CATEGORY THEORY

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1. Basic category theory

Definition 1.0.1 (Category)**.** A *category* $\mathcal{C}$ consists of the following data:

- A class of *objects* denoted $\mathrm{Ob}(\mathcal{C})$.
- For each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a class

$$\mathrm{Hom}_{\mathcal{C}}(X, Y)$$

  of *morphisms* (also called *arrows* or *homs*). If the category $\mathcal{C}$ is clear, then this *hom-class* is also denoted by $\mathrm{Hom}(X, Y)$. It may also be denoted by $\mathrm{hom}_{\mathcal{C}}(X, Y)$ or $\mathrm{hom}(X, Y)$, especially to distinguish from other types of hom's (e.g. internal hom's)
- For each triple of objects X, Y, Z, a composition law

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z),$$

  denoted $(g, f) \mapsto g \circ f$.
- For each object X, an *identity morphism*

$$\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X).$$

These data satisfy the following axioms:

- (Associativity) For all morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$, and $h \in \mathrm{Hom}_{\mathcal{C}}(Z,W)$,
$$h \circ (g \circ f) = (h \circ g) \circ f.$$
- (Identity) For all $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$,
$$\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X.$$

One often writes $X \in \mathcal{C}$ synonymously with $X \in \mathrm{Ob}(\mathcal{C})$, i.e. to denote that X is an object of of $\mathcal{C}$.

We may call a category as above an *ordinary category* to distinguish this notion from the notions of *categories enriched in monoidal categories* or higher/n-categories. (♠ TODO: TODO: define n-categories)

A category as defined above may be called called a *large category* or a *class category* to emphasize that the hom-classes may be proper classes rather than sets (note, however, that the possibility that hom-classes are sets is not excluded for large categories). Accordingly, a *category* may often refer to a locally small category (Definition 1.0.9), which is a category whose hom-classes are all sets.

Definition 1.0.2. We introduce the following common examples of categories, together with some frequently used notational variants:

- The *category of sets* (Definition 1.0.3), denoted **Set**, also written Set, **Sets**, or (**Set**), has objects all sets and morphisms all set-theoretic functions.
- The *category of groups* (Definition C.0.6), denoted **Grp**, also written Grp or **Groups**, has objects all groups (Definition C.0.5) and morphisms all group homomorphisms (Definition C.0.6).
- The *category of abelian groups*, denoted **Ab**, also written Ab or **AbGrp**, has objects all abelian groups (Definition C.0.6) and morphisms all group homomorphisms (Definition C.0.6).
- Given a not-necessarily commutative ring with unity R (Definition C.0.1), the *category of (left/right/two-sided)R-modules*, denoted **Mod**$_R$, also written R-Mod, has objects all (left/right/two-sided)R-modules and morphisms the R-module homomorphisms (Definition C.0.8).
- Given a field k, the *category of vector spaces over k*, denoted **Vect**$_k$, also written k-Vect or Vect(k), has objects all k-vector spaces and morphisms all k-linear maps.
- The *category of topological spaces*, denoted **Top**, also written Top or **TopSpaces**, has objects all topological spaces and morphisms all continuous maps.
- The *category of rings*, denoted **Ring**, also written Ring, has objects all rings with unity and morphisms all unital ring homomorphisms.
- The *category of commutative rings*, denoted **CRing**, also written CRing or **CommRing**, has objects all commutative rings with unity and morphisms all unital ring homomorphisms.
- The *category of fields*, denoted **Field**, also written Field or **Fields**, has objects all fields and morphisms all field homomorphisms.
- The *category of monoids*, denoted **Mon**, also written Mon or **Monoids**, has objects all monoids and morphisms all monoid homomorphisms.

- The *category of semigroups*, denoted **SemiGrp**, also written SGrp or **Semigroups**, has objects all semigroups and morphisms all semigroup homomorphisms.
- Given a topological space X, the *category of sheaves of sets on X*, denoted **Sh**(X), also written Sh(X), has objects all sheaves of sets on X and morphisms all morphisms of sheaves.
- The *category of small categories*, denoted **Cat**, also written Cat or **Categories**, has objects all small categories (Definition 1.0.9) and morphisms all functors (Definition 1.1.1) between them.
- Given categories $\mathcal{C}$ and $\mathcal{D}$, the *functor category* (Definition 1.2.1), denoted $[\mathcal{C}, \mathcal{D}]$, also written Fun$(\mathcal{C}, \mathcal{D})$, has objects all functors from $\mathcal{C}$ to $\mathcal{D}$ and morphisms all natural transformations between them.

Definition 1.0.3. The category of sets is the (locally small) (Definition 1.0.9) category (Definition 1.0.1)

- whose objects are sets (Definition A.0.1), and
- whose morphisms $X \to Y$ are set functions (Definition A.0.2) $X \to Y$.

The category of sets is often denoted by notations such as Set, **Set**, Sets, **Sets**, (Set), (**Set**), (Sets), (**Sets**).

Definition 1.0.4. 1. The *category of groups* is the locally small (Definition 1.0.9) category (Definition 1.0.1) whose objects are groups (Definition C.0.5) and whose morphisms are group homomorphisms (Definition C.0.6). It is often denoted by notations such as **Grp**.

2. The *category of abelian groups* is the locally small (Definition 1.0.9) category (Definition 1.0.1) whose objects are abelian groups (Definition C.0.5) and whose morphisms are group homomorphisms (Definition C.0.6). It is often denoted by notations such as **Ab**.

Definition 1.0.5. 1. The *category of rings* is the locally small (Definition 1.0.9) category (Definition 1.0.1) whose objects are rings (Definition C.0.1) R and whose morphisms $R \to S$ are ring homomorphisms (Definition C.0.4). The category of rings over R is often denoted by notations such as **Ring**.

2. The *category of commutative rings* is the full subcategory (Definition 1.3.2) of **Ring** consisting of the commutative rings (Definition C.0.3). It is denoted by notations such as **CommRing** or **CRing**.

Definition 1.0.6. The *category of topological spaces* is the (locally small) (Definition 1.0.9) category (Definition 1.0.1)

- whose objects are topological spaces (Definition B.0.1), and
- whose morphisms are continuous maps (Definition B.0.3).

The category of topological spaces is often denoted by notations such as Top, **Top**, etc.

Definition 1.0.7 (Category of opens of a topological space)**.** Let X be a topological space (Definition B.0.1). The *category of opens of X*, sometimes denoted **Open**(X) (or Open(X) or Ouv(X) (for the French word "ouvert", meaning open), etc.), is the small (Definition 1.0.9) category (Definition 1.0.1) defined as follows:

- The objects are the open subsets $U \subseteq X$.
- For two open sets $U, V \subseteq X$, the morphism set is

$$\mathrm{Hom}_{\mathbf{Open}(X)}(U, V) = \begin{cases} \{\iota_{U,V}\}, & \text{if } U \subseteq V, \\ \varnothing, & \text{otherwise,} \end{cases}$$

  where $\iota_{U,V}$ denotes the inclusion morphism $U \hookrightarrow V$.
- Composition of morphisms is given by composition of set-theoretic inclusions, i.e.

$$\iota_{V,W} \circ \iota_{U,V} = \iota_{U,W} \quad \text{whenever } U \subseteq V \subseteq W.$$

- The identity morphism on an object U is the inclusion $\iota_{U,U} = \mathrm{id}_U$.

Definition 1.0.8 (Isomorphism in a category)**.** Let $\mathcal{C}$ be a (large) category (Definition 1.0.1), and let $x, y \in \mathrm{Ob}(\mathcal{C})$. A morphism $f \in \mathcal{C}(x, y)$ is called an *isomorphism* if there exists a morphism $g \in \mathcal{C}(y, x)$ such that

$$g \circ f = 1_x \qquad \text{and} \qquad f \circ g = 1_y.$$

In this case, g is called the *inverse of f*, and x and y are said to be *isomorphic objects* in $\mathcal{C}$. It is standard to write $x \cong y$ if there exists an isomorphism $f : x \to y$.

In practice, isomorphisms in specific categories may be defined in different, yet equivalent, ways.

Definition 1.0.9 (Locally small category)**.** A (large) category (Definition 1.0.1) $\mathcal{C}$ is called a *locally small category* if for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the collection $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms between them is a (small) *set* (as opposed to a proper class). In other words, each hom-class is a set and may even be called a *hom-set*.

In some contexts, a locally small category may simply be called a *category*, especially when genuinely large categories are not considered.

A category $\mathcal{C}$ is called a *small category* if it is a locally small category and the class $\mathrm{Ob}(\mathcal{C})$ of objects is a set.

Given a universe (Definition E.0.1) U, we can define the notion of a *U-locally small category* and of a *U-small category* similarly. More explicitly,

1. a U-locally small category is a category such that for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the collection $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms between them is a U-set.
2. a U-small category is a category such that $\mathrm{Ob}(\mathcal{C})$ is a U-set and for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the collection $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms between them is a U-set; in particular the collection of all objects and morhpisms in a U-small category is a U-set.

Remark 1.0.10. Many "concrete" categories considered in "classical mathematics" or outside of more "abstract" category theory tend to be locally small. For example, the categories of sets, groups, R-modules, vector spaces, topological spaces, schemes, manifolds, sheaves on "small enough" sites are all locally small.

Convention 1.0.11. In this document, care should be taken to make it clear whether a category is locally small or large. However, currently such care has not been fully taken, so most statements that just speak of "categories" may not be applicable to large categories.

Definition 1.0.12 (Opposite category)**.** Let $\mathcal{C}$ be a (large) category (Definition 1.0.1). The *opposite category* of $\mathcal{C}$, denoted $\mathcal{C}^{\mathrm{op}}$, is defined as follows:

- The objects of $\mathcal{C}^{\mathrm{op}}$ are the same as those of $\mathcal{C}$.
- For any pair of objects $X, Y \in \mathcal{C}$, the morphisms from X to Y in $\mathcal{C}^{\mathrm{op}}$ are given by the morphisms from Y to X in $\mathcal{C}$:

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(Y, X).$$

- Composition in $\mathcal{C}^{\mathrm{op}}$ is defined by reversing the order of composition in $\mathcal{C}$. That is, for morphisms $f \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(Y, Z)$, their composition is

$$g \circ_{\mathcal{C}^{\mathrm{op}}} f := f \circ_{\mathcal{C}} g.$$

Intuitively, the category $\mathcal{C}^{\mathrm{op}}$ thus "reverses" the direction of all morphisms in $\mathcal{C}$.

1.1. Functors between categories.

Definition 1.1.1. Let $\mathcal{C}$ and $\mathcal{D}$ be (large) categories (Definition 1.0.1).

1. A *functor $F : \mathcal{C} \to \mathcal{D}$ (from $\mathcal{C}$ to $\mathcal{D}$)* consists of :
    - For each object X in $\mathcal{C}$, an object $F(X)$ in $\mathcal{D}$.
    - For each morphism $f : X \to Y$ in $\mathcal{C}$, a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$,
    such that:

    $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all objects X in $\mathcal{C}$,

    $F(g \circ f) = F(g) \circ F(f)$ for all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ and all $f : X \to Y, g : Y \to Z$ in $\mathcal{C}$.

    Functors as defined above are also referred to as *covariant functors* to distinguish them from contravariant functors

2. A *contravariant functor from $\mathcal{C}$ to $\mathcal{D}$* refers to a covariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. Equivalently, such a functor consists of
    - For each object X in $\mathcal{C}$, an object $F(X)$ in $\mathcal{D}$.
    - For each morphism $f : X \to Y$ in $\mathcal{C}$, a morphism $F(f) : F(Y) \to F(X)$ in $\mathcal{D}$,
    such that:

    $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all objects X in $\mathcal{C}$,

    $F(g \circ f) = F(f) \circ F(g)$ for all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ and all $f : X \to Y, g : Y \to Z$ in $\mathcal{C}$.

    A synonym for a "contravariant functor from $\mathcal{C}$ to $\mathcal{D}$" is a "presheaf on $\mathcal{C}$ with values in $\mathcal{D}$ (Definition 5.0.1)".

Note that declarations such as "Let $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ be a contravariant functor" can be common; such declarations usually mean "Let F be a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$" as opposed to "Let F be a contravariant functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$". further note that a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is equivalent to a covariant functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$.

Convention 1.1.2. By a "functor", without the qualifying adjectives of "covariant" (Definition 1.1.1) or "contravariant" (Definition 1.1.1), we will most usually mean a covariant functor.

1.1.1. Natural transformations between functors.

Definition 1.1.3. Let $\mathcal{C}$ and $\mathcal{D}$ be (large) categories (Definition 1.0.1). Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors (Definition 1.1.1).

A *natural transformation η between F and G* is a family of morphisms $\eta_X : F(X) \to G(X)$ in $\mathcal{D}$, one for each object X in $\mathcal{C}$, such that for every morphism $f : X \to Y$ in $\mathcal{C}$,

$$G(f) \circ \eta_X = \eta_Y \circ F(f)$$

in $\mathcal{D}$. In other words, the following diagram commutes:

$$
\begin{array}{ccc}
F(X) & \xrightarrow{F(f)} & F(Y) \\
\eta_X \downarrow & & \downarrow \eta_Y \\
G(X) & \xrightarrow[G(f)]{} & G(Y)
\end{array}
$$

We write such a natural transformation by $\eta : F \Rightarrow G$.

If η_X is an isomorphism (Definition 1.0.8) for all objects X of $\mathcal{C}$, then η is said to be a *natural isomorphism*.

Definition 1.1.4. Let $\mathcal{C}$ and $\mathcal{D}$ be (large)) categories (Definition 1.0.1). Let $F : \mathcal{C} \to \mathcal{D}$ be a functor (Definition 1.1.1).

1. F is called *full* if for every pair of objects $x, y \in \mathrm{Ob}(\mathcal{C})$, the induced rule/assignment/class function
$$F_{x,y} : \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$$
on Hom-collections is "surjective", i.e. for all morphisms $g : F(x) \to F(y)$, there exists some morphism $f : x \to y$ such that $F(f) = g$.
2. F is called *faithful* if for every pair of objects $x, y \in \mathrm{Ob}(\mathcal{C})$, the induced class function (assignment)
$$F_{x,y} : \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$$
on Hom-collections is "injective", i.e., for any morphisms $f_1, f_2 \in \mathrm{Hom}_{\mathcal{C}}(x, y)$, if $F(f_1) = F(f_2)$ in $\mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$, then $f_1 = f_2$.
3. F is called *fully faithful* if it is both full and faithful.

Definition 1.1.5 (Image of a functor)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor (Definition 1.1.1) between (large) categories (Definition 1.0.1). The *image of F* is the subcategory $\mathrm{Im}(F)$ of $\mathcal{D}$ defined by:

- $\mathrm{Ob}(\mathrm{Im}(F)) = \{\, F(c) \mid c \in \mathrm{Ob}(\mathcal{C}) \,\}$, the collection of objects of $\mathcal{D}$ that arise as images of objects of $\mathcal{C}$.

- For $F(x), F(y) \in \mathrm{Ob}(\mathrm{Im}(F))$, the hom-set is

$$\mathrm{Im}(F)(F(x), F(y)) = \{\, F(f) \mid f \in \mathcal{C}(x, y) \,\}.$$

- Composition and identities are inherited from $\mathcal{D}$ and are well-defined by functoriality of F.

Thus $\mathrm{Im}(F)$ is the smallest (not necessarily full (Definition 1.3.2)) subcategory of $\mathcal{D}$ containing all $F(c)$ for $c \in \mathrm{Ob}(\mathcal{C})$ and all $F(f)$ for f a morphism of $\mathcal{C}$.

Definition 1.1.6 (Essential image of a functor). Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between (large) categories (Definition 1.0.1). The *essential image of F* is the full subcategory (Definition 1.3.2) of $\mathcal{D}$ whose objects are those $d \in \mathrm{Ob}(\mathcal{D})$ for which there exists an object $c \in \mathrm{Ob}(\mathcal{C})$ such that

$$F(c) \cong d.$$

(Definition 1.0.8) Equivalently, the essential image is given by

$$\mathrm{EssIm}(F) = \{\, d \in \mathrm{Ob}(\mathcal{D}) \mid \exists c \in \mathrm{Ob}(\mathcal{C}),\ F(c) \cong d \,\},$$

endowed with all morphisms $\mathcal{D}(d, d')$ between such objects.

Equivalently, the essential image of F is the smallest replete (Definition 1.3.3) full subcategory (Definition 1.3.2) of $\mathcal{D}$ containing the image (Definition 1.1.5) of F

Definition 1.1.7. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between (large) categories (Definition 1.0.1). It is said to be essentially surjective if its essential image (Definition 1.1.6) coincides with $\mathcal{D}$.

Definition 1.1.8. An *equivalence of categories* between two (large) categories (Definition 1.0.1) $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of functors (Definition 1.1.1)

$$F : \mathcal{C} \to \mathcal{D} \quad \text{and} \quad G : \mathcal{D} \to \mathcal{C}$$

together with natural isomorphisms (Definition 1.1.3)

$$\eta : \mathrm{Id}_{\mathcal{C}} \xrightarrow{\sim} G \circ F \quad \text{and} \quad \epsilon : F \circ G \xrightarrow{\sim} \mathrm{Id}_{\mathcal{D}}.$$

Such functors F and G may be called *(natural) inverses of each other*.

When $\mathcal{C}$ and $\mathcal{D}$ are locally small categories (Definition 1.0.9), F is an equivalence of categories if and only if F is fully faithful (Definition 1.1.4) and essentially surjective (Definition 1.1.7)

Definition 1.1.9. A category $\mathcal{C}$ is called *essentially small* if it is equivalent (Definition 1.1.8) to a small category (Definition 1.0.9), i.e., there exists a small category $\mathcal{D}$ and an equivalence of categories

$$F : \mathcal{D} \to \mathcal{C}.$$

Note that an essentially small category is necessarily locally small (Definition 1.0.9).

1.1.2. *Yoneda's lemma.*

Theorem 1.1.10 (Yoneda Lemma)**.** Let $\mathcal{C}$ be a locally small category (Definition 1.0.9). Let A be an object of $\mathcal{C}$, and let $F : \mathcal{C} \to \mathbf{Set}$ be a covariant functor (Definition 1.1.1) to the category of sets (Definition 1.0.3). Let $h^A : \mathcal{C} \to \mathbf{Set}$ denote the covariant representable functor defined by $h^A(X) = \mathrm{Hom}_{\mathcal{C}}(A, X)$.

There exists a bijection
$$y_{A,F} : \mathrm{Nat}(h^A, F) \overset{\cong}{\to} F(A)$$
between the set of natural transformations (Definition 1.1.3) from h^A to F and the set $F(A)$. This bijection is given by the mapping
$$\alpha \mapsto \alpha_A(\mathrm{id}_A),$$
where $\alpha : h^A \to F$ is a natural transformation, $\alpha_A : h^A(A) \to F(A)$ is its component at A, and $\mathrm{id}_A \in h^A(A) = \mathrm{Hom}_{\mathcal{C}}(A, A)$ is the identity morphism.

Furthermore, this isomorphism is natural in both A and F. Explicitly:

1. For any morphism $f : A \to B$ in $\mathcal{C}$, the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{Nat}(h^B, F) & \overset{y_{B,F}}{\longrightarrow} & F(B) \\
{\scriptstyle -\circ h^f}\downarrow & & \downarrow{\scriptstyle F(f)} \\
\mathrm{Nat}(h^A, F) & \overset{y_{A,F}}{\longrightarrow} & F(A)
\end{array}
$$

   where $h^f : h^B \to h^A$ is the natural transformation induced by pre-composition with f.
2. For any natural transformation $\eta : F \to G$, the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{Nat}(h^A, F) & \overset{y_{A,F}}{\longrightarrow} & F(A) \\
{\scriptstyle \eta\circ -}\downarrow & & \downarrow{\scriptstyle \eta_A} \\
\mathrm{Nat}(h^A, G) & \overset{y_{A,G}}{\longrightarrow} & G(A)
\end{array}
$$

Corollary 1.1.11 (Yoneda Embedding)**.** Let $\mathcal{C}$ be a locally small category (Definition 1.0.9). The functor
$$h^{\bullet} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathcal{C}}$$
(Definition 1.0.12) (Definition 1.2.1) defined on objects by $A \mapsto h^A = \mathrm{Hom}_{\mathcal{C}}(A, -)$ and on morphisms by $f \mapsto h^f = (- \circ f)$ is fully faithful (Definition 1.1.4). That is, for any objects A, B in $\mathcal{C}$, the map
$$\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Nat}(h^B, h^A)$$
given by sending a morphism $f : A \to B$ to the natural transformation (Definition 1.1.3) $h^f : h^B \to h^A$ (pre-composition by f) is a bijection.

Consequently, $\mathcal{C}^{\mathrm{op}}$ embeds as a full subcategory (Definition 1.3.2) of the functor category $\mathbf{Set}^{\mathcal{C}}$.

Theorem 1.1.12 (Contravariant Yoneda Lemma)**.** Let $\mathcal{C}$ be a locally small category (Definition 1.0.9). Let A be an object of $\mathcal{C}$, and let $G : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ be a contravariant functor (Definition 1.1.1) (i.e. a presheaf (Definition 5.0.1)). Let $h_A : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ denote the contravariant representable functor defined by $h_A(X) = \mathrm{Hom}_{\mathcal{C}}(X, A)$.

There exists a bijection natural in A and G:

$$\mathrm{Nat}(h_A, G) \cong G(A)$$

given by $\alpha \mapsto \alpha_A(\mathrm{id}_A)$.

1.1.3. *Adjoint functors between categories.*

Definition 1.1.13. Let $\mathcal{C}$ and $\mathcal{D}$ be categories (Definition 1.0.1). Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors.

An *adjunction between F and G* consists of two natural transformations (Definition 1.1.3): $\eta : \mathrm{Id}_{\mathcal{C}} \implies GF$ (the *unit*), and $\varepsilon : FG \implies \mathrm{Id}_{\mathcal{D}}$ (the *counit*)

These must satisfy the triangle identities: For every object $X \in \mathcal{C}$ and $Y \in \mathcal{D}$,

$$\varepsilon_{FX} \circ F(\eta_X) = \mathrm{id}_{FX}$$

$$G(\varepsilon_Y) \circ \eta_{GY} = \mathrm{id}_{GY}.$$

In diagrammatic form, the triangle identities assert that the following are commutative diagrams:

$$
\begin{array}{ccc}
F(X) \xrightarrow{F(\eta_X)} FGF(X) & & G(Y) \xrightarrow{\eta_{G(Y)}} GFG(Y) \\
\quad{}_{\mathrm{id}_{F(X)}}\searrow \quad \downarrow{\varepsilon_{F(X)}} & & \quad{}_{\mathrm{id}_{G(Y)}}\searrow \quad \downarrow{G(\varepsilon_Y)} \\
F(X) & & G(Y)
\end{array}
$$

We say that F is a *left adjoint to G* and G is a *right adjoint to F* (written $F \dashv G$).

In the case that $\mathcal{C}$ and $\mathcal{D}$ are locally small (Definition 1.0.9) categories (or U-locally small categories if a universe (Definition E.0.1) U is available), we have an adjunction $F \dashv G$ if and only if for every object X in $\mathcal{C}$ and Y in $\mathcal{D}$ there is a natural isomorphism (Definition 1.1.3)

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

that is natural in both X and Y. In this case, the *unit of the adjunction* is the natural transformation $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ such that,

1. for every $X \in \mathcal{C}$, the morphism $\eta_X : X \to GF(X)$ (each called a *unit morphism*) in $\mathcal{C}$ is obtained as the image of $\mathrm{id}_{F(X)}$ via the adjoint isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(F(X), F(X)) \cong \mathrm{Hom}_{\mathcal{C}}(X, GF(X)).$$

2. for every $Y \in \mathcal{D}$, the morphism $\epsilon_Y : FG(Y) \to Y$ (each called a *counit morphism*) in $\mathcal{D}$ is obtained as the image of $\mathrm{id}_{G(Y)}$ via the adjoint isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(G(Y), G(Y)) \cong \mathrm{Hom}_{\mathcal{D}}(FG(Y), Y).$$

Proposition 1.1.14 (Factorization through the counit and the unit). Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let
$$L : \mathcal{C} \to \mathcal{D}, \qquad R : \mathcal{D} \to \mathcal{C}$$
be adjoint functors (Definition 1.1.13) with $L \dashv R$. Denote by
$$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow R \circ L, \qquad \varepsilon : L \circ R \Rightarrow \mathrm{Id}_{\mathcal{D}}$$
the unit and counit of the adjunction (Definition 1.1.13).

(1) For every $X' \in \mathrm{Ob}(\mathcal{C})$ and $X \in \mathrm{Ob}(\mathcal{D})$, a morphism
$$f : L(X') \to X \quad \text{in } \mathcal{D}$$
factors uniquely as
$$f = \varepsilon_X \circ L(g),$$
for some morphism $g : X' \to R(X)$ in $\mathcal{C}$. Equivalently, the assignment $g \mapsto \varepsilon_X \circ L(g)$ gives a bijection
$$\mathrm{Hom}_{\mathcal{C}}(X', R(X)) \;\cong\; \mathrm{Hom}_{\mathcal{D}}(L(X'), X),$$
which is precisely the adjunction isomorphism.

(2) Dually, for every $X \in \mathrm{Ob}(\mathcal{D})$ and $X' \in \mathrm{Ob}(\mathcal{C})$, a morphism
$$h : X \to R(X') \quad \text{in } \mathcal{C}$$
factors uniquely as
$$h = R(f) \circ \eta_X,$$
for some morphism $f : L(X) \to X'$ in $\mathcal{D}$. Equivalently, the assignment $f \mapsto R(f) \circ \eta_X$ gives a bijection
$$\mathrm{Hom}_{\mathcal{D}}(L(X), X') \;\cong\; \mathrm{Hom}_{\mathcal{C}}(X, R(X')),$$
again exhibiting the adjunction isomorphism.

1.2. Diagrams, systems, and limits in categories.

Definition 1.2.1 (Diagram in a category and category of diagrams). Let $\mathcal{C}$ be a (large) category (Definition 1.0.1), and let I be a (large) category (Definition 1.0.1).

1. A *diagram of shape I in $\mathcal{C}$* is a functor (Definition 1.1.1) $D : I \to \mathcal{C}$. We often denote such a diagram by the family $\{D(i)\}_{i \in \mathrm{Ob}(I)}$ with transition maps given by the functorial image of morphisms in I.

   A diagram is also synonymously called a *system*. Moreover, the category I is called the *index category* or the *indexing category of the diagram D*.
2. Given two diagrams $D, E : I \to \mathcal{C}$, a *morphism of diagrams* is a simply a natural transformation (Definition 1.1.3) $D \Rightarrow E$ of the functors D and E.
3. The *category of I-shaped diagrams in $\mathcal{C}$* or simply *diagram category (of I-shaped diagrams in $\mathcal{C}$)*, often denoted $\mathcal{C}^I$, $[I, \mathcal{C}]$, or $\mathrm{Fun}(I, \mathcal{C})$, is the (large) category whose objects are functors $I \to \mathcal{C}$ (that is, diagrams of shape I in $\mathcal{C}$) and whose morphisms are natural transformations (Definition 1.1.3) between such functors. The category $\mathcal{C}^I$ is also called the *functor category of functors $I \to \mathcal{C}$*. Equivalently, the functor category $\mathcal{C}^I$ is the category $\mathrm{PreShv}(I^{\mathrm{op}}, \mathcal{C})$ of presheaves (Definition 5.0.1) on I^{op} with

values in $\mathcal{C}$ and hence notations for presheaf categories are applicable as notations for functor categories.

If $\mathcal{C}$ is locally small (Definition 1.0.9) and I is small, then $\mathcal{C}^I$ is locally small by Lemma 5.0.2.

Intuitively, a limit of a diagram in a category is a "universal" object with a morphism to (all objects in) the diagram and a colimit, the dual notion, is a "universal" object with a morphism from (all objects in) the diagram.

