ALGEBRAIC GROUPS

January 23, 2026

Contents

1. Definitions

1.1. Algebraic groups and linear algebraic groups.

Definition 1.1.1. Let S be a scheme. An algebraic group scheme over S (or an S-group scheme) is a group object G in the category of schemes over S; that is, G is an S-scheme equipped with S-morphisms: $m : G \times_S G \to G$ (*multiplication*), $i : G \to G$ (*inverse*), and $e : S \to G$ (*identity*), satisfying the group axioms expressed by the commutativity of the following diagrams:

1. **Associativity**
$$\begin{tikzcd} G \times_S G \times_S G \arrow[r, "m\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times m"'] & G \times_S G \arrow[d, "m"] \\ G \times_S G \arrow[r, "m"] & G \end{tikzcd}$$

2. **Identity**
$$\begin{tikzcd} G \times_S S \arrow[r, "\mathrm{id}\times e"] \arrow[dr, "\simeq"'] & G \times_S G \arrow[d, "m"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} S \times_S G \arrow[r, "e\times \mathrm{id}"] \arrow[dr, "\simeq"'] & G \times_S G \arrow[d, "m"] \\ & G \end{tikzcd}$$

3. **Inverse**
$$\begin{tikzcd} G \arrow[r, "(\mathrm{id},i)"] \arrow[d, "\mathrm{id}"'] & G \times_S G \arrow[d, "m"] \\ G \arrow[r, "e\circ\pi"] & G \end{tikzcd}$$
where $\pi : G \to S$ is the structure morphism and $e \circ \pi$ sends g to the identity section.

If G is affine over (Definition B.0.10) S, we call it an *affine group scheme over S*.

If the base scheme S is the spectrum of a field k, then we call G a *k-algebraic group* or an *algebraic group (scheme) over k*. If G is additionally a k-variety, then we call G a *k-group variety*.

Theorem 1.1.2 (Cartier; see [Mil17, Theorem 3.23])**.** Every affine algebraic group (Definition 1.1.1) over a field of characteristic zero is smooth.

Definition 1.1.3. Let S be a scheme, and let G be an algebraic group scheme over S. A *subgroup scheme* (or synonymously an *algebraic subgroup*) H of G is an S-subscheme

$$H \subseteq G$$

such that for every S-scheme T, the subset $H(T) \subseteq G(T)$ is a subgroup of the group $G(T)$. Equivalently, H is a group object in the category of S-schemes equipped with a monomorphism of group schemes $H \to G$.

(♠ TODO: ARe all group schemes over a characteric zero field reduced?) (♠ TODO: ARe all subgroups of group schemes over a characteric zero field closed?)

Over positive characteristic fields, not all subgroup schemes are closed:

Example 1.1.4 (A subgroup scheme that is not closed)**.** Let k be an algebraically closed field of characteristic $p > 0$. Consider the additive group scheme $\mathbf{G}_a = \mathrm{Spec}(k[t])$ over k. Define the subgroup functor H by

$$H(R) = \{x \in \mathbf{G}_a(R) \mid x^p = 0\}$$

for any k-algebra R. Then H is a subgroup scheme of $\mathbf{G}_a$ known as the *infinitesimal subgroup of order p*.

However, H is not a closed subgroup scheme of $\mathbf{G}_a$ because it corresponds to the ideal (t^p) in $k[t]$, which is not radical and hence does not define a closed subscheme.

Definition 1.1.5. Let S be a scheme, and let G and H be S-algebraic groups. A morphism of S-schemes $f : G \to H$ is a *homomorphism of algebraic groups* if f is a group homomorphism, i.e.,

- $f(m_G(x, y)) = m_H(f(x), f(y))$ for all $x, y \in G$,
- $f(i_G(x)) = i_H(f(x))$ for all $x \in G$, and

- $f(e_G) = e_H$.

It is called an *isomorphism of algebraic groups (over S)* if it is additionally an isomorphism of S-schemes. If there exists an isomorphism $f : G \to H$ of algebraic groups over S, then G and H are said to be *isomorphic S-algebraic groups*.

Definition 1.1.6 (General linear group)**.** Let S be a base scheme and $n \geq 1$ an integer. The *general linear group over S*, denoted $\mathrm{GL}_{n,S}$ or GL_n/S or simply by GL_n if the base S is clear, is the group scheme over S defined by

$$\mathrm{GL}_{n,S}(T) = \{\text{ invertible } n \times n \text{ matrices with entries in } \mathcal{O}_T(T)\,\}$$

for any S-scheme T, where $\mathcal{O}_T$ is the structure sheaf of T.

Equivalently, $\mathrm{GL}_{n,S}$ may be defined as the general linear group scheme $\mathrm{GL}(\mathscr{O}_S^{\oplus n})$ associated to the free $\mathscr{O}_S$-module of rank n (Definition 2.0.1).

Definition 1.1.7 (Linear algebraic group over a scheme)**.** Let S be a base scheme. A *linear algebraic group over S* is an affine group scheme (Definition 1.1.1) G over S that is finitely presented and smooth over S, and such that for some integer $n \geq 1$, there exists a closed immersion of S-group schemes

$$G \hookrightarrow \mathrm{GL}_{n,S}\,.$$

(Definition 1.1.6)

1.2. **Normal subgroup schemes.**

Definition 1.2.1. Let S be a scheme, let G be an algebraic group over S (Definition 1.1.1), and let $H \subseteq G$ be a closed subgroup scheme (Definition 1.1.3) over S.

1. The subgroup scheme H is called a *normal subgroup scheme of G* if for every S-scheme T, the subgroup $H(T) \subseteq G(T)$ is normalized by $G(T)$, i.e.

$$ghg^{-1} \in H(T), \quad \forall\, g \in G(T),\, h \in H(T).$$

2. The subgroup scheme H is called a *characteristic subgroup scheme of G* if for every S-scheme T and automorphism $\varphi : G_T \to G_T$ of T-group schemes, one has

$$\varphi(H_T) = H_T.$$

In particular, all characteristic subgroup schemes are normal subgroup schemes.

Definition 1.2.2.
1. Let k be a field, and let G be a group scheme (Definition 1.1.1) over $\mathrm{Spec}(k)$ (Definition B.0.1). The *identity component or neutral component of G*, denoted G^0, is the unique connected component of G containing the identity point $e \in G(k)$.
2. Let S be a scheme, and let G be an algebraic group scheme over S. The *identity component of G*, denoted G^0, is the unique open and closed subgroup scheme of G such that for every geometric point $\bar{s} \to S$, the fiber $(G^0)_{\bar{s}}$ is the connected component of the identity element in the group scheme $G_{\bar{s}}$.

   Note that G^0/S might not be connected if S is not connected

Lemma 1.2.3 (cf. [Mil17, Proposition 1.52])**.** (♠ TODO: Does this hold for more general groups G/S?) Let G/k be an algebraic group (Definition 1.1.1) over a field. The identity component G^0 (Definition 1.2.2) is a characteristic subgroup (Definition 1.2.1) and hence a normal subgroup of G.

Definition 1.2.4 (Finite index subgroup scheme of an algebraic group scheme over a scheme)**.** Let S be a scheme and G an algebraic group scheme over S (Definition 1.1.1). A (not necessarily normal) closed subgroup scheme $H \subseteq G$ (i.e., a monomorphism of group schemes $H \to G$ over S) is called a *finite index subgroup scheme* if the fppf (or equivalently, the fpqc) quotient sheaf G/H (Definition 1.2.6) is representable (Definition B.0.6) by a finite S-scheme.

Equivalently, H is of finite index in G if H is a closed subgroup scheme and the morphism $G \to G/H$ exhibits G as a finite fppf cover of G/H.

Lemma 1.2.5 (see e.g. [Mil17, Between Definitions 6.5 and 6.6])**.** Let G/k be an algebraic group scheme (Definition 1.1.1) and let H be a subgroup scheme (Definition 1.1.3).

1. The subgroup H has finite index (Definition 1.2.4) if and only if $\dim H = \dim G$.
2. If G is smooth, then H has finite index if and only if H contains G^0 (Definition 1.2.2).

Definition 1.2.6 (Quotient of an algebraic group scheme by an algebraic subgroup)**.** Let S be a scheme. Let G be an algebraic group scheme over S (Definition 1.1.1), and let $H \subseteq G$ be a closed subgroup scheme over S, i.e., a monomorphism of group schemes $H \to G$ over S.

The *quotient sheaf* G/H is the fppf sheafification of the presheaf

$$T \mapsto G(T)/H(T)$$

on the category of S-schemes, where T is any S-scheme.

If G/H is representable by a scheme (Definition B.0.7) (or by an algebraic space (Definition B.0.8)) Q over S, then Q is called the *quotient of G by H*. In general, if H is normal (Definition 1.2.1), then G/H inherits the structure of a group scheme over S.

Otherwise, G/H is just a sheaf of sets (or algebraic space on $(\mathrm{Sch}/S)_{fppf}$ (Definition B.0.9)) over S without a natural group structure.

Proposition 1.2.7 (e.g. [Mil17, Proposition 5.23])**.** Let G be an algebraic group (Definition 1.1.1) over a field and let H be an algebraic subgroup such that the quotient G/H (Definition 1.2.6) is representable by a scheme (Definition B.0.7). We have $\dim G = \dim H + \dim G/H$.

Definition 1.2.8 (Strongly connected algebraic group scheme over a scheme)**.** Let S be a scheme. A group scheme G over S (Definition 1.1.1) is called a *strongly connected algebraic group scheme* if it has no proper algebraic gsubgroup over S of finite index.

Definition 1.2.9. Let S be a scheme and G an algebraic group scheme over S (Definition 1.1.1). The *strong identity component* G^{so} of G is the intersection of the algebraic subgroups over S of finite index.

Proposition 1.2.10 (see [Mil17, Between Definition 6.9 and Proposition 6.10])**.** Let G/k be a smooth algebraic group scheme over a field. The strong identity component G^{so} (Definition 1.2.9) coincides with the identity component G^0 (Definition 1.2.2).

1.3. **Major examples of linear algebraic groups.**

Definition 1.3.1 (Special linear group)**.** Let S be a base scheme and $n \geq 1$ an integer. The *special linear group* $\mathrm{SL}_{n,S}$ is the subgroup scheme of $\mathrm{GL}_{n,S}$ (Definition 1.1.6) defined by the condition

$$\mathrm{SL}_{n,S}(T) = \{\, g \in \mathrm{GL}_{n,S}(T) : \det(g) = 1 \,\}.$$

Equivalently, a linear algebraic group over S may be regarded as an affine group scheme over S equipped with a representation on (Definition 2.0.2) the free $\mathscr{O}_S$-module of rank n.

Definition 1.3.2 (Orthogonal group)**.** (♠ TODO: define quadratic form) Let S be a base scheme, $n \geq 1$ an integer, and let q be a quadratic form on the rank n free $\mathcal{O}_S$-module $\mathcal{O}_S^n$, i.e., a global section

$$q : \mathcal{O}_S^n \to \mathcal{O}_S,$$

satisfying the usual properties of a quadratic form. The *orthogonal group* $\mathrm{O}(q)$ over S is the subgroup scheme of $\mathrm{GL}_{n,S}$ (Definition 1.1.6) given by

$$\mathrm{O}(q)(T) = \{\, g \in \mathrm{GL}_{n,S}(T) : q_T(g(v)) = q_T(v) \text{ for all } v \in \mathcal{O}_T^n \,\},$$

where q_T is the pullback of q to T.

