimes called the *sheaf asscciated to P*.

Definition A.0.12 (Diagram in a category and category of diagrams)**.** Let $\mathcal{C}$ be a (large) category (Definition A.0.1), and let I be a (large) category (Definition A.0.1).

1. A *diagram of shape I in $\mathcal{C}$* is a functor (Definition A.0.2) $D : I \to \mathcal{C}$. We often denote such a diagram by the family $\{D(i)\}_{i \in \mathrm{Ob}(I)}$ with transition maps given by the functorial image of morphisms in I.

   A diagram is also synonymously called a *system*. Moreover, the category I is called the *index category* or the *indexing category of the diagram D*.
2. Given two diagrams $D, E : I \to \mathcal{C}$, a *morphism of diagrams* is a simply a natural transformation (Definition A.0.3) $D \Rightarrow E$ of the functors D and E.
3. The *category of I-shaped diagrams in $\mathcal{C}$*, often denoted $\mathcal{C}^I$, $[I, \mathcal{C}]$, or $\mathrm{Fun}(I, \mathcal{C})$, is the (large) category whose objects are functors $I \to \mathcal{C}$ (that is, diagrams of shape I in $\mathcal{C}$) and whose morphisms are natural transformations (Definition A.0.3) between such functors. The category $\mathcal{C}^I$ is also called the *functor category of functors $I \to \mathcal{C}$*. Equivalently, the functor category $\mathcal{C}^I$ is the category $\mathrm{PreShv}(I^{\mathrm{op}}, \mathcal{C})$ of presheaves (Definition A.0.8) on I^{op} with values in $\mathcal{C}$ and hence notations for presheaf categories are applicable as notations for functor categories.

   If $\mathcal{C}$ is locally small (Definition A.0.6) and I is small, then $\mathcal{C}^I$ is locally small by Lemma **??**.

Definition A.0.13 (Cones, limits and colimits in a category)**.** Let $\mathcal{C}$ be a (large) category (Definition A.0.1), let I be a (large) category, and let $D : I \to \mathcal{C}$ be a diagram (Definition A.0.12) (Definition A.0.12).

1. A *cone to the diagram D* is an object $L \in \mathcal{C}$ together with a family of morphisms
$$\{\pi_i : L \to D(i)\}_{i \in I}$$
such that for every morphism $f : i \to j$ in I, the diagram
$$\begin{tikzcd} & L \arrow[dl, "\pi_i"'] \arrow[dr, "\pi_j"] & \\ D(i) \arrow[rr, "D(f)"] & & D(j) \end{tikzcd}$$
commutes, i.e. $D(f) \circ \pi_i = \pi_j$.
2. A cone $(L, \{\pi_i\})$ is called a *limit of D* if it satisfies the following "universal property": for any cone $(C, \{f_i\})$ over D, there exists a *unique* morphism $u : C \to L$ such that
$$\pi_i \circ u = f_i \quad \text{for all } i \in I.$$
Visually, the following diagrams commute every morphism $f : i \to j$ in I:
$$\begin{tikzcd} & C \arrow[d, dashed, "\exists! u"] \arrow[ddl, bend right, "f_i"'] \arrow[ddr, bend left, "f_j"] & \\ & L \arrow[dl, "\pi_i"] \arrow[dr, "\pi_j"'] & \\ D(i) \arrow[rr, "D(f)"] & & D(j). \end{tikzcd}$$
If such a cone exists, then the object L is necessarily unique up to unique isomorphism by the universal property. In this case, L is denoted by $\lim_{i \in I} D$ or $\lim D$.
3. A *cocone from the diagram D* is an object $C \in \mathcal{C}$ together with a family of morphisms
$$\{\iota_i : D(i) \to C\}_{i \in I}$$
such that for every morphism $f : i \to j$ in I, the diagram
$$\begin{tikzcd} D(i) \arrow[rr, "D(f)"] \arrow[dr, "\iota_i"'] & & D(j) \arrow[dl, "\iota_j"] \\ & C & \end{tikzcd}$$
commutes, i.e. $\iota_j \circ D(f) = \iota_i$.
4. A cocone $(L, \{\iota_i\})$ is called a *colimit of D* if it satisfies the following "universal property": for any cocone $(C, \{g_i\})$ under D, there exists a *unique* morphism $u : L \to C$ such that
$$u \circ \iota_i = g_i \quad \text{for all } i \in I.$$
Visually, the following diagrams commute every morphism $f : i \to j$ in I:

$$\begin{tikzcd}
D(i) \arrow[rr, "D(f)"] \arrow[dr, "\iota_i"] \arrow[ddr, bend right, "g_i"'] & & D(j) \arrow[dl, "\iota_j"'] \arrow[ddl, bend left, "g_j"] \\
& L \arrow[d, dashed, "\exists! u"] & \\
& C &
\end{tikzcd}.$$

If such a cocone exists, then the object L is necessarily unique up to unique isomorphism by the universal property. In this case, L is denoted by $\operatorname{colim}_{i\in I} D$ or $\operatorname{colim} D$.

Some authors use the terms *projective limit* or *inverse limit* to refer to what is defined here as a limit, Similarly, the terms *inductive limit* or *direct limit* are sometimes used to mean a colimit. However, these phrases can have more specific meanings to other authors: a *projective* or *inverse limit* may refer to a limit over a diagram indexed by a codirected poset. Likewise, an *inductive* or *direct limit* may refer to a colimit over a directed poset (see Definition D.0.12) .

Thus, while the terms are sometimes used interchangeably with "limit" and "colimit," they may also emphasize particular indexing shapes and directions, distinguishing them from general limits and colimits taken over arbitrary small categories.

