MANIFOLDS

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Contents

1. Basic definitions

1.1. Topological manifolds.

Definition 1.1.1 (Topological Manifold). 1. A *topological manifold of dimension n* is a Hausdorff, second-countable topological space (Definition 1.1.1) M such that each point $p \in M$ has an open neighborhood (Definition D.0.4) $U_p \subseteq M$ homeomorphic to an open subset of $\mathbb{R}^n$. That is, there exists a homeomorphism (a chart (Definition 1.1.2))

$$\varphi_p : U_p \to V_p \subseteq \mathbb{R}^n,$$

where V_p is open in $\mathbb{R}^n$. Some synonyms of the notion of "topological manifold" include: *manifold* or *real manifold*.

2. A *topological manifold with boundary of dimension n* is a Hausdorff (Definition D.0.7), second-countable (Definition D.0.8) topological space (Definition D.0.1) M such that each point $p \in M$ has an open neighborhood $U_p \subseteq M$ homeomorphic (Definition D.0.6) to an open subset of either $\mathbb{R}^n$ or the closed half-space (Definition D.0.3)

$$\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n \geq 0\}.$$

That is, there exists a homeomorphism (chart)

$$\varphi_p : U_p \to V_p,$$

where V_p is open in $\mathbb{R}^n$ or $\mathbb{H}^n$. A point $p \in M$ is called a *boundary point* if it corresponds under such a chart to a point in $\{x \in \mathbb{H}^n : x_n = 0\}$, and an *interior point* otherwise. The set of all boundary points of M is denoted by ∂M and is called the *boundary of M*. Boundary points and an interior points of M coincide with boundary points and interior points (Definition D.0.9) of M as a topological space. Some synonyms of the notion of "topological manifold with boundary" include: *manifold with boundary* or *real manifold with boundary*

By a *topological manifold without boundary*, we mean a topological manifold. In particular, we may speak of a *topological manifold with or without boundary*; many properties and attributes can be spoken for both topological manifolds and topological manifolds without boundary.

Note that by the above standard definition, every topological manifold is technically a topological manifold with boundary (but not vice versa). Thus, in principal, definitions concerning topological manifolds with boundary should be applicable to topological manifolds without boundary.

Definition 1.1.2 (Chart).

1. A *(coordinate) chart on a topological manifold M of dimension n* is a pair (U, φ) where
   - $U \subseteq M$ is an open subset;
   - $\varphi : U \to V \subseteq \mathbb{R}^n$ is a homeomorphism (Definition D.0.6) onto an open subset V of $\mathbb{R}^n$.

   The map φ is called a *coordinate map*, and the image $\varphi(p)$ of a point $p \in U$ gives the *coordinates of p* in this chart.
2. A *(coordinate) chart on a topological manifold with boundary M of dimension n* is a pair (U, φ) where
   - $U \subseteq M$ is an open subset;
   - $\varphi : U \to V \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$ is a homeomorphism onto an open subset V of either $\mathbb{R}^n$ or the closed half-space $\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n \geq 0\}$ (Definition D.0.3); equivalently, we may just specify φ to be a homeomorphism onto an open subet of $\mathbb{H}^n$.

   The map φ is called a *coordinate map*, and the image $\varphi(p)$ of a point $p \in U$ gives the *coordinates of p* in this chart. A point $p \in M$ is called a *boundary point* if for some chart (U, φ) containing p, the coordinates satisfy $\varphi(p)_n = 0$; otherwise, p is an *interior point*. Boundary points and an interior points of M coincide with boundary points and interior points (Definition D.0.9) of M as a topological space.

Definition 1.1.3 (Atlas)**.** An *atlas* on a topological manifold M with or without boundary (Definition 1.1.1) of dimension n is a collection $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ of charts such that the sets U_α cover M, i.e.

$$\bigcup_{\alpha \in A} U_\alpha = M.$$

Two atlases are said to be *compatible* if their union is also an atlas. An atlas is said to be *maximal* if it is not properly contained in any larger atlas.

Definition 1.1.4 (Chart Transition Map)**.** Given two charts (U, φ) and (V, ψ) on a topological manifold with boundary (Definition 1.1.1) M such that $U \cap V \neq \emptyset$, the *chart transition map* or *change of coordinates map* (from U to V) is the map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V),$$

which is a homeomorphism (Definition D.0.6) between open subsets of $\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n \geq 0\}$.

In particular, we may speak of this notion when M is a topological manifold without boundary (Definition 1.1.1).

1.2. C^k **manifolds.** C^k-manifolds are manifolds whose transition maps are C^k-maps between open subsets of Euclidean spaces.

Definition 1.2.1. Let $U \subseteq \mathbb{H}^n$ and $V \subseteq \mathbb{H}^m$ be open subsets of the closed half-spaces $\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n \geq 0\}$ and $\mathbb{H}^m = \{y \in \mathbb{R}^m : y_m \geq 0\}$ respectively, and let $k \in \mathbb{N}_0 \cup \{\infty\}$ be fixed.

1. A *C^k-morphism* (or *C^k-map*) from U to V is a function $f : U \to V$ such that f extends to a k-times continuously differentiable function on an open neighborhood of U in $\mathbb{R}^n$. Equivalently, all partial derivatives
$$D^\alpha f : U \to \mathbb{R}^m, \quad |\alpha| \leq k,$$
exist and are continuous up to the boundary on U. We denote the set of all such maps by $C^k(U, V)$.
2. A *C^k-function on U* is a C^k-map from U to $\mathbb{R}$. We let $C^k(U)$ denote the space of C^k-functions on U.
3. A *C^k-diffeomorphism* $f : U \to V$ is a C^k-morphism that is a bijection whose inverse f^{-1} is also a C^k-morphism between open subsets of half-spaces.
4. A *smooth morphism/map/function/diffeomorphism* is one that is C^∞ in the above sense.

In fact, when U and V are open subsets of $\mathbb{R}^n$ without boundary points, the notions of C^k-morphisms coincide with the classical ones of k-times continuously differentiable maps between open subsets of $\mathbb{R}^n$. In this case, the extension condition is trivially satisfied by restricting to an open neighborhood in $\mathbb{R}^n$. For open subsets with boundary points in $\mathbb{H}^n$, the extension requirement ensures differentiability up to the boundary.

Definition 1.2.2. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. Let M be a topological manifold (Definition 1.1.1) (with or without boundary) of dimension n, and let $k \in \mathbb{N}_0 \cup \{\infty\}$. Two charts (Definition 1.2.2) (U, φ) and (V, ψ) on M are said to be *C^k-compatible* (or *smoothly compatible* if $k = \infty$) if the transition maps

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

$$\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$$

are C^k-maps between open subsets of $\mathbb{H}^n$ (Definition 1.2.1); in particular, both transition maps are C^k-diffeomorphisms (Definition 1.2.1).

Definition 1.2.3. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ be fixed. Let M be a topological manifold with or without boundary (Definition 1.1.1) of dimension n.

A *C^k-atlas* (or *smooth atlas* if $k = \infty$) on M is an atlas (Definition 1.1.3)

$$\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$$

such that for every pair $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$, the charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are C^k-compatible (Definition 1.2.2).

Definition 1.2.4. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ be fixed. An *n-dimensional C^k/k-differentiable-(real)manifold with boundary (resp. without boundary)* is a pair $(M, \mathcal{A})$, where M is a topological manifold with boundary (resp. without boundary) (Definition 1.1.1) of dimension n and $\mathcal{A}$ is a C^k-atlas (Definition 1.2.3) on M.

The atlas $\mathcal{A}$ is usually taken to be maximal (Definition 1.1.3) with respect to C^k-compatibility (Definition 1.2.2), meaning it contains every C^k-chart compatible with all charts in $\mathcal{A}$.

Note that a C^0-manifold is simply a topological manifold (Definition 1.1.1) and that a C^∞-manifold is synonymously referred to as a *smooth/differentiable (real) manifold*.

Convention 1.2.5. In manifold theory, there are many notions describable or definable via C^k, i.e. k-differentiability. In the case of $k = 0$, the adjective/adverb of C^0 is omitted. In the case of $k = \infty$, one can synonymously describe that notion as "smooth" or simply "differentiable". In other cases, one can say "k-differentiable" instead of "C^k". For example, a C^0-manifold is simply a (real) topological manifold, a k-differentiable manifold refers to a C^k-manifold, and a smooth/differentiable (real) manifold refers to a C^∞-manifold.

Definition 1.2.6. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ be fixed. Let $(M, \mathcal{A}_M)$ and $(N, \mathcal{A}_N)$ be C^k-manifolds with boundary (Definition 1.2.4) of dimensions n and m, respectively, where M, N are topological manifolds with boundary (Definition 1.1.1) and $\mathcal{A}_M$ and $\mathcal{A}_N$ are C^k-atlases (Definition 1.2.3) whose charts map to open subsets of the closed half-spaces (Definition D.0.3) $\mathbb{H}^n$ and $\mathbb{H}^m$.

A *C^k-morphism* (or *C^k-map*) between M and N is a continuous map (Definition D.0.5)

$$f : M \to N$$

such that for every $p \in M$ there exist charts $(U, \varphi) \in \mathcal{A}_M$ with $p \in U$ and $(V, \psi) \in \mathcal{A}_N$ with $f(p) \in V$ satisfying

$$\psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \to \psi(V)$$

is a C^k-map (Definition 1.2.1) between open subsets of the closed half-spaces $\mathbb{H}^n$ and $\mathbb{H}^m$, i.e.,

$$\psi \circ f \circ \varphi^{-1} \in C^k(\varphi(U \cap f^{-1}(V)), \psi(V)).$$

If f is a homeomorphism and its inverse $f^{-1} : N \to M$ is also a C^k-morphism, then f is called a *C^k-diffeomorphism*. We let $C^k(M, N)$ denote the space of C^k-maps $M \to N$. We let $C^k(M)$ denote the space of *C^k-functions*, i.e., the C^k-maps $M \to \mathbb{R}$.

In particular, we may speak of these notions when M and N are C^k-manifolds without boundary (Definition 1.2.4).

Remark 1.2.7. The notations $C^k(M, N)$ (and $C^k(M)$) agrees with the usual notations $C^k(M, N)$ and $C^k(M)$ in the case that M is an open subset of $\mathbb{R}^n$ (Definition 1.2.1).