Definition 1.3.3 (Special orthogonal group)**.** (♠ TODO: quadratic form) Let S be a base scheme, $n \geq 1$ an integer, and let q be a quadratic form on the rank n free $\mathcal{O}_S$-module $\mathcal{O}_S^n$, i.e., a global section

$$q : \mathcal{O}_S^n \to \mathcal{O}_S,$$

satisfying the usual properties of a quadratic form. The *special orthogonal group* $\mathrm{SO}(q)$ is the subgroup scheme of $\mathrm{O}(q)$ consisting of elements with determinant 1:

$$\mathrm{SO}(q)(T) = \{\, g \in \mathrm{O}(q)(T) : \det(g) = 1 \,\}.$$

(♠ TODO: check the definition of unitary group)

Definition 1.3.4 (Unitary group)**.** Let S be a base scheme (Definition B.0.2). Suppose A is an $\mathcal{O}_S$-algebra (Definition B.0.18) equipped with an involution $* : A \to A$, and let h be a hermitian form (Definition A.0.25) on a locally free (Definition B.0.19) A-module M of finite rank over S. The *unitary group over S* is the subgroup scheme $\mathrm{U}(h)$ of (♠ TODO: general linear group of sheaf algebra) $\mathrm{GL}_A(M)$ defined by

$$\mathrm{U}(h)(T) = \{\, g \in \mathrm{GL}_A(M)(T) : h_T(g(x), g(y)) = h_T(x, y) \text{ for all } x, y \in M_T \,\},$$

where h_T and M_T are the base changes of h and M to T.

Definition 1.3.5 (Special unitary group)**.** Let S be a base scheme (Definition B.0.2). Suppose A is an $\mathcal{O}_S$-algebra (Definition B.0.18) equipped with an involution $* : A \to A$, and let h be a hermitian form (Definition A.0.25) on a locally free (Definition B.0.19) A-module M of finite rank over S. The *special unitary group* $\mathrm{SU}(h)$ is the subgroup scheme of $\mathrm{U}(h)$ (Definition 1.3.4) consisting of elements of reduced norm 1, i.e., (♠ TODO: reduced norm)

$$\mathrm{SU}(h)(T) = \{\, g \in \mathrm{U}(h)(T) : \mathrm{Nrd}(g) = 1 \,\},$$

where Nrd is the reduced norm.

Definition 1.3.6 (Multiplicative group scheme)**.** Let S be a scheme (Definition B.0.2). The *multiplicative group scheme over S*, denoted $\mathbb{G}_{m,S}$, is the group scheme over S defined as the open subscheme

$$\mathbb{G}_{m,S} := \mathbb{A}^1_S \setminus \{0\}_S$$

of the affine line (Definition B.0.28) $\mathbb{A}^1_S$, equipped with the group law given by multiplication of functions:

$$m : \mathbb{G}_{m,S} \times_S \mathbb{G}_{m,S} \to \mathbb{G}_{m,S}, \quad (x, y) \mapsto xy.$$

The identity section is the morphism

$$e : S \to \mathbb{G}_{m,S}, \quad s \mapsto 1,$$

and the inversion morphism is given by

$$i : \mathbb{G}_{m,S} \to \mathbb{G}_{m,S}, \quad x \mapsto x^{-1}.$$

1.4. Unipotent groups, reductive groups, tori, and Borel subgroups.

Definition 1.4.1 (Unipotent element in a linear algebraic group)**.** Let k be a field, and let $G \subseteq \mathrm{GL}_n(k)$ be a linear algebraic group given by a faithful representation. An element $g \in G(k)$ is called a *unipotent element* if its image in $\mathrm{GL}_n(k)$ is a unipotent matrix, i.e. if all eigenvalues of g in an algebraic closure $\overline{k}$ of k are equal to 1. Equivalently, g is unipotent if $g - I_n$ is nilpotent.

Definition 1.4.2. Let R be a ring. The group of $n \times n$ square matrices over R that are upper triangular and whose diagonal entries are all 1 is called the *unitriangular group* or the *group of unit upper triangular matrices*. It is often denoted by U_n[1] or UT_n.

Definition 1.4.3. Let S be a scheme and let $n \geq 1$ be an integer. The *linear algebraic group of unit upper triangular $n \times n$ matrices over S* (Definition 1.1.7) is the subgroup scheme $U_{n,S} = UT_{n,S}$ (or $U_n/S = UT_n/S$ or $U_n = UT_n$ if the base S is clear) of $\mathrm{GL}_{n,S}$ (Definition 1.1.6) defined by

$$UT_{n,S}(T) = \{A \in \mathrm{GL}_{n,S}(T) : A \text{ is upper triangular with diagonal entries } 1\}$$

for any S-scheme T.

Definition 1.4.4 (Unipotent group)**.**
1. An algebraic group scheme (Definition 1.1.1) U/k over a field is said to be *unipotent* if the following equivalent conditions hold: (♠ TODO: state some of these conditions more precisely)
   - every nonzero representation of the group has a nonzero fixed vector.
   - there exists a faithful representation (Definition 2.0.3) $U \hookrightarrow \mathrm{GL}_n$ whose image consists entirely of unipotent matrices (Definition 1.4.1).
   - U is isomorphic (Definition 1.1.5) to a closed subgroup scheme of $UT_n/\operatorname{Spec} k$ for some integer $n \geq 1$.
   - U is a linear algebraic group (Definition 1.1.7) and for any k-algebra R, the matrix group $U(R)$ consists entirely of unipotent matrices.

[1]Note that this notation can be confused for or conflicts with notation for the unitary group (Definition 1.3.4).

2. Let U be an algebraic group scheme over a base scheme S. We say that U is *unipotent* if it satisfies one of the following equivalent conditions:
   - For every geometric point $s \in S$, the fiber U_s is a unipotent algebraic group over the residue field $\kappa(s)$.
   - There exists a faithful representation (Definition 2.0.3)
   $$\rho : U \hookrightarrow \mathrm{GL}_n$$
   over S such that for every S-algebra R, the image $\rho(U(R)) \subseteq \mathrm{GL}_n(R)$ consists entirely of unipotent matrices (i.e., matrices whose eigenvalues are all equal to 1).
   - U admits a finite central composition series
   $$1 = U_0 \triangleleft U_1 \triangleleft \cdots \triangleleft U_r = U,$$
   where each successive quotient U_i/U_{i-1} is isomorphic to a subgroup scheme of the additive group scheme $\mathbb{G}_{a,S}$.
   - U is isomorphic to a closed subgroup scheme of the group scheme UT_n of upper unitriangular $n \times n$ matrices over S for some integer n.

Example 1.4.5. Let k be a field.

(1) Characteristic zero case: Consider the additive group scheme $\mathbb{G}_a$ over k. Over a field of characteristic zero, $\mathbb{G}_a$ is a unipotent group scheme.

(2) Positive characteristic case: For k a field of characteristic $p > 0$, consider the finite group scheme α_p defined as the kernel of the Frobenius morphism on $\mathbb{G}_a$. Then α_p is a unipotent group scheme over k.

These examples illustrate fundamental instances of unipotent group schemes in different characteristic settings.

Definition 1.4.6. (♠ TODO: define smooth scheme, finitely presented, geometric fibers) Let G be a group scheme (Definition 1.1.1) that is smooth, finitely presented, with connected geometric fibers over a scheme S. The *unipotent radical of G*, denoted $\mathrm{Rad}_u(G)$, is the largest smooth, finitely presented, normal (Definition 1.2.1), unipotent subgroup of G with connected geometric fibers.

(♠ TODO: show why the unipotent radical exists)

Definition 1.4.7 (Reductive group)**.** Let G be a linear algebraic group (Definition 1.1.7) that is smooth, finitely presented, and with connected geometric fibers over a scheme S.

- In the case that S is $\operatorname{Spec} k$ for a field k, the group G is called *reductive* if $\mathrm{Rad}_u(G)$ is trivial, i.e. $\mathrm{Rad}_u(G) = \{e\}$, and G is smooth, connected, and affine over S.
- For general S, the group G is called *reductive* if it is a smooth affine group scheme G/S whose geometric fibers are connected reductive algebraic groups.

Lemma 1.4.8. Let G/k be a reductive group (Definition 1.4.7) over a field. Any normal subgroup scheme (Definition 1.2.1) of G over k is a reductive group.

Proof. Let N be a normal subgroup scheme of G. The unipotent radical $\mathrm{Rad}_u(N)$ (Lemma 1.4.9) is a characteristic subgroup of N (Lemma 1.4.9) and hence it is a normal subgroup of G. In particular, $\mathrm{Rad}_u(N) \leq \mathrm{Rad}_u(G) = 1$ and hence $\mathrm{Rad}_u(N)$ is trivial, i.e. N is reductive. □

Lemma 1.4.9. Let G/k be a smooth, finitely presented scheme over a field. The unipotent radical $\mathrm{Rad}_u(G)$ (Definition 1.4.6) is a characteristic subgroup (Definition 1.2.1) of G.

Proof. For any k-scheme T, given a T-automorphism $\varphi : G \times_k T \to G \times_k T$, (♠ TODO: continue;) □

Theorem 1.4.10. (♠ TODO: define the characteristic of a ring) The category of finite-dimensional representations of a reductive group (Definition 1.4.7) over a field of characteristic 0 is semisimple (Definition B.0.5)

Definition 1.4.11 (Borel subgroup scheme)**.**

1. Let G be a connected reductive group over an algebraically closed field k. A *Borel subgroup* $B \subseteq G$ is a maximal connected closed solvable subgroup of G. Equivalently, a Borel subgroup is a minimal parabolic subgroup of G.
2. Let $G \to S$ be a connected reductive group scheme (Definition 1.4.7) over a base scheme S. A *Borel subgroup scheme of G* is a closed subgroup scheme $B \subseteq G$ such that:
   (a) $B \to S$ is smooth and affine,
   (b) for every geometric point $s \to S$, the fiber $B_s \subseteq G_s$ is connected and is a Borel subgroup of the connected reductive algebraic group G_s.

One also says *maximal Borel subgroup* to emphasize the maximality.

Proposition 1.4.12. Let G be a connected reductive group (Definition 1.4.7) over an algebraically closed field k. For any two Borel subgroups $B_1, B_2 \subseteq G$, there exists some $g \in G(k)$ such that $B_2 = gB_1g^{-1}$.

In certain contexts, maximal compact subgroups and maximal tori of reductive linear algebraic groups may be of interest; in nice enough cases, maximal compact subgroups of reductive groups are unique up to conjugation.

Proposition 1.4.13.

1. Let G be a connected reductive group (Definition 1.4.7) over $k = \mathbb{R}$ or $\mathbb{C}$. For any two maximal compact subgropus $K_1, K_2 \subseteq G$, there exists some $g \in G(k)$ such that $K_2 = gK_1g^{-1}$.
   (♠ TODO: define a p-adic field)
2. Let F be a p-adic field and let $\mathcal{O}_F$ be its ring of integers. For any maximal compact subgroup $K \subseteq \mathrm{GL}_n/F$, there exists some $g \in \mathrm{GL}_n(F)$ such that $\mathrm{GL}_n(\mathcal{O}_F) = gKg^{-1}$.

Proposition 1.4.14. Let G be a connected reductive group (Definition 1.4.7) over an algebraically closed field k. for any two maximal tori $T_1, T_2 \subseteq G$, there exists some $g \in G(k)$ such that $T_2 = gT_1g^{-1}$.

1.4.1. *Examples.*

Example 1.4.15 (Examples of reductive linear algebraic groups)**.** Some standard examples of reductive linear algebraic groups over fields include:

1. The general linear group GL_n (Definition 1.1.6), the group of invertible $n \times n$ matrices.
2. The special linear group SL_n (Definition 1.3.1), consisting of matrices with determinant 1, which is connected and semisimple.
3. The multiplicative group $\mathbb{G}_m \cong \mathrm{GL}_1$ (Definition 1.3.6), a one-dimensional torus.
4. Products of multiplicative groups, i.e., *algebraic tori* $\mathbb{G}_m^r$.
5. The special orthogonal groups SO_n (Definition 1.3.3), preserving nondegenerate quadratic forms (connected for $n \geq 3$). (♠ TODO: define the symplectic groups)
6. The *symplectic groups* Sp_{2n}, preserving nondegenerate alternating bilinear forms.

Non-examples include the additive group $\mathbb{G}_a$, since it is unipotent, and Borel subgroups that have nontrivial unipotent radical.