Definition A.0.14 (Product in a category)**.** Let $\mathcal{C}$ be a category and let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$ indexed by a class I.

1. A *product of the family* $\{X_i\}$ is an object P of $\mathcal{C}$ together with a "universal" family of morphisms
$$\pi_i : P \to X_i, \quad \text{for each } i \in I.$$
More precisely, for any object Y and any family of morphisms $\{f_i : Y \to X_i\}_{i\in I}$, there exists a unique morphism
$$f : Y \to P$$
making the following diagram commute for all $i \in I$, i.e. $\pi_i \circ f = f_i$:
$$\begin{tikzcd}
Y \arrow[d, dashed, "\exists! f"] \arrow[dr, "f_i"] & \\
\prod X_i \arrow[r, "\pi_i"'] & X_i
\end{tikzcd}$$
Such a product is often denoted by $\prod_{i\in I} X_i$. If $\prod_{i\in I} X_i$ exists in $\mathcal{C}$, then it is unique up to unique isomorphism by the universal property described above.

   Equivalently, the product $\prod_{i\in I} X_i$ is the limit (Definition A.0.13) of the diagram (Definition A.0.12) $I \to \mathcal{C}, i \mapsto X_i$, where I is made into a category whose objects are the members of I and whose morphisms are just the identity morphisms.
2. A *coproduct* (or synonymously *direct sum*) of the family $\{X_i\}$ is an object C of $\mathcal{C}$ together with a "universal" family of morphisms
$$\iota_i : X_i \to C, \quad \text{for each } i \in I.$$

More precisely, for any object Y and any family of morphisms $\{g_i : X_i \to Y\}_{i\in I}$, there exists a unique morphism

$$g : C \to Y$$

making the following diagram commute for all $i \in I$, i.e. $g \circ \iota_i = g_i$:

$$\begin{tikzcd} X_i \arrow[r, "\iota_i"] \arrow[dr, "g_i"'] & \coprod X_i \arrow[d, dashed, "\exists! g"] \\ & Y \end{tikzcd}$$

Such a coproduct is often denoted by $\coprod_{i\in I} X_i$ or $\oplus_{i\in I} X_i$. If $\coprod_{i\in I} X_i$ exists in $\mathcal{C}$, then it is unique up to unique isomorphism by the universal property described above.

Equivalently, the coproduct $\coprod_{i\in I} X_i$ is the colimit (Definition A.0.13) of the diagram (Definition A.0.12) $I \to \mathcal{C}, i \mapsto X_i$, where I is made into a category whose objects are the members of I and whose morphisms are just the identity morphisms.

Definition A.0.15. (♠ TODO: make it soe that the for excerpts using this, the categories involved are locally small) Let $\mathcal{C}$ and $\mathcal{D}$ be locally small (Definition A.0.6) categories (or U-locally small categories if a universe U is available).

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. F is a *left adjoint to G* and G is a *right adjoint to F* (written $F \dashv G$) if for every object A in $\mathcal{C}$ and B in $\mathcal{D}$ there is a natural isomorphism (Definition A.0.3)

$$\mathrm{Hom}_{\mathcal{D}}(F(A), B) \cong \mathrm{Hom}_{\mathcal{C}}(A, G(B))$$

that is natural in both A and B.

Definition A.0.16 (Monomorphism and Epimorphism in Categories)**.** Let $\mathcal{C}$ be a category (Definition A.0.1). For objects $A, B \in \mathcal{C}$, let $f : A \to B$ be a morphism in $\mathcal{C}$.

- The morphism f is called a *monomorphism* (or a *monic morphism*) if for every object X and every pair of morphisms $g_1, g_2 : X \to A$, the equality $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.
- The morphism f is called an *epimorphism* (or an *epic morphism*) if for every object Y and every pair of morphisms $h_1, h_2 : B \to Y$, the equality $h_1 \circ f = h_2 \circ f$ implies $h_1 = h_2$.

Definition A.0.17 (Category of objects over a fixed object)**.** Let $\mathcal{C}$ be a category (Definition A.0.1) and let $X \in \mathrm{Ob}(\mathcal{C})$ be a fixed object.

1. The *category of objects over X* (or synonymously the *slice category of X in $\mathcal{C}$* or the *over category of X in $\mathcal{C}$*), commonly denoted $\mathcal{C}/X$ or $\mathcal{C}_{/X}$, is the category defined as follows:
    - An object of $\mathcal{C}/X$ is a morphism $f\colon A \to X$ in $\mathcal{C}$, where $A \in \mathrm{Ob}(\mathcal{C})$.

- A morphism from $f\colon A \to X$ to $g\colon B \to X$ in $\mathcal{C}/X$ is a morphism $h\colon A \to B$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[dr, "f"'] & B \arrow[d, "g"] \\ & X \end{tikzcd}$$

  i.e. such that $g \circ h = f$.
- The identity morphisms and composition in $\mathcal{C}/X$ are inherited from $\mathcal{C}$.

2. The *category of objects under X* (or synonymously the *coslice category of X in $\mathcal{C}$* or the *under category of X in $\mathcal{C}$*), commonly denoted $X/\mathcal{C}$, $X\backslash\mathcal{C}$ or $\mathcal{C}_{X/}$, is the category defined as follows:
   - An object of $X/\mathcal{C}$ is a morphism $f\colon X \to A$ in $\mathcal{C}$, where $A \in \mathrm{Ob}(\mathcal{C})$.
   - A morphism from $f\colon X \to A$ to $g\colon X \to B$ in $X/\mathcal{C}$ is a morphism $h\colon A \to B$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"] & A \arrow[d, "h"] \\ & B \end{tikzcd}$$

     i.e. such that $h \circ f = g$.
   - The identity morphisms and composition in $X/\mathcal{C}$ are inherited from $\mathcal{C}$.

