AUTOMORPHIC FORMS

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September 12, 2025

Contents

1. Modular forms

Definition 1.0.1. (♠ TODO: define congruence subgroups for more general arithmetic groups and describe how these are instances of those) Let N be a positive integer.

- The *principal congruence subgroup of* $\mathrm{SL}_2(\mathbb{Z})$ *of level* N is the group $\Gamma(N)$ defined by
$$\Gamma(N) := \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv I_2 \pmod N\}.$$
- The subgroup $\Gamma_1(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ is defined by
$$\Gamma_1(N) := \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod N,\ c \equiv 0 \pmod N\right\}.$$
- The subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ is defined by
$$\Gamma_0(N) := \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod N\right\}.$$
- A *congruence subgroup of* $\mathrm{SL}_2(\mathbb{Z})$ is any subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ that contains $\Gamma(N)$ for some $N \geq 1$.

Note that
$$\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z}).$$

Moreover, the natural reduction homomorphism

$$\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$$

is surjective with kernel $\Gamma(N)$. Therefore, we have an isomorphism

$$\mathrm{SL}_2(\mathbb{Z})/\Gamma(N) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}).$$

In particular, $\Gamma(N)$ and hence $\Gamma_1(N)$ and $\Gamma_0(N)$ are all of finite index in $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.

Lemma 1.0.2. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ (Definition 1.0.1). There is some integer $h \in \mathbb{Z}_{>0}$ such that $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$.

Definition 1.0.3 (Modular group action)**.** Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a subgroup. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\tau \in \mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, define the action

$$\gamma \cdot \tau := \frac{a\tau + b}{c\tau + d}.$$

This is called the *modular action of Γ on the upper half-plane $\mathbb{H}$*.

Definition 1.0.4 (Weight-k slash operator)**.** Let $k \in \mathbb{Z}$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and any function $f : \mathbb{H} \to \mathbb{C}$, the *weight-$k$ slash operator* (or simply the *weight-k operator*) is defined as

$$(f|_k\gamma)(\tau) := (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$

$f|_k\gamma$ is also often denoted by $f[\gamma]_k$.

Definition 1.0.5 (Weakly modular form)**.** Let $k \in \mathbb{Z}$, and let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a subgroup. A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a *weakly modular form of weight k with respect to Γ* if

$$f|_k\gamma = f \quad \text{for all } \gamma \in \Gamma.$$

(Definition 1.0.4).

Lemma 1.0.6. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ (Definition 1.0.1). Let $h \in \mathbb{Z}_{>0}$ be such that $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ by Lemma 1.0.2. Every weakly modular form f (Definition 1.0.5) of some weight $k \in \mathbb{Z}$ is $h\mathbb{Z}$-periodic, i.e. $f(z+h) = f(z)$ for all $z \in \mathbb{H}$ (♠ TODO: cite the upper half plane).

In particular, there exists a function $g : D' \to \mathbb{C}$ where $D' \subset \mathbb{C}$ is the punctured unit disk such that $f(z) = g(q_h)$ where $q_h = e^{2\pi i z/h}$.

Definition 1.0.7. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ (Definition 1.0.1). Let $h \in \mathbb{Z}_{>0}$ be minimal such that $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ by Lemma 1.0.2.

Let $f : \mathbb{H} \to \mathbb{C}$ be a meromorphic function of the form $f(z) = g(q_h)$ for some meromorphic function $g : D' \to \mathbb{C}$ on the punctured unit disk D' where $q_h = e^{2\pi i z/h}$ (e.g. weakly modular forms (Definition 1.0.5) satisfy this condition, see Lemma 1.0.6). In particular, g has a Laurent expansion near $q_h = 0$ (which corresponds to $\mathrm{Im}(z) \to \infty$) (♠ TODO: cite this as a complex analytic fact) and hence f has a Fourier expansion:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n q_h^n.$$

The values $a_n \in \mathbb{C}$ are called the *Fourier coefficients of f*.