Example 1.4.16 (Borel subgroups of standard reductive groups)**.** For each of the following reductive groups over an algebraically closed field k, a Borel subgroup can be described as follows:

1. For the *general linear group* $\mathrm{GL}_n(k)$, a Borel subgroup is the subgroup of *invertible upper triangular matrices.*
2. For the *special linear group* $\mathrm{SL}_n(k)$, the Borel subgroup is given by the subgroup of upper triangular matrices with determinant 1.
3. For the *multiplicative group* $\mathbb{G}_m$, the group itself is a torus and hence a Borel subgroup.
4. For products of multiplicative groups $\mathbb{G}_m^r$, Borel subgroups are the groups themselves since they are tori.
5. For the *special orthogonal group* $\mathrm{SO}_n(k)$, a Borel subgroup can be realized as the stabilizer of a suitable isotropic flag of subspaces, often realized by certain block upper-triangular matrices preserving the quadratic form.
6. For the *symplectic group* $\mathrm{Sp}_{2n}(k)$, a Borel subgroup is typically the subgroup preserving a full isotropic flag with respect to the symplectic form, formed by upper-triangular block matrices in a suitable basis.

Example 1.4.17 (Maximal compact subgroups of standard reductive groups)**.** Some standard examples of maximal compact subgroups of reductive linear algebraic groups over $\mathbb{R}$ and $\mathbb{C}$ include:

1. **General linear group** GL_n:
   - Over $\mathbb{R}$, a maximal compact subgroup is the orthogonal group $\mathrm{O}(n)$.
   - Over $\mathbb{C}$, the maximal compact subgroup is the unitary group $\mathrm{U}(n)$.
2. **Special linear group** SL_n:
   - Over $\mathbb{R}$, a maximal compact subgroup is the special orthogonal group $\mathrm{SO}(n)$.
   - Over $\mathbb{C}$, the maximal compact subgroup is the special unitary group $\mathrm{SU}(n)$.
3. **Multiplicative group** $\mathbb{G}_m \cong \mathrm{GL}_1$:
   - Over $\mathbb{R}$, the maximal compact subgroup is $\{\pm 1\}$, the unit circle in $\mathbb{R}^*$.
   - Over $\mathbb{C}$, the maximal compact subgroup is the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.
4. **Algebraic tori** $\mathbb{G}_m^r$:
   - Over $\mathbb{R}$, a product of copies of $\{\pm 1\}$ and S^1.
   - Over $\mathbb{C}$, the maximal compact subgroup is $(S^1)^r$.
5. **Special orthogonal groups** SO_n:

- Over $\mathbb{R}$, $\mathrm{SO}_n(\mathbb{R})$ itself is compact (for $n \geq 2$) and forms its own maximal compact subgroup.
- Over $\mathbb{C}$, the maximal compact subgroup corresponds to the compact real form $\mathrm{SO}_n(\mathbb{R})$.

6. **Symplectic groups** Sp_{2n}:
   - Over $\mathbb{R}$, the maximal compact subgroup is $\mathrm{Sp}(n)$, the compact symplectic group (quaternionic unitary group).
   - Over $\mathbb{C}$, the maximal compact subgroup corresponds to the compact real form isomorphic to $\mathrm{Sp}(n)$.

2. Representations of linear algebraic groups

Definition 2.0.1. (♠ TODO: define locally free module on a scheme) Let S be a scheme (Definition B.0.2) and let $\mathcal{E}$ be a locally free $\mathcal{O}_S$-module of finite rank n. The *general linear group scheme assocciated to $\mathcal{E}$*, denoted $\mathrm{GL}(\mathcal{E})$, is the affine group scheme over S representing the functor

$$\mathrm{GL}(\mathcal{E})(T) = \mathrm{Aut}_{\mathcal{O}_T}(\mathcal{E}_T)$$

for every S-scheme T, where $\mathcal{E}_T := \mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{O}_T$ and $\mathrm{Aut}_{\mathcal{O}_T}(\mathcal{E}_T)$ denotes the group of $\mathcal{O}_T$-linear automorphisms of $\mathcal{E}_T$.

Definition 2.0.2. Let S be a scheme (Definition B.0.2), let G be an affine group scheme (Definition 1.1.1) over S, and let $\mathcal{E}$ be a locally free $\mathcal{O}_S$-module of finite rank. A *representation of G on $\mathcal{E}$* is a morphism of S-group schemes (Definition 1.1.5)

$$\rho : G \longrightarrow \mathrm{GL}(\mathcal{E}),$$

where $\mathrm{GL}(\mathcal{E})$ is the general linear group scheme associated to $\mathcal{E}$ (Definition 2.0.1).

In the case that $S = \operatorname{Spec} k$ for a field k, note that a representation of G is necessarily a morphism of k-group schemes

$$\rho : G \to \mathrm{GL}_{n,\operatorname{Spec} k}$$

(Definition 1.1.6) for some $n \geq 1$.

Definition 2.0.3. Let S be a scheme, and let G be an affine group scheme (Definition 1.1.1) over S. Let $\mathcal{E}$ be a locally free $\mathcal{O}_S$-module of finite rank. Let

$$\rho : G \longrightarrow \mathrm{GL}(\mathcal{E}),$$

(♠ TODO: define monomorphism of group schemes) be a representation of G (Definition 2.0.2). Such a representation ρ is called *faithful* if ρ is a monomorphism of group schemes, i.e., if for every S-scheme T, the induced group homomorphism

$$\rho_T : G(T) \longrightarrow \mathrm{GL}(\mathcal{E})(T)$$

is injective.

(♠ TODO: groupoid object)

Definition 3.0.1. A *groupoid* can be defined equivalently in categorical or set-theoretic terms:

1. **Categorical Definition**: A groupoid is a small category (Definition B.0.16) $\mathcal{G}$ in which every morphism is an isomorphism. That is, for every morphism $f : x \to y$ in $\mathcal{G}$, there exists a morphism $g : y \to x$ such that $g \circ f = \mathrm{id}_x$ and $f \circ g = \mathrm{id}_y$.
2. **Set-Theoretic Definition**: A groupoid consists of a pair of sets (G_0, G_1), called the *set of objects* and the *set of arrows* respectively, equipped with the following structure maps:
   - *Source* and *Target*: $s, t : G_1 \to G_0$,
   - *Identity*: $e : G_0 \to G_1$, assigning to each object $x \in G_0$ an identity arrow $e(x)$,
   - *Composition*: A partial map $m : G_1 \times_{s,G_0,t} G_1 \to G_1$, defined on the set of composable pairs
   $$G_1 \times_{s,G_0,t} G_1 := \{(g, h) \in G_1 \times G_1 \mid s(g) = t(h)\}$$
   and denoted by $m(g, h) = g \circ h$,
   - *Inverse*: $i : G_1 \to G_1$, denoted by $i(g) = g^{-1}$.

   These structure maps must satisfy the following axioms for all $g, h, k \in G_1$ and $x \in G_0$ where the operations are defined:
   (a) **Source and Target Compatibility**:
   $$s(g \circ h) = s(h), \quad t(g \circ h) = t(g).$$
   (b) **Associativity**: If $s(g) = t(h)$ and $s(h) = t(k)$, then
   $$(g \circ h) \circ k = g \circ (h \circ k).$$
   (c) **Identity**:
   $$s(e(x)) = x, \quad t(e(x)) = x,$$
   $$g \circ e(s(g)) = g, \quad e(t(g)) \circ g = g.$$
   (d) **Inverse**:
   $$s(g^{-1}) = t(g), \quad t(g^{-1}) = s(g),$$
   $$g \circ g^{-1} = e(t(g)), \quad g^{-1} \circ g = e(s(g)).$$

Definition 3.0.2. Let $\mathcal{C}$ be a (large) category (Definition B.0.15).

A *groupoid object in $\mathcal{C}$* consists of two objects X_0 (the "object of objects") and X_1 (the "object of morphisms"), together with five structure morphisms:

- *Source* and *Target*: $s, t : X_1 \to X_0$, such that the fiber product (Definition B.0.14) $X_1 \times_{s,X_0,t} X_1$ of the morphisms s and t exists in $\mathcal{C}$,
- *Identity*: $e : X_0 \to X_1$,

- *Composition*: $m : X_1 \times_{s,X_0,t} X_1 \to X_1$,
- *Inverse*: $i : X_1 \to X_1$,

such that the following conditions hold (expressing the axioms of a category where every morphism is invertible):

1. **Source/Target identities**:
$$s \circ e = \mathrm{id}_{X_0}, \quad t \circ e = \mathrm{id}_{X_0}$$
$$s \circ m = s \circ \pi_2, \quad t \circ m = t \circ \pi_1$$
2. **Associativity**: The following diagram of composition commutes:
$$m \circ (m \times \mathrm{id}_{X_1}) = m \circ (\mathrm{id}_{X_1} \times m)$$
3. **Unitality**:
$$m \circ (e \circ s, \mathrm{id}_{X_1}) = \mathrm{id}_{X_1}, \quad m \circ (\mathrm{id}_{X_1}, e \circ t) = \mathrm{id}_{X_1}$$
4. **Invertibility**:
$$m \circ (i, \mathrm{id}_{X_1}) = e \circ s, \quad m \circ (\mathrm{id}_{X_1}, i) = e \circ t$$

Definition 3.0.3. Let R be a commutative ring (Definition B.0.12).

A *Hopf algebra over R* is a R-module H equipped with the structure of a unital associative algebra (H, μ, η) and a counital coassociative coalgebra (H, Δ, ε), along with a R-linear map $S : H \to H$ called the *antipode*, satisfying the following compatibility axioms:

1. Δ and ε are algebra homomorphisms.
2. The antipode condition:
$$\mu \circ (S \otimes \mathrm{id}_H) \circ \Delta = \eta \circ \varepsilon = \mu \circ (\mathrm{id}_H \otimes S) \circ \Delta$$

In Sweedler notation, if $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$, the antipode condition is expressed as:
$$\sum S(h_{(1)})h_{(2)} = \varepsilon(h)1_H = \sum h_{(1)}S(h_{(2)})$$

Definition 3.0.4. Let R be a commutative ring (Definition B.0.12).

A *Hopf algebroid over R* is a pair of commutative R-algebras (A, Γ) together with structure maps that form a cogroupoid object in the category of commutative R-algebras. Specifically, it consists of:

- *Left and Right Units* (dual to source/target): $\eta_L : A \to \Gamma$ and $\eta_R : A \to \Gamma$. These induce a left A-module structure on Γ via η_L and a right A-module structure on Γ via η_R.
- *Counit* (dual to identity): $\varepsilon : \Gamma \to A$, such that $\varepsilon \circ \eta_L = \mathrm{id}_A = \varepsilon \circ \eta_R$.
- *Comultiplication* (dual to composition): $\Delta : \Gamma \to \Gamma \otimes_A \Gamma$, where the tensor product is formed using the right A-action on the left factor and the left A-action on the right factor.

- *Antipode* (dual to inverse): $S : \Gamma \to \Gamma$, which is an algebra anti-homomorphism (or homomorphism if Γ is commutative) satisfying the appropriate compatibility diagrams dual to the groupoid axioms.

The pair (A, Γ) represents the affine groupoid scheme with objects $\mathrm{Spec}(A)$ and morphisms $\mathrm{Spec}(\Gamma)$.

Definition 3.0.5. Let S be a scheme.

An *affine groupoid scheme over S* (or simply an *affine groupoid over S*) is a groupoid object (Definition 3.0.2) in the category of schemes affine (Definition B.0.10) over (Definition B.0.11) S. Explicitly, it consists of a pair of S-schemes (G_0, G_1) equipped with structural morphisms:

- *Source* and *Target*: $s, t : G_1 \to G_0$,
- *Identity*: $e : G_0 \to G_1$,
- *Multiplication* (or composition): $m : G_1 \times_{s,G_0,t} G_1 \to G_1$,
- *Inverse*: $i : G_1 \to G_1$,

such that:

1. The structure morphisms $G_1 \to S$ and $G_0 \to S$ are affine morphisms (Definition B.0.10).
2. The morphisms satisfy the standard axioms of a groupoid (associativity of m, unitality of e, invertibility via i).