1.3. **(Real) C^k-vector bundles on (real) C^k-manifolds.**

Definition 1.3.1. Let $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, and let M be a C^k manifold with or without boundary (Definition 1.2.4) of dimension m. A *C^k vector bundle of rank r over M* is a triple (E, π, M) where:

- E is a topological space (Definition D.0.1) called the *total space*,
- $\pi : E \to M$ is a continuous (Definition D.0.5) surjection (Definition A.0.1) called the *projection map*,
- For each $p \in M$, the fiber (Definition D.0.13) $E_p := \pi^{-1}(\{p\})$ is endowed with the structure of a vector space (Definition B.0.1) over $\mathbb{R}$ of dimension r,
- There exists an open cover (Definition D.0.10) $\{U_\alpha\}_{\alpha \in A}$ of M by open sets, and homeomorphisms (Definition D.0.6) (called *local trivializations*)

  $$\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$$

  such that:
  - Each ϕ_α is a C^k diffeomorphism (Definition 1.2.1) onto its image, where U_α is identified with an open subset of $\mathbb{R}^m_+$,
  - For every $p \in U_\alpha$, the restriction

    $$\phi_\alpha|_{E_p} : E_p \to \{p\} \times \mathbb{R}^r \cong \mathbb{R}^r$$

    is a vector space isomorphism (Definition B.0.3),
- For all indices α, β, define the *transition functions*

  $$t_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})$$

  uniquely by the relation

  $$\phi_\alpha \circ \phi_\beta^{-1}(p, v) = (p, t_{\alpha\beta}(p)v) \quad \text{for } p \in U_\alpha \cap U_\beta,\ v \in \mathbb{R}^r.$$

  Each $t_{\alpha\beta}$ is a C^k map respecting the boundary structure.

The total space E then in fact has a canonical structure as a C^k-manifold (without boundary if M is a C^k manifold without boundary) (Theorem 1.3.2)

Let E be a C^k vector bundle over a C^k manifold with boundary M. A *C^k-section of E over an open subset $U \subseteq M$* (where U may intersect the boundary) is a C^k map (Definition 1.2.6) $s : U \to E$ such that $\pi \circ s = \mathrm{id}_U$. We might denote by

$$\Gamma^{C^k}(U, E) = \Gamma^{C^k}(U, E; \mathbb{R}) = E^{C^k}(U) = E^{C^k}(U; \mathbb{R})$$

the space of C^k sections of E (as a vector space of M). It is a real vector space. When k is self-apparent, this space may also be without the superscript of C^k, i.e. by

$$\Gamma(U, E) = \Gamma(U, E; \mathbb{R}) = E(U) = E(U; \mathbb{R}).$$

A C^k-section of E over M itself may be referred to as a *global C^k-section of E*; the space of such sections may be shorthand-notated as $\Gamma_k(E)$, $\Gamma(E)$, or $\Gamma_k(E; \mathbb{R})$.

A C^0-section of E is simply called a *(continuous) section of E*, and a C^∞-section of E is called a *smooth section of E*.

Theorem 1.3.2 (Total space of a C^k vector bundle as a C^k manifold)**.** Let $k \in \mathbb{N} \cup \{\infty\}$, let M be a C^k manifold with or without boundary of dimension n, and let

$$\pi : E \to M$$

be a C^k vector bundle of rank r over M. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of M together with local trivializations (Definition 1.3.1)

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r.$$

Fix a C^k atlas (Definition 1.2.3) $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ on M, where each

$$\varphi_\alpha : U_\alpha \xrightarrow{\cong} V_\alpha \subseteq \mathbb{R}^n$$

is a homeomorphism onto an open subset V_α.

Define charts (Definition 1.2.2) on E by

$$\Psi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow V_\alpha \times \mathbb{R}^r, \qquad \Psi_\alpha(e) := \big(\varphi_\alpha(\pi(e)),\, v_\alpha(e)\big),$$

where $v_\alpha(e) \in \mathbb{R}^r$ is determined by the identity

$$\Phi_\alpha(e) = \big(\pi(e),\, v_\alpha(e)\big).$$

Then the following hold.

1. Each Ψ_α is a homeomorphism (Definition D.0.6) from $\pi^{-1}(U_\alpha)$ onto the open subset $V_\alpha \times \mathbb{R}^r$ of $\mathbb{R}^{n+r}$.
2. For any $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \varnothing$, the chart transition map (Definition 1.1.4)
$$\Psi_\beta \circ \Psi_\alpha^{-1} : \big(V_\alpha \cap \varphi_\alpha(U_\alpha \cap U_\beta)\big) \times \mathbb{R}^r \longrightarrow \big(V_\beta \cap \varphi_\beta(U_\alpha \cap U_\beta)\big) \times \mathbb{R}^r$$
is given by
$$(u, v) \longmapsto \Big(\varphi_\beta \circ \varphi_\alpha^{-1}(u),\, g_{\beta\alpha}\big(\varphi_\alpha^{-1}(u)\big)\, v\Big),$$

and is of class C^k (Definition 1.2.1), since $\varphi_\beta \circ \varphi_\alpha^{-1}$ and $g_{\beta\alpha}$ are C^k.

3. Hence the collection of charts

$$\left\{\left(\pi^{-1}(U_\alpha), \Psi_\alpha\right)\right\}_{\alpha \in A}$$

is a C^k atlas (Definition 1.2.3) on E, making E into a C^k manifold (Definition 1.2.4) (without boundary if M is without boundary) of dimension $n + r$.

4. With this C^k manifold structure on E, the projection

$$\pi : E \to M$$

(♠ TODO: submersion) is a C^k submersion.

In particular, the total space (Definition 1.3.1) of a C^k vector bundle of rank r over an n-dimensional C^k manifold is canonically a C^k manifold of dimension $n + r$.

Definition 1.3.3 (Morphism of C^k vector bundles over a C^k manifold with or without boundary)**.** Let $k \in \mathbb{N} \cup \{\infty\}$, let M be a C^k manifold with or without boundary (Definition 1.2.4), and let

$$\pi_E : E \to M, \qquad \pi_F : F \to M$$

be C^k vector bundles (Definition 1.3.1) of ranks r_E and r_F over M. A *morphism of C^k vector bundles over M* (or a *C^k vector bundle morphism covering the identity of M*) is a map $\Phi : E \to F$ satisfying:

1. Φ is of class C^k as a map (Definition 1.2.6) between C^k manifolds (with or without boundary); recall that E has the structure of a C^k-manifold (Theorem 1.3.2),
2. $\pi_F \circ \Phi = \pi_E$ (i.e. Φ is fiber-preserving over the identity on M),
3. for each $x \in M$, the restriction

$$\Phi_x := \Phi\big|_{E_x} : E_x \longrightarrow F_x$$

(Definition 1.3.1) is a linear map (Definition B.0.3) of real vector spaces.

In this situation Φ is also called a *bundle map* (of class C^k) from E to F over M.

Definition 1.3.4. Let $k \in \mathbb{N}_0 \cup \{\infty\}$, let M be a C^k-manifold with or without boundary (Definition 1.2.4), and let (E, π, M) be a C^k vector bundle of rank r over M. The *dual bundle of E* is the C^k vector bundle (E^*, π_{E^*}, M) defined of rank r as follows:

- The total space (Definition 1.3.1)

$$E^* := \bigsqcup_{p \in M} E_p^*$$

is the disjoint union of the dual vector spaces $E_p^* := \mathrm{Hom}_{\mathbb{R}}(E_p, \mathbb{R})$ (Definition C.0.9) of the fibers (Definition D.0.13) E_p.

- The projection map (Definition 1.3.1)

$$\pi_{E^*} : E^* \to M$$

sends each $\varphi \in E_p^*$ to its base point $p \in M$.

- If $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$ is a system of local trivializations (Definition 1.3.1) of E over coordinate charts (Definition 1.1.2) (U_α, ψ_α) compatible with the manifold structure (allowing charts modeled on open subsets of $\mathbb{R}^n_+$ near ∂M), where
$$\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r,$$
then the *induced local trivializations of E^** are defined by
$$\phi^*_\alpha : \pi^{-1}_{E^*}(U_\alpha) \to U_\alpha \times (\mathbb{R}^r)^*,$$
determined fiberwise by dualizing the linear isomorphisms
$$(\phi_\alpha)|_{E_p} : E_p \to \{p\} \times \mathbb{R}^r,$$
and hence $(\mathbb{R}^r)^* \cong \mathbb{R}^r$.
- The transition functions (Definition 1.3.1) of E^* relative to these trivializations are given by the inverse transpose of those of E:
$$t^*_{\alpha\beta}(p) = (t_{\alpha\beta}(p)^T)^{-1}, \quad p \in U_\alpha \cap U_\beta,$$
where $t_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})$ are the C^k transition functions of E.

Definition 1.3.5. Let M be a C^k-manifold with or without boundary (Definition 1.2.4) and let $E \to M$ be a C^k vector bundle.

1. For any (C^k-)section (Definition 1.3.1) $s \in \Gamma(E)$, define its *support*
$$\mathrm{supp}(s) = \overline{\{x \in M : s(x) \neq 0\}}$$
where 0 denotes the zero vector in the fiber (Definition D.0.13) E_x and the closure (Definition D.0.11) is taken in M.
2. A section $s \in \Gamma(E)$ is said to be *compactly supported* if its support is compact (Definition D.0.12).
3. Let $\Gamma_c(E) = \Gamma_c(M) = \Gamma_c(M; \mathbb{R})$ denote the space of compactly supported sections, i.e.
.
$$\Gamma_c(E) = \{s \in \Gamma(E) : \mathrm{supp}(s) \text{ is compact}\}$$

Lemma 1.3.6. Let M be a C^k-manifold with or without boundary and let $E \to M$ be a C^k vector bundle. If M is compact (Definition D.0.12), then $\Gamma_c(E) = \Gamma(E)$ (Definition 1.3.5).

1.4. **Tangent and cotangent bundles of (real) C_k manifolds.** In general, we may speak of tangent/cotangent bundles of C^k-manifolds; these have natural structures as C^{k-1}-manifolds. We are most interested in the case that $k = \infty$.

Definition 1.4.1. Let M be a topological manifold of dimension n with or without boundary (Definition 1.1.1), and let $p \in M$ be a point such that there is an open neighborhood of M that is a C^1-manifold (Definition 1.2.4) as a submanifold of M.