Definition 1.2.2 (Cones, limits and colimits in a category). Let $\mathcal{C}$ be a (large) category (Definition 1.0.1), let I be a (large) category, and let $D : I \to \mathcal{C}$ be a diagram (Definition 1.2.1) (Definition 1.2.1).

1. A *cone to the diagram D* is an object $L \in \mathcal{C}$ together with a family of morphisms

$$\{\pi_i : L \to D(i)\}_{i \in I}$$

such that for every morphism $f : i \to j$ in I, the diagram

$$
\begin{array}{ccc}
& L & \\
\pi_i \swarrow & & \searrow \pi_j \\
D(i) & \xrightarrow{D(f)} & D(j)
\end{array}
$$

commutes, i.e. $D(f) \circ \pi_i = \pi_j$.

2. A cone $(L, \{\pi_i\})$ is called a *limit of D* if it satisfies the following "universal property": for any cone $(C, \{f_i\})$ over D, there exists a *unique* morphism $u : C \to L$ such that

$$\pi_i \circ u = f_i \quad \text{for all } i \in I.$$

Visually, the following diagrams commute every morphism $f : i \to j$ in I:

$$
\begin{array}{ccc}
& C & \\
& \downarrow{\exists! u} & \\
f_i & L & f_j \\
& \pi_i \swarrow \quad \searrow \pi_j & \\
D(i) & \xrightarrow{D(f)} & D(j).
\end{array}
$$

If such a cone exists, then the object L is necessarily unique up to unique isomorphism by the universal property. In this case, L is denoted by $\lim_{i \in I} D$ or $\lim D$.

3. A *cocone from the diagram D* is an object $C \in \mathcal{C}$ together with a family of morphisms

$$\{\iota_i : D(i) \to C\}_{i \in I}$$

such that for every morphism $f : i \to j$ in I, the diagram

$$D(i) \xrightarrow{\quad D(f) \quad} D(j)$$

with arrows ι_i and ι_j going down to C.

commutes, i.e. $\iota_j \circ D(f) = \iota_i$.

4. A cocone $(L, \{\iota_i\})$ is called a colimit of D if it satisfies the following "universal property": for any cocone $(C, \{g_i\})$ under D, there exists a *unique* morphism $u : L \to C$ such that

$$u \circ \iota_i = g_i \quad \text{for all } i \in I.$$

Visually, the following diagrams commute every morphism $f : i \to j$ in I:

$$D(i) \xrightarrow{\quad D(f) \quad} D(j)$$

with ι_i, ι_j going to L, g_i, g_j going to C, and $\exists! u$ from L to C.

If such a cocone exists, then the object L is necessarily unique up to unique isomorphism by the universal property. In this case, L is denoted by $\operatorname{colim}_{i \in I} D$ or $\operatorname{colim} D$.

A limit/colimit is called *finite* (resp. *small*) if the index category I is finite (resp. small).

Some authors use the terms *projective limit* or *inverse limit* to refer to what is defined here as a limit, Similarly, the terms *inductive limit* or *direct limit* are sometimes used to mean a colimit. However, these phrases can have more specific meanings to other authors: a *projective* or *inverse limit* may refer to a limit over a diagram indexed by a codirected poset (Definition 1.2.7). Likewise, an *inductive* or *direct limit* may refer to a colimit over a directed poset (Definition 1.2.7) (see Definition 1.2.12) .

Thus, while the terms are sometimes used interchangeably with "limit" and "colimit," they may also emphasize particular indexing shapes and directions, distinguishing them from general limits and colimits taken over arbitrary small categories.

Theorem 1.2.3. Let $\mathcal{C}$ be a category (Definition 1.0.1), and let $\mathcal{J}$ be a category such that the limits (resp. colimits) (Definition 1.2.2) indexed by $\mathcal{J}$ exist in $\mathcal{C}$. Then the process of taking limits (resp. colimits) defines a functor

$$\lim : \mathcal{C}^{\mathcal{J}} \to \mathcal{C} \quad (\text{resp. } \operatorname{colim} : \mathcal{C}^{\mathcal{J}} \to \mathcal{C})$$

from the functor category (Definition 1.2.1) $\mathcal{C}^{\mathcal{J}}$ to $\mathcal{C}$.

More precisely:

1. If $\mathcal{C}$ has all limits indexed by the category $\mathcal{J}$, then the assignment sending each diagram $D : \mathcal{J} \to \mathcal{C}$ to its limit $\lim D$ extends to a functor

$$\lim : \mathrm{Fun}(\mathcal{J}, \mathcal{C}) \to \mathcal{C}.$$

   This functor sends a natural transformation $\alpha : D \to D'$ to the unique morphism $\lim D \to \lim D'$ induced by the universal property of limits.

2. Similarly, if $\mathcal{C}$ has all colimits indexed by $\mathcal{J}$, then the assignment sending each diagram $D : \mathcal{J} \to \mathcal{C}$ to its colimit $\mathrm{colim}\, D$ extends to a functor

$$\mathrm{colim} : \mathrm{Fun}(\mathcal{J}, \mathcal{C}) \to \mathcal{C}.$$

   This functor sends a natural transformation $\alpha : D \to D'$ to the unique morphism $\mathrm{colim}\, D \to \mathrm{colim}\, D'$ induced by the universal property of colimits.

For a diagram indexed by an essentially small category I, we can identify their limits/colimits with those of diagrams indexed by a small category J equivalent to I.

Theorem 1.2.4 (Limits and Colimits indexed by essentially small categories)**.** Let $\mathcal{C}$ be a (large) category (Definition 1.0.1) and let I be an essentially small category (Definition 1.1.9). Suppose $D : I \to \mathcal{C}$ is a diagram indexed by I (Definition 1.2.1).

If J is a small category (Definition 1.0.9) that is equivalent (Definition 1.1.8) to I via an equivalence of categories

$$F : J \xrightarrow{\sim} I,$$

then the limit and colimit (Definition 1.2.2) of D indexed by I may be identified with the limit and colimit of the composed diagram

$$D \circ F : J \to \mathcal{C}.$$

In other words, if either $\lim D$ or $\mathrm{colim}\, D$ exists in $\mathcal{C}$, then so does $\lim(D \circ F)$ and $\mathrm{colim}(D \circ F)$, and these limits and colimits are canonically isomorphic as objects of $\mathcal{C}$.

Definition 1.2.5. Let $\mathcal{C}$ be a (large) category (Definition 1.0.1), let I be a small category (Definition 1.0.9), and let $D : I \to \mathcal{C}$ be a diagram (Definition 1.2.1).

A limit or colimit (Definition 1.2.2) is called *finite* (resp. *small*) if the indexing category I has finitely many objects and morphisms (resp. if I is a small category (Definition 1.0.9)).

Theorem 1.2.6. Let $\mathcal{C}$ be a category (Definition 1.0.1). Let $F : \mathcal{J} \to \mathcal{C}$ be a diagram (Definition 1.2.1) where $\mathcal{J}$ is a small category (Definition 1.0.9).

1. The limit (Definition 1.2.2) of F is constructed as the equalizer (Definition 1.2.31) of the pair of morphisms (s, t): assuming that the products (Definition 1.2.30) and equalizers below exist in $\mathcal{C}$, the limit $\lim F$ exists and

$$\lim F \cong \mathrm{eq}\left(\prod_{j \in \mathrm{Ob}(\mathcal{J})} F(j) \xrightarrow[t]{s} \prod_{\alpha \in \mathrm{Mor}(\mathcal{J})} F(\mathrm{cod}(\alpha)) \right)$$

   where $\mathrm{cod}(\alpha)$ stands for the codomain of the morphism α, and the morphisms s and t are induced by the universal property of the product, such that for any morphism $\alpha : i \to k$ in $\mathcal{J}$, the projection to the factor indexed by α is:

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- $\pi_\alpha \circ s = F(\alpha) \circ \pi_i$
- $\pi_\alpha \circ t = \pi_k$

2. The colimit (Definition 1.2.2) of F is constructed as the coequalizer (Definition 1.2.31) of the pair of morphisms (s, t): assuming that the coproducts (Definition 1.2.30) and coequalizers below exist in $\mathcal{C}$, the colimit $\operatorname{colim} F$ exists and

$$\operatorname{colim} F \cong \operatorname{coeq}\left(\coprod_{\alpha \in \operatorname{Mor}(\mathcal{J})} F(\operatorname{dom}(\alpha)) \underset{t}{\overset{s}{\rightrightarrows}} \coprod_{j \in \operatorname{Ob}(\mathcal{J})} F(j) \right)$$

where $\operatorname{dom}(\alpha)$ stands for the domain of the morphism α, and the morphisms s and t are induced by the universal property of the coproduct, such that for any morphism $\alpha : i \to k$ in $\mathcal{J}$, the injection from the summand indexed by α is:
- $s \circ \iota_\alpha = \iota_k \circ F(\alpha)$
- $t \circ \iota_\alpha = \iota_i$

In particular,

1. If $\mathcal{C}$ has all nonempty finite (resp. small) products and equalizers, then $\mathcal{C}$ has all nonempty finite (resp. small) limits.
2. If $\mathcal{C}$ has all nonempty finite (resp. small) coproducts and coequalizers, then $\mathcal{C}$ has all nonempty finite (resp. small) colimits.
3. If $\mathcal{C}$ has all finite (resp. small) products and equalizers, then $\mathcal{C}$ has all finite (resp. small) limits.
4. If $\mathcal{C}$ has all finite (resp. small) coproducts and coequalizers, then $\mathcal{C}$ has all finite (resp. small) colimits.

Definition 1.2.7 (Partially ordered set). 1. A *partially ordered set* (or *poset*), or *ordered set* is a pair $(P, \leq)$ where P is a set and

$$\leq : P \times P \to \{\text{true}, \text{false}\}$$

is a binary relation on P satisfying the following axioms for all $a, b, c \in P$:
- *Reflexivity:* $a \leq a$,
- *Antisymmetry:* if $a \leq b$ and $b \leq a$, then $a = b$,
- *Transitivity:* if $a \leq b$ and $b \leq c$, then $a \leq c$.

The relation $\leq$ is called an *order* or a *partial order*

2. A partially ordered set $(P, \leq)$ is called a *directed partially ordered set* if for every pair $a, b \in P$, there exists $c \in P$ such that

$$a \leq c \quad \text{and} \quad b \leq c.$$

3. A partially ordered set $(P, \leq)$ is called a *codirected partially ordered set* (or *downward directed poset*) if for every pair $a, b \in P$, there exists $d \in P$ such that

$$d \leq a \quad \text{and} \quad d \leq b.$$

Definition 1.2.8 (Filtered category). 1. A *filtered category* is a (nonempty, large) category $\mathcal{I}$ satisfying the following conditions:
- For every finite collection of objects $i_1, i_2, \ldots, i_n$ in $\mathcal{I}$, there exists an object j and morphisms

$$\phi_k : i_k \to j, \quad \text{for each } k = 1, \ldots, n.$$

- For every pair of morphisms $f, g : i \to j$ in $\mathcal{I}$, there exists an object k and a morphism
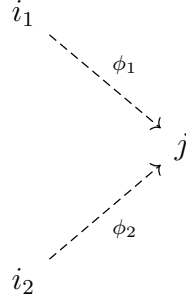$$h : j \to k$$
that satisfies
$$h \circ f = h \circ g.$$

FIGURE 1. *
Condition 1: Upper Bound

FIGURE 2. *
Condition 2: Coequalizing map

In other words, $\mathcal{I}$ is nonempty, any finite diagram of objects admits a cocone (Definition 1.2.2), and any pair of parallel morphisms become equal after post-composition with an appropriate morphism.

2. Dually, a *Cofiltered category* is a category whose opposite category (Definition 1.0.12) is filtered. More explicitly, A cofiltered category is a (nonempty, large) category $\mathcal{I}$ satisfying the following conditions:
   - For every finite collection of objects $i_1, i_2, \ldots, i_n$ in $\mathcal{I}$, there exists an object j and morphisms
   $$\phi_k : j \to i_k, \quad \text{for each } k = 1, \ldots, n.$$
   - For every pair of morphisms $f, g : j \to i$ in $\mathcal{I}$, there exists an object k and a morphism
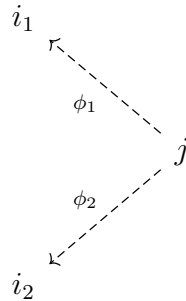 $$h : k \to j$$
   that satisfies
   $$f \circ h = g \circ h.$$

FIGURE 3. *
Condition 1: Lower Bound

FIGURE 4. *
Condition 2: Equalizing map

In other words, $\mathcal{I}$ is nonempty, any finite diagram of objects admits a cone, and any pair of parallel morphisms become equal after pre-composition with an appropriate morphism.

Definition 1.2.9 (Thin category). A *thin category* is a (large) category $\mathcal{C}$ such that for every pair of objects $X, Y \in \mathcal{C}$, the hom-collection

$$\mathrm{Hom}_{\mathcal{C}}(X, Y)$$

has at most one element.

Lemma 1.2.10. Let $(P, \leq)$ be a nonempty poset (Definition 1.2.7).

1. Regarding P as a category whose objects are the elements of P and such that there is a unique arrow $a \to b$ if and only if $a \leq b$, the category is filtered.
2. Every nonempty small (Definition 1.0.9) thin (Definition 1.2.9) filtered category (Definition 1.2.8) corresponds to a poset in this way.
3. Moreover, the poset P is directed (Definition 1.2.7) if and only if the category is filtered. The poset P is codirected (Definition 1.2.7) if and only if the category is cofiltered.

Definition 1.2.11 (Systems in a category). Let $\mathcal{C}$ be a (large) category. Let I be a (large) category.

1. A diagram/system (Definition 1.2.1) $I \to C$ is called *filtered* (resp. *cofiltered*) if I is a filtered (Definition 1.2.8) (resp. cofiltered (Definition 1.2.8)) category.
2. A diagram/system $I \to C$ is called *directed* (resp. *codirected* if I is small and thing, i.e. is regardable/comes from (Lemma 1.2.10) a directed (resp. codirected) partially ordered set (Definition 1.2.7). A *direct system* or *inductive system* is synonymous for a directed system and a *inverse system* or *projective system* is synonymous for a codirected system.

One might also speak of a *filtered direct/inductive system* synonymously for a filtered system to emphasize that the indexing caetgory is a general filtered category, rather than a directed poset.

Definition 1.2.12 (Special cases of limits). Let $\mathcal{C}$ be a (large) category. Let I be a (large) category. Let $I \to \mathcal{C}$ be a diagram/system.

- Suppose that the system is a cofiltered system (Definition 1.2.11), i.e. I is a cofiltered category. A limit (Definition 1.2.2) of this diagram is often denoted by

$$\varprojlim_{i \in I} D(i)$$

and may be called a *cofiltered (inverse/projective) limit*. In case that the system is more specifically an inverse/projective system (Definition 1.2.11), i.e. I is a cofiltered poset (Definition 1.2.7), the preferred term for such a limit is *inverse/projective limit*.

- Suppose that the system is a filtered system, i.e. I is a filtered category. A colimit of this diagram is often denoted by

$$\varinjlim_{i \in I} D(i)$$

and may be called a *filtered colimit* or a *direct/inductive/injective limit*. In case that the system is more specifically a direct/inductive system, i.e. I is a filtered poset (Definition 1.2.7), the preferred term for such a limit is *direct/inductive limit*.

Definition 1.2.13. Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

1. Assume $\mathcal{C}$ admits all finite limits (Definition 1.2.2). A functor $F : \mathcal{C} \to \mathcal{D}$ is called *left exact* if it preserves all finite limits. Explicitly, for every finite diagram $D : \mathcal{J} \to \mathcal{C}$, the limit

$$\varprojlim(F \circ D)$$

exists and the canonical map

$$F(\varprojlim D) \xrightarrow{\cong} \varprojlim(F \circ D)$$

is an isomorphism.
2. Assume $\mathcal{C}$ admits all finite colimits (Definition 1.2.2). A functor $F : \mathcal{C} \to \mathcal{D}$ is called *right exact* if it preserves all finite colimits. Explicitly, for every finite diagram $D : \mathcal{J} \to \mathcal{C}$, the colimit

$$\varinjlim(F \circ D)$$

exists and the canonical map

$$\varinjlim(F \circ D) \xrightarrow{\cong} F(\varinjlim D)$$

is an isomorphism.
3. If $\mathcal{C}$ admits both finite limits and finite colimits, a functor F is called *exact* if it is both left exact and right exact.

Lemma 1.2.14. Let $\mathcal{E}$ and $\mathcal{F}$ be topoi (Definition 5.1.45).

1. A functor $F : \mathcal{E} \to \mathcal{F}$ is left exact (Definition 1.2.13) if and only if it preserves terminal objects (Definition 2.0.6) and pullbacks (Definition 1.2.32).
2. A functor $F : \mathcal{E} \to \mathcal{F}$ is right exact (Definition 1.2.13) if and only if it preserves initial objects (Lemma 2.0.7) and pushouts (Definition 1.2.20).

Definition 1.2.15. Let $\mathcal{C}$ be a category (Definition 1.0.1), let Z be an object, and let X, Y be objects of $\mathcal{C}$ over (Definition 1.3.4) Z, i.e. morphisms $X \to Z$ and $Y \to Z$ are fixed. A *cartesian product of X and Y over Z in $\mathcal{C}$* (or *fiber product* or *pullback diagram*) is an object, often denoted by $X \times_Z Y$, with *projection morphisms* $X \times_Z Y \to X$ and $X \times_Z Y \to Y$ that are universal. More precisely, for any object T of $\mathcal{C}$ and morphisms $f_X : T \to X$, $f_Y : T \to Y$, there exists a unique morphism $u : T \to X \times_Z Y$ such that the following diagram commutes:

Equivalently, $X \times_Z Y$ is the limit (Definition 1.2.2) of the diagram (Definition 1.2.1)

$$
\begin{array}{ccc}
 & & X \\
 & & \downarrow \\
Y & \longrightarrow & Z
\end{array}
$$

in $\mathcal{C}$.

The commutative diagram

$$
\begin{array}{ccc}
X \times_Z Y & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Z
\end{array}
$$

may be referred to as a *cartesian square*.

Definition 1.2.16. Let $\mathcal{C}$ be a category (Definition 1.0.1) which admits finite products (Definition 1.2.30). Let Y, Z be objects of $\mathcal{C}$.

1. An *exponential object from Y to Z* is an object Z^Y (also denoted $[Y, Z]$ or $\underline{\mathrm{hom}}(Y, Z)$) equipped with a morphism

$$
\mathrm{ev} : Z^Y \times Y \to Z
$$

   called the *evaluation morphism*, satisfying the following universal property:
2. For any object X in $\mathcal{C}$ and any morphism $f : X \times Y \to Z$, there exists a unique morphism $\tilde{f} : X \to Z^Y$ (often called the *transpose* or *currying of f*) such that the following diagram commutes:

$$
\begin{array}{ccc}
X \times Y & \xrightarrow{\tilde{f} \times \mathrm{id}_Y} & Z^Y \times Y \\
 & \searrow{\scriptstyle f} & \downarrow{\scriptstyle \mathrm{ev}} \\
 & & Z
\end{array}
$$

   Equivalently, the functor $- \times Y : \mathcal{C} \to \mathcal{C}$ has a right adjoint (Definition 1.1.13), denoted $(-)^Y : \mathcal{C} \to \mathcal{C}$.

Definition 1.2.17. Let $\mathcal{C}$ be a category (Definition 1.0.1). $\mathcal{C}$ is called *cartesian closed* if it satisfies the following three conditions:

1. It has a terminal object (Definition 2.0.6) (empty product), often denoted $\mathbf{1}$.

2. It has all binary products (Definition 1.2.30) (and thus all finite products). That is, for any objects $A, B \in \mathcal{C}$, the product $A \times B$ exists.
3. It has exponential objects for all object $Y \to Z$.

Definition 1.2.18. Let $\mathcal{C}$ be a category with a terminal object (Lemma 2.0.7) 1. A *subobject classifier* is an object Ω together with a morphism $\text{true} : 1 \to \Omega$ satisfying the following universal property:

For any monomorphism (Definition 1.3.7) $m : U \hookrightarrow X$ in $\mathcal{C}$, there exists a unique morphism $\chi_m : X \to \Omega$ (called the *characteristic morphism of m*) such that the following diagram is a pullback square (Definition 1.2.32):

$$
\begin{array}{ccc}
U & \xrightarrow{\;\;!\;\;} & 1 \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \text{true}} \\
X & \xrightarrow[\chi_m]{} & \Omega
\end{array}
$$

where $U \to 1$ is the unique morphism to the terminal object. Intuitively, Ω acts as a "truth value object" where χ_m assigns "true" to the "points" of X that lie in the subobject U.

Remark 1.2.19. In the category of sets (**Set**), the subobject classifier is the set $\Omega = \{0, 1\}$ (often denoted 2), and "true" is the function picking out 1. The characteristic morphism χ_m corresponds to the characteristic function of the subset $U \subseteq X$.

Definition 1.2.20. Let $\mathcal{C}$ be a category (Definition 1.0.1), let Z be an object, and let X, Y be objects of $\mathcal{C}$ under (Definition 1.3.4) Z, i.e. morphisms $Z \to X$ and $Z \to Y$ are fixed. A *cocartesian sum of X and Y under Z in $\mathcal{C}$* (or *fiber coproduct* or *pushout diagram*) is an object, often denoted by $X \amalg_Z Y$, with *inclusion morphisms* $X \to X \amalg_Z Y$ and $Y \to X \amalg_Z Y$ that are universal. More precisely, for any object T of $\mathcal{C}$ and morphisms $j_X : X \to T$, $j_Y : Y \to T$, there exists a unique morphism $v : X \amalg_Z Y \to T$ such that the following diagram commutes:

$$
\begin{array}{ccc}
Z & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X \amalg_Z Y
\end{array}
$$

Equivalently, $X \amalg_Z Y$ is the colimit (Definition 1.2.2) of the diagram (Definition 1.2.1)

$$ X \longleftarrow Z \longrightarrow Y $$

in $\mathcal{C}$.

The commutative diagram

$$Z \longrightarrow X$$
$$\downarrow \qquad \downarrow$$
$$Y \longrightarrow X \amalg_Z Y$$

may be referred to as a *cocartesian square*.

Theorem 1.2.24 states the right adjoint functors commute with limits and left adjoint functors commute with colimits.

Definition 1.2.21. Let $\mathcal{C}$ and $\mathcal{D}$ be (large) categories (Definition 1.0.1), $F : \mathcal{C} \to \mathcal{D}$ a functor (Definition 1.1.1), and $D : \mathcal{J} \to \mathcal{C}$ a diagram (Definition 1.2.1) indexed by a small category $\mathcal{J}$.

1. Assume that the colimit (Definition 1.2.2) of D exists in $\mathcal{C}$, denoted by $\operatorname{colim}_{\mathcal{J}} D$, with universal cocone (Definition 1.2.2) $\eta : D \Rightarrow \Delta_{\operatorname{colim} D}$. The functor F maps this to a cocone $F(\eta) : F \circ D \Rightarrow \Delta_{F(\operatorname{colim} D)}$ in $\mathcal{D}$.
   If the colimit of the composite diagram $F \circ D$ exists in $\mathcal{D}$, denoted by $\operatorname{colim}_{\mathcal{J}}(F \circ D)$, then by the universal property of the colimit, there exists a unique morphism

   $$\phi : \operatorname{colim}_{j \in \mathcal{J}} F(D(j)) \longrightarrow F\left(\operatorname{colim}_{j \in \mathcal{J}} D(j)\right)$$

   mediating between the cocone of the composite colimit and the image cocone $F(\eta)$. This unique morphism is called the *canonical colimit comparison morphism*.
2. Dually, assume that the limit of D exists in $\mathcal{C}$, denoted by $\lim_{\mathcal{J}} D$, with universal cone $\varepsilon : \Delta_{\lim D} \Rightarrow D$. The functor F maps this to a cone $F(\varepsilon) : \Delta_{F(\lim D)} \Rightarrow F \circ D$ in $\mathcal{D}$.
   If the limit of the composite diagram $F \circ D$ exists in $\mathcal{D}$, denoted by $\lim_{\mathcal{J}}(F \circ D)$, then by the universal property of the limit, there exists a unique morphism

   $$\psi : F\left(\lim_{j \in \mathcal{J}} D(j)\right) \longrightarrow \lim_{j \in \mathcal{J}} F(D(j))$$

   mediating between the image cone $F(\varepsilon)$ and the cone of the composite limit. This unique morphism is called the *canonical limit comparison morphism*:

Definition 1.2.22. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor (Definition 1.1.1) between (large) categories (Definition 1.0.1) $\mathcal{C}$ and $\mathcal{D}$.

Let $D : \mathcal{J} \to \mathcal{C}$ be a diagram in $\mathcal{C}$ such that the limit $\lim_{\mathcal{J}} D$ exists in $\mathcal{C}$, with limiting cone $\lambda : \Delta_{\lim D} \Rightarrow D$. The functor F *preserves the limit of D* if the image cone

$$F(\lambda) : \Delta_{F(\lim D)} \cong F \circ \Delta_{\lim D} \Rightarrow F \circ D$$

exhibits $F(\lim_{\mathcal{J}} D)$ as a limit of the composite diagram $F \circ D$ in $\mathcal{D}$. Explicitly, the canonical comparison morphism (Definition 1.2.21)

$$F(\lim_{\mathcal{J}} D) \xrightarrow{\cong} \lim_{\mathcal{J}}(F \circ D)$$

must be an isomorphism.

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Dually, let $D : \mathcal{J} \to \mathcal{C}$ be a diagram such that the colimit $\operatorname{colim}_{\mathcal{J}} D$ exists in $\mathcal{C}$. The functor F *preserves the colimit of D* if the canonical comparison morphism (Definition 1.2.21)

$$\operatorname{colim}_{\mathcal{J}}(F \circ D) \overset{\cong}{\to} F(\operatorname{colim}_{\mathcal{J}} D)$$

is an isomorphism.

Definition 1.2.23. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor (Definition 1.1.1) between (large) categories (Definition 1.0.1) $\mathcal{C}$ and $\mathcal{D}$.

- The functor F is called *continuous* if it preserves all small limits that exist in $\mathcal{C}$. That is, for every small category $\mathcal{J}$ and every diagram $D : \mathcal{J} \to \mathcal{C}$ having a limit in $\mathcal{C}$, F preserves the limit (Definition 1.2.22) of D.
- The functor F is called *cocontinuous* if it preserves all small colimits that exist in $\mathcal{C}$. That is, for every small category $\mathcal{J}$ and every diagram $D : \mathcal{J} \to \mathcal{C}$ having a colimit in $\mathcal{C}$, F preserves the colimit (Definition 1.2.22) of D.

Theorem 1.2.24 (Continuity of Adjoint Functors). Let $\mathcal{C}$ and $\mathcal{D}$ be (large) categories (Definition 1.0.1), and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors forming an adjunction (Definition 1.1.13) $F \dashv G$, i.e. F is the left adjoint and G is the right adjoint. Let $\mathcal{J}$ be a small index category.

1. **Right Adjoints Preserve Limits:** Let $D : \mathcal{J} \to \mathcal{D}$ be a diagram (Definition 1.2.1) such that the limit (Definition 1.2.2) $\lim_{\mathcal{J}} D$ exists in $\mathcal{D}$. Then the limit of the composite diagram $G \circ D : \mathcal{J} \to \mathcal{C}$ exists in $\mathcal{C}$, and G preserves the limit (Definition 1.2.22), i.e. the comparison morphism (Definition 1.2.21) induced by the universal property of the limit,

$$\vartheta : G\left(\lim_{j \in \mathcal{J}} D(j)\right) \overset{\cong}{\to} \lim_{j \in \mathcal{J}} G(D(j)),$$

   is an isomorphism.
2. **Left Adjoints Preserve Colimits:** Let $D : \mathcal{J} \to \mathcal{C}$ be a diagram such that the colimit $\operatorname{colim}_{\mathcal{J}} D$ exists in $\mathcal{C}$. Then the colimit of the composite diagram $F \circ D : \mathcal{J} \to \mathcal{D}$ exists in $\mathcal{D}$, and F preserves the colimits (Definition 1.2.22), i.e. the comparison morphism (Definition 1.2.21) induced by the universal property of the colimit,

$$\varphi : \operatorname{colim}_{j \in \mathcal{J}} F(D(j)) \overset{\cong}{\to} F\left(\operatorname{colim}_{j \in \mathcal{J}} D(j)\right),$$

   is an isomorphism.