If $S = \mathrm{Spec}(R)$ for a commutative ring (Definition B.0.12) k, then $G_0 = \mathrm{Spec}(A)$ and $G_1 = \mathrm{Spec}(H)$ for some commutative R-algebras (Definition B.0.13) A and H, and the data is equivalent to a Hopf algebroid structure (Definition 3.0.4) on the pair (A, H).

Appendix A. Sesquilinear and Hermitian forms

Definition A.0.1 (Center of a ring)**.** Let R be a (not necessarily commutative) ring. The *center of R*, denoted by $Z(R)$, is the subset of elements in R that commute with every element of R:

$$Z(R) := \{z \in R \mid zr = rz \text{ for all } r \in R\}.$$

The center $Z(R)$ is a commutative (Definition B.0.12) subring of R.

Definition A.0.2. Let $\mathcal{C}$ be a category. An *involution on an object $X \in \mathrm{Ob}(\mathcal{C})$* is an endomorphism $i : X \to X$ such that $i \circ i = \mathrm{id}_X$.

Definition A.0.3. Let R be a unital associative ring. An *involution on R* is a ring homomorphism (Definition B.0.25) that is an involution (Definition A.0.2), i.e. a map $\sigma : R \to R$ satisfying the following conditions for all $a, b \in R$:

1. $\sigma(a + b) = \sigma(a) + \sigma(b)$;
2. $\sigma(ab) = \sigma(b)\sigma(a)$ (anti-homomorphism property);
3. $\sigma(1) = 1$;
4. $\sigma(\sigma(a)) = a$.

In many contexts, an involution may simply be notated by $\cdot \mapsto \bar{\cdot}$ A *ring with involution* is a ring eqiupped with an involution.

Definition A.0.4. Let (A, σ) be a ring with involution (Definition A.0.3). Let M be a right A-module (Definition B.0.20). A *σ-sesquilinear form* (or simply *sesquilinear form*) on M is a map $b : M \times M \to A$ such that for all $u, v, w \in M$ and $a \in A$:

1. $b(u + v, w) = b(u, w) + b(v, w)$ and $b(u, v + w) = b(u, v) + b(u, w)$;
2. $b(ua, v) = \sigma(a)b(u, v)$ (conjugate-linear in the first variable);
3. $b(u, va) = b(u, v)a$ (linear in the second variable).

Alternatively, one can impose the convention that the module M be a left module and that the sesquilinear form is so that the first variable is linear and the second variable is conjugate-linear. The above convention is used in [?].

We may denote the set of sesquilinear forms on M by $\mathrm{Sesq}_A(M)$. It is a module over the center of A (Definition A.0.1).

Definition A.0.5. Let (A, σ) be a ring with involution (Definition A.0.3). The construction $M \mapsto \mathrm{Sesq}_A(M)$ (Definition A.0.4) is functorial; there is a contravariant functor (Definition B.0.29)

$$\mathrm{Sesq}_A : \mathrm{Mod}_A^{\mathrm{op}} \to {}_{Z(A)}\mathrm{Mod}_{Z(A)}$$

(Definition B.0.30)(Definition A.0.1) that sends a homomorphism $\varphi : M_1 \to M_2$ of A-modules to the $Z(A)$-linear map

$$\mathrm{Sesq}_A(\varphi) : \mathrm{Sesq}_A(M_2) \to \mathrm{Sesq}_A(M_1), \quad \mathrm{Sesq}_A(\varphi)(b)(x, y) = b(\varphi(x), \varphi(y)).$$

Equivalently, the corresponding map

$$\mathrm{Hom}_A(M_2, M_2^*) \to \mathrm{Hom}_A(M_1, M_1^*)$$

for the adjoints is given by $h \mapsto \varphi^* h_\varphi$ (Definition A.0.14).

Definition A.0.6. Let (A, σ) be a ring with involution (Definition A.0.3). Let (M_1, b_1) and (M_2, b_2) be (right) modules (Definition B.0.20) equipped with sesquilinear forms (Definition A.0.4). A *morphism* $(M_1, b_1) \to (M_2, b_2)$ between the sesquilinear modules is a (right) A-module homomorphism (Definition B.0.26) such that $\mathrm{Sesq}(\varphi)(b_2) = b_1$, i.e.

$$b_2(\varphi(x), \varphi(y)) = b_1(x, y)$$

for all $x, y \in M_1$.

Definition A.0.7. Let (A, σ) be a ring with involution (Definition A.0.3). Let (M_1, b_1) and (M_2, b_2) be (right) modules (Definition B.0.20) equipped with sesquilinear forms (Definition A.0.4).

An *isometry* $\varphi : (M_1, b_1) \to (M_2, b_2)$ is a morphism of sesquilinear modules (Definition A.0.6) that is an isomorphism. If an isometry exists between the two sesquilinear modules, then the two forms are said to be *isometric*.

Definition A.0.8. Let (A, σ) be a ring with involution (Definition A.0.3). Let (M, b) be (right) module (Definition B.0.20) equipped with a sesquilinear form (Definition A.0.4). The isometries (Definition A.0.7) of (M, b) form a group with respect to composition called the *orthgonal* or *unitary group of (M,b)*. The group may be denoted by notations such as $O(M,b)$, $O_A(M,b)$, $U(M,b)$, and $U_A(M,b)$.

Definition A.0.9. Let R be a (not necessarily commutative) ring. Depending on the module structure of M, we define its dual module as follows:

1. If M is a left R-module (Definition B.0.20), then the *(right) dual module of M* is
$$M^* = M^\vee := \operatorname{Hom}_R(M, R).$$
Note that it is a right R-module, as M is a $R-\mathbb{Z}$-bimodule and R is an $R-R$-bimodule.
2. If M is a right R-module (Definition B.0.20), then the *(left) dual module of M* is
$$^*M = {}^\vee M := \operatorname{Hom}_R(M, R).$$
Note that it is a left R-module, as M is a $\mathbb{Z}-R$-bimodule and R is an $R-R$-bimodule.
3. If M is a two-sided R-module, then the *dual of M* usually refers to either the right or the left dual as above.

If a convention for the rightness or the leftness of modules is fixed, then the appropriate dual module is often denoted by $M^\vee$.

In any case, the functor $M \mapsto M^\vee$ is a contravariant functor (Definition B.0.29) from the appropriate category of modules (Definition B.0.30) to itself.

If R is a field F and V is an F-vector space, then the dual module
$$V^* = V^\vee := \operatorname{Hom}_F(V, F)$$
is called the *dual vector space of V*.

Definition A.0.10. Let R be a ring with involution (Definition A.0.3) $r \mapsto \bar{r}$.

1. Let M be a left/right R-module (Definition B.0.20). The *opposite module*, also called the *conjugate module* or the module by restriction of scalars along the involution (not to be confused with the notion of opposite modules (Definition B.0.35) for modules over general rings), is the right/left R-module, denoted by notations such as M^{op} or $\overline{M}$, with the same underlying abelian group as M, where the right R-action is defined as follows
   - $$m \cdot r = \bar{r}m$$
     for all $m \in M$ and $r \in R$ if M is a left R-module.

     $$r \cdot m = m\bar{r}$$
     for all $m \in M$ and $r \in R$ if M is a right R-module.

Proposition A.0.11. Let R be a ring with involution (Definition A.0.3). The opposite module constructions provide an equivalence of categories (Definition B.0.32) between the category of left R-modules (Definition B.0.30) and the category of right R-modules. In fact, if M is a left/right R-module, then $(M^{op})^{op}$ is naturally isomorphic to M as a left/right R-module via the identity map.

Definition A.0.12.

Definition A.0.13. Let (A, σ) be a ring with involution (Definition A.0.3). Let N be a right A-module (Definition B.0.20). The *twisted dual module*, denoted by notations such as N^*, (or sometimes $N^\vee$; not to be confused with the standard dual (Definition A.0.9) $N^\vee$) or $\mathrm{Hom}_A(N, A)_\sigma$, is the set of all A-linear maps (Definition B.0.26) $f : N \to A$, equipped with the left A-module structure given by

$$(f \cdot r)(n) = \sigma(r) \cdot f(n)$$

for all $r \in A$, $f \in N^\vee$, $n \in N$. Equivalently, it may be constructed as $\overline{M^\vee}$, where by $\bar{\cdot}$, we mean the opposite module (Definition A.0.10) and by $\cdot^\vee$, we mean the standard dual (Definition A.0.9). The construction $\overline{M^\vee}$ is naturally isomorphic (Definition B.0.31) to $(\overline{M})^\vee$.

Lemma A.0.14. (♠ TODO: knus 2.1.1)

Definition A.0.15. Let (A, σ) be a ring with involution (Definition A.0.3) and M a right A-module (Definition B.0.20) equipped with a sesquilinear form (Definition A.0.4) $b : M \times M \to A$. We define two A-linear mappings associated with b:

1. The *left adjoint map* is the morphism of A-modules:
$$\hat{b}_l : M \to M^*, \quad u \mapsto (v \mapsto b(u, v))$$
where $M^* = \mathrm{Hom}_A(M, A)_\sigma$ is the twisted dual module (Definition A.0.13).
2. The *right adjoint map* is the morphism of A-modules:
$$\hat{b}_r : M \to M^*, \quad v \mapsto (u \mapsto \sigma(b(u, v)))$$

If b is an ε-hermitian form (Definition A.0.17), these maps are related by $\hat{b}_l = \hat{b}_r \cdot \varepsilon$ (under the standard identification of the module and its dual).

When we simply talk about the *adjoint map* of b on the right A-module M, we will mean the left adjoint and denote it by $\hat{b}$.

Definition A.0.16. Let (A, σ) be a ring with involution (Definition A.0.3), and let M be a (left) A-module (Definition B.0.20). Let $h : M \to M^*$ be a (left) A-module homomorphism where M^* is the twisted dual module of M (Definition A.0.13). The *sesquilinear form associated to h* is $b_h : M \times M \to A$ given by $b_h(x, y) = h(x)(y)$.

Lemma A.0.17. Let (A, σ) be a ring with involution (Definition A.0.3), and let M be a (left) A-module (Definition B.0.20). A sesquilinear form (Definition A.0.4) $b : M \times M \to A$ on M is determined by its adjoint (Definition A.0.14) $\hat{b}$. Conversely, A-linear homomorphisms $b : M \to M^*$ (Definition A.0.13) are determined by their associated sesquilinear forms (Definition A.0.15) b_h. In fact, the map $b \mapsto \hat{b}$ is a (two-sided) $Z(A)$ (Definition A.0.1)-module isomorphism

$$\mathrm{Sesq}_A(M) \cong \mathrm{Hom}_A(M, M^*).$$

Definition A.0.18. Let (A, σ) be a ring with involution (Definition A.0.3), let $\varepsilon \in Z(A)$ be an element such that $\varepsilon \cdot \sigma(\varepsilon) = 1$, and let M be a right A-module (Definition B.0.20). A *ε-hermitian form on M* (or simply *hermitian form on M*, especially when ε is understood or is just 1) is a σ-sesquilinear form (Definition A.0.4) $\phi : M \times M \to A$ satisfying the symmetry condition:

$$\phi(y, x) = \varepsilon \cdot \sigma(\phi(x, y))$$

for all $x, y \in M$. The pair (M, ϕ) is often called a *hermitian module*.

1-hermitian forms are may simply be called *hermitian forms*, and (-1)-hermitian forms are called *skew-hermitian forms*.

Definition A.0.19. Let (A, σ) be a ring with involution (Definition A.0.3) and M a right A-module (Definition B.0.20) equipped with a sesquilinear form (Definition A.0.4) $\phi : M \times M \to A$. Let

$$\hat{\phi} : M \to M^*, \quad x \mapsto \phi(x, \cdot)$$

be the induced (left) adjoint map (Definition A.0.14).

1. The form ϕ is said to be *nondegenerate* if $\hat{\phi}$ is injective.
   In the case where ϕ is an ε-hermitian form (Definition A.0.17), this condition is equivalent to requiring that $\phi(x, y) = 0$ for all $y \in M$ implies $x = 0$.
2. The form ϕ is said to be *nonsingular* or *regular* if $\hat{\phi}$ is an isomorphism.