1. The *tangent space of M at the point p*, denoted T_pM, is defined as follows: (♠ TODO: TODO: define the derivative/jacobian matrix of a self map of R^n) (♠ TODO: TODO: justify why having the derivative of the transition map between charts implies that T_pM is well defined and independent of the choice of chart.) Choose a chart (Definition 1.1.2) (U, φ) around p with $p \in U \subseteq M$, where $\varphi : U \to V$ is a homeomorphism

(Definition D.0.6) onto an open subset $V \subseteq \mathbb{R}^n$ or the upper half-space $\mathbb{H}$ (Definition D.0.3) (when p is a boundary point).

We identify T_pM with the vector space $\mathbb{R}^n$ via the differential of φ at p. More precisely,

$$T_pM := \{\, (U, \varphi), v \mid v \in \mathbb{R}^n \,\}/\sim$$

where two pairs $((U, \varphi), v)$ and $((U', \varphi'), v')$ are equivalent if $p \in U \cap U'$ and

$$v' = d(\varphi' \circ \varphi^{-1})_{\varphi(p)}(v),$$

with $d(\varphi' \circ \varphi^{-1})_{\varphi(p)} : \mathbb{R}^n \to \mathbb{R}^n$ the derivative (Jacobian matrix) of the transition map at $\varphi(p)$.

A *tangent vector of M at p* is then an element of T_pM.

2. The *cotangent space of M at p*, denoted T_p^*M, is the dual space $(T_pM)^* = \mathrm{Hom}_{\mathbb{R}}(T_pM, \mathbb{R})$ (Definition C.0.9).

   A *cotangent vector of M at p* is then an element of T_p^*M.

Definition 1.4.2. Let $k \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$, and let M be a C^k n-dimensional manifold with or without boundary (Definition 1.2.4).

The *tangent bundle of M* is the vector bundle (Definition 1.3.1)

$$(TM, \pi, M)$$

where:

- The total space (Definition 1.3.1)

  $$TM := \bigsqcup_{p \in M} T_pM$$

  is the disjoint union of tangent spaces (Definition 1.4.1) of M at all points, defined via equivalence classes of C^k-compatible curves or derivations of C^k functions at p, including points at the boundary.
- The projection map

  $$\pi : TM \to M$$

  sends each tangent vector to its base point.
- Locally, for any chart (Definition 1.1.2) (U, φ) on M with

  $$\varphi : U \to V \subset \mathbb{R}^n \text{ or } \varphi : U \to V \subset \mathbb{H} \text{ if } U \text{ contians boundary points},$$

  the tangent bundle trivializes (Definition 1.4.2) as

  $$\pi^{-1}(U) \cong U \times \mathbb{R}^n.$$

  This reflects the identification

  $$T_pM \cong \mathbb{R}^n$$

  via the differential (pushforward) of the C^k chart φ.

The total space TM carries a natural C^{k-1} vector bundle structure over M and hence (Theorem 1.3.2) a C^{k-1} manifold structure (without boundary if M is without boundary), and the projection π is a C^{k-1} map (Definition 1.2.6).

Definition 1.4.3. Let $k \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$, and let M be a C^k n-dimensional manifold.

The *cotangent bundle of M* is the dual vector bundle

$$(T^*M, \pi_{T^*M}, M)$$

of the tangent bundle TM. It is also denoted by ΩM. In particular, ΩM carries a carries a natural C^{k-1} vector bundle structure over M and hence (Theorem 1.3.2) a C^{k-1} manifold structure (without boundary if M is without boundary), cf. Definition 1.4.2.

1.5. **Exterior powers of real vector bundles.** In general, one can define exterior powers of (real) vector bundles on real manifolds. A case of interest would be the exterior powers of the (co)tangent bundle of a smooth manifold.

Definition 1.5.1. Let R be a (not necessarily commutative) ring (Definition C.0.1), and let M an two-sided R-module.

1. The *symmetric power of M of degree n*, denoted by $S^n_R(M)$ or $\mathrm{Sym}^n(M) = \mathrm{Sym}^n_R(M)$, is the quotient two-sided module (Definition D.0.16)
$$S^n_R(M) := M^{\otimes_R n}/I_{\mathrm{sym}},$$
(Definition D.0.15) where I_{sym} is the two-sided (Definition C.0.5) submodule of $M^{\otimes_R n}$ generated by (Definition D.0.14) all elements of the form
$$x_1 \otimes \cdots \otimes x_n - (x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}), \qquad x_i \in M,\ \sigma \in \mathfrak{S}_n.$$
2. The *exterior power of M of degree n*, denoted by $\Lambda^n_R(M)$, is the quotient two-sided module (Definition D.0.16)
$$\Lambda^n_R(M) := M^{\otimes_R n}/I_{\mathrm{alt}},$$
where I_{alt} is two-sided submodule of $M^{\otimes_R n}$ generated by (Definition D.0.14) all elements of the form
$$x_1 \otimes \cdots \otimes x_n - \mathrm{sgn}(\sigma)\,(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}), \qquad x_i \in M,\ \sigma \in \mathfrak{S}_n.$$

In particular, we often speak of symmetric powers of exterior powers of modules over commutative rings (Definition C.0.3) and even vector spaces (Definition B.0.1) over fields (Definition C.0.4).

Definition 1.5.2. (♠ TODO: do this definition for manifolds with boundary) Let $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Let $\pi : E \to M$ be a real C^k-vector bundle of rank r over a C^k-topological manifold (Definition 1.2.4) M with or without boundary, and let $l \in \{0, 1, \ldots, r\}$.

(♠ TODO: TODO: describe how to define a vector bundle fiberwise) The *lth exterior power of E*, denoted by $\bigwedge^l E$, is the vector bundle over M defined fiberwise by

$$(\bigwedge^l E)_p := \bigwedge^l (E_p)$$

for each $p \in M$, where $E_p = \pi^{-1}(\{p\})$ is the fiber of E at p and $\bigwedge^l(E_p)$ is the lth exterior power of the vector space (Definition 1.5.1) E_p.

The total space of $\bigwedge^l E$ is given the unique vector bundle structure for which local trivializations of E induce local trivializations

$$\bigwedge^l \Phi_U : \pi^{-1}(U) \to U \times \bigwedge^l (\mathbb{R}^r)$$

making $\bigwedge^l E$ a vector bundle of rank $\binom{r}{l}$ over M.

Proposition 1.5.3. Let M be a n-dimensional (real) C^k-manifold for $k \in \mathbb{N}_0 \cup \{\infty\}$ and let E be a C^k-vector bundle of rank r on M. The l-th exterior power $\bigwedge^l E$ is a C^k-vector bundle of rank $\binom{r}{k}$.

1.6. k-forms on a smooth real manifold.

Definition 1.6.1. Let $k \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$, let M be a n-dimensional C^k-manifold with or without boundary, and let $l \in \{0, \ldots, n\}$. Recall that TM and T^*M carry C^{k-1}-vector bundle structures over M and hence are C^{k-1}-manifolds themselves.

The lth exterior power $\bigwedge^l \Omega M$ of the cotangent bundle $T^*M = \Omega M$ is often denoted by $\Omega^l M$ and referred to as the *bundle of l-forms on M*; if $k = \infty$, then we might refer to such a l-form as a *smooth/differentiable l-form*.

We use notation such as $\Omega^l(M)$ and $\Omega^l(M;\mathbb{R})$ to denote a space of sections over M, i.e. $\Gamma(M, \Omega^l M)$, and call sections of such a space a *l-form on M*. — unless otherwise specified, we take $\Omega^l(M)$ be the space $\Gamma^{C^{k-1}}(M, \Omega^l M)$ of C^{k-1} l-forms.

In other words, a l-form on M is a C^{k-1}-section ω of $\Omega^l M$, i.e. a C^{k-1} assignment $p \mapsto \omega_p$ where $\omega_p : (T_pM)^l \to \mathbb{R}$ is an alternating (Definition D.0.18) multilinear form (Definition D.0.17).

By convention, we let $\Omega^0(M)$ equal the space $C^{k-1}(M)$ of C^{k-1} real-valued functions on M.

Let $\Omega^l_c(M) = \Omega^l_c(M;\mathbb{R}) \subseteq \Omega^l(M)$ be the space of compactly supported (Definition 1.3.5) (C^{k-1}) l-forms on M, i.e.

$$\Omega^l_c(M) = \Omega^l_c(M;\mathbb{R}) := \Gamma_c\left(M, \Omega^l M\right).$$

(♠ TODO: TODO: read the following four definitions)

Definition 1.6.2. Let $k \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$, let M be a n-dimensional C^k manifold with or without boundary, and let $U \subseteq M$ be open.

For each $0 \leq m$, the *exterior derivative* is a map

$$d : \Omega^m(U) \to \Omega^{m+1}(U)$$

with $\Omega^m(U)$ and $\Omega^{m+1}(U)$ consisting of C^{k-1}-sections (Definition 1.3.1) of $\Omega^m M$ and $\Omega^{m+1} M$ over U respectively (Definition 1.6.1); we note however, that the range of d consists of C^{k-2}-sections of $\Omega^{m+1} M$. The exterior derivative is the unique $\mathbb{R}$-linear map satisfying:

- For any function $f \in C^{k-1}(U) = \Omega^0(U)$, the exterior derivative $df \in \Omega^1(U)$ is the usual differential of f: for every smooth vector field X on U,
$$df(X) = X(f),$$
and, in local coordinates $(x^1, \ldots, x^n)$, (♠ TODO: partial derivatives)
$$df = \frac{\partial f}{\partial x^1}dx^1 + \cdots + \frac{\partial f}{\partial x^n}dx^n.$$
- For all $\omega \in \Omega^m(U)$, $\eta \in \Omega^n(U)$,
$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^m \omega \wedge d\eta$$
(graded Leibniz rule).
- Assuming that $k \geq 3$, for all $\omega \in \Omega^m(U)$,
$$d(d\omega) = 0.$$

Moreover, if $\omega \in \Omega_c^m(U)$, then $d\omega \in \Omega_c^{m+1}(U)$, i.e., the exterior derivative preserves compactly supported forms (Definition 1.3.5).

Definition 1.6.3. Let $k \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$, let M be a C^k manifold with or without boundary, and $U \subset M$ open.