Theorem 1.2.25 states the colimits commutes with colimits and limits commute with limits.

Theorem 1.2.25 (Fubini Theorem for Colimits and Limtis). Let $\mathcal{C}$ be a category and let I and J be small categories. Let $D : I \times J \to \mathcal{C}$ be a bifunctor.

1. Assume that the following colimits (Definition 1.2.2) exist in $\mathcal{C}$:
   - for every object $i \in I$, $\varinjlim_{j \in J} D(i, j)$
   - $\varinjlim_{i \in I} \left(\varinjlim_{j \in J} D(i, j)\right)$
   - for every object $j \in J$, $\varinjlim_{i \in I} D(i, j)$

- $\varinjlim_{j \in J} \left(\varinjlim_{i \in I} D(i, j) \right)$.

Then there is a canonical isomorphism:

$$\varinjlim_{i \in I} \left(\varinjlim_{j \in J} D(i, j) \right) \cong \varinjlim_{(i,j) \in I \times J} D(i, j) \cong \varinjlim_{j \in J} \left(\varinjlim_{i \in I} D(i, j) \right).$$

2. Assume that the following limits (Definition 1.2.2) exist in $\mathcal{C}$:
   - for every object $i \in I$, $\varprojlim_{j \in J} D(i, j)$
   - $\varprojlim_{i \in I} \left(\varprojlim_{j \in J} D(i, j) \right)$
   - for every object $j \in J$, $\varprojlim_{i \in I} D(i, j)$
   - $\varprojlim_{j \in J} \left(\varprojlim_{i \in I} D(i, j) \right)$.

   Then there is a canonical isomorphism:

$$\varprojlim_{i \in I} \left(\varprojlim_{j \in J} D(i, j) \right) \cong \varprojlim_{(i,j) \in I \times J} D(i, j) \cong \varprojlim_{j \in J} \left(\varprojlim_{i \in I} D(i, j) \right).$$

Remark 1.2.26. These results assert that the colimit functor $\mathrm{colim} : \mathcal{C}^{I \times J} \to \mathcal{C}$ (assuming $\mathcal{C}$ is cocomplete) is isomorphic to the composition of partial colimit functors $\mathrm{colim}_I \circ \mathrm{colim}_J$ and $\mathrm{colim}_J \circ \mathrm{colim}_I$.

At times, we may want to consider the limit/colimit of a system of functors. Such limits/colimits can be computed "pointwise", whenever the pointwise limits/colimits exist.

Proposition 1.2.27 (Pointwise Computation of Limits and Colimits in Functor Categories). Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Let $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ denote the category of functors from $\mathcal{C}$ to $\mathcal{D}$ (also denoted $\mathcal{D}^{\mathcal{C}}$). Let $\mathcal{J}$ be a small category and let $F : \mathcal{J} \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be a diagram of functors, denoted by $j \mapsto F_j$.

1. **Limits are computed pointwise:** Suppose that for every object $C \in \mathcal{C}$, the limit of the diagram $j \mapsto F_j(C)$ exists in $\mathcal{D}$. Then the limit of the diagram F exists in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and is computed pointwise. That is, there is an isomorphism in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$:

$$\left(\varprojlim_{j \in \mathcal{J}} F_j \right)(C) \cong \varprojlim_{j \in \mathcal{J}} (F_j(C)).$$

   The action of this limit functor on a morphism $f : C \to C'$ in $\mathcal{C}$ is the unique morphism induced by the family $\{F_j(f)\}_{j \in \mathcal{J}}$ via the universal property of limits in $\mathcal{D}$.

2. **Colimits are computed pointwise:** Suppose that for every object $C \in \mathcal{C}$, the colimit of the diagram $j \mapsto F_j(C)$ exists in $\mathcal{D}$. Then the colimit of the diagram F exists in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and is computed pointwise. That is, there is an isomorphism in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$:

$$\left(\mathrm{colim}_{j \in \mathcal{J}} F_j \right)(C) \cong \mathrm{colim}_{j \in \mathcal{J}} (F_j(C)).$$

   The action of this colimit functor on a morphism $f : C \to C'$ in $\mathcal{C}$ is the unique morphism induced by the family $\{F_j(f)\}_{j \in \mathcal{J}}$ via the universal property of colimits in $\mathcal{D}$.

Definition 1.2.28. Let $\mathcal{C}$ and $\mathcal{J}$ be categories (Definition 1.0.1). The *diagonal functor*

$$\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$$

(Definition 1.2.1) (♠ TODO: constant functor) is the functor sending an object $X \in \mathcal{C}$ to the constant functor $\Delta(X) : \mathcal{J} \to \mathcal{C}$, which maps every object in $\mathcal{J}$ to X and every morphism in $\mathcal{J}$ to the identity morphism 1_X. For a morphism $f : X \to Y$ in $\mathcal{C}$, $\Delta(f)$ is the natural transformation (Definition 1.1.3) whose component at every $j \in \mathcal{J}$ is f.

Theorem 1.2.29. Let $\mathcal{C}$ be a locally small (Definition 1.0.9) category and $\mathcal{J}$ be a small index category (Definition 1.0.9).

1. If $\mathcal{C}$ admits all colimits (Definition 1.2.2) of shape (Definition 1.2.1) $\mathcal{J}$, then the colimit functor (Theorem 1.2.3)

   $$\mathrm{colim} : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$$

   is left adjoint (Definition 1.1.13) to the diagonal functor Δ. That is, for any functor $F : \mathcal{J} \to \mathcal{C}$ and any object $X \in \mathcal{C}$, there is a natural bijection:

   $$\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}\, F, X) \cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{J}}}(F, \Delta(X)).$$

2. If $\mathcal{C}$ admits all limits of shape $\mathcal{J}$, then the limit functor (Theorem 1.2.3)

   $$\lim : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$$

   is right adjoint (Definition 1.1.13) to the diagonal functor (Definition 1.2.28) Δ. That is, for any object $X \in \mathcal{C}$ and any functor $F : \mathcal{J} \to \mathcal{C}$, there is a natural bijection:

   $$\mathrm{Hom}_{\mathcal{C}}(X, \lim F) \cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{J}}}(\Delta(X), F).$$

These adjunctions characterize limits and colimits via their universal properties.

1.2.1. *Some specific kinds of limits.*

Definition 1.2.30 (Product in a category). Let $\mathcal{C}$ be a category and let $\{X_i\}_{i \in I}$ be a family of objects in $\mathcal{C}$ indexed by a class I.

1. A *product of the family* $\{X_i\}$ is an object P of $\mathcal{C}$ together with a "universal" family of morphisms

   $$\pi_i : P \to X_i, \quad \text{for each } i \in I.$$

   More precisely, for any object Y and any family of morphisms $\{f_i : Y \to X_i\}_{i \in I}$, there exists a unique morphism

   $$f : Y \to P$$

   making the following diagram commute for all $i \in I$, i.e. $\pi_i \circ f = f_i$:

Such a product is often denoted by $\prod_{i \in I} X_i$. If $\prod_{i \in I} X_i$ exists in $\mathcal{C}$, then it is unique up to unique isomorphism by the universal property described above.

Equivalently, the product $\prod_{i \in I} X_i$ is the limit (Definition 1.2.2) of the diagram (Definition 1.2.1) $I \to \mathcal{C}, i \mapsto X_i$, where I is made into a category whose objects are the members of I and whose morphisms are just the identity morphisms.

2. A *coproduct* (or synonymously *direct sum*) of the family $\{X_i\}$ is an object C of $\mathcal{C}$ together with a "universal" family of morphisms

$$\iota_i : X_i \to C, \quad \text{for each } i \in I.$$

More precisely, for any object Y and any family of morphisms $\{g_i : X_i \to Y\}_{i \in I}$, there exists a unique morphism

$$g : C \to Y$$

making the following diagram commute for all $i \in I$, i.e. $g \circ \iota_i = g_i$:

$$
\begin{array}{ccc}
X_i & \xrightarrow{\ \iota_i\ } & \coprod X_i \\
& {\scriptstyle g_i} \searrow & \downarrow {\scriptstyle \exists! g} \\
& & Y
\end{array}
$$

Such a coproduct is often denoted by $\coprod_{i \in I} X_i$ or $\oplus_{i \in I} X_i$. If $\coprod_{i \in I} X_i$ exists in $\mathcal{C}$, then it is unique up to unique isomorphism by the universal property described above.

Equivalently, the coproduct $\coprod_{i \in I} X_i$ is the colimit (Definition 1.2.2) of the diagram (Definition 1.2.1) $I \to \mathcal{C}, i \mapsto X_i$, where I is made into a category whose objects are the members of I and whose morphisms are just the identity morphisms.

Definition 1.2.31 (Equalizer in a category). Let $\mathcal{C}$ be a (large) category (Definition 1.0.1) and let $f, g : X \to Y$ be morphisms in $\mathcal{C}$.

1. An *equalizer of f and g* is an object E together with a morphism

$$e : E \to X$$

such that

$$f \circ e = g \circ e$$

and for any object Z with morphism $z : Z \to X$ satisfying

$$f \circ z = g \circ z,$$

there exists a unique morphism $u : Z \to E$ making the diagram commute:

$$e \circ u = z.$$

$$
\begin{array}{ccc}
Z & & \\
\downarrow {\scriptstyle \exists! u} & \searrow {\scriptstyle z} & \\
E \xrightarrow{\ e\ } & X \underset{g}{\overset{f}{\rightrightarrows}} Y
\end{array}
$$

If such an equalizer of f and g exists, then we say that the following *equalizer diagram is exact*:

$$E \xrightarrow{\ e\ } X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

2. A *coequalizer of f and g* is an object Q together with a morphism

$$q : Y \to Q$$

such that

$$q \circ f = q \circ g$$

and for any object Z with morphism $w : Y \to Z$ satisfying

$$w \circ f = w \circ g,$$

there exists a unique morphism $v : Q \to Z$ making the diagram commute:

$$v \circ q = w.$$

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{\ q\ } Q$$

If such a coequalizer of f and g exists, then we say that the following *coequalizer diagram is exact*:

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{\ q\ } Q$$

Definition 1.2.32. Let $\mathcal{C}$ be a category (Definition 1.0.1), let Z be an object, and let X, Y be objects of $\mathcal{C}$ over (Definition 1.3.4) Z, i.e. morphisms $X \to Z$ and $Y \to Z$ are fixed. A *cartesian product of X and Y over Z in $\mathcal{C}$* (or *fiber product* or *pullback diagram*) is an object, often denoted by $X \times_Z Y$, with *projection morphisms* $X \times_Z Y \to X$ and $X \times_Z Y \to Y$ that are universal. More precisely, for any object T of $\mathcal{C}$ and morphisms $f_X : T \to X$, $f_Y : T \to Y$, there exists a unique morphism $u : T \to X \times_Z Y$ such that the following diagram commutes:

Equivalently, $X \times_Z Y$ is the limit (Definition 1.2.2) of the diagram (Definition 1.2.1)

$$
\begin{array}{ccc}
 & & X \\
 & & \downarrow \\
Y & \longrightarrow & Z
\end{array}
$$

in $\mathcal{C}$.

The commutative diagram

$$X \times_Z Y \longrightarrow X$$
$$\downarrow \qquad\qquad \downarrow$$
$$Y \longrightarrow Z$$

may be referred to as a *cartesian square*.

Definition 1.2.33. Let $\mathcal{C}$ be a category

1. Let X be an object in $\mathcal{C}$. If the product $X \times X$ exists, then the *diagonal morphism of X* is the morphism

$$\Delta_X : X \to X \times X$$

   induced by the universal property of $X \times X$ for the identity morphisms $\mathrm{id}_X : X \to X$ from X to each of the first and second factors of X in $X \times X$. In other words, Δ_f is the unique morphism such that the following equalities hold:

$$p_1 \circ \Delta_X = \mathrm{id}_X, \qquad p_2 \circ \Delta_X = \mathrm{id}_X$$

   where $p_1, p_2 : X \times X \to X$ are the two canonical projections.

2. Let $f : X \to Z$ be a morphism in $\mathcal{C}$. If the cartesian product (Definition 1.2.32) $X \times_Z X$ exists, then the *diagonal morphism of f* is the morphism

$$\Delta_f = \Delta_X : X \to X \times_Z X$$

   induced by the universal property of $X \times_Z X$ for the identity morphisms $\mathrm{id}_X : X \to X$ from X to each of the first and second factors of X in $X \times_Z X$. In other words, Δ_f is the unique morphism such that the following equalities hold:

$$p_1 \circ \Delta_f = \mathrm{id}_X, \qquad p_2 \circ \Delta_f = \mathrm{id}_X$$

   where $p_1, p_2 : X \times X \to X$ are the two canonical projections.

Definition 1.2.34. Let $\mathcal{C}$ be a category (Definition 1.0.1), let Z be an object, and let X, Y be objects of $\mathcal{C}$ under (♠ TODO: under category) Z, i.e. morphisms $f : Z \to X$ and $g : Z \to Y$ are fixed. A *pushout of X and Y under Z in $\mathcal{C}$* is an object, often denoted by $X \amalg_Z Y$, with *inclusion morphisms* $i_X : X \to X \amalg_Z Y$ and $i_Y : Y \to X \amalg_Z Y$ that are universal. More precisely, the morphisms satisfy $i_X \circ f = i_Y \circ g$, and for any object Q of $\mathcal{C}$ with morphisms $u_X : X \to Q$, $u_Y : Y \to Q$ satisfying $u_X \circ f = u_Y \circ g$, there exists a unique morphism $w : X \amalg_Z Y \to Q$ such that the following diagram commutes:

Equivalently, $X \amalg_Z Y$ is the colimit of the diagram (Definition 1.2.1)

$$
\begin{array}{c}
\\
Z\\
\end{array}
\quad
\begin{array}{c}
\nearrow^{\;X}\\
{\scriptstyle f}\\
\\
{\scriptstyle g}\\
\searrow_{\;Y}\\
\end{array}
$$

in $\mathcal{C}$.
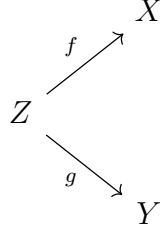
The commutative diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{\;f\;} & X \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle i_X} \\
Y & \xrightarrow[\;i_Y\;]{} & X \amalg_Z Y
\end{array}
$$

may be referred to as a *cocartesian square*.

Note that the notion of cocartesian squares is dual ot the notion of cartesian squares (Definition 1.2.32).

1.3. Miscellaneous categorical constructions and definitions.

1.3.1. *Subcategory of a category.*

Definition 1.3.1 (Subcategory). Let $\mathcal{C}$ be a (large) category (Definition 1.0.1). A *subcategory* $\mathcal{D}$ of $\mathcal{C}$ consists of:

- a subclass of objects $\mathrm{Ob}(\mathcal{D}) \subseteq \mathrm{Ob}(\mathcal{C})$,
- for each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{D})$, a subclass of morphisms

$$\mathrm{Hom}_{\mathcal{D}}(X, Y) \subseteq \mathrm{Hom}_{\mathcal{C}}(X, Y),$$

such that

- for every object $X \in \mathrm{Ob}(\mathcal{D})$, the identity morphism id_X of X in $\mathcal{C}$ lies in $\mathrm{Hom}_{\mathcal{D}}(X, X)$,
- the composition of morphisms in $\mathcal{D}$ is inherited from $\mathcal{C}$ and is closed in $\mathcal{D}$: for morphisms $f \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{D}}(Y, Z)$, their composition $g \circ f$, which is a priori in $\mathrm{Hom}_{\mathcal{C}}(X, Z)$, is in $\mathrm{Hom}_{\mathcal{D}}(X, Z)$.

Definition 1.3.2 (Full subcategory). Let $\mathcal{C}$ be a (large) category (Definition 1.0.1). A *full subcategory* $\mathcal{D}$ of $\mathcal{C}$ is a subcategory (Definition 1.3.1) such that for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{D})$, the morphism classes coincide:

$$\mathrm{Hom}_{\mathcal{D}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

In other words, a full subcategory includes all morphisms between its objects that exist in the ambient category $\mathcal{C}$.

Definition 1.3.3 (Replete subcategory)**.** Let $\mathcal{C}$ be a (large) category (Definition 1.0.1). A full subcategory (Definition 1.3.2) $\mathcal{D} \subseteq \mathcal{C}$ is called *replete* if it satisfies the following property:

- For all $x \in \mathrm{Ob}(\mathcal{C})$ and $y \in \mathrm{Ob}(\mathcal{D})$, if $x \cong y$ in $\mathcal{C}$, then $x \in \mathrm{Ob}(\mathcal{D})$ as well.

Equivalently, a replete subcategory of $\mathcal{C}$ is a full subcategory closed under isomorphisms in $\mathcal{C}$.

Definition 1.3.4 (Category of objects over a fixed object)**.** Let $\mathcal{C}$ be a category (Definition 1.0.1) and let $X \in \mathrm{Ob}(\mathcal{C})$ be a fixed object.

1. The *category of objects over X* (or synonymously the *slice category of X in $\mathcal{C}$* or the *over category of X in $\mathcal{C}$*), commonly denoted $\mathcal{C}/X$, $\mathcal{C}_{/X}$, or $(\mathcal{C} \downarrow X)$ is the category defined as follows:
   - An object of $\mathcal{C}/X$ is a morphism $f\colon A \to X$ in $\mathcal{C}$, where $A \in \mathrm{Ob}(\mathcal{C})$.
   - A morphism from $f\colon A \to X$ to $g\colon B \to X$ in $\mathcal{C}/X$ is a morphism $h\colon A \to B$ in $\mathcal{C}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\ h\ } & B \\
 & {\scriptstyle f}\searrow & \downarrow{\scriptstyle g} \\
 & & X
\end{array}
$$

   i.e. such that $g \circ h = f$.
   - The identity morphisms and composition in $\mathcal{C}/X$ are inherited from $\mathcal{C}$.
2. The *category of objects under X* (or synonymously the *coslice category of X in $\mathcal{C}$* or the *under category of X in $\mathcal{C}$*), commonly denoted $X/\mathcal{C}$, $X\backslash\mathcal{C}$, $\mathcal{C}_{X/}$, or $(X \downarrow \mathcal{C})$, is the category defined as follows:
   - An object of $X/\mathcal{C}$ is a morphism $f\colon X \to A$ in $\mathcal{C}$, where $A \in \mathrm{Ob}(\mathcal{C})$.
   - A morphism from $f\colon X \to A$ to $g\colon X \to B$ in $X/\mathcal{C}$ is a morphism $h\colon A \to B$ in $\mathcal{C}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & A \\
 & {\scriptstyle g}\searrow & \downarrow{\scriptstyle h} \\
 & & B
\end{array}
$$

   i.e. such that $h \circ f = g$.
   - The identity morphisms and composition in $X/\mathcal{C}$ are inherited from $\mathcal{C}$.

Both notions are special cases of comma categories (Definition 1.3.12).

Lemma 1.3.5. Let $\mathcal{C}$ be a category (Definition 1.0.1) and let $X \in \mathrm{Ob}(\mathcal{C})$ be a fixed object. The slice category $\mathcal{C}/X$ (Definition 1.3.4) has X as its final object (Definition 2.0.6).

Proof. This is clear. $\qquad\square$

Definition 1.3.6 (Reflective subcategory)**.** Let $\mathcal{C}$ be a (large) category (Definition 1.0.1) and let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory (Definition 1.3.2). The subcategory $\mathcal{D}$ is called *reflective*

if the inclusion functor

$$I : \mathcal{D} \hookrightarrow \mathcal{C}$$

admits a left adjoint

$$L : \mathcal{C} \to \mathcal{D}.$$

In this case, L is called the *reflector*, and for each $c \in \mathrm{Ob}(\mathcal{C})$, the unit morphism of the adjunction

$$\eta_c : c \to I(L(c))$$

is called the *reflection arrow of c into $\mathcal{D}$*.

($\spadesuit$ TODO: describe what monomorphisms and epimorphisms are like in additive or abelian categories)

1.3.2. *Types of morphisms in a category.*

Definition 1.3.7 (Monomorphism and Epimorphism in Categories)**.** Let $\mathcal{C}$ be a category (Definition 1.0.1). For objects $A, B \in \mathcal{C}$, let $f : A \to B$ be a morphism in $\mathcal{C}$.

- The morphism f is called a *monomorphism* (or a *monic morphism*) if for every object X and every pair of morphisms $g_1, g_2 : X \to A$, the equality $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.
- The morphism f is called an *epimorphism* (or an *epic morphism*) if for every object Y and every pair of morphisms $h_1, h_2 : B \to Y$, the equality $h_1 \circ f = h_2 \circ f$ implies $h_1 = h_2$.

Definition 1.3.8. Let $\mathcal{C}$ be a (large) (Definition 1.0.1) pointed category, i.e. a category with a zero object (Definition 2.0.6) 0. Let $X, Y \in \mathrm{Ob}(\mathcal{C})$ be an object and let $f : X \to Y$ be a morphism.

1. A morphism $i : K \to X$ is called the *kernel of f* if:
   (a) $f \circ i = 0$, where 0 is the zero morphism $K \to Y$,
   (b) for any morphism $g : Z \to X$ such that $f \circ g = 0$, there exists a unique morphism $u : Z \to K$ such that $g = i \circ u$.
   The kernel, if it exists, is unique up to unique isomorphism (Definition 1.0.8). $\ker(f)$ denotes the object K determined (up to isomorphism) by a kernel of f.
2. a morphism $p : Y \to Q$ is called the *cokernel of f* if:
   (a) $p \circ f = 0$, where 0 is the zero morphism (Definition 2.0.6) $X \to Q$,
   (b) for any morphism $g : Y \to Z$ such that $g \circ f = 0$, there exists a unique morphism $v : Q \to Z$ such that $g = v \circ p$.
   The cokernel, if it exists, is unique up to unique isomorphism. $\mathrm{coker}(f)$ denotes the object Q determined (up to isomorphism) by a cokernel of f.

1.3.3. *Essential image of a functor between categories.*

1.3.4. *Types of categories.*

Definition 1.3.9 (Complete and Cocomplete Category). Let $\mathcal{C}$ be a category (Definition 1.0.1).

- The category $\mathcal{C}$ is called *complete* (resp. *finitely complete*) if all small limits (Definition 1.2.5) (resp. finite limits) exist in $\mathcal{C}$; that is, for every small diagram $D : J \to \mathcal{C}$ (with J a small category), the limit $\lim D$ exists and is an object of $\mathcal{C}$.
- The category $\mathcal{C}$ is called *cocomplete* (resp. *finitely cocomplete*) if all small colimits (Definition 1.2.5) (resp. finite colimits) exist in $\mathcal{C}$; that is, for every small diagram $D : J \to \mathcal{C}$, the colimit $\operatorname{colim} D$ exists and is an object of $\mathcal{C}$.

1.3.5. *Categories constructed from other categories.*

Definition 1.3.10 (Ind-category). Let $\mathcal{C}$ be a locally small category (Definition 1.0.9).

1. The *Ind-category of $\mathcal{C}$*, denoted $\operatorname{Ind}(\mathcal{C})$, is defined as follows:
   - Objects of $\operatorname{Ind}(\mathcal{C})$ are formal filtered colimits (Definition 1.2.12) of objects in $\mathcal{C}$. More precisely, an object is given by a filtered (Definition 1.2.8) small category I and a functor
   $$X : I \to \mathcal{C}.$$
   - Morphisms between objects $X : I \to \mathcal{C}$ and $Y : J \to \mathcal{C}$ are defined by
   $$\operatorname{Hom}_{\operatorname{Ind}(\mathcal{C})}(X, Y) \ := \ \varprojlim_{i \in I} \varinjlim_{j \in J} \operatorname{Hom}_{\mathcal{C}}(X_i, Y_j),$$
   (Definition 1.2.12) where X_i and Y_j denote the images of $i \in I$ and $j \in J$ under X and Y, respectively.

   The composition of morphisms is induced naturally from composition in $\mathcal{C}$. Hence, $\operatorname{Ind}(\mathcal{C})$ is the completion of $\mathcal{C}$ under filtered colimits. Objects of $\operatorname{Ind}(\mathcal{C})$ are called *Ind-objects of $\mathcal{C}$*.
2. The *Pro-category of $\mathcal{C}$*, denoted $\operatorname{Pro}(\mathcal{C})$, is defined as follows:
   - Objects of $\operatorname{Pro}(\mathcal{C})$ are formal cofiltered limits (Definition 1.2.12) of objects in $\mathcal{C}$. More precisely, an object is given by a cofiltered small category I and a functor
   $$X : I \to \mathcal{C}.$$
   - Morphisms between objects $X : I \to \mathcal{C}$ and $Y : J \to \mathcal{C}$ are defined by
   $$\operatorname{Hom}_{\operatorname{Pro}(\mathcal{C})}(X, Y) := \varinjlim_{j \in J} \varprojlim_{i \in I} \operatorname{Hom}_{\mathcal{C}}(X_i, Y_j),$$
   where X_i and Y_j denote the images of $i \in I$ and $j \in J$ under X and Y, respectively.

   The composition of morphisms is induced naturally from composition in $\mathcal{C}$.

   Hence, $\operatorname{Pro}(\mathcal{C})$ is the completion of $\mathcal{C}$ under cofiltered limits. Objects of $\operatorname{Pro}(\mathcal{C})$ are called *Pro-objects of $\mathcal{C}$*.

Since **Sets** has all limits and colimits (♠ TODO:) and hence has all projective and inductive limits and since $\mathcal{C}$ is locally small, $\operatorname{Ind}(\mathcal{C})$ and $\operatorname{Pro}(\mathcal{C})$ are locally small.

Definition 1.3.11 (Product Category of a Family of Categories)**.** Let $\{\mathcal{C}_i\}_{i \in I}$ be a family of (large) categories (Definition 1.0.1) indexed by a class I. The *product category of the family*, denoted

$$\prod_{i \in I} \mathcal{C}_i,$$

is the very large category (♠ TODO: define very large categories) defined as follows:

- The class of objects is

$$\mathrm{Ob}\Big(\prod_{i \in I} \mathcal{C}_i\Big) = \prod_{i \in I} \mathrm{Ob}(\mathcal{C}_i),$$

  i.e., an object is a family $(A_i)_{i \in I}$ with $A_i \in \mathrm{Ob}(\mathcal{C}_i)$.
- For two objects $(A_i)_i$ and $(B_i)_i$, the morphism class is

$$\mathrm{Hom}_{\prod_{i \in I} \mathcal{C}_i}((A_i)_i, (B_i)_i) = \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}_i}(A_i, B_i).$$

  In other words, a morphism $(f_i)_i : (A_i)_i \to (B_i)_i$ consists of morphisms $f_i : A_i \to B_i$ in each $\mathcal{C}_i$.
- For morphisms $(f_i)_i : (A_i)_i \to (B_i)_i$ and $(g_i)_i : (B_i)_i \to (C_i)_i$, composition is defined componentwise:

$$(g_i)_i \circ (f_i)_i = (g_i \circ_i f_i)_i.$$

- For each object $(A_i)_i$, the identity morphism is given by the family

$$(\mathrm{id}_{A_i})_i.$$

If I is a set, then $\prod_{i \in I} \mathcal{C}_i$ is a large category. If I is a set and if each $\mathcal{C}_i$ is locally small (Definition 1.0.9), then $\prod_{i \in I} \mathcal{C}_i$ is locally small.

In case that I is finite, the notation of $\times$ may be used for product categories, e.g. $\mathcal{C}_i \times \mathcal{C}_j$ denotes the product of two categories $\mathcal{C}_i \times \mathcal{C}_j$.