Definition A.0.18 (Category enriched in a monoidal category)**.** Let $(\mathcal{V}, \otimes, \mathbf{1})$ be a monoidal category. A *category enriched in $\mathcal{V}$* (or a *$\mathcal{V}$-enriched category* or a *$\mathcal{V}$-category*) $\mathcal{C}$ consists of the following data:

- A class $\mathrm{Ob}(\mathcal{C})$ of *objects*. As with regular categories, we may write $X \in \mathrm{Ob}(\mathcal{C})$ or $X \in \mathcal{C}$ to mean that X is an object of $\mathcal{C}$.
- For each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, an object $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \in \mathrm{Ob}(\mathcal{V})$ of *morphisms*; it is an object of the monoidal category $\mathcal{V}$. It is also often denoted by notations such as $\mathcal{C}(X,Y)$, $\mathrm{Hom}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$, or $\mathrm{Mor}(X,Y) = \mathrm{Mor}_{\mathcal{C}}(X,Y)$.
- For each triple $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a *composition morphism*

$$\mu_{X,Y,Z} : \underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Z) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Z).$$

  It is a morphism in $\mathcal{V}$.
- For each object X, a *unit morphism* $\eta_X : \mathbf{1} \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X,X)$ in $\mathcal{V}$.

These data satisfy the following axioms:

- (Associativity) For all $W, X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, the following diagram in $\mathcal{V}$ commutes:

$$\begin{tikzcd}
\big(\underline{\mathrm{Hom}}_{\mathcal{C}}(Z,W) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Z)\big) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \arrow[r, "\alpha"] \arrow[d, "\mu\otimes\mathrm{id}"] & \underline{\mathrm{Hom}}_{\mathcal{C}}(Z,W) \otimes \big(\underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Z) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)\big) \arrow[d, "\mathrm{id}\otimes\mu"] \\
\underline{\mathrm{Hom}}_{\mathcal{C}}(Y,W) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \arrow[d, "\mu"] & \underline{\mathrm{Hom}}_{\mathcal{C}}(Z,W) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Z) \arrow[d, "\mu"] \\
\underline{\mathrm{Hom}}_{\mathcal{C}}(X,W) \arrow[r, equal] & \underline{\mathrm{Hom}}_{\mathcal{C}}(X,W)
\end{tikzcd}$$

  where α is the associativity constraint in $\mathcal{V}$.
- (Unit) For all $X, Y \in \mathrm{Ob}(\mathcal{C})$, the following diagrams commute:

$$\begin{tikzcd}
\mathbf{1} \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \arrow[r, "\eta_Y\otimes\mathrm{id}"] \arrow[dr, "\lambda"'] & \underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Y) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \arrow[d, "\mu"] \\
& \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \otimes \mathbf{1} \arrow[r, "\mathrm{id}\otimes\eta_X"] \arrow[dr, "\rho"'] & \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X,X) \arrow[d, "\mu"] \\
& \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)
\end{tikzcd}$$

  where λ and ρ are the left and right unit constraints in $\mathcal{V}$.

Definition A.0.19 (n-ary (Multivariable) Functor)**.** Let I be a finite set with $|I| = n$, and let $\{\mathcal{C}_i\}_{i\in I}$ be (large) categories (Definition A.0.1), together with another category $\mathcal{D}$. An *n-ary functor* (also called a *multivariable functor*) from the categories $\{\mathcal{C}_i\}_{i\in I}$ to $\mathcal{D}$ is a functor (Definition A.0.2)

$$F : \prod_{i\in I} \mathcal{C}_i \to \mathcal{D}.$$

That is, F assigns:

- to each object $((A_i)_{i\in I})$ in $\prod_{i\in I} \mathcal{C}_i$, an object $F((A_i)_{i\in I})$ in $\mathcal{D}$,
- to each morphism $((f_i)_{i\in I}) : (A_i)_i \to (B_i)_i$, a morphism $F((f_i)_i) : F((A_i)_i) \to F((B_i)_i)$ in $\mathcal{D}$,

so that F preserves identities and composition componentwise. For instance, a *bifunctor* is an n-ary functor when $n = 2$, a *ternary functor/trifunctor* is an n-ary functor when $n = 3$, etc.

Definition A.0.20 (Localization by a multiplicative system)**.** Given a category $\mathcal{C}$ and a multiplicative system $S \subseteq \mathrm{Mor}(\mathcal{C})$, the *localization* $\mathcal{C}[S^{-1}]$ is a category equipped with a functor

$$Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$$

that sends every morphism in S to an isomorphism satisfying the universal property: any functor from $\mathcal{C}$ sending the morphisms in S to isomorphisms factors uniquely through Q.

Proposition A.0.21. Let $\mathcal{C}$ be an essentially small category and let $\mathcal{A}$ be an abelian category.

1. Let $U \in \mathrm{Ob}(\mathcal{C})$ be some fixed object. The sections functor
$$\Gamma(U, -) : \mathrm{PSh}(\mathcal{C}, \mathcal{A}) \to \mathcal{A}$$
is left exact. (♠ TODO: state that presheaves and sheaves valued in an abelian category form abelian categories)
2. Assume that $\Gamma(\mathcal{F})$ exists for all $\mathcal{F}$ in $\mathrm{PSh}(\mathcal{C}, \mathcal{A})$ so that Γ is a functor
$$\Gamma : \mathrm{PSh}(\mathcal{C}, \mathcal{A}) \to \mathcal{A}.$$
The functor Γ is left exact.