Lemma A.0.20. (♠ TODO: define S_ε)

(♠ TODO: construct the space of hermitian forms and the space of even hermtiian forms as in Knus)

Definition A.0.21. (♠ TODO: define the even ε hermitian forms)

Definition A.0.22. (♠ TODO: complete this) Let (A, σ) be a ring with involution (Definition A.0.3).

1. A *(sesquilinear) space* is a pair (M, b) consisting of a (right) A-module (Definition B.0.20) equipped with a sesquilinear form (Definition A.0.4) that is nonsingular (Definition A.0.18).
2. A *ε-hermitian space* is a ε-hermitian

Proposition A.0.23. Let (R, σ) be a ring with involution (Definition A.0.3) and $\phi : M \times M \to R$ a sesquilinear form (Definition A.0.4). If M is a finitely generated (Definition B.0.23) projective (Definition B.0.24) R-module, then the adjoint map $\hat{\phi} : M \to \mathrm{Hom}_R(M, R)_\sigma$ is an isomorphism if and only if ϕ is non-degenerate (Definition A.0.18).

Definition A.0.24. Let $(\mathcal{S}, \tau)$ be a site (Definition B.0.21). Let $\mathcal{O}$ be a sheaf (Definition B.0.22) of rings on $\mathcal{S}$. An *involution on $\mathcal{O}$* is a morphism of sheaves of rings $\sigma : \mathcal{O} \to \mathcal{O}^{\mathrm{op}}$ such that $\sigma \circ \sigma = \mathrm{id}_{\mathcal{O}}$, i.e. an involution (Definition A.0.2) in the category of sheaves of rings on $\mathcal{S}$.

Definition A.0.25. Let $(\mathcal{S}, \tau)$ be a site (Definition B.0.21). Let $(\mathcal{O}, \sigma)$ be a sheaf of rings (Definition B.0.22) on $\mathcal{S}$ with involution (Definition A.0.23) $\sigma : \mathcal{O} \to \mathcal{O}$. Let $\mathcal{E}$ be a sheaf of left $\mathcal{O}$-modules (Definition B.0.27) on $\mathcal{S}$.

A *sesquilinear form on $\mathcal{E}$* is a morphism of sheaves (Definition B.0.22) of abelian groups

$$\phi : \mathcal{E} \times \mathcal{E} \to \mathcal{O}$$

such that for every object $U \in \mathcal{S}$, the induced map on sections

$$\phi_U : \mathcal{E}(U) \times \mathcal{E}(U) \to \mathcal{O}(U)$$

is a σ_U-sesquilinear form (Definition A.0.4) on the $\mathcal{O}(U)$-module $\mathcal{E}(U)$, where $\sigma_U : \mathcal{O}(U) \to \mathcal{O}(U)$ is the involution (Definition A.0.3) on global sections.

Definition A.0.26. Let $(\mathcal{O}, \sigma)$ be a sheaf (Definition B.0.22) of rings with involution (Definition A.0.23) on a site (Definition B.0.21) $\mathcal{S}$. Let $\mathcal{E}$ be a sheaf of left $\mathcal{O}$-modules (Definition B.0.27).

A *hermitian form on $\mathcal{E}$* is a morphism of sheaves of sets

$$\phi : \mathcal{E} \times \mathcal{E} \to \mathcal{O}$$

such that for every object U in $\mathcal{S}$, the map on sections

$$\phi_U : \mathcal{E}(U) \times \mathcal{E}(U) \to \mathcal{O}(U)$$

is a hermitian form (Definition A.0.17) on the $\mathcal{O}(U)$-module $\mathcal{E}(U)$ with respect to the involution (Definition A.0.3) $\sigma_U : \mathcal{O}(U) \to \mathcal{O}(U)$.

Appendix B. Miscellaneous definitions

Definition B.0.1 (Affine scheme)**.** Let A be a commutative ring with unity (Definition B.0.12). Define the set $\operatorname{Spec}(A)$ to be the set of all prime ideals of A. Equip it with the *Zariski topology*, which is the topology whose closed sets are given by *vanishing loci*

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec}(A) : I \subseteq \mathfrak{p}\}$$

for ideals $I \subseteq A$. Define the sheaf $\mathcal{O}_{\operatorname{Spec}(A)}$, called the *structure sheaf of* $\operatorname{Spec} A$, by

$$\mathcal{O}_{\operatorname{Spec}(A)}(U) = \{ \text{ locally defined fractions of elements of } A \text{ on } U \ \},$$

for each open set $U \subseteq \operatorname{Spec}(A)$. It is the case that the stalk at $\mathfrak{p} \in \operatorname{Spec}(A)$ is canonically the localization $A_{\mathfrak{p}}$. Then $(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)})$ is a locally ringed space, called the *affine scheme associated to A*.

Moreover, given $f \in A$, we define the locus $D(f)$ by

$$D(f) = \operatorname{Spec} A \setminus V((f)) = \{\mathfrak{p} \in \operatorname{Spec} A : f \notin \mathfrak{p}\}$$

Definition B.0.2 (Scheme)**.** A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ that admits an open cover $\{U_i\}_{i \in I}$ such that each $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic (as a locally ringed space) to an affine scheme $(\operatorname{Spec}(A_i), \mathcal{O}_{\operatorname{Spec}(A_i)})$ (Definition B.0.1) for some commutative ring A_i. In other words, a scheme is a locally ringed space locally isomorphic to affine schemes.

Definition B.0.3 (Additive category)**.** Let $\mathcal{A}$ be a locally small category (Definition B.0.16).

1. $\mathcal{A}$ is said to be a *preadditive category* if the following hold:
   - For any two objects A, B in $\mathcal{A}$, the set $\operatorname{Hom}_{\mathcal{A}}(A, B)$ is an abelian group, and composition of morphisms is bilinear.
   - There is a zero object 0 in $\mathcal{A}$.
2. If $\mathcal{A}$ is preadditive, then it is called *additive* if it additionally satisfies the following:

- For any two objects A, B in $\mathcal{A}$, there exists a product object $A \times B$, often written $A \oplus B$, called the *direct sum of A and B*. In fact, $A \oplus B$ is not only a product but also a coproduct of A and B.

Given a finite collection $\{A_i\}_i$ of objects A_i in an additive category $\mathcal{A}$, we may more generally speak of the *direct sum* $\bigoplus_i A_i$; it has canonical injections from and projections to each A_i.

Definition B.0.4. Let $\mathcal{C}$ be an additive category (Definition B.0.3). An object $X \in \mathrm{Ob}(\mathcal{C})$ is called *semisimple* if it is isomorphic to a finite direct sum (Definition B.0.3) of simple objects in $\mathcal{C}$.

Definition B.0.5. An additive category (Definition B.0.3) $\mathcal{C}$ is called a *semisimple category* if every object of $\mathcal{C}$ is semisimple (Definition B.0.4).

Definition B.0.6. Let C be a category enriched in a monidal category $\mathcal{V}$. Given an object X of C, the *functor of points* h_X is the functor (Definition B.0.29)/presheaf $C^{\mathrm{op}} \to \mathcal{V}$ given by $T \mapsto \mathrm{Hom}_C(T, X)$. A functor $C^{\mathrm{op}} \to \mathcal{V}$ (or equivalently, a presheaf on C valued in $\mathcal{V}$) is said to be *representable* if it is naturally isomorphic (Definition B.0.31) to some functor h_X of points for an object X of C.

Dually, a functor $C \to \mathcal{V}$ is called *co-representable* if it is naturally isomorphic to a functor $T \mapsto \mathrm{Hom}_C(X, T)$ for an object X in C.

For instance, we may speak of these notions when $\mathcal{V}$ is the monoidal category **Sets**, i.e. C is a locally small category (Definition B.0.16).

Definition B.0.7. (♠ TODO: make this definition be for more general presheaves valued in more general caetgories) Let S be a scheme and let $f : F \to G$ be a morphism of presheaves valued in sets on Sch/S. We say that f is *representable by schemes* if for every scheme T over S and every morphism $h_T \to G$ of sheaves (i.e., every T-point of G), the fiber product

$$h_T \times_G F$$

is representable (Definition B.0.6) by a scheme over T, i.e. there exists a scheme X over T such that $h_T \times_G F \cong h_X$ as presheaves.

Definition B.0.8. (♠ TODO: TODO: read) (♠ TODO: TODO: state that a sheaf gives a stack) (♠ TODO: TODO: find out if it is equivalent to have U be an algebraic space instead.) Let S be a scheme (Definition B.0.2). Let τ be a Grothendieck topology (Definition B.0.21) on Sch/S. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over the site $(Sch/S)_\tau$. We say that f is *representable by algebraic spaces* if for every scheme U over S and every 1-morphism $y : U \to \mathcal{Y}$ (i.e., every object $y \in \mathcal{Y}(U)$), the 2-fiber product

$$\mathcal{X} \times_{\mathcal{Y}} U$$

is equivalent (as a stack over $(Sch/U)_\tau$) to an algebraic space over U (Definition B.0.9).

Definition B.0.9. Let S be a scheme. Let τ be a Grothendieck topology (Definition B.0.21) on Sch/S (Definition B.0.7) for which surjectivity and étaleness are preserved under base change and τ-local on the base. An *algebraic space over S for the topology τ* is a sheaf F of sets for $(Sch/S)_\tau$ such that

1. The diagonal morphism $F \to F \times F$ is representable by schemes (Definition B.0.7);
2. There exists a scheme U over S and a morphism of sheaves $h_U \to F$ which is surjective and étale.

Definition B.0.10. Let $f : X \to Y$ be a morphism of schemes. We say that f is an *affine morphism* if for every affine open $V = \operatorname{Spec} B \subseteq Y$, the preimage $U = f^{-1}(V)$ is an affine scheme (Definition B.0.1).

Definition B.0.11 (Scheme over a scheme)**.** Let $(S, \mathcal{O}_S)$ be a scheme. A *scheme over S* (or an *S-scheme*) is a scheme $(X, \mathcal{O}_X)$ together with a morphism of schemes

$$\pi : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S).$$

This morphism π is called the *structure morphism of the scheme X over S*.

If $S = \operatorname{Spec}(R)$ is an affine scheme for a commutative ring R, then an S-scheme is synonymously called an *R-scheme* or a *scheme over R*.

Let $(S, \mathcal{O}_S)$ be a scheme, and let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be schemes over S with structure morphisms

$$\pi_X : X \to S, \quad \pi_Y : Y \to S.$$

A *morphism of S-schemes* (or synonymously a *S-scheme morphism*) is a morphism of schemes

$$(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ S & = & S \end{array}$$

In other words,

$$\pi_Y \circ f = \pi_X.$$

Given a fixed scheme S, there is a category, often denoted by Sch_S, $\mathrm{Sch}_{/S}$, Sch/S, $\mathbf{Sch}_S$, $\mathbf{Sch}_{/S}$, $\mathbf{Sch}/S$ etc. whose objects are schemes T over S and whose morphisms $T_1 \to T_2$ are morphisms of schemes over S. If $S = \operatorname{Spec} R$ for some commutative ring R, then we may instead write Sch_R to denote $\mathrm{Sch}_{\operatorname{Spec} R}$, etc. It is noteworthy that $\mathrm{Sch}_{\mathbb{Z}}$ coincides with the category Sch of all schemes. In other words, a $\mathbb{Z}$-scheme can be identified simply with a scheme.

Definition B.0.12. A *commutative (unital) ring* is a ring $(R, +, \cdot)$ such that $\cdot$ is a commutative operation, i.e. $a \cdot b = b \cdot a$.

For many writers (e.g. "commutative" algebraists or number theorists), a *ring* refers to a commutative ring as above.

Definition B.0.13. Let R be a (not-necessarily commutative) ring with unity. An *R-algebra* is a ring A together with a ring homomorphism (Definition B.0.25)

$$\varphi : R \to A$$

into the center $Z(A)$ (Definition A.0.1) of A (so that $\varphi(r)$ commutes with every element of A for all $r \in R$), such that $\varphi(1_R) = 1_A$. The ring homomorphism φ is called the *structure map* of the algebra.

