ÉTALE SHEAVES ON SCHEMES

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Contents

1. Definitions

1.1. Étale morphisms of schemes.

Notation 1.1.1. Let S be a scheme. Let Sch/S denote the category of schemes over S (Definition .3.3).

Definition 1.1.2 (Locally of finite presentation morphism of schemes)**.** Let $f : X \to Y$ be a morphism of schemes.

1. We say that f is *locally of finite presentation* if for every affine open subset $\operatorname{Spec}(B) \subseteq Y$ (Definition .3.4), and every affine open subset $\operatorname{Spec}(A) \subseteq f^{-1}(\operatorname{Spec}(B))$, the induced ring homomorphism $B \to A$ presents A as a B-algebra of finite presentation (Definition .3.6); that is, A is isomorphic to a quotient of a polynomial ring in finitely many variables over B by a finitely generated ideal:
$$A \cong B[x_1, \ldots, x_n]/(f_1, \ldots, f_m),$$
for some finite n, m.
2. A morphism of schemes $f : X \to Y$ is of *finite presentation* if it is locally of finite presentation and quasi-compact and quasi-separated; in particular, f can be covered by finitely many affine opens satisfying the finite presentation condition above.

Definition 1.1.3 (Flat module over a ring)**.** Let R be a (not necessarily commutative) ring (Definition C.0.8).

1. Let M be a left R-module. The module M is said to be *flat (with respect to the left R-module structure)* if the functor
$$- \otimes_R M : \mathrm{Mod}_R \to \mathbf{Ab}$$
(Definition C.0.12) from the category of right R-modules to abelian groups is exact; that is, for every exact sequence of right R-modules
$$0 \to N' \to N \to N'' \to 0,$$
the induced sequence
$$0 \to N' \otimes_R M \to N \otimes_R M \to N'' \otimes_R M \to 0$$
is exact.
(♠ TODO: tor) Equivalently, M is flat if $\operatorname{Tor}_1^R(-, M) = 0$.
2. Let M be a right R-module. The module M is said to be *flat (with respect to the right R-module structure)* if the functor
$$M \otimes_R - : {}_R\mathrm{Mod} \to \mathbf{Ab}$$
from the category of left R-modules to abelian groups is exact; that is, for every exact sequence of right R-modules
$$0 \to N' \to N \to N'' \to 0,$$
the induced sequence
$$0 \to M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0$$

is exact.

Definition 1.1.4 (Flat morphism of schemes)**.** Let $f : X \to Y$ be a morphism of schemes (Definition .3.2).

1. Let $x \in X$ be a point and let $y = f(x)$. We say that f is *flat at x* if the induced ring homomorphism on local rings (Definition C.0.21)
$$\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$$
makes $\mathcal{O}_{X,x}$ (Definition C.0.23) into a flat (Definition 1.1.3) $\mathcal{O}_{Y,y}$-module.
2. We say f is *flat* if it is flat at every point $x \in X$.
3. f is *faithfully flat* if it is flat and surjective.

Definition 1.1.5 (Unramified morphism of schemes)**.** (♠ TODO: sheaf of relative differentials) A morphism of schemes (Definition .3.2) $f : X \to Y$ is *unramified* if it is locally of finite type (Definition C.0.19) and the sheaf of relative differentials $\Omega_{X/Y}$ is zero. Equivalently:

- For every $x \in X$, the induced ring map on stalks $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is of finite type,
- and the module of Kähler differentials $\Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}}$ is 0.

Definition 1.1.6. A morphism of schemes (Definition .3.2) $f : X \to Y$ is called *étale* if it satisfies the following conditions: (♠ TODO: sheaf of relative differentials)

- f is locally of finite presentation (Definition 1.1.2),
- f is flat (Definition 1.1.4),
- f is unramified (Definition 1.1.5), i.e., the sheaf of relative differentials $\Omega_{X/Y}$ equals 0.

(♠ TODO: relative dimension) Equivalently, a morphism of schemes is étale if and only if it is smooth of relative dimension 0. A finite (Definition 1.1.8) étale morphism is synonymously called a *finite étale cover*.

Definition 1.1.7. Let S be a scheme and let $f : X \to S$ be a morphism of schemes over S. The *automorphism group of X over S*, denoted $\mathrm{Aut}(X/S) = \mathrm{Aut}_S(X)$, is the group of all S-scheme (Definition .3.3) automorphisms of X, i.e., all isomorphisms of schemes $\varphi : X \to X$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "\varphi", "\sim"'] \arrow[dr, "f"'] & & X \arrow[dl, "f"] \\ & S & \end{tikzcd}$$

commutes. The group operation is composition of morphisms.

Definition 1.1.8. Let $f : X \to Y$ be an affine morphism of schemes (Definition .3.4). We say that f is a *finite morphism* if for every affine open $V = \operatorname{Spec} B \subseteq Y$ with $U = f^{-1}(V) = \operatorname{Spec} A$, the ring A is a finite B-algebra.

Definition 1.1.9. Let S be a scheme. Let $f : X \to Y$ over S be a finite morphism (Definition 1.1.8). It is called *Galois* (or synonymously a *Galois cover/Galois covering*) if there exists a finite group G acting on X over Y such that

- f is identified with the quotient map $X \to X/G$.

- the group action realizes Y as the categorical quotient under the group action.

If f is Galois, then the finite group G is isomorphic to the automorphism group $\mathrm{Aut}_Y(X)$. By a *Galois G-cover/morphism*, we mean a Galois morphism whose automorphism group is (isomorphic to) G, often equipped with a fixed isomorphism $\phi : G \to \mathrm{Aut}_Y(X)$.

In case f is an étale morphism, then f is equivalently Galois if there exists a finite group G acting on X over Y such that

- f is identified with the quotient map $X \to X/G$.
- G acts simply transitively on geometric fibres.

2. The étale fundamental group of a scheme at a geometric point

Definition 2.0.1. Let S be a scheme. A *geometric point of S* is a morphism $\overline{s} : \mathrm{Spec}(\Omega) \to S$ where Ω is an algebraically closed field.

Definition 2.0.2. (♠ TODO: TODO: define profinite topology) Let S be a connected scheme. The *étale fundamental group of S at a geometric point $\overline{s} : \mathrm{Spec}(\Omega) \to S$* is the profinite group

$$\pi_1^{\text{ét}}(S, \overline{s}) := \varprojlim_{X \to S} \mathrm{Aut}_S(X),$$

(Definition 1.1.7) where the inverse limit is taken over all finite étale covers $X \to S$ pointed above $\overline{s}$. Equivalently, one may take the limit over all finite étale Galois covers $X \to S$ pointed above $\overline{s}$. The étale fundamental group is equipped with the profinite topology.

Theorem 2.0.3. Let S be a connected scheme with a chosen geometric point $\overline{s}$.

1. The functor
$$X \mapsto \mathrm{Hom}_S(\overline{s}, X).$$
is an equivalence of categories between the category of finite étale covers of S and the category of finite sets (equipped with the discrete topology) with a continuous (left) action of the étale fundamental group $\pi_1^{\text{et}}(S, \overline{s})$ (Definition 2.0.2). We also note that the discrete set $\mathrm{Hom}_S(\overline{s}, X)$ is identifiable with the fiber $X \times_S \overline{s}$.
2. There is a natural bijection between isomorphism classes of finite étale Galois covers (Definition 1.1.9) $f : X \to S$ with Galois group isomorphic to G and continuous group homomorphisms

(A)
$$\varphi : \pi_1^{\text{et}}(S, \overline{s}) \to G$$

up to conjugation in G. More explicitly, two homomorphism φ, φ' correspond to equivalent finite étale Galois G-covers of S if and only if there exists some $g \in G$ such that
$$\varphi'(\gamma) = g\varphi(\gamma)g^{-1}$$
for all $\gamma \in \pi_1^{\text{ét}}(S, \overline{s})$.

Concretely, given a finite étale Galois cover $f : X \to S$, the fiber $f^{-1}(\overline{s}) \cong X \times_S \overline{s}$ has an action of $\pi_1^{\text{ét}}(S, \overline{s})$. Since G acts simply transitively on this fiber, there is an induced group homomorphism (A); the conjugacy class of this group homomorphism corresponds to f.

Furthemore, the covering scheme X is connected if and only if the homomorphism φ is surjective.

3. Sheaves on the small étale site of a scheme

We define presheaves generally.

Definition 3.0.1 (Presheaf on a category)**.** Let C and $\mathcal{A}$ be (large) categories (Definition .1.1).

1. A *presheaf $\mathcal{F}$ on C with values in $\mathcal{A}$* is a functor
$$\mathcal{F} : C^{\mathrm{op}} \to \mathcal{A}.$$
In other words, a presheaf $\mathcal{F}$ on C with values in $\mathcal{A}$ is simply a contravariant functor from C to $\mathcal{A}$. Explicitly, for every object U in C, one has an object $\mathcal{F}(U)$ in $\mathcal{A}$ (called the *U-valued sections/sections evaluated at U of $\mathcal{F}$*, cf. Definition 6.1.1), and for every morphism $f : V \to U$ in C, one has a morphism (called the *restriction map*)
$$\mathcal{F}(f) : \mathcal{F}(U) \to \mathcal{F}(V)$$
in $\mathcal{A}$, such that for all composable morphisms $W \xrightarrow{g} V \xrightarrow{f} U$ in C, the following diagram in $\mathcal{A}$ commutes:
$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\mathcal{F}(f)"] \arrow[rr, bend left, "\mathcal{F}(f\circ g)"] & \mathcal{F}(V) \arrow[r, "\mathcal{F}(g)"] & \mathcal{F}(W) \end{tikzcd}$$
That is,
$$\mathcal{F}(g) \circ \mathcal{F}(f) = \mathcal{F}(f \circ g),$$
and for every object U in C, $\mathcal{F}(\mathrm{id}_U) = \mathrm{id}_{\mathcal{F}(U)}$.
2. Let $\mathcal{F}, \mathcal{G} : C^{\mathrm{op}} \to \mathcal{A}$ be two presheaves on C with values in $\mathcal{A}$. A *morphism of presheaves*
$$\varphi : \mathcal{F} \to \mathcal{G}$$
is a natural transformation of functors: for each object U of C, one has a morphism
$$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$
in $\mathcal{A}$, such that for every morphism $f : V \to U$ in C, the diagram
$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\varphi_U"'] & \mathcal{F}(V) \arrow[d, "\varphi_V"] \\ \mathcal{G}(U) \arrow[r, "\mathcal{G}(f)"'] & \mathcal{G}(V) \end{tikzcd}$$
commutes, i.e.,
$$\varphi_V \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \varphi_U$$
for all objects and morphisms in C.

3. Given a universe (Definition .1.2) U, a *U-presheaf on $\mathcal{C}$* typically refers to a presheaf of U-sets on C.
4. The *presheaf category/category of $\mathcal{A}$-valued presheaves on $\mathcal{C}$* is the (large) category whose objects are the presheaves on C with values in $\mathcal{A}$ and whose morphisms are the presheaf morphisms. Common notations for the presheaf category include, but are not limited to: $\mathcal{A}^{\mathcal{C}^{\mathrm{op}}}$, $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$, $[\mathcal{C}^{\mathrm{op}}, \mathcal{A}]$. If the value category $\mathcal{A}$ is clear from context, then notations such as $\mathrm{PreShv}(\mathcal{C})$ are also common. Note that the presheaf category $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ is equivalent to the category of functors (Definition .1.25) $\mathcal{C}^{\mathrm{op}} \to \mathcal{A}$ and hence notations for the functor categories are applicable as notations for presheaf categories.

We can speak of sheaves on a general site.

We will be interested in sheaves of sets/abelian groups/R-modules on the small étale site $X_{\text{ét}}$ (Definition 3.1.3) of any scheme as $X_{\text{ét}}$ is essentially small (Definition .1.18).

Definition 3.0.2 (Sheaf on a site)**.** Let $(\mathcal{C}, J)$ be a site (Definition A.0.4). Let $\mathcal{A}$ be a (large) category (Definition .1.1).

1. A presheaf (Definition 3.0.1) $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$ is called a *sheaf on the site $(\mathcal{C}, J)$ valued in $\mathcal{A}$* if, for every object U of $\mathcal{C}$ and every covering sieve (Definition A.0.4) $S \in J(U)$, the limit (Definition .1.12)
$$\varprojlim_{(V \to U) \in (\mathcal{D}_S)^{\mathrm{op}}} \mathcal{F}|_{\mathcal{D}_S}(V),$$
exists and the canonical natural morphism
$$\mathcal{F}(U) \to \varprojlim_{(V \to U) \in (\mathcal{D}_S)^{\mathrm{op}}} \mathcal{F}|_{\mathcal{D}_S}(V)$$
is an isomorphism. Here, $\mathcal{D}_S \hookrightarrow \mathcal{C}/U$ (Definition A.0.5) is the full downward-closed subcategory such that $\mathrm{Ob}(\mathcal{D}_S) = \{(f : V \to U) : f \in S(V)\}$,
   In particular, when we are working with a Grothendieck pretopology (Definition 3.2.2) K on a category $\mathcal{C}$, we may speak of sheaves on the site whose Grothendieck topology is the one generated by (Definition 3.2.1) K.
2. Given sheaves $\mathcal{F}, \mathcal{G} : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$ on the site $(\mathcal{C}, J)$, a *morphism between the sheaves* is a morphism (Definition 3.0.1) between $\mathcal{F}$ and $\mathcal{G}$ as presheaves.
3. Let U be a universe (Definition .1.2). A *U-sheaf* typically refers to a U-presheaf that is a sheaf for a U-site. In other words, a U-sheaf is a sheaf on a site whose underlying category is U-locally small (Definition .1.4) and which has a U-small topologically generating family such that the sheaf is valued in U-sets.
4. The *sheaf category/category of $\mathcal{A}$-valued sheaves on $\mathcal{C}$* is the (large) category defined as the full subcategory of $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ whose objects are the sheaves on $\mathcal{C}$ with values in $\mathcal{A}$. Common notations for the sheaf category include $\mathrm{Shv}(\mathcal{C}, \mathcal{A})$, $\mathrm{Shv}(\mathcal{C}, J, \mathcal{A})$, $\mathrm{Sh}(\mathcal{C}, \mathcal{A})$, $\mathrm{Sh}(\mathcal{C}, J, \mathcal{A})$. If the value category $\mathcal{A}$ is clear from context, then notations such as $\mathrm{Shv}(\mathcal{C})$, $\mathrm{Shv}(\mathcal{C}, J)$, $\mathrm{Sh}(\mathcal{C})$, $\mathrm{Sh}(\mathcal{C}, J)$ are also common.

3.1. **The small and big Étale sites of a scheme.**

Definition 3.1.1 (e.g. see [Mil80, II §1]). Let S be a scheme (Definition .3.1). Let E be some class of morphisms in Sch/S (Definition .3.3) satisfying the following:

- All isomorphisms are in E.
- E is closed under compositions.
- Any base change of a morphism in E is in E.

1. Let C/S be some full subcategory of Sch/S that is closed under fiber products such that for any $X \to S$ in C/S and any E-morphism $U \to X$, the composite $U \to S$ is in C/S. An *E-covering of an object X of C/S* is a family $(U_i \xrightarrow{g_i} X)_{i \in I}$ of E-morphisms such that $Y = \bigcup_i g_i(X_i)$. The class of all such coverings of all such objects is the *E-topology on C/S*; it is a Grothendieck pretopology (Definition 3.2.2). The category C/S equipped with the E-topology (more precisely, the Grothendieck topology (Definition A.0.4) generated by (Definition 3.2.1) the E-topology) is the *E-site over S*.
2. Assuming that all morphisms in E are locally of finite-type, the *big E-site on S* is the site (Definition A.0.4) whose underlying category is the category of locally of finite type schemes (Definition C.0.19) over S and whose Grothendieck topology (Definition A.0.4) is the one generated by (Definition 3.2.1) the pretopology (Definition 3.2.2) whose coverings are the E-coverings.
3. The *small E-site on S* is the site whose underlying category is the full subcategory of Sch/S of S-schemes whose structure morphisms are E-morphisms and whose coverings are E-coverings.

Definition 3.1.2 (Big étale site of a scheme). Let S be a fixed scheme. The *big étale site on S*, denoted by $(\mathrm{Sch}/S)_{\text{ét}}$, is defined as the following site (Definition A.0.4):

- The underlying category is the category Sch/S of all schemes over S (Notation 1.1.1). That is, objects are morphisms of schemes $X \to S$, and morphisms are S-morphisms between such X.
- The Grothendieck topology (Definition A.0.4) is the one generated by (Definition 3.2.1) the pretopology (Definition 3.2.2) whose covering families are families $\{f_i : X_i \to X\}_{i \in I}$ in Sch/S such that each f_i is an étale morphism (Definition 1.1.6) and the family is jointly surjective on the underlying topological spaces. Such a cover is called an *étale cover of X*.

Equivalently, the big étale site on S is the big site for étale morphisms on S (Definition 3.1.1).

Definition 3.1.3 (Small étale site of a scheme). Let X be a fixed scheme. The *small étale site on X*, commonly denoted by notations including $X_{\text{ét}}$, $X_{\text{étale}}$, or $\mathrm{Et}_{/X}$, is defined as the following site (Definition A.0.4):

- The underlying category is the full subcategory of the big étale site $(\mathrm{Sch}/X)_{\text{ét}}$ (Definition 3.1.2) whose objects are schemes U equipped with an étale morphism $U \to X$.
- The Grothendieck topology is the one generated by (Definition 3.2.1) the pretopology (Definition 3.2.2) whose covering families are families $\{g_j : U_j \to U\}_{j \in J}$ of morphisms such that each g_j is an étale (Definition 1.1.6) and the family is jointly surjective on the underlying topological spaces.

Equivalently, the small étale site on X is the small site for étale morphisms on X (Definition 3.1.1). (♠ TODO: state this as a fact) $X_{\text{ét}}$ is an essentially small category (Definition .1.18).

3.2. **Other common sites of schemes.**

Definition 3.2.1. Let $\mathcal{C}$ be a category equipped with a Grothendieck pretopology (Definition 3.2.2) K. The *Grothendieck topology generated by K*, denoted J_K, is the smallest Grothendieck topology (Definition A.0.4) on $\mathcal{C}$ such that every family in $K(U)$ is a covering family for J_K.

Explicitly, a sieve (Definition A.0.1) S on an object U belongs to $J_K(U)$ if and only if there exists a covering family (Definition 3.2.2) $\{U_i \to U\}_{i \in I} \in K(U)$ such that for every $i \in I$, the morphism $U_i \to U$ belongs to S.

The condition that S contains the family $\{U_i \to U\}$ is equivalent to saying that the sieve generated by this family is a sub-sieve of S.

Definition 3.2.2. Let $\mathcal{C}$ be a category (Definition .1.1). A *basis for a Grothendieck topology* (also called a *Grothendieck pretopology* or simply a *pretopology*) on $\mathcal{C}$ is a collection of families $K(U)$ of morphisms for each object $U \in \mathcal{C}$, called *coverings* or *covering families*, satisfying the following axioms:

1. **(Identity)** For every isomorphism $U' \to U$, the singleton family $\{U' \to U\}$ is in $K(U)$.
2. **(Base Change)** If $\{U_i \to U\}_{i \in I}$ is a covering family in $K(U)$ and $V \to U$ is any morphism in $\mathcal{C}$, then the fiber products (Definition C.0.15) $U_i \times_U V$ exist, and the family of projections $\{U_i \times_U V \to V\}_{i \in I}$ is in $K(V)$.
3. **(Composition)** If $\{U_i \to U\}_{i \in I}$ is in $K(U)$ and for each $i \in I$, $\{V_{ij} \to U_i\}_{j \in J_i}$ is in $K(U_i)$, then the composite family $\{V_{ij} \to U_i \to U\}_{i \in I, j \in J_i}$ is in $K(U)$.

Definition 3.2.3 (Big Zariski site)**.** Let S be a scheme. The *big Zariski site of S*, denoted by $(\mathrm{Sch}/S)_{\mathrm{Zar}}$, is the site (Definition A.0.4) whose underlying category is the cateogry of S-schemes and whose Grothendieck topology is the one generated by (Definition 3.2.1) the pretopology (Definition 3.2.2) whose coverings are families of morphisms

$$\{f_i : U_i \to U\}_{i \in I}$$

such that each f_i is an open immersion (Definition C.0.14) and the images $\{f_i(U_i)\}_{i \in I}$ form an open cover of U. Such a covering is called a *Zariski covering of U*.

Equivalently, the big Zariski site of S is the big site on the category of schemes over S (Definition 3.1.1) for the class of open immersions.

Definition 3.2.4 (Small Zariski site)**.** Let X be a scheme. The *small Zariski site of X*, denoted by X_{Zar}, is the site (Definition A.0.4)

whose underlying category consists of open subschemes $U \subseteq X$, with inclusions as morphisms, and whose Grothendieck topology is the one generated by (Definition 3.2.1) the pretopology (Definition 3.2.2) whose coverings are families of open immersions (Definition C.0.14)

$$\{U_i \to U\}_{i\in I}$$

such that the U_i form an open cover of U in the usual topological sense.

Equivalently, the small Zariski site of X is the small site on the category of schemes over X (Definition 3.1.1) for the class of open immersions.

Also equivalently, the small Zariski site of X is the Site of opens of X (Definition .1.9) as a topological space.

Definition 3.2.5 (Faithfully flat morphism of schemes)**.** Let $f : X \to Y$ be a morphism of schemes.

The morphism f is *faithfully flat* if it is *flat* and *surjective* on the underlying topological spaces.

More precisely:

- f is *flat*, meaning for every $x \in X$ with $y = f(x)$, the local ring homomorphism
$$\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$$
makes $\mathcal{O}_{X,x}$ a flat $\mathcal{O}_{Y,y}$-module.
- f is *surjective* at the level of topological spaces, i.e., the continuous map $f : |X| \to |Y|$ is surjective.

Equivalently, f is faithfully flat if the functor $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ on quasi-coherent sheaves is both exact and faithful.

Definition 3.2.6 (big fppf site)**.** Let S be a scheme. The *big fppf site of S*, denoted by $(\mathrm{Sch}/S)_{\mathrm{fppf}}$, is the following site (Definition A.0.4):

- The underlying category is the category Sch/S of all schemes over S.
- The Grothendieck topology is the one generated by (Definition 3.2.1) the pretopology (Definition 3.2.2) whose coverings are families of morphisms $\{f_i : X_i \to X\}_{i\in I}$ in Sch/S is a covering family in $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ if each f_i is a flat (Definition 1.1.4) morphism of locally of finite presentation (Definition .3.6) such that $\bigcup_i f_i(X_i) = f(X)$. Such a cover is called an *fppf cover of X*.

Equivalently, the big fppf site of S is the big site on S (Definition 3.1.1) for the class of flat (Definition 1.1.4) and locally of finite presentation (Definition 1.1.2) morphisms.

fppf stands for *fidèlement plate de présentation finie*, which translates to faithfully flat and of finite presentation; note however, that the morphisms in an fppf cover are flat and locally of finite presentation, not faithfully flat (Definition 3.2.5) and of finite presentation (Definition .3.6).

Definition 3.2.7 (small fppf site)**.** Let X be a fixed scheme. The *small fppf site on X*, commonly denoted by notations including X_{fppf}, or $\mathrm{fppf}_{/X}$, is defined as the following site (Definition A.0.4):

- The underlying category is the full subcategory of the big étale site $(\mathrm{Sch}/X)_{\mathrm{fppf}}$ (Definition 3.2.6) whose objects are schemes U equipped with a (♠ TODO:)
  (♠ TODO:)
- The Grothendieck topology is the one generated by (Definition 3.2.1) the pretopology (Definition 3.2.2) whose coverings are families of morphisms $\{g_j : U_j \to U\}_{j\in J}$ in this category is a covering family in $X_{\text{ét}}$ if each g_j is étale (Definition 1.1.6) and the family is jointly surjective on the underlying topological spaces.

Equivalently, the small fppfsite on X is the small site for morphisms that are flat and locally of finite presentation on X (Definition 3.1.1). (♠ TODO: state this as a fact) X_{fppf} is an essentially small category (Definition .1.18).

Definition 3.2.8 (big fpqc site)**.** (♠ TODO:) Let S be a scheme. The *big fpqc site of S*, denoted by $(\mathrm{Sch}/S)_{\mathrm{fpqc}}$, is following site (Definition A.0.4):

1. The underlying category is the category Sch/S of all schemes over S.
2. A family of morphisms $\{f_i : X_i \to X\}_{i\in I}$ in Sch/S is a covering family in $(\mathrm{Sch}/S)_{\mathrm{fpqc}}$ if
   - each f_i is a flat (Definition 1.1.4) morphism
   - for each affine open $U \subseteq X$, there exists a finite subset $J \subseteq I$ and affineo pen subsets $U_j \subseteq X_j$ for each $j \in J$ such that $U = \bigcup_{j\in J} f_j(U_j)$.

   Such a cover is called an *fpqc cover of X*.

Definition 3.2.9 (Small fpqc site)**.** (♠ TODO:) Let X be a scheme. The *small fpqc site of X*, denoted by X_{fpqc}, is the site whose underlying category consists of schemes U equipped with a morphism

$$U \longrightarrow X,$$

such that the morphism is *flat* and *quasi-compact* and whose Grothendieck topology is the one generated by (Definition 3.2.1) the pretopology (Definition 3.2.2) whose coverings are families of morphisms

$$\{f_i : U_i \to U\}_{i\in I}$$

over X such that

1. Each f_i is *faithfully flat*.
2. Each f_i is *quasi-compact*.
3. The images of the family jointly cover U; that is,
$$\bigcup_{i\in I} f_i(U_i) = U.$$
4. For every affine open $V \subseteq U$, there exists a finite subset $J \subseteq I$ and affine opens $V_j \subseteq U_j$, for $j \in J$, such that
$$V = \bigcup_{j\in J} f_j(V_j).$$

This topology is called the *fpqc topology* (faithfully flat and quasi-compact).

Definition 3.2.10. [See [Voe98, Definition 2.1], [MV99, Section 3 Definition 1.3]] Let S be a scheme (Definition .3.1). An elementary distinguished square in the category Sm/S of smooth schemes over S is a square of the form

$$\text{(B)}\qquad \begin{tikzcd} p^{-1}(U) \arrow[r] \arrow[d, "p"] & V \arrow[d, "p"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

(♠ TODO: open embedding, reduced subscheme, support) such that p is an étale morphism (Definition 1.1.6), j is an open embedding, and $p^{-1}(X-U) \to X-U$ is an isomorphism (where $X-U$ is the maximal reduced subscheme with support in the closed subset $X-U$).

Proposition 3.2.11 ([MV99, Section 3 Proposition 1.1])**.** Let S be a Noetherian (Definition C.0.16) scheme of finite dimension (Definition C.0.20). Let X be a scheme of finite type (Definition C.0.19) over S and let $\{U_i \to X\}$ be a finite family of étale morphisms (Definition 1.1.6) in Sch/S (Definition .3.3). The following conditions are equivalent: (♠ TODO: residue field)

1. For any point x of X there is an i and a point u of U_i over x such that the corresponding morphism of residue fields is an isomorphism which maps to x with the same residue field.
2. For any point $x \in X$, the morphism
$$\coprod_i (U_i \times_X \operatorname{Spec} \mathscr{O}^h_{X,x}) \to \operatorname{Spec} \mathscr{O}^h_{X,x}$$
of S-schemes admits a section.

Moroever, the collection of families of étale morphisms $\{U_i \to X\}$ in Sm/S satisfying the equivalent conditions above forms a pretopology on Sm/S.

Definition 3.2.12. [See [MV99, Section 3 Definition 1.2]] Let S be a Noetherian (Definition C.0.16) scheme of finite dimension (Definition C.0.20). The Grothendieck topology (Definition A.0.4) generated by the pretopology (Definition 3.2.2) of Proposition 3.2.11 is called the *Nisnevich topology on* Sm/S. The site whose underlying category is Sm/S and whose Grothendieck topology is the Nisnevich topology is called the *(big) Nisnevich site of* S and is denoted by notations such as $(\mathrm{Sm}/S)_{\mathrm{Nis}}$, $(\mathbf{Sm}/S)_{\mathrm{Nis}}$, etc. A covering family in the Nisnevich site of S is called a *Nisnevich covering*.

(♠ TODO: establish that a nisnevich covering is a family of etale moprhisms such that there is an isomorphism of residue fields at every point)

Definition 3.2.13 (Small Nisnevich site)**.** Let X be a scheme. The *small Nisnevich site of* X, denoted by

$$X_{\mathrm{Nis}},$$

is the site whose underlying category consists of schemes U equipped with an étale morphism (Definition 1.1.6)

$$U \to X,$$

and whose Grothendieck topology is the one generated by (Definition 3.2.1) the Grothendieck pretopology (Definition 3.2.2) whose coverings are given by Nisnevich coverings (Definition 3.2.12).

Definition 3.2.14 (crystalline site)**.** (♠ TODO:) Let S be a scheme and p a prime number. Let $\mathcal{C} = \mathrm{CRIS}(S/\mathbf{Z}_p)$ denote the category whose objects are triples (U, T, δ) where

- U is an open subscheme of S,
- T is a scheme equipped with a divided power structure δ on an ideal $\mathcal{I} \subseteq \mathcal{O}_T$,
- together with a closed immersion $U \hookrightarrow T$ compatible with δ (called a PD-thickening).

The *crystalline site* $(S/\mathbf{Z}_p)_{\mathrm{cris}}$ is this category equipped with the Grothendieck topology (Definition A.0.4) generated by (Definition 3.2.1) families of morphisms $\{(U_i, T_i, \delta_i) \to (U, T, \delta)\}$ such that the underlying morphisms of the $T_i \to T$ form a Zariski open cover of T.

Theorem 3.2.15 (Hierarchy of Grothendieck topologies on Sch/S)**.** Let S be a scheme and consider the following Grothendieck topologies (Definition A.0.4) on the category Sch/S of schemes over S (Definition .3.3):

$$\mathrm{Zar}\quad (\text{Zariski}),\quad \mathrm{Nis}\quad (\text{Nisnevich}),\quad \text{Ét}\quad (\text{étale}),\quad \mathrm{fppf},\quad \mathrm{fpqc}.$$

(Definition 3.2.3, Definition 3.2.12, Definition 3.1.2, Definition 3.2.6, Definition 3.2.8)

Then these topologies satisfy the chain of refinements (fineness) relations

$$\mathrm{Zar} \prec \mathrm{Nis} \prec \text{Ét} \prec \mathrm{fppf} \prec \mathrm{fpqc},$$

meaning that each topology is strictly finer (Definition .1.3) than the previous one, i.e., every covering in a coarser topology is a covering in any finer topology.

More explicitly, for every object $X \in \mathrm{Sch}/S$,

$$\mathrm{Cov}_{\mathrm{Zar}}(X) \subsetneq \mathrm{Cov}_{\mathrm{Nis}}(X) \subsetneq \mathrm{Cov}_{\text{Ét}}(X) \subsetneq \mathrm{Cov}_{\mathrm{fppf}}(X) \subsetneq \mathrm{Cov}_{\mathrm{fpqc}}(X).$$

Theorem 3.2.16 (Essential smallness of small sites)**.** Let X be a scheme. Consider the following sites associated to X:

- small Zariski site X_{Zar} (Definition 3.2.4),
- small Nisnevich site X_{Nis} (Definition 3.2.13),
- small étale site $X_{\text{Ét}}$ (Definition 3.1.3),
- small fppf site X_{fppf} (Definition 3.2.7).
- small fpqc site X_{fpqc} (Definition 3.2.9).

These sites are essentially small (Definition .1.18).

3.3. **Sheafication of a presheaf on a site.**

Definition 3.3.1. Let $\mathcal{C}$ be a site (Definition A.0.4) and let $\mathcal{A}$ be a (large) category (Definition .1.1).

Assuming that the presheaf (Definition 3.0.1) category $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ (and hence the sheaf (Definition 3.0.2) category $\mathrm{Shv}(\mathcal{C}, \mathcal{A})$) is locally small (Definition .1.4) (or U-locally small if a Grothendieck universe (Definition .1.2) U is available), a *sheafification functor* refers to a functor

$$a : \mathrm{PreShv}(\mathcal{C}, \mathcal{A}) \to \mathrm{Shv}(\mathcal{C}, \mathcal{A})$$

that is left adjoint (Definition .1.8) to the inclusion functor

$$i : \mathrm{Shv}(\mathcal{C}, \mathcal{A}) \hookrightarrow \mathrm{PreShv}(\mathcal{C}, \mathcal{A}).$$

If such a sheafification functor exists, then it is unique up to unique natural isomorphism. Given a presheaf P, the sheafification $a(P)$ is also sometimes called the *sheaf asscciated to P*. See Theorem 3.3.2 for common conditions under which sheafification exists.

Theorem 3.3.2. cf. [GV72, Exposé II, Théorème 3.4]

1. Let U be a universe. Let $\mathcal{C}$ be a U-site (Definition A.0.4). A sheafification functor

$$a : \mathrm{PreShv}(\mathcal{C}, U{-}\mathbf{Sets}) \to \mathrm{Shv}(\mathcal{C}, U{-}\mathbf{Sets}).$$

   exists.
2. Let $\mathcal{C}$ be a site whose underlying category is locally small (Definition .1.4) and which has a topologically generating family (Definition A.0.4) that is a set (rather than a proper class). A sheafification functor

$$a : \mathrm{PreShv}(\mathcal{C}, \mathbf{Sets}) \to \mathrm{Shv}(\mathcal{C}, \mathbf{Sets})$$

   exists.
3. (see e.g. [nLa25, 3]) Let $(\mathcal{C}, J)$ be a site (Definition A.0.4) on an essentially small category (Definition .1.18) $\mathcal{C}$. Suppose that the category $\mathcal{A}$ is complete, cocomplete, that small filtered colimits (Definition .1.13) in $\mathcal{A}$ are exact, and that $\mathcal{A}$ satisfies the IPC-property. A sheafification functor (Definition 3.3.1)

$$a : \mathrm{PreShv}(\mathcal{C}, \mathcal{A}) \to \mathrm{Shv}(\mathcal{C}, \mathcal{A})$$

   exists. (♠ TODO: IPC-property, exactess in this context.)

   (♠ TODO: state as a fact that these categories are complete, cocomplete, with small filtered colimits that are exact) This is true for instance of $\mathcal{A} = \mathbf{Set}, \mathbf{Grp}$, $k{-}\mathbf{Alg}$ for a field k, or $\mathbf{Mod}_R$ for a (not necessarily commutative unital) ring R (Definition C.0.8).

Remark 3.3.3. If the presheaf is valued in nice "algebraic category", e.g. groups, abelian groups, rings, modules over a ring, etc., then the sheafification is again valued in that category. (♠ TODO: Make this more precise.)

Corollary 3.3.4. Let X be a scheme. For the categories $\mathcal{A} = \mathbf{Set}, \mathbf{Grp}$, $k - \mathbf{Alg}$ for a field k, or $\mathbf{Mod}_R$ for a (not necessarily commutative unital) ring R (Definition C.0.8), and for the sites $\mathcal{C} = X_{\mathrm{Zar}}, X_{\mathrm{Nis}}, X_{\text{Ét}}, X_{\mathrm{fppf}}, X_{\mathrm{fpqc}}$, there exists a sheafification functor (Theorem 3.3.2)

$$a : \mathrm{PreShv}(\mathcal{C}, \mathcal{A}) \to \mathrm{Shv}(\mathcal{C}, \mathcal{A}).$$

Proof. This follows from Theorem 3.3.2 and Theorem 3.2.16. □

3.4. **Examples of sheaves on the étale site of a scheme.**

Constant and locally constant sheaves are discussed not just for $X_{\text{ét}}$, but also for more general sites.

Definition 3.4.1 (Constant sheaf on a site)**.** Let $\mathcal{C}$ be a (large) category (Definition .1.1), let $\mathcal{A}$ be a (large category), and let A be an object of $\mathcal{A}$.

1. The *constant presheaf on $\mathcal{C}$ with value A* is the presheaf (Definition 3.0.1) P defined by
$$P(U) = A$$
for every object U of $\mathcal{C}$ such that every morphism $f : V \to U$ in $\mathcal{C}$ induces the identity map $A = P(U) \to P(V) = A$.
2. Let $\mathcal{C}$ be a site (Definition A.0.4) and assume that a sheafification functor (Definition 3.3.1)
$$a : \mathrm{Shv}(\mathcal{C}, \mathcal{A}) \to \mathrm{PreShv}(\mathcal{C}, \mathcal{A})$$
exists (e.g. see Theorem 3.3.2) . The *constant sheaf on $\mathcal{C}$ with value A*, or the *constant sheaf on $\mathcal{C}$ associated to A* commonly denoted $\underline{A}$ or sometimes just A by abuse of notation, is the sheaf associated to (Theorem 3.3.2) the constant presheaf P with value A above.
3. Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of (not-necessarily commutative) rings on $\mathcal{C}$. Assume that the global sections ring $\Gamma(\mathcal{O})$ (Definition 6.1.1) exists. A *constant $\mathcal{O}$-module* is an $\mathcal{O}$-module (Definition 6.1.3) $\mathcal{F}$ which is isomorphic as a sheaf to the constant sheaf on $\mathcal{C}$ with value M where M is a module of the ring $\Gamma(\mathcal{O})$. Note that sheafification functors exist for presheaves/sheaves valued in Ab (Theorem 3.3.2).

   In case that $\mathcal{O}$ is the constant sheaf associated to A for some (not-necessarily commutative) ring A, a constant $\mathcal{O}$-module is simply called a *constant A-module*.

Definition 3.4.2 (Locally constant sheaf on a site)**.** Let $(\mathcal{C}, J)$ be a site (Definition A.0.4).

1. Let $\mathcal{A}$ be a (large) category, and let A be an object of $\mathcal{A}$. (♠ TODO: If such a sheafification functor exist, does a sheafification functor exist when restricted to an object U?) Assume that a sheafification functor (Definition 3.3.1)
$$a : \mathrm{PreShv}(\mathcal{C}, J, \mathcal{A}) \to \mathrm{Shv}(\mathcal{C}, J, \mathcal{A})$$
(Definition 3.0.2) (Definition 3.0.1) exists (e.g. see Theorem 3.3.2) . Let U be an object of $\mathcal{C}$.

   A sheaf $\mathcal{F}$ on C with values in $\mathcal{A}$ is said to be a *locally constant sheaf on U with value A* if there exists a covering sieve (Definition A.0.4) $\{U_i \to U\}$ of every object U in C such that for each i, the restriction $\mathcal{F}|_{U_i}$ is isomorphic to the constant sheaf (Definition C.0.10) $\underline{A}$ on the slice site $\mathcal{C}_{/U_i}$ (Definition A.0.7).