Proof. Recall that $\mathrm{PSh}(\mathcal{C}, \mathcal{D})$ is an abelian category (**??**) since $\mathcal{C}$ is essentially small. (♠ TODO: Talk about how limits are left exact and) □

Theorem A.0.22 (e.g. see [Sta25, Tag 01DU])**.** For any site (Definition A.0.9) $(\mathcal{C}, J)$ on an essentially small category $\mathcal{C}$ and a sheaf of rings (Definition A.0.10) $\mathcal{O}$ on $\mathcal{C}$, the category $\mathbf{Mod}(\mathcal{O})$ of $\mathcal{O}$-modules is an abelian category that has enough injectives. In fact, there is a functorial injective embedding (♠ TODO: what does this mean?)

Definition A.0.23. Let $(\mathcal{C}, J)$ be a site (Definition A.0.9) on a locally small category (Definition A.0.6) or a U-site for some universe U. Let $\mathcal{O}$ be a sheaf of rings (Definition A.0.10) on $\mathcal{C}$, so that $(\mathcal{C}, J, \mathcal{O})$ is a ringed site. Recall that the category $\mathbf{Mod}(\mathcal{O})$ of $\mathcal{O}$-modules is abelian and has enough injectives (Theorem A.0.22).

Assume that global sections objects $\Gamma(\mathcal{G})$ exist for all objects $\mathcal{G}$ of $\mathrm{Sh}(\mathcal{C}, \mathbf{Ab})$[1] so that Γ is a functor
$$\mathrm{Sh}(\mathcal{C}, \mathbf{Ab}) \to \mathbf{Ab},$$
which is a left exact functor (Proposition A.0.21). Note that Γ restricts to a left exact functor
$$\mathbf{Mod}(\mathcal{O}) \to \mathbf{Ab}.$$
If $\mathcal{C}$ has a final object (Definition A.0.4) $*$ as well, then recall that $\Gamma(\mathcal{F}) = \mathcal{F}(*)$.

Let $\mathcal{F}$ be an object of $\mathbf{Mod}(\mathcal{O})$. For each integer $n \geq 0$, the *n-th (abelian) sheaf cohomology group of $\mathcal{F}$* is
$$H^n(\mathcal{C}, J; \mathcal{F}) := R^n\Gamma(\mathcal{F}),$$
where $R^n\Gamma$ is the n-th right derived functor of the global sections functor Γ.

In particular, each H^n is a functor
$$H^n : \mathbf{Mod}(\mathcal{O}) \to \mathbf{Ab}.$$

Theorem A.0.24. cf. [GV72, Exposé II, Théorème 3.4]

[1] for example, this occurs when $\mathcal{C}$ is essentially small

1. Let U be a universe. Let $\mathcal{C}$ be a U-site (Definition A.0.9). A sheafification functor
$$a : \mathrm{Shv}(\mathcal{C}, U{-}\mathbf{Sets}) \to \mathrm{PreShv}(\mathcal{C}, U{-}\mathbf{Sets}).$$
exists.
(♠ TODO: topologically generating family)
2. Let $\mathcal{C}$ be a site whose underlying category is locally small and which has a topologically generating family that is a set (rather than a proper class). A sheafification functor
$$a : \mathrm{Shv}(\mathcal{C}, \mathbf{Sets}) \to \mathrm{PreShv}(\mathcal{C}, \mathbf{Sets})$$
exists.
3. (see e.g. [nLa25c, 3]) Let $(\mathcal{C}, J)$ be a site (Definition A.0.9) on an essentially small category $\mathcal{C}$. Suppose that the category $\mathcal{A}$ is complete, cocomplete, that small filtered colimits (Definition D.0.12) in $\mathcal{A}$ are exact, and that $\mathcal{A}$ satisfies the IPC-property. A sheafification functor (Definition A.0.11)
$$a : \mathrm{PreShv}(\mathcal{C}, \mathcal{A}) \to \mathrm{Shv}(\mathcal{C}, \mathcal{A})$$
exists.
This is true for instance of $\mathcal{A} = \mathbf{Set}, \mathbf{Grp}$, $k - \mathbf{Alg}$ for a field k, or $\mathbf{Mod}_R$ for a (not necessarily commutative unital) ring R.

Remark A.0.25. If the presheaf is valued in nice "algebraic category", e.g. groups, abelian groups, rings, modules over a ring, etc., then the sheafification is again valued in that category. (♠ TODO: Make this more precise.)

Definition A.0.26 (Constant sheaf on a site)**.** Let $\mathcal{C}$ be a (large) category (Definition A.0.1), let $\mathcal{A}$ be a (large category), and let A be an object of $\mathcal{A}$.

1. The *constant presheaf on $\mathcal{C}$ with value A* is the presheaf (Definition A.0.8) P defined by
$$P(U) = A$$
for every object U of $\mathcal{C}$ such that every morphism $f : V \to U$ in $\mathcal{C}$ induces the identity map $A = P(U) \to P(V) = A$.
2. Let $\mathcal{C}$ be a site (Definition A.0.9) and assume that a sheafification functor (Definition A.0.11)
$$a : \mathrm{Shv}(\mathcal{C}, \mathcal{A}) \to \mathrm{PreShv}(\mathcal{C}, \mathcal{A})$$
exists (e.g. see Theorem A.0.24) . The *constant sheaf on $\mathcal{C}$ with value A*, or the *constant sheaf on $\mathcal{C}$ asscciated to A* commonly denoted $\underline{A}$ or sometimes just A by abuse of notation, is the sheaf associated to (Theorem A.0.24) the constant presheaf P with value A above.
3. Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of (not-necessarily commutative) rings on $\mathcal{C}$. Assume that the global sections ring $\Gamma(\mathcal{O})$ exists. A *constant $\mathcal{O}$-module* is an $\mathcal{O}$-module $\mathcal{F}$ which is isomorphic as a sheaf to the constant sheaf on $\mathcal{C}$ with value M where M is a module of the ring $\Gamma(\mathcal{O})$. Note that sheafification functors exist for presheaves/sheaves valued in Ab (Theorem A.0.24).
In case that $\mathcal{O}$ is the constant sheaf associated to A for some (not-necessarily commutative) ring A, a constant $\mathcal{O}$-module is simply called a *constant A-module*.