Equivalently, an R-algebra consists of a ring A endowed with a two-sided R-module (Definition B.0.20) structure for which the scalar multiplication satisfies

$$r \cdot (ab) = (r \cdot a)b = a(r \cdot b) \quad \text{for all } r \in R,\ a, b \in A.$$

In particular, any ring homomorphism between commutative rings (Definition B.0.12) specifies an algebra structure.

Definition B.0.14. Let $\mathcal{C}$ be a category (Definition B.0.15), let Z be an object, and let X, Y be objects of $\mathcal{C}$ over Z, i.e. morphisms $X \to Z$ and $Y \to Z$ are fixed. A *cartesian product of X and Y over Z in $\mathcal{C}$* (or *fiber product* or *pullback diagram*) is an object, often denoted by $X \times_Z Y$, with *projection morphisms* $X \times_Z Y \to X$ and $X \times_Z Y \to Y$ that are universal. More precisely, for any object T of $\mathcal{C}$ and morphisms $f_X : T \to X$, $f_Y : T \to Y$ such that the compositions $T \xrightarrow{f_X} X \to Z$ and $T \xrightarrow{f_Y} Y \to Z$ agree, there exists a unique morphism $u : T \to X \times_Z Y$ such that the following diagram commutes:

$$\begin{tikzcd}
T \arrow[dr, dotted, "u"] \arrow[drr, bend left, "f_X"] \arrow[ddr, bend right, "f_Y"'] & & \\
& X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d] \\
& Y \arrow[r] & Z
\end{tikzcd}$$

Equivalently, $X \times_Z Y$ is the limit of the diagram

$$\begin{tikzcd}
& X \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}$$

in $\mathcal{C}$.

The commutative diagram

$$\begin{tikzcd}
X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}$$

may be referred to as a *cartesian square*.

Definition B.0.15 (Category)**.** A *category* $\mathcal{C}$ consists of the following data:

- A class of *objects* denoted $\mathrm{Ob}(\mathcal{C})$.

- For each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a class

$$\mathrm{Hom}_{\mathcal{C}}(X, Y)$$

of *morphisms* (also called *arrows* or *homs*). If the category $\mathcal{C}$ is clear, then this *hom-class* is also denoted by $\mathrm{Hom}(X, Y)$. It may also be denoted by $\mathrm{hom}_{\mathcal{C}}(X, Y)$ or $\mathrm{hom}(X, Y)$, especially to distinguish from other types of hom's (e.g. internal hom's)
- For each triple of objects X, Y, Z, a composition law

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z),$$

denoted $(g, f) \mapsto g \circ f$.
- For each object X, an *identity morphism*

$$\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X).$$

These data satisfy the following axioms:

- (Associativity) For all morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, and $h \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

- (Identity) For all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$,

$$\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X .$$

One often writes $X \in \mathcal{C}$ synonymously with $X \in \mathrm{Ob}(\mathcal{C})$, i.e. to denote that X is an object of of $\mathcal{C}$.

We may call a category as above an *ordinary category* to distinguish this notion from the notions of *categories enriched in monoidal categories* or higher/n-categories. (♠ TODO: TODO: define n-categories)

A category as defined above may be called called a *large category* or a *class category* to emphasize that the hom-classes may be proper classes rather than sets (note, however, that the possibility that hom-classes are sets is not excluded for large categories). Accordingly, a *category* may often refer to a locally small category (Definition B.0.16), which is a category whose hom-classes are all sets.

Definition B.0.16 (Locally small category)**.** A (large) category (Definition B.0.15) $\mathcal{C}$ is called a *locally small category* if for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the collection $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms between them is a (small) *set* (as opposed to a proper class). In other words, each hom-class is a set and may even be called a *hom-set*.

In some contexts, a locally small category may simply be called a *category*, especially when genuinely large categories are not considered.

A category $\mathcal{C}$ is called a *small category* if it is a locally small category and the class $\mathrm{Ob}(\mathcal{C})$ of objects is a set.

Remark B.0.17. Many "concrete" categories considered in "classical mathematics" or outside of more "abstract" category theory tend to be locally small. For example, the categories of sets, groups, R-modules, vector spaces, topological spaces, schemes, manifolds, sheaves on "small enough" sites are all locally small.

Definition B.0.18. Let $(\mathcal{C}, J)$ be a site (Definition B.0.21). Let $\mathcal{O}$ be a sheaf of commutative rings on $(\mathcal{C}, J)$ (Definition B.0.22), i.e., $((\mathcal{C}, J), \mathcal{O})$ is a ringed site.

1. An *$\mathcal{O}$-algebra* consists of the following data:
   - A sheaf $\mathcal{A}$ of (not necessarily commutative) rings on $(\mathcal{C}, J)$,
   - A morphism of sheaves of rings $\eta : \mathcal{O} \to \mathcal{A}$ such that for every object $U \in \mathcal{C}$, the image of $\eta_U : \mathcal{O}(U) \to \mathcal{A}(U)$ is contained in the center of $\mathcal{A}(U)$.

   This makes $\mathcal{A}(U)$ an $\mathcal{O}(U)$-algebra for every $U \in \mathcal{C}$, such that for every morphism $f : V \to U$ in $\mathcal{C}$, the restriction map
   $$\rho_{U,V} : \mathcal{A}(U) \to \mathcal{A}(V)$$
   is a homomorphism of $\mathcal{O}(U)$-algebras (where the $\mathcal{O}(U)$-algebra structure on $\mathcal{A}(V)$ is induced via restriction $\mathcal{O}(U) \to \mathcal{O}(V)$).
2. Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{O}$-algebras (Definition B.0.18).

   A *morphism of $\mathcal{O}$-algebras* $\varphi : \mathcal{A} \to \mathcal{B}$ is a morphism of sheaves (Definition B.0.22) of rings such that, for every object $U \in \mathcal{C}$, the component map
   $$\varphi_U : \mathcal{A}(U) \to \mathcal{B}(U)$$
   is a homomorphism of $\mathcal{O}(U)$-algebras, i.e., it is a ring homomorphism that commutes with the structure maps $\eta_{\mathcal{A}}$ and $\eta_{\mathcal{B}}$:
   $$\varphi_U(\eta_{\mathcal{A},U}(r)) = \eta_{\mathcal{B},U}(r) \quad \text{for all } r \in \mathcal{O}(U).$$

The collection of all $\mathcal{O}$-algebras together with their morphisms forms the *category of $\mathcal{O}$-algebras*, denoted by notations such as $\mathbf{Alg}(\mathcal{O})$.

Definition B.0.19. Let $((\mathcal{C}, J), \mathcal{O})$ be a ringed site, where $\mathcal{O}$ is a sheaf (Definition B.0.22) of rings on the site $(\mathcal{C}, J)$. (♠ TODO: This needs some genuine fixing to be definable on a general site, not just one given by a pretopology)

1. Let I be an indexing set. The *free sheaf of $\mathcal{O}$-modules of rank I* (or simply a *free sheaf*), denoted by $\mathcal{O}^{\oplus I}$, is the sheaf associated to the presheaf $U \mapsto \mathcal{O}(U)^{\oplus I}$. If I is finite with cardinality n, we usually write $\mathcal{O}^{\oplus n}$.
2. Let $\mathcal{A}$ be a (large) category, and let A be an object of $\mathcal{A}$. Assume that a sheafification functor (♠ TODO: If such a sheafification functor exist, does a sheafification functor exist when restricted to an object U?)
   $$a : \mathrm{PreShv}(\mathcal{C}, J, \mathcal{A}) \to \mathrm{Shv}(\mathcal{C}, J, \mathcal{A})$$
   (Definition B.0.22) exists. Let U be an object of $\mathcal{C}$.

   An $\mathcal{O}$-module (Definition B.0.27) $\mathcal{F}$ is called *locally free of rank n on U* (for an integer $n \geq 0$) if there exists a covering sieve (Definition B.0.21) $\{U_i \to U\}_{i \in I}$ such that for each i, the restriction $\mathcal{F}|_{U_i}$ is isomorphic to the free sheaf $(\mathcal{O}|_{U_i})^{\oplus n}$ as an $\mathcal{O}|_{U_i}$-module.

We might call a locally free $\mathcal{O}$-module of rank n an *algebraic vector bundle of rank n*. A *(algebraic) line bundle* or *invertible sheaf* or is then an algebraic vector bundle of rank 1.

Definition B.0.20. Let R be a not-necessarily commutative ring.

1. A *left R-module* is an abelian group $(M,+)$ together with an operation $R \times M \to M$, denoted $(r,m) \mapsto rm$, such that for all $r,s \in R$ and $m,n \in M$:
   - $r(m+n) = rm + rn$,
   - $(r+s)m = rm + sm$,
   - $(rs)m = r(sm)$,
   - $1_R m = m$ where 1_R is the multiplicative identity of R.
2. A *right R-module* is defined similarly as an abelian group $(M,+)$ with an operation $M \times R \to M$, denoted $(m,r) \mapsto mr$, such that for all $r,s \in R$ and $m,n \in M$:
   - $(m+n)r = mr + nr$,
   - $m(r+s) = mr + ms$,
   - $m(rs) = (mr)s$,
   - $m1_R = m$.
3. Let R and S be (not necessarily commutative) rings.

   An *R-S-bimodule* (or an *R-S-module* or an (R,S)-module, etc.) is an abelian group $(M,+)$ equipped with
   (a) a left action of R:
   $$R \times M \to M, \quad (r,m) \mapsto r \cdot m,$$
   making M a left R-module (Definition B.0.20),
   (b) a right action of S:
   $$M \times S \to M, \quad (m,s) \mapsto m \cdot s,$$
   making M a right S-module,

   such that the left and right actions commute; that is, for all $r \in R$, $s \in S$, and $m \in M$,
   $$r \cdot (m \cdot s) = (r \cdot m) \cdot s.$$
4. A *two-sided R-module* (or *R-bimodule*) is an R-R-bimodule.

If R is a commutative ring (Definition B.0.12), then a left/right R-module can automatically be regarded as a two-sided R-module. As such, we simply talk about *R-modules* in this case.

Any abelian group is equivalent to a two-sided $\mathbb{Z}$-module. Moreover, any left R-module is equivalent to an $R-\mathbb{Z}$-bimodule (Definition B.0.20) and any right R-module is equivalent to an $\mathbb{Z} - R$-bimodule (Definition B.0.20). Given a left/right/two-sided R-module, its *natural bimodule structure* will refer to its structure as a R-$\mathbb{Z}$/$\mathbb{Z}$-R/R-R bimodule. In this way, many definitions associated with the notions of left/right/two-sided R-modules can be defined as special cases for definitions for R-S-bimodules.

Definition B.0.21 (Grothendieck topology)**.** Let $\mathscr{U}$ be a universe.

1. (See [**?**, Exposé II, Définition 1.1]) Let $\mathcal{C}$ be a category (Definition B.0.15). A *Grothendieck topology on $\mathcal{C}$* assigns to each object U of $\mathcal{C}$ a collection $J(U)$ of sieves $\{U_i \to U\}_{i \in I}$, each called a *covering sieve of U*, satisfying:

(a) (Stability under "base change"): If $S \in J(U)$ is a covering sieve of an object U, and $f : V \to U$ is any morphism in $\mathcal{C}$, then the pullback sieve f^*S is a covering sieve of U.
(b) (Local character condition) If S is a sieve on U, and if there exists a covering sieve $R \in J(U)$ such that for all $f : V \to U$ in R the pullback sieve f^*S is in $J(V)$, then $S \in J(U)$.
(c) The maximal sieve is a covering sieve.

Some will refer to a Grothendieck topology as simply a *topology*, not to be confused with the related, but less general, notion of a topology on a set.

2. (See [**?**, Exposé II, 1.1.5]) A *site* is a category $\mathcal{C}$ equipped with a Grothendieck topology.