   If $\mathcal{C}$ has a final object, then we may say that $\mathcal{F}$ is a *locally constant sheaf with value A* if it is a locally constants sheaf on the final object with value A.
2. Let $\mathcal{O}$ be a sheaf of (not-necessarily commutative) rings on $\mathcal{C}$. A *locally constant $\mathcal{O}$-module* is a $\mathcal{O}$-module (Definition 6.1.3) $\mathcal{F}$ such that there exists a covering sieve

(Definition A.0.4) $\{U_i \to U\}$ for every object U in C such that for each i, the restriction $\mathcal{F}|_{U_i}$ is isomorphic, as an $\mathcal{O}|_{U_i}$-module, to a constant $\mathcal{O}|_{U_i}$-module (Definition C.0.10)[1].

We additionally say that $\mathcal{F}$ is

(a) *locally free of rank r over $\mathcal{O}$* if there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_i}$ is isomorphic as a $\mathcal{O}|_{U_i}$-module to $(\mathcal{O}|_{U_i})^{\oplus r}$ for each i.

(b) *free of rank r over $\mathcal{O}$* if $\mathcal{F} \cong \mathcal{O}^{\oplus r}$ as $\mathcal{O}$-modules.

(c) *of finite type* if there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_i}$ generated by finitely many sections over U_i as an $\mathcal{O}|_{U_i}$-module. In other words, there is an epimorphism

$$(\mathcal{O}|_{U_i})^{\oplus n_i} \to \mathcal{F}|_{U_i}$$

of $\mathcal{O}|_{U_i}$-modules for each i.

In case that $\mathcal{O}$ is the constant sheaf on $\mathcal{C}$ associated to A (Definition C.0.10) for some (not-necessarily commutative) ring A, a locally constant $\mathcal{O}$-module is simply called a *locally constant A-module*.

Definition 3.4.3. Let C be a category enriched in a monidal category $\mathcal{V}$. Given an object X of C, the *functor of points* h_X is the functor/presheaf (Definition 3.0.1) $C^{\mathrm{op}} \to \mathcal{V}$ given by $T \mapsto \mathrm{Hom}_C(T, X)$. A functor $C^{\mathrm{op}} \to \mathcal{V}$ (or equivalently, a presheaf on C valued in $\mathcal{V}$) is said to be *representable* if it is naturally isomorphic to some functor h_X of points for an object X of C.

Dually, a functor $C \to \mathcal{V}$ is called *co-representable* if it is naturally isomorphic to a functor $T \mapsto \mathrm{Hom}_C(X, T)$ for an object X in C.

For instance, we may speak of these notions when $\mathcal{V}$ is the monoidal category **Sets**, i.e. C is a locally small category (Definition .1.4).

Proposition 3.4.4 (Representable functors are étale sheaves)**.** (♠ TODO: This is probably more generally true for fppf, fpqc sites, etc.) Let X be a scheme (Definition .3.1) and consider the étale site $X_{\text{ét}}$ (Definition 3.1.3).

If a functor

$$F : (X_{\text{ét}})^{\mathrm{op}} \to \mathbf{Sets}$$

is representable (Definition 3.4.3) by an X-scheme Y, i.e.,

$$F(U) \cong \mathrm{Hom}_X(U, Y)$$

for all étale $U \to X$, then the presheaf F is an étale (Definition 3.1.3) sheaf (Definition 3.0.2).

Definition 3.4.5. (♠ TODO: regular function on a scheme) For a scheme (Definition .3.1) X, the *additive sheaf* $\mathbb{G}_a$ on $X_{\text{ét}}$ is defined by

$$\mathbb{G}_a(U) := \mathcal{O}_U(U),$$

the ring of regular functions on U for each étale $U \to X$. It may also be called the *additive group scheme*, on account of the fact that the sheaf $\mathbb{G}_a$ is representable by (Definition 3.4.3)

[1]The global sections rings $\Gamma(\mathcal{O}|_{U_i})$ (Definition 6.1.1) of the sheaves $\mathcal{O}|_{U_i}$ on the slice sites $\mathcal{C}_{/U_i}$ (Definition A.0.7) exist because each $\mathcal{C}_{/U_i}$ has a final object (Definition .1.10)(Lemma A.0.6), so we may speak of constant $\mathcal{O}|_{U_i}$-modules.

the scheme $\mathbb{G}_a = \mathbb{A}^1_X$ (Definition .3.7). (♠ TODO: affine line over a scheme) In particular, it is indeed a sheaf (Proposition 3.4.4).

Definition 3.4.6. For a scheme X, the *multiplicative sheaf* $\mathbb{G}_m$ on $X_{\text{ét}}$ is defined by

$$\mathbb{G}_m(U) := \mathcal{O}_U(U)^\times,$$

the group of invertible regular functions on U for each étale $U \to X$. It may also be called the *multiplicative group scheme*, on account of the fact that the sheaf $\mathbb{G}_m$ is representable by (Definition 3.4.3) the scheme $\mathbb{G}_m = \mathbb{A}^1_X \setminus \{0\}$. (♠ TODO: affine line over a scheme) In particular, it is indeed a sheaf (Proposition 3.4.4).

Definition 3.4.7 (The sheaf of n-th roots of unity μ_n)**.** (♠ TODO: invertible integer on a scheme) (♠ TODO: kernel for sheaves) Let n be a positive integer invertible on the scheme (Definition .3.1) X. The sheaf (Definition 3.0.2) μ_n on $X_{\text{ét}}$ (Definition 3.1.3) is defined as the kernel of the multiplication-by-n map on $\mathbb{G}_m$:

$$\mu_n := \ker\left(\mathbb{G}_m \xrightarrow{(\cdot)^n} \mathbb{G}_m\right).$$

In other words,

$$\mu_n(U) = \{f \in \mathcal{O}_U(U)^\times \mid f^n = 1\}$$

for each étale (Definition 1.1.6) $U \to X$.

We can use the sheaves μ_{ℓ^n} for a prime ℓ to construct ℓ-adic sheaves $\mathbb{Z}_\ell(d)$ (Definition 7.5.11) called *Tate twists*; we note that ℓ-adic sheaves (Definition 7.5.9) are not actual sheaves on $X_{\text{ét}}$, but rather a certain type of inverse system of sheaves on $X_{\text{ét}}$.

3.5. **Stalk of a (pre)sheaf on the small étale site of a scheme.** As with Zariski (pre)sheaves (or more generally locally ringed spaces), one can talk about stalks at points — for the étale site the stalks are not just as closed points, but rather at *geometric* points.

Definition 3.5.1 (Stalk of an étale sheaf at a geometric point)**.** Let X be a scheme (Definition .3.1), $\bar{x} : \mathrm{Spec}(\Omega) \to X$ a geometric point (Definition 2.0.1), and let $\mathcal{F}$ be a presheaf (Definition 3.0.1) of sets (resp. abelian groups, etc.) on the étale site $X_{\text{ét}}$ (Definition 3.1.3).

The *stalk of $\mathcal{F}$ at $\bar{x}$* is defined as the colimit (Definition .1.12)

$$\mathcal{F}_{\bar{x}} := \varinjlim_{(U,u)} \mathcal{F}(U),$$

where the limit runs over the filtering category (Definition .1.11) of pairs (U, u) consisting of an étale morphism (Definition 1.1.6) $U \to X$ together with a lift $u : \mathrm{Spec}(\Omega) \to U$ of $\bar{x}$, i.e., such that the composed morphism $u : \mathrm{Spec}(\Omega) \to U \to X$ equals $\bar{x}$.

This construction makes $\mathcal{F}_{\bar{x}}$ a set (resp. an abelian group, etc.), intuitively representing the "germ" of sections of $\mathcal{F}$ near the geometric point $\bar{x}$.

Theorem 3.5.9 describes stalks of étale sheaves in terms of strict henselizations (Definition 3.5.7). We define the notions of Henselian and strictly Henselian local rings.

Definition 3.5.2 (Henselian ring)**.** Let $(R, \mathfrak{m})$ be a local ring (Definition C.0.21) with maximal ideal $\mathfrak{m}$ (Definition C.0.4). The ring R is called *henselian* if for every monic polynomial $f(x) \in R[x]$ and factorization

$$\overline{f}(x) = \overline{g}(x) \cdot \overline{h}(x)$$

in $(R/\mathfrak{m})[x]$ into coprime monic polynomials $\overline{g}, \overline{h}$, there exist monic polynomials $g(x), h(x) \in R[x]$ lifting $\overline{g}, \overline{h}$ respectively such that

$$f(x) = g(x) \cdot h(x).$$

Theorem 3.5.3. (♠ TODO: complete local ring) All complete local rings are henselian (Definition 3.5.2).

Definition 3.5.4 (Strictly henselian ring)**.** (♠ TODO: residue field, separably closed) A *strictly henselian ring* is a henselian local ring (Definition 3.5.2) $(R, \mathfrak{m})$ whose residue field $k = R/\mathfrak{m}$ is separably closed.

Theorem 3.5.5. (♠ TODO: complete local ring) (♠ TODO: separably closed) (♠ TODO: residue field of local ring) All complete local rings with separably closed residue fields are strictly henselian.

Definition 3.5.6 (Henselization)**.** Let $(R, \mathfrak{m})$ be a local ring (Definition C.0.21). The *henselization of R* is the initial object among henselian local (Definition 3.5.2) R-algebras $(R^h, \mathfrak{m}^h)$ equipped with a local homomorphism (Definition C.0.5) $R \to R^h$ inducing an isomorphism on residue fields. Concretely, it is a henselian local ring R^h together with a morphism $R \to R^h$ such that any local ring homomorphism from R to a henselian local ring factors uniquely through R^h.

Definition 3.5.7 (Strict henselization)**.** (♠ TODO: separable closure) (♠ TODO: residue field) Let $(R, \mathfrak{m})$ be a local ring (Definition C.0.21) with residue field k. Fix a separable closure k^{sep} of k. The *strict henselization* R^{sh} of R is the henselization of the local ring of the étale site of $\operatorname{Spec}(R)$ at the geometric point corresponding to k^{sep}. It is a strictly henselian local ring equipped with a local homomorphism $R \to R^{sh}$ inducing the embedding $k \subseteq k^{\text{sep}}$ on residue fields.

Theorem 3.5.8 (Examples of Henselizations and Strict Henselizations of Specific Local Rings)**.** Let p be a prime number

1. The ring of integers localized at p, $\mathbb{Z}_{(p)}$, has Henselization (Definition 3.5.6) given by the p-adic integers $\mathbb{Z}_p$. More precisely,

   $$\mathbb{Z}_p \cong \text{Henselization of } \mathbb{Z}_{(p)}$$

   and is a complete discrete valuation ring that is Henselian.
2. The strict Henselization (Definition 3.5.7) of $\mathbb{Z}_{(p)}$ is the maximal unramified extension of $\mathbb{Z}_p$, obtained by adjoining all roots of unity of order prime to p. It is strictly Henselian and can be viewed as the ring of integers of the maximal unramified extension of the p-adic field $\mathbb{Q}_p$.

Theorem 3.5.9 (Stalks of étale sheaves and strict henselizations)**.** Let X be a scheme (Definition .3.1), and let $\bar{x} : \operatorname{Spec}(\Omega) \to X$ be a geometric point (Definition 2.0.1), where Ω is an algebraically closed field. Let $\mathcal{F}$ be a sheaf on the étale site $X_{\text{ét}}$ (Definition 3.1.3).

Then the stalk of $\mathcal{F}$ at $\bar{x}$ (Definition 3.5.1) is isomorphic to the value of the pullback sheaf on the strict henselization of the local ring of X at the image of $\bar{x}$.

More precisely, if $(\mathcal{O}_{X,x})^{sh}$ denotes the strict henselization (Definition 3.5.7) of the local ring at the point $x \in X$ which is the image of $\bar{x}$, then

$$\mathcal{F}_{\bar{x}} \cong \mathcal{F}((\mathcal{O}_{X,x})^{sh}),$$

where the right-hand side denotes the sections of $\mathcal{F}$ over $\mathrm{Spec}((\mathcal{O}_{X,x})^{sh})$ viewed as an object in the étale site.

Consequently, the notion of stalks at geometric points connects the local behavior of étale sheaves with the algebraic and topological properties of strictly henselian local rings.

3.6. **Constant and locally constant sheaves.**

4. Locally constant sheaves and local systems

Definition 4.0.1. (♠ TODO: TODO: distinguish between the case where Λ is finite and where Λ is a limit of finite rings) Let X be a scheme and let Λ be a commutative ring. A *local system of Λ-modules on X* is a locally constant sheaf of finite free Λ-modules (Definition 3.4.2) on the (small) étale site $X_{\text{ét}}$ (Definition 3.1.3).

Theorem 4.0.2 (See [Sta25, Tags 0DV5, 0GIY])**.** (♠ TODO: Work out a statement for $\mathbb{Q}_\ell$-coefficients) Let X be a connected scheme and let $\overline{x} \in X$ be a geometric point (Definition 2.0.1).

1. There is an equivalence of categories (♠ TODO: define finite in this context)
$$\left\{\begin{matrix}\text{finite locally constant}\\ \text{sheaves of sets on } X_{\acute{e}tale}\end{matrix}\right\} \longleftrightarrow \left\{\text{finite } \pi_1^{\text{ét}}(X,\overline{x})\text{-sets}\right\}.$$
(Definition 3.4.2) (Definition 3.1.3) (Definition 2.0.2)
2. There is an equivalence of categories
$$\left\{\begin{matrix}\text{finite locally constant}\\ \text{sheaves of abelian groups on } X_{\acute{e}tale}\end{matrix}\right\} \longleftrightarrow \left\{\text{finite } \pi_1^{\text{ét}}(X,\overline{x})\text{-modules}\right\}.$$
3. For a finite ring Λ, there is an equivalence of categories
$$\left\{\begin{matrix}\text{finite type, locally constant}\\ \text{sheaves of } \Lambda\text{-modules on } X_{\acute{e}tale}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{finite } \pi_1^{\text{ét}}(X,\overline{x})\text{-modules endowed}\\ \text{with commuting } \Lambda\text{-module structure}\end{matrix}\right\}.$$
((Definition 3.4.2))
4. Assume that X is irreducible and geometrically unibranch. For a ring Λ, there is an equivalence of categories
$$\left\{\begin{matrix}\text{finite type, locally constant}\\ \text{sheaves of } \Lambda\text{-modules on } X_{\acute{e}tale}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{finite } \Lambda\text{-modules } M \text{ endowed}\\ \text{with a continuous } \pi_1^{\text{ét}}(X,\overline{x})\text{-action}\end{matrix}\right\}.$$

5.1. The functors on sheaves.

5.1.1. *Inverse and direct images of sheaves via continuous functors of general sites.*

Definition 5.1.1. Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be sites (Definition A.0.4).

A functor $u : \mathcal{C} \to \mathcal{D}$ is said to be a *continuous functor of sites* if, for every object $U \in \mathrm{Ob}(\mathcal{D})$ and every covering sieve (Definition A.0.4) $S \in K(U)$, the pullback sieve u^*S (Definition A.0.2) belongs to $J(V)$ for all $V \in \mathcal{C}$ with a morphism $u(V) \to U$ in $\mathcal{D}$.

Equivalently, u is continuous if for every sheaf (Definition 3.0.2) of sets F on $\mathcal{D}$, the presheaf (Definition 3.0.1) $\mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}, X \mapsto F(u(X))$ is a sheaf on $\mathcal{C}$. (♠ TODO: show these are equivalent) (♠ TODO: define morphism of sites and recheck ref's to this definition)

Definition 5.1.2. Let (X, τ_X) and (Y, τ_Y) be topological spaces (Definition C.0.6), and let $f : X \to Y$ be a continuous map. Let $\mathrm{Open}(X)$ and $\mathrm{Open}(Y)$ be their respective categories of open sets with inclusion morphisms, equipped with the canonical (Definition .1.9) Grothendieck topologies (Definition A.0.4) given by open coverings.

Define the functor

$$f^{-1} : \mathrm{Open}(Y) \to \mathrm{Open}(X), \quad U \mapsto f^{-1}(U).$$

It is a continuous functor of sites (Definition 5.1.1) from $\mathrm{Open}(Y)$ to $\mathrm{Open}(Y)$ which induces a site morphism

$$f : (\mathrm{Open}(X), \mathrm{can}) \to (\mathrm{Open}(Y), \mathrm{can})$$

.

Definition 5.1.3. Let $f : X \to Y$ be a morphism of schemes, and consider one of the common Grothendieck topologies on schemes such as the Zariski, étale, Nisnevich, fppf, fpqc, or crystalline topology. Denote by $\mathbf{C}(X)$ and $\mathbf{C}(Y)$ the corresponding small sites (Definition A.0.4) of X and Y (i.e., categories of morphisms to X and Y respectively equipped with one of these topologies).

Then the base change functor

$$f^{-1} : \mathbf{C}(Y) \to \mathbf{C}(X), \quad (V \to Y) \mapsto (V \times_Y X \to X)$$

(Definition C.0.15) is a continuous functor. It in fact induces a morphism of sites

$$f : (\mathbf{C}(X), \tau_{\mathbf{C}}) \to (\mathbf{C}(Y), \tau_{\mathbf{C}})$$

where $\tau_{\mathbf{C}}$ denotes the chosen topology (Zariski, étale, Nisnevich, fppf, fpqc, crystalline).

Definition 5.1.4. Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be sites (Definition A.0.4) with small topological generating families (Definition A.0.4) (or U-small topologically generating families if a universe (Definition .1.2) U is available), and let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor of sites (Definition 5.1.1).

For any sheaf (Definition 3.0.2)

$$\mathcal{F} \in \mathrm{Sh}(\mathcal{D}, K; \mathcal{A}),$$

Define the *pushforward/direct image sheaf* $u^s\mathcal{F}$ by

$$u^s\mathcal{F} := \mathcal{F} \circ u : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}.$$

Because u is continuous, $u^s\mathcal{F}$ is a sheaf on $(\mathcal{C}, J)$ valued in $\mathcal{A}$. The assignment $\mathcal{F} \mapsto u^s\mathcal{F}$ defines the *direct image/pushforward functor*

$$u^s : \mathrm{Sh}(\mathcal{D}, K; \mathcal{A}) \to \mathrm{Sh}(\mathcal{C}, J; \mathcal{A}).$$

If u is the functor underlying a site morphism $f : (\mathcal{D}, K) \to (\mathcal{C}, J)$, we may alternatively denote $u^s\mathcal{F}$ by $f_*\mathcal{F}$ and call it the *direct image/pushforward of $\mathcal{F}$ under f*; the assignment $\mathcal{F} \mapsto f_*\mathcal{F}$ is then the *direct image/pushforward functor*.

$$f_* : \mathrm{Sh}(\mathcal{D}, K; \mathcal{A}) \to \mathrm{Sh}(\mathcal{C}, J; \mathcal{A}).$$

Note that while the continuous functor u and the site morphism f point in opposite directions, the definition $f_* := u^s$ ensures that f_* corresponds to the standard geometric pushforward used in topology and algebraic geometry.

Definition 5.1.5. Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be sites (Definition A.0.4) with small topological generating families (Definition A.0.4), and let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor of sites (Definition 5.1.1). Let $\mathcal{A}$ be a (large) category such that the presheaf category (Definition 3.0.1) $\mathrm{PreSh}(\mathcal{D}, K; \mathcal{A})$ has sheafification (Definition 3.3.1).

For any sheaf (Definition 3.0.2)

$$\mathcal{G} \in \mathrm{Sh}(\mathcal{C}, J; \mathcal{A}),$$

the *inverse image/pullback sheaf of $\mathcal{G}$ under u* is defined, assuming that all colimits below exist, as:

$$u_s\mathcal{G} : \mathcal{D}^{\mathrm{op}} \to \mathcal{A}, \quad V \mapsto a\left(\varinjlim_{(V\downarrow u)} \mathcal{G}(U)\right),$$

where a is the sheafification functor of presheaves and the colimit (Definition .1.12) is taken over the comma category $(V \downarrow u)$ of pairs $(U, V \to u(U))$ with $U \in \mathcal{C}$.

The assignment $\mathcal{G} \mapsto u_s\mathcal{G}$ defines the *inverse image/pullback functor*

$$u_s : \mathrm{Sh}(\mathcal{C}, J; \mathcal{A}) \to \mathrm{Sh}(\mathcal{D}, K; \mathcal{A}).$$

If u is the functor underlying a site morphism $f : (\mathcal{D}, K) \to (\mathcal{C}, J)$, we may alternatively denote $u_s\mathcal{G}$ by $f^*\mathcal{G}$ (or sometimes by $f^{-1}\mathcal{G}$) and call it the *inverse image/pullback of $\mathcal{G}$ under f*.

Note that while the continuous functor u and the site morphism f point in opposite directions, the identification $f^* := u_s$ ensures that f^* corresponds to the standard geometric pullback. In the case of topological spaces, this recovers the usual construction involving colimits over open neighborhoods to obtain stalks followed by sheafification.

Theorem 5.1.6. Let (C, J) and (D, K) be sites (Definition A.0.4) whose underlying categories are essentially small (Definition .1.18) (or essentially U-small if a universe (Definition .1.2) U is available), and let $u : C \to D$ be a continuous functor of sites (Definition 5.1.1). Let $\mathcal{A}$ be a locally small (Definition .1.4) (or U-locally small) category which has all small (or U-small) products (Definition .1.15). In particular, $\mathrm{Sh}(D, K; \mathcal{A})$ and $\mathrm{Sh}(C, J; \mathcal{A})$ are locally small[2].

The inverse image (Definition 5.1.4) and direct image (Definition 5.1.5) functors

$$u^* : \mathrm{Sh}(D, K; \mathcal{A}) \to \mathrm{Sh}(C, J; \mathcal{A})$$

$$u_* : \mathrm{Sh}(C, J; \mathcal{A}) \to \mathrm{Sh}(D, K; \mathcal{A})$$

are adjoint (Definition .1.8) For any sheaves (Definition 3.0.2),

$$\mathcal{F} \in \mathrm{Sh}(C, J; \mathcal{A}), \quad \mathcal{G} \in \mathrm{Sh}(D, K; \mathcal{A}),$$

there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Sh}(C,J;\mathcal{A})}(u^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathrm{Sh}(D,K;\mathcal{A})}(\mathcal{G}, u_*\mathcal{F}).$$

Definition 5.1.7 (Pullback (inverse image) of a sheaf)**.** Let $f : X \to Y$ be a continuous map between topological spaces (Definition C.0.6). Let $\mathcal{D}$ be a category with a terminal object.

1. Let $\mathcal{G}$ be a presheaf on Y valued in a $\mathcal{D}$ (Definition C.0.22). The *pullback* or *inverse image presheaf* $f^{-1}\mathcal{G}$ on X is defined as the presheaf

   $$U \mapsto \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$$

   where U ranges over open subsets of X and the colimit is taken over all open subsets $V \subseteq Y$ containing $f(U)$. This construction admits a natural isomorphism

   $$(f^{-1}\mathcal{G})_x \to \mathcal{G}_{f(x)}$$

   of stalks (Definition C.0.23) for every $x \in X$.
2. If $\mathcal{G}$ is a sheaf valued in $\mathcal{D}$, then we can define the *pullback* or *inverse image sheaf* $f^*\mathcal{G}$ on X as the sheaf associated to the presheaf (Definition 3.3.1) $f^{-1}\mathcal{G}$, assuming it exists.

   Assuming that a sheafification functor (Definition 3.3.1) exists, one may equivalently define $f^*\mathcal{G}$ via Definition 5.1.5 — More concretely, $f^*\mathcal{G}$ is the following equivalent constructions:
   - The direct image (Definition 5.1.5) $(f^{-1})_s\mathcal{G}$ of $\mathcal{G}$ under the continuous functor $f^{-1} : \mathrm{Open}\, Y \to \mathrm{Open}\, X, \quad W \mapsto f^{-1}(W)$ (Definition .1.9) (Definition 5.1.2).
   - The inverse image (Definition 5.1.5) of $\mathcal{G}$ under the site morphism $\mathrm{Open}\, X \to \mathrm{Open}\, Y$ whose underlying continuous functor (Definition 5.1.1) is f^{-1}

Definition 5.1.8 (Pushforward (direct image) of a sheaf)**.** Let $f : X \to Y$ be a continuous map between topological spaces (Definition C.0.6), and let $\mathcal{F}$ be a presheaf on X valued in a category $\mathcal{D}$ (Definition C.0.22) with a terminal object. The *pushforward* or *direct image*

[2] as a consequence of Lemma .1.7

presheaf $f_*\mathcal{F}$ on Y is the presheaf valued in $\mathcal{D}$ on Y (Definition 3.0.1) defined as follows: For every open set $V \subseteq Y$, the value of the pushforward is given by

$$f_*\mathcal{F}(V) := \mathcal{F}(f^{-1}(V)).$$

For an inclusion of open sets $V' \subseteq V$ in Y, the restriction morphism

$$\mathrm{res}^{f_*\mathcal{F}}_{V,V'} : f_*\mathcal{F}(V) \to f_*\mathcal{F}(V')$$

is defined as the restriction morphism of $\mathcal{F}$ associated with the inclusion of preimages $f^{-1}(V') \subseteq f^{-1}(V)$ in X:

$$\mathrm{res}^{\mathcal{F}}_{f^{-1}(V),f^{-1}(V')} : \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(f^{-1}(V')).$$

If $\mathcal{F}$ is a sheaf, then so is $f_*\mathcal{F}$. In this case, it is equivalent to define $f_*\mathcal{F}$ as the direct image (Definition 5.1.4) $(f^{-1})^s\mathcal{F}$ of $\mathcal{F}$ under the continuous functor $f^{-1} : \mathrm{Open}\, Y \to \mathrm{Open}\, X$ (Definition 5.1.2) (Definition 5.1.2) (Definition .1.9), which is also equivalent to the direct image (Definition 5.1.4) of $\mathcal{F}$ under the site morphism $\mathrm{Open}\, X \to \mathrm{Open}\, Y$ whose underlying continuous functor is f^{-1}.

Definition 5.1.9. (♠ TODO: think about if assumptions on the small site and $\mathcal{A}$ are needed) Let $f : X \to Y$ be a morphism of schemes, and let $\mathbf{C}(X)$ and $\mathbf{C}(Y)$ be small sites (Definition 3.1.1) associated to X and Y respectively, equipped with any common Grothendieck topologies such as Zariski, étale, Nisnevich, fppf, fpqc, or crystalline. Let $\mathcal{A}$ be a (large) category.

1. Given a sheaf $\mathcal{G} \in \mathrm{Sh}(\mathbf{C}(Y); \mathcal{A})$, the *inverse image* $f^*\mathcal{G}$ along f is defined to be the inverse image (Definition 5.1.5) of $\mathcal{G}$ under the continuous functor (Definition 5.1.1) $f^{-1} : \mathbf{C}(Y) \to \mathbf{C}(X)$ (Definition 5.1.3) on sites. In particular, it is an object of $\mathrm{Sh}(\mathbf{C}(X); \mathcal{A})$ and f^* yields a functor
$$f^* : \mathrm{Sh}(\mathbf{C}(Y); \mathcal{A}) \to \mathrm{Sh}(\mathbf{C}(X); \mathcal{A}),$$
2. Given a sheaf $\mathcal{F} \in \mathrm{Sh}(\mathbf{C}(X); \mathcal{A})$, the *direct image* $f_*\mathcal{F}$ along f is defined to be the direct image (Definition 5.1.4) of $\mathcal{F}$ under the continuous functor (Definition 5.1.1) $f^{-1} : \mathbf{C}(Y) \to \mathbf{C}(X)$ (Definition 5.1.3) on sites. In particular, it is an object of $\mathrm{Sh}(\mathbf{C}(Y); \mathcal{A})$ if it exists. If $f_*\mathcal{F}$ exists for all sheaves $\mathcal{F} \in \mathrm{Sh}(\mathbf{C}(X); \mathcal{A})$, then f_* yields a functor
$$f_* : \mathrm{Sh}(\mathbf{C}(X); \mathcal{A}) \to \mathrm{Sh}(\mathbf{C}(Y); \mathcal{A}).$$

Definition 5.1.10. (♠ TODO: there are places where sites and sheaves of rings on them are used, but it would be better to just have them be ringed sites.)

A *ringed site* is a site (Definition A.0.4) $(\mathcal{C}, J)$ with a small topological generating family (Definition A.0.4) equipped with a sheaf (Definition 3.0.2) of (not necessarily commutative) rings $\mathcal{O}$. If the Grothendieck topology J is clear in context, one may even write that $(C, \mathcal{O})$ is a ringed site.

A *morphism of ringed sites*

$$((\mathcal{C}, J), \mathcal{O}) \to ((\mathcal{C}', J'), \mathcal{O}')$$

consists of a morphism of sites $f : (\mathcal{C}, J) \to (\mathcal{C}', J')$ and a morphism of sheaves (Definition 3.0.2) of rings $f^\# : \mathcal{O}' \to f_*\mathcal{O}$ (Definition 5.1.5).

5.1.2. *Extension by zero of sheaves via open immersions of schemes.*

Definition 5.1.11 (Extension by zero functor for open immersions of topological spaces)**.** Let X be a topological space and let $j : U \hookrightarrow X$ be an open immersion, where U is an open subset of X. Consider the category $\mathrm{Sh}(X)$ of sheaves of abelian groups on X and $\mathrm{Sh}(U)$ on U. The *extension by zero functor*

$$j_! : \mathrm{Sh}(U) \to \mathrm{Sh}(X)$$

is defined as follows: for any sheaf $\mathcal{F}$ on U, the sheaf $j_!\mathcal{F}$ on X is given by

$$(j_!\mathcal{F})(V) = \{s \in \mathcal{F}(V \cap U) \mid s \text{ extends by zero outside } U\}$$

for each open subset $V \subseteq X$. Concretely, sections over V are sections over $V \cap U$, and sections supported outside U are identified with zero.

Definition 5.1.12 (Extension by zero functor for open immersions and the étale site)**.** Let X be a scheme (Definition .3.1), $j : U \hookrightarrow X$ an open immersion of schemes, and let Et/X (resp. Et/U) denote the small étale site (Definition 3.1.3) of X (resp. U). The *extension by zero functor*

$$j_! : \mathrm{Sh}(Et/U) \to \mathrm{Sh}(Et/X)$$

is defined as follows: for any sheaf $\mathcal{F}$ of abelian groups on Et/U and any object $V \to X$ of Et/X,

- if $V \to X$ factors through U (i.e., $V \times_X U \cong V$), then $(j_!\mathcal{F})(V) = \mathcal{F}(V \times_X U)$,
- otherwise, $(j_!\mathcal{F})(V) = 0$.

The assignment $j_!$ defines an exact functor called the extension by zero of sheaves via the open immersion j.

5.1.3. *Hom and tensor product functors of sheaves.* (♠ TODO:)

5.1.4. *Exactness properties on sheaves of abelian groups.*

Theorem 5.1.13 (Exactness of stalk functors on étale sheaves of abelian groups)**.** Let X be a scheme (Definition .3.1). For any geometric point (Definition 2.0.1) $\bar{x} : \mathrm{Spec}(\Omega) \to X$, the stalk (Definition 3.5.1) functor

$$\mathcal{F} \mapsto \mathcal{F}_{\bar{x}}$$

is an exact functor (Definition .1.28) from the category of sheaves (Definition 3.0.2) of abelian groups on $X_{\text{ét}}$ (Definition 3.1.3) (Proposition B.5.10) to the category of abelian groups.

That is, for every short exact sequence of sheaves of abelian groups

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

the induced sequence of stalks at $\bar{x}$

$$0 \to \mathcal{F}'_{\bar{x}} \to \mathcal{F}_{\bar{x}} \to \mathcal{F}''_{\bar{x}} \to 0$$

is exact.

Proposition 5.1.14 (Exactness of inverse image functors)**.** Let $f : Y \to X$ be a morphism of schemes. Then the inverse image functor (Definition 5.1.9)

$$f^{-1} : \mathrm{Sh}(X_{\text{ét}}, \mathrm{Ab}) \to \mathrm{Sh}(Y_{\text{ét}}, \mathrm{Ab})$$

on sheaves of abelian groups on the small étale sites is exact (Definition .1.28).

Proposition 5.1.15 (Left exactness and exactness criteria for direct image functors)**.** Let $f : Y \to X$ be a morphism of schemes. The direct image functor (Definition 5.1.9)

$$f_* : \mathrm{Sh}(Y_{\text{ét}}, \mathrm{Ab}) \to \mathrm{Sh}(X_{\text{ét}}, \mathrm{Ab})$$

is left exact (Definition .1.28) in general.

(♠ TODO: pin down precise conditions under which f_* is exact.) Moreover, if f is an étale morphism, or more generally if f is a morphism satisfying certain finiteness and flatness conditions (e.g., finite étale), then f_* is exact. In particular, if f is an open immersion (Definition C.0.14) or étale morphism (Definition 1.1.6), then f_* is exact.

5.2. The functors on derived categories of sheaves. (♠ TODO: Take the derived functors here and incorporate them in the definition of things for $D^b_c(X, R)$.)

Definition 5.2.1 (Torsion sheaf of abelian groups)**.** (♠ TODO: must the site have a small topological generating family?) Let $(\mathcal{C}, J)$ be a site (Definition A.0.4) with a small topological generating family (Definition A.0.4) (or a U-small topologically generating family if a universe (Definition .1.2) U is available). A sheaf (Definition 6.1.10) of abelian groups $\mathcal{F}$ on $(\mathcal{C}, J)$ is called a *torsion sheaf of abelian groups* if for every object U in $\mathcal{C}$ and every section $s \in \mathcal{F}(U)$, there exists a nonzero integer n such that $ns = 0$ in $\mathcal{F}(U)$.

Notation 5.2.2. Let X/S be a scheme over a base scheme S, and choose some small site (Definition 3.1.1) $C(X)$ on X associated to a big site on (Sch/S). It is standard to write the following:

- $D(X)$ for the derived category (Definition B.3.1) of the category of sheaves of abelian groups on $C(X)$.
- $D_{\mathrm{tor}}(X)$ or $D_{\mathrm{tors}}(X)$ for the full subcategory of $D(X)$ of complexes whose cohomology objects (Definition .2.3) are torsion sheaves (Definition 5.2.1) (♠ TODO: continue)
- (♠ TODO: continue notation)

Definition 5.2.3. e.g. see [Fu15, 6.3] Let $f : X \to Y$ be a morphism of schemes. (♠ TODO: define $Rf_* : D^+(X) \to D^+(Y)$, f^*)

(♠ TODO: state adjunction)

Definition 5.2.4. e.g. see (♠ TODO:) Let $f : X \to Y$ be an S-compactifiable morphism of schemes. (♠ TODO: define $Rf_! : D^+(X) \to D^+(Y)$, $f^!$)

(♠ TODO: state adjunction)

Definition 5.2.5. e.g. see (♠ TODO:) Let X/S be a scheme over a base scheme. (♠ TODO: define Ext, sheaf Ext, Tor)

(♠ TODO: state adjunction)

6. Cohomology of literal sheaves for the small étale site on a scheme

In this section, we define the étale cohomology of a *literal* sheaf (Definition 3.0.2) of abelian groups; we emphasize that the sheaves here as *literal* because λ-adic sheaves (Definition 7.5.9) as defined later are not actual sheaves but rather projective systems of them.

6.1. Sheaf cohomology of a general module over a sheaf of rings on a site.

Definition 6.1.1. Let $\mathcal{C}$ be a (large) category (Definition .1.1), and let $\mathcal{D}$ be a (large) category (Definition .1.1). Let $\mathcal{F} : \mathcal{C}^{op} \to \mathcal{D}$ be a presheaf valued in $\mathcal{D}$ (Definition 3.0.1).

1. For an object $U \in \mathcal{C}$, the *sections functor evaluated at U* is the functor
$$\Gamma(U, -) : \mathrm{PSh}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$$
defined by
$$\Gamma(U, \mathcal{F}) := \mathcal{F}(U),$$
i.e., the value of the presheaf $\mathcal{F}$ at the object U.
2. The *global sections of $\mathcal{F}$* is the object $\Gamma(\mathcal{F})$ of $\mathcal{D}$ defined as the limit (Definition .1.12)
$$\Gamma(\mathcal{F}) = \varprojlim_{U \in \mathcal{C}^{\mathrm{op}}} \mathcal{F}(U)$$
assuming that such a limit exists, where the limit is taken over objects $U \in \mathcal{C}$ and the restriction morphisms $\mathcal{F}(V) \to \mathcal{F}(U)$ in $\mathcal{D}$ for morphisms $U \to V$ in $\mathcal{C}$.
   If a final object (Definition .1.10) $* \in \mathcal{C}$ exists, then $\Gamma(\mathcal{F})$ exists and coincides with $\Gamma(*, \mathcal{F}) = \mathcal{F}(*)$. The construction $\Gamma(\mathcal{F})$ is functorial; in particular, if $\Gamma(\mathcal{F})$ exists for all $\mathcal{F}$ in $\mathrm{PSh}(\mathcal{C}, \mathcal{D})$, e.g. if limits of (Definition .1.12) diagrams in $\mathcal{D}$ indexed by $\mathcal{C}$ exist, then Γ is a functor
$$\Gamma : \mathrm{PSh}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$$
called the *global sections functor on* $\mathrm{PSh}(\mathcal{C}, \mathcal{D})$.

Proposition 6.1.2. Let $\mathcal{C}$ be an essentially small category (Definition .1.18) and let $\mathcal{A}$ be an abelian category (Definition .1.26).

1. Let $U \in \mathrm{Ob}(\mathcal{C})$ be some fixed object. The sections functor (Definition 6.1.1)
$$\Gamma(U, -) : \mathrm{PSh}(\mathcal{C}, \mathcal{A}) \to \mathcal{A}$$
is left exact (Definition .1.28). (♠ TODO: state that presheaves and sheaves valued in an abelian category form abelian categories)
2. Assume that $\Gamma(\mathcal{F})$ exists for all $\mathcal{F}$ in $\mathrm{PSh}(\mathcal{C}, \mathcal{A})$ so that Γ is a functor
$$\Gamma : \mathrm{PSh}(\mathcal{C}, \mathcal{A}) \to \mathcal{A}.$$
The functor Γ is left exact.

Proof. Recall that $\mathrm{PSh}(\mathcal{C}, \mathcal{D})$ is an abelian category (Proposition B.5.10) since $\mathcal{C}$ is essentially small. (♠ TODO: Talk about how limits are left exact and) □

Definition 6.1.3. 1. Let $\mathcal{C}$ be a site (Definition A.0.4), and let $\mathcal{A}$ and $\mathcal{B}$ be sheaves (Definition 3.0.2) of (not necessarily commutative) rings (Definition C.0.8) on $\mathcal{C}$.