(♠ TODO: ordinal, U_α) If α is an ordinal such that $\mathcal{C}_i$ and I are U_α-large (i.e. they live in $U_{\alpha+1}$), then $\prod_{i \in I} \mathcal{C}_i$ is $U_{\alpha+1}$-large.

Definition 1.3.12 (Comma Category)**.** Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be categories, and let $S : \mathcal{A} \to \mathcal{C}$ and $T : \mathcal{B} \to \mathcal{C}$ be functors.

The *comma category* $(S \downarrow T)$ (also denoted (S, T) or S/T) is defined as follows:

- **Objects:** Triples (A, B, h), where A is an object of $\mathcal{A}$, B is an object of $\mathcal{B}$, and $h : S(A) \to T(B)$ is a morphism in $\mathcal{C}$.
- **Morphisms:** A morphism from (A, B, h) to (A', B', h') consists of a pair (f, g), where $f : A \to A'$ is a morphism in $\mathcal{A}$ and $g : B \to B'$ is a morphism in $\mathcal{B}$, such that the following diagram in $\mathcal{C}$ commutes:

$$\begin{array}{ccc} S(A) & \xrightarrow{\;h\;} & T(B) \\ {\scriptstyle S(f)}\Big\downarrow & & \Big\downarrow{\scriptstyle T(g)} \\ S(A') & \xrightarrow{\;h'\;} & T(B') \end{array}$$

That is, $T(g) \circ h = h' \circ S(f)$.
- **Composition:** Composition is defined component-wise: $(f', g') \circ (f, g) = (f' \circ f, g' \circ g)$.

The general definition of the comma category subsumes several common categorical constructions:

1. **Slice Category (Over-category):** If $\mathcal{A} = \mathcal{C}$, $S = \mathrm{Id}_{\mathcal{C}}$, $\mathcal{B} = \mathbf{1}$ (the terminal category with one object $*$), and $T : \mathbf{1} \to \mathcal{C}$ is the functor selecting an object $X \in \mathcal{C}$ (i.e., $T(*) = X$), then $(S \downarrow T)$ is the slice category $\mathcal{C}/X$ (Definition 1.3.4).
2. **Coslice Category (Under-category):** If $\mathcal{A} = \mathbf{1}$, $S(*) = X$, $\mathcal{B} = \mathcal{C}$, and $T = \mathrm{Id}_{\mathcal{C}}$, then $(S \downarrow T)$ is the coslice category $X/\mathcal{C}$ (Definition 1.3.4).
3. **Relative Slice Category:** If $S = u : \mathcal{D} \to \mathcal{C}$ is a functor and $T : \mathbf{1} \to \mathcal{C}$ picks out an object $U \in \mathcal{C}$, then $(S \downarrow T)$ is the relative slice category (Definition 5.3.1).
4. **Arrow Category:** If $\mathcal{A} = \mathcal{B} = \mathcal{C}$ and $S = T = \mathrm{Id}_{\mathcal{C}}$, then $(S \downarrow T)$ is the arrow category $\mathcal{C}^{\to}$ (Definition 1.3.13) of $\mathcal{C}$. Its objects are all morphisms in $\mathcal{C}$.

Definition 1.3.13 (Arrow Category). Let $\mathcal{C}$ be a category (Definition 1.0.1). The arrow category of $\mathcal{C}$, denoted $\mathcal{C}^{\to}$ (or sometimes $\mathrm{Arr}(\mathcal{C})$ or $\mathcal{C}^{\mathbf{2}}$), is defined as follows:

- **Objects:** Morphisms $f : A \to B$ in $\mathcal{C}$. (We may view such an object as a triple (A, B, f)).
- **Morphisms:** A morphism from an object $f : A \to B$ to an object $f' : A' \to B'$ is a pair of morphisms (h_0, h_1) in $\mathcal{C}$, where $h_0 : A \to A'$ and $h_1 : B \to B'$, such that the following square commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B \\
{\scriptstyle h_0}\downarrow & & \downarrow{\scriptstyle h_1} \\
A' & \xrightarrow{\ f'\ } & B'
\end{array}
$$

That is, $h_1 \circ f = f' \circ h_0$.
- **Composition:** Composition is defined component-wise: $(k_0, k_1) \circ (h_0, h_1) = (k_0 \circ h_0, k_1 \circ h_1)$.

Arrow categories are special cases of comma categories (Definition 1.3.12).

1.3.6. *Types of functors.*

Definition 1.3.14 (Reflecting a type of morphism). Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between (large) categories (Definition 1.1.1), and let $\mathcal{P}$ be a property of morphisms (or more generally a property of sequences or families of morphisms) that is stable under isomorphism (Definition 1.0.8) (e.g. monomorphism, epimorphism (Definition 1.3.7), isomorphism, etc.). We say that F reflects $\mathcal{P}$-morphisms if for every morphism $f : x \to y$ in $\mathcal{C}$, whenever $F(f)$ has property $\mathcal{P}$ in $\mathcal{D}$, it follows that f has property $\mathcal{P}$ in $\mathcal{C}$.

Proposition 1.3.15 (Reflection of morphism properties by full or faithful functors). Let $F : \mathcal{C} \to \mathcal{D}$ be a functor (Definition 1.1.1) between locally small categories (Definition 1.0.9).

(1) If F is faithful (Definition 1.1.4), then F reflects (Definition 1.3.14) monomorphisms and epimorphisms (Definition 1.3.7). That is, if $f : x \to y$ in $\mathcal{C}$ is such that $F(f)$ is a

monomorphism (resp. epimorphism) in $\mathcal{D}$, then f is a monomorphism (resp. epimorphism) in $\mathcal{C}$.

(♠ TODO: define split monos and epis) (2) If F is fully faithful, then F reflects isomorphisms, split monomorphisms, and split epimorphisms. That is, if $f : x \to y$ in $\mathcal{C}$ is such that $F(f)$ is an isomorphism in $\mathcal{D}$, then f is an isomorphism in $\mathcal{C}$.

Definition 1.3.16 (n-ary (Multivariable) Functor)**.** Let I be a finite set with $|I| = n$, and let $\{\mathcal{C}_i\}_{i \in I}$ be (large) categories (Definition 1.0.1), together with another category $\mathcal{D}$. An *n-ary functor* (also called a *multivariable functor*, a *multivariate functor*, or a *multifunctor*) from the categories $\{\mathcal{C}_i\}_{i \in I}$ to $\mathcal{D}$ is a functor (Definition 1.1.1)

$$F : \prod_{i \in I} \mathcal{C}_i \to \mathcal{D}.$$

(Definition 1.3.11) That is, F assigns:

- to each object $((A_i)_{i \in I})$ in $\prod_{i \in I} \mathcal{C}_i$, an object $F((A_i)_{i \in I})$ in $\mathcal{D}$,
- to each morphism $((f_i)_{i \in I}) : (A_i)_i \to (B_i)_i$, a morphism $F((f_i)_i) : F((A_i)_i) \to F((B_i)_i)$ in $\mathcal{D}$,

so that F preserves identities and composition componentwise. For instance, a *bifunctor* is an n-ary functor when $n = 2$, a *ternary functor/trifunctor* is an n-ary functor when $n = 3$, etc.

1.3.7. *Algebraic objects in a category.*

Definition 1.3.17. Let $\mathcal{C}$ be a (large) category (Definition 1.0.1). A *semigroup object in $\mathcal{C}$* is an object $A \in \mathcal{C}$ such that the product (Definition 1.2.30) $A \times A$ exists in $\mathcal{C}$ together with a morphism

$$\mu : A \times A \to A,$$

called the *multiplication morphism* such that the associativity diagram

$$
\begin{array}{ccc}
A \times A \times A & \xrightarrow{\mu \times \mathrm{id}_A} & A \times A \\
{\scriptstyle \mathrm{id}_A \times \mu} \downarrow & & \downarrow {\scriptstyle \mu} \\
A \times A & \xrightarrow{\quad \mu \quad} & A
\end{array}
$$

commutes.

The semigroup object structure (A, μ, η, ι) is said to be *abelian* or *commutative* if the morphisms $\mu : A \times A \to A$ and $\mu \circ \tau_{A,A} : A \times A \to A$ coincide, where $\tau_{A,A} : A \times A \to A \times A$ is the symmetry morphism swapping the two factors.

Definition 1.3.18. Let $\mathcal{C}$ be a (large) category (Definition 1.0.1) with a final object (Definition 2.0.6). A *monoid object in $\mathcal{C}$* is a semigroup object (Definition 1.3.17) (A, μ) together with a *unit morphism*

$$\eta : 1 \to A$$

such that the products (Definition 1.2.30) $1 \times A$ and $A \times 1$ exist and the unit diagrams

$$
\begin{array}{ccc}
1 \times A & \xrightarrow{\ \eta \times \mathrm{id}_A\ } & A \times A \\
& {\scriptstyle \mathrm{pr}_2} \searrow \quad \swarrow {\scriptstyle \mu} & \\
& A &
\end{array}
$$

$$
\begin{array}{ccc}
A \times 1 & \xrightarrow{\ \mathrm{id}_A \times \eta\ } & A \times A \\
& {\scriptstyle \mathrm{pr}_1} \searrow \quad \swarrow {\scriptstyle \mu} & \\
& A &
\end{array}
$$

commute.

Definition 1.3.19. Let $\mathcal{C}$ be a (large) category (Definition 1.0.1) with a final object (Definition 2.0.6). A *group object in $\mathcal{C}$* is a monoid object (Definition 1.3.18) (A, μ, η) together with a *inverse morphism*

$$
\iota : A \to A
$$

such that the diagrams

$$
\begin{array}{ccc}
A & \xrightarrow{\ \Delta\ } & A \times A \\
& {\scriptstyle \eta \circ \,!_A} \searrow & \ \downarrow {\scriptstyle \mu \circ (\mathrm{id}_A \times \iota)} \\
& & A
\end{array}
$$

$$
\begin{array}{ccc}
A & \xrightarrow{\ \Delta\ } & A \times A \\
& {\scriptstyle \eta \circ \,!_A} \searrow & \ \downarrow {\scriptstyle \mu \circ (\iota \times \mathrm{id}_A)} \\
& & A
\end{array}
$$

commute, where $\Delta : A \to A \times A$ is the diagonal and $!_A : A \to 1$ is the unique morphism.

Definition 1.3.20. Let $\mathcal{C}$ be a (large) category (Definition 1.0.1) with a terminal object (Definition 2.0.6) 1.

A *ring object in $\mathcal{C}$* is an object $R \in \mathcal{C}$ equipped with:

- an abelian (Definition 1.3.17) group object (Definition 1.3.19) structure $(R, +, 0, -)$ (written additively), i.e. morphisms

$$
+ : R \times R \to R, \quad 0 : 1 \to R, \quad - : R \to R,
$$

  making $(R, +, 0, -)$ a group object;
- a monoid object (Definition 1.3.18) structure $(R, \cdot, 1_R)$ (written multiplicatively), i.e. morphisms

$$
\cdot : R \times R \to R, \quad 1_R : 1 \to R,
$$

  making $(R, \cdot, 1_R)$ a monoid object;

such that the usual distributivity and absorption axioms hold, expressed by the commutativity of the diagrams corresponding to

$$a \cdot (b + c) = a \cdot b + a \cdot c, \qquad (a + b) \cdot c = a \cdot c + b \cdot c,$$

and $0 \cdot a = a \cdot 0 = 0$, for all a, b, c (interpreted as morphisms in $\mathcal{C}$).

Definition 1.3.21 (Module object over a ring object in a category). Let $\mathcal{C}$ be a (large) category (Definition 1.0.1) with a terminal object (Definition 2.0.6) 1. Let R be a ring object (Definition 1.3.20) in $\mathcal{C}$. An R-*module object in* $\mathcal{C}$ is an object $M \in \mathcal{C}$ together with:

- an abelian (Definition 1.3.17) group object (Definition 1.3.19) structure $(M, +_M, 0_M, -_M)$ on M;
- an action morphism
$$\alpha : R \times M \to M,$$

such that the usual module axioms hold, expressed by commutative diagrams corresponding to

$$r \cdot (m_1 +_M m_2) = r \cdot m_1 +_M r \cdot m_2, \qquad (r_1 + r_2) \cdot m = r_1 \cdot m +_M r_2 \cdot m,$$

$$(r_1 \cdot r_2) \cdot m = r_1 \cdot (r_2 \cdot m), \qquad 1_R \cdot m = m, \qquad 0_R \cdot m = 0_M = r \cdot 0_M,$$

for all $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$, interpreted as morphisms and equalities in $\mathcal{C}$.

Definition 1.3.22. Let $\mathcal{C}$ be a (large) category (Definition 1.0.1).

A *groupoid object in* $\mathcal{C}$ consists of two objects X_0 (the "object of objects") and X_1 (the "object of morphisms"), together with five structure morphisms:

- *Source* and *Target*: $s, t : X_1 \to X_0$, such that the fiber product (Definition 1.2.32) $X_1 \times_{s, X_0, t} X_1$ of the morphisms s and t exists in $\mathcal{C}$,
- *Identity*: $e : X_0 \to X_1$,
- *Composition*: $m : X_1 \times_{s, X_0, t} X_1 \to X_1$,
- *Inverse*: $i : X_1 \to X_1$,

such that the following conditions hold (expressing the axioms of a category where every morphism is invertible):

1. **Source/Target identities**:
$$s \circ e = \mathrm{id}_{X_0}, \quad t \circ e = \mathrm{id}_{X_0}$$
$$s \circ m = s \circ \pi_2, \quad t \circ m = t \circ \pi_1$$

2. **Associativity**: The following diagram of composition commutes:
$$m \circ (m \times \mathrm{id}_{X_1}) = m \circ (\mathrm{id}_{X_1} \times m)$$

3. **Unitality**:
$$m \circ (e \circ s, \mathrm{id}_{X_1}) = \mathrm{id}_{X_1}, \quad m \circ (\mathrm{id}_{X_1}, e \circ t) = \mathrm{id}_{X_1}$$

4. **Invertibility**:
$$m \circ (i, \mathrm{id}_{X_1}) = e \circ s, \quad m \circ (\mathrm{id}_{X_1}, i) = e \circ t$$

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(♠ TODO:)

Definition 1.3.23 (Semigroup, monoid and group objects in an ∞-category). Let $\mathcal{C}$ be an ∞-category with finite products.

1. A *semigroup object* in $\mathcal{C}$ is a functor
    $$A : \mathrm{N}(\Delta_{\mathrm{inj}}^{\mathrm{op}}) \to \mathcal{C}$$
    from the nerve of the subcategory of Δ consisting of injective maps, satisfying the Segal condition: for each $[n] \in \Delta$ with $n \geq 2$, the canonical map
    $$A([n]) \to A([1]) \times \cdots \times A([1])$$
    (induced by the n face maps $[1] \to [n]$) is an equivalence in $\mathcal{C}$.
2. A *monoid object* in $\mathcal{C}$ is a functor
    $$A : \mathrm{N}(\Delta^{\mathrm{op}}) \to \mathcal{C}$$
    satisfying the Segal condition as above, together with a choice of unit encoded by the image of the unique degeneracy map $[0] \to [1]$, such that the usual associativity and unit diagrams hold up to coherent homotopy (i.e. A is a unital Segal object in $\mathcal{C}$).
3. A *group object* in $\mathcal{C}$ is a monoid object A such that the induced monoid object $\pi_0(A)$ in the homotopy category $h\mathcal{C}$ is a group object (equivalently, the underlying E_1-space is grouplike).

Definition 1.3.24 (Ring objects in an ∞-category). Let $\mathcal{C}$ be an ∞-category with finite products.

1. A *ring object* in $\mathcal{C}$ is an object $R \in \mathcal{C}$ equipped with:
    * a group object structure $(R, +, 0, -)$ in $\mathcal{C}$ (additive structure),
    * a monoid object structure $(R, \cdot, 1_R)$ in $\mathcal{C}$ (multiplicative structure),
    such that the distributivity, absorption and unital axioms for a ring hold up to coherent homotopy, expressed by the homotopy-commutativity of the diagrams corresponding to
    $$a \cdot (b + c) \simeq a \cdot b + a \cdot c, \qquad (a + b) \cdot c \simeq a \cdot c + b \cdot c,$$
    and $0 \cdot a \simeq 0 \simeq a \cdot 0$, for all $a, b, c \in R$.
2. A *commutative ring object* in $\mathcal{C}$ is a ring object whose multiplicative monoid structure is commutative up to coherent homotopy (i.e. a commutative monoid object in the sense above).

Definition 1.3.25 (Module objects over a ring object in an ∞-category). Let $\mathcal{C}$ be an ∞-category with finite products and let R be a ring object in $\mathcal{C}$. An *R-module object in $\mathcal{C}$* is an object $M \in \mathcal{C}$ together with

* a group object structure $(M, +_M, 0_M, -_M)$ in $\mathcal{C}$,
* an action morphism in $\mathcal{C}$
    $$\alpha : R \times M \to M,$$

such that the usual module axioms hold up to coherent homotopy, i.e. the diagrams expressing
$$r \cdot (m_1 +_M m_2) \simeq r \cdot m_1 +_M r \cdot m_2, \qquad (r_1 + r_2) \cdot m \simeq r_1 \cdot m +_M r_2 \cdot m,$$

$$(r_1 \cdot r_2) \cdot m \simeq r_1 \cdot (r_2 \cdot m), \qquad 1_R \cdot m \simeq m, \qquad 0_R \cdot m \simeq 0_M \simeq r \cdot 0_M$$

commute in the homotopy-coherent sense in $\mathcal{C}$.

2. Monoidal categories

(♠ TODO: Carefully check what statements need to assume that the categories are locally small and which do not)

Definition 2.0.1. A *monoidal category* is a (large) category (Definition 1.0.1) $\mathcal{C}$ equipped with:

- a bifunctor (Definition 1.3.16) $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (Definition 1.3.11) (called the *tensor product*);
- an object $\mathbb{I} \in \mathrm{Ob}(\mathcal{C})$ (often called the *unit object*); common notations for the unit object include $\mathbb{I}$ and $\mathbf{1}$.
- natural isomorphisms (*associator*) $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ for all $X, Y, Z \in \mathcal{C}$;
- natural isomorphisms (*left and right unitors*) $\lambda_X : \mathbb{I} \otimes X \to X$, $\rho_X : X \otimes \mathbb{I} \to X$ for all $X \in \mathcal{C}$;

such that the following coherence diagrams commute:

Pentagon coherence: For all $W, X, Y, Z \in \mathcal{C}$, the diagram

$$
\begin{array}{ccccc}
((W \otimes X) \otimes Y) \otimes Z & \xrightarrow{\alpha_{W \otimes X, Y, Z}} & (W \otimes X) \otimes (Y \otimes Z) & \xrightarrow{\alpha_{W, X, Y \otimes Z}} & W \otimes (X \otimes (Y \otimes Z)) \\
{\scriptstyle \alpha_{W,X,Y} \otimes \mathrm{id}_Z} \downarrow & & & & \uparrow {\scriptstyle \mathrm{id}_W \otimes \alpha_{X,Y,Z}} \\
(W \otimes (X \otimes Y)) \otimes Z & & \xrightarrow{\alpha_{W, X \otimes Y, Z}} & & W \otimes ((X \otimes Y) \otimes Z).
\end{array}
$$

Triangle coherence: For all $X, Y \in \mathcal{C}$, the diagram

$$
\begin{array}{ccc}
(X \otimes \mathbb{I}) \otimes Y & \xrightarrow{\alpha_{X, \mathbb{I}, Y}} & X \otimes (\mathbb{I} \otimes Y) \\
& {\scriptstyle \rho_X \otimes \mathrm{id}_Y} \searrow \quad & \downarrow {\scriptstyle \mathrm{id}_X \otimes \lambda_Y} \\
& & X \otimes Y.
\end{array}
$$

Definition 2.0.2. Let $(\mathcal{C}, \otimes, \mathbb{I})$ be a monoidal category. Given objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, an *internal hom object from X to Y* is an object $\underline{\mathrm{Hom}}(X, Y) \in \mathrm{Ob}(\mathcal{C})$ together with a morphism

$$\mathrm{ev}_{X,Y} : \underline{\mathrm{Hom}}(X, Y) \otimes X \to Y$$

such that for every object $Z \in \mathcal{C}$, the assignment

$$\mathrm{hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(X, Y)) \to \mathrm{hom}_{\mathcal{C}}(Z \otimes X, Y), \quad f \mapsto \mathrm{ev}_{X,Y} \circ (f \otimes \mathrm{id}_X)$$

is a natural isomorphism of sets.

Definition 2.0.3. A *closed category* is a category $\mathcal{C}$ equipped with the following data:

- A functor

$$\underline{\mathrm{Hom}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$$

called the *internal hom-functor*.
- A *left evaluation morphism*

$$j_X : I \to \underline{\mathrm{Hom}}(X, X)$$

for each object $X \in \mathcal{C}$, where I is a fixed object serving as a "unit" of the internal hom-structure.
- A family of natural *left composition morphisms*

$$L^X_{Y,Z} : \underline{\mathrm{Hom}}(Y, Z) \otimes \underline{\mathrm{Hom}}(X, Y) \to \underline{\mathrm{Hom}}(X, Z)$$

(♠ TODO: TODO: the axioms) satisfying appropriate associativity and unit axioms that generalize composition.

Definition 2.0.4. A *closed monoidal category* is a monoidal category (Definition 2.0.1) that is a closed category (Definition 2.0.3) where the internal hom (Definition 2.0.2) is right adjoint to the monoidal tensor.

In other words, for all objects $Y, Z \in \mathrm{Ob}(\mathcal{C})$ there is a natural isomorphism

$$\mathrm{hom}_\mathcal{C}(Y \otimes X, Z) \cong \mathrm{hom}_\mathcal{C}(Y, \underline{\mathrm{Hom}}(X, Z))$$

of Hom-sets natural in Y and Z.

Definition 2.0.5. A *symmetric monoidal category* is a monoidal category $(\mathcal{C}, \otimes, \mathbb{I})$ together with a natural isomorphism (symmetry)

$$\gamma_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$$

for all $X, Y \in \mathcal{C}$, such that for all $X, Y, Z \in \mathcal{C}$ the following holds:

- $\gamma_{Y,X} \circ \gamma_{X,Y} = \mathrm{id}_{X \otimes Y}$ (involutivity);
- the **hexagon coherence diagrams** commute:

$$
\begin{array}{ccccc}
(X \otimes Y) \otimes Z & \xrightarrow{\alpha_{X,Y,Z}} & X \otimes (Y \otimes Z) & \xrightarrow{\gamma_{X,Y \otimes Z}} & (Y \otimes Z) \otimes X \\
{\scriptstyle \gamma_{X,Y} \otimes \mathrm{id}_Z} \downarrow & & & & \uparrow {\scriptstyle \mathrm{id}_Y \otimes \gamma_{X,Z}} \\
(Y \otimes X) \otimes Z & & \xrightarrow{\alpha_{Y,X,Z}} & & Y \otimes (X \otimes Z)
\end{array}
$$

and the analogous hexagon with inverse braiding:

$$
\begin{array}{ccccc}
X \otimes (Y \otimes Z) & \xrightarrow{\alpha^{-1}_{X,Y,Z}} & (X \otimes Y) \otimes Z & \xrightarrow{\gamma_{X \otimes Y,Z}} & Z \otimes (X \otimes Y) \\
{\scriptstyle \mathrm{id}_X \otimes \gamma_{Y,Z}} \downarrow & & & & \uparrow {\scriptstyle \gamma_{X,Z} \otimes \mathrm{id}_Y} \\
X \otimes (Z \otimes Y) & & \xrightarrow{\alpha^{-1}_{X,Z,Y}} & & (X \otimes Z) \otimes Y
\end{array}
$$

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- the **symmetry coherence diagram** commutes:

$$X \otimes Y \xrightarrow{\gamma_{X,Y}} Y \otimes X$$

with $\mathrm{id}_{X \otimes Y}$ and $\gamma_{Y,X}$ to $X \otimes Y$

A *closed symmetric monoidal category* usually refers to a symmetric monoidal category that is closed as a monoidal category.

2.0.1. *Types of objects in a category.*

Definition 2.0.6. Let $\mathcal{C}$ be a (large) category (Definition 1.0.1).

1. An object $I \in \mathcal{C}$ is called an *initial object* if for every object $X \in \mathcal{C}$ there exists a unique morphism
$$I \to X.$$
Equivalently, an initial object is a limit (Definition 1.2.2) of the empty diagram (Definition 1.2.1), if such a limit exists.
2. An object $F \in \mathcal{C}$ is called a *final object* (or *terminal object*) if for every object $X \in \mathcal{C}$ there exists a unique morphism
$$X \to F.$$
Equivalently, a final object is a colimit (Definition 1.2.2) of the empty diagram (Definition 1.2.1), if such a colimit exists.
3. An object $Z \in \mathcal{C}$ is called a *zero object* if Z is both initial and final in $\mathcal{C}$. In particular, for every object $X \in \mathcal{C}$ there exist unique morphisms
$$Z \to X \quad \text{and} \quad X \to Z.$$

In particular, if initial/final/zero objects exist in a cateogry, then they are unique up to unique isomorphism.

Lemma 2.0.7. Let $\mathcal{C}$ be a full subcategory (Definition 1.3.2) of a (large) category (Definition 1.0.1) $\mathcal{D}$. Suppose that $\mathcal{D}$ hsa an initial object (Definition 2.0.6) I (resp. a final object (Definition 2.0.6) F) and that this object also belongs to $\mathcal{C}$. The object is initial (resp. final) in $\mathcal{C}$.

Definition 2.0.8. Let C and D be categories (Definition 1.0.1), let $S : C^{op} \times C \to D$ be a functor, and let d be an object of D.

A *wedge from d to S* is a family of morphisms $\omega = \{\omega_c\}_{c \in \mathrm{Ob}(C)}$ in D, denoted

$$\omega : d \to S,$$

where each component is a morphism $\omega_c : d \to S(c, c)$, satisfying the wedge condition: for every morphism $f : c \to c'$ in C, the diagram

$$S(1_c, f) \circ \omega_c = S(f, 1_{c'}) \circ \omega_{c'}$$

commutes in D. (Explicitly, both composites are morphisms $d \to S(c, c')$).

Dually, a *cowedge from S to d* is a family of morphisms $\beta = \{\beta_c\}_{c \in \mathrm{Ob}(C)}$ in D, denoted

$$\beta : S \to d,$$

where each component is a morphism $\beta_c : S(c, c) \to d$, satisfying the *cowedge condition*: for every morphism $f : c \to c'$ in C, the equality

$$\beta_c \circ S(f, 1_c) = \beta_{c'} \circ S(1_{c'}, f)$$

holds in D (as morphisms from $S(c', c)$ to d).

Definition 2.0.9. Let C and D be categories (Definition 1.0.1) and let $S : C^{op} \times C \to D$ be a functor.

An *end of S* is a pair (E, ω), where E is an object of D and $\omega : E \to S$ is a wedge (Definition 2.0.8), which is universal among all wedges to S. That is, for any object $d \in D$ and any wedge $\nu : d \to S$, there exists a unique morphism $h : d \to E$ in D such that $\nu_c = \omega_c \circ h$ for all $c \in \mathrm{Ob}(C)$. The object E is uniquely determined up to unique isomorphism and is denoted by

$$\int_{c \in C} S(c, c) \quad \text{or} \quad \int_C S.$$

Dually, a *coend of S* is a pair (K, β), where K is an object of D and $\beta : S \to K$ is a cowedge, which is universal among all cowedges from S. That is, for any object $d \in D$ and any cowedge $\gamma : S \to d$, there exists a unique morphism $g : K \to d$ in D such that $\gamma_c = g \circ \beta_c$ for all $c \in \mathrm{Ob}(C)$. The object K is denoted by

$$\int^{c \in C} S(c, c) \quad \text{or} \quad \int^C S.$$

3. Additive and abelian categories

(♠ TODO: Carefully check what statements need to assume that the categories are locally small and which do not)

Definition 3.0.1 (Additive category). Let $\mathcal{A}$ be a locally small category (Definition 1.0.9).