1. The *de Rham complex* on U is the sequence of $\mathbb{R}$-vector spaces and maps:
$$0 \to \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{\min(\dim M, k-1)}(U) \to 0$$
where each d is the exterior derivative.
2. The *de Rham complex of compactly supported differential forms* on U is the sequence of $\mathbb{R}$-vector spaces and maps:
$$0 \to \Omega_c^0(U) \xrightarrow{d} \Omega_c^1(U) \xrightarrow{d} \Omega_c^2(U) \xrightarrow{d} \cdots \xrightarrow{d} \Omega_c^{\min(\dim M, k-1)}(U) \to 0$$
where each d is the exterior derivative as in the previous notation, and the image of d is contained in forms with compact support because d preserves compact support.

Both are chain complexes (Definition D.0.19) of $\mathbb{R}$-vector spaces.

2. Complex Manifolds

Definition 2.0.1 (Holomorphic function from a subset of $\mathbb{C}^n$ to $\mathbb{C}^m$)**.** Let $n, m \geq 1$ be integers, and consider $\mathbb{C}^n$ and $\mathbb{C}^m$ as complex vector spaces endowed with their standard Euclidean topologies. Let $U \subseteq \mathbb{C}^n$ be an open subset, and let $f : U \to \mathbb{C}^m$ be a function.

1. We say that f is *holomorphic/(complex) analytic/complex differentiable function at a point $a \in U$* if there exists a $\mathbb{C}$-linear map
$$L_a : \mathbb{C}^n \to \mathbb{C}^m$$

(♠ TODO: norms on $\mathbb{C}^n$ are equivalent to one another) such that, for some (and hence any) norm $\|\cdot\|$ on $\mathbb{C}^n$ and $\mathbb{C}^m$, one has

$$\lim_{h\to 0} \frac{\|f(a+h) - f(a) - L_a(h)\|}{\|h\|} = 0,$$

where the limit is taken over $h \in \mathbb{C}^n$ with $a + h \in U$. In this case, the map L_a is uniquely determined and is called the *complex derivative* (or *complex Fréchet derivative*) of f at a.

In the case of $n, m = 1$, we f is holomorphic at a if and only if the limit

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists in $\mathbb{C}$. In that case the limit is denoted $f'(z_0)$ and is called the *complex derivative of f at z_0*.

the value $L_a(0)$ complex derivative

2. The function f is called *holomorphic/analytic on U* if it is holomorphic at every point $a \in U$.
3. A holomorphic function $\mathbb{C}^n \to \mathbb{C}^m$ is called *entire*.

Definition 2.0.2 (Biholomorphic map of subsets of $\mathbb{C}$)**.** Let $U \subseteq \mathbb{C}^n$ and $V \subseteq \mathbb{C}^m$ be open subsets for integer $n, m \geq 1$. A map $f : U \to V$ is called a *biholomorphic map of subsets of $\mathbb{C}$*, or simply a *biholomorphism between U and V*, if:

1. f is bijective (Definition A.0.1) as a map of sets,
2. f and f^{-1} (Definition A.0.1) are holomorphic (Definition 2.0.1) on U.

In this case, we say that U and V are *(complex-)analytically isomorphic as open subsets of $\mathbb{C}$*. Moreover, n and m must necessarily be equal in this case.

Definition 2.0.3 (Complex Manifold)**.**

1. A *complex (analytic) manifold M of complex dimension n (without boundary)* is a topological manifold (Definition 1.1.1) of dimension $2n$ equipped with an atlas (Definition 1.1.3) of charts (Definition 1.1.2) $\{(U_\alpha, \varphi_\alpha)\}$ where:
   - Each chart $\varphi_\alpha : U_\alpha \to \Omega_\alpha \subseteq \mathbb{R}^{2n}$ is regarded as a homeomorphism (Definition D.0.6) onto an open subset of $\mathbb{C}^n$ by homeomorphically identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$,
   - The transition maps (Definition 1.1.4) $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ are biholomorphic maps (Definition 2.0.2) between open subsets of $\mathbb{C}^n$.

   An equivalent definition of a complex manifold M of complex dimension n is a Hausdorff (Definition D.0.7) second countable (Definition D.0.8) topological space equipped with an atlas of charts $\{(U_\alpha, \varphi_\alpha)\}$ where:
   - Each $U_\alpha \subseteq M$ is open and covers M,
   - Each chart $\varphi_\alpha : U_\alpha \to \Omega_\alpha \subseteq \mathbb{C}^n$ is a homeomorphism (Definition D.0.6) onto an open subset of $\mathbb{C}^n$,
   - The transition maps $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ are biholomorphic maps (Definition 2.0.2) between open subsets of $\mathbb{C}^n$.

Equivalently, M is a locally ringed space locally isomorphic to $(\Omega, \mathcal{O}_\Omega)$ where $\Omega \subseteq \mathbb{C}^n$ is open and $\mathcal{O}_\Omega$ is the sheaf of holomorphic functions.

2. A *complex manifold M of complex dimension n with boundary* is a topological manifold with boundary (Definition 1.1.1) of dimension $2n$ eqiupped with an atlas (Definition 1.1.3) of charts (Definition 1.1.2) $\{(U_\alpha, \varphi_\alpha)\}$ where
   - Each chart $\varphi_\alpha : U_\alpha \to \Omega_\alpha \subseteq \mathbb{R}^{2n}_+$, which is a map into the $2n$-dimensional Euclidean closed half-space (Definition D.0.3) $\mathbb{R}^{2n}_+$, is regarded as a homeomorphism (Definition D.0.6) onto an open subset of the complex closed upper half-space $\mathbb{H}^n$ by homeomorphically identifying $\mathbb{H}^n$ with $\mathbb{R}^{2n}_+$,
   - The transition maps (Definition 1.1.4) $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ are extendable to biholomorphic maps (Definition 2.0.2) between open subsets of $\mathbb{C}^n$. More precisely, there exist open subsets $\Omega_\alpha, \Omega_\beta$ of $\mathbb{C}^n$ containing $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ respectively and a biholomorphic map $f_{\alpha\beta} : \Omega_\alpha \to \Omega_\beta$ such that $f_{\alpha\beta}$ restricts to $\varphi_\beta \circ \varphi_\alpha^{-1}$ on $\varphi_\alpha(U_\alpha \cap U_\beta)$.

Definition 2.0.4. Let M be a smooth manifold of real dimension $2n$ with or without boundary. An *almost complex structure on M* is a smooth vector bundle endomorphism (Definition 1.3.3)

$$J : TM \to TM$$

of the tangent bundle TM (Definition 1.4.2) of M such that

$$J^2 = -\mathrm{id}_{TM}.$$

Definition 2.0.5. (♠ TODO: TODO: Lie bracket of vector fields) Let M be a smooth manifold of real dimension $2n$ with or without boundary equipped with an almost complex structure

$$J : TM \to TM, \quad J^2 = -\mathrm{id}_{TM}.$$

We say that J is *integrable* if the Nijenhuis tensor

$$N_J(X, Y) := [X, Y] + J([JX, Y]) + J([X, JY]) - [JX, JY]$$

vanishes identically for all smooth vector fields $X, Y \in \Gamma(TM)$, where $[\cdot, \cdot]$ is the Lie bracket of vector fields.

Proposition 2.0.6. Let M be a topological manifold without boundary (Definition 1.1.1). The following are equivalent:

1. M is a complex manifold (Definition 2.0.3).
2. M is a smooth manifold (Definition 1.2.4) equipped with an integrable (Definition 2.0.5) almost complex structure (Definition 2.0.4).

3. Vector fields

Definition 3.0.1 (Local and global vector fields on a C^k manifold)**.** Let $k \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$, let M be a C^k manifold (possibly with boundary) (Definition 1.2.4), and let

$$\pi : TM \to M$$

denote its tangent bundle (Definition 1.4.2), which is a C^{k-1}-manifold.

1. Let $U \subseteq M$ be an open subset. A *local C^{k-1} vector field on U* is a C^{k-1}-section (Definition 1.3.1) over U, i.e. a C^{k-1} map (Definition 1.2.6)
$$X : U \to TM$$
of C^{k-1}-manifolds such that $\pi \circ X = \iota_U$, where $\iota_U : U \hookrightarrow M$ is the inclusion map. Equivalently, X is a C^k section of the restricted bundle $TM|_U \to U$.
2. A *global C^{k-1} vector field on M* (or simply a *C^{k-1} vector field on M*) is a local C^k vector field on $U = M$, i.e. a global C^{k-1}-section (Definition 1.3.1).

Definition 3.0.2 (Local and global frames of the tangent bundle of a differentiable manifold)**.** Let $k \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$, let M be a C^k manifold of (pure) dimension n, and let TM be its tangent bundle (Definition 1.4.2).

1. Let $U \subseteq M$ be a nonempty open subset. A *local C^k frame of TM over U* (or simply a *local frame over U*) is an ordered n-tuple of local C^k vector fields (Definition 3.0.1)
$$(X_1, \ldots, X_n) \in \Gamma^k(TU)^n$$
(Definition 1.3.1) such that for every point $p \in U$ the n-tuple
$$(X_1(p), \ldots, X_n(p))$$
is a basis (Definition B.0.2) of the n-dimensional real vector space T_pM.
2. A *global C^k frame of TM* (or simply a *global frame*) is a local C^k frame over $U = M$, i.e. an ordered n-tuple of global C^k vector fields
$$(X_1, \ldots, X_n) \in \Gamma^k(TM)^n$$
such that $(X_1(p), \ldots, X_n(p))$ is a basis of T_pM for every $p \in M$.

4. Orientation on manifolds

Definition 4.0.1 (Orientation of a real vector space)**.** Let V be a finite-dimensional real vector space of dimension $n \geq 1$. An *orientation of V* is an equivalence class of ordered bases of V under the following equivalence relation: two ordered bases
$$(v_1, \ldots, v_n) \quad \text{and} \quad (w_1, \ldots, w_n)$$
of V are declared equivalent if the unique linear automorphism
$$T : V \to V$$
(♠ TODO: determinant) satisfying $T(v_i) = w_i$ for all $i = 1, \ldots, n$ has positive determinant with respect to (equivalently, in any) choice of identification of V with $\mathbb{R}^n$.