(a) An *$(\mathcal{A}, \mathcal{B})$-bimodule* (or a *bimodule over $(\mathcal{A}, \mathcal{B})$*) is a sheaf (Definition 3.0.2) $\mathcal{M}$ of abelian groups on $\mathcal{C}$ equipped with a left $\mathcal{A}$-module structure given by a morphism of sheaves (Definition 3.0.2) of sets

$$\lambda : \mathcal{A} \times \mathcal{M} \longrightarrow \mathcal{M},$$

and a right $\mathcal{B}$-module structure given by a morphism of sheaves of sets

$$\rho : \mathcal{M} \times \mathcal{B} \longrightarrow \mathcal{M},$$

such that the actions are compatible. Specifically, for every object U in $\mathcal{C}$, every section $m \in \mathcal{M}(U)$, every $a \in \mathcal{A}(U)$, and every $b \in \mathcal{B}(U)$, the equality

$$\lambda_U(a, \rho_U(m, b)) = \rho_U(\lambda_U(a, m), b)$$

holds in $\mathcal{M}(U)$. In standard multiplicative notation where $\lambda(a, m)$ is denoted $a \cdot m$ and $\rho(m, b)$ is denoted $m \cdot b$, this condition is the associativity axiom

$$(a \cdot m) \cdot b = a \cdot (m \cdot b).$$

In particular, for every object $U \in \mathcal{C}$, the abelian group $\mathcal{M}(U)$ has the structure of an $\mathcal{A}(U) - \mathcal{B}(U)$-bimodule (Definition C.0.11).

(b) Let $\mathcal{M}$ and $\mathcal{N}$ be $(\mathcal{A}, \mathcal{B})$-bimodules. A *homomorphism of $(\mathcal{A}, \mathcal{B})$-bimodules* (or an *$(\mathcal{A}, \mathcal{B})$-linear morphism*) is a morphism of sheaves of abelian groups $f : \mathcal{M} \to \mathcal{N}$ such that for every object U of $\mathcal{C}$, every section $m \in \mathcal{M}(U)$, every $a \in \mathcal{A}(U)$, and every $b \in \mathcal{B}(U)$, the following compatibility conditions hold:

$$f_U(a \cdot m) = a \cdot f_U(m) \quad \text{and} \quad f_U(m \cdot b) = f_U(m) \cdot b.$$

We denote the category of $(\mathcal{A}, \mathcal{B})$-bimodules, with morphisms being morphisms of sheaves of abelian groups compatible with both the left $\mathcal{A}$-action and the right $\mathcal{B}$-action, by $\mathcal{A}\text{-}\mathcal{B}\text{-}\mathsf{Mod}$ or sometimes by ${}_{\mathcal{A}}\mathsf{Mod}_{\mathcal{B}}$ (♠ TODO: talk about how bimodules can be identifies with left/right modules)

2. Let $(\mathcal{C}, J)$ be a site (Definition A.0.4). Let $\mathcal{O}$ be a sheaf of (not necessarily commutative) rings on $(\mathcal{C}, J)$ (Definition 3.0.2), i.e. $((\mathcal{C}, J), \mathcal{O})$ is a ringed site (Definition 5.1.10).

(a) An *(left/right/two-sided) $\mathcal{O}$-module* consists of the following data:
- A sheaf $\mathcal{F}$ of abelian groups on $(\mathcal{C}, J)$,
- for every object $U \in \mathcal{C}$, the structure of an (left/right/two-sided) $\mathcal{O}(U)$-module on $\mathcal{F}(U)$,

such that for every morphism $f : V \to U$ in $\mathcal{C}$, the restriction map

$$\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$$

is $\mathcal{O}(U)$-linear when the $\mathcal{O}(U)$-action on $\mathcal{F}(V)$ is defined via the natural ring homomorphism

$$\mathcal{O}(U) \to \mathcal{O}(V)$$

induced by f.

(b) Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}$-modules (Definition 6.1.3).
A *morphism of $\mathcal{O}$-modules* $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves (Definition 3.0.2) of abelian groups such that, for every object $U \in \mathcal{C}$, the component map

$$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

is $\mathcal{O}(U)$-linear, i.e. it satisfies

$$\varphi_U(r \cdot s) = r \cdot \varphi_U(s) \quad \text{for all } r \in \mathcal{O}(U),\ s \in \mathcal{F}(U).$$

The collection of all $\mathcal{O}$-modules together with their morphisms of $\mathcal{O}$-modules forms the *category of $\mathcal{O}$-modules*, denoted $\mathbf{Mod}(\mathcal{O})$.

In case that a sheafification functor (Definition 3.3.1)

$$\mathrm{PreShv}(\mathcal{C}, \mathbf{Rings}) \to \mathrm{Shv}(\mathcal{C}, \mathbf{Rings})$$

exists, a left, right, two-sided $\mathcal{O}$-module (and morphisms thereof) is equivalent to a $(\mathcal{O}, \mathbb{Z})$-bimodule, $(\mathbb{Z}, \mathcal{O})$-bimodule, and $(\mathcal{O}, \mathcal{O})$-bimodule (and morphisms thereof) respectively, where $\mathbb{Z}$ is the constant sheaf (Definition C.0.10) of the integer ring $\mathbb{Z}$.

Definition 6.1.4 (Tensor product of sheaves of bimodules)**.** Let $\mathcal{C}$ be a site (Definition A.0.4). Let $\mathcal{R}, \mathcal{S}, \mathcal{T}$ be (not necessarily commutative) sheaves (Definition 3.0.2) of rings (Definition C.0.8) on $\mathcal{C}$. Let $\mathcal{M}$ be an $(\mathcal{R}, \mathcal{S})$-bimodule (Definition 6.1.3), and let $\mathcal{N}$ be an $(\mathcal{S}, \mathcal{T})$-bimodule.

1. Define the *presheaf tensor product* $\mathcal{M} \otimes_{p,\mathcal{S}} \mathcal{N}$ to be the presheaf (Definition 3.0.1) of abelian groups given by

$$U \longmapsto \mathcal{M}(U) \otimes_{\mathcal{S}(U)} \mathcal{N}(U).$$

(Definition C.0.12) As a presheaf, it has the structure of a $\mathcal{R}-\mathcal{T}$-bimodule (♠ TODO: bimodule for a presheaf).

2. Assume that a sheafification functor (Definition 3.3.1)

$$\mathrm{Shv}(\mathcal{C}, \mathbf{Sets}) \to \mathrm{PreShv}(\mathcal{C}, \mathbf{Sets})$$

exists. The *tensor product of sheaves $\mathcal{M}$ and $\mathcal{N}$*, denoted $\mathcal{M} \otimes_{\mathcal{S}} \mathcal{N}$, is defined to be the sheafification of the presheaf $\mathcal{M} \otimes_{p,\mathcal{S}} \mathcal{N}$. This tensor product becomes naturally an $(\mathcal{R}, \mathcal{T})$-bimodule.

The left $\mathcal{R}$-action and right $\mathcal{T}$-action are induced by the presheaf actions defined section-wise by

$$r \cdot (m \otimes n) = (r \cdot m) \otimes n,$$
$$(m \otimes n) \cdot t = m \otimes (n \cdot t),$$

for all $r \in \mathcal{R}(U)$, $t \in \mathcal{T}(U)$, $m \in \mathcal{M}(U)$, and $n \in \mathcal{N}(U)$. The sheafification functor preserves these actions, yielding the required bimodule structure on $\mathcal{M} \otimes_{\mathcal{S}} \mathcal{N}$.

Inductively, given sheaves of rings $\mathcal{R}_0, \ldots, \mathcal{R}_k$ and $(\mathcal{R}_{i-1}, \mathcal{R}_i)$-bimodules $\mathcal{M}_i$ for $i = 1, \ldots, k$, we may speak of the tensor product

$$\mathcal{M}_0 \otimes_{\mathcal{R}_1} \mathcal{M}_1 \otimes_{\mathcal{R}_2} \cdots \otimes_{\mathcal{R}_{k-1}} \mathcal{M}_k.$$

Tensor products of sheaves are associative up to canonical isomorphism(♠ TODO:), allowing us to omit parentheses. This iterated tensor product carries a natural $(\mathcal{R}_0, \mathcal{R}_k)$-bimodule structure.

In general, the assignment $(\mathcal{M}_0, \dots, \mathcal{M}_k) \mapsto \mathcal{M}_0 \otimes_{\mathcal{R}_1} \cdots \otimes_{\mathcal{R}_{k-1}} \mathcal{M}_k$ defines a $(k+1)$-ary additive functor (Definition C.0.13)

$$_{\mathcal{R}_0}\mathbf{Mod}_{\mathcal{R}_1} \times \cdots \times {}_{\mathcal{R}_{k-1}}\mathbf{Mod}_{\mathcal{R}_k} \to {}_{\mathcal{R}_0}\mathbf{Mod}_{\mathcal{R}_k}$$

()Proposition B.5.10).

Given a sheaf of rings $\mathcal{R}$ and a two-sided $\mathcal{R}$-module $\mathcal{M}$, we may also speak of the *n-fold tensor product* $\mathcal{M}^{\otimes n} = \mathcal{M}^{\otimes_{\mathcal{R}} n}$.

Definition 6.1.5. Let $(\mathcal{C}, J)$ be a site (Definition 3.0.2), and let $\mathcal{A}$ be a category in which sheaves (Definition 3.0.2) on $(\mathcal{C}, J)$ can be defined.

For a sheaf $\mathcal{F} \in \mathrm{Sh}(\mathcal{C}, J; \mathcal{A})$, a *subsheaf $\mathcal{G}$ of $\mathcal{F}$* is a sheaf equipped with a monomorphism (Definition .1.19) of sheaves

$$\iota : \mathcal{G} \hookrightarrow \mathcal{F}$$

in $\mathrm{Sh}(\mathcal{C}, J; \mathcal{A})$.

Concretely, for every object $U \in \mathcal{C}$, the morphism

$$\iota(U) : \mathcal{G}(U) \to \mathcal{F}(U)$$

is a monomorphism in $\mathcal{A}$, and the collection $\{\iota(U)\}_U$ is compatible with restriction morphisms so that $\mathcal{G}$ is a subsheaf in the usual sense.

Definition 6.1.6. Let $(\mathcal{C}, J)$ be a site (Definition A.0.4), and let $\mathcal{O}$ be a sheaf of (not necessarily comutative) rings (Definition 3.0.2) on $(\mathcal{C}, J)$ with values in a category $\mathcal{A}$ for which sheaves are defined.

A *sheaf of (left/right/two-sided) ideals* $\mathcal{I}$ of the sheaf of rings $\mathcal{O}$ is a subsheaf of $\mathcal{O}$ (Definition 6.1.5) in the category of sheaves of (left/right/two-sided) $\mathcal{O}$-modules, i.e., a sheaf of $\mathcal{O}$-modules $\mathcal{I}$ together with a monomorphism (Definition .1.19) of sheaves of $\mathcal{O}$-modules $\iota : \mathcal{I} \hookrightarrow \mathcal{O}$ such that for every object U in $\mathcal{C}$, the morphism $\iota(U) : \mathcal{I}(U) \to \mathcal{O}(U)$ identifies $\mathcal{I}(U)$ as an ideal of the ring $\mathcal{O}(U)$.

Equivalently, a sheaf of ideals $\mathcal{I}$ is a subsheaf (Definition 6.1.5) of modules of $\mathcal{O}$ as a sheaf of modules (Definition 6.1.3) of $\mathcal{O}$

Theorem 6.1.7 (e.g. see [Sta25, Tag 01DU])**.** For any site (Definition A.0.4) $(\mathcal{C}, J)$ on an essentially small category (Definition .1.18) $\mathcal{C}$ and a sheaf of rings (Definition 3.0.2) $\mathcal{O}$ on $\mathcal{C}$, the category $\mathbf{Mod}(\mathcal{O})$ of $\mathcal{O}$-modules (Definition 6.1.3) is an abelian category that has enough injectives (Definition .1.29). In fact, there is a functorial injective embedding (♠ TODO: what does this mean?)

We can describe the sections of a sheaf over an object by the direct image from the object to the final object, if it exists.

Theorem 6.1.8. Let $(\mathcal{C}, J)$ be a site (Definition 3.0.2) with final object x (Definition .1.10), and let F be a sheaf on $(\mathcal{C}, J)$ (Definition 3.0.2). Let $f : X \to x$ be any object X over x (regarded as a morphism in $\mathcal{C}$), and let f_*F denote the direct image (pushforward) of F along f (Definition 5.1.5). Then

$$(f_*F)(x) = F(X) = \Gamma(X, F).$$

(Definition 6.1.1)

Corollary 6.1.9. (♠ TODO: the sites) Let $f : X \to S$ be a scheme over a scheme S and let $\mathbf{C}(X)$ and $\mathbf{C}(S)$ be small sites associated to X and S respectively, equipped with any common Grothendieck topologies such as Zariski, étale, Nisnevich, fppf, fpqc, or crystalline. Let $\mathcal{A}$ be a (large) category and let $\mathcal{F} \in \mathrm{Sh}(\mathbf{C}(X), \mathcal{A})$ (Definition 3.0.2). We have a natural isomorphism

$$\Gamma(\mathcal{F}) \cong (f_* \mathcal{F})(S)$$

(Definition 5.1.9) (♠ TODO: natural in both $\mathcal{F}$ and S perhaps?)

Definition 6.1.10. Let $(\mathcal{C}, J)$ be a site (Definition A.0.4) on a locally small category (Definition .1.4) or a U-site for some universe (Definition .1.2) U. Let $\mathcal{O}$ be a sheaf of rings (Definition 3.0.2) on $\mathcal{C}$, so that $(\mathcal{C}, J, \mathcal{O})$ is a ringed site (Definition 5.1.10). Recall that the category $\mathbf{Mod}(\mathcal{O})$ (Definition 6.1.3) of $\mathcal{O}$-modules is abelian and has enough injectives (Definition .1.29) (Theorem 6.1.7).

Assume that global sections objects $\Gamma(\mathcal{G})$ (Definition 6.1.1) exist for all objects $\mathcal{G}$ of $\mathrm{Sh}(\mathcal{C}, \mathbf{Ab})$[3] so that Γ is a functor

$$\mathrm{Sh}(\mathcal{C}, \mathbf{Ab}) \to \mathbf{Ab},$$

which is a left exact functor (Definition .1.28) (Proposition 6.1.2). Note that Γ restricts to a left exact functor

$$\mathbf{Mod}(\mathcal{O}) \to \mathbf{Ab}.$$

If $\mathcal{C}$ has a final object (Definition .1.10) $*$ as well, then recall that $\Gamma(\mathcal{F}) = \mathcal{F}(*)$.

Let $\mathcal{F}$ be an object of $\mathbf{Mod}(\mathcal{O})$.

1. For each integer $n \geq 0$, the *n-th (abelian) (global) sheaf cohomology group of $\mathcal{F}$* is

$$H^n(\mathcal{C}, J; \mathcal{F}) := R^n\Gamma(\mathcal{F}),$$

   where $R^n\Gamma$ is the n-th right derived functor (Definition .1.31) of the global sections functor Γ (Definition 6.1.1).

   In particular, each H^n is a functor

$$H^n : \mathbf{Mod}(\mathcal{O}) \to \mathbf{Ab}.$$

2. Given an object $U \in \mathcal{C}$ and for each integer $n \geq 0$, the *n-th (abelian) sheaf cohomology group of $\mathcal{F}$ of sections at U* is

$$H^n(U, \mathcal{F}) := (R^n\Gamma(U, -))(\mathcal{F}),$$

   where $R^n\Gamma(U, -)$ is the n-th right derived functor (Definition .1.31) of the sections functor $\Gamma(U, -)$ evaluated at U (Definition 6.1.1).

   In particular, $H^n(U, \mathcal{F})$ can be regarded as the nth global sheaf cohomology group of the restriction $\mathcal{F}|_U$ of $\mathcal{F}$ to U.

[3]for example, this occurs when $\mathcal{C}$ is essentially small (Definition .1.18)

6.2. **Étale cohomology of literal sheaves.**

Definition 6.2.1. Let X be a scheme, let $\mathcal{O}$ be a sheaf of rings (Definition 3.0.2) on $X_{\text{ét}}$ (Definition 3.1.3) and let $\mathcal{F}$ be an $\mathcal{O}$-module (Definition 6.1.3), which is sheaf of abelian groups on the small étale site $X_{\text{ét}}$ (Definition 3.1.3). Note that the category of $\mathcal{O}$-modules has enough injectives (Definition .1.29) (Theorem 6.1.7)

The *étale cohomology groups of X with coefficients in $\mathcal{F}$* are defined as the right derived functors (Definition .1.31) of the global sections functor (Definition 6.1.1), which is left exact (Definition .1.28) (Proposition 6.1.2). Explicitly, they are the abelian groups

$$H^i_{\text{ét}}(X, \mathcal{F}) \coloneqq H^i(X_{\text{ét}}, \mathcal{F}) = R^i\Gamma(X, \mathcal{F}), \quad i \geq 0.$$

Equivalently, $H^i_{\text{ét}}(X, \mathcal{F})$ is defined as the sheaf cohomology groups of the sheaf $\mathcal{F}$ on the essentially small site $X_{\text{ét}}$ (Definition 6.1.10)

Definition 6.2.2. Let X be a scheme. Let A be an abelian group and write $\underline{A}$ for the constant sheaf (Definition C.0.10) on $X_{\text{ét}}$ (Definition 3.1.3) associated to A. In this case, the *étale cohomology groups of X with coefficients in A* are defined as

$$H^i_{\text{ét}}(X, A) \coloneqq H^i_{\text{ét}}(X, \underline{A}).$$

(Definition 6.2.1); here $\underline{A}$ is regarded as a module of the constant sheaf $\mathbb{Z}$ on $X_{\text{ét}}$.

7. Derived categories of λ-adic sheaves constructible through limits of categories

(♠ TODO: Give an outline of how $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ is defined) Classically, given a noetherian scheme X and a prime ℓ invertible on X, the category $D^b_c(X, \overline{\mathbb{Q}}_\ell)$, which is intuitively treated as a cohomologically bounded derived category of constructible complexes of sheaves with $\overline{\mathbb{Q}}_\ell$ coefficients, is constructed via the following steps:

(♠ TODO: Cite the statements of the constructions)

1. Write E for a finite extension of $\mathbb{Q}_\ell$, and write R for the integral closure of $\mathbb{Z}_\ell$ in E. Note that R is a complete discrete valuation ring. Write λ for a uniformizer of R.
2. Construct $D^b_c(X, R)$ (♠ TODO: Explain how $D^b_c(X, R)$ is constructed.).
3. Define $D^b_c(X, E) = D^b_c(X, R)[\lambda^{-1}]$, i.e. construct $D^b_c(X, E)$ as the localization of $D^b_c(X, R)$ by the maps $\lambda^m : \mathcal{F} \to \mathcal{F}$ in $D^b_c(X, R)$.
4. Observe that for finite extensions E'/E, there are functors
$$D^b_c(X, E) \to D^b_c(X, E')$$
5. Construct $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ as the 2-direct limit of the $D^b_c(X, E)$ as E runs over the finite extensions of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$:
$$D^b_c(X, \overline{\mathbb{Q}}_\ell) = \varinjlim_{\substack{\overline{\mathbb{Q}}_\ell \supset E \supseteq \mathbb{Q}_\ell \\ E/\mathbb{Q}_\ell \text{ fintie}}} D^b_c(X, E)$$

7.1. **ℓ-adic sheaves.** ℓ-adic sheaves do not refer to literal sheaves on $X_{\text{ét}}$ with coefficients in ℓ-adic rings such as $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$. In a purely formal sense, one could talk about sheaves on $X_{\text{ét}}$ with such coefficients, but the categories of such sheaves are too ill-behaved to be of arithmetic use.

7.2. **ℓ-adic cohomology.** ℓ-adic cohomology does not refer to étale cohomology with coefficients (Definition 6.2.2) in the rings $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$.

(♠ TODO: define ℓ-adic sheaves (say with R, E, and $\overline{\mathbb{Q}}_\ell$-coefficients)) (♠ TODO: define ℓ-adic cohomology in R-coefficients, E-coefficients, and $\overline{\mathbb{Q}}_\ell$-coefficients)

7.3. **The categories of λ-adic sheaves and A-R λ-adic sheaves.** Historically, the theory of λ-adic sheaves (as opposed to "derived categories"/categories of complexes of λ-adic sheaves) was first studied in SGA5 [Gro77, Exposé V]. Note that SGA5 [Gro77, Exposé III] did study derived categories of sheaves with *finite* coefficients.

This subsection considers four abelian categories whose objects are inverse systems of sheaves:

1. The category of inverse systems of λ-torsion sheaves (Definition 7.5.1)
2. The A-R category of inverse systems of λ-torsion sheaves (Definition 7.5.8)
3. The category of λ-adic sheaves (Definition 7.5.9)
4. The category of A-R λ-adic sheaves (Definition 7.5.9)

These categories are related via the following commutative diagram of functors:

$$\begin{tikzcd}
\text{(inverse system of } \lambda\text{-torsion sheaves)} \arrow[r, "\text{localization}"] & \text{(A-R category)} \\
(\lambda\text{-adic sheaves}) \arrow[u, hook, "\text{full subcategory}"] \arrow[r, "\cong"] & \text{(A-R } \lambda\text{-adic sheaves)} \arrow[u, hook, "\text{full subcategory}"']
\end{tikzcd}$$

7.4. **Constructible sheaves.**

Definition 7.4.1 (Constructible sheaf)**.** Let X be either

1. A topological space (Definition C.0.6) (made into a site (Definition A.0.4) with the natural Grothendieck topology (Definition .1.9)) or (♠ TODO: This second case needs to be formulated more precisely; in particular, a notion of "restriction" is necessary, and there needs to be a more precise way to formulate that the site is a site associated to X.)
2. A scheme (Definition .3.1) along with a small site $(\mathcal{C}, J)$ on X (Definition 3.1.1). Assume that the there is a sheafification functor (Definition 3.3.1) for the presheaves/sheaves valued in abelian groups so that locally constant sheaves (Definition 3.4.2) of abelian groups are defined. For example, it suffices for the underlying category of the site to be essentially small (Definition .1.18)(Theorem 3.3.2).

Let $\mathcal{O}$ be a sheaf of rings (Definition 3.0.2) on the chosen site $(\mathcal{C}, J)$ associated to X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules (Definition 6.1.3) on X.

The sheaf $\mathcal{F}$ is called *constructible* if there exists a finite partition of X into locally closed subsets (Definition C.0.7)

$$X = \bigsqcup_{\alpha \in A} X_\alpha$$

such that the restriction $\mathcal{F}|_{X_\alpha}$ is a locally constant sheaf of $\mathcal{O}|_{X_\alpha}$-modules of finite type (Definition 3.4.2) on the site on X_α induced by $(\mathcal{C}, J)$ and the embedding $X_\alpha \hookrightarrow X$ (♠ TODO: The site on X_α needs to be elaborated on.) for each $\alpha \in A$.

Definition 7.4.2 (Full subcategory)**.** Let $\mathcal{C}$ be a (large) category (Definition .1.1). A *full subcategory* $\mathcal{D}$ of $\mathcal{C}$ is a subcategory such that for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{D})$, the morphism classes coincide:

$$\mathrm{Hom}_{\mathcal{D}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

In other words, a full subcategory includes all morphisms between its objects that exist in the ambient category $\mathcal{C}$.

Definition 7.4.3. Let X be either

1. A topological space (Definition C.0.6) (made into a site (Definition A.0.4) with the natural Grothendieck topology (Definition .1.9)) or
2. A scheme (Definition .3.1) along with a small site $(\mathcal{C}, J)$ on X (Definition 3.1.1). Assume that the there is a sheafification functor (Definition 3.3.1) for the presheaves/sheaves valued in abelian groups so that locally constant sheaves (Definition 3.4.2) of abelian groups are defined. For example, it suffices for the underlying category of the site to be essentially small (Definition .1.18)(Theorem 3.3.2).

Note that the sites are essentially small (Definition .1.18) (see Theorem 3.2.16).

The category $D_c(X, \mathcal{O})$ is the full subcategory (Definition 7.4.2) of the derived category $D(X, \mathcal{O})$ (Notation 9.0.7) of objects $K^\bullet$ whose (co)homology objects (Definition .2.3) are constructible sheaves (Definition 7.4.1). It is customary to speak of the subcategories $D^?(\mathcal{C}, \mathcal{O})$ for $? \in \{+, -, b\}$ as usual.

7.5. λ-adic sheaves.

Definition 7.5.1 (λ-torsion sheaf on a scheme)**.** Let $(\mathcal{C}, J)$ be a site (Definition A.0.4). Let $\mathcal{O}$ be a sheaf of commutative rings (Definition 3.0.2) on $(\mathcal{C}, J)$. Let $\lambda \subseteq \mathcal{O}$ be a sheaf of ideals (Definition 6.1.6).

A sheaf $\mathcal{F}$ of $\mathcal{O}$-modules (Definition 6.1.3) on $(\mathcal{C}, J)$ is called a *λ-torsion sheaf* if

$$\mathcal{F} = \bigcup_{n \geq 1} \ker(\lambda^n : \mathcal{F} \to \mathcal{F})$$

or equivalently if for every section (Definition 6.1.1) $s \in \mathcal{F}(U)$ for some object $U \in \mathrm{Ob}(\mathcal{C})$, there exists some $n \geq 1$ such that $\lambda^n \cdot s = 0$.

For instance, we may speak of λ-torsion sheaves by letting $\mathcal{O} = \underline{R}$ (Definition C.0.10) where R is a commutative ring, $\lambda = \underline{I}$ where $I \subseteq R$ is an ideal, and $(\mathcal{C}, J)$ be the small étale site of a scheme (Definition 3.1.3).

λ-adic sheaves (Definition 7.5.9) are specific inverse systems of torsion sheaves; imposing the Mittag-Leffler condition (Definition 7.5.4) ensures the existence of the limit as a sheaf and the stronger Artin-Rees-Mittag-Leffler condition (Definition 7.5.7) yields an abelian category.

Definition 7.5.2. Let $\mathcal{C}$ be a category (Definition .1.1) and let I be the poset (Definition .1.20) of non-negative integers $\mathbb{N}$ with the standard ordering $\geq$ (i.e. I has arrows $\cdots \to 2 \to 1 \to 0$). A *tower in $\mathcal{C}$* is a functor $X : I \to \mathcal{C}$. Explicitly, a tower consists of a sequence of objects $\{A_i\}_{i\in\mathbb{N}}$ and morphisms $f_i : A_i \to A_{i-1}$ for each $i \geq 1$:

$$\cdots \xrightarrow{f_3} A_2 \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0.$$

Moreover a *morphism of towers* is a morphism of towers as objects of the functor category (Definition .1.25) $\mathcal{C}^I$.

Definition 7.5.3. Let $\mathcal{C}$ be a category (Definition .1.1) and let I be the poset (Definition .1.20) of non-negative integers $\mathbb{N}$ with the standard ordering $\geq$ (i.e. I has arrows $\cdots \to 2 \to 1 \to 0$) (or more generally let I be the poset of integers $\mathbb{Z}$ with similar ordering).

Given a tower (Definition 7.5.2) $A = (A_i, u_i)_{i\in I}$ where $u_i : A_i \to A_{i-1}$ (for $i \geq 1$ if $\mathbb{N}$ is the collection of objects of I) and $r \geq 0$, the *shifted tower* $A[r]$ is the tower $A[r] = (A[r]_i, u[r]_i)$ given by $A[r]_i = A_{r+i}$ and whose transition maps are given by $u[r]_i : A[r]_i = Ar + i \xrightarrow{u_{r+i}} A_{r+i-1} = A[r]_{i-1}$.

Note that there is a morphism of towers (Definition 7.5.2) $A[r] \to A$ where $A[r]_i \to A_i$ is given by the transition morphism $A_{r+i} \to A_i$.

Definition 7.5.4 (cf. [Wei94, Definition 3.5.6])**.** Let I be a directed set (Definition .1.11) and let $\{A_i, \phi_{ji}\}_{i\in I}$ be an inverse system (Definition .1.17) of objects in a category $\mathcal{C}$ where images are well-defined (such as the category of sets, abelian groups, or modules).

The system is said to satisfy the *Mittag-Leffler condition* if for every index $i \in I$, there exists an index $j \geq i$ such that for all $k \geq j$, the image of the transition map $\phi_{ki} : A_k \to A_i$ is equal to the image of $\phi_{ji} : A_j \to A_i$.

For a fixed i, let $I_{k,i} = \mathrm{im}(\phi_{ki}) \subseteq A_i$ for all $k \geq i$. The condition states that the decreasing family of subobjects

$$A_i \supseteq I_{i,i} \supseteq I_{i+1,i} \supseteq I_{i+2,i} \supseteq \cdots$$

becomes stationary.

Definition 7.5.5 (cf. [Wei94, Definition 3.5.6])**.** Let I be a directed set (Definition .1.11) and let $\{A_i, \phi_{ji}\}_{i\in I}$ be an inverse system (Definition .1.17) of objects in a pointed category $\mathcal{C}$.

The system is said to satisfy the *trivial Mittag-Leffler condition* if for every index $i \in I$, there exists an index $j > i$ such that the map $A_j \to A_i$ is a zero morphism.

Definition 7.5.6. Let I be the poset (Definition .1.20) of non-negative integers $\mathbb{N}$ with the standard ordering $\geq$ (i.e. I has arrows $\cdots \to 2 \to 1 \to 0$) and let $\{A_i, \phi_{ji}\}_{i\in I}$ be an inverse system (Definition .1.17) of objects (so a tower (Definition 7.5.2)) in a pointed category $\mathcal{C}$ where images are well-defined (such as the category of sets, abelian groups, or modules).

The system is said to be a *null system* if there exists $r \geq 0$ such that for every $i \in I$, the transition map $\phi_{(i+r)i} : A_{i+r} \to A_i$ is the zero morphism.

Equivalently, the tower is a null system if there exists an integer $r \geq 0$ such that the morphism of towers (Definition 7.5.2) $A[r] \to A$ (Definition 7.5.3) is the zero morphism. (♠ TODO: The category of towers $\mathcal{C}^I$ is pointed as $\mathcal{C}$ is pointed; the zero morphism is defined component-wise.)

Definition 7.5.7 (See e.g. [Gro77, Exposé V, VI], [Del80, 1.1] [Fu15, 10.1])**.** Let $A = \{A_i, \varphi_{ji}\}$ be a tower (Definition 7.5.2) in a category $\mathcal{C}$ where images are well-defined.

The tower is said to satisfy the *Artin-Rees-Mittag-Leffler (ARML) condition* if there exists an integer $r \geq 0$ such that for every index $i \in \mathbb{N}$ and every $k \geq i + r$, the image of the transition map $\phi_{ki} : A_k \to A_i$ is equal to the image of $\phi_{(i+r)i} : A_{i+r} \to A_i$.

Equivalently, the tower satisfies the ARML condition if and only if for any integer n, there exists an integer $r \geq 0$ such that

$$\operatorname{im}(A[r] \to A) = \operatorname{im}(A[t] \to A)$$

Definition 7.5.3 for all $t \geq r$.

Definition 7.5.8 (A–R category of inverse systems of λ-torsion sheaves)**.** Let X be a scheme. Let

$$\mathcal{F} = (\mathcal{F}_n, u_n)_{n \in \mathbb{Z}}, \quad u_n : \mathcal{F}_n \to \mathcal{F}_{n-1}$$

be an inverse system (Definition .1.17) of sheaves of abelian groups on the (small) étale site $X_{\text{ét}}$. Let R be a commutative ring and let $\lambda \subseteq R$ be an ideal.

The *A–R category of inverse systems of λ-torsion sheaves on $X_{ét}$*, denoted $\mathrm{AR}(\lambda\text{-tors}(X))$, is the localization of the multiplicative system consisting of the family of morphisms of the form $\mathcal{F}[r] \to \mathcal{F}$. In other words, the objects of the category are inverse systems of λ-torsion sheaves (Definition 7.5.1) and for any objects $\mathcal{F}$ and $\mathcal{G}$, we have

$$\mathrm{Hom}_{\mathrm{AR}(\lambda\text{-tors}(X))} = \varinjlim_{r \geq 0} \mathrm{Hom}(\mathcal{F}[r], \mathcal{G}).$$

(♠ TODO: morphism of inverse systems; more generally diagrams) In particular, a morphism $\mathcal{F} \to \mathcal{G}$ in $\mathrm{AR}(\lambda\text{-tors}(X))$ is represented by a morphsm $\mathcal{F}[r] \to \mathcal{G}$ of inverse systems for some $r \geq 0$, and the kernel and cokernel of $\mathcal{F}[r] \to \mathcal{G}$ are the kernel and cokernel of $\mathcal{F} \to \mathcal{G}$ in $\mathrm{AR}(\lambda\text{-tors}(X))$

It is worth noting that $\mathrm{AR}(\lambda\text{-tors}(X))$ is an abelian category (Definition .1.26) and that $\mathcal{F} \in \mathrm{AR}(\lambda\text{-tors}(X))$ is zero if and only if it is a null system (Definition 7.5.6).

Definition 7.5.9 (λ-adic sheaf on a scheme)**.** Let R be a commutative ring and let $\lambda \subseteq R$ be an ideal. Let X be a scheme. Let

$$\mathcal{F} = (\mathcal{F}_n, u_n)_{n \in \mathbb{Z}}, \quad u_n : \mathcal{F}_n \to \mathcal{F}_{n-1}$$

be an inverse system (Definition .1.17) of λ-torsion sheaves (Definition 7.5.1) on the (small) étale site $X_{\text{ét}}$.

1. The inverse system is called a *λ-adic sheaf on X* if

- $\mathcal{F}_n$ is constructible (Definition 7.4.1) for all n,
- $\mathcal{F}_n$ is annihilated by λ^n for all n,
- there are isomorphisms $\mathcal{F}_n \cong \mathcal{F}_{n+1}/\lambda^n \mathcal{F}_{n+1}$ compatible with transition maps.

2. A λ-adic sheaf is called *lisse* if each $\mathcal{F}_n$ is locally constant (Definition 3.4.2).
3. A λ-adic sheaf is called an *A–R λ-adic sheaf* if it satisfies the Artin–Rees–Mittag–Leffler condition (Definition 7.5.7).

Under nice enough circumstances, the categories of λ-adic sheaves and $A - R$ λ-adic sheaves are equivalent (Theorem 7.5.13).

Here is a basic example of a lambda adic sheaf:

Definition 7.5.10 (ℓ-adic sheaf $\mathbb{Z}_\ell$ on the small étale site)**.** Let X be a scheme and let ℓ be a prime invertible on X.

The *ℓ-adic sheaf* $\mathbb{Z}_\ell$ on the small étale site $X_{\text{ét}}$ (Definition 3.1.3) is defined as the inverse system (Definition .1.17)

$$\mathbb{Z}_\ell := \varprojlim_k \mathbb{Z}/\ell^k\mathbb{Z},$$

where each $\mathbb{Z}/\ell^k\mathbb{Z}$ is the constant sheaf (Definition C.0.10) of abelian groups on $X_{\text{ét}}$, and the transition maps are the canonical projections.

These inverse limits define objects in the category of *ℓ-adic sheaves*, rather than single sheaves in the classical étale topology.

Definition 7.5.11 (The ℓ-adic Tate twists $\mathbb{Z}_\ell(1)$ and $\mathbb{Z}_\ell(d)$)**.** Let ℓ be a prime invertible on X, and set

$$\mathbb{Z}_\ell(1) := \varprojlim_k \mu_{\ell^k},$$

the inverse limit (projective system) (Definition .1.12) of sheaves of ℓ^k-th roots of unity (Definition 3.4.7) on $X_{\text{ét}}$ (Definition 3.1.3), equipped with its natural $\mathbb{Z}_\ell$-module structure.

For an integer $d \geq 1$, define the *d-th Tate twist*

$$\mathbb{Z}_\ell(d) := \underbrace{\mathbb{Z}_\ell(1) \otimes_{\mathbb{Z}_\ell} \cdots \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(1)}_{d \text{ times}}.$$

For $d = 0$ set $\mathbb{Z}_\ell(0) := \mathbb{Z}_\ell$ (Definition 7.5.10).

Lemma 7.5.12. Let $(\mathcal{C}, J)$ be a site (Definition A.0.4). Let $\mathcal{O}$ be a sheaf of commutative rings (Definition 3.0.2) on $(\mathcal{C}, J)$. Let $\lambda \subseteq \mathcal{O}$ be a sheaf of ideals (Definition 6.1.6). Let $\{\mathcal{G}_n\}_{n\geq 1}$ be an inverse system of constructible (♠ TODO: generalize the notion of constructible sheaves to a sheaf of $\mathcal{O}$-modules)

Let R be a commutative ring and let $\lambda \subseteq R$ be an ideal. (♠ TODO: Try to generalize stacks 03UN as an exercise)

Theorem 7.5.13. See e.g. [Fu15, 10.1] Let X be a noetherian scheme. Let ℓ be a prime number that is invertible on X. Let R be the integral closure of $\mathbb{Z}_\ell$ in a finite extension E of $\mathbb{Q}_\ell$. Note that R is a complete discrete valuation ring and let λ be the maximal ideal of R. The category of λ-adic sheaves (Definition 7.5.9) is equivalent to the category of A-R λ-adic sheaves (Definition 7.5.9).

7.6. **The category $D_c^b(X, R)$ of integral coefficients.** For a scheme X, a prime ℓ invertible on X, and the ring of integers R in a finite extension E of $\mathbb{Q}_\ell$, Deligne [Del80, I.I.2] defined "derived" categories $D_c^b(X, R)$ (Definition 7.6.1), $D_c^b(X, E)$, and $D_c^b(X, \overline{\mathbb{Q}}_\ell)$, intuitively of "integral", "rational", and $\overline{\mathbb{Q}}_\ell$ coefficients respectively. These categories are not derived categories themselves but constructed λ-adically through the genuine derived categories $D_c^b(X, R/\lambda^n)$ of finite coefficients. loc. cit. also established that the six functor formalisms on the categories $D_c^b(X, R/\lambda^n)$ induce a six functor formalism on $D_c^b(X, R)$ under suitable finiteness conditions, e.g. X is of finite type over a regular base S of dimension ≤ 1, and that $D_c^b(X, R)$ is a triangulated category under other finiteness conditions, e.g. X is of finite type over a finite field or an algebraically closed field.