We say that f is *meromorphic at ∞* if g is meromorphic at $q_h = 0$, i.e. the Fourier expansion of f has the form

$$f(z) = \sum_{n=n_0}^{\infty} a_n q_h^n.$$

for some $n_0 \in \mathbb{Z}$. We say that f is *holomorphic at ∞* if g is holomorphic at $q_h = 0$, i.e. the Fourier expansion of f has the form

$$f(z) = \sum_{n=0}^{\infty} a_n q_h^n.$$

If f is holomorphic at ∞ and we additionally have $a_0 = 0$, then we say that f has a *cusp at ∞*.

Definition 1.0.8. (♠ TODO: define holomorphic function, upper half plane) (♠ TODO: define cusps of Γ) Let k be an integer and $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ a congruence subgroup (Definition 1.0.1). A *modular form of weight k with respect to Γ* is a function $f : \mathbb{H} \to \mathbb{C}$, where $\mathbb{H}$ is the upper half-plane, such that

- The function f is holomorphic on $\mathbb{H}$.
- The funciton f is a weakly modular form of weight k with respect to Γ (Definition 1.0.5). Equivalently, for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and all $z \in \mathbb{H}$, we have
$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z).$$
- f is holomorphic at the cusps of Γ (i.e., its Fourier expansion at all cusps of Γ is holomorphic). More precisely, $f|_k\alpha$ (Definition 1.0.4) is holomorphic at ∞ (Definition 1.0.7) for all $\alpha \in \Gamma$.

A *cusp form* is a modular form f whose Fourier expansion at every cusp of Γ has constant term 0, i.e. $f|_k\alpha$ has a cusp at ∞ for every $\alpha \in \Gamma$.

Definition 1.0.9. (♠ TODO: define holomorphic function, upper half plane) (♠ TODO: define cusps of Γ) Let k be an integer and $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ a congruence subgroup (Definition 1.0.1). A *modular form of weight k with respect to Γ* is a function $f : \mathbb{H} \to \mathbb{C}$, where $\mathbb{H}$ is the upper half-plane, such that

- The function f is meromorphic on $\mathbb{H}$.

- The funciton f is a weakly modular form of weight k with respect to Γ (Definition 1.0.5). Equivalently, for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and all $z \in \mathbb{H}$, we have

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z).$$

- f is meromorphic at the cusps of Γ (i.e., its Fourier expansion at all cusps of Γ is holomorphic). More precisely, $f|_k\alpha$ (Definition 1.0.4) is meromorphic at ∞ (Definition 1.0.7) for all $\alpha \in \Gamma$.

We define the L-function of a modular form by putting the Fourier coefficients of the modular form into a Dirichlet series.

Definition 1.0.10 (L-function of a modular form)**.** Let $k \in \mathbb{Z}_{>0}$, and let $f : \mathbb{H} \to \mathbb{C}$ be a modular form (Definition 1.0.8) of weight k with respect to a congruence subgroup (Definition 1.0.1) $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$. Let

$$f(\tau) = \sum_{n=0}^{\infty} a_n e^{\frac{2\pi i n\tau}{h}}$$

be the Fourier expansion of f (Definition 1.0.7) where $h > 0$ is an appropriate integer.

The *L-function of the modular form f* is the complex function defined for $s \in \mathbb{C}$ with sufficiently large $\mathrm{Re}(s)$ by the Dirichlet series

$$L(f, s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

(♠ TODO: define the completed L-function)

The construction of the L-function of a modular form is not arbitrary — a Fourier transform involving the modular form yields the completed L-function.

Theorem 1.0.11. Let $k \in \mathbb{Z}_{>0}$, and let $f : \mathbb{H} \to \mathbb{C}$ be a modular form (Definition 1.0.8) of weight k with respect to a congruence subgroup (Definition 1.0.1) $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$. Let

$$f(\tau) = \sum_{n=0}^{\infty} a_n e^{\frac{2\pi i n\tau}{h}}$$

be the Fourier expansion of f (Definition 1.0.7) where $h > 0$ is an appropriate integer.