1. $\mathcal{A}$ is said to be a *preadditive category* if the following hold:
   - For any two objects A, B in $\mathcal{A}$, the set $\mathrm{Hom}_{\mathcal{A}}(A, B)$ is an abelian group (Definition C.0.5), and composition of morphisms is bilinear.
   - There is a zero object (Definition 2.0.6) 0 in $\mathcal{A}$.

   Equivialently, a preadditive cateogry $\mathcal{A}$ is a (necessarily locally small) category enriched in (Definition 4.0.1) the monoidal category (Definition 2.0.1) Ab that also possesses a zero object.
2. If $\mathcal{A}$ is preadditive, then it is called *additive* if it additionally satisfies the following:

- For any two objects A, B in $\mathcal{A}$, there exists a product object $A \times B$ (Definition 1.2.30), often written $A \oplus B$, called the *direct sum of A and B*. In fact, $A \oplus B$ is not only a product but also a coproduct of A and B.

Given a finite collection $\{A_i\}_i$ of objects A_i in an additive category $\mathcal{A}$, we may more generally speak of the *direct sum* $\bigoplus_i A_i$; it has canonical injections from and projections to each A_i.

Definition 3.0.2 (Additive functor). 1. Let $\mathcal{A}$ and $\mathcal{B}$ be pre-additive categories. A functor

$$F : \mathcal{A} \to \mathcal{B}$$

is an *additive functor* if for every pair of objects $A, A' \in \mathcal{A}$, the induced map

$$F_{A,A'} : \mathrm{Hom}_{\mathcal{A}}(A, A') \to \mathrm{Hom}_{\mathcal{B}}(F(A), F(A'))$$

is a group homomorphism of abelian groups, or equvialently if it is enriched over the category Ab of abelian groups (Definition 4.0.1).

2. Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories. A functor

$$F : \mathcal{A} \to \mathcal{B}$$

is an *additive functor* if it an additive functor of pre-additive categories and satisfies the following:

- F sends the zero object $0_{\mathcal{A}}$ of $\mathcal{A}$ to the zero object $0_{\mathcal{B}}$ of $\mathcal{B}$, i.e.,

$$F(0_{\mathcal{A}}) = 0_{\mathcal{B}}.$$

- F preserves finite direct sums: For any finite family of objects $\{A_i\}_{i=1}^n$ in $\mathcal{A}$,

$$F\left(\bigoplus_{i=1}^n A_i\right) \cong \bigoplus_{i=1}^n F(A_i)$$

via the canonical isomorphism induced by F applied to the canonical injections and projections.

In other words, F is a functor that is compatible with the additive structures on $\mathcal{A}$ and $\mathcal{B}$.

Definition 3.0.3 (Abelian category). Let $\mathcal{A}$ be a category. The category $\mathcal{A}$ is an *abelian category* if:

- $\mathcal{A}$ is an additive category (Definition 3.0.1).
- Every morphism $f : A \to B$ has a kernel $\ker(f)$ and a cokernel $\mathrm{coker}(f)$ (Definition 1.3.8).
- For every morphism $f : A \to B$, the canonical morphism $\mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism, where

$$\mathrm{coim}(f) = \mathrm{coker}(\ker(f) \to A), \quad \mathrm{im}(f) = \ker(B \to \mathrm{coker}(f)).$$

(♠ TODO: I think I need to re-check this defintion) (♠ TODO: coimage)

.

It is also worth considering Grothendieck's additional axioms for abelian categories (Definition 3.0.4).

Definition 3.0.4 (Grothendieck's axioms for abelian categories (Ab1–Ab5))**.** Let $\mathcal{A}$ be an abelian category (Definition 3.0.3).

Grothendieck introduced the following hierarchy of additional axioms to express stronger completeness and exactness properties in $\mathcal{A}$ — we note that Ab1, Ab2, and Ab2* are already satisfied for any abelian category:

- *Ab1*: Every morphism in $\mathcal{A}$ has a kernel and a cokernel (Definition 1.3.8).
- *Ab2*: Every monic (Definition 1.3.7) in $\mathcal{A}$ is the kernel of its cokernel.
- *Ab2***: Every epi in $\mathcal{A}$ is the cokernel of its kernel.
- *AB3*: The category $\mathcal{A}$ is cocomplete (Definition 1.3.9).
   - Since $\mathcal{A}$ is abelian (and hence admits equalizers (Definition 1.2.31) as kernels), this is equivalent to requiring that $\mathcal{A}$ has all small coproducts (Definition 1.2.30) (direct sums).
- *AB4*: The category $\mathcal{A}$ satisfies AB3, and coproducts are *exact*.
   - That is, the coproduct of a family of short exact sequences is a short exact sequence. Explicitly, for any family of short exact sequences $0 \to A_i \to B_i \to C_i \to 0$ indexed by a set I, the sequence
   $$0 \to \bigoplus_{i \in I} A_i \to \bigoplus_{i \in I} B_i \to \bigoplus_{i \in I} C_i \to 0$$
   is exact in $\mathcal{A}$.
- *AB5*: The category $\mathcal{A}$ satisfies AB3, and filtered colimits (Definition 1.2.12) are *exact*.
   - Equivalently, for any filtered (Definition 1.2.8) index category J and any directed system (Definition 1.2.11) of short exact sequences $0 \to A_j \to B_j \to C_j \to 0$, the colimit sequence
   $$0 \to \varinjlim A_j \to \varinjlim B_j \to \varinjlim C_j \to 0$$
   is exact.
   - Note: AB5 implies AB4. An abelian category satisfying AB5 and having a generator (Definition 5.1.49) is called a *Grothendieck category*.
- *AB6*: The category $\mathcal{A}$ satisfies AB3, and for any object X and any family of filtered subobjects $\{F_i\}_{i \in I}$ of X (where each F_i is a filter of subobjects), the intersection commutes with the limit:
   $$\bigcap_{i \in I} (\varinjlim_{j \in F_i} U_{i,j}) = \varinjlim_{(j_i) \in \prod F_i} (\bigcap_{i \in I} U_{i,j_i}).$$
   (This axiom is less commonly cited but appears in Grothendieck's Tohoku paper).
- *AB3***: The category $\mathcal{A}$ is complete (Definition 1.3.9) (i.e., has all small products).
- *AB4***: The category $\mathcal{A}$ satisfies AB3* and products are exact.
   - Note: This is rarely satisfied for module categories (e.g., it fails for Abelian groups), but it is satisfied for the category of sheaves on a space.
- *AB5***: The category $\mathcal{A}$ satisfies AB3* and filtered limits (inverse limits) are exact.

Notes:

- AB5 implies AB4, and AB4 implies AB3.

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- AB5* implies AB4*, and AB4* implies AB3*.

Proposition 3.0.5. The following are examples of abelian categories (Definition 3.0.3):

1. The category of R-S bimodules where R,S are (not necessarily commutative) rings (Definition C.0.1) (**??**).
2. The category **Ab** of abelian groups and group homomorphisms is abelian.
3. The category Vect_k of vector spaces over a field k and k-linear maps is abelian.
4. More generally, if R is a noetherian ring, then the category of finitely generated R-modules is abelian.
5. For a ringed space $(X, \mathcal{O}_X)$, the category of $\mathcal{O}_X$-modules (Definition 5.3.9) is abelian. (♠ TODO: a quasi-coherent sheaf on a locally ringed space)
6. If X is a scheme (or more generally a locally ringed space), the category of quasi-coherent sheaves on X is abelian.
7. For any essentially small category (Definition 1.1.9) $\mathcal{C}$ and any abelian category $\mathcal{A}$, the functor category $[\mathcal{C}, \mathcal{A}]$ (Definition 1.2.1) and the category $\text{PreShv}(\mathcal{C}, \mathcal{A})$ of presheaves (Definition 5.0.1) are abelian. (♠ TODO: apparently, the essentially smallness condition is removable, provided that the sheafification functor exists. However, the essentially small assumption is needed to show that the category of sheaves of O-modules is a Grothendieck abelian caetgory. Verify all this. Moreover, when working with a big site of a scheme, one typically fixes a unvierse or work relative to a cardinal cutoff to treat it as essentially small)
8. For any site (Definition 5.1.9) $(\mathcal{C}, J)$ on an essentially small category (Definition 1.1.9) $\mathcal{C}$ and any abelian category $\mathcal{A}$, the category $\text{Shv}(\mathcal{C}, \mathcal{A})$ of sheaves (Definition 5.1.25) is abelian.
9. For any site (Definition 5.1.9) $(\mathcal{C}, J)$ on an essentially small category (Definition 1.1.9) $\mathcal{C}$ and a sheaf of rings (Definition 5.1.25) $\mathcal{O}$ on $\mathcal{C}$, the category $\mathbf{Mod}(\mathcal{O})$ of $\mathcal{O}$-modules (Definition 5.3.9) is an abelian category.

Definition 3.0.6. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor (Definition 3.0.2) between abelian categories (Definition 3.0.3).

1. F is called *left exact* if it preserves all finite limits (Definition 1.2.2), or equivalently it preserves kernels (Definition 1.3.8) and any finite limit diagrams. Equivalently, for every left exact sequence in $\mathcal{A}$

$$0 \to A' \xrightarrow{f} A \xrightarrow{g} A''$$

   the sequence

$$0 \to F(A') \xrightarrow{F(f)} F(A) \xrightarrow{F(g)} F(A'')$$

   is exact at $F(A')$ and $F(A)$ (i.e., F preserves monomorphisms (Definition 1.3.7) and exactness at the first two terms).

2. Dually, F is called *right exact* if it preserves all finite colimits (Definition 1.2.2), or equivalently it preserves cokernels (Definition 1.3.8) and any finite colimit diagrams. Equivalently, for every right exact sequence in $\mathcal{A}$

$$A' \xrightarrow{f} A \xrightarrow{g} A'' \to 0,$$

the sequence

$$F(A') \xrightarrow{F(f)} F(A) \xrightarrow{F(g)} F(A'') \to 0$$

is exact at $F(A)$ and $F(A'')$ (i.e., F preserves epimorphisms (Definition 1.3.7) and exactness at the last two terms).

3. F is called *exact* if it is both left and right exact.

The additive functor F is left/right exact if and only if it is left/right exact (Definition 1.2.13) in the more general sense, i.e. if it preserves all finite (Definition 1.2.5) limits/colimits (Definition 1.2.2)

Lemma 3.0.7. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor (Definition 3.0.2) between abelian categories (Definition 3.0.3). The functor F is left exact/right exact/exact (Definition 3.0.6) as an additive functor if and only if it is left exact/right exact/exact (Definition 1.2.13) as a functor.

3.1. Simple and semisimple objects in an additive category.

Definition 3.1.1. Let $\mathcal{C}$ be an additive category (Definition 3.0.1). Let $X \in \mathrm{Ob}(\mathcal{C})$ be an object. A *subobject of X* refers to a monomorphism (Definition 1.3.7) $i : Y \hookrightarrow X$ in $\mathcal{C}$. We regard two subobjects (Y, i) and (Y', i') of X as isomorphic if there exists an isomorphism $f : Y \to Y'$ such that $i = i' \circ f$. One often leaves the monomorphism i implicit, suppressing it from the notation.

Unlike the notion of a subobject, the notion of a quotient object is more appropriate to speak of for an abelian category rather than a more general additive category.

Definition 3.1.2. Let $\mathcal{C}$ be an abelian category (Definition 3.0.3). Let $X \in \mathrm{Ob}(\mathcal{C})$ be an object. Let $i : A \hookrightarrow X$ be a subobject (Definition 3.1.1). The cokernel (Definition 1.3.8) $\pi : X \twoheadrightarrow X/A := \mathrm{coker}(i)$ is called the *quotient object of X by A*. The object X/A is determined up to canonical isomorphism.

Definition 3.1.3. Let $\mathcal{C}$ be an abelian category (Definition 3.0.3). Let $X \in \mathrm{Ob}(\mathcal{C})$, and let $A \hookrightarrow B \hookrightarrow X$ be subobjects of X. The quotient object (Definition 3.1.2) $B/A := \mathrm{coker}(A \hookrightarrow B)$ is called a *subquotient object of X*.

Definition 3.1.4. Let $\mathcal{C}$ be an additive category (Definition 3.0.1) and $X \in \mathrm{Ob}(\mathcal{C})$. A subobject (Definition 1.3.1) $i : Y \hookrightarrow X$ is called a *direct summand of X* if there exists a morphism $p : X \to Y$ such that $p \circ i = \mathrm{id}_Y$. In this case, X is isomorphic to the direct sum (Definition 3.0.1) $Y \oplus Z$ for some object Z, with i the canonical inclusion.

Definition 3.1.5. Let $\mathcal{C}$ be an additive category (Definition 3.0.1). An object $S \in \mathrm{Ob}(\mathcal{C})$ is called *simple* (or *irreducible*) if $S \neq 0$ and the only subobjects (Definition 3.1.1) of S are 0 and S itself (up to isomorphism of subobjects).

Definition 3.1.6. Let $\mathcal{C}$ be an additive category (Definition 3.0.1). An object $X \in \mathrm{Ob}(\mathcal{C})$ is called *semisimple* if it is isomorphic to a finite direct sum (Definition 3.0.1) of simple objects (Definition 3.1.5) in $\mathcal{C}$.

Definition 3.1.7. An additive category (Definition 3.0.1) $\mathcal{C}$ is called a *semisimple category* if every object of $\mathcal{C}$ is semisimple (Definition 3.1.6).

(♠ TODO: TODO: Give examples of monoidal categoriess)

4. Categories enriched in monoidal categories

(♠ TODO: Carefully check what statements need to assume that the categories are locally small and which do not)

We might want to study categories whose hom-sets carry additional structure beyond that of a set (e.g. an abelian group, vector space); the theory of categories enriched (Definition 4.0.1) in monoidal categories accommodates such desires. In fact, enriched categories can have hom-objects which are not sets.

Definition 4.0.1 (Category enriched in a monoidal category). Let $(\mathcal{V}, \otimes, \mathbf{1})$ be a monoidal category (Definition 2.0.1). A *category enriched in $\mathcal{V}$* (or a *$\mathcal{V}$-enriched category* or a *$\mathcal{V}$-category*) $\mathcal{C}$ consists of the following data:

- A class $\mathrm{Ob}(\mathcal{C})$ of *objects*. As with regular categories, we may write $X \in \mathrm{Ob}(\mathcal{C})$ or $X \in \mathcal{C}$ to mean that X is an object of $\mathcal{C}$.
- For each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, an object $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \in \mathrm{Ob}(\mathcal{V})$ of *morphisms*; it is an object of the monoidal category $\mathcal{V}$. It is also often denoted by notations such as $\mathcal{C}(X, Y)$, $\mathrm{Hom}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$, or $\mathrm{Mor}(X, Y) = \mathrm{Mor}_{\mathcal{C}}(X, Y)$.
- For each triple $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a *composition morphism*

$$\mu_{X,Y,Z} : \underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Z).$$

  It is a morphism in $\mathcal{V}$.
- For each object X, a *unit morphism* $\eta_X : \mathbf{1} \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, X)$ in $\mathcal{V}$.

These data satisfy the following axioms:

- (Associativity) For all $W, X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, the following diagram in $\mathcal{V}$ commutes:

$$
\begin{array}{ccc}
\big(\underline{\mathrm{Hom}}_{\mathcal{C}}(Z, W) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z)\big) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) & \xrightarrow{\ \alpha\ } & \underline{\mathrm{Hom}}_{\mathcal{C}}(Z, W) \otimes \big(\underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)\big) \\
\downarrow{\scriptstyle \mu \otimes \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id} \otimes \mu} \\
\underline{\mathrm{Hom}}_{\mathcal{C}}(Y, W) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) & & \underline{\mathrm{Hom}}_{\mathcal{C}}(Z, W) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Z) \\
\downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\
\underline{\mathrm{Hom}}_{\mathcal{C}}(X, W) & =\!\!=\!\!=\!\!=\!\!=\!\!=\!\!= & \underline{\mathrm{Hom}}_{\mathcal{C}}(X, W)
\end{array}
$$

  where α is the associativity constraint in $\mathcal{V}$.
- (Unit) For all $X, Y \in \mathrm{Ob}(\mathcal{C})$, the following diagrams commute:

$$1 \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \xrightarrow{\eta_Y \otimes \mathrm{id}} \underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Y) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)$$

with diagonal map λ and vertical map μ to $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)$

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \otimes 1 \xrightarrow{\mathrm{id} \otimes \eta_X} \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X,X)$$

with diagonal map ρ and vertical map μ to $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)$

where λ and ρ are the left and right unit constraints in $\mathcal{V}$.

Example 4.0.2.   1. A category enriched in **Sets** is equivalent to a locally small category.
2. A category enriched in Ab is equivalent to a pre-additive category. (♠ TODO: TODO: define an R-linear category and a k-linear category)
3. Let R be a commutative ring. A category enriched in $R-\mathrm{Mod}$ is equivalent to a R-linear category.
4. Let k be a field. A category enriched in $k-\mathrm{Vect}$ is equivalent to a k-linear category.

Definition 4.0.3. Let $\mathcal{V}$ be a monoidal category (Definition 2.0.1) and let $\mathcal{C}$ and $\mathcal{D}$ be categories enriched in $\mathcal{V}$ (Definition 4.0.1). A *$\mathcal{V}$-functor between $\mathcal{C}$ and $\mathcal{D}$* or an *enriched functor between $\mathcal{C}$ and $\mathcal{D}$*, written

$$F : \mathcal{C} \to \mathcal{D},$$

consists of:

- a function on objects $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$,
- for all $A, B \in \mathrm{Ob}(\mathcal{C})$, morphisms in $\mathcal{V}$,

$$F_{A,B} : \mathcal{C}(A,B) \to \mathcal{D}(FA, FB),$$

such that the following diagrams in $\mathcal{V}$ commute:

$$\mathcal{C}(B,C) \otimes \mathcal{C}(A,B) \xrightarrow{\circ_{\mathcal{C}}} \mathcal{C}(A,C)$$
$$\downarrow F_{B,C} \otimes F_{A,B} \qquad \downarrow F_{A,C}$$
$$\mathcal{D}(FB,FC) \otimes \mathcal{D}(FA,FB) \xrightarrow{\circ_{\mathcal{D}}} \mathcal{D}(FA,FC)$$

and for all $A \in \mathrm{Ob}(\mathcal{C})$, the unit compatibility condition holds:

$$\iota_{FA} = F_{A,A} \circ \iota_A.$$

Definition 4.0.4. Let $\mathcal{V}$ be a monoidal category (Definition 2.0.1), let $\mathcal{A}, \mathcal{B}$ be $\mathcal{V}$-categories (Definition 4.0.1), and let $F, G : \mathcal{A} \to \mathcal{B}$ be $\mathcal{V}$-functors (Definition 4.0.3).

A *$\mathcal{V}$-natural transformation* $\alpha : F \Rightarrow G$ consists of a family of morphisms in $\mathcal{V}_0$:

$$\alpha_x : I \to \mathcal{B}(Fx, Gx)$$

indexed by objects $x \in \mathrm{Ob}(\mathcal{A})$, satisfying the $\mathcal{V}$-naturality axiom.

For every pair of objects $x, y \in \mathrm{Ob}(\mathcal{A})$, the following hexagonal diagram in $\mathcal{V}_0$ commutes (expressing the enriched analogue of $\alpha_y \circ F(f) = G(f) \circ \alpha_x$):

$$\begin{array}{ccccc}
\mathcal{A}(x,y) & \xrightarrow{\ F_{xy}\ } & \mathcal{B}(Fx,Fy) & \xrightarrow[\cong]{\ \lambda^{-1}\ } & I \otimes \mathcal{B}(Fx,Fy) \\
\downarrow{\scriptstyle G_{xy}} & & & & \downarrow{\scriptstyle \alpha_y \otimes 1} \\
\mathcal{B}(Gx,Gy) & & & & \mathcal{B}(Fy,Gy) \otimes \mathcal{B}(Fx,Fy) \\
\downarrow{\scriptstyle \rho^{-1}}{\scriptstyle \cong} & & & & \downarrow{\scriptstyle M^{\mathcal{B}}_{Fx,Fy,Gy}} \\
\mathcal{B}(Gx,Gy) \otimes I & \xrightarrow{\ 1 \otimes \alpha_x\ } & \mathcal{B}(Gx,Gy) \otimes \mathcal{B}(Fx,Gx) & \xrightarrow{\ M^{\mathcal{B}}_{Fx,Gx,Gy}\ } & \mathcal{B}(Fx,Gy)
\end{array}$$
Here,

$\lambda : I \otimes X \cong X$ and $\rho : X \otimes I \cong X$ denote the left and right unit isomorphisms of the monoidal category $\mathcal{V}$, and $M^{\mathcal{B}}$ denotes the composition in $\mathcal{B}$.

Definition 4.0.5. Let $\mathcal{V} = (\mathcal{V}_0, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category (Definition 2.0.1), and let $\mathcal{A}$ be a $\mathcal{V}$-category (Definition 4.0.1) with object set $\mathrm{Ob}(\mathcal{A})$ and hom-objects $\mathcal{A}(X,Y) \in \mathrm{Ob}(\mathcal{V}_0)$.

The *underlying ordinary category* of $\mathcal{A}$, denoted $\mathcal{A}_0$, is the category (Definition 1.0.1) defined as follows:

1. The objects of $\mathcal{A}_0$ are the objects of $\mathcal{A}$, i.e., $\mathrm{Ob}(\mathcal{A}_0) = \mathrm{Ob}(\mathcal{A})$.
2. For any two objects $X, Y \in \mathrm{Ob}(\mathcal{A}_0)$, the set of morphisms is defined by
$$\mathrm{Hom}_{\mathcal{A}_0}(X,Y) = \mathrm{Hom}_{\mathcal{V}_0}(I, \mathcal{A}(X,Y)),$$
   where $\mathrm{Hom}_{\mathcal{V}_0}$ denotes the hom-set in the ordinary category $\mathcal{V}_0$.
3. The identity morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{A}_0}(X,X)$ is the element corresponding to the identity morphism $j_X : I \to \mathcal{A}(X,X)$ in $\mathcal{V}$.
4. The composition of morphisms $f \in \mathrm{Hom}_{\mathcal{A}_0}(X,Y)$ and $g \in \mathrm{Hom}_{\mathcal{A}_0}(Y,Z)$ (represented by morphisms $f : I \to \mathcal{A}(X,Y)$ and $g : I \to \mathcal{A}(Y,Z)$ in $\mathcal{V}_0$) is defined to be the composition $g \circ f$ in $\mathcal{V}_0$ given by the following diagram in $\mathcal{V}_0$:
$$I \xrightarrow{\cong} I \otimes I \xrightarrow{g \otimes f} \mathcal{A}(Y,Z) \otimes \mathcal{A}(X,Y) \xrightarrow{M_{XYZ}} \mathcal{A}(X,Z),$$
   where M_{XYZ} is the composition morphism of the enriched category $\mathcal{A}$.

Definition 4.0.6. Let C be a category enriched in a monidal category (Definition 4.0.1) $\mathcal{V}$. Given an object X of C, the *functor of points* h_X is the functor (Definition 1.1.1)/presheaf (Definition 5.0.1) $C^{\mathrm{op}} \to \mathcal{V}$ given by $T \mapsto \mathrm{Hom}_C(T,X)$. A functor $C^{\mathrm{op}} \to \mathcal{V}$ (or equivalently, a presheaf on C valued in $\mathcal{V}$) is said to be *representable* if it is naturally isomorphic (Definition 1.1.3) to some functor h_X of points for an object X of C.

Dually, a functor $C \to \mathcal{V}$ is called *co-representable* if it is naturally isomorphic to a functor $T \mapsto \mathrm{Hom}_C(X,T)$ for an object X in C.

For instance, we may speak of these notions when $\mathcal{V}$ is the monoidal category **Sets**, i.e. C is a locally small category (Definition 1.0.9).

Theorem 4.0.7 ((Enriched) Yoneda lemma)**.** (♠ TODO: functor of enriched categories, natural transform) Let C be a $\mathcal{V}$-enriched category (Definition 4.0.1) where $\mathcal{V}$ is a monoidal category (Definition 2.0.1), and let $F : C^{\mathrm{op}} \to \mathcal{V}$ be a $\mathcal{V}$-functor. For any object $X \in C$, there is a natural isomorphism:

$$\mathrm{Nat}_{\mathcal{V}}(h_X, F) \cong F(X),$$

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where $h_X = \mathrm{Hom}_C(-, X)$ is the representable functor associated to X (Definition 4.0.6), and $\mathrm{Nat}_{\mathcal{V}}(h_X, F)$ is the $\mathcal{V}$-object of $\mathcal{V}$-natural transformations.

(♠ TODO: end) Alternatively, if $\mathcal{V}$ is closed (Definition 2.0.4) and locally small (Definition 1.0.9), then

$$\mathrm{Nat}_{\mathcal{V}}(h_X, F) \cong \int_{T \in C} [\mathrm{Hom}_C(T, X), F(T)],$$

where $[\_, \_]$ denotes the internal hom (Definition 2.0.2) in $\mathcal{V}$ and $\int$ is the end.

In particular, when $\mathcal{V} = \mathbf{Sets}$ (so C is simply a locally small category (Definition 1.0.9)), the (classical) Yoneda lemma states the following: let $F : C^{\mathrm{op}} \to \mathbf{Sets}$ a functor, and let X be an object of C. Then there is a natural isomorphism

$$\mathrm{Nat}(h_X, F) \cong F(X).$$

5. Sites, sheaves and topoi

Definition 5.0.1 (Presheaf on a category). Let C and $\mathcal{A}$ be (large) categories (Definition 1.0.1).

1. A *presheaf $\mathcal{F}$ on C with values in $\mathcal{A}$* is a functor

$$\mathcal{F} : C^{\mathrm{op}} \to \mathcal{A}.$$

In other words, a presheaf $\mathcal{F}$ on C with values in $\mathcal{A}$ is simply a contravariant functor (Definition 1.1.1) from C to $\mathcal{A}$. Explicitly, for every object U in C, one has an object $\mathcal{F}(U)$ in $\mathcal{A}$ (called the *U-valued sections/sections evaluated at U of $\mathcal{F}$*, cf. Definition 5.3.11), and for every morphism $f : V \to U$ in C, one has a morphism (called the *restriction map*)

$$\mathcal{F}(f) : \mathcal{F}(U) \to \mathcal{F}(V)$$

in $\mathcal{A}$, such that for all composable morphisms $W \xrightarrow{g} V \xrightarrow{f} U$ in C, the following diagram in $\mathcal{A}$ commutes:

$$\mathcal{F}(U) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(V) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(W)$$

with the arc $\mathcal{F}(f \circ g)$ from $\mathcal{F}(U)$ to $\mathcal{F}(W)$.

That is,

$$\mathcal{F}(g) \circ \mathcal{F}(f) = \mathcal{F}(f \circ g),$$

and for every object U in C, $\mathcal{F}(\mathrm{id}_U) = \mathrm{id}_{\mathcal{F}(U)}$.

2. Let $\mathcal{F}, \mathcal{G} : C^{\mathrm{op}} \to \mathcal{A}$ be two presheaves on C with values in $\mathcal{A}$. A *morphism of presheaves*

$$\varphi : \mathcal{F} \to \mathcal{G}$$

is a natural transformation of functors (Definition 1.1.3): for each object U of C, one has a morphism

$$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

in $\mathcal{A}$, such that for every morphism $f : V \to U$ in C, the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(V) \\ \varphi_U \downarrow & & \downarrow \varphi_V \\ \mathcal{G}(U) & \xrightarrow[\mathcal{G}(f)]{} & \mathcal{G}(V) \end{array}$$

commutes, i.e.,

$$\varphi_V \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \varphi_U$$

for all objects and morphisms in C.