Definition 7.6.1 (see [Del80, I.I.2], cf. [Fu15, Between Propositions 10.1.16 and 10.1.17], for a discussion in the case that R is the ring of integers in a finite extension of $\mathbb{Q}_\ell$ and λ is the maximal ideal of R)**.** (♠ TODO: tensor product) Let R be a commutative ring (Definition C.0.1) and let $\lambda \subseteq R$ be an ideal (Definition C.0.2) such that R/λ^n is of finite cardinality for all $n \geq 0$. Let X be a scheme, and consider sheaves on some site on X whose underlying category is a (not necessarily full) subcategory of the category of X-schemes (e.g. the small Zariski site (Definition 3.2.4), the small Nisnevich site (Definition 3.2.13), the small étale site (Definition 3.1.3), the small fpqc site, or the small fppf site (Definition 3.2.7) on X). (♠ TODO: It might be necessary to but more conditions on λ, particularly that R/λ is finite and has exponent that is invertible on X)

(Definition B.3.1)

1. Let $D^-(X, R)$ be the following category:
   - Objects are families $K = (K_n, u_n)_{n\geq 0}$ where $K_n \in \operatorname{Ob} D^-(X, R/\lambda^{n+1})$ (Notation B.5.11)[4] and u_n are isomorphisms
   $$u_n : K_{n+1} \overset{\mathrm{L}}{\otimes}_{R/\lambda^{n+2}} R/\lambda^{n+1} \cong K_n$$
   (Definition B.5.15) (Definition 6.1.4) in $D^-(X, R/\lambda^{n+1})$.
   - Morphisms $f : K = (K_n, u_n)_{n\geq 0} \to K' = (K'_n, u'_n)_{n\geq 0}$ are families (f_n) of morphisms $f_n : K_n \to K_n$ in $D(X, R/\lambda^{n+1})$ such that
   $$f_n u_n = u'_n(f_{n+1} \overset{\mathrm{L}}{\otimes}_{R/\lambda^{n+2}} \mathrm{id}_{R/\lambda^{n+1}}).$$

   K_n is also often denoted by $K \overset{\mathrm{L}}{\otimes} R/\lambda^{n+1}$.
2. Let $D_c^b(X, R)$ be the full subcategory of $D^-(X, R)$ (Notation B.5.11) whose objects $K = (K_n, u_n)_{n\geq 0}$ satisfy $K_0 \in \operatorname{Ob} D_c^b(X, R/(\lambda))$ (Definition 7.4.3).

[4] Note that [Del80, I.I.2] uses slightly different numbering conventions, letting K_n be an object of $D^-(X, R/\lambda^n)$ instead.

Proposition 7.6.2 (See, e.g. [Fu15, Proposition 10.1.16])**.** Let X be a noetherian scheme. Let ℓ be a prime number that is invertible on X. Let R be the integral closure of $\mathbb{Z}_\ell$ in a finite extension E of $\mathbb{Q}_\ell$. Note that R is a complete discrete valuation ring and let λ be the maximal ideal of R.

For $K = (K_n, u_n)_{n\geq 0}$ in $D_c^b(X, R)$ (Definition 7.6.1), we have that $K_n \in D^b_{\mathrm{ctf}}(X, R/(\lambda^{n+1}))$(♠ TODO: ctf) for all n, and the inverse systems $(H^i(K_n)_{n\in\mathbb{Z}})$ are A-R λ-adic (Definition 7.5.9).

Remark 7.6.3. Let X be a noetherian scheme. Let ℓ be a prime number that is invertible on X. Let R be the integral closure of $\mathbb{Z}_\ell$ in a finite extension E of $\mathbb{Q}_\ell$. If $\mathcal{F} = (\mathcal{F}_n)$ is a torsion free λ-adic sheaf (♠ TODO: define torsion free), then $\mathcal{F}$ defines an objects in $D_c^b(X, R)$.

Definition 7.6.4. (♠ TODO: cite these things) Let X be a noetherian (Definition C.0.16) scheme. Let ℓ be a prime number that is invertible on X. Let R be the integral closure (Definition C.0.17) of $\mathbb{Z}_\ell$ in a finite extension E of $\mathbb{Q}_\ell$.

(♠ TODO: Define the six functors on $D_c^b(X, R/lambda^n)$) Let $f : X \to Y$ be a morphism between noetherian-schemes over a base scheme S. Consider sheaves (Definition 3.0.2) on the small étale sites (Definition 3.1.3) on X and Y. Take objects $K = (K_n)$ and $L = (L_n)$ of $D_c^b(X, R)$ (Definition 7.6.1) and $M = (M_n)$ of $D_c^b(Y, R)$. Define the following:

1. $$f^*K = (f^*K_n)_n$$
   (Definition 5.1.9) (Definition 5.1.4)
   (♠ TODO: compactifiable morphism)
2. If f is an S-compactifiable morphism over some base scheme S,
   $$Rf_!K = (Rf_!K_n)$$
   (♠ TODO:)
3. If f is an S-compactifiable morphism where S is a noetherian regular scheme (Definition C.0.18) of dimension (Definition C.0.20) ≤ 1 and X and Y are S-schemes of finite type (Definition C.0.19),
   $$\begin{aligned} Rf_*K &= (Rf_*K_n)_n \\ Rf^!K &= (Rf^!K_n)_n \quad \text{(assuming that } f \text{ is compactifiable over } S) \\ K \overset{\mathrm{L}}{\otimes}_R L &= (K_n \overset{\mathrm{L}}{\otimes}_{R/\lambda^{n+1}} L_n) \\ R\mathscr{H}om(K, L) &= (R\mathscr{H}om)(K_n, L_n). \end{aligned}$$
   (Definition 5.1.5) (♠ TODO: exceptional inverse image) (Definition B.5.15)(♠ TODO: derived sheaf hom)

When they are defined, Rf_*K and $Rf_!K$ are objects in $D_c^b(Y, R)$ and $f^*M, Rf^!M, K \otimes_R^L L$ and $R\mathscr{H}om(K, L)$ are objects in $D_c^b(X, R)$. $Rf^!K$, $K \overset{\mathrm{L}}{\otimes}_R L$, and $R\mathscr{H}om(K, L)$ may alternatively be denoted by notations such as $f^!K$, $K \otimes_R L$, and $\mathrm{RHom}(K, L)$ respectively.

Proposition 7.6.5. Let $f : X \to Y$ be a morphism between noetherian schemes. Let ℓ be a prime number that is invertible on X and Y. Let R be the integral closure of $\mathbb{Z}_\ell$ in a finite extension E of $\mathbb{Q}_\ell$. Note that R is a complete discrete valuation ring and let λ be the maximal ideal of R.

1. For any $K = (K_n) \in D_c^b(Y, R)$, we have $f^*k = (f^*K_n) \in D_c^b(X, R)$.

Definition 7.6.6. Let $(R, +, \cdot)$ be a not-necessarily commutative ring (Definition C.0.8).

1. An element $a \in R$ is a *left zero-divisor* if there exists a nonzero $x \in R$ such that $ax = 0$. Otherwise, a is called *left regular* or *left cancellable*.
2. An element $a \in R$ is a *right zero-divisor* if there exists a nonzero $x \in R$ such that $xa = 0$. Otherwise, a is called *right regular* or *right cancellable*.
3. An element $a \in R$ is a *zero-divisor* if it is a left zero-divisor or a right zero-divsor.
4. An element $a \in R$ is a *two-sided zero-divisor* if it is both a left zero-divisor and a right zero-divsor.
5. An element $a \in R$ is *regular*, *cancellable*, or a *non-zero-divisor* if it is both left and right regular.

A zero-divisor of any kind that is not itself 0 is said to be a *nonzero zero divisor* or a *nontrivial zero divisor* of its kind.

A non-zero ring with no nontrivial zero divisors is called a *domain*. A domain that it also a commutative ring (Definition C.0.1) is also called an *integral domain*.

Definition 7.6.7 (cf. [Fu15, Between Corollary 10.1.21 and Lemma 10.1.22])**.** Let R be a integral domain with fraction field E, and let $\lambda \subseteq R$ be an ideal (Definition C.0.2) such that R/λ^n is of finite cardinality for all $n \geq 0$. Let X be a scheme, and consider sheaves on some site on X whose underlying category is a (not necessarily full) subcategory of the category of X-schemes (e.g. the small Zariski site (Definition 3.2.4), the small Nisnevich site (Definition 3.2.13), the small étale site (Definition 3.1.3), the small fpqc site, or the small fppf site (Definition 3.2.7) on X). (♠ TODO: It might be necessary to but more conditions on λ, particularly that R/λ is finite and has exponent that is invertible on X)

1. On the category $D_c^b(X, R)$ (Definition 7.6.1), the morphisms defined by multiplications by λ^m for $m \geq 0$ form a multiplicative system (Definition B.2.1) (♠ TODO: why). Define the category $D_c^b(X, E)$ as the localization (Definition B.2.2) of $D_c^b(X, R)$ by this system. In particular, the objects of $D_c^b(X, E)$ are those of $D_c^b(X, R)$ and
$$\mathrm{Hom}_{D_c^b(X,R)}(K, L) \otimes_R E \cong \mathrm{Hom}_{D_c^b(X,E)}(K, L)$$
for all objects K, L of $D_c^b(X, R)$.
2. Let K/E be some algebraic extension (Definition 7.6.8). The category $D_c^b(X, K)$ is defined as the 2-direct limit of the categories $D_c^b(X, L)$ where L run over the finite extensions of K:
$$D_c^b(X, K) = \varinjlim_{L/K \text{ finite extension}} D_c^b(X, E).$$
In other words, each object of $D_c^b(X, K)$ is represented by some object of $D_c^b(X, L)$ for some L/K and given two objects $M_1 \in \mathrm{Ob}(D_c^b(X, L_1))$ and $M_2 \in \mathrm{Ob}(D_c^b(X, L_2))$, we

have

$$\mathrm{Hom}_{D^b_c(X,K)}(M_1, M_2) \cong \varinjlim_{L/K \text{ finite extension},\ L_1,L_2\subseteq L} \mathrm{Hom}(M_1 \otimes_{L_1} L, M_2 \otimes_{L_2} L).$$

(♠ TODO: define base change)

Definition 7.6.8 (Algebraic Element, Algebraic Extension)**.** Let L/K be a field extension and let $x \in L$.

- If there exists a nonzero polynomial $f(t) \in K[t]$ such that $f(x) = 0$, then x is called an *algebraic element over K*. There exists a unique such monic irreducible polynomial $f(t)$, which is called the *minimal polynomial of x over K*.
- Otherwise, x is called a *transcendental element over K*.

If every $x \in L$ is algebraic over K, then L/K is called an *algebraic extension*.

Definition 7.6.9. Let R be a integral domain with fraction field E, and let $\lambda \subseteq R$ be an ideal (Definition C.0.2) such that R/λ^n is of finite cardinality for all $n \geq 0$. Let K/E be some algebraic extension (Definition 7.6.8). Let X be a scheme, and consider sheaves on some site on X whose underlying category is a (not necessarily full) subcategory of the category of X-schemes (e.g. the small Zariski site (Definition 3.2.4), the small Nisnevich site (Definition 3.2.13), the small étale site (Definition 3.1.3), the small fpqc site, or the small fppf site (Definition 3.2.7) on X). (♠ TODO: It might be necessary to but more conditions on λ, particularly that R/λ is finite and has exponent that is invertible on X)

(♠ TODO: probably need to talk about base changes of ell-adic sheaves, $\overline{\mathbb{Q}}_\ell$-sheaves, etc.)
(♠ TODO: $D^b_c(X, \overline{\mathbb{Q}}_\ell)$)

8. Theorems

Corollary 8.0.1 (Finiteness of cohomology)**.** (♠ TODO: read) Let X be a variety of finite type over a separably closed field k, and let $\ell \neq \mathrm{char}(k)$. Then each $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$ is a finite-dimensional $\mathbb{Q}_\ell$-vector space, and vanishes for $i > 2\dim(X)$.

8.1. Base change theorem.

Proposition 8.1.1 (Smooth base change)**.** (♠ TODO: read) Let $f : X \to S$ be a smooth, proper morphism of schemes, and let ℓ be a prime invertible on S. Then the formation of étale cohomology commutes with base change: for any cartesian square

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow f \\ T & \longrightarrow & S \end{array}$$

one has natural isomorphisms

$$H^i_{\text{ét}}(Y_{\overline{t}}, \mathbb{Q}_\ell) \ \cong\ H^i_{\text{ét}}(X_{\overline{s}}, \mathbb{Q}_\ell),$$

where $\bar{t}$ lies over $t \in T$, and $\bar{s}$ its image in S.

Theorem 8.1.2 (Proper base change theorem)**.** Let $f : X \to Y$ be a proper morphism of schemes, and let ℓ be a prime invertible on Y. Then for any sheaf of $\mathbb{Z}_\ell$-modules $\mathcal{F}$ on $X_{\text{ét}}$, the higher direct image sheaves $R^i f_* \mathcal{F}$ on $Y_{\text{ét}}$ satisfy

$$(R^i f_* \mathcal{F})_y \cong H^i_{\text{ét}}(X_y, \mathcal{F}|_{X_y})$$

for each geometric point y of Y, where X_y is the fiber over y.

Theorem 8.1.3 (Smooth and proper base change compatibility)**.** Under suitable smooth and proper morphisms, the formation of étale cohomology commutes with base change on the target scheme. More precisely, given a cartesian square

$$\begin{array}{ccc} X' & \to & X \\ \downarrow & & \downarrow \\ Y' & \to & Y \end{array}$$

with $f : X \to Y$ smooth and proper, and a prime ℓ invertible on Y, the natural base change morphism induces isomorphisms in étale cohomology

$$H^i_{\text{ét}}(X_y, \mathbb{Q}_\ell) \cong H^i_{\text{ét}}(X'_{y'}, \mathbb{Q}_\ell)$$

for geometric points y of Y and y' over y in Y'.

8.2. **Poincaré duality.**

Theorem 8.2.1 (Generalized Poincaré Duality in Étale Cohomology)**.** (♠ TODO: read this) Let k be a separably closed field, and let X be a smooth, separated scheme of finite type over k, of pure dimension d. Let ℓ be a prime different from $\text{char}(k)$.

Let Λ be one of the following coefficient rings:

- a finite extension of $\mathbb{Z}_\ell$,
- the ring of integers $\mathcal{O}_E$ of a finite extension E of $\mathbb{Q}_\ell$,
- E itself, or
- an algebraic closure $\overline{\mathbb{Q}}_\ell$.

Let $\mathcal{F}^\bullet$ be a bounded constructible complex of Λ-modules on the étale site $X_{\text{ét}}$.

Then there is a canonical perfect pairing in the derived category of Λ-modules:

$$R\Gamma_{\text{ét}}(X, \mathcal{F}^\bullet) \ \otimes^{\mathbf{L}}_\Lambda \ R\Gamma_c^{\text{ét}}(X, R\mathcal{H}om_\Lambda(\mathcal{F}^\bullet, \Lambda(d)[2d])) \ \longrightarrow \ \Lambda[-2d],$$

where $R\Gamma_c^{\text{ét}}$ denotes étale cohomology with compact support, and

$$\Lambda(d) := \underbrace{\Lambda(1) \otimes_\Lambda \cdots \otimes_\Lambda \Lambda(1)}_{d \text{ times}}$$

is the d-th Tate twist.

In particular, this induces a perfect duality between finite generated Λ-modules

$$H^i_{\text{ét}}(X, \mathcal{F}^\bullet) \cong \text{Hom}_\Lambda(H_c^{2d-i}(X, R\mathcal{H}om_\Lambda(\mathcal{F}^\bullet, \Lambda(d)[2d])), \Lambda).$$

Remarks:

- For projective X, $R\Gamma_c^{\text{ét}}$ can be replaced by the usual cohomology $R\Gamma_{\text{ét}}$.
- The duality is a manifestation of Verdier duality and requires the formalism of the six operations and dualizing complexes in étale cohomology.
- This generalizes classical Poincaré duality by allowing more general coefficient complexes and non-projective varieties with compact support.

8.3. **Lefschetz trace formula.**

Theorem 8.3.1 (Lefschetz trace formula in étale cohomology)**.** (♠ TODO: read) Let X be a separated scheme of finite type over a finite field $\mathbb{F}_q$, and let ℓ be a prime number not dividing q. Then the number of $\mathbb{F}_q$-rational points of X is given by the Lefschetz trace formula:

$$|X(\mathbb{F}_q)| = \sum_{i=0}^{2\dim(X)} (-1)^i \operatorname{Tr}(\mathrm{Frob}_q^* \mid H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)),$$

where Frob_q is the geometric Frobenius acting on the ℓ-adic étale cohomology groups of X.

Theorem 8.3.2 (Grothendieck–Lefschetz trace formula in terms of traces)**.** Let X_0 be a separated scheme of finite type over a finite field $\mathbb{F}_q$. Denote by $\overline{X} = X_0 \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ the base change to an algebraic closure, and let $K_0 \in D_c^b(X_0, \overline{\mathbb{Q}}_\ell)$ be a bounded constructible complex of étale sheaves.

Then the number of fixed points of the n-th iterate of the geometric Frobenius on X_0, weighted by the trace of the induced endomorphism on stalks of K_0, satisfies the formula

$$\sum_{x \in X_0(\mathbb{F}_{q^n})} \operatorname{tr}\left((\mathrm{Frob}_{q^n})_x \mid K_{0,\overline{x}}\right) = \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{tr}\left(\mathrm{Frob}_{q^n} \mid H^i_c(\overline{X}, K)\right),$$

where K denotes the pullback of K_0 to $\overline{X}$, and $(\mathrm{Frob}_{q^n})_x$ is the induced action on the stalk of K_0 at a geometric point over x.

In particular, this expresses the weighted count of $\mathbb{F}_{q^n}$-rational points on X_0 in terms of traces of Frobenius on étale cohomology with compact support, encapsulating the Grothendieck–Lefschetz trace formula in terms of traces rather than directly using determinant or Frobenius polynomials.

8.4. **Weil conjectures.**

Theorem 8.4.1 (Weil Conjectures via Étale Cohomology)**.** Let X be a smooth projective variety of pure dimension d defined over a finite field $\mathbb{F}_q$. Let ℓ be a prime number different from the characteristic of $\mathbb{F}_q$. Let F denote the geometric Frobenius endomorphism acting on the étale cohomology groups $H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$, where $\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$.

Then the Weil conjectures state the following:

1. (Rationality) The zeta function
$$Z(X,t) = \exp\left(\sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_{q^n})}{n} t^n\right)$$
is a rational function which can be expressed as
$$Z(X,t) = \prod_{i=0}^{2d} P_i(t)^{(-1)^{i+1}},$$
where each $P_i(t) = \det(1 - tF \mid H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell))$ is a polynomial with coefficients in $\mathbb{Q}_\ell$.
2. (Functional Equation) The zeta function satisfies a functional equation relating $Z(X,t)$ and $Z(X, q^{-d}t^{-1})$, induced by Poincaré duality on cohomology and the action of F.
3. (Betti Numbers) Each polynomial $P_i(t)$ has degree equal to the i-th Betti number of X (i.e., $\dim_{\mathbb{Q}_\ell} H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$).
4. (Riemann Hypothesis) The eigenvalues of the geometric Frobenius F acting on $H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ are algebraic numbers, all of whose complex absolute values equal $q^{i/2}$.

This theorem links the counting of points over finite fields to the action of Frobenius on étale cohomology and provides a cohomological interpretation and proof of the Weil conjectures.

Here is another, more general, formulation of the rationality statement of the Weil conjectures:

Theorem 8.4.2 (Grothendieck, see [DBG+77, Rapport 3 Théorème 3.1]; see also [Fu15, Theorem 10.5.1] for a statement and cf. [KW13, Theorem I.1.1])**.** Let X_0 be a scheme of finite type over the finite field $\mathbb{F}_q$, and let $K_0 \in D^b_c(X_0, \overline{\mathbb{Q}}_\ell)$. Write $X = X_0 \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ for the base change, and K for the the pullback of K_0 to X.

We have
$$L(X_0, K_0, t) = \prod_{i \in \mathbb{Z}} \det\left(1 - t \cdot \mathrm{Frob}_q \mid H^i_c(X, K)\right)^{(-1)^{i+1}}.$$

9. Ekedahl's formalism

(♠ TODO: discuss what [Eke07] does) We will discuss Ekedahl's formalism on the small étale site (Definition 3.1.3) of finite type and seperated schemes over regular bases of dimension 0 or 1

Remark 9.0.1. Ekedahl's definitions and notations [Eke07] are developed in terms of a ringed topos (S, R). We will develop definitions and notations of his theory in terms of a ringed site (Definition 5.1.10).

Definition 9.0.2 (Categories of Inverse and Direct Sequences of objects in a category)**.** Let $\mathcal{C}$ be a category (Definition .1.1).

1. The *category of inverse sequences* or *Pro-category indexed by* $\mathbb{N}$, denoted by $\mathrm{Pro}_{\mathbb{N}}(\mathcal{C})$, is defined as follows:

- Objects are sequences $(M_n, p_n)_{n\geq 1}$ where for each $n \geq 1$, $M_n \in \mathrm{Ob}(\mathcal{C})$ and $p_n : M_{n+1} \to M_n$ is a morphism in $\mathcal{C}$.
- A morphism $f : (M_n, p_n) \to (N_n, q_n)$ consists of a collection of morphisms $\{f_n : M_n \to N_n\}_{n\geq 1}$ in $\mathcal{C}$ satisfying the compatibility condition

$$\forall n \geq 1, \quad q_n \circ f_{n+1} = f_n \circ p_n.$$

Composition and identities are defined componentwise.

2. The *category of direct sequences* or *Ind-category indexed by* $\mathbb{N}$, denoted by $\mathrm{Ind}_{\mathbb{N}}(\mathcal{C})$, is defined as follows:
   - Objects are sequences $(M_n, i_n)_{n\geq 1}$ where for each $n \geq 1$, $M_n \in \mathrm{Ob}(\mathcal{C})$ and $i_n : M_n \to M_{n+1}$ is a morphism in $\mathcal{C}$.
   - A morphism $f : (M_n, i_n) \to (N_n, j_n)$ consists of a collection of morphisms $\{f_n : M_n \to N_n\}_{n\geq 1}$ in $\mathcal{C}$ satisfying the compatibility condition

$$\forall n \geq 1, \quad j_n \circ f_n = f_{n+1} \circ i_n.$$

Composition and identities are defined componentwise.

Equivalently, $\mathrm{Pro}_{\mathbb{N}}(\mathcal{C})$ and $\mathrm{Ind}_{\mathbb{N}}(\mathcal{C})$ may be regarded as the functor categories $\mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathcal{C})$ and $\mathrm{Fun}(\mathbb{N}, \mathcal{C})$ respectively, where $\mathbb{N}$ is regarded (Lemma .1.21) as a poset (Definition .1.20) category with objects $1, 2, \ldots$ with a unique morphism $n \to m$ if and only if $n \leq m$.

Context 9.0.3. Let $(\mathcal{C}, J, R)$ be a ringed site (Definition 5.1.10) and let $m \subseteq R$ be a two-sided ideal. Write S for the topos (Definition .1.23) of $(\mathcal{C}, J)$. Let $\mathrm{Pro}_{\mathbb{N}}(S)$ be the category of inverse sequences of S (Definition 9.0.2); this is equivalent to the functor category $[\mathbb{N}^{\mathrm{op}}, S]$ (Definition .1.25) and is a topos (Theorem .1.24) itself. Regard it as a ringed topos equipped with $R_\bullet$ (Definition 9.0.5).

Context 9.0.4. Assume Context 9.0.3. Write $\pi : \mathrm{Pro}_{\mathbb{N}}(S) \to S$ for the topos morphism (♠ TODO: topos morphism) given by $\pi_* M_\bullet = \varprojlim M_\bullet$ and $\pi^* M = \{M, \mathrm{id}\}_{n\geq 1}$, i.e. the system consisting of the objects M at each level and the identity maps between them.

Definition 9.0.5. Assume Context 9.0.4. There is a ring object, which we might sometimes denote by $R_\bullet$, given by the projective system

$$\cdots \to R/m^{n+1} \to R/m^n \to \cdots .$$

When $\mathrm{Pro}_{\mathbb{N}}(S)$ is equipped with $R_\bullet$, π is a morphism of ringed topoi. (♠ TODO: morphism of ringed topoi). In particular, for an object M of the ringed topos (S, R), its pullback $\pi^* M$ is the inverse system $\{M \otimes_R R_n\}_{n\geq 1}$.

Definition 9.0.6. Assume Context 9.0.3

1. We say that an abelian group object $M_\bullet$ of $\mathrm{Pro}_{\mathbb{N}}(S)$ is *essentially zero* if there is a covering of the final object (Definition .1.10) of S such that, when $M_\bullet$ is restricted to each element of the covering, there is for all n an $m \geq n$ such that the map $M_m \to M_n$ in $M_\bullet$ is zero.
2. Let $M^\bullet$ be a complex (Definition .2.1) of abelian group objects of $\mathrm{Pro}_{\mathbb{N}}(S)$. We say that $M^\bullet$ is *essentially zero* if $H^i(M^\bullet)$ (Definition .2.3) is essentially zero for all i.

   The category of essentially zero complexes form a triangulated subcategory (Definition B.1.2) of the category of complexes of abelian group objects of $\mathrm{Pro}_{\mathbb{N}}(S)$.

One may say that $M^\bullet$ is *essentially bounded*, *essentially bounded from below*, or *essentially bounded from above*, if $H^m(M^\bullet)$ is essentially zero for all m with $|m| \gg 0$, $-m \gg 0$, and $m \gg 0$ respectively.

The (derived) categories (Definition B.3.1) of essentially bounded, essentially bounded from below, and essentially bounded from above complexes of R-module objects of $\mathrm{Pro}_{\mathbb{N}}(S)$ may be denoted by $D^{eb}(\mathrm{Pro}_{\mathbb{N}}(S), R_\bullet)$, $D^{e+}(\mathrm{Pro}_{\mathbb{N}}(S), R_\bullet)$, and $D^{e-}(\mathrm{Pro}_{\mathbb{N}}(S), R_\bullet)$ respectively[5].

Notation 9.0.7. Let (S, R) be a ringed site (Definition 5.1.10). Write $D(S, R)$ for the derived category (Definition B.3.1) of the category of R-modules (Definition 6.1.3). Accordingly, $D^+(S, R)$, $D^-(S, R)$, and $D^b(A)$ may be used to denote the full subcategories of cohomologically bounded below, cohomologically bounded above, and cohomologically bounded complexes (Definition .2.2) respectively, cf. Definition B.3.1

Definition 9.0.8 ([Eke07, Section 1])**.** Assume Context 9.0.4.

1. We may say that $(\mathcal{C}, J, R)$ or (S, R) *satisfies Ekedahl's condition A* if the following holds: there exists a class S^{gen} of generators of S (Definition .1.22) and an integer N such that for all $T \in S^{\mathrm{gen}}$, all R/m-modules (Definition 6.1.3) M, and $i > N$, we have $H^i(T, M) = 0$.
2. We may say that $(\mathcal{C}, J, R)$ or (S, R) *satisfies Ekedahl's condition B* if the following hold:
   - there is an integer N such that for every $n \geq 1$ and locally on $(\mathcal{C}, J)$, there is a left resolution (Definition .2.7) $F_n^\bullet \to R/m^n$ of R/m^n by a complex of (right) R-modules of finitely generated free objects such that $\pi^* F_n^\bullet \to \pi^*(R/m^n)$ is a resolution modulo essentially zero systems (♠ TODO: essentially zero system), where $\pi : (\mathrm{Pro}_{\mathbb{N}}(S), R_\bullet) \to (S, R)$ is considered as a morphism of ringed topoi.
   - m^n/m^{n+1} is, locally on $(\mathcal{C}, J)$, of finite Tor-dimension over R/m. (♠ TODO: finite Tor dimension)

Notation 9.0.9 ([Eke07, After Lemma 1.1])**.** Assume Context 9.0.3

Given $M \in D^{e+}(\mathrm{Pro}_{\mathbb{N}}(S), \mathbb{Z}_\bullet)$, [Eke07] lets $\tau(M)$ denote the following:

- If $(\mathcal{C}, J, R)$ satisifes Ekedahl's condition A (Definition 9.0.8), then $\tau(M) = M$.
- Otherwise, $\tau(M)$ is the pro-object
$$(\cdots \to \tau_{i-1} M \to \tau_{\geq i} M \to \tau_{i+1} M \to \cdots)$$
(Definition .2.6).

(♠ TODO: Talk about $R\pi_* M$)

Definition 9.0.10 ([Eke07, Definition 2.1])**.** Assume Context 9.0.4

Let M, N be objects $D(\mathrm{Pro}_{\mathbb{N}}(S), R_\bullet)$ (Notation 9.0.7).

1. We may say that M is an *R-complex* if $\pi^*(R_1) \otimes^L_{R_\bullet} M$ (♠ TODO: derived tensor product) is essentially constant (♠ TODO: essentially constant)

[5] In [Eke07], these are denoted by $D^{eb}(S^{\mathbb{N}} - R_\bullet)$, etc.

2. M is *negligible* if $\pi^*(R_1) \otimes^L_{R_\bullet} M$ is essentially zero (Definition 9.0.6).
3. M is *normalised* if $\tau(\widehat{M}) \to \tau(M)$ (Notation 9.0.9)(♠ TODO: $\widehat{M}$; what is the morphism $\tau(\widehat{M}) \to \tau(M)$) is a quasi-isomorphism (Definition .2.4).
4. A morphism $M \to N$ is *essentially an isomorphism* if it has a negligible mapping cone (Definition .2.5).

Context 9.0.11. Assume Context 9.0.4; in particular, Context 9.0.3 is also assumed. Assume that Ekedahl's condition A or B holds (Definition 9.0.8). Let

$$* = \begin{cases} - & \text{if Ekedahl's condition A holds} \\ e+ & \text{if Ekedahl's condition B holds} \\ \text{(blank)} & \text{if both Ekedahl's condition A and condition B hold.} \end{cases}$$

In particular, we will consider the category $D^-(\mathrm{Pro}_{\mathbb{N}}, R_\bullet)$ (Notation 9.0.7), $D^{e+}(\mathrm{Pro}_{\mathbb{N}}, R_\bullet)$ (Definition 9.0.6), or $D(\mathrm{Pro}_{\mathbb{N}}, R_\bullet)$ (Notation 9.0.7), depending on which combination of A and B holds.

Proposition 9.0.12 ([Eke07, Proposition 2.2])**.** Assume Context 9.0.11.

1. M is an R-complex (Definition 9.0.10) if and only if $\tau(\widehat{M}) \to \tau(M)$ (Notation 9.0.9) (♠ TODO: $\widehat{M}$) is essentially an isomorphism (Definition 9.0.10).
2. M is normalised (Definition 9.0.10) if and only if $R_n \otimes^L_{R_{n+1}} i^*_{n+1} M \to i^*_n M$ is an isomorphism for all n. (♠ TODO: tensor product, i^*_n)
3. Let ? be $-$, $+$, or blank if Ekedahl's condition A, condition B, or conditions A and B hold respectively. Let $M' \in D^?(S, R)$ (Notation 9.0.7). The object $L\pi^* M'$ (♠ TODO: $L\pi^*$) belongs to $D^?(\mathrm{Pro}_{\mathbb{N}}, R_\bullet)$. In particular, $\widehat{M}$ is normalised.

Definition 9.0.13 (cf. [Eke07, Definition 2.5])**.** (♠ TODO: I don't think this is quite the right definition of Ekedahl's category; somehow, the conditoin that the transition maps induce isomorphisms need to be incorporated.) Assume Context 9.0.4.

Denote by $D_{\mathrm{Ek}}(S, R)$ or $D_{\mathrm{Ek}}(\mathcal{C}, R)$ the category whose objects are R-complexes (Definition 9.0.10) and morphisms are the morphisms $\tau(M) \to \tau(N)$ (Notation 9.0.9) with essential isomorphisms (Definition 9.0.6) inverted. For $* \in \{b, +, -\}$, Denote by $D^*_{\mathrm{Ek}}(S, R)$ or $D^*_{\mathrm{Ek}}(\mathcal{C}, R)$ the full subcategory whose objects are the complexes which are essentially bounded, essentially bounded from below, and essentially bounded from above (Definition 9.0.6) respectively.

It is appropriate to call these categories *"Ekedahl's adic categories"*. By abuse of language in conflict with the terminology of Definition 9.0.10, [Eke07] calls the objects of $D_{\mathrm{Ek}}(S, R)$ *R-complexes*.

Theorem 9.0.14 ([Eke07, Proposition 2.7])**.** Assume Context 9.0.4. Let $* \in \{b, +, -\}$.

1. $R_1 \otimes^L_R (-) : D_{\mathrm{Ek}}(S, R) \to D(S, R_1)$ (♠ TODO: tensor product) (Context 9.0.3, Definition 9.0.13, Notation 9.0.7) is conservative. (♠ TODO: conservative functor)
2. Letting $D^*_{\mathrm{norm}}(\mathrm{Pro}_{\mathbb{N}}(S), R_\bullet) \subset D^*(\mathrm{Pro}_{\mathbb{N}}(S), R_\bullet)$ (Definition 9.0.2, Notation 9.0.7) be the subcategory of normalised R-complexes (Definition 9.0.10). The canonical functor

$$D^*_{\mathrm{norm}}(\mathrm{Pro}_{\mathbb{N}}(S), R_\bullet) \to D^*_{\mathrm{Ek}}(S - R)$$

(♠ TODO: What is this canonical functor) and the functor

$$(\widehat{-}) : D^*_{\mathrm{Ek}}(S - R) \to D^*_{\mathrm{norm}}(\mathrm{Pro}_{\mathbb{N}}(S), R_\bullet)$$

are inverse equivalences of categories (Definition .1.6).

Theorem 9.0.15 ([Eke07, Theorem 6.3])**.** (♠ TODO: regular ring, scheme) Let S be a regular scheme of dimension (Definition C.0.20) at most 1. Let R be a commutative (Definition C.0.1), local (Definition C.0.21), regular ring with maximal ideal m such that the residue field $R_1 = R/m$ is of positive characteristic invertible in $\mathcal{O}_s$. Write R also for the constant sheaf (Definition C.0.10) of R on $S_{\text{ét}}$ (Definition 3.1.3). Let c be the full subcategory of the category of sheaves of R_1-modules (Definition 6.1.3) whose objects are constructible (Definition 7.4.1).

Let X, Y be S-schemes

1. The category (♠ TODO: Ekedahl's constructible, bounded category)$D^b_c(X_{\text{ét}}, R)$ and the functor
$$R_1 \otimes^L_R (-) : D^b_c(X_{\text{ét}}, R) \to D^b_c(X_{\text{ét}}, R_1)$$
is a conservative triangulated functor (♠ TODO: conservative triangulated functor)
2. $D^b_c(X_{\text{ét}}, R)$ has a t-structure (♠ TODO:) whose heart is equivalent to the category of m-adic constructible sheaves and for which every object of $D^b_c(X_{\text{ét}}, R)$ has finite amplitude, i.e. $D^b_c(X_{\text{ét}}, R) = \bigcup_{i\in\mathbb{Z}} D^b_c(X_{\text{ét}}, R)^{\geq i} = \bigcup_{i\in\mathbb{Z}} D^b_c(X_{\text{ét}}, R)^{\leq i}$.
3. (♠ TODO: tensor and RHom)
4. (♠ TODO: pushforward, pullback)
5. (♠ TODO: perverse sheaves)

10. Pro-étale topology

Definition 10.0.1 (Diagonal morphism of a morphism of schemes)**.** Let $f : X \to Y$ be a morphism of schemes (Definition .3.2).

The *diagonal morphism associated to f* is the morphism $\Delta_f : X \to X \times_Y X$ (Definition C.0.15) which is the diagonal morphism associated to the morphism f in the category of schemes.

In other words, Δ_f is defined as the unique morphism induced by the universal property of the fiber product (Definition C.0.15) making the following diagram commute:

$$\begin{tikzcd}
X, \arrow[dr, "\Delta_f"] \arrow[drr, bend left, "\mathrm{id}_X"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\
& X \times_Y X \arrow[r, "p_2"] \arrow[d, "p_1"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

where p_1 and p_2 are the natural projections from the fiber product.

In other words, Δ_f is given by the pair of identity morphisms $(\mathrm{id}_X, \mathrm{id}_X)$ over Y:

$$\Delta_f := (\mathrm{id}_X, \mathrm{id}_X) : X \to X \times_Y X.$$

Definition 10.0.2 (Weakly étale morphism of schemes, [BS15, Definition 2.2])**.** A morphism $f : X \to Y$ of schemes (Definition .3.2) is *weakly étale* if it and its diagonal (Definition 10.0.1) are both flat (Definition 1.1.4).

Definition 10.0.3 (Pro-étale site of a scheme, [BS15, Definition 1.2])**.** Let X be a scheme (Definition .3.1). The *pro-étale site* $X_{\text{proét}}$ is the site whose underlying category is the category of weakly étale (Definition 10.0.2) X-schemes and whose covers are covers in the fpqc topology (Definition 3.2.8), i.e. a family $\{Y_i \to Y\}$ of maps on $X_{\text{proét}}$ is a covering family if any open affine in Y is mapped onto by an open affine in $\coprod_i Y_i$.

Theorem 10.0.4. (♠ TODO: the categories) Let ℓ be a prime number. Let E be an algebraic extension of $\mathbb{Q}_\ell$. The triangulated category $D_{\text{cons}}(X_{\text{proét}}, E)$ is equivalent to the triangulated category traditionally called $D^b_c(X, E)$

.1. Categorical definitions.

Definition .1.1 (Category)**.** A *category* $\mathcal{C}$ consists of the following data:

- A class of *objects* denoted $\mathrm{Ob}(\mathcal{C})$.
- For each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a class

  $$\mathrm{Hom}_{\mathcal{C}}(X, Y)$$

  of *morphisms* (also called *arrows* or *homs*). If the category $\mathcal{C}$ is clear, then this *hom-class* is also denoted by $\mathrm{Hom}(X, Y)$. It may also be denoted by $\mathrm{hom}_{\mathcal{C}}(X, Y)$ or $\mathrm{hom}(X, Y)$, especially to distinguish from other types of hom's (e.g. internal hom's)
- For each triple of objects X, Y, Z, a composition law

  $$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z),$$

  denoted $(g, f) \mapsto g \circ f$.
- For each object X, an *identity morphism*

  $$\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X).$$

These data satisfy the following axioms:

- (Associativity) For all morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, and $h \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$,

  $$h \circ (g \circ f) = (h \circ g) \circ f.$$
- (Identity) For all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$,

  $$\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X .$$

One often writes $X \in \mathcal{C}$ synonymously with $X \in \mathrm{Ob}(\mathcal{C})$, i.e. to denote that X is an object of of $\mathcal{C}$.

We may call a category as above an *ordinary category* to distinguish this notion from the notions of *categories enriched in monoidal categories* or higher/n-categories. (♠ TODO: TODO: define n-categories)

A category as defined above may be called called a *large category* or a *class category* to emphasize that the hom-classes may be proper classes rather than sets (note, however, that the possibility that hom-classes are sets is not excluded for large categories). Accordingly, a *category* may often refer to a locally small category (Definition .1.4), which is a category whose hom-classes are all sets.

Definition .1.2 (Grothendieck Universe)**.** Let U be a set. We say U is a *Grothendieck universe* (or just a *universe*) if the following conditions hold:

1. If $x \in U$ and $y \in x$, then $y \in U$ (transitivity).
2. If $x, y \in U$, then $\{x, y\} \in U$ (closed under pair formation).
3. If $x \in U$, then the power set $\mathcal{P}(x) \in U$.
4. If $I \in U$ and $(x_\alpha)_{\alpha \in I}$ is a family with each $x_\alpha \in U$, then $\bigcup_{\alpha \in I} x_\alpha \in U$.