The Mellin transform of $f(i\tau)$ equals the completed L-function $\Lambda(f, s)$.

Proof. Let $f : \mathbb{H} \to \mathbb{C}$ be a meromorphic function periodic with period $h > 0$, so

$$f(z + h) = f(z).$$

This implies that f factors through the variable

$$q_h := e^{2\pi i z/h}.$$

Define

$$\Phi(t) := f(it), \quad t > 0.$$

Using the periodicity, f has a Fourier expansion in q_h:

$$f(z) = \sum_{n=n_0}^{\infty} a_n q_h^n = \sum_{n=n_0}^{\infty} a_n e^{2\pi i n z/h}.$$

Substituting $z = it$, we have

$$\Phi(t) = f(it) = \sum_{n=n_0}^{\infty} a_n e^{-2\pi n t/h}.$$

Consider the Mellin transform integral with complex parameter s:

$$I(s) := \int_0^{\infty} \Phi(t) t^s \frac{dt}{t}.$$

For $\mathrm{Re}(s)$ sufficiently large, interchange sum and integral (justified by absolute convergence):

$$I(s) = \sum_{n=n_0}^{\infty} a_n \int_0^{\infty} e^{-2\pi n t/h} t^s \frac{dt}{t}.$$

Evaluate the integral inside for each fixed n. Change variables by setting $u = \frac{2\pi n t}{h}$, so $t = \frac{hu}{2\pi n}$, and

$$dt = \frac{h}{2\pi n} du.$$

Hence,

$$\int_0^{\infty} e^{-2\pi n t/h} t^s \frac{dt}{t} = \int_0^{\infty} e^{-u} \left(\frac{hu}{2\pi n}\right)^s \frac{du}{u} = \left(\frac{h}{2\pi n}\right)^s \int_0^{\infty} e^{-u} u^s \frac{du}{u}.$$

Recognizing the Gamma function, we get

$$\int_0^{\infty} e^{-u} u^s \frac{du}{u} = \Gamma(s).$$

Therefore,

$$I(s) = \Gamma(s) \left(\frac{h}{2\pi}\right)^s \sum_{n=n_0}^{\infty} a_n n^{-s}.$$

By definition, the completed L-function $\Lambda(f, s)$ associated to f is proportional to this Mellin transform expression:

$$\Lambda(f, s) = \Gamma(s) \left(\frac{h}{2\pi}\right)^s L(f, s),$$

where

$$L(f, s) := \sum_{n=n_0}^{\infty} \frac{a_n}{n^s}.$$

This establishes the integral representation of $\Lambda(f,s)$ as the Mellin transform of $\Phi(t) = f(it)$, adapted to period h. $\square$

1.1. L-function of a modular form.

2. Automorphic forms

Definition 2.0.1 (Adelic Quotient)**.** Let F be a global field (Definition B.0.1), and let $\mathbb{A}_F$ be the ring of adeles (Definition B.0.4) of F. Let G be a linear algebraic group (Definition B.0.7) defined over F. The *adelic quotient of G over F* is the quotient space

$$G(F)\backslash G(\mathbb{A}_F).$$

(Definition B.0.5) where $G(F)$ is given the discrete topology. (♠ TODO: What topology does $G(F)$ have?)

Definition 2.0.2 (Automorphy)**.** Let F be a global field (Definition B.0.1), and let $\mathbb{A}_F$ be the ring of adeles of F (Definition B.0.4). Let G be a linear algebraic group (Definition B.0.7) defined over F.

Let $f : G(\mathbb{A}_F) \to \mathbb{C}$ be a function. We say f satisfies *automorphy* if it is left $G(F)$-invariant, i.e.,

$$f(\gamma g) = f(g) \quad \text{for all } \gamma \in G(F),\ g \in G(\mathbb{A}_F).$$

Equivalently, f is a function which descends to a well defined function $f : G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C}$ on the adelic quotient (Definition 2.0.1).