3. Given a universe (Definition E.0.1) U, a U-*presheaf on* $\mathcal{C}$ typically refers to a presheaf of U-sets on C.

4. The *presheaf category/category of $\mathcal{A}$-valued presheaves on $\mathcal{C}$* is the (large) category whose objects are the presheaves on C with values in $\mathcal{A}$ and whose morphisms are the presheaf morphisms. Common notations for the presheaf category include, but are not limited to: $\mathcal{A}^{\mathcal{C}^{\mathrm{op}}}$, $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$, $[\mathcal{C}^{\mathrm{op}}, \mathcal{A}]$. If the value category $\mathcal{A}$ is clear from context, then notations such as $\mathrm{PreShv}(\mathcal{C})$ are also common. Note that the presheaf category $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ is equivalent to the category of functors (Definition 1.2.1) $\mathcal{C}^{\mathrm{op}} \to \mathcal{A}$ and hence notations for the functor categories are applicable as notations for presheaf categories.

Lemma 5.0.2. Let $\mathcal{C}$ be a small category (Definition 1.0.9) (resp. U-small category where U is some universe (Definition E.0.1)) and let $\mathcal{A}$ be a locally small (Definition 1.0.9) category (resp. U-locally small category). The presheaf category $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ (Definition 5.0.1) is locally small (resp. U-locally small).

Proof. A morphism $\mathcal{F} \to \mathcal{G}$ in $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ is a natural transformation (Definition 1.1.3) of the functors $\mathcal{F}, \mathcal{G} : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$. Such a natural transformation is encoded by a family $(\eta_C)_C$ of morphisms (satisfying certain conditions) $\eta_C : \mathcal{F}(C) \to \mathcal{G}(C)$ in $\mathcal{A}$ over objects C of $\mathcal{C}^{\mathrm{op}}$. The product $\prod_{C \in \mathrm{Ob}\,\mathcal{C}^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}(C), \mathcal{G}(C))$ is a product of (U-small) sets indexed by a (U-small) set, and the collection of natural transformations is a subset of this set. Therefore, $\mathrm{Hom}_{\mathrm{PreShv}(\mathcal{C}, \mathcal{A})}(\mathcal{F}, \mathcal{G})$ is a (U-small) set. $\square$

Remark 5.0.3. Even when $\mathcal{C}$ is (U-)locally small and ($\mathcal{A}$ is (U-)locally small), $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ may not be locally small.

Remark 5.0.4. In practice, one might treat $\mathcal{C}$ as if it were a small category, even when it is technically not a small category, to treat $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ as if it were a locally small category. For example, in algebraic geometry, we might take $\mathcal{C}$ to be the small étale site $X_{\text{ét}}$ of some scheme X; this is not a small category. Instead, we might technically replace $X_{\text{ét}}$ with a small category, e.g. the set of étale maps $U \to X$ of the following form: U is obtained by patching the schemes attached to quotients of rings of the form $A[T_1, T_2, \ldots]$ where $A = \Gamma(V, \mathcal{O}_X)$ for some open affine $V \subset X$ and $\{T_1, T_2, \ldots\}$ is a fixed countable set of symbols. Such technicalities of replacing $\mathcal{C}$ with a small category are often glossed over.

Lemma 5.0.5. Let $\mathcal{C}$ be a (large) category (Definition 1.0.1). The category of set-valued presheaves (Definition 5.0.1) on $\mathcal{C}$ has a final object (Definition 2.0.6), namely the presheaf $\mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}$ which sends any object T of $\mathcal{C}$ to a singleton set $\{*\}$.

5.1. **Sieves, Grothendieck topologies, and sites.**

5.1.1. *Sieves.*

Definition 5.1.1 ([GV72, Exposé I Définition 4.1])**.** Let C be a (large) category (Definition 1.0.1).

1. A *sieve S on the category C* is a full subcategory (Definition 1.3.2) D of C such that for any object U of C there exists an object V of (🔥 TODO: correctly parse the definiton)
2. A *sieve S on an object $U \in \mathrm{Ob}(C)$* is a collection of morphisms in C with codomain U that is closed under precomposition by any compatible morphism in C. In other words, S is a sieve if for every $f : V \to U$ in S and morphism $g : W \to V$ in C, the composition $f \circ g : W \to U$ is also in S.

    Given a morphism $f : V \to U$ in a sieve S, we also say that *f factors through U*.

Definition 5.1.2 (Downward/upward closed subcategory)**.** Let $\mathcal{C}$ be a (large) category (Definition 1.0.1) and $\mathcal{D} \hookrightarrow \mathcal{C}$ a full subcategory (Definition 1.3.2).

1. We say that $\mathcal{D}$ is *downward closed* if for every object $d \in \mathrm{Ob}(\mathcal{D})$ and every morphism $f : c \to d$ in $\mathcal{C}$ (with $c \in \mathrm{Ob}(\mathcal{C})$), the domain object c belongs to $\mathrm{Ob}(\mathcal{D})$ (and hence $f \in \mathcal{D}$ by fullness of the embedding).
2. We say that $\mathcal{D}$ is *upward closed* if for every object $d \in \mathrm{Ob}(\mathcal{D})$ and every morphism $f : d \to c$ in $\mathcal{C}$ (with $c \in \mathrm{Ob}(\mathcal{C})$), the codomain object c belongs to $\mathrm{Ob}(\mathcal{D})$ (and hence $f \in \mathcal{D}$ by fullness of the embedding).

Proposition 5.1.3. Let $\mathcal{C}$ be a category (Definition 1.0.1). Let X be an object of $\mathcal{C}$. The sieves (Definition 5.1.1) on X are in bijection with the downward closed (Definition 5.1.2) full subcategories of the slice category $\mathcal{C}/X$ (Definition 1.3.4).

Definition 5.1.4. Let C be a category and let U be na object of C. The *maximal sieve on U* is the collection of all morphisms with target U; clearly, it is a sieve on U.

Definition 5.1.5. Let $\mathcal{C}$ be a category (Definition 1.0.1) and $U \in \mathcal{C}$ an object. Let $\mathcal{S} = \{f_i : U_i \to U\}_{i \in I}$ be a family of morphisms with codomain U.

The *sieve generated by $\mathcal{S}$*, denoted $(\mathcal{S})$ or $\langle \mathcal{S} \rangle$, is the smallest sieve on U (Definition 5.1.1) containing all the morphisms in $\mathcal{S}$.

Explicitly, a morphism $h : V \to U$ belongs to the generated sieve if and only if h factors through some morphism in $\mathcal{S}$. That is, there exists an index $i \in I$ and a morphism $g : V \to U_i$ such that
$$h = f_i \circ g.$$

Definition 5.1.6. Let C be a category, let $U \in \mathrm{Ob}(C)$, and let S be a sieve on U (Definition 5.1.1). For a morphism $f : V \to U$ in C, the *pullback sieve f^*S* (or *basechange sieve $S \times_U V$*) on V is defined by
$$f^*S = \{g : W \to V \mid f \circ g \in S\}.$$

In other words, f^*S consists of all morphisms into V whose composite with f belongs to the sieve S on U.

Lemma 5.1.7. Let C be a category, let $U \in \text{Ob}(C)$, and let S be a sieve on U (Definition 5.1.1). Let $f : V \to U$ be a morphism in C. The pullback sieve (Definition 5.1.6) f^*S is indeed a sieve (Definition 5.1.1) on V.

Proof. Let $g : W \to V$ be a morphism in f^*S, and let $h : X \to W$ be a morphism in C. The composition $g \circ h : X \to V$ is also in f^*S because $f \circ (g \circ h) = (f \circ g) \circ h$, the map $(f \circ g)$ is in S, and S is a sieve. $\square$

5.1.2. *Grothendieck pretopologies.*

Definition 5.1.8. Let $\mathcal{C}$ be a category (Definition 1.0.1). A *basis for a Grothendieck topology* (also called a *Grothendieck pretopology* or simply a *pretopology*) on $\mathcal{C}$ is a collection of families $K(U)$ of morphisms for each object $U \in \mathcal{C}$, called *coverings* or *covering families*, satisfying the following axioms:

1. **(Identity)** For every isomorphism $U' \to U$, the singleton family $\{U' \to U\}$ is in $K(U)$.
2. **(Base Change)** If $\{U_i \to U\}_{i \in I}$ is a covering family in $K(U)$ and $V \to U$ is any morphism in $\mathcal{C}$, then the fiber products (Definition 1.2.32) $U_i \times_U V$ exist, and the family of projections $\{U_i \times_U V \to V\}_{i \in I}$ is in $K(V)$.
3. **(Composition)** If $\{U_i \to U\}_{i \in I}$ is in $K(U)$ and for each $i \in I$, $\{V_{ij} \to U_i\}_{j \in J_i}$ is in $K(U_i)$, then the composite family $\{V_{ij} \to U_i \to U\}_{i \in I, j \in J_i}$ is in $K(U)$.

5.1.3. *Coverages.*

5.1.4. *Grothendieck topologies.*

Definition 5.1.9 (Grothendieck topology)**.** Let $\mathscr{U}$ be a universe (Definition E.0.1).

1. (See [GV72, Exposé II, Définition 1.1]) Let $\mathcal{C}$ be a category (Definition 1.0.1). A *Grothendieck topology on $\mathcal{C}$* assigns to each object U of $\mathcal{C}$ a collection $J(U)$ of sieves (Definition 5.1.1) $\{U_i \to U\}_{i \in I}$, each called a *covering sieve of U*, satisfying:
   (a) (Stability under "base change"): If $S \in J(U)$ is a covering sieve of an object U, and $f : V \to U$ is any morphism in $\mathcal{C}$, then the pullback sieve (Definition 5.1.6) f^*S is a covering sieve of U.
   (b) (Local character condition) If S is a sieve on U, and if there exists a covering sieve $R \in J(U)$ such that for all $f : V \to U$ in R the pullback sieve (Definition 5.1.6) f^*S is in $J(V)$, then $S \in J(U)$.
   (c) The maximal sieve (Definition 5.1.4) is a covering sieve.
   Some will refer to a Grothendieck topology as simply a *topology*, not to be confused with the related, but less general, notion of a topology on a set (Definition B.0.1).
2. (See [GV72, Exposé II, 1.1.5]) A *site* is a category $\mathcal{C}$ equipped with a Grothendieck topology.
   When we are working with a Grothendieck pretopology (Definition 5.1.8) K on a category $\mathcal{C}$, we may regard $\mathcal{C}$ as a site by equipping it with the Grothendieck topology generated by (Definition 5.1.10) K.

3. (See [GV72, Exposé II, Définition 1.2]) Let $(\mathcal{C}, J)$ be a site. A family of morphisms $(U_i \to U)_{i \in I}$ is called a *covering family of U (with respect to the site/topology)* or a *cover of U (with respect to the site/topology)* if the sieve generated by (Definition 5.1.5) the family is a covering sieve of U.

4. (See [GV72, Exposé II, Définition 3.0.1]) Let $(\mathcal{C}, J)$ be a site (Definition 5.1.9), where J is a Grothendieck topology on $\mathcal{C}$.

   A family G of objects $\mathcal{C}$ is called a *topologically generating family of the site/topology* or a *generating family/collection of the site/topology* if for every object $X \in \mathcal{C}$, there is a covering family $\{X_\alpha \to X\}_{\alpha \in A}$ of X such that every X_α is a member of G.

   Equivalently, the Grothendieck topology J is the smallest Grothendieck topology containing all covers of the U_i. Also equivalently, for any $S \in J(X)$, the sieve S contains a covering family $\{V_i \to X\}$ such that each morphism $V_i \to X$ factors through some member of G. (♠ TODO: Verify that these claimed equivalences are indeed equivalences)

5. (See [GV72, Exposé II, Définition 3.0.2]) A *$\mathscr{U}$-site* is a site whose underlying category $\mathcal{C}$ is $\mathscr{U}$-locally small (Definition 1.0.9) and which has a $\mathscr{U}$-small topologically generating family. A $\mathscr{U}$-site is called *$\mathscr{U}$-small* if its underlying category is $\mathscr{U}$-small. Similarly, a *small site* is a site whose underlying category is a set and a *locally small site* is a site whose underlying category is locally small (Definition 1.0.9).

Definition 5.1.10. Let $\mathcal{C}$ be a category equipped with a Grothendieck pretopology (Definition 5.1.8) K. The *Grothendieck topology generated by K*, denoted J_K, is the smallest Grothendieck topology (Definition 5.1.9) on $\mathcal{C}$ such that every family in $K(U)$ is a covering family for J_K.

Explicitly, a sieve (Definition 5.1.1) S on an object U belongs to $J_K(U)$ if and only if there exists a covering family (Definition 5.1.8) $\{U_i \to U\}_{i \in I} \in K(U)$ such that for every $i \in I$, the morphism $U_i \to U$ belongs to S.

The condition that S contains the family $\{U_i \to U\}$ is equivalent to saying that the sieve generated by this family is a sub-sieve of S.

Definition 5.1.11. Let $(\mathcal{C}, J)$ and $(\mathcal{C}, J')$ be two sites (Definition 5.1.9) on the same underlying category $\mathcal{C}$. We say that the topology J is *finer than J'* (or equivalently, that *J' is coarser than J*) if for every object U in $\mathcal{C}$, there is an inclusion

$$J'(U) \subseteq J(U)$$

That is, every sieve (Definition 5.1.1) on U which is a covering sieve for J' is also a covering sieve for J.

Definition 5.1.12 (cf. [GV72, Exposé III, 3.1]). Let $\mathcal{C}, \mathcal{D}$ be categories (Definition 1.0.1), let $u : \mathcal{C} \to \mathcal{D}$ be a functor (Definition 1.1.1), and let $(\mathcal{D}, K)$ be a site (Definition 5.1.9). The *induced topology $u^* K$* (also called the *pullback topology)* on $\mathcal{C}$ is the finest (Definition 5.1.11) Grothendieck topology (Definition 5.1.9) making u a continuous functor (Definition 5.2.1). Equivalently, it is the Grothendieck topology given as follows: for every object $V \in \mathcal{C}$, a sieve (Definition 5.1.1) $S \in J_{u^* K}(V)$ is a covering sieve if and only if its image $u(S)$ under u is a covering sieve in $K(u(V))$.

Proposition 5.1.13. Let $\mathcal{C}, \mathcal{D}$ be categories (Definition 1.0.1), let $u : \mathcal{C} \to \mathcal{D}$ be a functor (Definition 1.1.1), and let $(\mathcal{D}, K)$ be a site (Definition 5.1.9).

The functor $u : (\mathcal{C}, u^*K) \to (\mathcal{D}, K)$ (Definition 5.1.12) is continuous (Definition 5.2.1).

Theorem 5.1.14 (Comparison Lemma, cf. [GV72, Exposé III, Théorème 4.1])**.** Let $\mathcal{C}$ be a small category (Definition 1.0.9), let $(\mathcal{D}, K)$ be a site (Definition 5.1.9), let $u : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor (Definition 1.1.4), and equip $\mathcal{C}$ with the Grothendieck topology (Definition 5.1.9) J induced by (Definition 5.1.12) u. Let $\mathcal{A}$ be a locally small (Definition 1.0.9) complete category (Definition 1.3.9), i.e. a category which admits all small limits (Definition 1.2.5).

If the objects of $\mathcal{C}$ form a topologically generating family (Definition 5.1.9)(♠ TODO: I need to check that the condition of covering in SGA really matches the notion of topologically generating family) of $\mathcal{D}$, then the direct image (Definition 5.3.2)/restriction functor

$$u_s : \mathrm{Sh}(\mathcal{D}, K; \mathcal{A}) \to \mathrm{Sh}(\mathcal{C}, J; \mathcal{A})$$

is an equivalence of categories (Definition 1.1.8). In particular, $\mathrm{Sh}(\mathcal{D}, K; \mathcal{A})$ is a locally small category (Definition 1.0.9).

Proof. We explicitly construct the inverse functor and prove the equivalence without assuming *a priori* that $\mathrm{Sh}(\mathcal{D}, K)$ is locally small.

Step 1: Construction of the Inverse Functor u_*

We define the functor $u_* : \mathrm{Sh}(\mathcal{C}, J) \to \mathrm{Sh}(\mathcal{D}, K)$ as the **Right Kan Extension** of a sheaf F along u.

For a sheaf $F \in \mathrm{Sh}(\mathcal{C}, J)$ and an object $d \in \mathcal{D}$, define:

$$u_*F(d) = \lim_{(c,f) \in (u \downarrow d)^{op}} F(c)$$

Here, the limit is taken over the opposite of the comma category $(u \downarrow d)$, whose objects are pairs $(c, f : u(c) \to d)$.

Verification of Well-Definedness (Size): Since $\mathcal{C}$ is a small category and $\mathcal{D}$ is locally small, the collection of objects in $(u \downarrow d)$ is a set (indexed by objects of $\mathcal{C}$ and hom-sets of $\mathcal{D}$). Therefore, the limit defining $u_*F(d)$ is a **small limit of sets**, which exists in **Set**. Thus, u_*F takes values in **Set** rather than proper classes.

Verification that u_*F is a Sheaf: Since limits commute with limits, and the sheaf condition is a limit condition, the Right Kan Extension of a sheaf along a continuous functor is a sheaf. The density condition ensures that covers in $\mathcal{D}$ are "seen" by $\mathcal{C}$, ensuring the sheaf condition is preserved.

Step 2: The Unit of Adjunction ($u^* u_* \cong \mathrm{Id}$)

We examine $u^* u_* F$ for $F \in \mathrm{Sh}(\mathcal{C}, J)$. Evaluating at an object $c_0 \in \mathcal{C}$:

$$(u^* u_* F)(c_0) = u_* F(u(c_0)) = \lim_{(c,f) \in (u \downarrow u(c_0))^{op}} F(c)$$

Since u is **fully faithful**, the comma category $(u \downarrow u(c_0))$ has a terminal object: $(c_0, \mathrm{id}_{u(c_0)})$. The limit over a category with a terminal object is isomorphic to the value at that object. Thus, $(u^* u_* F)(c_0) \cong F(c_0)$, concluding $u^* u_* \cong \mathrm{Id}_{\mathrm{Sh}(\mathcal{C})}$.

Step 3: The Counit of Adjunction ($H \cong u_* u^* H$)

This is the critical step that uses the **Density Condition** to control the size of sheaves on $\mathcal{D}$. Let $H \in \mathrm{Sh}(\mathcal{D}, K)$. We construct a map $\eta_d : H(d) \to u_*(u^* H)(d)$. By definition:

$$u_*(u^* H)(d) = \lim_{u(c) \to d} H(u(c))$$

There is a canonical map η_d induced by the morphisms $H(f) : H(d) \to H(u(c))$ for each $f : u(c) \to d$.

To see that η_d is an isomorphism:

1. **Density as a Cover:** By hypothesis, the family of morphisms $\mathcal{S} = \{f : u(c) \to d \mid c \in \mathcal{C}\}$ generates a covering sieve S on d.
2. **Sheaf Property:** Since H is a sheaf on $(\mathcal{D}, K)$, $H(d) \cong \mathrm{Match}(S, H)$.
3. **Matching Families:** The limit over the comma category $(u \downarrow d)$ is exactly the set of compatible families indexed by the generators of the sieve. Because $\mathcal{S}$ generates S, the data of a matching family on $\mathcal{S}$ extends uniquely to the whole sieve S.

The canonical map $H(d) \to \lim_{u(c) \to d} H(u(c))$ is therefore an isomorphism, so $H \cong u_*(u^* H)$.

Step 4: Conclusion of Equivalence and Local Smallness

We have established natural isomorphisms $u^* u_* \cong \mathrm{Id}$ and $\mathrm{Id} \cong u_* u^*$, establishing an equivalence. Since $\mathcal{C}$ is small, $\mathrm{Sh}(\mathcal{C}, J)$ is a locally small category. Since $\mathrm{Sh}(\mathcal{D}, K)$ is equivalent to it, $\mathrm{Sh}(\mathcal{D}, K)$ is itself locally small. Specifically:

$$\mathrm{Hom}_{\mathcal{D}}(H, G) \cong \mathrm{Hom}_{\mathcal{C}}(u^* H, u^* G)$$

where the latter is a set.

Since $\mathcal{C}$ is small and $\mathcal{A}$ is locally small, the category of presheaves on $\mathcal{C}$ valued in $\mathcal{A}$ is locally small by Lemma 5.0.2. Therefore, the category of sheaves is locally small. $\square$

Proposition 5.1.15 ([GV72, Exposé II, Proposition 6.3.1]). Let $\mathcal{C}$ be a site (Definition 5.1.9), and let γ be a "kind of algebraic structure defined by finite projective limits" (♠ TODO: Probably the correct terminology for this is "essentially algebraic categories" or "Lawvere theories") (e.g. γ can be the notion of group objects or abelian group objects (Definition 1.3.19), ring objects (Definition 1.3.20), or module objects (Definition 1.3.21)). The following categories are equivalent (Definition 1.1.8):

1. The category of sheaves (Definition 5.1.25) on $\mathcal{C}$ with values in the category of γ-sets,
2. The category of γ-objects of the category of sheaves of sets on $\mathcal{C}$, and
3. The category of γ-objects of the category of presheaves (Definition 5.0.1) of sets on $\mathcal{C}$ whose underlying presheaf is a sheaf of sets.

Definition 5.1.16. Let $\mathcal{C}$ be a locally small (Definition 1.0.9) category (Definition 1.0.1). An object X in $\mathcal{C}$ is called *quasi-compact* (or *compact*, or *finitely presentable* depending on context) if the functor $h^X = \hom_{\mathcal{C}}(X, -)$ represented by (Definition 4.0.6) X, preserves filtered colimits (Definition 1.2.12).

Explicitly, this means that for any filtered diagram (Definition 1.2.1) $D : \mathcal{I} \to \mathcal{C}$ such that the colimit (Definition 1.2.2) $\operatorname{colim}_{i \in \mathcal{I}} D_i$ exists in $\mathcal{C}$, the canonical map

$$\operatorname{colim}_{i \in \mathcal{I}} \hom_{\mathcal{C}}(X, D_i) \xrightarrow{\cong} \hom_{\mathcal{C}}(X, \operatorname{colim}_{i \in \mathcal{I}} D_i)$$

is a bijection.

Definition 5.1.17. Let $(\mathcal{C}, J)$ be a site (Definition 5.1.9). Assume that $\mathcal{C}$ is locally small (Definition 1.0.9).

An object U in $\mathcal{C}$ is called *quasi-compact (relative to the topology J)* if every covering sieve (Definition 5.1.9) $S \in J(U)$ contains a finite set of morphisms $\{f_i : V_i \to U\}_{i=1}^n$ which generates (Definition 5.1.5) a covering sieve (Definition 5.1.9).

Equivalently, if coverings are defined via a basis (or pretopology) (Definition 5.1.8), an object U is quasi-compact if every covering family $\{U_i \to U\}_{i \in I}$ admits a finite subfamily $\{U_i \to U\}_{i \in I_0}$ (with $I_0 \subseteq I$ finite) which is also a covering family.

Definition 5.1.18. (♠ TODO: does this need local smallness) Let $(\mathcal{C}, J)$ be a locally small (Definition 1.0.9) site (Definition 5.1.9). A morphism $f : X \to Y$ in $\mathcal{C}$ is called *quasi-compact* if for every morphism $Z \to Y$ where the source Z is a quasi-compact object with respect to J (Definition 5.1.17), the fiber product (Definition 1.2.32) $X \times_Y Z$ exists and is also a quasi-compact object with respect to J.

Definition 5.1.19. Let $(\mathcal{C}, J)$ be a locally small (Definition 1.0.9) site (Definition 5.1.9).

An object X in $\mathcal{C}$ is called *quasi-separated with respect to J* if the product (Definition 1.2.30) $X \times X$ exists and the diagonal morphism (Definition 1.2.33) $\Delta : X \to X \times X$ is a quasi-compact morphism with respect to J.

Explicitly, this means that for any two morphisms $U \to X$ and $V \to X$ from quasi-compact objects U and V, the fiber product $U \times_X V$ is a quasi-compact object, assuming that the relevant fiber products exist.

Definition 5.1.20. Let $(\mathcal{C}, J)$ be a site where $\mathcal{C}$ is locally small (Definition 1.0.9). An object $X \in \mathcal{C}$ is called a *coherent object* if it satisfies two conditions:

1. X is quasi-compact (Definition 5.1.17) relative to the topology J.
2. X is quasi-separated (Definition 5.1.19) relative to the topology J.

Definition 5.1.21. A site (Definition 5.1.9) $(\mathcal{C}, J)$ is called a *coherent site* if it satisfies the following conditions:

1. The category $\mathcal{C}$ is locally small with a small topologically generating family (Definition 5.1.9).
2. The category $\mathcal{C}$ admits all finite limits (Definition 1.2.2).

3. Every object in $\mathcal{C}$ is quasi-compact relative to (Definition 5.1.17) the topology J.
4. The collection of quasi-compact objects relative to J is closed under fiber products (Definition 1.2.32). Explicitly, if $X \to Z$ and $Y \to Z$ are morphisms where X, Y, Z are quasi-compact, then $X \times_Z Y$ is also quasi-compact.

Definition 5.1.22 (Slice site)**.** Let $(\mathcal{C}, \tau)$ be a site (Definition 5.1.9), where τ is a Grothendieck topology on the (locally small or U-locally small (Definition 1.0.9), if a universe (Definition E.0.1) U is available) category $\mathcal{C}$. For a fixed object X in $\mathcal{C}$, the *slice site* (or the *over site*, the *site on the slice category* $\mathcal{C}_{/X}$, the *site induced on the over category* $\mathcal{C}_{/X}$, the *localization of the site* $\mathcal{C}$ *at the object* X, etc.) $(\mathcal{C}_{/X}, \tau_{/X})$ is the site whose underlying category is the slice category $\mathcal{C}_{/X}$ (Definition 1.3.4), and whose Grothendieck topology $\tau_{/X}$ (also denoted by notations such as $\tau|_X$ or τ/X) is defined by declaring a family of morphisms $\{f_i : Y_i \to Y\}$ in $\mathcal{C}_{/X}$ to be a covering if and only if the family $\{f_i : Y_i \to Y\}$ is a covering in $(\mathcal{C}, \tau)$.

Definition 5.1.23. Let (X, τ_X) be a topological space. The *small site associated to* X or *the site of open covers of* X or *the canonical site on* $\mathrm{Open}\, X$ is the category $\mathrm{Open}(X)$ of open subsets (Definition 1.0.7) of X with inclusion morphisms, equipped with the canonical Grothendieck topology (Definition 5.1.9) generated by (Definition 5.1.10) the Grothendieck pretopology (Definition 5.1.8) whose covering families $\{U_i \to U\}_{i \in I}$, for $U \in \mathrm{Open}(X)$ are families of morphisms in $\mathrm{Open}(X)$ such that $\bigcup_{i \in I} U_i = U$. In other words, $\{U_i \to U\}_{i \in I}$ is a covering for the pretopology if it is an open coverings (Definition B.0.2).

Proposition 5.1.24. (♠ TODO: many definitions) The following are common examples of sites:

1. Let X be a scheme. The following categories equipped with their respective Grothendieck topologies form *small sites associated to* X:
   - The *small Zariski site* $\mathrm{Zar}(X)$: the category of open immersions $U \to X$ with the Zariski topology given by open covers.
   - The *small étale site* $\mathrm{\acute{E}t}(X)$: the category of étale morphisms (Definition D.0.1) $U \to X$ equipped with the étale topology defined by étale coverings.
   - The *small Nisnevich site* $\mathrm{Nis}(X)$: the category of étale morphisms $U \to X$ with the Nisnevich topology given by Nisnevich covers.
   - The *small fppf site* $\mathrm{fppf}(X)$: the category of flat morphisms locally of finite presentation $U \to X$ equipped with the fppf topology.
   - The *small fpqc site* $\mathrm{fpqc}(X)$: the category of flat morphisms $U \to X$ (not necessarily locally of finite presentation), with the fpqc topology.
   - The *small crystalline site* $\mathrm{Cris}(X)$: the category of PD-thickenings over X equipped with the crystalline topology.

5.1.5. *Sheaf on a site.*

Definition 5.1.25 (Sheaf on a site)**.** Let $(\mathcal{C}, J)$ be a site (Definition 5.1.9). Let $\mathcal{A}$ be a (large) category (Definition 1.0.1).