Definition A.0.27 (Nonabelian Sheaf Cohomology: H^0 and H^1)**.** Let $(\mathcal{C}, J)$ be a site (Definition A.0.9) and $\mathcal{G}$ a sheaf (Definition A.0.10) of groups on $(\mathcal{C}, J)$. For $U \in \mathcal{C}$, we define the *0th and 1st (nonabelian) sheaf cohomology sets* as follows:

- The zeroth sheaf cohomology group $H^0(\mathcal{C}, J, \mathcal{G}) := \Gamma(\mathcal{G})$ is the group of global sections of $\mathcal{G}$, assuming that it exists (which is always the case when $\mathcal{C}$ is essentially small for example since the category of groups is closed under small projective limits (♠ TODO:)).
- The first sheaf cohomology set $H^1(\mathcal{C}, J, \mathcal{G})$ is the pointed set of isomorphism classes of $\mathcal{G}$-torsors on the site $(\mathcal{C}, J)$.

Definition A.0.28 (Slice site)**.** Let $(\mathcal{C}, \tau)$ be a site (Definition A.0.9), where τ is a Grothendieck topology on the (locally small or U-locally small (Definition A.0.6), if a universe U is available) category $\mathcal{C}$. For a fixed object X in $\mathcal{C}$, the *slice site* (or the *over site* or the *site on the slice category* $\mathcal{C}_{/X}$ or the *site induced on the over category* $\mathcal{C}_{/X}$, etc.) $(\mathcal{C}_{/X}, \tau_{/X})$ is the site whose underlying category is the slice category $\mathcal{C}_{/X}$ (Definition A.0.17), and whose Grothendieck topology $\tau_{/X}$ (also denoted by notations such as $\tau|_X$) is defined by declaring a family of morphisms $\{f_i : Y_i \to Y\}$ in $\mathcal{C}_{/X}$ to be a covering if and only if the family $\{f_i : Y_i \to Y\}$ is a covering in $(\mathcal{C}, \tau)$.

Proposition A.0.29. See [Sta25, Tag 03AJ] for a statement Let $(\mathcal{C}, J)$ be a site (Definition A.0.9). Assume that global sections objects $\Gamma(\mathcal{G})$ exist for all objects $\mathcal{G}$ of $\mathrm{Sh}(\mathcal{C}, \mathbf{Ab})$[2] so that Γ is a functor

$$\mathrm{Sh}(\mathcal{C}, \mathbf{Ab}) \to \mathbf{Ab}.$$

Let $\mathcal{G}$ be a sheaf (Definition A.0.10) of abelian groups on $(\mathcal{C}, J)$. For $i = 0, 1$, there is a canonical bijection between the set of isomorphism classes of $\mathcal{G}$-torsors and the (abelian) sheaf cohomology group $H^i(\mathcal{C}, J; \mathcal{G})$ (Definition A.0.23). In other words, for $i = 0, 1$, there is a canonical bijection between the ith nonabelian sheaf cohomology (Definition A.0.27) and the ith abelian sheaf cohomology (Definition A.0.23) of $\mathcal{G}$.

Appendix B. points of topoi

We work in a fixed universe.

Definition B.0.1.

Definition B.0.2 (Topos)**.** There are a multitude of notions of topos. Here are some that we consider; more notions may be added later.

1. A *(sheaf/Grothendieck) topos* is a category (Definition A.0.1) equivalent (Definition A.0.5) to the category of sheaves (Definition A.0.10) of sets on some site (Definition A.0.9). That is, there exists a site (C, J) such that the category is equivalent to $\mathrm{Sh}(C, J)$, the category of sheaves of sets on (C, J).
2. Let U be a universe. A *U-(sheaf)topos* is a category equivalent to the category of U-sheaves (Definition A.0.10) (valued in U-sets) [GV72, Exposé IV Définition 1.1]

[2]for example, this occurs when $\mathcal{C}$ is essentially small

3. An *elementary topos* is a cateogry which has all finite limits (Definition A.0.13), is cartesian closed, and has a subobject classifier (♠ TODO: cartesian closed, subobject classifier)

Proposition B.0.3 (See [GV72, Exposé II Proposition 4.8])**.** Let $\mathscr{U}$ be a universe and let T be a $\mathscr{U}$-site. The category $\mathrm{Shv}(T)$ of sheaves on T has the following properties:

1. $\mathrm{Shv}(T)$ is closed under finite projective limits.
2. $\mathrm{Shv}(T)$ is closed under arbitrary coproducts (indexed by $\mathscr{U}$-small set).

Lemma B.0.4. Let C be a category. A final object, if it exists, of C is the projective limit of the empty diagram. In particular, any category that is closed under finite products (including finite projective limits), then it has a final object.

Proof. This follows by considering the universal property of the final object. □

Lemma B.0.5. Let $\mathscr{U}$ be a universe and let T be a $\mathscr{U}$-site. The category $\mathrm{Shv}(T)$ of sheaves on T has a final object.