Notation 2.0.3 (Right-regular action of $G(\mathbb{A}_F)$)**.** Let F be a global field (Definition B.0.1), and let $\mathbb{A}_F$ be the ring of adeles (Definition B.0.4) of F. Let G be a linear algebraic group (Definition B.0.7) defined over F. For $g \in G(\mathbb{A}_F)$ (Definition B.0.5) and a function $f : G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C}$, the *right-regular action/representation of $G(\mathbb{A}_F)$* is the representation

$$R : G(\mathbb{A}_F) \to \operatorname{Aut}(\operatorname{Fun}(G(F)\backslash G(\mathbb{A}_F), \mathbb{C}))$$

on the space of complex valued functions on $G(F)\backslash G(\mathbb{A}_F)$ given by

$$(R(g)f)(x) = f(xg)$$

for $g \in G(\mathbb{A}_F)$, $f : G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C}$, and $x \in G(F)\backslash G(\mathbb{A}_F)$.

Definition 2.0.4 (Smooth function)**.** Let F be a global field (Definition B.0.1), and let $\mathbb{A}_F$ be the ring of adeles of F (Definition B.0.4). Let G be a linear algebraic group (Definition B.0.7) defined over F.

Let $f : G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C}$ be a function. We say f is *smooth* if it is (♠ TODO: define $G(\mathbb{A}_F)$, $G(\mathbb{A}_F^{\mathrm{fin}})$, $G(F_\infty)$)

1. locally constant in the non-archimedean directions, i.e. for every point $g = (g_v)_v \in G(\mathbb{A}_F)$, there exists an open compact subgroup $U = \prod_v U_v \subset G(\mathbb{A}_F^{\mathrm{fin}})$ (Definition B.0.5) such that for all $u \in U$, we have

$$f(gu) = f(g),$$

and

2. C^∞ in each archimedean direction, i.e. for every point $g = (g_v)_v \in G(\mathbb{A}_F)$, the map
$$G(F_\infty) \to \mathbb{C}, (h_v)_{v|\infty} \mapsto f(g \cdot (h_v)_{v|\infty})$$
(Definition B.0.5) is C^∞ as a smooth manifold; note that $G(F_\infty) = \prod_{v|\infty} G(F_v)$ is a product of real and complex Lie groups. (♠ TODO: define real and omcplex lie groups)

Definition 2.0.5 (K-finiteness)**.** Let F be a global field (Definition B.0.1), and let $\mathbb{A}_F$ be the ring of adeles of F (Definition B.0.4). Let G be a linear algebraic group (Definition B.0.7) defined over F.

Let $K \subseteq G(\mathbb{A}_F)$ be a maximal compact subgroup. A smooth function (Definition 2.0.4) $f : G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C}$ is called *K-finite* if the vector space
$$\langle R(k)f : \ k \in K\rangle$$
(Notation 2.0.3) spanned by the K-translates of f under the right-regular action (Notation 2.0.3) is finite-dimensional.

Definition 2.0.6 (Moderate growth)**.** Let F be a global field (Definition B.0.1), and let $\mathbb{A}_F$ be the ring of adeles of F (Definition B.0.4). Let G be a linear algebraic group (Definition B.0.7) defined over F. Let $\|\cdot\|$ be a fixed height function (or norm-like function) on $G(\mathbb{A}_F)$ compatible with F and G (e.g. the height function of Definition B.0.6).

A smooth function (Definition 2.0.4) $f : G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C}$ is said to have *moderate growth* if there exist constants $C > 0$ and $N \geq 0$ such that
$$|f(g)| \leq C \cdot \|g\|^N \quad \text{for all } g \in G(\mathbb{A}_F),$$

Definition 2.0.7 (Automorphic form)**.** Let F be a global field (Definition B.0.1), and let $\mathbb{A}_F$ be the ring of adeles of F (Definition B.0.4). Let G be a linear algebraic group (Definition B.0.7) defined over F.