Equivalently, fix any ordered basis $(e_1, \ldots, e_n)$ of V, and declare that an ordered basis $(v_1, \ldots, v_n)$ of V is *positively oriented with respect to $(e_1, \ldots, e_n)$* if the determinant of the change-of-basis matrix from $(e_1, \ldots, e_n)$ to $(v_1, \ldots, v_n)$ is positive. This defines an equivalence relation on the set of ordered bases of V with exactly two equivalence classes, called the *orientations of V*. A choice of one of these two classes is an orientation of V.

A *oriented real vector space* is a pair (V, o) where V is a finite-dimensional real vector space and o is a chosen orientation of V in the above sense.

(♠ TODO: TODO: overall defining orientations is still very confusing to me, so I will have to fix definitions)

Definition 4.0.2 (Standard Orientation on $\mathbb{R}^n$)**.** Let $n \in \mathbb{N}$ and consider the Euclidean space $\mathbb{R}^n$ equipped with the standard ordered basis

$$\mathcal{E} = (e_1, e_2, \ldots, e_n),$$

where each e_i is the vector with a 1 in the ith coordinate and 0 elsewhere.

The *standard orientation on $\mathbb{R}^n$* is the orientation (Definition 4.0.1) defined by declaring the ordered basis $\mathcal{E}$ to be positively oriented. More precisely, since for n-dimensional vector spaces the orientation is given by equivalence classes of ordered bases modulo the sign of the determinant of the change of basis matrix, an ordered basis $(v_1, \ldots, v_n)$ of $\mathbb{R}^n$ is said to be *positively oriented* if and only if

$$\det([v_1 \; v_2 \; \cdots \; v_n]) > 0,$$

where $[v_1 \; v_2 \; \cdots \; v_n]$ is the matrix whose columns are the vectors v_i written in the standard basis $\mathcal{E}$.

Thus, the standard orientation on $\mathbb{R}^n$ is the equivalence class of ordered bases that yield a positive determinant with respect to the standard basis $\mathcal{E}$.

Definition 4.0.3 (Pointwise orientation of a manifold)**.** Let $n \geq 0$ be an integer, let $k \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$, and let M be a C^k-manifold (Definition 1.2.4) of (pure) dimension n with or without boundary.

1. An *orientation on M at p* is a choice of orientation (Definition 4.0.1) of the tangent space T_pM (Definition 1.4.1).
2. A *pointwise orientation of M* is a choice of orientation of M at each $p \in M$.
3. Say that M is equipped with a pointwise orientation. A local frame (Definition 3.0.2) (E_i) for TM is said to be
   (a) *(positively) oriented* if $(E_1|_p, \ldots, E_n|_p)$ is positively oriented (Definition 4.0.1) basis for T_pM at each $p \in U$ with respect to the pointwise orientation.
   (b) *negative oriented* if $(E_1|_p, \ldots, E_n|_p)$ is negatively oriented (Definition 4.0.1) basis for T_pM at each $p \in U$ with respect to the pointwise orientation.
4. A pointwise orientation of M is said to be *continuous* if every point of M is in the domain of an oriented local frame.
5. An *orientation of M* is a continuous pointwise orientation.
6. M is said to be *orientable* if there exists an orientation of M. Otherwise, M is said to be *nonorientable*
7. An *oriented manifold (with or without boundary)* is an ordered pair $(M, \mathcal{O})$ of an orientable C^k manifold and $\mathcal{O}$ is a choice of orientation for M.

5. Poincaré duality

(♠ TODO: read the following formulations of Poincare duality) (♠ TODO: give formulations of poincare duality on a non-compact manifold)

Theorem 5.0.1 (Classical Poincaré Duality)**.** (♠ TODO: define oriented manifold, cap product, fundamental class, singular homology, cohomology) Let M be a connected, closed (i.e., compact without boundary), oriented topological manifold (Definition 1.1.1) of dimension n, and let R be a commutative ring with unity. Then for each $k \in \{0, \dots, n\}$, there is an isomorphism

$$H^k(M; R) \cong H_{n-k}(M; R)$$

induced by the cap product with the fundamental class $[M] \in H_n(M; R)$.

Theorem 5.0.2 (Perfect Pairing)**.** (♠ TODO: define $\smile$, $\langle , \rangle$, singular homology, cohomology) Under the hypotheses of the previous theorem, for each $k \in \{0, \dots, n\}$, the bilinear pairing $H^k(M; R) \times H^{n-k}(M; R) \to R, \quad (\alpha, \beta) \mapsto \langle \alpha \smile \beta, [M] \rangle$ is nondegenerate (i.e., perfect).

Proposition 5.0.3 (Relative Poincaré Duality)**.** (♠ TODO: define relative cohomology, relative fundamental class) Let M be a compact, oriented n-manifold with (possibly nonempty) boundary ∂M, and let R be a commutative ring with unity. For each $k \in \{0, \dots, n\}$, there is an isomorphism $H^k(M, \partial M; R) \cong H_{n-k}(M; R)$ again induced via cap product with the relative fundamental class $[M, \partial M] \in H_n(M, \partial M; R)$.

Proposition 5.0.4 (Poincaré Duality for Compact Supports)**.** Let X be a connected, oriented n-dimensional manifold (not necessarily compact) and R a commutative ring with unity. Then for each $k \in \{0, \dots, n\}$, the cap product with the fundamental class in Borel–Moore homology induces an isomorphism $H_c^k(X; R) \cong H_{n-k}^{BM}(X; R)$.

Proposition 5.0.5 (Poincaré Duality Groups)**.** Let G be a group of type FP (i.e., admits a finite projective resolution as a $\mathbb{Z}[G]$-module), and let n be an integer. G is said to be an n-dimensional Poincaré duality group over a commutative ring R with dualizing module D if there exists a class $\mu \in H^n(G; D)$ such that cap product with μ induces isomorphisms $H^k(G; M) \cong H_{n-k}(G; D \otimes_R M)$ for all (left) $R[G]$-modules M and all $k \geq 0$.

Corollary 5.0.6 (Betti Number Symmetry)**.** Let M be a closed, connected, oriented n-manifold. Then the k-th Betti number equals the $(n-k)$-th Betti number: $\operatorname{rank}_R H^k(M; R) = \operatorname{rank}_R H^{n-k}(M; R)$.

Appendix A. Set Theory

Definition A.0.1. Let X and Y be sets and let $f : X \to Y$ be a function.

- The function f is said to be *injective* (or *one-to-one*) if for all $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
- The function f is said to be *surjective* (or *onto*) if for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$.
- The map f is *bijective* if it is both injective and surjective. In this case, there exists a unique *inverse map* $f^{-1} : Y \to X$ such that for all $x \in X$ and $y \in Y$,

$$f^{-1}(f(x)) = x \text{ and } f(f^{-1}(y)) = y.$$

Definition B.0.1 (Vector space over a field)**.** Let $(k, +, \cdot)$ be a field (Definition C.0.4). A *vector space over k* or a *k-vector space* is a triple $(V, +, \cdot)$[1] where

1. $(V, +)$ is an abelian group, and
2. $\cdot$ is a map $k \times V \to V$, called *scalar multiplication*

such that the following axioms hold for all $a, b \in k$ and all $u, v \in V$:

1. (Compatibility with field multiplication)
$$(ab) \cdot v = a \cdot (b \cdot v).$$
2. (Identity scalar)
$$1 \cdot v = v.$$
3. (Distributivity over vector addition)
$$a \cdot (u + v) = a \cdot u + a \cdot v.$$
4. (Distributivity over scalar addition)
$$(a + b) \cdot v = a \cdot v + b \cdot v.$$

Definition B.0.2. Let F be a field (Definition C.0.4), and let V be an F-vector space (Definition B.0.1). A subset $B \subseteq V$ is called a *basis of V* if: (i) B is linearly independent over F, and (ii) B spans V.

If B is a basis, we define the *dimension of V over F* (or *rank of V over F*), denoted by
$$\dim_F(V),$$
(♠ TODO: cardinality) to be the cardinality of B. This value is uniquely determined by V and F.

Definition B.0.3. Let F be a field (Definition C.0.4), and let V and W be F-vector spaces (Definition B.0.1). A function $T : V \to W$ is called a *(homo)morphism of vector spaces over F*, or an *F-linear map*, if for all $u, v \in V$ and all $a, b \in F$, we have
$$T(au + bv) = aT(u) + bT(v).$$
The set of all such morphisms from V to W is denoted by
$$\mathrm{Hom}_F(V, W).$$
The F-vector spaces with vector space morphisms form a locally small category. The F-vector spaces of finite dimension (Definition B.0.2) form a full subcategory. These categories are often denoted by Vec_F or $\mathbf{Vec}_F$, although these notations are more often used for the category of finite dimensional F-vector spaces.

[1] Note that $+$ and $\cdot$ are abuse of notation here as these are already used for the addition and multiplication of $\cdot$

Definition C.0.1. A *ring* is a triple $(R, +, \cdot)$ where

1. $(R, +)$ is a commutative group, and
2. $(R, \cdot)$ is a monoid.
3. $\cdot$ is distributive over $+$, i.e. for all $a, b, c \in R$, we have

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and} \quad (a + b) \cdot c = a \cdot c + b \cdot c.$$

Equivalently, a ring is a triple $(R, +, \cdot)$ where $+, \cdot : R \times R \to R$ are binary operations satisfying

1. $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$ for all $a, b, c \in R$
2. There exists an element $0 \in R$ such that $a + 0 = a = 0 + a$ for all $a \in R$.
3. For every $a \in R$, there exists an element $-a \in R$ such that $a + (-a) = 0 = (-a) + a$ for all $a \in R$.
4. There exists an element $1 \in R$ such that $a \cdot 1 = a = 1 \cdot a$ for all $a \in R$.
5. For all $a, b, c \in R$, we have

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and} \quad (a + b) \cdot c = a \cdot c + b \cdot c.$$

The operation $+$ is often called *addition* and the operation $\cdot$ is often called *multiplication*. Accordingly, the identity element 0 of $+$ is often called the *additive identity* and the identity element 1 of $\cdot$ is often called the *multiplicative identity*.

Remark C.0.2. Some writers might not require a ring to have a multiplicative identity element, i.e. would define a ring so that $(R, +)$ is a commutative group, $(R, \cdot)$ is a semigroup, and $\cdot$ is distributive over $+$. Such writers would call the notion of ring in Definition C.0.1 a *unitary ring* to emphasize the existence of the multiplicative identity 1.