Proof. This follows from Lemma B.0.4 and Proposition B.0.3. □

Definition B.0.6. [GV72, Exposé IV Définition 3.1], see [nLa25a, Definition 2.1] Let E and E' be topoi (in a universe $\mathscr{U}$). A *(continuous) morphism $f : E \to E'$* is a triple $f = (f_*, f^*, \varphi)$ consisting of functors

$$f_* : E \to E', \quad f^* : E' \to E$$

and an *adjunction isomorphism*

$$\varphi : \mathrm{Hom}_E(u^*(X'), Y) \xrightarrow{\sim} \mathrm{Hom}_{E'}(X', u_*(Y))$$

of bifunctors (Definition A.0.19) in $X' \in \mathrm{Ob}\, E'$ and $Y \in \mathrm{Ob}\, E$ such that f^* commutes with finite (projective) limits. The functors f_* and f^* are respectively called the *direct image functor of f* and the *inverse image functor of f*.

This notion of morphism of topoi is also referred to as *geometric morphism of topoi*.

Note that **Sets** is a topos by virtue of being the category of sheaves on the site consisting of a single point.

Definition B.0.7 ([nLa25b, Definition 1.1])**.** A *point of a topos E* is a (geometric) morphism $x : \mathbf{Sets} \to E$.

For an object $A \in E$, its inverse image $x^*A \in \mathbf{Sets}$ under such a point x is the *stalk of A at x of A at x*.

Definition B.0.8 ([nLa25b, Definition 1.3])**.** A topos E is said to have *enough points* if for any morphism $f : A \to B$ in E, the following are equivalent:

1. f is an isomorphism.
2. for every geometric point $p : \mathbf{Sets} \to E$, the morphism of stalks $p^*f : p^*A \to p^*B$ is an isomorphism (of sets)

A site (Definition A.0.9) T is said to have *enough points* if $\mathrm{Shv}(T)$ has enough points.

Lemma B.0.9. Let $\mathscr{U}$ be a universe. Let E be a $\mathscr{U}$-topos. Write e_E for the final object of E, which exists by Lemma B.0.5. Given a point $x : \mathbf{Sets} \to E$, (♠ TODO: TODO)

(♠ TODO: TODO: understand the relationship betewen big/small topologies)

Appendix C. Topology

Definition C.0.1 (Topology)**.** Let X be a set. A *topology on X* is a collection $\mathcal{T}$ of subsets of X such that:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
2. For any collection $\{U_i\}_{i \in I} \subseteq \mathcal{T}$ (with I arbitrary), the union $\bigcup_{i \in I} U_i \in \mathcal{T}$,
3. For any finite collection $\{U_1, \ldots, U_n\} \subseteq \mathcal{T}$, the intersection $U_1 \cap \cdots \cap U_n \in \mathcal{T}$.

If $\mathcal{T}$ is a topology on X, the pair $(X, \mathcal{T})$ is called a *topological space*. Members of $\mathcal{T}$ are called *open sets*.

A subset $C \subseteq X$ is *closed* if its complement $X \setminus C$ is an open set in $\mathcal{T}$

One very often refers to X as a topological spcae, omitting the notation of the topology $\mathcal{T}$.

The collection of all topologies on a set X may be denoted by notations such as $\mathrm{Top}(X)$, $\mathbf{Top}(X)$, or $\mathsf{Top}(X)$.

Appendix D. Miscellaneous definitions

Definition D.0.1. Let $X : \Delta^{\mathrm{op}} \to \mathbf{Sets}$ be a simplicial set. The *free simplicial abelian group* $\mathbb{Z}[X]$ (which we may also denote by $\mathbb{Z}(X)$) is the simplicial abelian group $\Delta^{\mathrm{op}} \to \mathrm{Ab}$ given by sending $[n]$ to the free abelian group $\mathbb{Z}[X_n]$ generated by the set X_n; the morphisms $[n] \to [m]$ naturally induce morphisms $X_m \to X_n$ which in turn induce natural morphisms $\mathbb{Z}[X_m] \to \mathbb{Z}[X_n]$. Note that $\mathbb{Z}[X]$ is functorial in X.

Definition D.0.2 (Scheme)**.** A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ that admits an open cover $\{U_i\}_{i \in I}$ such that each $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic (as a locally ringed space) to an affine scheme $(\mathrm{Spec}(A_i), \mathcal{O}_{\mathrm{Spec}(A_i)})$ (Definition D.0.15) for some commutative ring A_i. In other words, a scheme is a locally ringed space locally isomorphic to affine schemes.

Definition D.0.3 (Morphism of schemes)**.** Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be schemes (Definition D.0.2). A *morphism of schemes* is a morphism as locally ringed spaces.

In particular, there is a category (Definition A.0.1), often denoted by Sch, $\mathbf{Sch}$ etc., whose objects are schemes and whose morphisms are morphisms of schemes.

Definition D.0.4 (Locally Noetherian Scheme and Noetherian Scheme)**.** Let X be a scheme (Definition D.0.2).

- X is called *locally Noetherian* if it admits an open cover $\{U_i\}$ such that for each i, the ring $\mathcal{O}_X(U_i)$ of regular functions on U_i is a Noetherian ring. Equivalently, X is locally Noetherian if it is covered by open affine subschemes $\operatorname{Spec} A_i$ with each A_i a Noetherian ring.
- X is called *Noetherian* if it is locally Noetherian and quasi-compact, i.e., X can be covered by finitely many affine opens $\operatorname{Spec} A_i$ where each A_i is Noetherian.

Definition D.0.5 (Smooth Morphism of Schemes)**.** Let $f : X \to S$ be a morphism of schemes (Definition D.0.2).