A set X is called *U-small* or a *U-set* if $X \in U$.

Definition .1.3 (Coarser and finer Grothendieck topologies)**.** Let $\mathcal{C}$ be a category (Definition .1.1) and let τ, τ' be two Grothendieck topologies (Theorem 8.4.2) on $\mathcal{C}$.

We say that *τ is finer than τ'* (or equivalently, *τ' is coarser than τ*) if for every object $U \in \mathcal{C}$ every τ-covering of U is also a τ'-covering of U.

If τ and τ' are not comparable by inclusion, i.e., neither is finer or coarser than the other, they are said to be *incomparable*.

Definition .1.4 (Locally small category)**.** A (large) category (Definition .1.1) $\mathcal{C}$ is called a *locally small category* if for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the collection $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms between them is a (small) *set* (as opposed to a proper class). In other words, each hom-class is a set and may even be called a *hom-set*.

In some contexts, a locally small category may simply be called a *category*, especially when genuinely large categories are not considered.

A category $\mathcal{C}$ is called a *small category* if it is a locally small category and the class $\mathrm{Ob}(\mathcal{C})$ of objects is a set.

Given a universe (Definition .1.2) U, we can define the notion of a *U-locally small category* and of a *U-small category* similarly.

Remark .1.5. Many "concrete" categories considered in "classical mathematics" or outside of more "abstract" category theory tend to be locally small. For example, the categories of sets, groups, R-modules, vector spaces, topological spaces, schemes, manifolds, sheaves on "small enough" sites are all locally small.

Definition .1.6. An *equivalence of categories* between two (large) categories (Definition .1.1) $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of functors

$$F : \mathcal{C} \to \mathcal{D} \quad \text{and} \quad G : \mathcal{D} \to \mathcal{C}$$

together with natural isomorphisms

$$\eta : \mathrm{Id}_{\mathcal{C}} \xrightarrow{\sim} G \circ F \quad \text{and} \quad \epsilon : F \circ G \xrightarrow{\sim} \mathrm{Id}_{\mathcal{D}}.$$

Such functors F and G may be called *(natural) inverses of each other*.

When $\mathcal{C}$ and $\mathcal{D}$ are locally small categories (Definition .1.4), F is an equivalence of categories if and only if F is fully faithful and essentially surjective

Lemma .1.7. Let $\mathcal{C}$ be a small category (Definition .1.4) (resp. U-small category where U is some universe (Definition .1.2)) and let $\mathcal{A}$ be a locally small (Definition .1.4) category (resp. U-locally small category). The presheaf category $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ (Definition 3.0.1) is locally small (resp. U-locally small).

Proof. A morphism $\mathcal{F} \to \mathcal{G}$ in $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ is a natural transformation of the functors $\mathcal{F}, \mathcal{G} : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$. Such a natural transformation is encoded by a family $(\eta_C)_C$ of morphisms (satisfying certain conditions) $\eta_C : \mathcal{F}(C) \to \mathcal{G}(C)$ in $\mathcal{A}$ over objects C of $\mathcal{C}^{\mathrm{op}}$. The product $\prod_{C \in \mathrm{Ob}\, \mathcal{C}^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}(C), \mathcal{G}(C))$ is a product of (U-small) sets indexed by a (U-small) set, and the collection of natural transformations is a subset of this set. Therefore, $\mathrm{Hom}_{\mathrm{PreShv}(\mathcal{C}, \mathcal{A})}(\mathcal{F}, \mathcal{G})$ is a (U-small) set. $\square$

Definition .1.8. Let $\mathcal{C}$ and $\mathcal{D}$ be categories (Definition .1.1). Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors.

An *adjunction between F and G* consists of two natural transformations: $\eta : \mathrm{Id}_{\mathcal{C}} \implies GF$ (the *unit*), and $\varepsilon : FG \implies \mathrm{Id}_{\mathcal{D}}$ (the *counit*)

These must satisfy the triangle identities: For every object $X \in \mathcal{C}$ and $Y \in \mathcal{D}$,

$$\varepsilon_{FX} \circ F(\eta_X) = \mathrm{id}_{FX}$$

$$G(\varepsilon_Y) \circ \eta_{GY} = \mathrm{id}_{GY}.$$

In diagrammatic form, the triangle identities assert that the following are commutative diagrams:

$$\begin{tikzcd} F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "\mathrm{id}_{F(X)}"'] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \\ & F(X) \end{tikzcd} \qquad \begin{tikzcd} G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "\mathrm{id}_{G(Y)}"'] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] \\ & G(Y) \end{tikzcd}$$

We say that F is a *left adjoint to G* and G is a *right adjoint to F* (written $F \dashv G$).

In the case that $\mathcal{C}$ and $\mathcal{D}$ are locally small (Definition .1.4) categories (or U-locally small categories if a universe (Definition .1.2) U is available), we have an adjunction $F \dashv G$ if and only if for every object X in $\mathcal{C}$ and Y in $\mathcal{D}$ there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

that is natural in both X and Y. In this case, the *unit of the adjunction* is the natural transformation $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ such that,

1. for every $X \in \mathcal{C}$, the morphism $\eta_X : X \to GF(X)$ (each called a *unit morphism*) in $\mathcal{C}$ is obtained as the image of $\mathrm{id}_{F(X)}$ via the adjoint isomorphism
$$\mathrm{Hom}_{\mathcal{D}}(F(X), F(X)) \cong \mathrm{Hom}_{\mathcal{C}}(X, GF(X)).$$
2. for every $Y \in \mathcal{D}$, the morphism $\epsilon_Y : FG(Y) \to Y$ (each called a *counit morphism*) in $\mathcal{D}$ is obtained as the image of $\mathrm{id}_{G(Y)}$ via the adjoint isomorphism
$$\mathrm{Hom}_{\mathcal{C}}(G(Y), G(Y)) \cong \mathrm{Hom}_{\mathcal{D}}(FG(Y), Y).$$

Definition .1.9. Let (X, τ_X) be a topological space. The *small site assocciated to X* or *the site of open covers of X* or *the canonical site on* $\mathrm{Open}\, X$ is the category $\mathrm{Open}(X)$ of open subsets of X with inclusion morphisms, equipped with the canonical Grothendieck topology (Definition A.0.4) generated by (Definition 3.2.1) the Grothendieck pretopology (Definition 3.2.2) whose covering families $\{U_i \to U\}_{i \in I}$, for $U \in \mathrm{Open}(X)$ are families of morphisms in $\mathrm{Open}(X)$ such that $\bigcup_{i \in I} U_i = U$. In other words, $\{U_i \to U\}_{i \in I}$ is a covering for the pretopology if it is an open coverings.

Definition .1.10. Let $\mathcal{C}$ be a (large) category (Definition .1.1).

1. An object $I \in \mathcal{C}$ is called an *initial object* if for every object $X \in \mathcal{C}$ there exists a unique morphism
$$I \to X.$$
Equivalently, an initial object is a limit (Definition .1.12) of the empty diagram (Definition .1.25), if such a limit exists.
2. An object $F \in \mathcal{C}$ is called a *final object* (or *terminal object*) if for every object $X \in \mathcal{C}$ there exists a unique morphism
$$X \to F.$$
Equivalently, a final object is a colimit (Definition .1.12) of the empty diagram (Definition .1.25), if such a colimit exists.
3. An object $Z \in \mathcal{C}$ is called a *zero object* if Z is both initial and final in $\mathcal{C}$. In particular, for every object $X \in \mathcal{C}$ there exist unique morphisms
$$Z \to X \quad \text{and} \quad X \to Z.$$

In particular, if initial/final/zero objects exist in a cateogry, then they are unique up to unique isomorphism.

Definition .1.11 (Filtered category)**.** 1. A *filtered category* is a (nonempty, large) category $\mathcal{I}$ satisfying the following conditions:
- For every finite collection of objects $i_1, i_2, \ldots, i_n$ in $\mathcal{I}$, there exists an object j and morphisms
$$\phi_k : i_k \to j, \quad \text{for each } k = 1, \ldots, n.$$
- For every pair of morphisms $f, g : i \to j$ in $\mathcal{I}$, there exists an object k and a morphism
$$h : j \to k$$
that satisfies
$$h \circ f = h \circ g.$$

$$\begin{tikzcd} i_1 \arrow[dr, dashed, "\phi_1"] & \\ & j \\ i_2 \arrow[ur, dashed, "\phi_2"'] & \end{tikzcd}$$

Figure 1. *
Condition 1: Upper Bound

$$\begin{tikzcd} i \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & j \arrow[r, dashed, "h"] & k \end{tikzcd}$$

Figure 2. *
Condition 2: Coequalizing map

In other words, $\mathcal{I}$ is nonempty, any finite diagram of objects admits a cocone (Definition .1.12), and any pair of parallel morphisms become equal after post-composition with an appropriate morphism.

2. Dually, a *Cofiltered category* is a category whose opposite category is filtered. More explicitly, A cofiltered category is a (nonempty, large) category $\mathcal{I}$ satisfying the following conditions:
   - For every finite collection of objects $i_1, i_2, \ldots, i_n$ in $\mathcal{I}$, there exists an object j and morphisms
   $$\phi_k : j \to i_k, \quad \text{for each } k = 1, \ldots, n.$$
   - For every pair of morphisms $f, g : j \to i$ in $\mathcal{I}$, there exists an object k and a morphism
   $$h : k \to j$$
   that satisfies
   $$f \circ h = g \circ h.$$

$$\begin{tikzcd} i_1 & \\ & j \arrow[ul, dashed, "\phi_1"'] \arrow[dl, dashed, "\phi_2"] \\ i_2 & \end{tikzcd}$$

Figure 3. *
Condition 1: Lower Bound

$$\begin{tikzcd} k \arrow[r, dashed, "h"] & j \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & i \end{tikzcd}$$

Figure 4. *
Condition 2: Equalizing map

In other words, $\mathcal{I}$ is nonempty, any finite diagram of objects admits a cone, and any pair of parallel morphisms become equal after pre-composition with an appropriate morphism.

Definition .1.12 (Cones, limits and colimits in a category)**.** Let $\mathcal{C}$ be a (large) category (Definition .1.1), let I be a (large) category, and let $D : I \to \mathcal{C}$ be a diagram (Definition .1.25) (Definition .1.25).

1. A *cone to the diagram D* is an object $L \in \mathcal{C}$ together with a family of morphisms

$$\{\pi_i : L \to D(i)\}_{i \in I}$$

such that for every morphism $f : i \to j$ in I, the diagram

$$\begin{tikzcd} & L \arrow[dl, "\pi_i"'] \arrow[dr, "\pi_j"] & \\ D(i) \arrow[rr, "D(f)"] & & D(j) \end{tikzcd}$$

commutes, i.e. $D(f) \circ \pi_i = \pi_j$.

2. A cone $(L, \{\pi_i\})$ is called a *limit of D* if it satisfies the following "universal property": for any cone $(C, \{f_i\})$ over D, there exists a *unique* morphism $u : C \to L$ such that

$$\pi_i \circ u = f_i \quad \text{for all } i \in I.$$

Visually, the following diagrams commute every morphism $f : i \to j$ in I:

$$\begin{tikzcd} & C \arrow[d, dashed, "\exists ! u"] \arrow[ddl, bend right, "f_i"'] \arrow[ddr, bend left, "f_j"] & \\ & L \arrow[dl, "\pi_i"'] \arrow[dr, "\pi_j"] & \\ D(i) \arrow[rr, "D(f)"] & & D(j). \end{tikzcd}$$

If such a cone exists, then the object L is necessarily unique up to unique isomorphism by the universal property. In this case, L is denoted by $\lim_{i \in I} D$ or $\lim D$.

3. A *cocone from the diagram D* is an object $C \in \mathcal{C}$ together with a family of morphisms

$$\{\iota_i : D(i) \to C\}_{i \in I}$$

such that for every morphism $f : i \to j$ in I, the diagram

$$\begin{tikzcd} D(i) \arrow[rr, "D(f)"] \arrow[dr, "\iota_i"'] & & D(j) \arrow[dl, "\iota_j"] \\ & C & \end{tikzcd}$$

commutes, i.e. $\iota_j \circ D(f) = \iota_i$.

4. A cocone $(L, \{\iota_i\})$ is called a *colimit of D* if it satisfies the following "universal property": for any cocone $(C, \{g_i\})$ under D, there exists a *unique* morphism $u : L \to C$ such that

$$u \circ \iota_i = g_i \quad \text{for all } i \in I.$$

Visually, the following diagrams commute every morphism $f : i \to j$ in I:

$$\begin{tikzcd}
D(i) \arrow[rr, "D(f)"] \arrow[dr, "\iota_i"] \arrow[ddr, bend right, "g_i"'] & & D(j) \arrow[dl, "\iota_j"'] \arrow[ddl, bend left, "g_j"] \\
& L \arrow[d, dashed, "\exists! u"] & \\
& C &
\end{tikzcd}.$$

If such a cocone exists, then the object L is necessarily unique up to unique isomorphism by the universal property. In this case, L is denoted by $\operatorname{colim}_{i \in I} D$ or $\operatorname{colim} D$.

A limit/colimit is called *finite* (resp. *small*) if the index category I is finite (resp. small).

Some authors use the terms *projective limit* or *inverse limit* to refer to what is defined here as a limit, Similarly, the terms *inductive limit* or *direct limit* are sometimes used to mean a colimit. However, these phrases can have more specific meanings to other authors: a *projective* or *inverse limit* may refer to a limit over a diagram indexed by a codirected poset (Definition .1.20). Likewise, an *inductive* or *direct limit* may refer to a colimit over a directed poset (Definition .1.20) (see Definition .1.13) .

Thus, while the terms are sometimes used interchangeably with "limit" and "colimit," they may also emphasize particular indexing shapes and directions, distinguishing them from general limits and colimits taken over arbitrary small categories.

Definition .1.13 (Special cases of limits)**.** Let $\mathcal{C}$ be a (large) category. Let I be a (large) category. Let $I \to \mathcal{C}$ be a diagram/system.

- Suppose that the system is a cofiltered system (Definition .1.17), i.e. I is a cofiltered category. A limit (Definition .1.12) of this diagram is often denoted by

$$\varprojlim_{i \in I} D(i)$$

  and may be called a *cofiltered (inverse/projective) limit*. In case that the system is more specifically an inverse/projective system (Definition .1.17), i.e. I is a cofiltered poset (Definition .1.20), the preferred term for such a limit is *inverse/projective limit*.
- Suppose that the system is a filtered system, i.e. I is a filtered category. A colimit of this diagram is often denoted by

$$\varinjlim_{i \in I} D(i)$$

  and may be called a *filtered colimit* or a *direct/inductive/injective limit*. In case that the system is more specifically a direct/inductive system, i.e. I is a filtered poset (Definition .1.20), the preferred term for such a limit is *direct/inductive limit*.

Definition .1.14 (Ind-category)**.** Let $\mathcal{C}$ be a locally small category (Definition .1.4).

1. The *Ind-category of $\mathcal{C}$*, denoted $\operatorname{Ind}(\mathcal{C})$, is defined as follows:

- Objects of $\mathrm{Ind}(\mathcal{C})$ are formal filtered colimits (Definition .1.13) of objects in $\mathcal{C}$. More precisely, an object is given by a filtered (Definition .1.11) small category I and a functor
$$X : I \to \mathcal{C}.$$
- Morphisms between objects $X : I \to \mathcal{C}$ and $Y : J \to \mathcal{C}$ are defined by
$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(X, Y) \ := \varprojlim_{i \in I} \varinjlim_{j \in J} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j),$$
(Definition .1.13) where X_i and Y_j denote the images of $i \in I$ and $j \in J$ under X and Y, respectively.

The composition of morphisms is induced naturally from composition in $\mathcal{C}$. Hence, $\mathrm{Ind}(\mathcal{C})$ is the completion of $\mathcal{C}$ under filtered colimits. Objects of $\mathrm{Ind}(\mathcal{C})$ are called *Ind-objects of $\mathcal{C}$*.

2. The *Pro-category of $\mathcal{C}$*, denoted $\mathrm{Pro}(\mathcal{C})$, is defined as follows:
   - Objects of $\mathrm{Pro}(\mathcal{C})$ are formal cofiltered limits (Definition .1.13) of objects in $\mathcal{C}$. More precisely, an object is given by a cofiltered small category I and a functor
   $$X : I \to \mathcal{C}.$$
   - Morphisms between objects $X : I \to \mathcal{C}$ and $Y : J \to \mathcal{C}$ are defined by
   $$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X, Y) := \varinjlim_{j \in J} \varprojlim_{i \in I} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j),$$
   where X_i and Y_j denote the images of $i \in I$ and $j \in J$ under X and Y, respectively.

   The composition of morphisms is induced naturally from composition in $\mathcal{C}$.

   Hence, $\mathrm{Pro}(\mathcal{C})$ is the completion of $\mathcal{C}$ under cofiltered limits. Objects of $\mathrm{Pro}(\mathcal{C})$ are called *Pro-objects of $\mathcal{C}$*.

Since **Sets** has all limits and colimits (♠ TODO:) and hence has all projective and inductive limits and since $\mathcal{C}$ is locally small, $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C})$ are locally small.

Definition .1.15 (Product in a category)**.** Let $\mathcal{C}$ be a category and let $\{X_i\}_{i \in I}$ be a family of objects in $\mathcal{C}$ indexed by a class I.

1. A *product of the family $\{X_i\}$* is an object P of $\mathcal{C}$ together with a "universal" family of morphisms
$$\pi_i : P \to X_i, \quad \text{for each } i \in I.$$
More precisely, for any object Y and any family of morphisms $\{f_i : Y \to X_i\}_{i \in I}$, there exists a unique morphism
$$f : Y \to P$$
making the following diagram commute for all $i \in I$, i.e. $\pi_i \circ f = f_i$:
$$\begin{tikzcd} Y \arrow[d, dashed, "\exists! f"'] \arrow[dr, "f_i"] & \\ \prod X_i \arrow[r, "\pi_i"'] & X_i \end{tikzcd}$$

Such a product is often denoted by $\prod_{i \in I} X_i$. If $\prod_{i \in I} X_i$ exists in $\mathcal{C}$, then it is unique up to unique isomorphism by the universal property described above.

Equivalently, the product $\prod_{i \in I} X_i$ is the limit (Definition .1.12) of the diagram (Definition .1.25) $I \to \mathcal{C}, i \mapsto X_i$, where I is made into a category whose objects are the members of I and whose morphisms are just the identity morphisms.

2. A *coproduct* (or synonymously *direct sum*) of the family $\{X_i\}$ is an object C of $\mathcal{C}$ together with a "universal" family of morphisms

$$\iota_i : X_i \to C, \quad \text{for each } i \in I.$$

More precisely, for any object Y and any family of morphisms $\{g_i : X_i \to Y\}_{i \in I}$, there exists a unique morphism

$$g : C \to Y$$

making the following diagram commute for all $i \in I$, i.e. $g \circ \iota_i = g_i$:

$$\begin{tikzcd} X_i \arrow[r, "\iota_i"] \arrow[dr, "g_i"'] & \coprod X_i \arrow[d, dashed, "\exists! g"] \\ & Y \end{tikzcd}$$

Such a coproduct is often denoted by $\coprod_{i \in I} X_i$ or $\oplus_{i \in I} X_i$. If $\coprod_{i \in I} X_i$ exists in $\mathcal{C}$, then it is unique up to unique isomorphism by the universal property described above.

Equivalently, the coproduct $\coprod_{i \in I} X_i$ is the colimit (Definition .1.12) of the diagram (Definition .1.25) $I \to \mathcal{C}, i \mapsto X_i$, where I is made into a category whose objects are the members of I and whose morphisms are just the identity morphisms.

Definition .1.16 (Equalizer in a category)**.** Let $\mathcal{C}$ be a (large) category (Definition .1.1) and let $f, g : X \to Y$ be morphisms in $\mathcal{C}$.

1. An *equalizer of f and g* is an object E together with a morphism

$$e : E \to X$$

such that

$$f \circ e = g \circ e$$

and for any object Z with morphism $z : Z \to X$ satisfying

$$f \circ z = g \circ z,$$

there exists a unique morphism $u : Z \to E$ making the diagram commute:

$$e \circ u = z.$$

$$\begin{tikzcd} Z \arrow[d, dashed, "\exists! u"] \arrow[dr, "z"] & & \\ E \arrow[r, "e"'] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

If such an equalizer of f and g exists, then we say that the following *equalizer diagram is exact*:

$$E \xrightarrow{e} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

2. A *ccequalizer of f and g* is an object Q together with a morphism

$$q : Y \to Q$$

such that

$$q \circ f = q \circ g$$

and for any object Z with morphism $w : Y \to Z$ satisfying

$$w \circ f = w \circ g,$$

there exists a unique morphism $v : Q \to Z$ making the diagram commute:

$$v \circ q = w.$$

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] \arrow[dr, "w"'] & Q \arrow[d, dashed, "\exists! v"] \\ & & Z \end{tikzcd}$$

If such a coequalizer of f and g exists, then we say that the following *ccequalizer diagram is exact*:

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{q} Q$$

Definition .1.17 (Systems in a category)**.** Let $\mathcal{C}$ be a (large) category. Let I be a (large) category.

1. A diagram/system (Definition .1.25) $I \to C$ is called *filtered* (resp. *cofiltered*) if I is a filtered (Definition .1.11) (resp. cofiltered (Definition .1.11)) category.
2. A diagram/system $I \to C$ is called *directed* (resp. *codirected* if I is small and thing, i.e. is regardable/comes from (Lemma .1.21) a directed (resp. codirected) partially ordered set (Definition .1.20). A *direct system* or *inductive system* is synonymous for a directed system and a *inverse system* or *projective system* is synonymous for a codirected system.

One might also speak of a *filtered direct/inductive system* synonymously for a filtered system to emphasize that the indexing caetgory is a general filtered category, rather than a directed poset.

Definition .1.18. A category $\mathcal{C}$ is called *essentially small* if it is equivalent (Definition .1.6) to a small category (Definition .1.4), i.e., there exists a small category $\mathcal{D}$ and an equivalence of categories

$$F : \mathcal{D} \to \mathcal{C}.$$

Note that an essentially small category is necessarily locally small (Definition .1.4).

Definition .1.19 (Monomorphism and Epimorphism in Categories)**.** Let $\mathcal{C}$ be a category (Definition .1.1). For objects $A, B \in \mathcal{C}$, let $f : A \to B$ be a morphism in $\mathcal{C}$.

- The morphism f is called a *monomorphism* (or a *monic morphism*) if for every object X and every pair of morphisms $g_1, g_2 : X \to A$, the equality $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.
- The morphism f is called an *epimorphism* (or an *epic morphism*) if for every object Y and every pair of morphisms $h_1, h_2 : B \to Y$, the equality $h_1 \circ f = h_2 \circ f$ implies $h_1 = h_2$.

Definition .1.20 (Partially ordered set)**.** 1. A *partially ordered set* (or *poset*), or *ordered set* is a pair $(P, \leq)$ where P is a set and

$$\leq : P \times P \to \{\text{true}, \text{false}\}$$

is a binary relation on P satisfying the following axioms for all $a, b, c \in P$:
- *Reflexivity:* $a \leq a$,
- *Antisymmetry:* if $a \leq b$ and $b \leq a$, then $a = b$,
- *Transitivity:* if $a \leq b$ and $b \leq c$, then $a \leq c$.

The relation $\leq$ is called an *order* or a *partial order*

2. A partially ordered set $(P, \leq)$ is called a *directed partially ordered set* if for every pair $a, b \in P$, there exists $c \in P$ such that

$$a \leq c \quad \text{and} \quad b \leq c.$$

3. A partially ordered set $(P, \leq)$ is called a *codirected partially ordered set* (or *downward directed poset*) if for every pair $a, b \in P$, there exists $d \in P$ such that

$$d \leq a \quad \text{and} \quad d \leq b.$$

Lemma .1.21. Let $(P, \leq)$ be a nonempty poset (Definition .1.20).

1. Regarding P as a category whose objects are the elements of P and such that there is a unique arrow $a \to b$ if and only if $a \leq b$, the category is filtered.
2. Every nonempty small (Definition .1.4) thin filtered category (Definition .1.11) corresponds to a poset in this way.
3. Moreover, the poset P is directed (Definition .1.20) if and only if the category is filtered. The poset P is codirected (Definition .1.20) if and only if the category is cofiltered.

Definition .1.22 (Generator of a category)**.** Let $\mathcal{C}$ be a category (Definition .1.1).

1. An object $G \in \mathcal{C}$ is called a *generator* (or *separator*) if for every pair of distinct morphisms $f, g : X \to Y$ in $\mathcal{C}$, there exists a morphism $h : G \to X$ such that

$$f \circ h \neq g \circ h.$$

In case that $\mathcal{C}$ is locally small (Definition .1.4), this is equivalent to the condition that the representable functor (Definition 3.4.3)

$$\mathrm{Hom}_{\mathcal{C}}(G, -) : \mathcal{C} \to \mathbf{Set}$$

is faithful, which in turn is equivalent to the condition that for every object $X \in \mathcal{C}$, there exists an epimorphism

$$\bigoplus_{i \in I} G \twoheadrightarrow X$$

for some indexing set I, where $\bigoplus$ denotes the coproduct (Definition .1.15) in $\mathcal{C}$.

2. A family $\{G_i\}_{i \in I}$ is called a *generating family* if for every pair of distinct morphisms $f, g : X \to Y$ in $\mathcal{C}$, there exists some index $i \in I$ and a morphism $h : G_i \to X$ such that

$$f \circ h \neq g \circ h.$$

In case $\mathcal{C}$ is locally small, this is equivalent to the condition that the collection of representable functors

$$\{\mathrm{Hom}_{\mathcal{C}}(G_i, -) : \mathcal{C} \to \mathbf{Set}\}_{i \in I}$$

is jointly faithful, which in turn is equivalent to the condition that for every object $X \in \mathcal{C}$, there exists a family of objects $\{G_i\}_{i \in J}$ from the generating set indexed by some set J, and an epimorphism

$$\bigoplus_{i \in J} G_i \twoheadrightarrow X.$$

Definition .1.23 (Topos)**.** There are multiple notions of a topos depending on the context (geometric vs. logical).

1. A *Grothendieck topos* (or *sheaf topos*) is a category (Definition .1.1) equivalent (Definition .1.6) to the category of sheaves (Definition 3.0.2) of sets on a *small* site (Definition A.0.4). That is, there exists a small site $(\mathcal{C}, J)$ such that the category is equivalent to $\mathrm{Sh}(\mathcal{C}, J)$.
2. Let $\mathscr{U}$ be a universe (Definition .1.2). A *$\mathscr{U}$-topos* is a category equivalent to the category of sheaves of sets on a $\mathscr{U}$-small site $(\mathcal{C}, J)$, where the sheaves take values in the category of $\mathscr{U}$-sets ($\mathbf{Set}_{\mathscr{U}}$). [GV72, Exposé IV Définition 1.1]
3. An *elementary topos* is a category which has all finite limits (Definition .1.12), is cartesian closed, and has a subobject classifier.

Remark: Every Grothendieck topos is an elementary topos, but the converse is not true (e.g., the category of finite sets is an elementary topos but not a Grothendieck topos).

Theorem .1.24 (Functor Category of a Grothendieck Topos is a Topos)**.** Let I be a small category (Definition .1.4), and let $\mathcal{S}$ be a Grothendieck topos (Definition .1.23). Then the functor category $\mathrm{Fun}(I, \mathcal{S})$ (Definition .1.25) is a Grothendieck topos.

Definition .1.25 (Diagram in a category and category of diagrams)**.** Let $\mathcal{C}$ be a (large) category (Definition .1.1), and let I be a (large) category (Definition .1.1).

1. A *diagram of shape I in $\mathcal{C}$* is a functor $D : I \to \mathcal{C}$. We often denote such a diagram by the family $\{D(i)\}_{i \in \mathrm{Ob}(I)}$ with transition maps given by the functorial image of morphisms in I.

   A diagram is also synonymously called a *system*. Moreover, the category I is called the *index category* or the *indexing category of the diagram D*.
2. Given two diagrams $D, E : I \to \mathcal{C}$, a *morphism of diagrams* is a simply a natural transformation $D \Rightarrow E$ of the functors D and E.
3. The *category of I-shaped diagrams in $\mathcal{C}$* or simply *diagram category (of I-shaped diagrams in $\mathcal{C}$)*, often denoted $\mathcal{C}^I$, $[I, \mathcal{C}]$, or $\mathrm{Fun}(I, \mathcal{C})$, is the (large) category whose

objects are functors $I \to \mathcal{C}$ (that is, diagrams of shape I in $\mathcal{C}$) and whose morphisms are natural transformations between such functors. The category $\mathcal{C}^I$ is also called the *functor category of functors $I \to \mathcal{C}$*. Equivalently, the functor category $\mathcal{C}^I$ is the category $\mathrm{PreShv}(I^{\mathrm{op}}, \mathcal{C})$ of presheaves (Definition 3.0.1) on I^{op} with values in $\mathcal{C}$ and hence notations for presheaf categories are applicable as notations for functor categories.

If $\mathcal{C}$ is locally small (Definition .1.4) and I is small, then $\mathcal{C}^I$ is locally small by Lemma .1.7.

.1.1. *Abelian categories.*

Definition .1.26 (Abelian category)**.** Let $\mathcal{A}$ be a category. The category $\mathcal{A}$ is an *abelian category* if:

- $\mathcal{A}$ is an additive category.
- Every morphism $f : A \to B$ has a kernel $\ker(f)$ and a cokernel $\mathrm{coker}(f)$.
- For every morphism $f : A \to B$, the canonical morphism $\mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism, where

$$\mathrm{coim}(f) = \mathrm{coker}(\ker(f) \to A), \quad \mathrm{im}(f) = \ker(B \to \mathrm{coker}(f)).$$

(♠ TODO: I think I need to re-check this defintion) (♠ TODO: coimage)

.

It is also worth considering Grothendieck's additional axioms for abelian categories.

Proposition .1.27. The following are examples of abelian categories (Definition .1.26):

1. The category of R-S bimodules where R,S are (not necessarily commutative) rings (Definition C.0.8) (**??**).
2. The category $\mathbf{Ab}$ of abelian groups and group homomorphisms is abelian.
3. The category Vect_k of vector spaces over a field k and k-linear maps is abelian.
4. More generally, if R is a noetherian ring, then the category of finitely generated R-modules is abelian.
5. For a ringed space $(X, \mathcal{O}_X)$, the category of $\mathcal{O}_X$-modules (Definition 6.1.3) is abelian. (♠ TODO: a quasi-coherent sheaf on a locally ringed space)
6. If X is a scheme (Definition .3.1) (or more generally a locally ringed space), the category of quasi-coherent sheaves on X is abelian.
7. For any essentially small category (Definition .1.18) $\mathcal{C}$ and any abelian category $\mathcal{A}$, the functor category $[\mathcal{C}, \mathcal{A}]$ (Definition .1.25) and the category $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ of presheaves (Definition 3.0.1) are abelian. (♠ TODO: apparently, the essentially smallness condition is removable, provided that the sheafification functor exists. However, the essentially small assumption is needed to show that the category of sheaves of O-modules is a Grothendieck abelian caetgory. Verify all this. Moreover, when working with a big site of a scheme, one typically fixes a unvierse or work relative to a cardinal cutoff to treat it as essentially small)
8. For any site (Definition A.0.4) $(\mathcal{C}, J)$ on an essentially small category (Definition .1.18) $\mathcal{C}$ and any abelian category $\mathcal{A}$, the category $\mathrm{Shv}(\mathcal{C}, \mathcal{A})$ of sheaves (Definition 3.0.2) is abelian.

9. For any site (Definition A.0.4) $(\mathcal{C}, J)$ on an essentially small category (Definition .1.18) $\mathcal{C}$ and a sheaf of rings (Definition 3.0.2) $\mathcal{O}$ on $\mathcal{C}$, the category $\mathbf{Mod}(\mathcal{O})$ of $\mathcal{O}$-modules (Definition 6.1.3) is an abelian category.

Definition .1.28. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories (Definition .1.26).

1. F is called *left exact* if it preserves all finite limits (Definition .1.12), or equivalently it preserves kernels and any finite limit diagrams. Equivalently, for every left exact sequence in $\mathcal{A}$
$$0 \to A' \xrightarrow{f} A \xrightarrow{g} A''$$
the sequence
$$0 \to F(A') \xrightarrow{F(f)} F(A) \xrightarrow{F(g)} F(A'')$$
is exact at $F(A')$ and $F(A)$ (i.e., F preserves monomorphisms (Definition .1.19) and exactness at the first two terms).
2. Dually, F is called *right exact* if it preserves all finite colimits (Definition .1.12), or equivalently it preserves cokernels and any finite colimit diagrams. Equivalently, for every right exact sequence in $\mathcal{A}$
$$A' \xrightarrow{f} A \xrightarrow{g} A'' \to 0,$$
the sequence
$$F(A') \xrightarrow{F(f)} F(A) \xrightarrow{F(g)} F(A'') \to 0$$
is exact at $F(A)$ and $F(A'')$ (i.e., F preserves epimorphisms (Definition .1.19) and exactness at the last two terms).
3. F is called *exact* if it is both left and right exact.

Definition .1.29. Let $\mathcal{A}$ be an abelian category (Definition .1.26).

1. $\mathcal{A}$ is said to *have enough injectives* if for every object A in $\mathcal{A}$, there is an monomorphism (Definition .1.19) $A \to I$ with I an injective object of $\mathcal{A}$.
2. $\mathcal{A}$ is said to *have enough projectives* if for every object A in $\mathcal{A}$, there is a epimorphism (Definition .1.19) $P \to A$ with P a projective object of $\mathcal{A}$.

Lemma .1.30 (cf. [Wei94, Lemma 2.2.5, Lemma 2.3.6])**.** Let $\mathcal{A}$ be an abelian category (Definition .1.26) and let $\mathcal{X}$ be a class of objects in $\mathcal{A}$.

1. If $\mathcal{A}$ has enough objects of class $\mathcal{X}$ on the right, then for every object $A \in \mathcal{A}$ there exists an $\mathcal{X}$-right resolution of A (Definition .2.7).
2. If $\mathcal{A}$ has enough objects of class $\mathcal{X}$ on the left, then for every object $A \in \mathcal{A}$ there exists an $\mathcal{X}$-left resolution of A (Definition .2.7).

Note that this is a special case of **??** obtained by letting the complex $M^\bullet$ be the complex such that
$$M^i = \begin{cases} A & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}.$$

In particular,

- If $\mathcal{A}$ has enough injective objects (Definition .1.29), then for every object $A \in \mathcal{A}$ there exists an injective resolution of A (Definition .2.7).
- If $\mathcal{A}$ has enough projective objects (Definition .1.29), then for every object $A \in \mathcal{A}$ there exists a projective resolution of A (Definition .2.7).
- If $F : \mathcal{A} \to \mathcal{B}$ is a left (resp. right) exact functor (Definition .1.28) between abelian categories and $\mathcal{A}$ has enough F-acyclic objects on the right (resp. left), then for every object $A \in \mathcal{A}$, there exists an right (resp. left) F-acyclic resolution of A.

Proof.

1. Let $A \in \mathcal{A}$ be an object. Since $\mathcal{A}$ has enough objects of class $\mathcal{X}$ of the right, there is an object X_0 of $\mathcal{X}$ and a monomorphism $\varepsilon_0 : A \to X_0$. Let $A_0 = \operatorname{coker} \varepsilon_0$. Inductively, given an object A_{n-1} of $\mathcal{A}$, choose an object X_n of $\mathcal{X}$ and a monomorphism $\varepsilon_n : A_{n-1} \hookrightarrow X_n$. Let $A_n = \operatorname{coker} \varepsilon_n$. In particular, there is a surjection $X_n \twoheadrightarrow A_n$. Let d_n be the composition
$$X_{n-1} \twoheadrightarrow A_{n-1} \xrightarrow{\varepsilon_n} X_n.$$
The chain complex
$$0 \to A \xrightarrow{\varepsilon_0} X_0 \xrightarrow{d_0} X_1 \xrightarrow{d_1} \cdots$$
is thus an $\mathcal{X}$-right resolution of A.
2. This is simply the dual statement of the next statement.

□

Definition .1.31. (♠ TODO: I think that the definition of derived categories might be doable for more general kinds of resolutions? Perhaps it is that if I have a right exact functor F, then $L^i F$ can be computed with resolutions of F-acyclic objects?) (♠ TODO: Apparently, left/right derived functors may be defined for functors that are additive and preserve finite coproducts, and not necessarily right/left exact; the exactness condition ensures that the zeroth derived functor agrees with F.) Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories (Definition .1.26), and let
$$F : \mathcal{A} \to \mathcal{B}$$
be an additive functor.

1. Suppose that the functor F is right exact (Definition .1.28) and suppose that $A \in \mathcal{A}$ is an object for which a projective resolution (Definition .2.7)
$$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0$$
exists in $\mathcal{A}$. We define the *left derived object* $L_n F A \in \mathcal{B}$ by applying F to obtain a complex
$$\cdots \to F(P_2) \to F(P_1) \to F(P_0) \to 0$$
and letting $L_n F(A)$ be the n-th homology object (Definition .2.3) of this complex in $\mathcal{B}$:
$$L_n F(A) := H_n(F(P_\bullet)).$$
The object $L_n F(A)$ is independent of the choice of projective resolution up to natural isomorphism (Proposition .1.32).

   By convention, set $L_n F = 0$ for $n < 0$.

   The *higher left derived objects* refer to the object $L_n F(A)$ for $n > 0$.

2. Suppose that the functor F is right exact (Definition .1.28) and that $\mathcal{A}$ has enough projectives (Definition .1.29). The *left derived functors* refer to the family of functors

$$L_nF : \mathcal{A} \to \mathcal{B}, \quad A \mapsto L_nF(A).$$

The *higher left derived functors* refer to the functors L_nF for $n > 0$.
3. Suppose that the functor F is right exact (Definition .1.28) and suppose that $A \in \mathcal{A}$ is an object for which a injective resolution (Definition .2.7)

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$$

exists in $\mathcal{A}$. We define the *right derived object* $R_nFA \in \mathcal{B}$, also often denoted by R^nFA, by applying F to obtain a complex

$$0 \to F(I^0) \to F(I^1) \to F(I^2) \to \cdots .$$

and letting $R_nF(A)$ be the n-th cohomology object (Definition .2.3) of this complex in $\mathcal{B}$:

$$R_nF(A) := H^n(F(I_\bullet)).$$

The object $R_nF(A)$ is independent of the choice of injective resolution up to natural isomorphism (Proposition .1.32).
By convention, set $R_nF = 0$ for $n < 0$.
The *higher right derived objects* refer to the object $R_nF(A)$ for $n > 0$.
4. Suppose that the functor F is right exact (Definition .1.28) and that $\mathcal{A}$ has enough injectives (Definition .1.29). The *right derived functors* refer to the family of functors

$$R_nF : \mathcal{A} \to \mathcal{B}, \quad A \mapsto R_nF(A).$$

The right derived functors are also often denoted by R^nF. The *higher right derived functors* refer to the functors R_nF for $n > 0$.