   When we are working with a Grothendieck pretopology K on a category $\mathcal{C}$, we may regard $\mathcal{C}$ as a site by equipping it with the Grothendieck topology generated by K.
3. (See [**?**, Exposé II, Définition 1.2]) Let $(\mathcal{C}, J)$ be a site. A family of morphisms $(U_i \to U)_{i\in I}$ is called a *covering family of U (with respect to the site/topology)* or a *cover of U (with respect to the site/topology)* if the sieve generated by the family is a covering sieve of U.
4. (See [**?**, Exposé II, Définition 3.0.1]) Let $(\mathcal{C}, J)$ be a site (Definition B.0.21), where J is a Grothendieck topology on $\mathcal{C}$.

   A family G of objects $\mathcal{C}$ is called a *topologically generating family of the site/topology* or a *generating family/collection of the site/topology* if for every object $X \in \mathcal{C}$, there is a covering family $\{X_\alpha \to X\}_{\alpha\in A}$ of X such that every X_α is a member of G.

   Equivalently, the Grothendieck topology J is the smallest Grothendieck topology containing all covers of the U_i. Also equivalently, for any $S \in J(X)$, the sieve S contains a covering family $\{V_i \to X\}$ such that each morphism $V_i \to X$ factors through some member of G. (♠ TODO: Verify that these claimed equivalences are indeed equivalences)
5. (See [**?**, Exposé II, Définition 3.0.2]) A *$\mathscr{U}$-site* is a site whose underlying category $\mathcal{C}$ is $\mathscr{U}$-locally small (Definition B.0.16) and which has a $\mathscr{U}$-small topologically generating family. A $\mathscr{U}$-site is called *$\mathscr{U}$-small* if its underlying category is $\mathscr{U}$-small. Similarly, a *small site* is a site whose underlying category is a set and a *locally small site* is a site whose underlying category is locally small (Definition B.0.16).

Definition B.0.22 (Sheaf on a site)**.** Let $(\mathcal{C}, J)$ be a site (Definition B.0.21). Let $\mathcal{A}$ be a (large) category (Definition B.0.15).

1. A presheaf $\mathcal{F} : \mathcal{C}^{\text{op}} \to \mathcal{A}$ is called a *sheaf on the site $(\mathcal{C}, J)$ valued in $\mathcal{A}$* if, for every object U of $\mathcal{C}$ and every covering sieve (Definition B.0.21) $S \in J(U)$, the limit

$$\varprojlim_{(V\to U)\in(\mathcal{D}_S)^{\text{op}}} \mathcal{F}|_{\mathcal{D}_S}(V),$$

   exists and the canonical natural morphism

$$\mathcal{F}(U) \to \varprojlim_{(V\to U)\in(\mathcal{D}_S)^{\text{op}}} \mathcal{F}|_{\mathcal{D}_S}(V)$$

   is an isomorphism. Here, $\mathcal{D}_S \hookrightarrow \mathcal{C}/U$ is the full downward-closed subcategory such that $\text{Ob}(\mathcal{D}_S) = \{(f : V \to U) : f \in S(V)\}$, In particular, when we are working with a Grothendieck pretopology K on a category $\mathcal{C}$, we may speak of sheaves on the site whose Grothendieck topology is the one generated by K.

2. Given sheaves $\mathcal{F}, \mathcal{G} : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$ on the site $(\mathcal{C}, J)$, a *morphism between the sheaves* is a morphism between $\mathcal{F}$ and $\mathcal{G}$ as presheaves.
3. Let U be a universe. A *U-sheaf* typically refers to a U-presheaf that is a sheaf for a U-site. In other words, a U-sheaf is a sheaf on a site whose underlying category is U-locally small (Definition B.0.16) and which has a U-small topologically generating family such that the sheaf is valued in U-sets.
4. The *sheaf category/category of $\mathcal{A}$-valued sheaves on $\mathcal{C}$* is the (large) category defined as the full subcategory of $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ whose objects are the sheaves on $\mathcal{C}$ with values in $\mathcal{A}$. Common notations for the sheaf category include $\mathrm{Shv}(\mathcal{C}, \mathcal{A})$, $\mathrm{Shv}(\mathcal{C}, J, \mathcal{A})$, $\mathrm{Sh}(\mathcal{C}, \mathcal{A})$, $\mathrm{Sh}(\mathcal{C}, J, \mathcal{A})$. If the value category $\mathcal{A}$ is clear from context, then notations such as $\mathrm{Shv}(\mathcal{C})$, $\mathrm{Shv}(\mathcal{C}, J)$, $\mathrm{Sh}(\mathcal{C})$, $\mathrm{Sh}(\mathcal{C}, J)$ are also common.

Definition B.0.23 (Finitely generated modules and bimodules)**.** Let R and S be (not necessarily commutative) rings.

1. An R-S-bimodule M is *finitely generated* if it has a finite spanning set.
2. A left/right/two-sided R-module is *finitely generated* if has a finite spanning set, or equivalently if its natural bimodule structure (Definition B.0.20) is finitely generated.

Definition B.0.24. Let R and S be (not necessarily commutative) rings. A *projective R-S-bimodule* is an (R, S)-bimodule (Definition B.0.20) P that satisfies any of the following equivalent conditions:

1. The functor
$$\mathrm{Hom}_{{}_R\mathsf{Mod}_S}(P, -) : {}_R\mathsf{Mod}_S \to \mathsf{Ab}$$
is an exact functor between the abelian categories ${}_R\mathsf{Mod}_S$ (Definition B.0.30) and Ab.
2. P is a projective left module over the ring $R \otimes_{\mathbb{Z}} S^{\mathrm{op}}$ (Definition B.0.33).
3. P is a direct summand of a free (R, S)-bimodule. (A free (R, S)-bimodule is a direct sum of copies of the tensor product $R \otimes_{\mathbb{Z}} S$, equipped with the natural left R-action and right S-action).
4. P is a projective object in the category ${}_R\mathsf{Mod}_S$. That is, for every surjective homomorphism of (R, S)-bimodules $f : M \to N$ and every homomorphism $g : P \to N$, there exists a homomorphism $h : P \to M$ such that $f \circ h = g$.

Being a projective bimodule is a strictly stronger condition than being projective as a left or right module.

- A bimodule ${}_R P_S$ may be projective as a left R-module (i.e., projective in ${}_R\mathsf{Mod}$) without being a projective bimodule.
- Similarly, it may be projective as a right S-module (i.e., projective in Mod_S) without being a projective bimodule.
- A bimodule that is projective on both sides is sometimes called *biprojective*, but this does not imply it is a projective object in ${}_R\mathsf{Mod}_S$. For example, if $R = S = \mathbb{Z}$, the

bimodule $\mathbb{Z}$ is free (hence projective) on both sides, but it is *not* a projective $(\mathbb{Z},\mathbb{Z})$-bimodule because $\mathbb{Z}$ is not a projective $\mathbb{Z}[\mathbb{Z}]$-module (the augmentation ideal is not projective).

Definition B.0.25. Let $(R,+,\cdot)$ and $(S,+,\cdot)$ be rings, not assumed to be commutative. A function $f : R \to S$ is called a *ring homomorphism* if for all $r_1, r_2 \in R$ the following properties hold:

1. $f(r_1 + r_2) = f(r_1) + f(r_2)$,
2. $f(r_1 r_2) = f(r_1) f(r_2)$,
3. $f(1_R) = 1_S$ where 1_R and 1_S denote the multiplicative identities in R and S, respectively.

A ring homomorphism is said to be a *ring isomorphism* if it is invertible as a map of sets.

An *R-ring* refers to a ring S equipped with a ring homomorphism $f : R \to S$.

We note that a ring homomorphism $f : R \to S$ yields a natural left R-module (Definition B.0.20) structure on S and a natural right R-module structure on S respectively as follows for $r \in R$ and $s \in S$:

$$r \cdot s = f(r) \cdot s$$
$$s \cdot r = s \cdot f(r).$$

However, these left and right module structures need not yield a two-sided R–module structure.

Definition B.0.26. Let R, S be (not-necessarily commutative) rings.

1. Let M and N be R-S-bimodules (Definition B.0.20). A function $\varphi : M \to N$ is called an *R-S-bimodule homomorphism* or *R-S-linear* if it is a group homomorphism of the underlying abelian groups of M and N and respects the scalar actions as follows: for all $m_1, m_2 \in M$, $r \in R$, and $s \in S$,
$$\varphi(r \cdot m_1) = r \cdot \varphi(m_1),$$
$$\varphi(m_1 \cdot s) = \varphi(m_1) \cdot s.$$
2. Let M and N be left/right/two-sided R-modules (Definition B.0.20). A function $\varphi : M \to N$ is called a *left/right/two-sided R-module homomorphism* if it is an bimodule homomorphism on the natural bimodule structures (Definition B.0.20) of M and N. Such a function is also called *R-linear*.

Modules and homomorphisms of a fixed type (i.e. R-S-bimodules or left/righ/two-sided R-modules) form a locally small (Definition B.0.16) category (Definition B.0.15).

Definition B.0.27. 1. Let $\mathcal{C}$ be a site (Definition B.0.21), and let $\mathcal{A}$ and $\mathcal{B}$ be sheaves (Definition B.0.22) of (not necessarily commutative) rings on $\mathcal{C}$.

(a) An *$(\mathcal{A},\mathcal{B})$-bimodule* (or a *bimodule over $(\mathcal{A},\mathcal{B})$*) is a sheaf (Definition B.0.22) $\mathcal{M}$ of abelian groups on $\mathcal{C}$ equipped with a left $\mathcal{A}$-module structure given by a morphism of sheaves (Definition B.0.22) of sets

$$\lambda : \mathcal{A} \times \mathcal{M} \longrightarrow \mathcal{M},$$

and a right $\mathcal{B}$-module structure given by a morphism of sheaves of sets

$$\rho : \mathcal{M} \times \mathcal{B} \longrightarrow \mathcal{M},$$

such that the actions are compatible. Specifically, for every object U in $\mathcal{C}$, every section $m \in \mathcal{M}(U)$, every $a \in \mathcal{A}(U)$, and every $b \in \mathcal{B}(U)$, the equality

$$\lambda_U(a, \rho_U(m, b)) = \rho_U(\lambda_U(a, m), b)$$

holds in $\mathcal{M}(U)$. In standard multiplicative notation where $\lambda(a, m)$ is denoted $a \cdot m$ and $\rho(m, b)$ is denoted $m \cdot b$, this condition is the associativity axiom

$$(a \cdot m) \cdot b = a \cdot (m \cdot b).$$

In particular, for every object $U \in \mathcal{C}$, the abelian group $\mathcal{M}(U)$ has the structure of an $\mathcal{A}(U) - \mathcal{B}(U)$-bimodule (Definition B.0.20).

(b) Let $\mathcal{M}$ and $\mathcal{N}$ be $(\mathcal{A}, \mathcal{B})$-bimodules. A *homomorphism of $(\mathcal{A}, \mathcal{B})$-bimodules* (or an *$(\mathcal{A}, \mathcal{B})$-linear morphism*) is a morphism of sheaves of abelian groups $f : \mathcal{M} \to \mathcal{N}$ such that for every object U of $\mathcal{C}$, every section $m \in \mathcal{M}(U)$, every $a \in \mathcal{A}(U)$, and every $b \in \mathcal{B}(U)$, the following compatibility conditions hold:

$$f_U(a \cdot m) = a \cdot f_U(m) \quad \text{and} \quad f_U(m \cdot b) = f_U(m) \cdot b.$$

We denote the category of $(\mathcal{A}, \mathcal{B})$-bimodules, with morphisms being morphisms of sheaves of abelian groups compatible with both the left $\mathcal{A}$-action and the right $\mathcal{B}$-action, by $\mathcal{A}$-$\mathcal{B}$-Mod or sometimes by ${}_{\mathcal{A}}\mathsf{Mod}_{\mathcal{B}}$ (♠ TODO: talk about how bimodules can be identifies with left/right modules)

2. Let $(\mathcal{C}, J)$ be a site (Definition B.0.21). Let $\mathcal{O}$ be a sheaf of (not necessarily commutative) rings on $(\mathcal{C}, J)$ (Definition B.0.22), i.e. $((\mathcal{C}, J), \mathcal{O})$ is a ringed site.