A function $f : G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C}$ is called an *automorphic form* if it satisfies all of the following conditions:

1. f is smooth (Definition 2.0.4),
2. f is K-finite (Definition 2.0.5) for some maximal compact subgroup $K \subseteq G(\mathbb{A}_F)$,
3. f has moderate growth (Definition 2.0.6),

(♠ TODO: read the following AI generated statements and make them precise) (♠ TODO: Formulate how modular forms are automorphic forms)

Theorem 2.0.8 (Basic Properties of Automorphic Forms)**.** Let F be a global field, $\mathbb{A}_F$ its ring of adeles, and G a linear algebraic group defined over F. Let $K \subseteq G(\mathbb{A}_F)$ be a maximal compact subgroup. Suppose
$$f : G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C}$$
is an automorphic form, i.e., f is smooth, K-finite, and of moderate growth. Then:

1. f is left-invariant under $G(F)$ (automorphy condition).

2. f generates a $(\mathfrak{g}, K)$-module under the right regular representation of $G(\mathbb{A}_F)$, where $\mathfrak{g}$ is the Lie algebra of $G(F_\infty)$.
3. The space of automorphic forms is preserved by Hecke operators arising from elements in the spherical Hecke algebra associated with $G(\mathbb{A}_F^\infty)$.

Proposition 2.0.9 (Equivalent Characterizations of Automorphic Forms)**.** Under standard assumptions on G, the following conditions on a function $f : G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C}$ are equivalent:

1. f is smooth, K-finite, of moderate growth, and left $G(F)$-invariant.
2. f corresponds to a vector in a smooth admissible representation of $G(\mathbb{A}_F)$ realized as a subrepresentation of the space of L^2 automorphic forms.

Corollary 2.0.10 (Automorphic Forms as Sections)**.** Automorphic forms can be identified with smooth sections of certain automorphic vector bundles over the quotient space $G(F)\backslash G(\mathbb{A}_F)/K$ satisfying growth conditions.

Remark 2.0.11 (Intuition)**.** Automorphic forms generalize classical modular forms by encoding arithmetic and representation-theoretic data through functions on adelic quotients. The conditions of smoothness, K-finiteness, and moderate growth control analytic and algebraic properties needed for harmonic analysis and spectral theory on these quotients.

Appendix A. Miscellaneous definitions

Definition A.0.1 (Topological groups)**.** (♠ TODO: Product topology) A *topological group* is a group $(G, \cdot)$ together with a topology $\mathcal{T}$ on G such that the maps

$$\begin{aligned} \mu : G \times G &\to G, & (g,h) &\mapsto g \cdot h, \\ \iota : G &\to G, & g &\mapsto g^{-1}, \end{aligned}$$

are continuous with respect to the product topology on $G \times G$ and the topology $\mathcal{T}$ on G.

Appendix B. Adèles and idéles of global fields

Definition B.0.1. A *global field* is a field K that is either:

- a finite extension of the field of rational numbers $\mathbb{Q}$ (i.e., a number field), or
- a finite extension of the field of rational functions $\mathbb{F}_q(t)$ in one variable over a finite field $\mathbb{F}_q$ (i.e., a global function field).

Definition B.0.2. Let K be a global field (Definition B.0.1) and let v be a place of K. Write $|\cdot|_v$ for an absolute value representing v. The *completion of K at v*, often denoted K_v, is the completion of K with respect to the metric induced by $|\cdot|_v$.

Definition B.0.3. Let $\{X_i\}_{i\in I}$ be a family of topological spaces indexed by a set I. For each $i \in I$, let $K_i \subseteq X_i$ be a topological subspace.

The *restricted product topology* on the restricted product

$$\prod_{i\in I}{}' X_i := \left\{ (x_i)_{i\in I} \in \prod_{i\in I} X_i \;\middle|\; x_i \in K_i \text{ for all but finitely many } i \in I \right\},$$

with respect to the subsets $\{K_i\}_{i\in I}$, is the coarsest topology such that:

- The natural inclusion maps $X_j \to \prod'_{i\in I} X_i$, defined by $x_j \mapsto (y_i)$ where $y_j = x_j$ and $y_i = k_i$ (a fixed element in K_i) for all $i \neq j$, are continuous for all $j \in I$.
- The subspace topology on the product $\prod_{i\in F} X_i$ for any finite subset $F \subseteq I$ (where coordinates outside F are fixed in K_i) coincides with the product topology on finitely many factors.