1. A presheaf (Definition 5.0.1) $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$ (Definition 1.0.12) is called a *sheaf on the site* $(\mathcal{C}, J)$ *valued in* $\mathcal{A}$ if, for every object U of $\mathcal{C}$ and every covering sieve (Definition 5.1.9)

$S \in J(U)$, the limit (Definition 1.2.2)

$$\varprojlim_{(V \to U) \in (\mathcal{D}_S)^{\mathrm{op}}} \mathcal{F}|_{\mathcal{D}_S}(V),$$

exists and the canonical natural morphism

$$\mathcal{F}(U) \to \varprojlim_{(V \to U) \in (\mathcal{D}_S)^{\mathrm{op}}} \mathcal{F}|_{\mathcal{D}_S}(V)$$

is an isomorphism. Here, $\mathcal{D}_S \hookrightarrow \mathcal{C}/U$ (Definition 1.3.4) is the full downward-closed subcategory (Definition 5.1.2) such that $\mathrm{Ob}(\mathcal{D}_S) = \{(f : V \to U) : f \in S(V)\}$,

In particular, when we are working with a Grothendieck pretopology (Definition 5.1.8) K on a category $\mathcal{C}$, we may speak of sheaves on the site whose Grothendieck topology is the one generated by (Definition 5.1.10) K.

2. Given sheaves $\mathcal{F}, \mathcal{G} : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$ on the site $(\mathcal{C}, J)$, a *morphism between the sheaves* is a morphism (Definition 5.0.1) between $\mathcal{F}$ and $\mathcal{G}$ as presheaves.

3. Let U be a universe (Definition E.0.1). A *U-sheaf* typically refers to a U-presheaf that is a sheaf for a U-site. In other words, a U-sheaf is a sheaf on a site whose underlying category is U-locally small (Definition 1.0.9) and which has a U-small topologically generating family such that the sheaf is valued in U-sets.

4. The *sheaf category/category of $\mathcal{A}$-valued sheaves on $\mathcal{C}$* is the (large) category defined as the full subcategory of $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ whose objects are the sheaves on $\mathcal{C}$ with values in $\mathcal{A}$. Common notations for the sheaf category include $\mathrm{Shv}(\mathcal{C}, \mathcal{A})$, $\mathrm{Shv}(\mathcal{C}, J, \mathcal{A})$, $\mathrm{Sh}(\mathcal{C}, \mathcal{A})$, $\mathrm{Sh}(\mathcal{C}, J, \mathcal{A})$. If the value category $\mathcal{A}$ is clear from context, then notations such as $\mathrm{Shv}(\mathcal{C})$, $\mathrm{Shv}(\mathcal{C}, J)$, $\mathrm{Sh}(\mathcal{C})$, $\mathrm{Sh}(\mathcal{C}, J)$ are also common.

Definition 5.1.26 (Sheaf on a topological space). Let X be a topological space (Definition B.0.1), let $\mathcal{D}$ be a category (Definition 1.0.1) with a terminal object (Lemma 2.0.7), and let $\mathcal{F}$ be a presheaf valued in $\mathcal{D}$ on X (Definition 5.1.28). Then $\mathcal{F}$ is a *sheaf* if it satisfies the following additional condition (known as the *sheaf axioms*):

For every open set $U \subseteq X$ and every open cover (Definition B.0.2) $\{U_i\}_{i \in I}$ of U, let $\mathcal{J}$ be the diagram (Definition 1.2.1) in the category of opens (Definition 1.0.7) of U consisting of the inclusions $U_i \cap U_j \hookrightarrow U_i$ for all $i, j \in I$. Then $\mathcal{F}$ is a sheaf if the limit (Definition 1.2.2) of the diagram $\mathcal{F} \circ \mathcal{J}$ exists in $\mathcal{D}$ and the natural morphism

$$\mathcal{F}(U) \to \lim_{j \in \mathcal{J}} \mathcal{F}(j)$$

is an isomorphism. More precisely, $\mathcal{J} : J \to \mathrm{Open}(U)$ should be the functor whose index category J consists of

1. An object i for every $i \in I$ and an object (i, j) for every pair $i, j \in I$,
2. Morphisms $p_1 : (i, j) \to i$ and $p_2 : (i, j) \to j$ for every pair $i, j \in I$

and which sends the objects and morphisms as follows:

1. $\mathcal{J}(i) = U_i$
2. $\mathcal{J}(i, j) = U_i \cap U_j$
3. $\mathcal{J}(p_1) : U_i \cap U_j \hookrightarrow U_i$
4. $\mathcal{J}(p_2) : U_i \cap U_j \hookrightarrow U_j$.

In particular, taking $U = \emptyset$ and taking the empty open cover of the empty set, $\mathcal{F}(\emptyset)$ must be the terminal object (Lemma 2.0.7) of $\mathcal{D}$

In the case that $\mathcal{D}$ admits all small limits (Definition 1.2.5), the sheaf condition is equivalent to the following: For every open set $U \subset X$ and every open cover $\{U_i\}_{i \in I}$ of U, the following equalizer diagram is exact (Definition 1.2.31):

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j).$$

Here, the morphism $\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i)$ and the two morphisms $\prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$ are induced by the restriction maps (Definition 5.1.28) $\mathcal{F}(U) \to \mathcal{F}(U_i)$ and $\mathcal{F}(U_i) \to \mathcal{F}(U_i \cap U_j)$.

In the case that $\mathcal{D}$ is some subcategory (Definition 1.3.1) of the category of sets (Definition 1.0.3), the sheaf condition is equivalent to the following: For every open set $U \subseteq X$ and every open cover (Definition B.0.2) $\{U_i\}_{i \in I}$ of U,

- (Locality) If $s, t \in \mathcal{F}(U)$ are such that $s|_{U_i} = t|_{U_i}$ for all i, then $s = t$.
- (Gluing) If for each i there is $s_i \in \mathcal{F}(U_i)$ such that for all i, j one has $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all i.

Equivalently, a sheaf on a topological space X may be defined as a sheaf on (Definition 5.1.25) the site (Definition 5.1.9) of opens on X (Definition 5.1.23).

Definition 5.1.27 (Morphism of presheaves on a topological space)**.** Let X be a topological space (Definition B.0.1), let $\mathcal{D}$ be a category (Definition 1.0.1) and let $\mathcal{F}$, and $\mathcal{G}$ be presheaves valued in $\mathcal{D}$ on X (Definition 5.1.28).

A *morphism of presheaves* $\varphi : \mathcal{F} \to \mathcal{G}$ is a collection of maps

$$\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$$

in $\mathcal{D}$ defined for every open set $U \subseteq X$, such that the maps are compatible with restriction: for every inclusion $V \subseteq U$ of open sets, the diagram

$$
\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\
\downarrow \rho_V^U & & \downarrow \rho_V^U \\
\mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V)
\end{array}
$$

commutes.

Equivalently, φ is a natural transformation (Definition 1.1.3) of $\mathcal{F}, \mathcal{G}$, regarded as (Definition 5.1.28) functors

$$\mathcal{F}, \mathcal{G} : \mathbf{Open}(X)^{\mathrm{op}} \to \mathcal{D}.$$

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then a *morphism of sheaves* is a morphism between $\mathcal{F}$ and $\mathcal{G}$ as presheaves.

Definition 5.1.28 (Presheaf on a topological space)**.** Let X be a topological space (Definition B.0.1). Let $\mathcal{D}$ be a category.

A *presheaf (of objects of $\mathcal{D}$/valued in $\mathcal{D}$) on X* is a rule $\mathcal{F}$ that assigns:

- to each open set $U \subseteq X$, an object $\mathcal{F}(U) \in \mathrm{Ob}\,\mathcal{D}$, called the *sections of $\mathcal{F}$ over U*,
- to each inclusion of open sets $V \subseteq U$, a morphism

$$\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V), \quad s \mapsto s|_V,$$

in the category $\mathcal{D}$ called the *restriction map* such that the following conditions hold:

- (Identity) For every open set $U \subseteq X$, the restriction map ρ_U^U is the identity on $\mathcal{F}(U)$.
- (Transitivity) For inclusions $W \subseteq V \subseteq U$ of open sets, one has

$$\rho_W^U = \rho_W^V \circ \rho_V^U.$$

For instance, we may speak of a *presheaf of sets/groups/rings/etc. on the topological space X*.

Equivalently, a presheaf on X (of objects in a category $\mathcal{D}$) is a functor (Definition 1.1.1)

$$\mathbf{Open}(X)^{\mathrm{op}} \to \mathcal{D}$$

from the opposite of the category $\mathbf{Open}(X)$ (Definition 1.0.7) of open subsets of X (see also Definition 5.0.1) .

Equivalently, a presheaf on X is a presheaf on the category $\mathbf{Open}(X)$ in the sense of Definition 5.0.1 .

The sections object $\mathcal{F}(U)$ is also denoted by $\Gamma(U, \mathcal{F})$ (see Definition 5.3.11) . Moreover, the object $\mathcal{F}(X) = \Gamma(X, \mathcal{F})$ is called the *global sections object of $\mathcal{F}$*. This agrees with the notion of global sections as discussed in Definition 5.3.11.

Definition 5.1.29 ($\mathcal{P}$-torsion sheaf of abelian groups). (♠ TODO: must the site have a small topological generating family?) Let $(\mathcal{C}, J)$ be a site (Definition 5.1.9) with a small topological generating family (Definition 5.1.9) (or a U-small topologically generating family if a universe (Definition E.0.1) U is available), and let $\mathcal{P}$ be a collection of prime numbers. A sheaf (Definition 5.1.25) of abelian groups $\mathcal{F}$ on $(\mathcal{C}, J)$ is called a *$\mathcal{P}$-torsion sheaf* if for every object U of $\mathcal{C}$, every section $s \in \mathcal{F}(U)$, and every $p \in \mathcal{P}$, there exists some $n \geq 1$ such that $p^n s = 0$ in $\mathcal{F}(U)$.

Definition 5.1.30 (Torsion sheaf of abelian groups). (♠ TODO: must the site have a small topological generating family?) Let $(\mathcal{C}, J)$ be a site (Definition 5.1.9) with a small topological generating family (Definition 5.1.9) (or a U-small topologically generating family if a universe (Definition E.0.1) U is available). A sheaf (Definition 5.4.10) of abelian groups $\mathcal{F}$ on $(\mathcal{C}, J)$ is called a *torsion sheaf of abelian groups* if for every object U in $\mathcal{C}$ and every section $s \in \mathcal{F}(U)$, there exists a nonzero integer n such that $ns = 0$ in $\mathcal{F}(U)$.

5.1.6. *Sheafy functors.*

Definition 5.1.31. Let $\mathcal{A}$, $\mathcal{B}$ be categories (Definition 1.0.1), let $F : \mathcal{A} \to \mathcal{B}$ be a functor (Definition 1.1.1), and let $(\mathcal{D}, K)$ be a site (Definition 5.1.9).

Assume at least one of the following:

1. A sheafification functor (Definition 5.2.6)

$$a : \mathrm{PreSh}(\mathcal{D}, K; \mathcal{B}) \to \mathrm{Sh}(\mathcal{D}, K; \mathcal{B})$$

exists.
2. F is a continuous functor (Definition 1.2.23) and $\mathcal{B}$ admits limits (Theorem 1.2.29) indexed by covering sieves (Definition 5.1.9) of K.

Define the *sheafy functor* $\mathscr{F}$ of F under each of the above assumptions as follows:

1. As the functor

$$\mathscr{F} : \mathrm{PreSh}(\mathcal{D}, \mathcal{A}) \to \mathrm{Sh}(\mathcal{D}, K; \mathcal{B})$$

given by $S \mapsto a(F \circ S)$. By restriction, there is an induced functor

$$\mathrm{Sh}(\mathcal{D}, K; \mathcal{A}) \to \mathrm{Sh}(\mathcal{D}, K; \mathcal{B})$$

on the category of sheaves on $\mathcal{D}$ valued in $\mathcal{A}$; we usually denote this restricted functor by $\mathscr{F}$ as well.
2. As the functor

$$\mathscr{F} : \mathrm{Sh}(\mathcal{D}, K; \mathcal{A}) \to \mathrm{Sh}(\mathcal{D}, K; \mathcal{B})$$

given by $S \mapsto F \circ S$. Due to the assumptions on F and $\mathcal{B}$, $F \circ S$ is a sheaf.

Note that if both assumptions hold, then the two functors

$$\mathrm{Sh}(\mathcal{D}, K; \mathcal{A}) \to \mathrm{Sh}(\mathcal{D}, K; \mathcal{B})$$

are naturally isomorphic.

Sheafy functors are often notated with scripted letters.

Proposition 5.1.32. Let $(\mathcal{D}, K)$ be a site (Definition 5.1.9). Let $\mathcal{A}, \mathcal{B}$ be categories (Definition 1.0.1). There is an equivalence (Definition 1.1.8)

$$\mathrm{Sh}(\mathcal{D}, K; \mathcal{A} \times \mathcal{B}) \cong \mathrm{Sh}(\mathcal{D}, K; \mathcal{A}) \times \mathrm{Sh}(\mathcal{D}, K; \mathcal{B})$$

(Definition 5.1.25) (Definition 1.3.11).

Proof. (♠ TODO:) □

Theorem 5.1.33. Let $(\mathcal{D}, K)$ be a site (Definition 5.1.9). Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be categories (Definition 1.0.1). Let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ and $G : \mathcal{A}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{B}$ be functors such that for any fixed $A \in \mathcal{A}$, there is an adjunction (Definition 1.1.13) $F(A, -) \dashv G(A, -)$. Assume the following:

- there is a sheafification functor (Definition 5.2.6)

$$a : \mathrm{PreSh}(\mathcal{D}, \mathcal{C}) \to \mathrm{Sh}(\mathcal{D}, K; \mathcal{C}),$$

and
- $\mathcal{B}$ admits limits (Definition 1.2.2) indexed by the covering sieves in the Grothendieck topology (Definition 5.1.9) K.

Let $\mathscr{F}$ be the composition
$$\mathrm{Sh}(D, K; \mathcal{A}) \times \mathrm{Sh}(D, K; \mathcal{B}) \cong \mathrm{Sh}(D, K; \mathcal{A} \times \mathcal{B}) \to \mathrm{Sh}(D, K; \mathcal{C})$$
of the equivalence of Proposition 5.1.32 with the sheafy functor (Definition 5.1.31) of F.

For a fixed sheaf $\mathcal{R} \in \mathrm{Sh}(\mathcal{D}, K; \mathcal{A})$, let $\mathscr{G}_{\mathcal{R}} : \mathrm{Sh}(\mathcal{D}, K; \mathcal{C}) \to \mathrm{Sh}(\mathcal{D}, K; \mathcal{B})$ be the functor defined section-wise by
$$\mathcal{T} \mapsto (U \mapsto G(\mathcal{R}(U), \mathcal{T}(U))) \,.$$
Note that since $G(\mathcal{R}(U), -)$ is a right adjoint (Definition 1.1.13), it preserves (Theorem 1.2.24) the limits (Definition 1.2.2) indexed by the covering sieves of K; thus, $\mathscr{G}_{\mathcal{R}}(\mathcal{T})$ is a sheaf whenever $\mathcal{T}$ is a sheaf.

Furthermore, the assignment $(\mathcal{R}, \mathcal{T}) \mapsto \mathscr{G}_{\mathcal{R}}(\mathcal{T})$ defines a functor
$$\mathscr{G} : \mathrm{Sh}(\mathcal{D}, K; \mathcal{A})^{\mathrm{op}} \times \mathrm{Sh}(\mathcal{D}, K; \mathcal{C}) \to \mathrm{Sh}(\mathcal{D}, K; \mathcal{B}).$$

Then for all $\mathcal{R} \in \mathrm{Sh}(\mathcal{D}, K; \mathcal{A})$, there is an adjunction
$$\mathscr{F}(\mathcal{R}, -) \dashv \mathscr{G}_{\mathcal{R}}(-)$$
induced by the pointwise adjunctions $F(\mathcal{R}(U), -) \dashv G(\mathcal{R}(U), -)$ and the universal property of sheafification.

Proof. (♠ TODO:) $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Definition 5.1.34. 1. Let $\mathcal{C}$ be a site (Definition 5.1.9), and let $\mathcal{A}$ and $\mathcal{B}$ be sheaves (Definition 5.1.25) of (not necessarily commutative) rings (Definition C.0.1) on $\mathcal{C}$.

  (a) An $(\mathcal{A}, \mathcal{B})$-*bimodule* (or a *bimodule over $(\mathcal{A}, \mathcal{B})$*) is a sheaf (Definition 5.1.25) $\mathcal{M}$ of abelian groups on $\mathcal{C}$ equipped with a left $\mathcal{A}$-module structure given by a morphism of sheaves (Definition 5.1.25) of sets
  $$\lambda : \mathcal{A} \times \mathcal{M} \longrightarrow \mathcal{M},$$
  and a right $\mathcal{B}$-module structure given by a morphism of sheaves of sets
  $$\rho : \mathcal{M} \times \mathcal{B} \longrightarrow \mathcal{M},$$
  such that the actions are compatible. Specifically, for every object U in $\mathcal{C}$, every section $m \in \mathcal{M}(U)$, every $a \in \mathcal{A}(U)$, and every $b \in \mathcal{B}(U)$, the equality
  $$\lambda_U(a, \rho_U(m, b)) = \rho_U(\lambda_U(a, m), b)$$
  holds in $\mathcal{M}(U)$. In standard multiplicative notation where $\lambda(a, m)$ is denoted $a \cdot m$ and $\rho(m, b)$ is denoted $m \cdot b$, this condition is the associativity axiom
  $$(a \cdot m) \cdot b = a \cdot (m \cdot b).$$
  In particular, for every object $U \in \mathcal{C}$, the abelian group $\mathcal{M}(U)$ has the structure of an $\mathcal{A}(U) - \mathcal{B}(U)$-bimodule (Definition C.0.7).

  (b) Let $\mathcal{M}$ and $\mathcal{N}$ be $(\mathcal{A}, \mathcal{B})$-bimodules. A *homomorphism of $(\mathcal{A}, \mathcal{B})$-bimodules* (or an *$(\mathcal{A}, \mathcal{B})$-linear morphism*) is a morphism of sheaves of abelian groups $f : \mathcal{M} \to \mathcal{N}$ such that for every object U of $\mathcal{C}$, every section $m \in \mathcal{M}(U)$, every $a \in \mathcal{A}(U)$, and every $b \in \mathcal{B}(U)$, the following compatibility conditions hold:
  $$f_U(a \cdot m) = a \cdot f_U(m) \quad \text{and} \quad f_U(m \cdot b) = f_U(m) \cdot b.$$

We denote the category of $(\mathcal{A}, \mathcal{B})$-bimodules, with morphisms being morphisms of sheaves of abelian groups compatible with both the left $\mathcal{A}$-action and the right $\mathcal{B}$-action, by $\mathcal{A}$-$\mathcal{B}$-Mod or sometimes by $_\mathcal{A}$Mod$_\mathcal{B}$ (♠ TODO: talk about how bimodules can be identifies with left/right modules)

2. Let $(\mathcal{C}, J)$ be a site (Definition 5.1.9). Let $\mathcal{O}$ be a sheaf of (not necessarily commutative) rings on $(\mathcal{C}, J)$ (Definition 5.1.25), i.e. $((\mathcal{C}, J), \mathcal{O})$ is a ringed site (Definition 5.4.9).

   (a) An *(left/right/two-sided) $\mathcal{O}$-module* consists of the following data:
      - A sheaf $\mathcal{F}$ of abelian groups on $(\mathcal{C}, J)$,
      - for every object $U \in \mathcal{C}$, the structure of an (left/right/two-sided) $\mathcal{O}(U)$-module on $\mathcal{F}(U)$,

      such that for every morphism $f : V \to U$ in $\mathcal{C}$, the restriction map

      $$\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$$

      is $\mathcal{O}(U)$-linear when the $\mathcal{O}(U)$-action on $\mathcal{F}(V)$ is defined via the natural ring homomorphism

      $$\mathcal{O}(U) \to \mathcal{O}(V)$$

      induced by f.

   (b) Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}$-modules (Definition 5.3.9).
      A *morphism of $\mathcal{O}$-modules* $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves (Definition 5.1.25) of abelian groups such that, for every object $U \in \mathcal{C}$, the component map

      $$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

      is $\mathcal{O}(U)$-linear, i.e. it satisfies

      $$\varphi_U(r \cdot s) = r \cdot \varphi_U(s) \quad \text{for all } r \in \mathcal{O}(U),\ s \in \mathcal{F}(U).$$

      The collection of all $\mathcal{O}$-modules together with their morphisms of $\mathcal{O}$-modules forms the *category of $\mathcal{O}$-modules*, denoted $\mathbf{Mod}(\mathcal{O})$.

   In case that a sheafification functor (Definition 5.2.6)

   $$\mathrm{PreShv}(\mathcal{C}, \mathbf{Rings}) \to \mathrm{Shv}(\mathcal{C}, \mathbf{Rings})$$

   exists, a left, right, two-sided $\mathcal{O}$-module (and morphisms thereof) is equivalent to a $(\mathcal{O}, \mathbb{Z})$-bimodule, $(\mathbb{Z}, \mathcal{O})$-bimodule, and $(\mathcal{O}, \mathcal{O})$-bimodule (and morphisms thereof) respectively, where $\mathbb{Z}$ is the constant sheaf (Definition 5.2.15) of the integer ring $\mathbb{Z}$.

Definition 5.1.35 (Tensor product of bimodules). Let R, S, T be (not necessarily commutative) rings (Definition C.0.1), let M be an R-S bimodule (Definition C.0.7), and let N be an S-T bimodule. In the free abelian group $\mathbb{Z}[M \times N]$ generated by the Cartesian product $M \times N$, let U be the subgroup generated by elements of the form (♠ TODO: subgroup generated)

$$(m + m', n) - (m, n) - (m', n),$$
$$(m, n + n') - (m, n) - (m, n'),$$
$$(m \cdot s, n) - (m, s \cdot n),$$

for all $m, m' \in M$, $n, n' \in N$, and $s \in S$. The *tensor product of M and N over S* is the quotient abelian group

$$M \otimes_S N := \mathbb{Z}[M \times N]/U.$$

The image of an element of the form $(m, n) \in M \times N$ in $M \otimes_S N$ is denoted $\boxed{m \otimes n}$ and called a *pure tensor*. In general, the elements of $M \otimes_S N$ are finite sums

$$\sum_{i=1}^{n} m_i \otimes n_i \quad m_i \in M, n_i \in N$$

of pure tensors. Thus, the pure tensors satisfy the following relations:

$$(m + m') \otimes n = m \otimes n + m' \otimes n$$
$$m \otimes (n + n') = m \otimes n + m \otimes n'$$
$$(m \cdot s) \otimes n = m \otimes (s \cdot n)$$

This tensor product becomes naturally an R-T bimodule with left action and right action defined by

$$r \cdot (m \otimes n) = (r \cdot m) \otimes n,$$
$$(m \otimes n) \cdot t = m \otimes (n \cdot t),$$

for all $r \in R$, $t \in T$, $m \in M$, and $n \in N$.

Inductively, given rings $R_0, \ldots, R_k$ and $R_{i-1} - R_i$-bimodules M_i for $i = 1, \ldots, k$, we may speak of the tensor product

$$M_0 \otimes_{R_1} M_1 \otimes_{R_2} \cdots \otimes_{R_{k-1}} M_k;$$

tensor products are associative(♠ TODO:), so parentheses are not strictly needed to notate them. Its *pure tensors* are elements of the form $m_0 \otimes m_1 \otimes \cdots \otimes m_k$ for $m_i \in M_i$, and its general elements are finite sums

$$\sum_{j=1}^{n} m_{0j} \otimes m_{1j} \otimes \cdots m_{kj} \quad m_{ij} \in M_i.$$

of pure tensors. It also has a natural $R_0 - R_k$-bimodule structure.

Given a ring R and a two-sided R-module M, we may also speak of the *n-fold tensor product* $M^{\otimes n} = M^{\otimes_R n}$

Definition 5.1.36. Let $\mathcal{R}ing$ be the category of (associative, with identity) rings (Definition 1.0.5). Let (R, S) and (R', S') be objects in $\mathcal{R}ing^2$ (Definition 1.3.11). Let M be an (R, S)-bimodule (Definition C.0.7) and M' be an (R', S')-bimodule. A *bimodule morphism over a morphism* $(f, g) \in Hom_{\mathcal{R}ing^2}((R, S), (R', S'))$ is a group homomorphism $\phi : M \to M'$ such that for all $r \in R, s \in S, m \in M$:

$$\phi(r \cdot m \cdot s) = f(r) \cdot \phi(m) \cdot g(s)$$

Definition 5.1.37. Let $\mathcal{R}ing$ be the category of (associative, with identity) rings (Definition 1.0.5). Define the *category of all bimodules over all rings* as the category, denoted by notations such as $\mathcal{B}imod$, as follows:

- Objects: Triples (R, S, M) where (R, S) is an object of $\mathcal{R}ing^2$ (Definition 1.3.11) and M is an (R, S)-bimodule (Definition C.0.7).

- Morphisms: A morphism $(f, g, \phi) : (R, S, M) \to (R', S', M')$ where (f, g) is a morphism in $\mathcal{R}ing^2$ and $\phi : M \to M'$ is a bimodule morphism over (f, g).
- Composition: $(f', g', \phi') \circ (f, g, \phi) = (f' \circ f, g' \circ g, \phi' \circ \phi)$.

Note that there is a canonical projection functor $\mathcal{B}imod \to \mathcal{R}ing^2$ defined by $(R, S, M) \mapsto (R, S)$ and $(f, g, \phi) \mapsto (f, g)$. This functor exhibits $\mathcal{B}imod$ as a fibered category (♠ TODO: define fibered category) over $\mathcal{R}ing^2$. For any morphism $(f, g) : (R, S) \to (R', S')$, the cartesian lift of an object (R', S', M') is given by the (R, S)-bimodule ${}_f M'_g$ where actions are defined via restriction of scalars: $r \cdot m \cdot s = f(r) \cdot m \cdot g(s)$.

Definition 5.1.38. The *category of (tensor) composable bimodules*, denotable by notations such as $\mathcal{B}imod \times_{\mathcal{R}ing} \mathcal{B}imod$, is the (♠ TODO: fiber product in what sense) fiber product of the category $\mathcal{B}imod$ (Definition 5.1.37) of all bimodules over all rings with itself over the category $\mathcal{R}ing$ (Definition 1.0.5) of all (associative, unital) rings with respect to the functors

$$\mathcal{B}imod \to \mathcal{R}ing^2 \xrightarrow{\mathrm{pr}_1} \mathcal{R}ing$$

$$\mathcal{B}imod \to \mathcal{R}ing^2 \xrightarrow{\mathrm{pr}_2} \mathcal{R}ing$$

where pr_1 and pr_2 are the projection functors.

Specifically, it is the category whose objects are pairs of bimodules $((R, S, M), (S, T, N))$ sharing a common middle ring S, and whose morphisms are pairs of morphisms $((f, g, \phi), (g, h, \psi))$ such that the second ring component of the first morphism and the first ring component of the second morphism coincide as $g : S \to S'$.

Definition 5.1.39. Let $\mathcal{B}imod \times_{\mathcal{R}ing} \mathcal{B}imod$ be the category of tensor composable bimodules (Definition 5.1.38). The *tensor product functor* $\otimes : \mathcal{B}imod \times_{\mathcal{R}ing} \mathcal{B}imod \to \mathcal{B}imod$ is defined as follows:

- On objects: $\otimes((R, S, M), (S, T, N)) = (R, T, M \otimes_S N)$ (Definition 5.1.35).
- On morphisms: $\otimes((f, g, \phi), (g, h, \psi)) = (f, h, \phi \otimes \psi)$, where $\phi \otimes \psi$ is the unique group homomorphism satisfying $(\phi \otimes \psi)(m \otimes n) = \phi(m) \otimes \psi(n)$.