(a) An *(left/right/two-sided) $\mathcal{O}$-module* consists of the following data:
- A sheaf $\mathcal{F}$ of abelian groups on $(\mathcal{C}, J)$,
- for every object $U \in \mathcal{C}$, the structure of an (left/right/two-sided) $\mathcal{O}(U)$-module on $\mathcal{F}(U)$,

such that for every morphism $f : V \to U$ in $\mathcal{C}$, the restriction map

$$\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$$

is $\mathcal{O}(U)$-linear when the $\mathcal{O}(U)$-action on $\mathcal{F}(V)$ is defined via the natural ring homomorphism

$$\mathcal{O}(U) \to \mathcal{O}(V)$$

induced by f.

(b) Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}$-modules (Definition B.0.27).
A *morphism of $\mathcal{O}$-modules* $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves (Definition B.0.22) of abelian groups such that, for every object $U \in \mathcal{C}$, the component map

$$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

is $\mathcal{O}(U)$-linear, i.e. it satisfies

$$\varphi_U(r \cdot s) = r \cdot \varphi_U(s) \quad \text{for all } r \in \mathcal{O}(U),\ s \in \mathcal{F}(U).$$

The collection of all $\mathcal{O}$-modules together with their morphisms of $\mathcal{O}$-modules forms the *category of $\mathcal{O}$-modules*, denoted $\mathbf{Mod}(\mathcal{O})$.
See also Definition B.0.18.

In case that a sheafification functor

$$\mathrm{PreShv}(\mathcal{C}, \mathbf{Rings}) \to \mathrm{Shv}(\mathcal{C}, \mathbf{Rings})$$

exists, a left, right, two-sided $\mathcal{O}$-module (and morphisms thereof) is equivalent to a $(\mathcal{O}, \mathbb{Z})$-bimodule, $(\mathbb{Z}, \mathcal{O})$-bimodule, and $(\mathcal{O}, \mathcal{O})$-bimodule (and morphisms thereof) respectively, where $\mathbb{Z}$ is the constant sheaf of the integer ring $\mathbb{Z}$.

Definition B.0.28. Let S be a scheme (Definition B.0.2) and let $n \geq 0$ be an integer. We define the *affine space of dimension n over S*, denoted by $\mathbb{A}^n_S$, as follows:

1. If $S = \operatorname{Spec} A$ is an affine scheme (Definition B.0.1), then $\mathbb{A}^n_S$ is the affine scheme defined by the polynomial ring in n variables over A:
$$\mathbb{A}^n_{\operatorname{Spec} A} = \operatorname{Spec}(A[T_1, \ldots, T_n]).$$
2. For a general scheme S, let $\{U_i = \operatorname{Spec} A_i\}_{i \in I}$ be an affine open covering of S. For each i, let $X_i = \mathbb{A}^n_{U_i} = \operatorname{Spec}(A_i[T_1, \ldots, T_n])$. Since polynomial rings behave well under localization, for any open immersion $U_{ij} = U_i \cap U_j \hookrightarrow U_i$, there is a canonical isomorphism on the overlaps:
$$\phi_{ij} : X_i|_{U_{ij}} \xrightarrow{\sim} X_j|_{U_{ij}}.$$
The scheme $\mathbb{A}^n_S$ is obtained by gluing the family $\{X_i\}_{i \in I}$ along these isomorphisms.

Definition B.0.29. Let $\mathcal{C}$ and $\mathcal{D}$ be (large) categories (Definition B.0.15).

1. A *functor $F : \mathcal{C} \to \mathcal{D}$ (from $\mathcal{C}$ to $\mathcal{D}$)* consists of :
   - For each object X in $\mathcal{C}$, an object $F(X)$ in $\mathcal{D}$.
   - For each morphism $f : X \to Y$ in $\mathcal{C}$, a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$,

   such that:
$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \text{for all objects } X \text{ in } \mathcal{C},$$
$$F(g \circ f) = F(g) \circ F(f) \quad \text{for all } X, Y, Z \in \mathrm{Ob}(\mathcal{C}) \text{ and all } f : X \to Y, g : Y \to Z \text{ in } \mathcal{C}.$$
   Functors as defined above are also referred to as *covariant functors* to distinguish them from contravariant functors
2. A *contravariant functor from $\mathcal{C}$ to $\mathcal{D}$* refers to a covariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. Equivalently, such a functor consists of
   - For each object X in $\mathcal{C}$, an object $F(X)$ in $\mathcal{D}$.
   - For each morphism $f : X \to Y$ in $\mathcal{C}$, a morphism $F(f) : F(Y) \to F(X)$ in $\mathcal{D}$,

   such that:
$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \text{for all objects } X \text{ in } \mathcal{C},$$
$$F(g \circ f) = F(f) \circ F(g) \quad \text{for all } X, Y, Z \in \mathrm{Ob}(\mathcal{C}) \text{ and all } f : X \to Y, g : Y \to Z \text{ in } \mathcal{C}.$$

Note that declarations such as "Let $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ be a contravariant functor" can be common; such declarations usually mean "Let F be a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$" as opposed to "Let F be a contravariant functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$". further note that a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is equivalent to a covariant functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$.

Definition B.0.30. Let R and S be (not necessarily commutative) rings.

1. The *category of (R,S)-bimodules* (or R-S-bimodules), denoted by notations such as ${}_R\mathsf{Mod}_S$, is the category whose objects are (R,S)-bimodules (Definition B.0.20) and whose R-S-bimodule homomorphisms (Definition B.0.26).
2. The *category of left R-modules*, denoted by notations such as ${}_R\mathsf{Mod}$ or $R-\mathbf{Mod}$, is the category ${}_R\mathsf{Mod}_{\mathbb{Z}}$, i.e. the category whose objects are left R-modules (Definition B.0.20) and whose morphisms are left R-linear maps (Definition B.0.26).
3. The *category of right R-modules*, denoted by notations such as Mod_R or $\mathbf{Mod}-R$, is the category ${}_{\mathbb{Z}}\mathsf{Mod}_R$, i.e. the category whose objects are right R-modules (Definition B.0.20) and whose morphisms are right R-linear maps (Definition B.0.26).

The category of bimodules can be canonically identified with module categories over tensor product rings:

- ${}_R\mathsf{Mod}_S$ is isomorphic to the category of left modules over the ring $R\otimes_{\mathbb{Z}} S^{\mathrm{op}}$.
- ${}_R\mathsf{Mod}_S$ is isomorphic to the category of right modules over the ring $R^{\mathrm{op}}\otimes_{\mathbb{Z}} S$.

Consequently, standard module-theoretic concepts (such as projective objects, injective objects, and flat objects) in ${}_R\mathsf{Mod}_S$ correspond exactly to the respective concepts in ${}_{R\otimes S^{\mathrm{op}}}\mathsf{Mod}$.

Note that there are canonical isomorphisms of categories:

$$ {}_R\mathsf{Mod}\cong {}_R\mathsf{Mod}_{\mathbb{Z}} \quad \text{and} \quad \mathsf{Mod}_S\cong {}_{\mathbb{Z}}\mathsf{Mod}_S. $$

That is, left R-modules are exactly $(R,\mathbb{Z})$-bimodules, and right S-modules are exactly $(\mathbb{Z},S)$-bimodules.

Definition B.0.31. Let $\mathcal{C}$ and $\mathcal{D}$ be (large) categories (Definition B.0.15). Let $F,G:\mathcal{C}\to\mathcal{D}$ be functors (Definition B.0.29).

A *natural transformation η between F and G* is a family of morphisms $\eta_X:F(X)\to G(X)$ in $\mathcal{D}$, one for each object X in $\mathcal{C}$, such that for every morphism $f:X\to Y$ in $\mathcal{C}$,

$$ G(f)\circ\eta_X=\eta_Y\circ F(f) $$

in $\mathcal{D}$. In other words, the following diagram commutes:

$$ \begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \eta_X}\big\downarrow & & \big\downarrow{\scriptstyle \eta_Y} \\ G(X) & \xrightarrow[G(f)]{} & G(Y) \end{array} $$

We write such a natural transformation by $\eta:F\Rightarrow G$.

If η_X is an isomorphism for all objects X of $\mathcal{C}$, then η is said to be a *natural isomorphism*.

Definition B.0.32. An *equivalence of categories* between two (large) categories (Definition B.0.15) $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of functors (Definition B.0.29)

$$ F:\mathcal{C}\to\mathcal{D} \quad \text{and} \quad G:\mathcal{D}\to\mathcal{C} $$

together with natural isomorphisms (Definition B.0.31)

$$\eta : \mathrm{Id}_{\mathcal{C}} \xrightarrow{\sim} G \circ F \quad \text{and} \quad \epsilon : F \circ G \xrightarrow{\sim} \mathrm{Id}_{\mathcal{D}}.$$

Such functors F and G may be called *(natural) inverses of each other*.

When $\mathcal{C}$ and $\mathcal{D}$ are locally small categories (Definition B.0.16), F is an equivalence of categories if and only if F is fully faithful and essentially surjective

Definition B.0.33 (Opposite ring)**.** Let $R = (R, +, \cdot, 0, 1)$ be a ring with addition $+$, multiplication $\cdot$, additive identity 0, and multiplicative identity 1 (not necessarily commutative).

The *opposite ring of R*, denoted R^{op}, is the ring with the same underlying set R and the same addition $+$ and additive identity 0, but with multiplication defined by

$$r \star s := s \cdot r$$

for all $r, s \in R$.

That is, multiplication in R^{op} is the multiplication of R reversed in order.

If R is commutative (Definition B.0.12), then R and R^{op} are naturally isomorphic to each other.

Definition B.0.34 (Additive functor)**.**

1. Let $\mathcal{A}$ and $\mathcal{B}$ be pre-additive categories. A functor

   $$F : \mathcal{A} \to \mathcal{B}$$

   is an *additive functor* if for every pair of objects $A, A' \in \mathcal{A}$, the induced map

   $$F_{A,A'} : \mathrm{Hom}_{\mathcal{A}}(A, A') \to \mathrm{Hom}_{\mathcal{B}}(F(A), F(A'))$$

   is a group homomorphism of abelian groups, or equvialently if it is enriched over the category Ab of abelian groups.
2. Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories. A functor

   $$F : \mathcal{A} \to \mathcal{B}$$

   is an *additive functor* if it an additive functor of pre-additive categories and satisfies the following:
   - F sends the zero object $0_{\mathcal{A}}$ of $\mathcal{A}$ to the zero object $0_{\mathcal{B}}$ of $\mathcal{B}$, i.e.,

     $$F(0_{\mathcal{A}}) = 0_{\mathcal{B}}.$$

   - F preserves finite direct sums: For any finite family of objects $\{A_i\}_{i=1}^n$ in $\mathcal{A}$,

     $$F\left(\bigoplus_{i=1}^n A_i\right) \cong \bigoplus_{i=1}^n F(A_i)$$

     via the canonical isomorphism induced by F applied to the canonical injections and projections.

   In other words, F is a functor that is compatible with the additive structures on $\mathcal{A}$ and $\mathcal{B}$.

Definition B.0.35. Let R and S be rings. Let M be an R-S-bimodule (Definition B.0.20). The *opposite module* M^{op} is the abelian group M viewed as an S^{op}-R^{op}-bimodule. For $s \in S^{op}$ (Definition B.0.33), $r \in R^{op}$, and $m \in M^{op}$, the left S^{op}-action and right R^{op}-action are defined by:

$$s \cdot m = ms$$
$$m \cdot r = rm$$

where the operations on the right-hand side are the original actions of S and R on M.

Proposition B.0.36. Let M be an R-S-bimodule. The construction of the opposite module M^{op} satisfies the following properties:

1. M^{op} is a well-defined S^{op}-R^{op}-bimodule.
2. The assignment $M \mapsto M^{op}$ defines an additive functor (Definition B.0.34) from the category of R-S-bimodules (Definition B.0.30) to the category of S^{op}-R^{op}-bimodules. (♠ TODO: define category of bimodules)
3. $(M^{op})^{op}$ and M are naturally isomorphic (Definition B.0.31) as R-S-bimodules.

References

[Mil17] James S. Milne. *Algebraic Groups*, volume 170 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2017.