Equivalently, the restricted product topology is generated by the base consisting of sets of the form

$$\prod_{i\in F} U_i \times \prod_{i\notin F} K_i,$$

where $F \subseteq I$ is finite, U_i are open sets in X_i, and outside F the coordinates lie in K_i.

Definition B.0.4. Let K be a global field. Write M_K for the set of all places of K and write M_K^∞ for the set of archimedean places of K. Let $S \subseteq M_K$ be some subset of places of K (typically, S is a finite set). For each $v \in M_K$, write $\mathcal{O}_v$ for the ring of integers in the completion K_v (Definition B.0.2)

- The *adèle ring of K*, denoted $\mathbb{A}_K$, is the restricted direct product of the K_v (over all places v of K), with respect to the $\mathcal{O}_v$ at non-archimedean v:

$$\mathbb{A}_K = \left\{ (x_v)_v \in \prod_{v\in M_K} K_v \;\middle|\; x_v \in \mathcal{O}_v \text{ for all but finitely many non-archimedean } v \right\}.$$

- The *idèle group of K*, commonly denoted $\mathbb{A}_K^\times$ or $\mathbb{I}_K$, is the group of invertible elements of $\mathbb{A}_K$:

$$\mathbb{I}_K = \mathbb{A}_K^\times = \left\{ (x_v)_v \in \prod_{v\in M_K} K_v^\times \;\middle|\; x_v \in \mathcal{O}_v^\times \text{ for all but finitely many non-archimedean } v \right\},$$

  where $\mathcal{O}_v^\times$ denotes the group of units of $\mathcal{O}_v$ for non-archimedean v.
- The *adèle ring outside S of K*, commonly denoted $\mathbb{A}_K^S$ or $\mathbb{A}_{K,S}$, is the restricted product of the completions K_v over all places $v \in M_K \setminus S$, with respect to the rings of integers $\mathcal{O}_v$ at non-archimedean places:

$$\mathbb{A}_{K,S} = \mathbb{A}_K^S = \left\{ (x_v)_v \in \prod_{v\in M_K\setminus S} K_v \;\middle|\; x_v \in \mathcal{O}_v \text{ for all but finitely many non-archimedean } v \right\}.$$

- The *idèle group outside S of K*, commonly denoted $(\mathbb{A}_K^\times)^S$, $(\mathbb{A}_{K,S}^\times)$, $\mathbb{I}_K^S$, or $\mathbb{I}_{K,S}$ is the group of invertible elements of $\mathbb{A}_K^S$:

$$(\mathbb{A}_K^\times)^S = \left\{ (x_v)_v \in \prod_{v\in M_K\setminus S} K_v^\times \;\middle|\; x_v \in \mathcal{O}_v^\times \text{ for all but finitely many non-archimedean } v \right\}.$$

- The *ring of finite adèles of K*, commonly denoted $\mathbb{A}_{K,\mathrm{fin}}$, $\mathbb{A}_K^{\mathrm{fin}}$, $\mathbb{A}_{K,\mathrm{f}}$, $\mathbb{A}_K^{\mathrm{f}}$, is the adèle ring outside $S = M_K^\infty$, the set of archimedean places of K:

$$\mathbb{A}_{K,\mathrm{fin}} := \mathbb{A}_K^{M_K^\infty} = \left\{ (x_v)_v \in \prod_{v\notin M_K^\infty} K_v \;\middle|\; x_v \in \mathcal{O}_v \text{ for all but finitely many non-archimedean } v \right\}.$$