Proposition .1.32 (cf.[Wei94, Lemma 2.4.1])**.** Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories (Definition .1.26). Let A be an object of $\mathcal{A}$.

1. Suppose that F is right exact (Definition .1.28), and suppose that a projective resolution (Definition .2.7)

$$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0$$

of A exists in $\mathcal{A}$. Let

$$\cdots \to Q_2 \to Q_1 \to Q_0 \to A \to 0$$

be any projective resolution of A in $\mathcal{A}$. For all n, there are natural isomorphisms

$$H_n(F(P_\bullet)) \cong H_n(F(Q_\bullet)).$$

In other words, the left derived objects $L_nF(A)$ (Definition .1.31) is well defined.
2. Suppose that F is left exact (Definition .1.28), and suppose that a injective resolution (Definition .2.7)

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$$

of A exists in $\mathcal{A}$. Let

$$0 \to A \to Q^0 \to Q^1 \to Q^2 \to \cdots$$

be any injective resolution of A in $\mathcal{A}$. For all n, there are natural isomorphisms

$$H_n(F(I^\bullet)) \cong H_n(F(Q^\bullet)).$$

In other words, the right derived objects $R_nF(A)$ (Definition .1.31) is well defined.

Proof. 1. By Lemma .1.33, there is a lift $f : P_\bullet \to Q_\bullet$ of the identity map $A \to A$ unique up to chain homotopy. There are then induced natural maps $H_n(F(f)) : H_n(F(P_\bullet)) \to H_n(F(Q_\bullet))$. There is also a lift $f' : Q_\bullet \to P_\bullet$ of the identity map $A \to A$ unique up to chain homotopy, and this also induces natural maps $H_n(F(f')) : H_n(F(Q_\bullet)) \to H_n(F(P_\bullet))$. The chain maps f and f' are in fact chain homotopy inverses because Lemma .1.33 also implies that any lifts $P_\bullet \to P_\bullet$ and $Q_\bullet \to Q_\bullet$ of the identity map $A \to A$ are chain homotopic to the identity chain maps. Therefore, $H_n(F(f))$ and $H_n(F(f'))$ are inverses of each other as morphisms in $\mathcal{B}$. (♠ TODO: prove basic facts about the fucntoriality of homology/cohomology of chain complexes)

2. This is dual to the previous part.

□

Lemma .1.33 (cf. [Wei94, Porism 2.2.7]). Let $\mathcal{A}$ be an abelian category (Definition .1.26).

1. Let
$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$
be a chain complex (Definition .2.1) with P_i projective. For every left resolution (Definition .2.7) $Q_\bullet \to N$ of an object N, every map $M \to N$ lifts to a complex map (Definition .2.1) $P_\bullet \to Q_\bullet$ unique up to chain homotopy.
2. Let
$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$$
be a (co)chain complex (Definition .2.1) with I^i injective. For every right resolution (Definition .2.7) $N \to Q^\bullet$ of an object N, every map $N \to M$ lifts to a complex map (Definition .2.1) $Q^\bullet \to I^\bullet$ unique up to chain homotopy.

Proof. 1. The map $P_0 \to M \to N$ lifts to a map $P_0 \to Q_0$ because P_0 is projective and $Q_0 \to N$ is an epimorphism. Inductively suppose that there are morphisms $P_i \to Q_i$ for $0 \leq i \leq n$, where $n \geq 0$ that make

$$\begin{tikzcd}
P_n \arrow[r] \arrow[d] & P_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
Q_n \arrow[r] & Q_{n-1} \arrow[r] & \cdots \arrow[r] & Q_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

into a commuting diagram are established. The morphism $Q_n \to Q_{n-1}$ (where we let $Q_{-1} = N$ and $P_{-1} = M$ here in case that $n = 0$) acts as 0 when restricted to $\mathfrak{I} := \mathrm{im}(P_{n+1} \to P_n \to Q_n)$ because the composition

$$P_{n+1} \to P_n \to Q_n \to Q_{n-1}$$

equals the composition

$$P_{n+1} \to P_n \to P_{n-1} \to Q_{n-1}.$$

In other words, $\mathfrak{I}$ is a subobject of $\ker(Q_n \to Q_{n-1})$, which is isomorphic to $\operatorname{im}(Q_{n+1} \to Q_n)$ by the acyclicity of the sequence of the Q_i's. Therefore, we have a map $P_{n+1} \twoheadrightarrow \mathfrak{I} \hookrightarrow \operatorname{im}(Q_{n+1} \to Q_n)$ along with an epimorphism $Q_{n+1} \twoheadrightarrow \operatorname{im}(Q_{n+1} \to Q_n)$. Since P_{n+1} is projective, the former map lifts to a map $P_{n+1} \to Q_{n+1}$ in a way that is compatible with the latter, i.e. the following commutes:

$$\begin{tikzcd} P_{n+1} \arrow[d, dotted] \arrow[dr] & \\ Q_{n+1} \arrow[r] & \operatorname{im}(Q_{n+1} \to Q_n). \end{tikzcd}$$

By induction, this shows thta $M \to N$ lifts to a morphism $P_\bullet \to Q_\bullet$ of complexes.

We show that the morphism of complexes is unique up to chain homotopy, i.e. if $f_1, f_2 : P_\bullet \to Q_\bullet$ are two morphisms of complexes, then $h \coloneqq f_1 - f_2$ is null homotopic. We construct a chain contraction $\{s_n : P_n \to Q_{n+1}\}$ of h by induction on n. If $n < 0$, then set $s_n = 0$. If $n = 0$, note that the composition $P_0 \xrightarrow{h_0} Q_0 \to N$ equals the composition $P_0 \to M \xrightarrow{0} N$, so $\operatorname{im}(h_0)$ is a subobject of $\ker(Q_0 \to N) \cong \operatorname{im}(Q_1 \to Q_0)$. The projectivitiy of P_0 thus yields a lift $s_0 : P_0 \to Q_1$ such that h_0 equals the composition $P_0 \xrightarrow{s_0} X_1 \xrightarrow{d} Q_0$:

$$\begin{tikzcd} & P_0 \arrow[dl, dotted, "s_0"'] \arrow[d, "h_0"] \\ Q_1 \arrow[r, "d"] & Q_0 \end{tikzcd}$$

Note moreover that $h_0 = ds_0 + s_{-1}d$ because $s_{-1} = 0$. Inductively suppose that we have maps s_i for $i \le n$ such that $h_n = ds_n + s_{n-1}d$ or equivalently that $ds_n = h_n - s_{n-1}d$. Consider the map $h_{n+1} - s_n d : P_{n+1} \to Q_{n+1}$. Compute

$$d(h_{n+1} - s_n d) = dh_{n+1} - ds_n d = dh_{n+1} - (h_n - s_{n-1}d)d = (dh_{n+1} - h_n d) + s_{n-1}dd = 0$$

Therefore, $\operatorname{im}(h_{n+1} - s_n d)$ is a subobject of $\ker(Q_{n+1} \to Q_n) \cong \operatorname{im}(Q_{n+2} \to Q_{n+1})$, which is in turn a quotient of Q_{n+2}. Since P_{n+1} is projective, there is a morphism $s_{n+1} : P_{n+1} \to Q_{n+2}$ such that $ds_{n+1} = h_{n+1} - s_n d$.

$$\begin{tikzcd} & P_{n+1} \arrow[dl, dotted, "s_{n+1}"'] \arrow[d, "h_n - s_{n-1}d = ds_n"] \\ Q_{n+2} \arrow[r, "d"] & \operatorname{im}(Q_{n+2} \to Q_{n+1}) \cong \ker(Q_{n+1} \to Q_n) \end{tikzcd}$$

The s_n thus form a chain contraction as needed.

2. This is simply dual to the previous part.

□

.2. **Homological algebra.**

Definition .2.1 (Chain complex in a preadditive category)**.** Let $\mathcal{A}$ be a preadditive category and let I be a totally ordered set (typically $\mathbb{Z}$, but $I \subseteq \mathbb{Z}$ is also allowed).

1. A *chain complex* $(K_\bullet, d_\bullet)$ in $\mathcal{A}$ indexed by I is the homological convention for sequences with decreasing degrees. It consists of:
   - Objects $\{K_i\}_{i \in I}$ in $\mathcal{A}$, called the *terms in degree i*,
   - Morphisms $d_i : K_i \to K_{i-1}$ in $\mathcal{A}$, called the *boundary maps* or *differentials in degree i*,

   such that for every $i \in I$, $d_{i-1} \circ d_i = 0$. That is,
   $$K_\bullet : \cdots \xrightarrow{d_{i+1}} K_i \xrightarrow{d_i} K_{i-1} \xrightarrow{d_{i-1}} K_{i-2} \to \cdots$$
   with $d_{i-1} d_i = 0$ for all i. We typically use the notation $K_\bullet = (K_i, d_i)_{i \in I}$.
2. Dually, a *cochain complex* $(K^\bullet, d^\bullet)$ in $\mathcal{A}$ follows the *cohomological convention* with increasing degrees. It consists of objects $\{K^i\}_{i \in I}$ and *coboundary maps* $d^i : K^i \to K^{i+1}$ such that $d^{i+1} \circ d^i = 0$:
   $$K^\bullet : \cdots \xrightarrow{d^{i-1}} K^i \xrightarrow{d^i} K^{i+1} \xrightarrow{d^{i+1}} K^{i+2} \to \cdots$$
   We typically use the notation $K^\bullet = (K^i, d^i)_{i \in I}$.
3. Let $K_\bullet = (K_i, d_i^K)$ and $L_\bullet = (L_i, d_i^L)$ be chain complexes (Definition .2.1) in $\mathcal{A}$ indexed by the same set I. A *morphism of chain complexes* (or *chain map*)
   $$f_\bullet : K_\bullet \to L_\bullet$$
   consists of morphisms $f_i : K_i \to L_i$ for all $i \in I$, such that for every $i \in I$, the following diagram commutes:
   $$\begin{array}{ccc} K_i & \xrightarrow{d_i^K} & K_{i-1} \\ \downarrow f_i & & \downarrow f_{i-1} \\ L_i & \xrightarrow{d_i^L} & L_{i-1} \end{array}$$
   i.e., $d_i^L \circ f_i = f_{i-1} \circ d_i^K$.

   A *morphism of cochain complexes* $f^\bullet : K^\bullet \to L^\bullet$ is defined similarly, satisfying the commutativity condition $d_L^i \circ f^i = f^{i+1} \circ d_K^i$.

The collection of these objects and morphisms forms a category. Notation for these categories is as follows:

- $\mathrm{Ch}(\mathcal{A})$ or $\mathbf{Ch}(\mathcal{A})$ is often used as a general term.
- To be explicit about the indexing convention, one uses $\mathrm{Ch}_\bullet(\mathcal{A})$ for chain complexes and $\mathrm{Ch}^\bullet(\mathcal{A})$ (or sometimes $\mathrm{CoCh}(\mathcal{A})$) for cochain complexes.
- The set of chain maps between two complexes is denoted by $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(K_\bullet, L_\bullet)$; it is an abelian group under pointwise addition $(f+g)_i = f_i + g_i$.

Definition .2.2 (Boundedness conditions on chain complexes)**.** Let $\mathcal{A}$ be an additive category.

Cohomological convention: Let $K^\bullet = (K^i, d^i)_{i \in \mathbb{Z}}$ be a cohomologically indexed chain complex in $\mathcal{A}$.

- $K^\bullet$ is *bounded above* if there exists $n \in \mathbb{Z}$ such that $K^i = 0$ for all $i > n$.
- $K^\bullet$ is *bounded below* if there exists $m \in \mathbb{Z}$ such that $K^i = 0$ for all $i < m$.
- $K^\bullet$ is *bounded* if it both bounded above and below, or equivalently if $K^i = 0$ for all but finitely many $i \in \mathbb{Z}$.
- Assuming that $\mathcal{A}$ is an abelian category,
  - $K^\bullet$ is *cohomologically bounded above* if there exists $n \in \mathbb{Z}$ such that $H^i(K^\bullet) = 0$ for all $i > n$.
  - $K^\bullet$ is *cohomologically bounded below* if there exists $m \in \mathbb{Z}$ such that $H^i(K^\bullet) = 0$ for all $i < m$.
  - $K^\bullet$ is *cohomologically bounded* if it is both cohomologically bounded above and below, or equivalently if the cohomology objects $H^i(K^\bullet)$ vanish for all but finitely many $i \in \mathbb{Z}$.

Homological convention: Let $K_\bullet = (K_i, d_i)_{i\in\mathbb{Z}}$ be a homologically indexed chain complex in $\mathcal{A}$.

- $K_\bullet$ is *bounded above* if there exists $n \in \mathbb{Z}$ such that $K_i = 0$ for all $i > n$.
- $K_\bullet$ is *bounded below* if there exists $m \in \mathbb{Z}$ such that $K_i = 0$ for all $i < m$.
- $K_\bullet$ is *bounded* if it both bounded above and below, or equivalently if $K_i = 0$ for all but finitely many $i \in \mathbb{Z}$.
- Assuming that $\mathcal{A}$ is an abelian category,
  - $K_\bullet$ is *homologically bounded above* if there exists $n \in \mathbb{Z}$ such that $H_i(K_\bullet) = 0$ for all $i > n$.
  - $K_\bullet$ is *homologically bounded below* if there exists $m \in \mathbb{Z}$ such that $H_i(K_\bullet) = 0$ for all $i < m$.
  - $K_\bullet$ is *homologically bounded* if it is both homologically bounded above and below, or equivalently if the homology objects $H_i(K_\bullet)$ vanish for all but finitely many $i \in \mathbb{Z}$.

Definition .2.3 (Chain complexes and their (co)homology objects)**.** Let $\mathcal{A}$ be an abelian category.

- For a cochain complex $K^\bullet$, its *cohomology object in degree i* is defined as the quotient of the object of i-cocycles by the object of i-coboundaries:

$$H^i(K^\bullet) := Z^i(K)/B^i(K) = \ker(d^i)/\operatorname{im}(d^{i-1}).$$

- For a chain complex $K_\bullet$, its *homology object in degree i* is defined as the quotient of the object of i-cycles by the object of i-boundaries:

$$H_i(K_\bullet) := Z_i(K)/B_i(K). = \ker(d_i)/\operatorname{im}(d_{i+1}).$$

Definition .2.4 (Quasi-isomorphism)**.** Let $\mathcal{A}$ be an abelian category (Definition .1.26), and let

$$f_\bullet : (C_\bullet, d^C_\bullet) \to (D_\bullet, d^D_\bullet)$$

be a chain map between complexes (Definition .2.1) in $\mathcal{A}$.

The morphism $f_\bullet$ is called a *quasi-isomorphism* if it induces isomorphisms on all cohomology objects, i.e., for every integer n, the induced morphism on homology (Definition .2.3) (or cohomology, depending on the convention)

$$H^n(f_\bullet) : H^n(C_\bullet) \to H^n(D_\bullet)$$

is an isomorphism in $\mathcal{A}$.

Note that all of these notions are applicable to the cohomological convention as well.

Definition .2.5. 1. Let $f : (C_\bullet, d^C_\bullet) \to (D_\bullet, d^D_\bullet)$ be a morphism of chain complexes (Definition .2.1) in an additive category $\mathcal{A}$.

The *mapping cone of f*, denoted $\mathrm{Cone}(f)$, is the chain complex defined by:

- Objects: For each n,

$$\mathrm{Cone}(f)_n = D_n \oplus C_{n-1}.$$

- Differential: For each n, define

$$d_n^{\mathrm{Cone}(f)} : \mathrm{Cone}(f)_n \to \mathrm{Cone}(f)_{n-1}$$

by the matrix morphism

$$d_n^{\mathrm{Cone}(f)} = \begin{pmatrix} d_n^D & f_{n-1} \\ 0 & -d_{n-1}^C \end{pmatrix} : D_n \oplus C_{n-1} \to D_{n-1} \oplus C_{n-2}.$$

This construction defines a chain complex, i.e., $d_{n-1}^{\mathrm{Cone}(f)} \circ d_n^{\mathrm{Cone}(f)} = 0$.

2. Dually, let $g : (C^\bullet, d_C^\bullet) \to (D^\bullet, d_D^\bullet)$ be a morphism of cochain complexes (Definition .2.1) in $\mathcal{A}$.

The *mapping cone of g*, denoted $\mathrm{Cone}(g)$, is the cochain complex (Definition .2.1) defined by:

- Objects: For each n,

$$\mathrm{Cone}(g)^n = D^n \oplus C^{n+1}.$$

- Differential: For each n, define

$$d^n_{\mathrm{Cone}(g)} : \mathrm{Cone}(g)^n \to \mathrm{Cone}(g)^{n+1}$$

by the matrix morphism

$$d^n_{\mathrm{Cone}(g)} = \begin{pmatrix} d_D^n & g^{n+1} \\ 0 & -d_C^{n+1} \end{pmatrix} : D^n \oplus C^{n+1} \to D^{n+1} \oplus C^{n+2}.$$

This construction defines a cochain complex, i.e., $d^{n+1}_{\mathrm{Cone}(g)} \circ d^n_{\mathrm{Cone}(g)} = 0$.

Definition .2.6. Let $\mathcal{A}$ be an abelian category (Definition .1.26).

1. The *canonical truncations* of a chain complex (Definition .2.1) $A_\bullet$ in $\mathbf{Ch}(\mathcal{A})$ (Definition .2.1) are defined by

$$(\tau_{\geq n} A)_i = \begin{cases} A_i, & i > n, \\ \ker(d_n : A_n \to A_{n-1}), & i = n, \\ 0, & i < n, \end{cases} \quad (\tau_{\leq n} A)_i = \begin{cases} 0, & i > n, \\ \mathrm{coker}(d_{n+1} : A_{n+1} \to A_n), & i = n, \\ A_i, & i < n. \end{cases}$$

The differentials are the restrictions and/or quotient maps induced from $A_\bullet$. In particular,

$$H_i(\tau_{\geq n}A_\bullet) = \begin{cases} H_i(A_\bullet), & i \geq n, \\ 0, & i < n, \end{cases} \quad \text{and} \quad H_i(\tau_{\leq n}A_\bullet) = \begin{cases} H_i(A_\bullet), & i \leq n, \\ 0, & i > n. \end{cases}$$

The assignments $A_\bullet \mapsto \tau_{\geq n}A_\bullet$ and $A_\bullet \mapsto \tau_{\leq n}A_\bullet$ extend to endofunctors

$$\tau_{\geq n},\ \tau_{\leq n} : \mathbf{Ch}(\mathcal{A}) \to \mathbf{Ch}(\mathcal{A}),$$

called the *truncation functors*. They are natural in both $A_\bullet$ and n, and fit into canonical morphisms of complexes

$$\tau_{\geq n}A_\bullet \longrightarrow A_\bullet \longrightarrow \tau_{\leq n}A_\bullet.$$

2. Similarly, let $A^\bullet$ be a cochain complex in $\mathbf{Ch}(\mathcal{A})$, i.e.

$$\cdots \xrightarrow{d^{n-2}} A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \xrightarrow{d^{n+1}} \cdots$$

with $d^{n+1} \circ d^n = 0$ for all $n \in \mathbb{Z}$. The *canonical truncations of $A^\bullet$* are defined by

$$(\tau_{\leq n}A)^i = \begin{cases} A^i, & i < n, \\ \ker(d^n : A^n \to A^{n+1}), & i = n, \\ 0, & i > n, \end{cases} \quad (\tau_{\geq n}A)^i = \begin{cases} 0, & i < n, \\ \operatorname{coker}(d^{n-1} : A^{n-1} \to A^n), & i = n, \\ A^i, & i > n. \end{cases}$$

The differentials are the restrictions or quotient maps induced by those of $A^\bullet$. These truncations satisfy

$$H^i(\tau_{\leq n}A^\bullet) = \begin{cases} H^i(A^\bullet), & i \leq n, \\ 0, & i > n, \end{cases} \quad \text{and} \quad H^i(\tau_{\geq n}A^\bullet) = \begin{cases} 0, & i < n, \\ H^i(A^\bullet), & i \geq n. \end{cases}$$

They also extend to endofunctors

$$\tau_{\leq n},\ \tau_{\geq n} : \mathbf{Ch}(\mathcal{A}) \to \mathbf{Ch}(\mathcal{A}),$$

natural in both $A^\bullet$ and n, fitting into canonical morphisms of cochain complexes

$$\tau_{\leq n}A^\bullet \longrightarrow A^\bullet \longrightarrow \tau_{\geq n}A^\bullet.$$

Definition .2.7. Let $\mathcal{A}$ be an abelian category (Definition .1.26) and let $\mathcal{X}$ be a class of objects in $\mathcal{A}$. Let M be an object of $\mathcal{A}$.

1. A *right resolution of M* is a cochain complex (Definition .2.1) $I^\bullet$ with $I^i = 0$ for $i < 0$ and a map $M \to I^0$ such that the augmented complex

$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$$

is exact.

2. A *left resolution of M* is a chain complex (Definition .2.1) $P_\bullet$ with $P_i = 0$ for $i < 0$ and a map $P_0 \to M$ such that the augmented complex

$$\cdots P_2 \to P_1 \to P_0 \to M \to 0$$

is exact.

3. An *$\mathcal{X}$-left resolution* of an object $M \in \mathcal{A}$ a left resolution (Definition .2.7) by objects of X, i.e. an exact complex
$$\cdots \to X_2 \to X_1 \to X_0 \to M \to 0$$
with each $X_i \in \mathcal{X}$.
4. An *$\mathcal{X}$-right resolution* of an object $M \in \mathcal{A}$ a right resolution (Definition .2.7) by objects of X, i.e. an exact complex
$$0 \to M \to X^0 \to X^1 \to X^2 \to \cdots$$
with each $X_i \in \mathcal{X}$.
5. A *projective resolution of M* is a left resolution $P^\bullet$ for which the objects P^i are all projective.
6. An *injective resolution of M* is a right resolution $I^\bullet$ for which the objects I^i are all injective.

.3. Schemes.

Definition .3.1 (Scheme)**.** A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ that admits an open cover $\{U_i\}_{i\in I}$ such that each $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic (as a locally ringed space) to an affine scheme $(\mathrm{Spec}(A_i), \mathcal{O}_{\mathrm{Spec}(A_i)})$ for some commutative ring A_i. In other words, a scheme is a locally ringed space locally isomorphic to affine schemes.

Definition .3.2 (Morphism of schemes)**.** Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be schemes (Definition .3.1). A *morphism of schemes* is a morphism as locally ringed spaces.

In particular, there is a category (Definition .1.1), often denoted by Sch, **Sch** etc., whose objects are schemes and whose morphisms are morphisms of schemes.

Definition .3.3 (Scheme over a scheme)**.** Let $(S, \mathcal{O}_S)$ be a scheme. A *scheme over S* (or an *S-scheme*) is a scheme $(X, \mathcal{O}_X)$ together with a morphism of schemes
$$\pi : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S).$$
This morphism π is called the *structure morphism of the scheme X over S*.

If $S = \mathrm{Spec}(R)$ is an affine scheme for a commutative ring R, then an S-scheme is synonymously called an *R-scheme* or a *scheme over R*.

Let $(S, \mathcal{O}_S)$ be a scheme, and let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be schemes over S with structure morphisms
$$\pi_X : X \to S, \quad \pi_Y : Y \to S.$$
A *morphism of S-schemes* (or synonymously a *S-scheme morphism*) is a morphism of schemes (Definition .3.2)
$$(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$
such that the following diagram commutes:
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \pi_X \downarrow & & \downarrow \pi_Y \\ S & = & S \end{array}$$

In other words,

$$\pi_Y \circ f = \pi_X.$$

Given a fixed scheme S, there is a category, often denoted by Sch_S, $\mathrm{Sch}_{/S}$, Sch/S, $\mathbf{Sch}_S$, $\mathbf{Sch}_{/S}$, $\mathbf{Sch}/S$ etc. whose objects are schemes T over S and whose morphisms $T_1 \to T_2$ are morphisms of schemes over S. If $S = \operatorname{Spec} R$ for some commutative ring R, then we may instead write Sch_R to denote $\mathrm{Sch}_{\operatorname{Spec} R}$, etc. It is noteworthy that $\mathrm{Sch}_{\mathbb{Z}}$ coincides with the category Sch (Definition .3.2) of all schemes. In other words, a $\mathbb{Z}$-scheme can be identified simply with a scheme.

Equivalently, the category $\mathrm{Sch}_{/S}$ is the category of schemes over S in the sense of Definition A.0.5.

Definition .3.4. Let $f : X \to Y$ be a morphism of schemes (Definition .3.2). We say that f is an *affine morphism* if for every affine open $V = \operatorname{Spec} B \subseteq Y$, the preimage $U = f^{-1}(V)$ is an affine scheme.

Definition .3.5. Let $f : X \to Y$ be a morphism of schemes (Definition .3.2). We say that f is a *finite type morphism* if for every affine open $V = \operatorname{Spec} B \subseteq Y$ with $U = f^{-1}(V)$ affine, say $U = \operatorname{Spec} A$, the ring A is a finitely generated B-algebra.

When X is equipped with a finite type morphism $f : X \to Y$, we say that X is a *finite type scheme over Y* or a *finite type Y-scheme* or a *Y-scheme of finite type* (Definition .3.3), etc.

Definition .3.6 (Finitely presented algebra over a ring)**.** Let R be a (not necessarily commutative) ring (Definition C.0.8). An R-algebra A is said to be *finitely presented* if there exists an integer $n \geq 0$ and a surjective R-algebra homomorphism

$$\varphi : R\langle x_1, \ldots, x_n\rangle \twoheadrightarrow A$$

where $R\langle x_1, \ldots, x_n\rangle$ is the free R-algebra on n generators, such that the kernel $\ker(\varphi)$ is a finitely generated two-sided ideal (Definition C.0.2) of $R\langle x_1, \ldots, x_n\rangle$.

In other words, A admits a presentation as

$$A \cong R\langle x_1, \ldots, x_n\rangle / I,$$

where I is a finitely generated two-sided ideal.

If R and A are commutative rings, this recovers the usual definition of a finitely presented commutative R-algebra by replacing $R\langle x_1, \ldots, x_n\rangle$ with the polynomial ring $R[x_1, \ldots, x_n]$ and I a finitely generated ideal.

Definition .3.7. (♠ TODO: TODO: define a (commutative) group scheme) (♠ TODO: TODO: define the multiplicative group scheme) Let S be a scheme. The *additive group scheme over S* is the group scheme $\mathbb{G}_a = \mathbb{G}_{a,S}$ over S defined by

$$\mathbb{G}_{a,S} = \operatorname{Spec} \mathcal{O}_S[T]$$

with group structure morphisms:

- **Comultiplication (Addition):** The morphism $\Delta : \mathbb{G}_{a,S} \to \mathbb{G}_{a,S} \times_S \mathbb{G}_{a,S}$ corresponding to the ring homomorphism $\mathcal{O}_S[T] \to \mathcal{O}_S[T] \otimes_{\mathcal{O}_S} \mathcal{O}_S[T]$ given by $T \mapsto T \otimes 1 + 1 \otimes T$.

- **Counit (Identity):** The morphism $\varepsilon : \mathbb{G}_{a,S} \to S$ corresponding to $T \mapsto 0$.
- **Coinverse (Inversion):** The morphism $\iota : \mathbb{G}_{a,S} \to \mathbb{G}_{a,S}$ corresponding to $T \mapsto -T$.

Thus, $(\mathbb{G}_{a,S}, \Delta, \varepsilon, \iota)$ is a commutative group scheme over S. Note that we may speak of the *additive group scheme over a ring R* as the additive group scheme over $\operatorname{Spec} R$.

Definition .3.8 (Dimension of a Scheme)**.** Let X be a scheme with underlying topological space $|X|$.

(♠ TODO: krull dimension)

- The *dimension at a point $x \in |X|$*, denoted $\dim_x(X)$, is the Krull dimension of the local ring $\mathcal{O}_{X,x}$ (Definition C.0.23). This is the supremum of the lengths n of chains of prime ideals
$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subseteq \mathcal{O}_{X,x}.$$
- The *dimension of the scheme X* is defined as
$$\dim(X) := \sup_{x \in |X|} \dim_x(X).$$
Equivalently, it is the supremum of the lengths of chains of distinct irreducible closed subsets of $|X|$ ordered by inclusion.

Appendix A. Presheaves and sheaves

Definition A.0.1 ([GV72, Exposé I Définition 4.1])**.** Let C be a (large) category (Definition .1.1).

1. A *sieve S on the category C* is a full subcategory (Definition 7.4.2) D of C such that for any object U of C there exists an object V of (♠ TODO: correctly parse the definiton)
2. A *sieve S on an object $U \in \mathrm{Ob}(C)$* is a collection of morphisms in C with codomain U that is closed under precomposition by any compatible morphism in C. In other words, S is a sieve if for every $f : V \to U$ in S and morphism $g : W \to V$ in C, the composition $f \circ g : W \to U$ is also in S.
   Given a morphism $f : V \to U$ in a sieve S, we also say that *f factors through U*.

Definition A.0.2. Let C be a category, let $U \in \mathrm{Ob}(C)$, and let S be a sieve on U (Definition A.0.1). For a morphism $f : V \to U$ in C, the *pullback sieve* *f^*S* (or *basechange sieve* *$S \times_U V$*) on V is defined by
$$f^*S = \{g : W \to V \mid f \circ g \in S\}.$$
In other words, f^*S consists of all morphisms into V whose composite with f belongs to the sieve S on U.

Definition A.0.3. Let $\mathcal{C}$ be a category (Definition .1.1) and $U \in \mathcal{C}$ an object. Let $\mathcal{S} = \{f_i : U_i \to U\}_{i \in I}$ be a family of morphisms with codomain U.

The *sieve generated by $\mathcal{S}$*, denoted $(\mathcal{S})$ or $\langle \mathcal{S} \rangle$, is the smallest sieve on U (Definition A.0.1) containing all the morphisms in $\mathcal{S}$.

Explicitly, a morphism $h : V \to U$ belongs to the generated sieve if and only if h factors through some morphism in $\mathcal{S}$. That is, there exists an index $i \in I$ and a morphism $g : V \to U_i$ such that

$$h = f_i \circ g.$$

Definition A.0.4 (Grothendieck topology)**.** Let $\mathscr{U}$ be a universe (Definition .1.2).

1. (See [GV72, Exposé II, Définition 1.1]) Let $\mathcal{C}$ be a category (Definition .1.1). A *Grothendieck topology on $\mathcal{C}$* assigns to each object U of $\mathcal{C}$ a collection $J(U)$ of sieves (Definition A.0.1) $\{U_i \to U\}_{i \in I}$, each called a *covering sieve of U*, satisfying:
   (a) (Stability under "base change"): If $S \in J(U)$ is a covering sieve of an object U, and $f : V \to U$ is any morphism in $\mathcal{C}$, then the pullback sieve (Definition A.0.2) f^*S is a covering sieve of U.
   (b) (Local character condition) If S is a sieve on U, and if there exists a covering sieve $R \in J(U)$ such that for all $f : V \to U$ in R the pullback sieve (Definition A.0.2) f^*S is in $J(V)$, then $S \in J(U)$.
   (c) The maximal sieve is a covering sieve.

   Some will refer to a Grothendieck topology as simply a *topology*, not to be confused with the related, but less general, notion of a topology on a set (Definition C.0.6).
2. (See [GV72, Exposé II, 1.1.5]) A *site* is a category $\mathcal{C}$ equipped with a Grothendieck topology.

   When we are working with a Grothendieck pretopology (Definition 3.2.2) K on a category $\mathcal{C}$, we may regard $\mathcal{C}$ as a site by equipping it with the Grothendieck topology generated by (Definition 3.2.1) K.
3. (See [GV72, Exposé II, Définition 1.2]) Let $(\mathcal{C}, J)$ be a site. A family of morphisms $(U_i \to U)_{i \in I}$ is called a *covering family of U (with respect to the site/topology)* or a *cover of U (with respect to the site/topology)* if the sieve generated by (Definition A.0.3) the family is a covering sieve of U.
4. (See [GV72, Exposé II, Définition 3.0.1]) Let $(\mathcal{C}, J)$ be a site (Definition A.0.4), where J is a Grothendieck topology on $\mathcal{C}$.

   A family G of objects $\mathcal{C}$ is called a *topologically generating family of the site/topology* or a *generating family/collection of the site/topology* if for every object $X \in \mathcal{C}$, there is a covering family $\{X_\alpha \to X\}_{\alpha \in A}$ of X such that every X_α is a member of G.

   Equivalently, the Grothendieck topology J is the smallest Grothendieck topology containing all covers of the U_i. Also equivalently, for any $S \in J(X)$, the sieve S contains a covering family $\{V_i \to X\}$ such that each morphism $V_i \to X$ factors through some member of G. (♠ TODO: Verify that these claimed equivalences are indeed equivalences)
5. (See [GV72, Exposé II, Définition 3.0.2]) A *$\mathscr{U}$-site* is a site whose underlying category $\mathcal{C}$ is $\mathscr{U}$-locally small (Definition .1.4) and which has a $\mathscr{U}$-small topologically generating family. A $\mathscr{U}$-site is called *$\mathscr{U}$-small* if its underlying category is $\mathscr{U}$-small. Similarly, a *small site* is a site whose underlying category is a set and a *locally small site* is a site whose underlying category is locally small (Definition .1.4).

Definition A.0.5 (Category of objects over a fixed object)**.** Let $\mathcal{C}$ be a category (Definition .1.1) and let $X \in \mathrm{Ob}(\mathcal{C})$ be a fixed object.

1. The *category of objects over X* (or synonymously the *slice category of X in $\mathcal{C}$* or the *over category of X in $\mathcal{C}$*), commonly denoted $\mathcal{C}/X$, $\mathcal{C}_{/X}$, or $(\mathcal{C} \downarrow X)$ is the category defined as follows:
   - An object of $\mathcal{C}/X$ is a morphism $f\colon A \to X$ in $\mathcal{C}$, where $A \in \mathrm{Ob}(\mathcal{C})$.
   - A morphism from $f\colon A \to X$ to $g\colon B \to X$ in $\mathcal{C}/X$ is a morphism $h\colon A \to B$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[dr, "f"'] & B \arrow[d, "g"] \\ & X \end{tikzcd}$$

     i.e. such that $g \circ h = f$.
   - The identity morphisms and composition in $\mathcal{C}/X$ are inherited from $\mathcal{C}$.
2. The *category of objects under X* (or synonymously the *coslice category of X in $\mathcal{C}$* or the *under category of X in $\mathcal{C}$*), commonly denoted $X/\mathcal{C}$, $X\backslash\mathcal{C}$, $\mathcal{C}_{X/}$, or $(X \downarrow \mathcal{C})$, is the category defined as follows:
   - An object of $X/\mathcal{C}$ is a morphism $f\colon X \to A$ in $\mathcal{C}$, where $A \in \mathrm{Ob}(\mathcal{C})$.
   - A morphism from $f\colon X \to A$ to $g\colon X \to B$ in $X/\mathcal{C}$ is a morphism $h\colon A \to B$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"] & A \arrow[d, "h"] \\ & B \end{tikzcd}$$

     i.e. such that $h \circ f = g$.
   - The identity morphisms and composition in $X/\mathcal{C}$ are inherited from $\mathcal{C}$.

Lemma A.0.6. Let $\mathcal{C}$ be a category (Definition .1.1) and let $X \in \mathrm{Ob}(\mathcal{C})$ be a fixed object. The slice category $\mathcal{C}/X$ (Definition A.0.5) has X as its final object (Definition .1.10).

Proof. This is clear. $\square$

Definition A.0.7 (Slice site)**.** Let $(\mathcal{C}, \tau)$ be a site (Definition A.0.4), where τ is a Grothendieck topology on the (locally small or U-locally small (Definition .1.4), if a universe (Definition .1.2) U is available) category $\mathcal{C}$. For a fixed object X in $\mathcal{C}$, the *slice site* (or the *over site*, the *site on the slice category $\mathcal{C}_{/X}$*, the *site induced on the over category $\mathcal{C}_{/X}$*, the *localization of the site $\mathcal{C}$ at the object X*, etc.) $(\mathcal{C}_{/X}, \tau_{/X})$ is the site whose underlying category is the slice category $\mathcal{C}_{/X}$ (Definition A.0.5), and whose Grothendieck topology $\tau_{/X}$ (also denoted by notations such as $\tau|_X$ or τ/X) is defined by declaring a family of morphisms $\{f_i : Y_i \to Y\}$ in $\mathcal{C}_{/X}$ to be a covering if and only if the family $\{f_i : Y_i \to Y\}$ is a covering in $(\mathcal{C}, \tau)$.

Appendix B. Derived categories

B.1. **Homotopy category of chain complexes of an additive category.** We will define homotopy categories and derived categories using the cohomological convention

Definition B.1.1 (Homotopy category $K(\mathcal{A})$)**.** Let $\mathcal{A}$ be an additive category. (♠ TODO: describe the distinguished triangles of this category)

1. The *homotopy category* $K(\mathcal{A})$ has as objects the (cochain) complexes over $\mathcal{A}$, and as morphisms the cochain maps (Definition .2.1) modulo cochain homotopy equivalence. More explicitly,
$$\mathrm{Hom}_{K(\mathcal{A})}(C^\bullet, D^\bullet) := \frac{\{\text{cochain maps } C^\bullet \to D^\bullet\}}{\{\text{cochain homotopies}\}},$$
where morphisms differing by a cochain homotopy are identified.
2. Let
$$F : \mathcal{A} \to \mathcal{B}$$
be an additive functor of additive categories.
   The functor F naturally induces a functor between the homotopy categories of complexes,
$$K(F) : K(\mathcal{A}) \to K(\mathcal{B}),$$
by applying F degreewise to complexes and cochain maps. That is, for a complex $(C^\bullet, d^\bullet)$ in $\mathcal{A}$, define
$$K(F)(C^\bullet) := (F(C^\bullet), F(d^\bullet)),$$
and for a cochain map $f^\bullet : C^\bullet \to D^\bullet$, define
$$K(F)(f^\bullet) := (F(f^\bullet)).$$
   The functor $K(F)$ respects cochain homotopies, thus it is well-defined on the homotopy category.
3. (See [DBG+77, Catégories dérivées Chapitre 1 2-3] for notation) We write $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ for the full subcategories of bounded below, bounded abaove, and bounded (Definition .2.2) complexes respectively.
4. (See [DBG+77, Catégories dérivées Chapitre 1 2-3] for notation) If $\mathcal{A}$ is abelian, we may also write $K^{\infty,+}(\mathcal{A})$, $K^{\infty,-}(\mathcal{A})$, $K^{\infty,b}(\mathcal{A})$, and $K^{\infty,\emptyset}(\mathcal{A})$ for the full subcategories of cohomologically bounded below, cohomologically bounded abaove, cohomologically bounded (Definition .2.2), and acyclic complexes respectively. We may further write $K^{?,??}(\mathcal{A})$ for the full subcategory where $? \in \{+, -, b\}$ indicates the boundedness of the complex and $?? \in \{+, -, b, (\text{blank})\}$ indicates the cohomological boundedness of the complex.