We say that f is *smooth*, and that X is a *smooth scheme over S*, if it satisfies the following conditions:

(♠ TODO: residue field)

- f is locally of finite presentation: for every point $x \in X$, there exists an open neighborhood $U \subseteq X$ of x and an open neighborhood $V \subseteq S$ of $f(x)$ such that the restriction $f|_U : U \to V$ corresponds to a morphism of rings $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ that is finitely presented.
- f is flat: the induced map on local rings is flat.
- For every point $x \in X$, the fiber $X_{f(x)} = X \times_S \operatorname{Spec} \kappa(f(x))$ is a smooth variety over the residue field $\kappa(f(x))$, equivalently, the sheaf of relative Kähler differentials $\Omega_{X/S}$ is locally free of finite rank.

Informally, a smooth morphism behaves like a submersion in differential geometry, providing "nice" fiber structures and descent properties.

Given a scheme S, the *category of smooth schemes over S* is the following locally small (Definition A.0.6) category (Definition A.0.1):

- The objects are smooth morphisms $X \to S$.
- The morphisms betewen objects $X_1 \to S$ and $X_2 \to S$ are S-morphisms (Definition D.0.8) $X_1 \to X_2$ such that the following commutes:

$$\begin{tikzcd} X_1 \arrow[rr] \arrow[dr] & & X_2 \arrow[dl] \\ & S & \end{tikzcd}$$

The category of smooth schemes over S is often denoted by notations such as Sm/S, $\mathbf{Sm}/S$, Sm_S, $\mathbf{Sm}_S$ etc.

Definition D.0.6. Let X be a scheme of finite type (Definition 2.1.2) over a field k.

1. The scheme X is said to be *smooth at a point $x \in X$* of dimension (Definition D.0.9) d if the local ring $\mathcal{O}_{X,x}$ is geometrically regular, or equivalently, if the rank of the sheaf of differentials $\Omega_{X/k}$ at x is equal to the dimension of X at x (Definition D.0.9).
2. The scheme X is said to be *smooth* if it is smooth at all of its points. The scheme X is smooth if and only if its structure morphism (Definition D.0.8) $X \to \operatorname{Spec} k$ is a smooth morphism (Definition D.0.5).

(♠ TODO: go into this more deeply)

Definition D.0.7. A morphism of schemes (Definition D.0.3) $f : X \to Y$ is called *étale* if it satisfies the following conditions: (♠ TODO: sheaf of relative differentials)

- f is locally of finite presentation,
- f is flat,
- f is unramified, i.e., the sheaf of relative differentials $\Omega_{X/Y}$ equals 0.

(♠ TODO: relative dimension) Equivalently, a morphism of schemes is étale if and only if it is smooth (Definition D.0.5) of relative dimension 0. A finite étale morphism is synonymously called a *finite étale cover*.

Definition D.0.8 (Scheme over a scheme)**.** Let $(S, \mathcal{O}_S)$ be a scheme. A *scheme over S* (or an *S-scheme*) is a scheme $(X, \mathcal{O}_X)$ together with a morphism of schemes

$$\pi : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S).$$

This morphism π is called the *structure morphism of the scheme X over S*.

If $S = \mathrm{Spec}(R)$ is an affine scheme for a commutative ring R, then an S-scheme is synonymously called an *R-scheme* or a *scheme over R*.

Let $(S, \mathcal{O}_S)$ be a scheme, and let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be schemes over S with structure morphisms

$$\pi_X : X \to S, \quad \pi_Y : Y \to S.$$

A *morphism of S-schemes* (or synonymously a *S-scheme morphism*) is a morphism of schemes (Definition D.0.3)

$$(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \pi_X} \downarrow & & \downarrow {\scriptstyle \pi_Y} \\ S & = & S \end{array}$$

In other words,

$$\pi_Y \circ f = \pi_X.$$

Given a fixed scheme S, there is a category, often denoted by Sch_S, $\mathrm{Sch}_{/S}$, Sch/S, $\mathbf{Sch}_S$, $\mathbf{Sch}_{/S}$, $\mathbf{Sch}/S$ etc. whose objects are schemes T over S and whose morphisms $T_1 \to T_2$ are morphisms of schemes over S. If $S = \mathrm{Spec}\, R$ for some commutative ring R, then we may instead write Sch_R to denote $\mathrm{Sch}_{\mathrm{Spec}\, R}$, etc. It is noteworthy that $\mathrm{Sch}_{\mathbb{Z}}$ coincides with the category Sch (Definition D.0.3) of all schemes. In other words, a $\mathbb{Z}$-scheme can be identified simply with a scheme.

Equivalently, the category $\mathrm{Sch}_{/S}$ is the category of schemes over S in the sense of Definition A.0.17.

Definition D.0.9 (Dimension of a Scheme)**.** Let X be a scheme with underlying topological space $|X|$.

(♠ TODO: krull dimension)

- The *dimension at a point $x \in |X|$*, denoted $\dim_x(X)$, is the Krull dimension of the local ring $\mathcal{O}_{X,x}$. This is the supremum of the lengths n of chains of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subseteq \mathcal{O}_{X,x}.$$

- The *dimension of the scheme X* is defined as

$$\boxed{\dim(X) := \sup_{x \in |X|} \dim_x(X).}$$

  Equivalently, it is the supremum of the lengths of chains of distinct irreducible closed subsets of $|X|$ ordered by inclusion.

Definition D.0.10 (Field)**.** A *field* is commutative division ring. In other words, a field is a commutative ring for which all nonzero elements have a multiplicative inverse.