Definition C.0.3. A *commutative (unital) ring* is a ring (Definition C.0.1) $(R, +, \cdot)$ such that $\cdot$ is a commutative operation, i.e. $a \cdot b = b \cdot a$.

For many writers (e.g. "commutative" algebraists or number theorists), a *ring* refers to a commutative ring as above.

Definition C.0.4 (Field)**.** A *field* is a A *field* is commutative division ring (Definition C.0.3).

Definition C.0.5. Let R be a not-necessarily commutative ring (Definition C.0.1).

1. A *left R-module* is an abelian group $(M, +)$ together with an operation $R \times M \to M$, denoted $(r, m) \mapsto rm$, such that for all $r, s \in R$ and $m, n \in M$:
   - $r(m + n) = rm + rn$,
   - $(r + s)m = rm + sm$,
   - $(rs)m = r(sm)$,
   - $1_R m = m$ where 1_R is the multiplicative identity of R.
2. A *right R-module* is defined similarly as an abelian group $(M, +)$ with an operation $M \times R \to M$, denoted $(m, r) \mapsto mr$, such that for all $r, s \in R$ and $m, n \in M$:
   - $(m + n)r = mr + nr$,

- $m(r+s) = mr + ms$,
- $m(rs) = (mr)s$,
- $m1_R = m$.

3. Let R and S be (not necessarily commutative) rings (Definition C.0.1).
   An *R-S-bimodule* (or an *R-S-module* or an (R,S)-module, etc.)is an abelian group $(M,+)$ equipped with
   (a) a left action of R:
   $$R \times M \to M, \quad (r,m) \mapsto r \cdot m,$$
   making M a left R-module (Definition C.0.5),
   (b) a right action of S:
   $$M \times S \to M, \quad (m,s) \mapsto m \cdot s,$$
   making M a right S-module,
   such that the left and right actions commute; that is, for all $r \in R$, $s \in S$, and $m \in M$,
   $$r \cdot (m \cdot s) = (r \cdot m) \cdot s.$$
4. A *two-sided R-module* (or *R-bimodule*) is an R-R-bimodule.

If R is a commutative ring (Definition C.0.3), then a left/right R-module can automatically be regarded as a two-sided R-module. As such, we simply talk about *R-modules* in this case.

Any abelian group is equivalent to a two-sided $\mathbb{Z}$-module. Moreover, any left R-module is equivalent to an $R-\mathbb{Z}$-bimodule (Definition C.0.5) and any right R-module is equivalent to an $\mathbb{Z}-R$-bimodule (Definition C.0.5). Given a left/right/two-sided R-module, its *natural bimodule structure* will refer to its structure as a R-$\mathbb{Z}$/$\mathbb{Z}$-R/R-R bimodule. In this way, many definitions associated with the notions of left/right/two-sided R-modules can be defined as special cases for definitions for R-S-bimodules.

Definition C.0.6. 1. Let $R_0, \dots, R_k$ be (not necessarily commutative) rings (Definition C.0.1). Let M_i be a $R_{i-1} - R_i$-bimodule (Definition C.0.5) for $i = 1, \dots, k$, and let N be an $R_0 - R_k$-bimodule. A function $\Phi : M_1 \times \cdots \times M_k \to N$ is called a *multilinear map* (or *$R_0 - R_k$-multilinear*) if
   - for each $j = 1, \dots, k$ and fixed $m_i \in M_i$ for $i \neq j$, the map $M_j \to N$ given by $m_j \mapsto \Phi(m_1, \dots, m_j, \dots, m_k)$ is a group homomorphism and
   - for all $m_i \in M_i$ for $i = 1, \dots, k$ and $r_j \in R_j$ where $j \in \{1, \dots, k-1\}$, we have
   $$\Phi(m_1, \dots, m_j r_j, m_{j+1}, \dots, m_k) = \Phi(m_1, \dots, m_j r_j, r_j m_{j+1}, \dots, m_k).$$
   - Φ is left R_0-linear in the first argument and right R_k-linear in the kth argument, i.e. for all $r_0 \in R_0$ and $r_k \in r_k$ we have
   $$\Phi(r_0 m_1, m_2, \dots, m_{k-1}, m_k r_k) = r_0 \cdot \Phi(m_1, \dots, m_k) \cdot r_k.$$
2. Let R be a (not necessarily commutative) ring and let M be a two-sided R-module. A *multilinear form* is a multilinear map $M^r \to R$ (where M^r here is the set theoretic Cartesian product, rather than a product of groups or modules) for some $r \geq 0$.

In particular, when R be a commutative ring (Definition C.0.3), and M_i for $i = 1, \ldots, k$ and N are R-modules, we may speak of a multilinear map $\Phi : M_1 \times \cdots \times M_k \to N$. We may thus also speak of multilinear maps $M^r \to R$ for $r \geq 0$

Additionally, we may speak of *bilinear maps/forms*, *trilinear maps/forms*, etc.

Definition C.0.7. Let R be a (not necessarily commutative) ring (Definition C.0.1), M an two-sided R-module, $k \in \mathbb{N}$ and $\Phi : M^k \to R$ a multilinear form (Definition D.0.17)

1. Φ is *symmetric* if for every permutation $\sigma \in S_k$ and all $x_1, \ldots, x_k \in M$,
$$\Phi(x_1, ..., x_k) = \Phi(x_{\sigma(1)}, ..., x_{\sigma(k)}).$$
2. Φ is *antisymmetric* if for all $\sigma \in S_k$ and all $x_1, \ldots, x_k \in M$,
$$\Phi(x_{\sigma(1)}, ..., x_{\sigma(k)}) = \operatorname{sgn}(\sigma)\Phi(x_1, ..., x_k),$$
where $\operatorname{sgn}(\sigma)$ is the sign of the permutation σ.
3. Φ is *alternating* if whenever $x_i = x_j$ for some $i \neq j$, $\Phi(x_1, ..., x_k) = 0$. Every alternating form is antisymmetric, since if $x_i = x_j$ and we swap coordinates, both terms are zero.

For $k = 2$, these definitions specialize to *bilinear forms*; in particular:

- Φ is symmetric if $\Phi(x, y) = \Phi(y, x)$.
- Φ is antisymmetric if $\Phi(x, y) = -\Phi(y, x)$.
- Φ is alternating if $\Phi(x, x) = 0$ for all x.

Definition C.0.8 (Hom of left/right/bi-modules)**.** Let R, S, T be (not necessarily commutative) rings (Definition C.0.1).

1. Let M and N be left R-modules (Definition C.0.5). The *homomorphism group of left R-modules from M to N* is the abelian group
$$\operatorname{Hom}(M, N) = \operatorname{Hom}_R(M, N) := \{f : M \to N \mid f \text{ is a left } R\text{-module homomorphism}\}.$$
2. Let M and N be right R-modules (Definition C.0.5). The *homomorphism group of right R-modules from M to N* is the abelian group
$$\operatorname{Hom}(M, N) = \operatorname{Hom}_R(M, N) := \{f : M \to N \mid f \text{ is a right } R\text{-module homomorphism}\}.$$
3. Let S be a (not necessarily commutative ring) and let M and N be $R-S$-bimodules (Definition C.0.5). The *homomorphism group of R-S-bimodules from M to N* is the abelian group
$$\operatorname{Hom}(M, N) = \operatorname{Hom}_{R-S}(M, N) \coloneqq \{f : M \to N | f \text{ is a } R-S\text{-bimodule homomorphism}\}$$

In each case, $\operatorname{Hom}(M, N)$ has a natural structure of an *abelian group* given by *pointwise addition*: for $f, g \in \operatorname{Hom}(M, N)$,
$$(f + g)(m) := f(m) + g(m),$$
and the zero morphism 0 given by $0(m) := 0_N$ acts as the identity element. The additive inverse $-f$ is defined by $(-f)(m) := -f(m)$. Moreover, depending on bi-module structures that M and N may be carrying, $\operatorname{Hom}(M, N)$ may itself carry additional module structures:

- In case that M is a $R-S$-bimodule and N is a $R-T$-bimodule, $\mathrm{Hom}_R(M,N)$, the group of left R-module homomorphisms, is an $S-T$-bimodule as follows:
$$(s\cdot f\cdot t)(m)=f(m\cdot s)\cdot t \quad f\in \mathrm{Hom}_R(M,N), s\in S, t\in T.$$
- Dually, in case that M is a $S-R$-bimodule and N is a $T-R$-bimodule, $\mathrm{Hom}_R(M,N)$, the group of right R-module homomorphisms, is an $S-T$-bimodule as follows:
$$(s\cdot f\cdot t)(m)=f(s\cdot m)\cdot t \quad f\in \mathrm{Hom}_R(M,N), s\in S, t\in T.$$

Some cases of interest may be when R, S, or T is in fact $\mathbb{Z}$ — these allow us to see module structures on $\mathrm{Hom}(M,N)$ even when M and N are one-sided modules.

We furthermore note that $\mathrm{Hom}(-,-)$ yields biadditive functors
$$\mathrm{Hom}(-,-): {}_R\mathbf{Mod}_S^{\mathrm{op}}\times {}_R\mathbf{Mod}_T\to {}_S\mathbf{Mod}_T$$
$$\mathrm{Hom}(-,-): {}_S\mathbf{Mod}_R^{\mathrm{op}}\times {}_T\mathbf{Mod}_R\to {}_S\mathbf{Mod}_T.$$

Definition C.0.9. Let R be a (not necessarily commutative) ring (Definition C.0.1). Depending on the module structure of M, we define its dual module as follows:

1. If M is a left R-module (Definition C.0.5), then the *(right) dual module of M* is
$$M^*=M^\vee:=\mathrm{Hom}_R(M,R)$$
(Definition C.0.8). Note that it is a right R-module, as M is a $R-\mathbb{Z}$-bimodule and R is an $R-R$-bimodule.
2. If M is a right R-module (Definition C.0.5), then the *(left) dual module of M* is
$${}^*M={}^\vee M:=\mathrm{Hom}_R(M,R)$$
(Definition C.0.8). Note that it is a left R-module, as M is a $\mathbb{Z}-R$-bimodule and R is an $R-R$-bimodule.
3. If M is a two-sided R-module, then the *dual of M* usually refers to the right or left dual as above.