The various categories here are (Proposition B.1.6) triangulated categories.

Definition B.1.2 (Triangulated category)**.** A *triangulated category* is a triple $(\mathcal{T}, [1], \Delta)$ where:

- $\mathcal{T}$ is an additive category,
- $[1] : \mathcal{T} \to \mathcal{T}$ (also denoted by notations such as Σ) is an additive auto-equivalence called the *shift* or *suspension functor*,
- Δ is a class of distinguished triangles of the form
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$
with objects $X, Y, Z \in \mathcal{T}$ and morphisms f, g, h in $\mathcal{T}$,

satisfying the following axioms:

(TR1) The class Δ is closed under isomorphisms of triangles and contains the triangles isomorphic to

$$A \xrightarrow{\mathrm{id}} A \to 0 \to A[1].$$

Moreover, for every morphism $f : X \to Y$ in $\mathcal{T}$, there exists a distinguished triangle

$$X \xrightarrow{f} Y \to Z \to X[1].$$

(TR2) (Rotation) If

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is a distinguished triangle, then so are

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

and

$$Z[-1] \xrightarrow{-h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z.$$

(TR3) (Octahedral axiom) Given morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ in $\mathcal{T}$, there exist distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{u} C(f) \xrightarrow{v} X[1],$$
$$Y \xrightarrow{g} Z \xrightarrow{w} C(g) \xrightarrow{x} Y[1],$$
$$X \xrightarrow{g\circ f} Z \xrightarrow{y} C(g \circ f) \xrightarrow{z} X[1],$$

and morphisms

$$s : C(f) \to C(g \circ f), \quad t : C(g \circ f) \to C(g)$$

such that the following diagram commutes and its rows and columns are distinguished triangles:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "u"] \arrow[d, "g"] & C(f) \arrow[r, "v"] \arrow[d, dashed, "s"] & X[1] \arrow[d, equal] \\
X \arrow[r, "g\circ f"] & Z \arrow[r, "y"] \arrow[d, "w"] & C(g \circ f) \arrow[r, "z"] \arrow[d, dashed, "t"] & X[1] \\
 & C(g) \arrow[r, equal] \arrow[d, "x"] & C(g) \arrow[d] & \\
 & Y[1] \arrow[r, "f[1]"] & X[1][1] &
\end{tikzcd}$$

In particular,

– The compositions $s \circ u = y$, $t \circ s = w$, and $x \circ t = v[1]$ hold,

– All rows and columns form distinguished triangles.

(TR4) The class Δ is closed under the shift functor [1]: if

$$X \to Y \to Z \to X[1]$$

is distinguished, then so is

$$X[1] \to Y[1] \to Z[1] \to X[2].$$

Definition B.1.3. A functor $T : D_1 \to D_2$ between triangulated categories is called *exact* if it is additive, is translation preserving, and transforms distinguished triangles to distinguished triangles. Oftentimes, a *morphism between triangulated categories* refers to an exact functor between triangulated categories.

Definition B.1.4. In the context of a triangulated category (Definition B.1.2) $\mathscr{D}$, a multiplicative system (Definition B.2.1) S is *compatible with the triangulation* if:

1. For any $s \in S$, $T(s) \in S$ where T is the translation functor.
2. For any commutative square between the first two steps of two distinguished triangles where the vertical arrows are in S, there exists a morphism in S between the third objects that makes the whole diagram of triangles commute.

Definition B.1.5. Let I be a finite set and $\varepsilon : I \to \{1, -1\}$ a variance function. Let $(\mathscr{A}_i)_{i \in I}$ and $\mathscr{A}$ be triangulated categories (Definition B.1.2). A multifunctor $F : \prod_{i \in I} \mathscr{A}_i \to \mathscr{A}$ is an *exact multifunctor of variance ε* if for every index $k \in I$, the partial functor

$$F_k : \mathscr{A}_k \to \mathscr{A}$$

(obtained by fixing all objects A_j for $j \neq k$) satisfies the following:

1. If $\varepsilon(k) = 1$, F_k is a covariant exact functor: it maps any distinguished triangle $X \to Y \to Z \to T_k X$ in $\mathscr{A}_k$ to a distinguished triangle
$$F_k(X) \to F_k(Y) \to F_k(Z) \to TF_k(X)$$
in $\mathscr{A}$.
2. If $\varepsilon(k) = -1$, F_k is a contravariant exact functor: it maps any distinguished triangle $X \to Y \to Z \to T_k X$ in $\mathscr{A}_k$ to a distinguished triangle
$$F_k(Z) \to F_k(Y) \to F_k(X) \to F_k(T_k^{-1} X)$$
in $\mathscr{A}$, using the isomorphism $F_k(T_k^{-1} X) \cong TF_k(X)$.

Furthermore, F must be a graded functor equipped with natural isomorphisms $\varphi_i : T \circ F \xrightarrow{\sim} F \circ T_i^{\varepsilon_i}$ for each $i \in I$. These must satisfy the anticommutativity condition: for any pair of distinct indices $i, j \in I$ ($i \neq j$), the following diagram of natural transformations commutes with a factor of -1:

(C)
$$\begin{array}{ccc} T^2 F & \xrightarrow{T\varphi_i} & TFT_i^{\varepsilon_i} \\ \downarrow{\scriptstyle T\varphi_j} & & \downarrow{\scriptstyle \varphi_j T_i^{\varepsilon_i}} \\ TFT_j^{\varepsilon_j} & \xrightarrow{-\varphi_i T_j^{\varepsilon_j}} & FT_i^{\varepsilon_i} T_j^{\varepsilon_j} \end{array}$$

Equivalently, the following square is anticommutative:

(D)
$$(\varphi_j T_i^{\varepsilon_i}) \circ (T\varphi_i) = -(\varphi_i T_j^{\varepsilon_j}) \circ (T\varphi_j)$$

as morphisms of functors $T^2F \to FT_i^{\varepsilon_i}T_j^{\varepsilon_j}$.

In particular in the case that $\epsilon(i) = 1$ for all i, we are reduced to the following definition: Let $(\mathscr{A}_i)_{1\leq i\leq n}$ and $\mathscr{A}$ be triangulated categories (Definition B.1.2). A multifunctor $F : \prod \mathscr{A}_i \to \mathscr{A}$ is called an *exact covariant multifunctor* if it is covariant in each argument and if for every index $i \in \{1, \ldots, n\}$ and every fixed collection of objects $(A_j)_{j\neq i}$, the partial functor $F(\ldots, A_{i-1}, -, A_{i+1}, \ldots) : \mathscr{A}_i \to \mathscr{A}$ is an exact functor (Definition B.1.3) of triangulated categories.

Proposition B.1.6. Let $\mathcal{A}$ be an additive category.

1. The homotopy category $K(\mathcal{A})$ (Definition B.1.1) is a triangulated category (Definition B.1.2).(♠ TODO: shift of chain complexes, describe what the distinguished triangles are)
2. The subcategory $K^?(\mathcal{A})$ is a triangulated subcategory of $K(\mathcal{A})$ for $? \in \{+, -, b\}$.
3. Let $\mathcal{A}$ be an abelian category. The subcategory $K^{?,??}(\mathcal{A})$ is a triangulated subcategory of $K(\mathcal{A})$ for $? \in \{\infty, +, -, b\}$ and $?? \in \{+, -, b, (\text{blank})\}$.

(♠ TODO: How homotopy categories of chain complexes relate to topological homotopy categories)

Theorem B.1.7 (Exact functor induces a functor on homotopy categories, cf. [DA73, Exposé XVII, Proposition 1.1.7.])**.** Let $F : \prod_{i=1}^n \mathscr{A}_i \to \mathscr{B}$ be an additive multifunctor (Definition C.0.13) of abelian categories (Definition .1.26). F induces exact multifunctors (Definition B.1.5)[6] between the corresponding homotopy categories (Definition B.1.1):

$$K(\mathscr{A}) \to K(\mathscr{B})$$
$$K^?(\prod \mathscr{A}_i) \to K^?(\mathscr{B})$$
$$K^{?,??}(\prod \mathscr{A}_i) \to K^{?,??}(\mathscr{B})$$

(Definition B.1.1). We write these functors by $K(F)$.

B.2. Multiplicative systems in categories.

Definition B.2.1 (Multiplicative system)**.** (♠ TODO: Write a precise definition) Let $\mathcal{C}$ be a category. A *multiplicative system S in $\mathcal{C}$* consists of a collection of morphisms in $\mathcal{C}$ satisfying:

- S contains all identity morphisms,
- S is closed under composition,
- (Ore conditions) For any morphism f in $\mathcal{C}$ and any morphism $s \in S$ with suitable domain and codomain, there exist morphisms to form commutative squares allowing localization.

The precise Ore condition varies depending on context but guarantees the localization exists.

[6]wlog covariant in each variable, but this can be made multi-variant by taking opposite categories

If F is contravariant in the j-th variable, the definitions are adjusted by replacing $\mathscr{A}_j$ with its opposite category $\mathscr{A}_j^{\mathrm{op}}$, thereby interchanging K^+ with K^- and RF with LF for that specific variable.

Definition B.2.2 (Localization by a multiplicative system)**.** Given a category $\mathcal{C}$ and a multiplicative system (Definition B.2.1) $S \subseteq \mathrm{Mor}(\mathcal{C})$, the *localization* $\mathcal{C}[S^{-1}]$ is a category equipped with a functor

$$Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$$

that sends every morphism in S to an isomorphism satisfying the universal property: any functor from $\mathcal{C}$ sending the morphisms in S to isomorphisms factors uniquely through Q.

Definition B.2.3. A multiplicative system (Definition B.2.1) S in a category (Definition .1.1) $\mathscr{C}$ is called *saturated* if it satisfies the following condition: any morphism f in $\mathscr{C}$ whose image $Q(f)$ in the localized category (Definition B.2.2) $\mathscr{C}(S^{-1})$ is an isomorphism belongs to S.

Proposition B.2.4. Let $\mathscr{C}$ be a category (Definition .1.1) and S a multiplicative system (Definition B.2.1). There exists a unique saturated multiplicative system (Definition B.2.3) $\bar{S}$ containing S such that the canonical functors $\mathscr{C}(S^{-1}) \to \mathscr{C}(\bar{S}^{-1})$ define an equivalence of categories (Definition .1.6). $\bar{S}$ consists of all morphisms f such that there exist morphisms g, h with $g \circ f \in S$ and $f \circ h \in S$.

B.3. **Derived categories of abelian categories.** (♠ TODO: describe the objects and morphisms in a derived category)

Definition B.3.1 (Derived category)**.** Let $\mathcal{A}$ be an abelian category. Consider the class S of quasi-isomorphisms (Definition .2.4) in the homotopy category $K(\mathcal{A})$ (Definition B.1.1).

The *derived category* $D(\mathcal{A})$ is the localization (Definition B.2.2) of the homotopy category at the multiplicative system (Definition B.2.1) S:

$$D(\mathcal{A}) := K(\mathcal{A})[S^{-1}],$$

We often write the localization functor (Definition B.2.2) $K(\mathcal{A}) \to D(\mathcal{A})$ by q or Q.

- We often write $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, and $D^b(\mathcal{A})$ for the full subcategories of cohomologically bounded below, cohomologically bounded above, and cohomologically bounded (Definition .2.2) complexes respectively.

In general, $D(\mathcal{A})$ (and the aforementioned subcategories) may only exist as a large category (Definition .1.1), rather than a locally small category. When $\mathcal{A}$ is a Grothendieck abelian category, however, $D(\mathcal{A})$ is a locally small category (see [Sta25, Tag 09PA]).

Convention B.3.2. Given an abelian category (Definition B.3.1) A, any object M of A may be identified with an object of the homotopy category $K(A)$ (Definition B.1.1) and the derived category $D(A)$ (Definition B.3.1) — simply take the chain complex concentrated at degree 0 with the object M. In fact, this derived object can often be regarded as an object of "nice" enough subcategories of $D(A)$, such as the bounded derived categories $D^+(A)$, $D^-(A)$, $D^b(A)$.

(♠ TODO: State how a derived category is a triangulated category)

Definition B.3.3. Let $\mathcal{A}$ be an additive category. For $? \in \{+, -, b, (\text{blank})\}$, write $K^?(\mathcal{I})$ (resp. $K^?(\mathcal{P})$) for the full subcategory of $K^?(\mathcal{A})$ (Definition B.1.1) whose objects are appropriately bounded (Definition .2.2) complexes of injectives (resp. projectives).

Even though derived categories of abelian categories may not be guaranteed to be locally small, the following gives sufficient conditions for the appropriately (cohomologically) bounded derived categories to be locally small. The following also shows that appropriately bounded complexes under these circumstances can be represented in the derived category by a complex of injective or projective objects; we might think of such complexes of injectives or projectives as acting like an injective/projective resolution.

Theorem B.3.4 (see e.g. [Wei94, Theorem 10.4.8])**.** Let $\mathcal{A}$ be an abelian category.

1. If $\mathcal{A}$ has enough injectives (Definition .1.29), then the category $D^+(\mathcal{A})$ (Definition B.3.1) is equivalent to the category $K^+(\mathcal{I})$ (Definition B.3.3), which is locally small (Definition .1.4). In fact, an equivalence is given by
$$K^+(\mathcal{I}) \xrightarrow{q} D^+(\mathcal{I}) \hookrightarrow D^+(\mathcal{A}).$$
2. If $\mathcal{A}$ has enough projectives (Definition .1.29), then the category $D^-(\mathcal{A})$ (Definition B.3.1) is equivalent to the category $K^-(\mathcal{P})$ (Definition B.3.3), which is locally small (Definition .1.4).
$$K^-(\mathcal{P}) \xrightarrow{q} D^-(\mathcal{P}) \hookrightarrow D^-(\mathcal{A}).$$

B.4. **Total derived functors.** The following subsection mostly takes the point of view of [Wei94, Chapter 10]

Definition B.4.1 (cf. [Wei94, Definition 10.5.1])**.** Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $F : K(\mathcal{A}) \to K(\mathcal{B})$ (Definition B.1.1) be an exact functor (Definition B.1.3) of triangulated categories. Write $q_{\mathcal{A}} : K(\mathcal{A}) \to D(\mathcal{A})$ and $q_{\mathcal{B}} : K(\mathcal{B}) \to D(\mathcal{B})$ for the localization functors (Definition B.3.1),

1. A *(total) right derived functor on F on $K(\mathcal{A})$* is an exact functor $RF : D(\mathcal{A}) \to D(\mathcal{B})$ together with a natural transformation
$$\xi : q_{\mathcal{B}}F \Rightarrow (RF)q_{\mathcal{A}},$$
which is universal in the sense that if $G : D(\mathcal{A}) \to D(\mathcal{B})$ is another morphism equipped with a natural transformation $\zeta : q_{\mathcal{B}}F \Rightarrow Gq_{\mathcal{A}}$, then there exists a unique natural transformation $\eta : RF \Rightarrow G$ so that $\zeta_A = \eta_{q_{\mathcal{A}}A} \circ \xi_A$ for all A in $K(\mathcal{A})$.
2. Dually, a *(total) left derived functor on F on $K(\mathcal{A})$* is an exact functor $LF : D(\mathcal{A}) \to D(\mathcal{B})$ together with a natural transformation
$$\xi : (LF)q_{\mathcal{A}} \Rightarrow q_{\mathcal{B}}F,$$
which is universal in the sense that if $G : D(\mathcal{A}) \to D(\mathcal{B})$ is another exact functor equipped with a natural transformation $\zeta : Gq_{\mathcal{A}} \Rightarrow q_{\mathcal{B}}F$, then there exists a unique natural transformation $\eta : G \Rightarrow LF$ such that
$$\zeta_A = \xi_A \circ \eta_{q_{\mathcal{A}}A}$$
for all A in $K(\mathcal{A})$.

The universal properties guarantee that if the total derived functors exist, then they are unique up to natural isomorphism, and that

These definitions/ideas may be extended to exact functors $K^?(\mathcal{A}) \to K(\mathcal{B})$ for $? \in \{+, -, b\}$ and to exact functors $K^{?,??}(\mathcal{A}) \to K(\mathcal{B})$ for $? \in \{\infty, +, -, b\}$ and $?? \in \{+, -, b, (\text{blank})\}$ (Definition B.1.1)

Proposition B.4.2. See e.g. [Wei94, 10.5.2] Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor (Definition .1.28) of abelian categories (Definition .1.26). There is an induced functor

$$D(\mathcal{A}) \to D(\mathcal{B})$$

(Definition B.3.1) also denoted by F by abuse of notation.

This functor is constructible in the "simple" way —

- For an object M of $D(\mathcal{A})$, take a chain complex K representing it. The object FM of $D(\mathcal{B})$ is the object represented by the chain complex FK.
- For a morphism $f : M \to N$ in $D(\mathcal{A})$, take a morphism $\tilde{f} : K \to L$ representing it. The morphism $Ff : FM \to FN$ is the morphism represented by $F\tilde{f} : FK \to FL$.

Due to the exactness of F, these do not depend on the choices of representatives.

The below shows that right and left total derived functors exist for nice enough abelian categories on sufficiently (cohomologically) boundeded objects and can be computed using injective/projective resolutions.

Theorem B.4.3 (See e.g. [Wei94, 10.5.6])**.** Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. Let $F : K^+(\mathcal{A}) \to K(\mathcal{B})$ (Definition B.1.1) be a morphism of triangulated categories (Definition B.1.3).

1. If $\mathcal{A}$ has enough injectives (Theorem B.1.7), then the right derived functor RF (Definition B.4.1) exists (as a functor $D^+(\mathcal{A}) \to D(\mathcal{B})$ (Definition B.3.1)), and if I is a bounded below (Definition .2.2) complex of injectives, then
$$RF(I) \cong qF(I)$$
where $q : K(\mathcal{B}) \to D(\mathcal{B})$ is the localization functor (Definition B.3.1).
2. Dually, if $\mathcal{A}$ has enough projectives, then the left derived functor LF exists (as a functor $D^-(\mathcal{A}) \to D(\mathcal{B})$), and if I is a bounded above complex of projectives, then
$$LF(P) \cong qF(P)$$

Corollary B.4.4. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories (Definition .1.26). Write $q : K(\mathcal{B}) \to D(\mathcal{B})$ (Definition B.1.1 Definition B.3.1) for the localization functor (Definition B.2.2).

1. Assuming that $\mathcal{A}$ has enough injectives (Definition .1.29), there exists a right derived functor (Definition B.4.1) $RF : D^+(\mathcal{A}) \to D(\mathcal{B})$ and if $I \in K^+(\mathcal{I})$ (Definition B.3.3) is a bounded below complex of injectives, then
$$RF(I) \cong q(K(F))(I).$$

where $K(F) : K^+(\mathcal{A}) \to K(\mathcal{B})$ is the exact functor (Definition B.1.3) induced by F (Theorem B.1.7).

If F is additionally left exact (Definition .1.28), then for all objects A of $\mathcal{A}$ we have

$$H^i(RF(A)) \cong R^i F(A)$$

(Definition .1.31) for all integers i where we regard A as the chain complex (Definition .2.1) object of $D^+(\mathcal{A})$ with A concentrated in degree 0 (Convention B.3.2).

2. Assuming that $\mathcal{A}$ has enough projectives (Definition .1.29), there exists a left derived functor (Definition B.4.1) $LF : D^-(\mathcal{A}) \to D(\mathcal{B})$ and if $P \in K^-(\mathcal{P})$ (Definition B.3.3) is a bounded above complex of projectives, then

$$LF(P) \cong q(K(F))(P).$$

where $K(F) : K^-(\mathcal{A}) \to K(\mathcal{B})$ is the exact functor (Definition B.1.3) induced by F (Theorem B.1.7).

If F is additionally right exact (Definition .1.28), then for all objects A of $\mathcal{A}$ we have

$$H^{-i}(LF(A)) \cong L_i F(A)$$

(Definition .1.31) for all integers i where we regard A as the chain complex (Definition .2.1) object of $D^+(\mathcal{A})$ with A concentrated in degree 0 (Convention B.3.2).

Proof. Apply Theorem B.4.3 to the exact functor $K(F)$ to obtain the isomorphisms

$$RF(I) \cong q(K(F))(I)$$
$$LF(P) \cong q(K(F))(P).$$

If F is left exact and A is an object of $\mathcal{A}$, then the right derived functors $R^i F(A)$ (Definition .1.31) are defined. Recall that they are defined first by taking an injective resolution (Definition .2.7)

$$0 \to A \to I^0 \to I^1 \to \cdots .$$

Since the injective resolution is acyclic, The object A and the complex $0 \to I^0 \to I^1 \to \cdots$ are equivalent as objects of $D^+(\mathcal{A})$ as an object of $D^+(\mathcal{A})$ are equivalent objects. Moreover, we have the isomorphism

$$RF(I) \cong q(K(F))(I).$$

In fact, the ith cohomology of the right hand side object precisely describes $R^i F(A)$ by definition. Therefore,

$$H^i(RF(A)) \cong R^i F(A)$$

as desired.

The analogous statement for $L_i F(A)$ holds similarly. □

(♠ TODO: define hyper-derived functors)

Corollary B.4.5 (See e.g. [Wei94, Corollary 10.5.7, Remark 10.5.8])**.** Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories (Definition .1.26).

1. If $\mathcal{A}$ has enough injectives (Definition .1.29), the hyper-derived functors $\mathbb{R}^i F(X)$ are the cohomology of $\mathbf{R}F(X)$: $\mathbb{R}^i F(X) \cong H^i \mathbf{R}F(X)$ for all i.

2. If $\mathcal{A}$ has enough projectives (Definition .1.29), the hyper-derived functors $\mathbb{L}_i F(X)$ are the cohomology of $\mathbf{L}F(X)$: $\quad \mathbb{L}_i F(X) \cong H^{-i}\mathbf{L}F(X)$ for all i.

B.5. **General derived functors for exact multifunctors between triangulated categories.** The following subsection mostly takes the point of view of [DA73, Exposé XVII, 1.2]

Definition B.5.1 (Ind-category)**.** Let $\mathcal{C}$ be a locally small category (Definition .1.4).

1. The *Ind-category of $\mathcal{C}$*, denoted $\mathrm{Ind}(\mathcal{C})$, is defined as follows:
   - Objects of $\mathrm{Ind}(\mathcal{C})$ are formal filtered colimits (Definition .1.13) of objects in $\mathcal{C}$. More precisely, an object is given by a filtered (Definition .1.11) small category I and a functor
   $$X : I \to \mathcal{C}.$$
   - Morphisms between objects $X : I \to \mathcal{C}$ and $Y : J \to \mathcal{C}$ are defined by
   $$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(X,Y) \ := \ \varprojlim_{i\in I} \varinjlim_{j\in J} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j),$$
   (Definition .1.13) where X_i and Y_j denote the images of $i \in I$ and $j \in J$ under X and Y, respectively.

   The composition of morphisms is induced naturally from composition in $\mathcal{C}$. Hence, $\mathrm{Ind}(\mathcal{C})$ is the completion of $\mathcal{C}$ under filtered colimits. Objects of $\mathrm{Ind}(\mathcal{C})$ are called *Ind-objects of $\mathcal{C}$*.
2. The *Pro-category of $\mathcal{C}$*, denoted $\mathrm{Pro}(\mathcal{C})$, is defined as follows:
   - Objects of $\mathrm{Pro}(\mathcal{C})$ are formal cofiltered limits (Definition .1.13) of objects in $\mathcal{C}$. More precisely, an object is given by a cofiltered small category I and a functor
   $$X : I \to \mathcal{C}.$$
   - Morphisms between objects $X : I \to \mathcal{C}$ and $Y : J \to \mathcal{C}$ are defined by
   $$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X,Y) := \varinjlim_{j\in J} \varprojlim_{i\in I} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j),$$
   where X_i and Y_j denote the images of $i \in I$ and $j \in J$ under X and Y, respectively.

   The composition of morphisms is induced naturally from composition in $\mathcal{C}$.

   Hence, $\mathrm{Pro}(\mathcal{C})$ is the completion of $\mathcal{C}$ under cofiltered limits. Objects of $\mathrm{Pro}(\mathcal{C})$ are called *Pro-objects of $\mathcal{C}$*.

Since **Sets** has all limits and colimits (♠ TODO:) and hence has all projective and inductive limits and since $\mathcal{C}$ is locally small, $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C})$ are locally small.

Definition B.5.2. Let $\mathscr{C}$ be a locally small category (Lemma .1.7).

1. An Ind-object (Definition B.5.1) $X = \varinjlim X_i$ is said to be *essentially constant* if it is isomorphic in $\mathrm{Ind}(\mathscr{C})$ to an object in the image of the canonical fully faithful functor $h : \mathscr{C} \to \mathrm{Ind}(\mathscr{C})$, which maps an object $A \in \mathrm{ob}(\mathscr{C})$ to the Ind-object represented by the constant diagram.

2. A Pro-object (Definition B.5.1) $X = \varprojlim X_i$ is said to be *essentially constant* if it is isomorphic in $\mathrm{Pro}(\mathscr{C})$ to an object in the image of the canonical fully faithful functor $h : \mathscr{C} \to \mathrm{Pro}(\mathscr{C})$, which maps an object $A \in \mathrm{ob}(\mathscr{C})$ to the Pro-object represented by the constant diagram.

In either definition, the object A may be referred to as the *value/limit/constant value of the Ind/Pro object* or the *object representing the Ind/Pro object*.

Proposition B.5.3. Let $\mathscr{C}$ be a locally small category (Lemma .1.7).

1. An Ind-object (Definition B.5.1) $X = \varinjlim_{i \in I} X_i$ (where I is a filtered (Definition .1.11) index category) is essentially constant (Definition B.5.2) and isomorphic to $A \in \mathrm{ob}(\mathscr{C})$ if and only if there exists a family of morphisms $(f_i : X_i \to A)_{i \in I}$ such that for every morphism $u : i \to j$ in I, the diagram

$$\begin{tikzcd} X_i \arrow[r, "X(u)"] \arrow[dr, "f_i"'] & X_j \arrow[d, "f_j"] \\ & A \end{tikzcd}$$

commutes, and the induced morphism $\varinjlim X_i \to A$ (Definition .1.13) is an isomorphism in $\mathrm{Ind}(\mathscr{C})$.

2. A Pro-object (Definition B.5.1) $X = \varprojlim_{i \in I} X_i$ (where I is a filtered (Definition .1.11) index category) is essentially constant (Definition B.5.2) and isomorphic to $A \in \mathrm{ob}(\mathscr{C})$ if and only if there exists a family of morphisms $(g_i : A \to X_i)_{i \in I}$ such that for every morphism $u : i \to j$ in I, the diagram

$$\begin{tikzcd} A \arrow[r, "g_i"] \arrow[dr, "g_j"'] & X_i \arrow[d, "X(u)"] \\ & X_j \end{tikzcd}$$

commutes, and the induced morphism $A \to \varprojlim X_i$ (Definition .1.13) is an isomorphism in $\mathrm{Pro}(\mathscr{C})$.

Definition B.5.4 (Mixed-Variance Derived Multifunctors)**.** [cf. [DA73, Exposé XVII, Définition 1.2.1., 1.2.5.]] Let $(\mathscr{A}_i)_{1 \le i \le n}$ and $(\mathscr{B}_j)_{1 \le j \le m}$ be families of triangulated categories (Definition B.1.2), and let $\mathscr{C}$ be a triangulated category. Let

$$F : \prod_{i=1}^{n} \mathscr{A}_i \times \prod_{j=1}^{m} \mathscr{B}_j \to \mathscr{C}$$

be an exact multifunctor (Definition B.1.5), covariant in A_i and contravariant in B_j. Let S_i, Σ_j, T be saturated multiplicative systems (Definition B.2.3) in $\mathscr{A}_i, \mathscr{B}_j, \mathscr{C}$ respectively.

1. The *right derived functor RF of F relative to (S_i, Σ_j) and T* is a functor

$$RF : \prod_{i=1}^{n} \mathscr{A}_i(S_i^{-1}) \times \prod_{j=1}^{m} \mathscr{B}_j(\Sigma_j^{-1}) \to \mathrm{Ind}(\mathscr{C}(T^{-1}))$$

which makes the following diagram commutative:

$$\begin{array}{ccc}\prod_{i=1}^n \mathscr{A}_i \times \prod_{j=1}^m \mathscr{B}_j & \xrightarrow{F} & \mathscr{C} \\ \downarrow^{\prod Q_{A_i} \times \prod Q_{B_j}} & & \downarrow^{Q_C} \\ \prod_{i=1}^n \mathscr{A}_i(S_i^{-1}) \times \prod_{j=1}^m \mathscr{B}_j(\Sigma_j^{-1}) & \xrightarrow{RF} & \mathrm{Ind}(\mathscr{C}(T^{-1}))\end{array}$$

2. The *left derived functor LF of F relative to (S_i, Σ_j) and T* is a functor

$$LF : \prod_{i=1}^n \mathscr{A}_i(S_i^{-1}) \times \prod_{j=1}^m \mathscr{B}_j(\Sigma_j^{-1}) \to \mathrm{Pro}(\mathscr{C}(T^{-1}))$$

which makes the following diagram commutative:

$$\begin{array}{ccc}\prod_{i=1}^n \mathscr{A}_i \times \prod_{j=1}^m \mathscr{B}_j & \xrightarrow{F} & \mathscr{C} \\ \downarrow^{\prod Q_{A_i} \times \prod Q_{B_j}} & & \downarrow^{Q_C} \\ \prod_{i=1}^n \mathscr{A}_i(S_i^{-1}) \times \prod_{j=1}^m \mathscr{B}_j(\Sigma_j^{-1}) & \xrightarrow{LF} & \mathrm{Pro}(\mathscr{C}(T^{-1}))\end{array}$$

Definition B.5.5 (Derivability and Adapted Families)**.** Let $(\mathscr{A}_i)_{1\le i\le n}$ and $(\mathscr{B}_j)_{1\le j\le m}$ be families of triangulated categories (Definition B.1.2), and let $\mathscr{C}$ be a triangulated category. Let

$$F : \prod_{i=1}^n \mathscr{A}_i \times \prod_{j=1}^m \mathscr{B}_j \to \mathscr{C}$$

be an exact multifunctor (Definition B.1.5), covariant in the variables $A_i \in \mathscr{A}_i$ and contravariant in the variables $B_j \in \mathscr{B}_j$. Let S_i, Σ_j, T be saturated multiplicative systems (Definition B.2.3) in $\mathscr{A}_i, \mathscr{B}_j, \mathscr{C}$ respectively.

1. Let RF be the right derived functor (Definition B.5.4) of F relative to the multiplicative systems.
   (a) RF is *defined at a family (A_i, B_j)* if the Ind-object (Definition B.5.1) $RF(A_i, B_j)$ is essentially constant. This object is the *value of RF at (A_i, B_j)*.
   (b) F is *right derivable* if RF is defined at all objects. In this case, we view it as a functor into $\mathscr{C}(T^{-1})$.
   (c) A family (A_i, B_j) is *adapted for RF* (or *deployed for RF* (*déployée pour RF* in French)) if the natural morphism $Q_C(F(A_i, B_j)) \to RF(Q_{A_i}A_i, Q_{B_j}B_j)$ is an isomorphism in $\mathrm{Ind}(\mathscr{C}(T^{-1}))$, i.e. the Ind-object $RF(A_i)$ of $A(S^{-1})$ is essentially constant (Definition B.5.2) with value $F((A_i), (B_j))$.
2. Let LF be the left derived functor (Definition B.5.4) of F relative to the multiplicative systems.
   (a) LF is *defined at a family (A_i, B_j)* if the Pro-object (Definition B.5.1) $LF(A_i, B_j)$ is essentially constant.
   (b) F is *left derivable* if LF is defined at all objects.
   (c) A family (A_i, B_j) is *adapted for LF* (or *deployed for LF* (*déployée pour LF* in French)) if the natural morphism $LF(Q_{A_i}A_i, Q_{B_j}B_j) \to Q_C(F(A_i, B_j))$ is an isomorphism in $\mathrm{Pro}(\mathscr{C}(T^{-1}))$, i.e. the Pro-object $LF(A_i)$ of $A(S^{-1})$ is essentially constant (Definition B.5.2) with value $F((A_i), (B_j))$.

B.5.1. *Derived multifunctors of additive multifunctors between abelian categories.*

Definition B.5.6. Let $F : \prod_{i=1}^n \mathscr{A}_i \to \mathscr{B}$ be an additive multifunctor (Definition C.0.13) of abelian categories (Definition .1.26).

The *right (resp. left) derived multifunctors of F* are the derived functors (Definition B.5.4) RF (resp. LF) of the induced multifunctors (Theorem B.1.7)

- $K(\prod \mathscr{A}_i) \to K(\mathscr{B})$
- $K^+(\prod \mathscr{A}_i) \to K^+(\mathscr{B})$
- $K^-(\prod \mathscr{A}_i) \to K^-(\mathscr{B})$

(Recall Proposition B.1.6) relative to the systems (Definition B.2.3) of quasi-isomorphisms (Definition .2.4) in the respective homotopy categories (Definition B.1.1). In particular, the functor RF refers to one of the functors

$$D^+(\prod \mathscr{A}_i) \to \operatorname{Ind} D^+(\mathscr{B}) D(\prod \mathscr{A}_i) \to \operatorname{Ind} D(\mathscr{B})$$

and the functor LF refers to one of the functors

$$D^-(\prod \mathscr{A}_i) \to \operatorname{Pro} D^-(\mathscr{B}) D(\prod \mathscr{A}_i) \to \operatorname{Pro} D(\mathscr{B})$$

(Definition B.3.1) (Definition B.5.1).

Definition B.5.7 ([DA73, Definition 1.2.3])**.** Let $F : \prod_{i=1}^n \mathscr{A}_i \to \mathscr{B}$ be an additive multifunctor (Definition C.0.13) of abelian categories (Definition .1.26).

1. An object $(A_1, \ldots, A_n)$ with $A_i \in \mathrm{ob}(\mathscr{A}_i)$ is *right acyclic for F* if the complex concentrated in degree 0 associated to (A_i) is adapted for (Definition B.5.5) RF .
2. Similarly, (A_i) is *left acyclic for F* if the complex concentrated in degree 0 is adapted for (Definition B.5.5) LF.

In the case that $n = 1$ and $A_1 \in \mathcal{A}_1$ has a projective resolution (Definition .2.7) (resp. injective resolution), the notion of left acyclic for F (resp. right acyclic for F) is equivalent to the notion of F-acyclicity.

Proposition B.5.8 (cf. [DA73, Exposé XVII, Proposition 1.2.4.])**.** Let $F : \prod_{i=1}^n \mathscr{A}_i \to \mathscr{B}$ be an additive multifunctor (Definition C.0.13) of abelian categories (Definition .1.26).

1. The following diagram of categories is commutative for the right derived multifunctor:

$$\begin{array}{ccc} D^+(\prod \mathscr{A}_i) & \longrightarrow & D(\prod \mathscr{A}_i) \\ \downarrow^{RF} & & \downarrow^{RF} \\ \operatorname{Ind} D^+(\mathscr{B}) & \longrightarrow & \operatorname{Ind} D(\mathscr{B}) \end{array}$$

   (Definition B.3.1) (**??**) (Definition B.5.6) where the horizontal arrows are the natural functors induced by inclusions. If RF is everywhere defined on $D(\prod \mathscr{A}_i)$, its restriction to $D^+(\prod \mathscr{A}_i)$ factors through $D^+(\mathscr{B})$ [DA73, Proposition 1.2.4].
2. Dually, the following diagram is commutative for the left derived multifunctor:

$$\begin{array}{ccc} D^-(\prod \mathscr{A}_i) & \longrightarrow & D(\prod \mathscr{A}_i) \\ \downarrow^{LF} & & \downarrow^{LF} \\ \operatorname{Pro} D^-(\mathscr{B}) & \longrightarrow & \operatorname{Pro} D(\mathscr{B}) \end{array}$$

If LF is everywhere defined on $D(\prod \mathscr{A}_i)$, its restriction to $D^-(\prod \mathscr{A}_i)$ factors through $D^-(\mathscr{B})$.

Theorem B.5.9 ([DA73, Exposé XVII, Proposition 1.2.7.]). (♠ TODO:) Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories (Definition .1.26).

If $\mathcal{A}$ has enough objects that are acyclic for (Definition B.5.7) LF (resp. for RF), the derived functor (Definition B.5.4) $LF : D^-(\mathcal{A}) \to D^-(\mathcal{B})$ (resp. $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$) is left (resp. right) derivable (Definition B.5.5) and every homologically upper bounded (Definition .2.2) (resp. homologically lower bounded) complex of acyclics for LF (resp. for RF) is adapted for LF (resp. for RF) (Definition B.5.5).