Definition D.0.11 (Filtered category)**.**

1. A *filtered category* is a (nonempty, large) category $\mathcal{I}$ satisfying the following conditions:
   - For every finite collection of objects $i_1, i_2, \ldots, i_n$ in $\mathcal{I}$, there exists an object j and morphisms

     $$\phi_k : i_k \to j, \quad \text{for each } k = 1, \ldots, n.$$

   - For every pair of morphisms $f, g : i \to j$ in $\mathcal{I}$, there exists an object k and a morphism

     $$h : j \to k$$

     (♠ TODO: equalizer) that is an equalizer of f and g, i.e. satisfies

     $$h \circ f = h \circ g.$$

$$\begin{tikzcd} i_1 \arrow[dr, dashed, "\phi_1"] & \\ & j \\ i_2 \arrow[ur, dashed, "\phi_2"'] & \end{tikzcd}$$

Figure 1. *
Condition 1: Upper Bound

$$\begin{tikzcd} i \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & j \arrow[r, dashed, "h"] & k \end{tikzcd}$$

Figure 2. *
Condition 2: Equalizer

   In other words, $\mathcal{I}$ is nonempty, any finite diagram of objects admits a cocone (Definition A.0.13), and any pair of parallel morphisms become equal after post-composition with an appropriate morphism.

2. Dually, a *Cofiltered category* is a category whose opposite category is filtered. More explicitly, A cofiltered category is a (nonempty, large) category $\mathcal{I}$ satisfying the following conditions:

- For every finite collection of objects $i_1, i_2, \dots, i_n$ in $\mathcal{I}$, there exists an object j and morphisms
$$\phi_k : j \to i_k, \quad \text{for each } k = 1, \dots, n.$$
- For every pair of morphisms $f, g : j \to i$ in $\mathcal{I}$, there exists an object k and a morphism
$$h : k \to j$$
that is a coequalizer of f and g, i.e. satisfies
$$f \circ h = g \circ h.$$

$$\begin{tikzcd}
i_1 & \\
& j \arrow[ul, dashed, "\phi_1"'] \arrow[dl, dashed, "\phi_2"] \\
i_2 &
\end{tikzcd}$$

Figure 3. *
Condition 1: Lower Bound

$$\begin{tikzcd}
k \arrow[r, dashed, "h"] & j \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & i
\end{tikzcd}$$

Figure 4. *
Condition 2: Equalizer

In other words, $\mathcal{I}$ is nonempty, any finite diagram of objects admits a cone, and any pair of parallel morphisms become equal after pre-composition with an appropriate morphism.

Definition D.0.12 (Special cases of limits)**.** Let $\mathcal{C}$ be a (large) category. Let I be a (large) category. Let $I \to \mathcal{C}$ be a diagram/system.

- Suppose that the system is a cofiltered system, i.e. I is a cofiltered category. A limit (Definition A.0.13) of this diagram is often denoted by
$$\varprojlim_{i \in I} D(i)$$
and may be called a *cofiltered (inverse/projective) limit*. In case that the system is more specifically an inverse/projective system, i.e. I is a cofiltered poset, the preferred term for such a limit is *inverse/projective limit*.
- Suppose that the system is a filtered system, i.e. I is a filtered category. A colimit of this diagram is often denoted by
$$\varinjlim_{i \in I} D(i)$$
and may be called a *filtered colimit* or a *direct/inductive/injective limit*. In case that the system is more specifically a direct/inductive system, i.e. I is a filtered poset, the preferred term for such a limit is *direct/inductive limit*.

Definition D.0.13. Let k be a field (Definition D.0.10). A *(algebraic) variety over k* is an integral, separated scheme of finite type (Definition 2.1.2) over k.

Definition D.0.14 (Field Extension)**.** Let K be a field and let L be a field such that $K \subseteq L$ and the operations of K are the restrictions of those of L. Then L is called a *extension field (or just an extension) of* K. The notation L/K is often used synonymously; we say that L/K is a *field extension*.

Definition D.0.15 (Affine scheme)**.** Let A be a commutative ring with unity. Define the set $\operatorname{Spec}(A)$ to be the set of all prime ideals of A. Equip it with the *Zariski topology*, which is the topology (Definition C.0.1) whose closed sets are given by *vanishing loci*

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec}(A) : I \subseteq \mathfrak{p}\}$$

for ideals $I \subseteq A$. Define the sheaf $\mathcal{O}_{\operatorname{Spec}(A)}$, called the *structure sheaf of* $\operatorname{Spec} A$, by

$$\mathcal{O}_{\operatorname{Spec}(A)}(U) = \{ \text{ locally defined fractions of elements of } A \text{ on } U \ \},$$

for each open set $U \subseteq \operatorname{Spec}(A)$. It is the case that the stalk at $\mathfrak{p} \in \operatorname{Spec}(A)$ is canonically the localization $A_{\mathfrak{p}}$. Then $(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)})$ is a locally ringed space, called the *affine scheme asscciated to* A.

Moreover, given $f \in A$, we define the locus $D(f)$ by

$$D(f) = \operatorname{Spec} A \setminus V((f)) = \{\mathfrak{p} \in \operatorname{Spec} A : f \notin \mathfrak{p}\}$$

Definition D.0.16 (Separable Element, Separable Extension)**.** Let L/K be a field extension (Definition D.0.14) and let $x \in L$ be algebraic over K with minimal polynomial $m_{x,K}(t) \in K[t]$.

- The element x is *separable over* K if $m_{x,K}(t)$ has distinct roots in a splitting field.
- The element x is *inseparable over* K otherwise.

An algebraic extension L/K is called *separable extension* if every element $x \in L$ is separable over K.

See also Definition D.0.17, which defines separable field extensions in greater generality.

Definition D.0.17 (Separable field extension)**.** Let E/F be a field extension (Definition D.0.14). In general (allowing transcendental extensions), E/F is called *separable* if there exists a separating transcendence basis for E over F. Equivalently, E/F is separable if every finitely generated intermediate field $F \subseteq K \subseteq E$ has a separating transcendence basis over F. In particular, a field extension E/F is algebraic and separable in the above sense if and only if it is a separable algebraic extension in the sense of Definition D.0.16.

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