If R is a field (Definition C.0.4) F and V is an F-vector space (Definition B.0.1), then the dual module
$$V^*=V^\vee:=\mathrm{Hom}_F(V,F)$$
is called the *dual vector space of V*.

Appendix D. Topology

Definition D.0.1 (Topology)**.** Let X be a set. A *topology on X* is a collection $\mathcal{T}$ of subsets of X such that:

1. $\emptyset\in\mathcal{T}$ and $X\in\mathcal{T}$,
2. For any collection $\{U_i\}_{i\in I}\subseteq\mathcal{T}$ (with I arbitrary), the union $\bigcup_{i\in I}U_i\in\mathcal{T}$,
3. For any finite collection $\{U_1,\dots,U_n\}\subseteq\mathcal{T}$, the intersection $U_1\cap\cdots\cap U_n\in\mathcal{T}$.

If $\mathcal{T}$ is a topology on X, the pair $(X, \mathcal{T})$ is called a *topological space*. Members of $\mathcal{T}$ are called *open sets*.

A subset $C \subseteq X$ is *closed* if its complement $X \setminus C$ is an open set in $\mathcal{T}$

One very often refers to X as a topological spcae, omitting the notation of the topology $\mathcal{T}$.

The collection of all topologies on a set X may be denoted by notations such as $\mathrm{Top}(X)$, $\mathbf{Top}(X)$, or $\mathsf{Top}(X)$.

Definition D.0.2. For a positive integer n, let $\mathbb{R}^n$ denote the n-fold Cartesian product of the real line $\mathbb{R}$ with itself:

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R} \text{ for all } i = 1, \ldots, n\}.$$

The set $\mathbb{R}^n$ is called Euclidean n-space. A point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is associated with the Euclidean norm

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

The corresponding metric $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ is given by

$$d(x, y) = \|x - y\|.$$

This metric induces the standard topology on $\mathbb{R}^n$, called the *Euclidean topology*.

Definition D.0.3. The *closed half-space* (in $\mathbb{R}^n$ (Definition D.0.2)) refers to the topological space (Definition D.0.1) $\mathbb{H}^n \subset \mathbb{R}^n$ defined by

$$\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n \geq 0\}$$

Other common notation for $\mathbb{H}^n$ include $\mathbb{R}^n_+$ and $\mathbb{R}^n_{\geq 0}$.

Definition D.0.4. Let $(X, \mathcal{T})$ be a topological space (Definition D.0.1) and let $x \in X$.

- An *open neighborhood of x* is any open set $U \in \mathcal{T}$ such that $x \in U$.
- A *neighborhood of x* is a set $N \subseteq X$ for which there exists an open neighborhood $U \in \mathcal{T}$ of x such that $U \subseteq N$.
- A *neighborhood basis* (or *local base*) at x is a nonempty collection $\mathcal{B}_x$ of neighborhoods of x such that for every neighborhood N of x, there exists $B \in \mathcal{B}_x$ with $B \subseteq N$.
  The elements of $\mathcal{B}_x$ are said to *form a base of neighborhoods* at x.

Definition D.0.5. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces (Definition D.0.1). A map $f : X \to Y$ is called *continuous* if for every open set $V \in \mathcal{T}_Y$, the preimage $f^{-1}(V)$ is an open set in X, that is,

$$\forall V \in \mathcal{T}_Y,\ f^{-1}(V) \in \mathcal{T}_X.$$

Equivalently, f is continuous if and only if for every closed set $C \subseteq Y$, the preimage $f^{-1}(C)$ is closed in X.

A *map of topological spaces* usually refers to a continuous map between the topological spaces.

The collection of topological spaces along with continuous maps form a locally small category, usually called the *category of topological spaces* and often denoted by notations such as Top,

Top, etc. The set of continuous maps from X to Y is sometimes denoted by $C(X,Y)$. Other standard notation include $\mathrm{Hom}_{\mathrm{Top}}(X,Y)$ or $\mathrm{Top}(X,Y)$ coming from more general notation for morphisms between objects in a category.

Definition D.0.6. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces (Definition D.0.1). A function $f : X \to Y$ is called a *homeomorphism* if it satisfies all of the following:

1. f is bijective (Definition A.0.1);
2. f is continuous (Definition D.0.5) with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$; and
3. the inverse map $f^{-1} : Y \to X$ (Definition A.0.1) is also continuous.

If such a function exists, the spaces X and Y are said to be *homeomorphic*.

Definition D.0.7 (Separation axioms)**.** Let $(X, \mathcal{T})$ be a topological space (Definition D.0.1).

- $(X, \mathcal{T})$ is T_0 (*Kolmogorov*) if for every pair of distinct points $x, y \in X$, there exists an open set $U \in \mathcal{T}$ such that, without loss of generality, $x \in U$ and $y \notin U$.
- $(X, \mathcal{T})$ is T_1 (*Fréchet*) if for every pair of distinct points $x, y \in X$, there exist open sets $U, V \in \mathcal{T}$ such that $x \in U$, $y \notin U$, and $y \in V$, $x \notin V$.
- $(X, \mathcal{T})$ is T_2 or *Hausdorff* if for every pair of distinct points $x, y \in X$, there exist disjoint open sets $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$.
- $(X, \mathcal{T})$ is *regular* if it is T_1 and for each point $x \in X$ and closed set $F \subseteq X$ with $x \notin F$, there exist disjoint open sets $U, V \in \mathcal{T}$ such that $x \in U$ and $F \subseteq V$.
- $(X, \mathcal{T})$ is T_3 (regular Hausdorff) if it is T_1 and regular.
- $(X, \mathcal{T})$ is *completely regular* if for each closed set $F \subseteq X$ and $x \notin F$, there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f|_F = 1$.
- $(X, \mathcal{T})$ is $T_{3\frac{1}{2}}$ (completely regular Hausdorff) if it is T_1 and completely regular.
- $(X, \mathcal{T})$ is *normal* if it is T_1 and for each pair of disjoint closed sets $A, B \subseteq X$, there exist disjoint open sets $U, V \in \mathcal{T}$ such that $A \subseteq U$ and $B \subseteq V$.
- $(X, \mathcal{T})$ is T_4 (normal Hausdorff) if it is T_1 and normal.
- $(X, \mathcal{T})$ is T_5 (completely normal Hausdorff) if it is T1 and completely normal.
- $(X, \mathcal{T})$ is *perfectly normal* if every closed set is a $G\delta$ (countable intersection of open sets) and the space is normal.
- $(X, \mathcal{T})$ is T_6 (perfectly normal Hausdorff) if it is T_1 and perfectly normal.

Definition D.0.8. Let $(X, \mathcal{T})$ be a topological space (Definition D.0.1).

- The space $(X, \mathcal{T})$ is said to be *first countable* if every point $x \in X$ has a countable neighborhood basis (Definition D.0.4), i.e., there exists a countable collection $\{U_n\}_{n \in \mathbb{N}}$ of open neighborhoods of x such that for any open neighborhood U of x, there exists $n \in \mathbb{N}$ with $U_n \subseteq U$.
- The space $(X, \mathcal{T})$ is said to be *second countable* if there exists a countable basis $\mathcal{B} \subseteq \mathcal{T}$ for the topology $\mathcal{T}$, i.e., every open set $U \in \mathcal{T}$ can be written as a union of elements of $\mathcal{B}$.

Definition D.0.9. Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$.

- The *interior of A*, denoted by $\mathrm{int}(A)$, is the union of all open sets contained in A:

$$\mathrm{int}(A) = \bigcup \{U \in \mathcal{T} : U \subseteq A\}.$$

- The *boundary of A*, denoted by ∂A, is defined as

$$\partial A = \overline{A} \setminus \mathrm{int}(A).$$

  (Definition D.0.11) Equivalently, a point $x \in X$ belongs to ∂A if every open neighborhood of x intersects both A and its complement $X \setminus A$.

Definition D.0.10. Let (X, τ) be a topological space (Definition D.0.1). An *open covering of X* is a family of open sets

$$\mathcal{U} = \{U_i\}_{i \in I}$$

such that

$$\bigcup_{i \in I} U_i = X.$$

Here, each $U_i \in \tau$ is an open subset of X indexed by a set I, which can be finite or infinite.

Definition D.0.11 (Closure of a subset)**.** Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$ a subset. The *closure of A in X*, denoted by $\overline{A}$, is defined as the intersection of all closed sets containing A, i.e.,

$$\overline{A} := \bigcap \{C \subseteq X : C \text{ is closed and } A \subseteq C\}.$$

Equivalently, $\overline{A}$ is the smallest closed set containing A.

Definition D.0.12 (Compact topological space)**.** A topological space $(X, \mathcal{T})$ is *compact* if every open cover of X admits a finite subcover; that is, for every collection $\{U_i\}_{i \in I}$ of open sets in $\mathcal{T}$ such that $X = \bigcup_{i \in I} U_i$, there exists a finite subcollection $\{U_{i_j}\}_{j=1}^n$ such that $X = \bigcup_{j=1}^n U_{i_j}$.

Some mathematicians, e.g. algebraic geometers, would refer to this property as *quasi-compactness*.

Definition D.0.13 (Fiber of a map of topological spaces)**.** Let X and Y be topological spaces (Definition D.0.1) and let $f : X \to Y$ be a continuous map (Definition D.0.5). For a point $y \in Y$, the *fiber of f over y* is the inverse image $f^{-1}(y) = f^{-1}(\{y\})$ endowed with the subspace topology induced from X. The fiber is also denoted by notations such as $\mathrm{Fib}_f(y)$ or X_y.

Definition D.0.14 (Submodule generated by elements in an (R, S)-bimodule)**.** Let R and S be (not necessarily commutative) rings (Definition C.0.1).

1. Let M be an (R, S)-bimodule (Definition C.0.5).

   Given a subset $X \subseteq M$, the *sub-bimodule of M generated by X* is the smallest (R, S)-sub-bimodule of M containing X. It is often denoted by notations such as $\langle X \rangle = \langle X \rangle_{R,S}$ and is more explicitly the intersection

$$\langle X \rangle_{R,S} = \bigcap_{X \subseteq T \subseteq M, T \text{ is a } (R,S)\text{-submodule of } M} T$$

of al (R,S)-submodules of M containing X.