(♠ TODO:

- **Address the Biadditive/Multifunctor Gap**
  - Add a Lemma or Remark following Definition 5.5.5 clarifying that for a biadditive functor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$, the condition of a pair of complexes $(M^\bullet, N^\bullet)$ being adapted (deployed) is what allows the representation of LF via the Total Complex construction.
  - Explicitly define the Total Complex functor $\mathrm{Tot} : \mathrm{K}(\mathcal{A}) \times \mathrm{K}(\mathcal{B}) \to \mathrm{K}(\mathcal{C})$ used to induce the multifunctor in Definition 5.5.6. Specify the sign convention for the total differential: $d(x \otimes y) = d_x(x) \otimes y + (-1)^p x \otimes d_y(y)$ for x of degree p.
- **Formulate the Balancing Results**
  - Add a Proposition (The Balancing Lemma) stating that if $\mathcal{P} \subset \mathcal{A}$ is a class of objects adapted for $F(-, N)$ and $\mathcal{Q} \subset \mathcal{B}$ is a class adapted for $F(M, -)$, there exist canonical isomorphisms in $D(\mathcal{C})$:

    $$\mathrm{Tot}(F(P^\bullet, N^\bullet)) \cong \mathrm{Tot}(F(M^\bullet, Q^\bullet)) \cong \mathrm{Tot}(F(P^\bullet, Q^\bullet))$$
  - Cite [SGA 4, XVII, 1.2.7] to justify that these isomorphisms are well-defined because complexes of adapted objects are "deployed".
- **Incorporate the "Everywhere Defined" Logic**
  - Include the "Finite Dimension" case from [SGA 4, XVII, Prop 1.2.10] to explain the conditions under which LF and RF can be extended to the unbounded derived category $D(\mathcal{A})$.
- **Technical Refinements**
  - Provide a precise proof or reference for the property: "If F is bi-right-exact, then projective objects are left-acyclic (adapted) for F."
  - Define the term "Deployed in the first variable": A complex $P^\bullet$ is deployed in the first variable if the family $(P^\bullet, N^\bullet)$ is deployed for the multifunctor F for every $N^\bullet \in \mathrm{K}(\mathcal{A}_2)$.
  - Resolve the ♠ TODO regarding the existence of limits/colimits in `Sets` to ensure Ind/Pro categories are locally small.
- **Categorical Equivalence**
  - Add a Remark confirming the equivalence of categories $D(\mathcal{A}_1) \times D(\mathcal{A}_2) \simeq D(\mathcal{A}_1 \times \mathcal{A}_2)$, noting that quasi-isomorphisms are defined component-wise.
- **derived tensor product**
  - Do proposition 4.1.7. expose XVII

)

Proposition B.5.10. The following are examples of abelian categories (Definition .1.26):

1. The category of R-S bimodules where R,S are (not necessarily commutative) rings (Definition C.0.8) (**??**).
2. The category **Ab** of abelian groups and group homomorphisms is abelian.
3. The category Vect_k of vector spaces over a field k and k-linear maps is abelian.
4. More generally, if R is a noetherian ring, then the category of finitely generated R-modules is abelian.
5. For a ringed space $(X, \mathcal{O}_X)$, the category of $\mathcal{O}_X$-modules (Definition 6.1.3) is abelian. (♠ TODO: a quasi-coherent sheaf on a locally ringed space)
6. If X is a scheme (Definition .3.1) (or more generally a locally ringed space), the category of quasi-coherent sheaves on X is abelian.
7. For any essentially small category (Definition .1.18) $\mathcal{C}$ and any abelian category $\mathcal{A}$, the functor category $[\mathcal{C}, \mathcal{A}]$ (Definition .1.25) and the category $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ of presheaves (Definition 3.0.1) are abelian. (♠ TODO: apparently, the essentially smallness condition is removable, provided that the sheafification functor exists. However, the essentially small assumption is needed to show that the category of sheaves of O-modules is a Grothendieck abelian caetgory. Verify all this. Moreover, when working with a big site of a scheme, one typically fixes a unvierse or work relative to a cardinal cutoff to treat it as essentially small)
8. For any site (Definition A.0.4) $(\mathcal{C}, J)$ on an essentially small category (Definition .1.18) $\mathcal{C}$ and any abelian category $\mathcal{A}$, the category $\mathrm{Shv}(\mathcal{C}, \mathcal{A})$ of sheaves (Definition 3.0.2) is abelian.
9. For any site (Definition A.0.4) $(\mathcal{C}, J)$ on an essentially small category (Definition .1.18) $\mathcal{C}$ and a sheaf of rings (Definition 3.0.2) $\mathcal{O}$ on $\mathcal{C}$, the category $\mathbf{Mod}(\mathcal{O})$ of $\mathcal{O}$-modules (Definition 6.1.3) is an abelian category.

Notation B.5.11. Let $((\mathcal{C}, J), \mathcal{O})$ be a ringed site (Definition 5.1.10) where $(\mathcal{C}, J)$ is essentially small. (♠ TODO: The essentially smallness hypothesis might not be strictly needed to have that the category of sheaves of O-modules is abelian.)

1. The homotopy category (Definition B.1.1) of the category of $\mathcal{O}$-modules (Definition 6.1.3), which is (Proposition B.5.10) abelian, is often denoted by notations such as $K(\mathcal{C}, \mathcal{O})$ or $K(\mathcal{O})$. All of the usual superscripts apply — we may thus speak of $K^{?,??}(\mathcal{C}, \mathcal{O})$ or $K^{?,??}(\mathcal{O})$ for $? \in \{+, -, b\}$ and $?? \in \{+, -, b, (\text{blank})\}$.

   If $(\mathcal{C}, J)$ is a site on some "space", e.g. a scheme or topological space, X, then it is also usual to denote the homotopy category by $K(X, \mathcal{O})$. We may also speak of $K^{?,??}(X, \mathcal{O})$ for $? \in \{+, -, b\}$ and $?? \in \{+, -, b, (\text{blank})\}$.
2. Similarly, it is customary to denote the derived category (Definition B.3.1) of the category of $\mathcal{O}$-modules (Definition 6.1.3) of the category of $\mathcal{O}$-modules by notations such as $D(\mathcal{C}, \mathcal{O})$ or $D(\mathcal{C}, \mathcal{O})$. All of the usual superscripts apply — we may thus speak of $D^?(\mathcal{C}, \mathcal{O})$ or $D^?(\mathcal{O})$ for $? \in \{+, -, b\}$.

   If $(\mathcal{C}, J)$ is a site on some "space", e.g. a scheme or topological space, X, then it is also usual to denote the homotopy category by $D(X, \mathcal{O})$. We may also speak of $D^?(X, \mathcal{O})$ for $? \in \{+, -, b\}$.

Lemma B.5.12. Let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a biadditive functor (Definition C.0.13) of abelian categories (Definition .1.26). Assume that (small) filtered colimits which exist in $\mathcal{C}$ are exact (e.g. which holds if $\mathcal{C}$ satisfies Ab5).

Let $A \in \mathcal{A}$ and $B \in \mathcal{B}$ be objects.

1. Suppose that left resolutions (Definition .2.7) $P_{A,\bullet} \to A$ and $P_{B,\bullet} \to B$ exist such that $P_{A,i}$ and $P_{B,i}$ are flat with respect to F on the left and right respectively, i.e. $F(P_{A,i}, -) : \mathcal{B} \to \mathcal{C}$ and $F(-, P_{B,i}) : \mathcal{A} \to \mathcal{C}$ are exact for all i.
   The complexes $F(P_{A,\bullet}, B)$ and $F(A, P_{B,\bullet})$ are quasi-isomorphic (Definition .2.4) to the complex $\mathrm{Tot}(F(P_{A,\bullet}, P_{B,\bullet}))$.
2. Suppose that right resolutions (Definition .2.7) $A \to I^{A,\bullet}$ and $B \to I^{B,\bullet}$ exist such that $I^{A,i}$ and $I^{B,i}$ are flat with respect to F on the left and right respectively, $F(I^{A,i}, -) : \mathcal{B} \to \mathcal{C}$ and $F(-, I^{B,i}) : \mathcal{A} \to \mathcal{C}$ are exact for all i.
   The complexes $F(I^{A,\bullet}, B)$ and $F(A, I^{B,\bullet})$ are quasi-isomorphic (Definition .2.4) to the complex $\mathrm{Tot}(F(I^{A,\bullet}, I^{B,\bullet}))$.

Proof. We prove 1. The other part is the dual statement.

Choose resolutions $P_{A,\bullet} \xrightarrow{\varepsilon} A$ and $P_{B,\bullet} \xrightarrow{\eta} B$ such that $F(P_{A,i}, -) : \mathcal{B} \to \mathcal{C}$ and $F(-, P_{B,i}) : \mathcal{A} \to \mathcal{C}$ are exact for all i. Identifying A and B with complexes concentrated in degree 0, we can form the three double complexes $F(P_{A,\bullet}, P_{B,\bullet})$, $F(A, P_{B,\bullet})$, and $F(P_{A,\bullet}, B)$. Note that the augmentation morphisms ε and η induce morphisms $P_{A,\bullet} \otimes P_{B,\bullet} \to A \otimes P_{B,\bullet}, P_{A,\bullet} \otimes B$.

Let C be the double complex of objects in $\mathcal{C}$ obtained from $F(P_{A,\bullet}, P_{B,\bullet})$ by adding $F(A, P_{B,\bullet}[-1])$ in the column $p = -1$. One can show that the translate $\mathrm{Tot}(C)[1]$ is the mapping cone (Definition .2.5) of the map

$$\mathrm{Tot}(F(P_{A,\bullet}, P_{B,\bullet})) \xrightarrow{\varepsilon\otimes\mathrm{id}} \mathrm{Tot}(F(A, P_{B,\bullet})) = F(A, P_{B,\bullet}).$$

Moreover, since each $F(-, P_{B,i})$ is an exact functor, every row of C is exact, so $\mathrm{Tot}(C)$ is exact by **??**. Therefore, $F(\varepsilon, \mathrm{id})$ is a quasi-isomorphism and hence

$$H_*(\mathrm{Tot}(F(P_{A,\bullet}, P_{B,\bullet}))) \xrightarrow{H_*(F(\varepsilon, P_{B,\bullet}))} H_*(F(A, P_{B,\bullet}))$$

is a natural isomorphism.

By symmetry, there is a natural isomorphism $H_*(\mathrm{Tot}(F(P_{A,\bullet} P_{B,\bullet}))) \to H_*(F(P_{A,\bullet}, B))$. $\square$

Proposition B.5.13. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories (Definition .1.26), and let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a biadditive functor (Definition C.0.13).

1. Suppose that $\mathcal{A}$ and $\mathcal{B}$ both have enough projectives (Definition .1.29). Given objects $M \in D^-(\mathcal{A})$ and $N \in D^-(\mathcal{A})$ (Definition B.3.1), the objects $LF(M, N)$ obtained as $(LF(M, -))(N)$ and $(LF(-, N))(M)$ are naturally isomorphic. Thus, the two definitions of $LF(M, N)$ in Definition B.5.14 are in agreement.
2. Dually, suppose that $\mathcal{A}$ and $\mathcal{B}$ both have enough injectives (Definition .1.29). Given objects $M \in D^+(\mathcal{A})$ and $N \in D^+(\mathcal{A})$ (Definition B.3.1), the objects $RF(M, N)$ obtained as $(RF(M, -))(N)$ and $(RF(-, N))(M)$ are naturally isomorphic. Thus, the two definitions of $RF(M, N)$ in Definition B.5.14 are in agreement.

Proof. We prove 1. The other part is dual. By Theorem B.3.4, note that $D^-(\mathcal{A})$ and $D^-(\mathcal{B})$ are respectively equivalent to the categories $K^-(\mathcal{P}_{\mathcal{A}})$ and $K^-(\mathcal{P}_{\mathcal{B}})$ (Definition B.3.3) of cohomologically bounded above complexes of projectives in $\mathcal{A}$ and $\mathcal{B}$ respectively. Let

$$P_\bullet \to M$$

and

$$Q_\bullet \to N$$

be projective resolutions. By Corollary B.4.4,

$$(LF(M,-)(N) \cong q(K(F(M,-)))(N)$$

$$(LF(-,N)(M) \cong q(K(F(-,N)))(M)$$

The former is represented by the complex $F(M, Q_\bullet)$ and the latter is represented by the complex $F(P_\bullet, N)$. These are quasi-isomorphic by Lemma B.5.12. □

Definition B.5.14. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories (Definition .1.26), and let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a biadditive functor (Definition C.0.13).

There are notions of derived functors that we may consider depending on whether $\mathcal{A}$ or $\mathcal{B}$ has enough projectives or injectives or flats with respect to F. (♠ TODO: define for flats)

(♠ TODO: I really should be letting M and N be complxes, not just objects.)

1.

1. (a) Let $M \in \mathcal{A}$. Assume that $\mathcal{B}$ has enough projectives (Definition .1.29). By Corollary B.4.4, the functor $F(M,-) : \mathcal{B} \to \mathcal{C}$ has a left derived functor (Definition B.4.1) $L(F(M,-)) : D^-(\mathcal{B}) \to D(\mathcal{C})$ (Definition B.3.1), which we may write as $LF(M,-)$.
   (b) Symmetrically, let $N \in \mathcal{B}$. Assume that $\mathcal{A}$ has enough projectives (Definition .1.29). By Corollary B.4.4, the functor $F(-,N) : \mathcal{A} \to \mathcal{C}$ has a left derived functor (Definition B.4.1) $L(F(-,N)) : D^-(\mathcal{A}) \to D(\mathcal{C})$ (Definition B.3.1), which we may write as $LF(-,N)$.

   Assuming that $\mathcal{A}$ and $\mathcal{B}$ have enough projectives, the notations $LF(M,N)$ above are in agreement (Proposition B.5.13)
2. (a) Let $M \in \mathcal{A}$. Assume that $\mathcal{B}$ has enough injectives (Definition .1.29). By Corollary B.4.4, the functor $F(M,-) : \mathcal{B} \to \mathcal{C}$ has a right derived functor (Definition B.4.1) $R(F(M,-)) : D^+(\mathcal{B}) \to D(\mathcal{C})$ (Definition B.3.1), which we may write as $RF(M,-)$.
   (b) Symmetrically, let $N \in \mathcal{B}$. Assume that $\mathcal{A}$ has enough injectives (Definition .1.29). By Corollary B.4.4, the functor $F(-,N) : \mathcal{A} \to \mathcal{C}$ has a right derived functor (Definition B.4.1) $R(F(-,N)) : D^+(\mathcal{A}) \to D(\mathcal{C})$ (Definition B.3.1), which we may write as $RF(-,N)$.

   Assuming that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives, the notations $RF(M,N)$ above are in agreement (Proposition B.5.13)
3. Let $M \in \mathcal{A}$. Assume that $\mathcal{B}$ has enough flats with respect to $\otimes$. Define $LF(M,-) : D^-(\mathcal{B}) \to D(\mathcal{C})$ as follows:

See Definition B.5.15 for notation used in the case that F is written as a tensor product.

Definition B.5.15. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories (Definition .1.26), and let $\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a biadditive functor (Definition C.0.13) written as tensor product.

(♠ TODO: need a definition with enough flats)

1. Let $M \in \mathcal{A}$. Assume that $\mathcal{B}$ has enough projectives (Definition .1.29). We write $M \otimes^L -$ for $LF(M, -) : D^-(\mathcal{B}) \to D(\mathcal{C})$ (Definition B.5.14) in the case that $F = M \otimes - : \mathcal{B} \to \mathcal{C}$.
2. Symmetrically, let $N \in \mathcal{B}$. Assume that $\mathcal{A}$ has enough projectives (Definition .1.29). We write $- \otimes^L N$ for $LF(-, N) : D^+(\mathcal{A}) \to D(\mathcal{C})$ (Definition B.5.14) in the case that $F = - \otimes N : \mathcal{A} \to \mathcal{C}$.
   (♠ TODO: show that projectives vs. flats yield the same thing) (♠ TODO: show that flats in each variable yield the same thing)
3. Let $M \in \mathcal{A}$. Assume that $\mathcal{B}$ has enough flats. We alternatively define $M \otimes^L - : D^-(\mathcal{B}) \to D(\mathcal{C})$ as follows — given an object $N \in D^-(\mathcal{B})$, say that $Q^\bullet$ is a complex of flat objects in $\mathcal{B}$ representing N (♠ TODO: show that such a thing exists), and let $M \otimes^L N$ be the object of $D(\mathcal{C})$ represented by the complex $M \otimes Q^\bullet$. (♠ TODO: show that this is well defined, i.e. does not depend on the choice of flat resolution)
4. Symmetrically, let $N \in \mathcal{B}$. Assume that $\mathcal{A}$ has enough flats. We alternatively define $- \otimes^L N : D^-(\mathcal{A}) \to D(\mathcal{C})$ as follows — given an object $M \in D^-(\mathcal{A})$, say that $P^\bullet$ is a complex of flat objects in $\mathcal{A}$ representing M (♠ TODO: show that such a thing exists), and let $M \otimes^L N$ be the object of $D(\mathcal{C})$ represented by the complex $P^\bullet \otimes N$. (♠ TODO: show that this is well defined, i.e. does not depend on the choice of flat resolution)

The first two notions agree assuming that $\mathcal{A}$ and $\mathcal{B}$ have enough projectives. (♠ TODO: comment on the next two notions agreeing)

Appendix C. Miscellaneous definitions

Definition C.0.1. A *commutative (unital) ring* is a ring (Definition C.0.8) $(R, +, \cdot)$ such that $\cdot$ is a commutative operation, i.e. $a \cdot b = b \cdot a$.

For many writers (e.g. “commutative” algebraists or number theorists), a *ring* refers to a commutative ring as above.

Definition C.0.2. Let R be a (not necessarily commutative, possibly nonunital) ring (Definition C.0.8). A *left ideal of R* is a subset $I \subseteq R$ such that

- $(I, +)$ is an additive subgroup of $(R, +)$,
- $RI \subseteq I$, i.e., for all $r \in R$ and $x \in I$, one has $rx \in I$.

Similarly, a *right ideal of R* is a subset $I \subseteq R$ such that

- $(I, +)$ is an additive subgroup of $(R, +)$,
- $IR \subseteq I$, i.e., for all $r \in R$ and $x \in I$, one has $xr \in I$.

A *two-sided ideal* (or simply an *ideal*) of R is a subset $I \subseteq R$ which is both a left ideal and a right ideal of R. We denote by $I \trianglelefteq R$ the relation expressing that I is a two-sided ideal of R.

Equivalently, an left/right/two-sided ideal of R is a submodule of R as an R-module (Definition C.0.11).

A left/right/two-sided ideal is said to be *proper* if it is strictly contained in R.

Note that every left or right ideal of a commutative ring is a two-sided ideal.

Definition C.0.3. Let R be a ring (Definition C.0.8) with unity, not necessarily commutative. The ring R is called a *local ring* if it has a unique maximal left ideal (Definition C.0.4). In this case, R also has a unique maximal right ideal, and these coincide with the Jacobson radical $J(R)$ of R. The unique maximal left (and right) ideal of a local ring R may sometimes be denoted by $\mathfrak{m}_R$.

Definition C.0.4. Let R be a (not necessarily commutative) ring (Definition C.0.8). A proper two-sided ideal $P \trianglelefteq R$ (Definition C.0.2) is called a *prime ideal* if the following equivalent conditions holds:

1. If I, J are left ideals and $IJ \subset P$, then $I \subset P$ or $J \subset P$.
2. If I, J are right ideals and $IJ \subset P$, then $I \subset P$ or $J \subset P$.
3. If I, J are two-sided idaels and $IJ \subset P$, then $I \subset P$ or $J \subset P$.
4. If $x, y \in R$ with $xRy \subset \mathfrak{p}$, then $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

A proper left/right/two-sided ideal $M \subsetneq R$ is called *maximal* if there exists no other left/right/two-sided ideal $J \trianglelefteq R$ such that $M \subsetneq J \subsetneq R$. Equivalently,

- a left/right ideal M of R is maximal if and only if the quotient module R/M is a simple left/right R-module.
- a two-sided ideal M of R is maximal if and only if the quotient ring R/M is a simple ring.

Definition C.0.5. Let $(R, \mathfrak{m}_R)$ and $(S, \mathfrak{m}_S)$ be local rings (Definition C.0.21), not necessarily commutative. A ring homomorphism $\varphi : R \to S$ is called a *local morphism* (or *local homomorphism*) if $\varphi(\mathfrak{m}_R) \subseteq \mathfrak{m}_S$.

Definition C.0.6 (Topology)**.** Let X be a set. A *topology on X* is a collection $\mathcal{T}$ of subsets of X such that:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
2. For any collection $\{U_i\}_{i\in I} \subseteq \mathcal{T}$ (with I arbitrary), the union $\bigcup_{i\in I} U_i \in \mathcal{T}$,
3. For any finite collection $\{U_1, \dots, U_n\} \subseteq \mathcal{T}$, the intersection $U_1 \cap \cdots \cap U_n \in \mathcal{T}$.

If $\mathcal{T}$ is a topology on X, the pair $(X, \mathcal{T})$ is called a *topological space*. Members of $\mathcal{T}$ are called *open sets*.

A subset $C \subseteq X$ is *closed* if its complement $X \setminus C$ is an open set in $\mathcal{T}$

One very often refers to X as a topological spcae, omitting the notation of the topology $\mathcal{T}$.

The collection of all topologies on a set X may be denoted by notations such as $\mathrm{Top}(X)$, $\mathbf{Top}(X)$, or $\mathsf{Top}(X)$.

Definition C.0.7. Let X be a topological space (Definition C.0.6). A subset $Z \subseteq X$ is called a *locally closed subset* if Z can be written as the intersection $U \cap C$, where U is an open subset of X and C is a closed subset of X. Equivalently, Z is a locally closed subset if it is an open subset of its closure $\overline{Z}$ endowed with the subspace topology.

Definition C.0.8. A *ring* is a triple $(R, +, \cdot)$ where

1. $(R, +)$ is a commutative group, and
2. $(R, \cdot)$ is a monoid.
3. $\cdot$ is distributive over $+$, i.e. for all $a, b, c \in R$, we have
$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and} \quad (a + b) \cdot c = a \cdot c + b \cdot c.$$

Equivalently, a ring is a triple $(R, +, \cdot)$ where $+, \cdot : R \times R \to R$ are binary operations satisfying

1. $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$ for all $a, b, c \in R$
2. There exists an element $0 \in R$ such that $a + 0 = a = 0 + a$ for all $a \in R$.
3. For every $a \in R$, there exists an element $-a \in R$ such that $a + (-a) = 0 = (-a) + a$ for all $a \in R$.
4. There exists an element $1 \in R$ such that $a \cdot 1 = a = 1 \cdot a$ for all $a \in R$.
5. For all $a, b, c \in R$, we have
$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and} \quad (a + b) \cdot c = a \cdot c + b \cdot c.$$

The operation $+$ is often called *addition* and the operation $\cdot$ is often called *multiplication*. Accordingly, the identity element 0 of $+$ is often called the *additive identity* and the identity element 1 of $\cdot$ is often called the *multiplicative identity*.

Remark C.0.9. Some writers might not require a ring to have a multiplicative identity element, i.e. would define a ring so that $(R, +)$ is a commutative group, $(R, \cdot)$ is a semigroup, and $\cdot$ is distributive over $+$. Such writers would call the notion of ring in Definition C.0.8 a *unitary ring* to emphasize the existence of the multiplicative identity 1.

Definition C.0.10 (Constant sheaf on a site)**.** Let $\mathcal{C}$ be a (large) category (Definition .1.1), let $\mathcal{A}$ be a (large category), and let A be an object of $\mathcal{A}$.

1. The *constant presheaf on $\mathcal{C}$ with value A* is the presheaf (Definition 3.0.1) P defined by
$$P(U) = A$$
for every object U of $\mathcal{C}$ such that every morphism $f : V \to U$ in $\mathcal{C}$ induces the identity map $A = P(U) \to P(V) = A$.
2. Let $\mathcal{C}$ be a site (Definition A.0.4) and assume that a sheafification functor (Definition 3.3.1)
$$a : \mathrm{Shv}(\mathcal{C}, \mathcal{A}) \to \mathrm{PreShv}(\mathcal{C}, \mathcal{A})$$
exists (e.g. see Theorem 3.3.2) . The *constant sheaf on $\mathcal{C}$ with value A*, or the *constant sheaf on $\mathcal{C}$ associated to A* commonly denoted $\underline{A}$ or sometimes just A by abuse

of notation, is the sheaf associated to (Theorem 3.3.2) the constant presheaf P with value A above.

3. Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of (not-necessarily commutative) rings on $\mathcal{C}$. Assume that the global sections ring $\Gamma(\mathcal{O})$ (Definition 6.1.1) exists. A *constant $\mathcal{O}$-module* is an $\mathcal{O}$-module (Definition 6.1.3) $\mathcal{F}$ which is isomorphic as a sheaf to the constant sheaf on $\mathcal{C}$ with value M where M is a module of the ring $\Gamma(\mathcal{O})$. Note that sheafification functors exist for presheaves/sheaves valued in Ab (Theorem 3.3.2).

   In case that $\mathcal{O}$ is the constant sheaf associated to A for some (not-necessarily commutative) ring A, a constant $\mathcal{O}$-module is simply called a *constant A-module*.

Definition C.0.11. Let R be a not-necessarily commutative ring (Definition C.0.8).

1. A *left R-module* is an abelian group $(M,+)$ together with an operation $R \times M \to M$, denoted $(r,m) \mapsto rm$, such that for all $r,s \in R$ and $m,n \in M$:
   - $r(m+n) = rm + rn$,
   - $(r+s)m = rm + sm$,
   - $(rs)m = r(sm)$,
   - $1_R m = m$ where 1_R is the multiplicative identity of R.
2. A *right R-module* is defined similarly as an abelian group $(M,+)$ with an operation $M \times R \to M$, denoted $(m,r) \mapsto mr$, such that for all $r,s \in R$ and $m,n \in M$:
   - $(m+n)r = mr + nr$,
   - $m(r+s) = mr + ms$,
   - $m(rs) = (mr)s$,
   - $m1_R = m$.
3. Let R and S be (not necessarily commutative) rings (Definition C.0.8).

   An *R-S-bimodule* (or an *R-S-module* or an (R,S)-module, etc.) is an abelian group $(M,+)$ equipped with
   (a) a left action of R:
   $$\boxed{R \times M \to M, \quad (r,m) \mapsto r \cdot m},$$
   making M a left R-module (Definition C.0.11),
   (b) a right action of S:
   $$\boxed{M \times S \to M, \quad (m,s) \mapsto m \cdot s},$$
   making M a right S-module,

   such that the left and right actions commute; that is, for all $r \in R$, $s \in S$, and $m \in M$,
   $$r \cdot (m \cdot s) = (r \cdot m) \cdot s.$$
4. A *two-sided R-module* (or *R-bimodule*) is an R-R-bimodule.

If R is a commutative ring (Definition C.0.1), then a left/right R-module can automatically be regarded as a two-sided R-module. As such, we simply talk about *R-modules* in this case.

Any abelian group is equivalent to a two-sided $\mathbb{Z}$-module. Moreover, any left R-module is equivalent to an $R-\mathbb{Z}$-bimodule (Definition C.0.11) and any right R-module is equivalent to an $\mathbb{Z} - R$-bimodule (Definition C.0.11). Given a left/right/two-sided R-module, its *natural bimodule structure* will refer to its structure as a R-$\mathbb{Z}$/$\mathbb{Z}$-R/R-R bimodule. In this way, many definitions associated with the notions of left/right/two-sided R-modules can be defined as special cases for definitions for R-S-bimodules.

Definition C.0.12 (Tensor product of bimodules)**.** Let R, S, T be (not necessarily commutative) rings (Definition C.0.8), let M be an R-S bimodule (Definition C.0.11), and let N be an S-T bimodule. In the free abelian group $\mathbb{Z}[M \times N]$ generated by the Cartesian product $M \times N$, let U be the subgroup generated by elements of the form (♠ TODO: subgroup generated)

$$\begin{aligned}&(m+m', n) - (m, n) - (m', n),\\&(m, n+n') - (m, n) - (m, n'),\\&(m \cdot s, n) - (m, s \cdot n),\end{aligned}$$

for all $m, m' \in M$, $n, n' \in N$, and $s \in S$. The *tensor product of M and N over S* is the quotient abelian group

$$M \otimes_S N := \mathbb{Z}[M \times N]/U.$$

The image of an element of the form $(m, n) \in M \times N$ in $M \otimes_S N$ is denoted $m \otimes n$ and called a *pure tensor*. In general, the elements of $M \otimes_S N$ are finite sums

$$\sum_{i=1}^{n} m_i \otimes n_i \quad m_i \in M, n_i \in N$$

of pure tensors. Thus, the pure tensors satisfy the following relations:

$$\begin{aligned}(m+m') \otimes n &= m \otimes n + m' \otimes n\\ m \otimes (n+n') &= m \otimes n + m \otimes n'\\ (m \cdot s) \otimes n &= m \otimes (s \cdot n)\end{aligned}$$

This tensor product becomes naturally an R-T bimodule with left action and right action defined by

$$\begin{aligned}r \cdot (m \otimes n) &= (r \cdot m) \otimes n,\\ (m \otimes n) \cdot t &= m \otimes (n \cdot t),\end{aligned}$$

for all $r \in R$, $t \in T$, $m \in M$, and $n \in N$.

Inductively, given rings $R_0, \ldots, R_k$ and $R_{i-1} - R_i$-bimodules M_i for $i = 1, \ldots, k$, we may speak of the tensor product

$$M_0 \otimes_{R_1} M_1 \otimes_{R_2} \cdots \otimes_{R_{k-1}} M_k;$$

tensor products are associative(♠ TODO:), so parentheses are not strictly needed to notate them. Its *pure tensors* are elements of the form $m_0 \otimes m_1 \otimes \cdots \otimes m_k$ for $m_i \in M_i$, and its general elements are finite sums

$$\sum_{j=1}^{n} m_{0j} \otimes m_{1j} \otimes \cdots m_{kj} \quad m_{ij} \in M_i.$$

of pure tensors. It also has a natural $R_0 - R_k$-bimodule structure.

In general, $(M_0, \ldots, M_k) \mapsto M_0 \otimes_{R_1} M_1 \otimes_{R_2} \cdots \otimes_{R_{k-1}} M_k$ defines a $(k+1)$-ary additive functor (Definition C.0.13)

$$ {}_{R_0}\mathbf{Mod}_{R_1} \times \cdots \times {}_{R_{k-1}}\mathbf{Mod}_{R_k} \to {}_{R_0}\mathbf{Mod}_{R_k} $$

(**??**).

Given a ring R and a two-sided R-module M, we may also speak of the *n-fold tensor product* $M^{\otimes n} = M^{\otimes_R n}$

Definition C.0.13 (n-ary Additive Functor)**.** Let I be a finite set with $|I| = n$. Let $\{\mathcal{A}_i\}_{i \in I}$ be additive categories and let $\mathcal{B}$ be an additive category. An *n-ary additive functor* (or *multilinear functor*)

$$ F : \prod_{i \in I} \mathcal{A}_i \to \mathcal{B} $$

is a functor such that for each fixed collection of all but one variable, the resulting functor in the remaining variable is additive. Equivalently, for every $j \in I$ and objects $(A_i)_{i \in I}$ and morphisms $f_1, f_2 : A_j \to A'_j$ in $\mathcal{A}_j$, we have

$$ \begin{aligned} &F(A_1, \ldots, A_{j-1}, f_1 + f_2, A_{j+1}, \ldots, A_n) \\ =&F(A_1, \ldots, A_{j-1}, f_1, A_{j+1}, \ldots, A_n) \\ &+ F(A_1, \ldots, A_{j-1}, f_2, A_{j+1}, \ldots, A_n), \end{aligned} $$

and F preserves zero morphisms componentwise:

$$ F(A_1, \ldots, 0_{A_j, A'_j}, \ldots, A_n) = 0_{F(A_1, \ldots), F(A'_1, \ldots)}. $$

A bifunctor that satisfies this property for $n = 2$ is simply called a *biadditive functor*.

Definition C.0.14. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of schemes (Definition .3.2).

- The morphism f is called an *open immersion* if the underlying map of topological spaces induces a homeomorphism from X onto an open subset $V \subseteq Y$, and the induced map of sheaves (Definition 3.0.2) $f^\sharp|_V : \mathcal{O}_Y|_V \to f_*\mathcal{O}_X$ is an isomorphism of sheaves of rings on V.
  (♠ TODO: surjective map of sheaves of sets)
- The morphism f is called a *closed immersion* if the underlying map of topological spaces induces a homeomorphism from X onto a closed subset $Z \subseteq Y$, and the induced map of sheaves $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective.

Definition C.0.15. Let $\mathcal{C}$ be a category (Definition .1.1), let Z be an object, and let X, Y be objects of $\mathcal{C}$ over (Definition A.0.5) Z, i.e. morphisms $X \to Z$ and $Y \to Z$ are fixed. A *cartesian product of X and Y over Z in $\mathcal{C}$* (or *fiber product* or *pullback diagram*) is an object, often denoted by $X \times_Z Y$, with *projection morphisms* $X \times_Z Y \to X$ and $X \times_Z Y \to Y$ that are universal. More precisely, for any object T of $\mathcal{C}$ and morphisms $f_X : T \to X$, $f_Y : T \to Y$, there exists a unique morphism $u : T \to X \times_Z Y$ such that the following diagram commutes:

$$\begin{tikzcd}
T \arrow[dr, dotted, "u"] \arrow[drr, bend left, "f_X"] \arrow[ddr, bend right, "f_Y"'] & & \\
& X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d] \\
& Y \arrow[r] & Z
\end{tikzcd}$$

Equivalently, $X \times_Z Y$ is the limit (Definition .1.12) of the diagram (Definition .1.25)

$$\begin{tikzcd}
& X \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}$$

in $\mathcal{C}$.

The commutative diagram

$$\begin{tikzcd}
X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}$$

may be referred to as a *cartesian square*.

Definition C.0.16 (Locally Noetherian Scheme and Noetherian Scheme)**.** Let X be a scheme (Definition .3.1).

- X is called *locally Noetherian* if it admits an open cover $\{U_i\}$ such that for each i, the ring $\mathcal{O}_X(U_i)$ of regular functions on U_i is a Noetherian ring. Equivalently, X is locally Noetherian if it is covered by open affine subschemes $\operatorname{Spec} A_i$ with each A_i a Noetherian ring.
- X is called *Noetherian* if it is locally Noetherian and quasi-compact, i.e., X can be covered by finitely many affine opens $\operatorname{Spec} A_i$ where each A_i is Noetherian.

Definition C.0.17 (Integral element over a ring)**.** Let R be a commutative ring with unity (Definition C.0.1).

1. Let A be an R-algebra. An element $a \in A$ is called *integral over R* if there exists a monic polynomial
$$p(x) = x^n + r_{n-1}x^{n-1} + \cdots + r_1 x + r_0$$
with coefficients $r_i \in R$ such that
$$p(a) = a^n + r_{n-1}a^{n-1} + \cdots + r_1 a + r_0 = 0 \quad \text{in } A.$$
2. Let A be an extension ring of R. The ring extension A/R is called an *integral extension* if every element of A is integral over R.

3. Let A be an extension ring of R. The *integral closure of R in A*, sometimes denoted by $\widetilde{A}$, is the subring
$$\widetilde{A} = \{a \in A : a \text{ is integral over } R\}.$$
We say that R is integrally closed in A if $\widetilde{A}$ coincides with A (considered as a subring of R).
4. Let R be an integral domain with field of fractions $K = \mathrm{Frac}(R)$. We say that R is *integrally closed* if it is integrally closed as a subring of K.

Definition C.0.18. Let X be a locally Noetherian scheme (Definition C.0.16).

- The scheme X is *regular at a point $x \in X$* (or *nonsingular at a point $x \in X$*) if the local ring $\mathcal{O}_{X,x}$ (Definition C.0.23) is a regular local ring. Otherwise, X is said to be *singular at x*, or synonymously at x is a *singularity of X*.
- The scheme X is called a *regular scheme* (or *nonsingular scheme*) if it is regular at every point $x \in X$. Otherwise, X is said to be *singular*.

Definition C.0.19. Let $f : X \to Y$ be a morphism of schemes (Definition .3.2). We say that f is a *finite type morphism* if for every affine open $V = \operatorname{Spec} B \subseteq Y$ with $U = f^{-1}(V)$ affine, say $U = \operatorname{Spec} A$, the ring A is a finitely generated B-algebra.

When X is equipped with a finite type morphism $f : X \to Y$, we say that X is a *finite type scheme over Y* or a *finite type Y-scheme* or a *Y-scheme of finite type* (Definition .3.3), etc.

Definition C.0.20 (Dimension of a Scheme)**.** Let X be a scheme with underlying topological space $|X|$.

(♠ TODO: krull dimension)

- The *dimension at a point $x \in |X|$*, denoted $\dim_x(X)$, is the Krull dimension of the local ring $\mathcal{O}_{X,x}$ (Definition C.0.23). This is the supremum of the lengths n of chains of prime ideals
$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subseteq \mathcal{O}_{X,x}.$$
- The *dimension of the scheme X* is defined as
$$\dim(X) := \sup_{x \in |X|} \dim_x(X).$$
Equivalently, it is the supremum of the lengths of chains of distinct irreducible closed subsets of $|X|$ ordered by inclusion.

Definition C.0.21. Let R be a ring (Definition C.0.8) with unity, not necessarily commutative. The ring R is called a *local ring* if it has a unique maximal left ideal (Definition C.0.4). In this case, R also has a unique maximal right ideal, and these coincide with the Jacobson radical $J(R)$ of R. The unique maximal left (and right) ideal of a local ring R may sometimes be denoted by $\mathfrak{m}_R$.

Definition C.0.22 (Presheaf on a topological space)**.** Let X be a topological space (Definition C.0.6). Let $\mathcal{D}$ be a category.

A *presheaf (of objects of $\mathcal{D}$/valued in $\mathcal{D}$) on X* is a rule $\mathcal{F}$ that assigns:

- to each open set $U \subseteq X$, an object $\mathcal{F}(U) \in \operatorname{Ob}\mathcal{D}$, called the *sections of $\mathcal{F}$ over U*,
- to each inclusion of open sets $V \subseteq U$, a morphism

$$\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V), \quad s \mapsto s|_V,$$

in the category $\mathcal{D}$ called the *restriction map* such that the following conditions hold:

- (Identity) For every open set $U \subseteq X$, the restriction map ρ_U^U is the identity on $\mathcal{F}(U)$.
- (Transitivity) For inclusions $W \subseteq V \subseteq U$ of open sets, one has

$$\rho_W^U = \rho_W^V \circ \rho_V^U.$$

For instance, we may speak of a *presheaf of sets/groups/rings/etc. on the topological space X*.

Equivalently, a presheaf on X (of objects in a category $\mathcal{D}$) is a functor

$$\mathbf{Open}(X)^{\mathrm{op}} \to \mathcal{D}$$

from the opposite of the category $\mathbf{Open}(X)$ of open subsets of X (see also Definition 3.0.1)
.

Equivalently, a presheaf on X is a presheaf on the category $\mathbf{Open}(X)$ in the sense of Definition 3.0.1 .

The sections object $\mathcal{F}(U)$ is also denoted by $\Gamma(U, \mathcal{F})$ (see Definition 6.1.1) . Moreover, the object $\mathcal{F}(X) = \Gamma(X, \mathcal{F})$ is called the *global sections object of $\mathcal{F}$*. This agrees with the notion of global sections as discussed in Definition 6.1.1.

Definition C.0.23 (Stalk of a sheaf)**.** Let X be a topological space, and let $\mathcal{D}$ be a category (Definition .1.1) Let $\mathcal{F}$ be a presheaf on X valued in $\mathcal{D}$ (Definition C.0.22). For a point $x \in X$, the *stalk of $\mathcal{F}$* at x, denoted $\mathcal{F}_x$, is defined as the direct limit (Definition .1.13)

$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U),$$

where the limit ranges over all open neighborhoods U of x in X ordered by inclusion, assuming that such a direct limit exists.

If $\mathcal{D}$ is some kind of category of sets (e.g. $\mathcal{D} = \mathbf{Sets}, \mathbf{Grps}, \mathbf{Rings}$), then an element of $\mathcal{F}_x$ is called a *germ of a section at x*. Concretely, a germ at x is given by a pair (U, s) with U an open neighborhood of x and $s \in \mathcal{F}(U)$, modulo the equivalence relation: $(U, s) \sim (V, t)$ if there exists an open neighborhood $W \subseteq U \cap V$ of x such that $s|_W = t|_W$.

If $\mathcal{F}$ is a sheaf of groups, rings, or modules, then each stalk $\mathcal{F}_x$ inherits the corresponding algebraic structure.

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