- The *finite idèle group of K*, commonly denoted $\mathbb{A}_{K,\mathrm{fin}}^\times$, $\mathbb{I}_{K,\mathrm{fin}}$, $\mathbb{I}_K^{\mathrm{fin}}$, $\mathbb{I}_{K,\mathrm{f}}$, $\mathbb{I}_K^{\mathrm{f}}$ etc. is the group of units of the ring of finite adèles:

$$\mathbb{A}_{K,\mathrm{fin}}^\times := (\mathbb{A}_K^\times)^{M_K^\infty} = \left\{ (x_v)_v \in \prod_{v\notin M_K^\infty} K_v^\times \;\middle|\; x_v \in \mathcal{O}_v^\times \text{ for all but finitely many non-archimedean } v \right\}.$$

All of these are equipped with the restricted product topology induced by the topologies of the local fields K_v and the subspace topologies thereof.

Definition B.0.5 (Adelic points of an algebraic group)**.** Let K be a global field (Definition B.0.1) with ring of integers $\mathcal{O}_K$, and let G be a linear algebraic group (Definition B.0.7) defined over K.

- The *group of adelic points of G* is the group $G(\mathbb{A}_K)$ defined by

$$G(\mathbb{A}_K) := \prod\nolimits_v' G(K_v),$$

(Definition B.0.2) where the restricted product is taken with respect to the compact open subgroups $G(\mathcal{O}_v)$ for almost all non-archimedean v.
- For a set S (usually finite) of places of K, the group of adelic points of G outside S is the group $G(\mathbb{A}_{K,S})$ defined by

$$G(\mathbb{A}_{K,S}) := \prod\nolimits_{v\notin S}' G(K_v).$$

- The group of finite adelic points is the group $G(\mathbb{A}_{K,\mathrm{fin}}) = G(\mathbb{A}_K^{\mathrm{fin}})$ defined by

$$G(\mathbb{A}_K^{\mathrm{fin}}) := \prod\nolimits_{v \text{ non-archimedean}}' G(K_v).$$

- The group of archimedean adelic points is the group $G(K_\infty)$ defined by

$$G(K_\infty) := \prod\nolimits_{v \text{ archimedean}} G(K_v).$$

Thus one has a natural factorization

$$G(\mathbb{A}_K) \;\cong\; G(\mathbb{A}_K^{\mathrm{fin}}) \times G(K_\infty).$$

The groups $G(\mathbb{A}_K)$, $G(\mathbb{A}_{K,S})$, and $G(\mathbb{A}_K^{\mathrm{fin}})$ are all topological groups (Definition A.0.1) under the restricted product topology (Definition B.0.3) induced by the topologies of the local fields K_v and the subspace topologies of $\mathcal{O}_v$ and become topological groups. The group $G(K_\infty)$ is a topological group under the direct product topology.

Definition B.0.6 (Height function on $G(\mathbb{A}_F)$)**.** Let F be a global field (Definition B.0.1), and let $\mathbb{A}_F$ be the ring of adeles of F (Definition B.0.4). Let G be a linear algebraic group (Definition B.0.7) defined over F embedded as an algebraic subgroup in GL_n.

For each place v of F, let $\|\cdot\|_v$ be a norm on $G(F_v) \subseteq \mathrm{GL}_n(F_v)$ defined as follows:

$$\|g_v\|_v := \max_{1 \leq i,j \leq n} |(g_v)_{ij}|_v,$$

(♠ TODO: define the usual absolute value on F_v) where $|\cdot|_v$ is the usual absolute value or valuation on F_v.

Then, for $g = (g_v)_v \in G(\mathbb{A}_F)$, define the adelic height function by the product

$$\|g\| := \prod_v \|g_v\|_v,$$

where the product converges due to $\|g_v\|_v = 1$ for all but finitely many non-archimedean places.

Definition B.0.7 (Linear algebraic group over a scheme)**.** Let S be a base scheme. A *linear algebraic group over S* is an affine group scheme G over S that is finitely presented and smooth over S, and such that for some integer $n \geq 1$, there exists a closed immersion of S-group schemes

$$G \hookrightarrow \mathrm{GL}_{n,S}\,.$$

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