Equivalently, $\langle X\rangle_{R,S}$ consists of all linear combinations of X.

2. If M is a left/right/two-sided R-module and given a subset $X \subseteq M$, the *submodule of M generated by X* is the submodule of the natural bimodule (Definition C.0.5) of M generated by X. It is denoted by notations such as $\langle X\rangle = \langle X\rangle_R$.

Definition D.0.15 (Tensor product of bimodules)**.** Let R, S, T be (not necessarily commutative) rings (Definition C.0.1), let M be an R-S bimodule (Definition C.0.5), and let N be an S-T bimodule. Consider the Cartesian product $M \times N$. Define the free abelian group generated by $M \times N$, denoted $\mathbb{Z}[M \times N]$. Let U be the subgroup generated by elements of the form

$$\begin{aligned}
&(m+m', n) - (m,n) - (m',n),\\
&(m, n+n') - (m,n) - (m,n'),\\
&(m\cdot s, n) - (m, s\cdot n),
\end{aligned}$$

for all $m, m' \in M$, $n, n' \in N$, and $s \in S$.

The quotient abelian group

$$M \otimes_S N := \mathbb{Z}[M \times N]/U$$

is called the *tensor product M and N*. It has elements denoted $m \otimes n$ for $m \in M$, $n \in N$ called *pure tensors*. In general, its elements are finite sums

$$\sum_{i=1}^{n} m_i \otimes n_i \quad m_i \in M, n_i \in N$$

of pure tensors.

This tensor product becomes naturally an R-T bimodule with left action and right action defined by

$$\begin{aligned}
r\cdot(m\otimes n) &= (r\cdot m)\otimes n,\\
(m\otimes n)\cdot t &= m\otimes(n\cdot t),
\end{aligned}$$

for all $r \in R$, $t \in T$, $m \in M$, and $n \in N$.

Inductively, given rings $R_0, \ldots, R_k$ and $R_{i-1} - R_i$-bimodules M_i for $i = 1, \ldots, k$, we may speak of the tensor product

$$M_0 \otimes_{R_1} M_1 \otimes_{R_2} \cdots \otimes_{R_{k-1}} M_k;$$

tensor products are associative(♠ TODO:), so parentheses are not needed in the above. Its *pure tensors* are elements of the form $m_0 \otimes m_1 \otimes \cdots \otimes m_k$ for $m_i \in M_i$, and its general elements are finite sums

$$\sum_{j=1}^{n} m_{0j} \otimes m_{1j} \otimes \cdots m_{kj} \quad m_{ij} \in M_i.$$

of pure tensors. It also has a natural $R_0 - R_k$-bimodule structure.

Given a ring R and a two-sided R-module M, we may also speak of the *n-fold tensor product $M^{\otimes n} = M^{\otimes_R n}$*

Definition D.0.16. (♠ TODO: define coset, kernel of R-module homomorphism) Let R, S be (not necessarily commutative) rings (Definition C.0.1).

1. Let M be an R-S-bimodule (Definition C.0.5). Let $N \subseteq M$ be a submodule of M.
   The quotient group M/N, which is well defined as M is an abelian group and hence N is a normal subgroup, has the structure of an R-S-bimodule — the (abelian) group structure is simply the group structure of M/N, whereas the R-S-bimodule structure is given as follows: for $m \in M$, $r \in R$, $s \in S$, we have
   $$r \cdot (m + N) \cdot s = r \cdot m \cdot s + N.$$
   This R-S-bidmodule structure on M/N is called the *quotient R-S-bidmodule of M by N* and is also denoted as M/N.
   The canonical projection map
   $$\pi : M \to M/N, \quad m \mapsto m + N,$$
   is a surjective R-module homomorphism with kernel N.
2. Let M be a left/right/two-sided R-module. Let $N \subseteq M$ be a submodule of M. The *quotient R-module* M/N is the quotient of M by N for their respective natural bimodule structures (Definition C.0.5).

Definition D.0.17. 1. Let $R_0, \ldots, R_k$ be (not necessarily commutative) rings (Definition C.0.1). Let M_i be a $R_{i-1} - R_i$-bimodule (Definition C.0.5) for $i = 1, \ldots, k$, and let N be an $R_0 - R_k$-bimodule. A function $\Phi : M_1 \times \cdots \times M_k \to N$ is called a *multilinear map* (or $R_0 - R_k$-*multilinear*) if
   - for each $j = 1, \ldots, k$ and fixed $m_i \in M_i$ for $i \neq j$, the map $M_j \to N$ given by $m_j \mapsto \Phi(m_1, \ldots, m_j, \ldots, m_k)$ is a group homomorphism and
   - for all $m_i \in M_i$ for $i = 1, \ldots, k$ and $r_j \in R_j$ where $j \in \{1, \ldots, k-1\}$, we have
     $$\Phi(m_1, \ldots, m_j r_j, m_{j+1}, \ldots, m_k) = \Phi(m_1, \ldots, m_j r_j, r_j m_{j+1}, \ldots, m_k).$$
   - Φ is left R_0-linear in the first argument and right R_k-linear in the kth argument, i.e. for all $r_0 \in R_0$ and $r_k \in r_k$ we have
     $$\Phi(r_0 m_1, m_2, \ldots, m_{k-1}, m_k r_k) = r_0 \cdot \Phi(m_1, \ldots, m_k) \cdot r_k.$$
2. Let R be a (not necessarily commutative) ring and let M be a two-sided R-module. A *multilinear form* is a multilinear map $M^r \to R$ (where M^r here is the set theoretic Cartesian product, rather than a product of groups or modules) for some $r \geq 0$.

In particular, when R be a commutative ring (Definition C.0.3), and M_i for $i = 1, \ldots, k$ and N are R-modules, we may speak of a multilinear map $\Phi : M_1 \times \cdots \times M_k \to N$. We may thus also speak of multilinear maps $M^r \to R$ for $r \geq 0$

Additionally, we may speak of *bilinear maps/forms*, *trilinear maps/forms*, etc.

Definition D.0.18. Let R be a (not necessarily commutative) ring (Definition C.0.1), M an two-sided R-module, $k \in \mathbb{N}$ and $\Phi : M^k \to R$ a multilinear form (Definition D.0.17)

1. Φ is *symmetric* if for every permutation $\sigma \in S_k$ and all $x_1, \ldots, x_k \in M$,
   $$\Phi(x_1, ..., x_k) = \Phi(x_{\sigma(1)}, ..., x_{\sigma(k)}).$$

2. Φ is *antisymmetric* if for all $\sigma \in S_k$ and all $x_1, \ldots, x_k \in M$,

$$\Phi(x_{\sigma(1)}, ..., x_{\sigma(k)}) = \operatorname{sgn}(\sigma)\Phi(x_1, ..., x_k),$$

where $\operatorname{sgn}(\sigma)$ is the sign of the permutation σ.
3. Φ is *alternating* if whenever $x_i = x_j$ for some $i \neq j$, $\Phi(x_1, ..., x_k) = 0$. Every alternating form is antisymmetric, since if $x_i = x_j$ and we swap coordinates, both terms are zero.

For $k = 2$, these definitions specialize to *bilinear forms*; in particular:

- Φ is symmetric if $\Phi(x, y) = \Phi(y, x)$.
- Φ is antisymmetric if $\Phi(x, y) = -\Phi(y, x)$.
- Φ is alternating if $\Phi(x, x) = 0$ for all x.

Definition D.0.19 (Chain complex in an additive category)**.** Let $\mathcal{A}$ be an additive category and let I be a totally ordered set (typically $\mathbb{Z}$, but $I \subseteq \mathbb{Z}$ is also allowed).

1. A *chain complex* $(K^\bullet, d^\bullet)$ in $\mathcal{A}$ indexed by I consists of:
   - Objects $\{K^i\}_{i \in I}$ in $\mathcal{A}$, called the *terms in degree i*,
   - Morphisms $d^i : K^i \to K^{i+1}$ in $\mathcal{A}$, called the *differentials in degree i*,

   such that for every $i \in I$, $d^{i+1} \circ d^i = 0$. That is,

$$K^\bullet : \cdots \xrightarrow{d^{i-2}} K^{i-1} \xrightarrow{d^{i-1}} K^i \xrightarrow{d^i} K^{i+1} \xrightarrow{d^{i+1}} \cdots$$

   with $d^{i+1} d^i = 0$ for all i. We might typically use notation such as $K^\bullet = (K^i, d^i)_{i \in I}$ to denote a chain complex in $\mathcal{A}$.

   A cochain complex can be defined similarly/dually.
2. Let $K^\bullet = (K^i, d^i_K)$ and $L^\bullet = (L^i, d^i_L)$ be chain complexes (Definition D.0.19) in $\mathcal{A}$ indexed by the same set I. A *morphism of chain complexes* (or *chain map*)

$$f^\bullet : K^\bullet \to L^\bullet$$

   consists of morphisms $f^i : K^i \to L^i$ for all $i \in I$, such that for every $i \in I$,

$$d^i_L \circ f^i = f^{i+1} \circ d^i_K,$$

   i.e., the following diagram commutes for all i:

$$\begin{array}{ccc} K^i & \xrightarrow{d^i_K} & K^{i+1} \\ \downarrow f^i & & \downarrow f^{i+1} \\ L^i & \xrightarrow{d^i_L} & L^{i+1} \end{array}.$$

   There is then a category, often denoted by $\mathrm{Ch}(\mathcal{A})$ or $\mathbf{Ch}(\mathcal{A})$, whose objects are chain complexes in $\mathcal{A}$ and whose morphisms are morphisms of chain complexes. In particular, we may denote by

$$\operatorname{Hom}(K^\bullet, L^\bullet) = \operatorname{Hom}_{\mathrm{Ch}(\mathcal{A})}(K^\bullet, L^\bullet)$$

   the set of chain maps $K^\bullet \to L^\bullet$; it is in fact an abelian group.

   A *morphism of cochain complexes* is defined similarly, and we similarly denote by $\mathrm{Ch}(\mathcal{A})$ or $\mathbf{Ch}(\mathcal{A})$ the caetgory of cochain complexes in $\mathcal{A}$.

If k is a commutative ring (Definition C.0.3) such that $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is enriched in the category of k-modules (Definition C.0.5), then $\mathrm{Ch}(\mathcal{A})$ can be equipped with the structure of a dg-category over k.

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