Proposition 5.1.40. The tensor product functor is associative up to a natural isomorphism. For any three composable bimodules ${}_R M_S$, ${}_S N_T$, and ${}_T P_U$, there exists a natural isomorphism in $\mathcal{B}imod$:

$$\alpha_{M,N,P} : (M \otimes_S N) \otimes_T P \xrightarrow{\cong} M \otimes_S (N \otimes_T P)$$

defined by the mapping $((m \otimes n) \otimes p) \mapsto (m \otimes (n \otimes p))$. This data, along with the unit bimodules ${}_R R_R$, endows $\mathcal{B}imod$ with the structure of a bicategory (Definition 9.1.2).

Definition 5.1.41. The *category of Hom composable bimodules*, (♠ TODO: there needs to be a discussion like Definition 5.1.38 but for Hom's)

Definition 5.1.42 (Sheafified Tensor Product)**.** Let $(\mathcal{D}, K)$ be a site (Definition 5.1.9) that admits a small topologically generating family (Definition 5.1.9) (e.g. by being essentially small) and let $\mathcal{R}, \mathcal{S}, \mathcal{T}$ be sheaves of rings (Definition 5.1.25) on $(\mathcal{D}, K)$. Let $\mathcal{M}$ be a sheaf (Definition 5.1.25) of $\mathcal{R}$-$\mathcal{S}$ bimodules (Definition 5.3.9) and $\mathcal{N}$ be a sheaf of $\mathcal{S}$-$\mathcal{T}$ bimodules. The *sheafified tensor product*, denoted $\mathcal{M} \otimes_{\mathcal{S}} \mathcal{N}$, is the sheaf of $\mathcal{R}$-$\mathcal{T}$ bimodules defined as

the sheafification (Definition 5.2.6) of the section-wise tensor product presheaf:

$$U \mapsto \mathcal{M}(U) \otimes_{\mathcal{S}(U)} \mathcal{N}(U).$$

(Definition 5.1.35) Explicitly, $\mathcal{M} \otimes_{\mathcal{S}} \mathcal{N} = a(U \mapsto \mathcal{M}(U) \otimes_{\mathcal{S}(U)} \mathcal{N}(U))$, where a is the sheafification functor, which exists by Theorem 5.2.13. (♠ TODO: to apply the theorem, it seems necessary to show that the category of sheaves of $\mathcal{R}$-$\mathcal{T}$ bimodules is complete, cocomplete (probably need sheafification of pointwise construction), has exact filtered colimits, and the IPC property)

In particular, we have a bi-additive functor

$$\otimes : {}_{\mathcal{R}}\mathsf{Mod}_{\mathcal{S}} \times {}_{\mathcal{S}}\mathsf{Mod}_{\mathcal{T}} \to {}_{\mathcal{R}}\mathsf{Mod}_{\mathcal{T}}$$

In the framework of Definition 5.1.31, the sheafified tensor product is the evaluation of the sheafy functor $\mathscr{F}$ induced by the tensor product functor (Definition 5.1.39) $F = \otimes : \mathcal{B}imod \times_{\mathcal{R}ing} \mathcal{B}imod \to \mathcal{B}imod$ (Definition 5.1.37) (Definition 5.1.38); (♠ TODO: must show that the category $\mathcal{B}imod$ is complete, cocomplete, has exact essentially small filtered limits, and satisfies the IPC property to argue that there is a sheafification functor by Theorem 5.2.13) (♠ TODO: show that the "composable" subcategory of sheaves of bimodules over all sheaves of rings on the site is equivalence to the category of sheaves valued in the caetgory of all composable bimodules) Specifically, if $\mathbb{M} := (\mathcal{R}, \mathcal{S}, \mathcal{M}) \in \mathrm{Sh}(\mathcal{D}, K; \mathcal{B}imod)$ and $\mathbb{N} := (\mathcal{S}, \mathcal{T}, \mathcal{N}) \in \mathrm{Sh}(\mathcal{D}, K; \mathcal{B}imod)$, then the sheafified tensor product is the module component of the sheaf $\mathscr{F}(\mathbb{M}, \mathbb{N}) \in \mathrm{Sh}(\mathcal{D}, K; \mathcal{B}imod)$, which is the triple $(\mathcal{R}, \mathcal{T}, \mathcal{M} \otimes_{\mathcal{S}} \mathcal{N})$.

Definition 5.1.43 (Sheafified Hom). Let $(\mathcal{D}, K)$ be a site (Definition 5.1.9) that admits a small topologically generating family (Definition 5.1.9) (e.g. by being essentially small) and let $\mathcal{R}, \mathcal{S}, \mathcal{T}$ be sheaves of rings on $(\mathcal{D}, K)$. Let $\mathcal{M}$ be a sheaf (Definition 5.1.25) of $\mathcal{R}$-$\mathcal{S}$ bimodules (Definition 5.3.9) and $\mathcal{N}$ be a sheaf of $\mathcal{R}$-$\mathcal{T}$ bimodules.

The *sheafified Hom*, denoted by notations such as $\mathcal{H}om_{\mathcal{R}}(\mathcal{M}, \mathcal{N})$, $\mathscr{H}om_{\mathcal{R}}(\mathcal{M}, \mathcal{N})$, or $\underline{\mathrm{Hom}}_{\mathcal{R}}(\mathcal{M}, \mathcal{N})$, is the sheaf of $\mathcal{S}$-$\mathcal{T}$ bimodules defined section-wise by:

$$\mathcal{H}om_{\mathcal{R}}(\mathcal{M}, \mathcal{N})(U) = \mathrm{Hom}_{\mathrm{Sh}(\mathcal{D}/U, K|_U; \mathrm{Mod}(\mathcal{R}|_U))}(\mathcal{M}|_U, \mathcal{N}|_U).$$

(Definition 1.3.4) (Definition 5.1.22) (Definition 5.2.16)

Equivalently, it is the unique sheaf which satisfies the right adjoint property relative to the sheafified tensor product (Theorem 5.1.44).

In the framework of Definition 5.1.31, the sheafified Hom corresponds to the right adjoint $\mathscr{G}$ to the sheafified tensor product $\mathscr{F}$. While the general framework defines sheafy functors via section-wise application, the internal Hom for sheaves of modules requires the restriction-based definition (using $\mathcal{D}/U$) to correctly represent the right adjoint in the category of sheaves, ensuring compatibility with the varying ring structure $\mathcal{R}$.

(♠ TODO: the following would actually need to be modified to have a notion of "Hom-composable" category of functors) Also in the framework of Definition 5.1.31, the sheafified Hom is the evaluation of the sheafy functor $\mathscr{G}$ induced by the algebraic Hom functor $G : \mathcal{A}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{B}$ under assumption (2). Given input sheaves $\mathbb{M} := (\mathcal{R}, \mathcal{S}, \mathcal{M})$ and $\mathbb{N} := (\mathcal{R}, \mathcal{T}, \mathcal{N})$, the result is the sheaf $\mathscr{G}(\mathbb{M}, \mathbb{N}) = (\mathcal{S}, \mathcal{T}, \mathcal{H}om_{\mathcal{R}}(\mathcal{M}, \mathcal{N}))$. Because G is a right

adjoint in the base categories, it is a continuous functor (i.e., it preserves limits); thus, $\mathscr{G}$ can be defined section-wise as in Case (2) of the framework. This section-wise construction $U \mapsto \mathrm{Hom}_{\mathcal{R}(U)}(\mathcal{M}(U), \mathcal{N}(U))$ is naturally isomorphic to the restriction-based internal Hom defined above, as both satisfy the same local-to-global universal property.

Theorem 5.1.44 (Tensor-Hom Adjunction for Sheaves of Bimodules)**.** Let $(\mathcal{D}, K)$ be a site (Definition 5.1.9) that admits a small topologically generating family (Definition 5.1.9) (e.g. by being essentially small). Let $\mathcal{R}, \mathcal{S}, \mathcal{T}$ be sheaves (Definition 5.1.25) of rings on $(\mathcal{D}, K)$. Let $\mathcal{M}$ be a sheaf of $\mathcal{R}$-$\mathcal{S}$ bimodules (Definition 5.3.9), $\mathcal{N}$ be a sheaf of $\mathcal{S}$-$\mathcal{T}$ bimodules, and $\mathcal{P}$ be a sheaf of $\mathcal{R}$-$\mathcal{T}$ bimodules.

Then there is a natural isomorphism:
$$\mathrm{Hom}_{\mathrm{Sh}(\mathcal{R}\text{-}\mathcal{T})}(\mathcal{M} \otimes_{\mathcal{S}} \mathcal{N}, \mathcal{P}) \cong \mathrm{Hom}_{\mathrm{Sh}(\mathcal{S}\text{-}\mathcal{T})}(\mathcal{N}, \mathcal{H}om_{\mathcal{R}}(\mathcal{M}, \mathcal{P}))$$
where $\mathcal{M} \otimes_{\mathcal{S}} \mathcal{N}$ is the sheafified tensor product and $\mathcal{H}om_{\mathcal{R}}(\mathcal{M}, \mathcal{P})$ is the sheaf of local $\mathcal{R}$-linear homomorphisms.

Proof. (♠ TODO: Verify this proof We apply Theorem 5.1.33 by specifying the categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and the functors F, G as follows:

1. **Categories:** Let $\mathcal{A}$ be the category whose objects are triples (R, S, M) where M is an R-S bimodule. Define $\mathcal{B}$ and $\mathcal{C}$ similarly for S-T and R-T bimodules.
2. **Base Functors:** Define $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ by $(M, N) \mapsto M \otimes_S N$ and $G : \mathcal{A}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{B}$ by $(M, P) \mapsto \mathrm{Hom}_R(M, P)$. At each component, $F(A, -) \dashv G(A, -)$ is the standard algebraic tensor-hom adjunction for bimodules.
3. **Sheaf Interpretation:** The sheaves $\mathcal{M}, \mathcal{N}, \mathcal{P}$ are viewed as objects in $\mathrm{Sh}(\mathcal{D}, K; \mathcal{A})$, $\mathrm{Sh}(\mathcal{D}, K; \mathcal{B})$, and $\mathrm{Sh}(\mathcal{D}, K; \mathcal{C})$ respectively, where the ring components of these sheaves are fixed to be the sheaves $\mathcal{R}, \mathcal{S}, \mathcal{T}$.

By Theorem 5.1.33, the sheafy functor $\mathscr{F}$ associated to the tensor product is:
$$\mathscr{F}(\mathcal{M}, \mathcal{N}) = a(U \mapsto \mathcal{M}(U) \otimes_{\mathcal{S}(U)} \mathcal{N}(U))$$
which is precisely the definition of the sheaf of $\mathcal{R}$-$\mathcal{T}$ bimodules $\mathcal{M} \otimes_{\mathcal{S}} \mathcal{N}$.

The right adjoint functor $\mathscr{G}_{\mathcal{M}}$ is defined section-wise by:
$$\mathscr{G}_{\mathcal{M}}(\mathcal{P}) : U \mapsto \mathrm{Hom}_{\mathcal{R}(U)}(\mathcal{M}(U), \mathcal{P}(U))$$
Per the theorem, since G is a right adjoint, $\mathscr{G}_{\mathcal{M}}(\mathcal{P})$ is a sheaf. In the category of modules, this section-wise Hom sheaf is naturally isomorphic to the internal sheaf $\mathcal{H}om_{\mathcal{R}}(\mathcal{M}, \mathcal{P})$ because any morphism of sheaves $\mathcal{M}|_U \to \mathcal{P}|_U$ is uniquely determined by its action on sections when the sheafification of the section-wise Hom agrees with the internal Hom.

Thus, the general adjunction $\mathscr{F}(\mathcal{M}, -) \dashv \mathscr{G}_{\mathcal{M}}(-)$ specializes to the required isomorphism of Hom-sets in the category of sheaves of bimodules.) □

5.1.7. *Topos.*

Definition 5.1.45 (Topos)**.** There are multiple notions of a topos depending on the context (geometric vs. logical).

1. A *Grothendieck topos* (or *sheaf topos*) is a category (Definition 1.0.1) equivalent (Definition 1.1.8) to the category of sheaves (Definition 5.1.25) of sets on a *small* site (Definition 5.1.9). That is, there exists a small site $(\mathcal{C}, J)$ such that the category is equivalent to $\mathrm{Sh}(\mathcal{C}, J)$.
2. Let $\mathscr{U}$ be a universe (Definition E.0.1). A *$\mathscr{U}$-topos* is a category equivalent to the category of sheaves of sets on a $\mathscr{U}$-small site $(\mathcal{C}, J)$, where the sheaves take values in the category of $\mathscr{U}$-sets ($\mathbf{Set}_{\mathscr{U}}$). [GV72, Exposé IV Définition 1.1]
3. An *elementary topos* is a category which has all finite limits (Definition 1.2.2), is cartesian closed (Definition 1.2.17), and has a subobject classifier (Definition 1.2.18).

Remark: Every Grothendieck topos is an elementary topos, but the converse is not true (e.g., the category of finite sets is an elementary topos but not a Grothendieck topos).

Definition 5.1.46. Let $\mathcal{E}$ and $\mathcal{F}$ be topoi (Definition 5.1.45). A *geometric morphism from $\mathcal{F}$ to $\mathcal{E}$*, denoted $f : \mathcal{F} \to \mathcal{E}$, is a pair of adjoint functors (f^*, f_*)

$$f^* : \mathcal{E} \rightleftarrows \mathcal{F} : f_*$$

where:

1. The functor f^* (called the *inverse image functor*) is left adjoint to the functor f_* (called the *direct image functor*).
2. The inverse image functor f^* is left exact, meaning it preserves finite limits (Definition 1.2.2).

A *morphism of topoi* usually refers to a geometric morphism between topoi.

Lemma 5.1.47 (see e.g. [GV72, Exposé IV 5.1]). Let E be a topos (Definition 5.1.45) and let X be an object of E. The slice category $E_{/X}$ (Definition 1.3.4) is itself a topos.

Definition 5.1.48. Let C be a (large) category (Definition 1.0.1). In view of Lemma 5.1.47, the slice category $E_{/X} = E/X$ (Definition 1.3.4) is often called the *slice topos* or *over topos*, etc.

Definition 5.1.49 (Generator of a category). Let $\mathcal{C}$ be a category (Definition 1.0.1).

1. An object $G \in \mathcal{C}$ is called a *generator* (or *separator*) if for every pair of distinct morphisms $f, g : X \to Y$ in $\mathcal{C}$, there exists a morphism $h : G \to X$ such that

$$f \circ h \neq g \circ h.$$

In case that $\mathcal{C}$ is locally small (Definition 1.0.9), this is equivalent to the condition that the representable functor (Definition 4.0.6)

$$\mathrm{Hom}_{\mathcal{C}}(G, -) : \mathcal{C} \to \mathbf{Set}$$

is faithful (Definition 1.1.4), which in turn is equivalent to the condition that for every object $X \in \mathcal{C}$, there exists an epimorphism

$$\bigoplus_{i \in I} G \twoheadrightarrow X$$

for some indexing set I, where $\bigoplus$ denotes the coproduct (Definition 1.2.30) in $\mathcal{C}$.

2. A family $\{G_i\}_{i \in I}$ is called a *generating family* if for every pair of distinct morphisms $f, g : X \to Y$ in $\mathcal{C}$, there exists some index $i \in I$ and a morphism $h : G_i \to X$ such that

$$f \circ h \neq g \circ h.$$

In case $\mathcal{C}$ is locally small, this is equivalent to the condition that the collection of representable functors

$$\{\mathrm{Hom}_{\mathcal{C}}(G_i, -) : \mathcal{C} \to \mathbf{Set}\}_{i \in I}$$

is jointly faithful, which in turn is equivalent to the condition that for every object $X \in \mathcal{C}$, there exists a family of objects $\{G_i\}_{i \in J}$ from the generating set indexed by some set J, and an epimorphism

$$\bigoplus_{i \in J} G_i \twoheadrightarrow X.$$

Proposition 5.1.50. (♠ TODO: find in SGA) Let $\mathcal{E}$ be a topos (Definition 5.1.45) that has all small coproducts (Definition 1.2.30) (which is true for any Grothendieck topos). A family of objects $\{G_i\}_{i \in I}$ is a generating family (Definition 5.1.49) if and only if for every object $X \in \mathcal{E}$, there exists a set J and an epimorphism

$$\coprod_{j \in J} G_{i_j} \twoheadrightarrow X$$

where each $i_j \in I$.

Theorem 5.1.51. (♠ TODO: find in SGA) Let $(\mathcal{C}, J)$ be a small (Definition 1.0.9) site (Definition 5.1.9), and let $\mathcal{E} = \mathrm{Sh}(\mathcal{C}, J)$ be the topos (Definition 5.1.45) of sheaves (Definition 5.1.25) of sets on it. Let $a \circ y : \mathcal{C} \to \mathcal{E}$ be the composite of the Yoneda embedding (Corollary 1.1.11) followed by the sheafification functor (Definition 5.2.6).

The family of sheaves $\{(a \circ y)(U) \mid U \in \mathrm{Ob}(\mathcal{C})\}$ is a generating family (Definition 5.1.49) for the topos $\mathcal{E}$.

(♠ TODO:)

Definition 5.1.52. A Grothendieck topos (Definition 5.1.45) $\mathcal{E}$ is called a *coherent topos* if there exists a coherent site (Definition 5.1.21) $(\mathcal{C}, J)$ such that $\mathcal{E}$ is equivalent to the category of sheaves (Definition 5.1.25) $\mathcal{E} \simeq \mathrm{Sh}(\mathcal{C}, J)$ on that site.

Alternatively, an intrinsic characterization [GSDV72, Exposé VI] is that $\mathcal{E}$ is coherent if there exists a generating family (Definition 5.1.49) $\mathcal{G}$ of objects in $\mathcal{E}$ such that:

1. Every object in $\mathcal{G}$ is coherent (Definition 5.1.20) (quasi-compact (Definition 5.1.17) and quasi-separated (Definition 5.1.19)).
2. The full subcategory generated by $\mathcal{G}$ is closed under fiber products (Definition 1.2.32).

It is a theorem that the full subcategory of *all* coherent objects in a coherent topos is closed under finite limits and finite colimits (forming a so-called *pretopos*).

Definition 5.1.53 ([nLa25a, Definition 1.1]). A *point of a topos E* is a geometric morphism (Definition 5.1.46) $x : \mathbf{Sets} \to E$ (Definition 1.0.3).

For an object $A \in E$, its inverse image $x^*A \in \mathbf{Sets}$ under such a point x is the *stalk of A at x of A at x*.

Definition 5.1.54. Let $\mathcal{E}$ be a topos (Definition 5.1.45), and let $p : \mathbf{Set} \to \mathcal{E}$ be a point of the topos (Definition 5.1.53). By definition, p is a geometric morphism, so it consists of an adjoint pair of functors (p^*, p_*) where the inverse image functor $p^* : \mathcal{E} \to \mathbf{Set}$ is left exact.

1. The inverse image functor $p^* : \mathcal{E} \to \mathbf{Set}$ is called the *stalk functor associated with the point p*.
2. For any object (sheaf) $F \in \mathcal{E}$, the set $p^*(F)$ is called the *stalk of F at the point p*.

Remark 5.1.55. If $\mathcal{E}$ is the topos of sheaves on a topological space X, a classical point $x \in X$ induces a point of the topos $p_x : \mathbf{Set} \to \mathrm{Sh}(X)$. In this case, the stalk functor p_x^* is equivalent to the classical stalk functor at x, which is defined as the colimit of sections over open neighborhoods of x.

$$p_x^*(F) \cong \varinjlim_{x \in U} F(U)$$

Definition 5.1.56. A topos (Definition 5.1.45) $\mathcal{E}$ is said to *have enough points* if the collection of all points of $\mathcal{E}$ is jointly faithful.

Explicitly, this means that for any two distinct morphisms $f, g : X \to Y$ in $\mathcal{E}$, there exists a point $p : \mathbf{Set} \to \mathcal{E}$ such that the induced maps on stalks are distinct:

$$p^*(f) \neq p^*(g).$$

Equivalently, a morphism f in $\mathcal{E}$ is an isomorphism if and only if for every point p, the map $p^*(f)$ is an isomorphism of sets.

Theorem 5.1.57. Let $\mathcal{E}$ be a coherent topos (Definition 5.1.52). Then $\mathcal{E}$ has enough points (Definition 5.1.56).

Theorem 5.1.58 (Deligne's Completeness Theorem for Coherent Topoi). Let $\mathcal{E}$ be a coherent topos (Definition 5.1.52). Then there exists a compact (Definition E.0.17) Hausdorff (Definition E.0.18) topological space X and an exact functor $\mathcal{E} \to \mathrm{Sh}(X)$ that reflects isomorphisms.

5.2. Morphism of sites and topoi.

Definition 5.2.1. Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be sites (Definition 5.1.9).

A functor $u : \mathcal{C} \to \mathcal{D}$ is said to be a *continuous functor of sites* if, for every object $U \in \mathrm{Ob}(\mathcal{D})$ and every covering sieve (Definition 5.1.9) $S \in K(U)$, the pullback sieve u^*S (Definition 5.1.6) belongs to $J(V)$ for all $V \in \mathcal{C}$ with a morphism $u(V) \to U$ in $\mathcal{D}$.

Equivalently, u is continuous if for every sheaf (Definition 5.1.25) of sets F on $\mathcal{D}$, the presheaf (Definition 5.0.1) $\mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}, X \mapsto F(u(X))$ is a sheaf on $\mathcal{C}$. (♠ TODO: show these are equivalent) (♠ TODO: define morphism of sites and recheck ref's to this definition)

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Definition 5.2.2. Let (X, τ_X) and (Y, τ_Y) be topological spaces (Definition B.0.1), and let $f : X \to Y$ be a continuous map (Definition B.0.3). Let $\mathrm{Open}(X)$ and $\mathrm{Open}(Y)$ be their respective categories of open sets (Definition 1.0.7) with inclusion morphisms, equipped with the canonical (Definition 5.1.23) Grothendieck topologies (Definition 5.1.9) given by open coverings.

Define the functor

$$f^{-1} : \mathrm{Open}(Y) \to \mathrm{Open}(X), \quad U \mapsto f^{-1}(U).$$

It is a continuous functor of sites (Definition 5.2.1) from $\mathrm{Open}(Y)$ to $\mathrm{Open}(Y)$ which induces a site morphism (Definition 5.2.4)

$$f : (\mathrm{Open}(X), \mathrm{can}) \to (\mathrm{Open}(Y), \mathrm{can})$$

.

Definition 5.2.3. Let $f : X \to Y$ be a morphism of schemes, and consider one of the common Grothendieck topologies on schemes such as the Zariski, étale, Nisnevich, fppf, fpqc, or crystalline topology. Denote by $\mathbf{C}(X)$ and $\mathbf{C}(Y)$ the corresponding small sites (Definition 5.1.9) of X and Y (i.e., categories of morphisms to X and Y respectively equipped with one of these topologies).

Then the base change functor

$$f^{-1} : \mathbf{C}(Y) \to \mathbf{C}(X), \quad (V \to Y) \mapsto (V \times_Y X \to X)$$

(Definition 1.2.32) is a continuous functor. It in fact induces a morphism of sites (Definition 5.2.4)

$$f : (\mathbf{C}(X), \tau_{\mathbf{C}}) \to (\mathbf{C}(Y), \tau_{\mathbf{C}})$$

where $\tau_{\mathbf{C}}$ denotes the chosen topology (Zariski, étale, Nisnevich, fppf, fpqc, crystalline).

Definition 5.2.4. Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be sites (Definition 5.1.9). A *morphism of sites* $f : (\mathcal{D}, K) \to (\mathcal{C}, J)$ consists of a continuous functor (Definition 5.2.1) $u : \mathcal{C} \to \mathcal{D}$ such that the inverse image (♠ TODO: carefully piece together in what sense this is an inverse image functor) functor $\mathrm{Sh}(\mathcal{D}, K; \mathbf{Sets}) \to \mathrm{Sh}(\mathcal{C}, J; \mathbf{Sets})$ (Definition 5.1.25) is left exact (♠ TODO: left exact), i.e. commutes with finite projective limits (Definition 1.2.12).

Proposition 5.2.5. Let $(\mathcal{C}, J)$ and $(\mathcal{C}, J')$ be two sites (Definition 5.1.9) on the same underlying category $\mathcal{C}$. If the topology J is finer (Definition 5.1.11) than J' on the category $\mathcal{C}$, then the identity functor

$$\mathrm{id}_{\mathcal{C}} : (\mathcal{C}, J) \to (\mathcal{C}, J')$$

is a continuous functor (Definition 5.2.1) between sites and in fact induces a site morphism (Definition 5.2.4).

$$(\mathcal{C}, J') \to (\mathcal{C}, J).$$

5.2.1. *Sheafification.*

Definition 5.2.6. Let $\mathcal{C}$ be a site (Definition 5.1.9) and let $\mathcal{A}$ be a (large) category (Definition 1.0.1).

Assuming that the presheaf (Definition 5.0.1) category $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ (and hence the sheaf (Definition 5.1.25) category $\mathrm{Shv}(\mathcal{C}, \mathcal{A})$) is locally small (Definition 1.0.9) (or U-locally small

if a Grothendieck universe (Definition E.0.1) U is available), a *sheafification functor* refers to a functor
$$a : \mathrm{PreShv}(\mathcal{C}, \mathcal{A}) \to \mathrm{Shv}(\mathcal{C}, \mathcal{A})$$
that is left adjoint (Definition 1.1.13) to the inclusion functor
$$i : \mathrm{Shv}(\mathcal{C}, \mathcal{A}) \hookrightarrow \mathrm{PreShv}(\mathcal{C}, \mathcal{A}).$$

If such a sheafification functor exists, then it is unique up to unique natural isomorphism. Given a presheaf P, the sheafification $a(P)$ is also sometimes called the *sheaf associated to P*. See Theorem 5.2.13 for common conditions under which sheafification exists.

Definition 5.2.7. Let $\mathcal{A}$ be a category (Definition 1.0.1) with a terminal object (Lemma 2.0.7). Let X be a topological space (Definition B.0.1).

1. Let $\mathcal{F}$ be a presheaf (Definition 5.1.28) on X valued in $\mathcal{A}$.
   A *sheafification of $\mathcal{F}$* is a pair $(\mathcal{F}^+, \theta)$, where $\mathcal{F}^+$ is a sheaf on X and $\theta : \mathcal{F} \to \mathcal{F}^+$ is a morphism of presheaves, satisfying the following universal property:
   For any sheaf $\mathcal{G}$ on X and any morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$, there exists a unique morphism of sheaves $\psi : \mathcal{F}^+ \to \mathcal{G}$ such that $\psi \circ \theta = \phi$.

$$
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\theta} & \mathcal{F}^+ \\
& \searrow{\phi} & \downarrow{\psi} \\
& & \mathcal{G}
\end{array}
$$

2. Assuming that the presheaf category $\mathrm{PreShv}(X, \mathcal{A})$ (and hence the sheaf (Definition 5.1.26) category $\mathrm{Shv}(X, \mathcal{A})$) are locally small (Definition 1.0.9), a *sheafification functor* refers to a functor
$$a : \mathrm{PreShv}(X, \mathcal{A}) \to \mathrm{Shv}(X, \mathcal{A})$$
that is left adjoint (Definition 1.1.13) to the inclusion functor
$$i : \mathrm{Shv}(X, \mathcal{A}) \hookrightarrow \mathrm{PreShv}(X, \mathcal{A}).$$

If such a sheafification functor exists, then it is unique up to unique natural isomorphism. Given a presheaf P, the sheafification $a(P)$ is also sometimes called the *sheaf associated to P*. The sheafification functor on $\mathrm{PreShv}(X, \mathcal{A})$ is a special case of the general definition of sheafification functor (Definition 5.2.6), applied to the site (Definition 5.1.9) whose underlying category is $\mathrm{Open}(X)$ (Definition 1.0.7) and whose Grothendieck topology is the canonical one (Definition 5.1.23).

Theorem 5.2.8. Let X be a topological space (Definition B.0.1).

1. Let $\mathcal{F}$ be a presheaf (Definition 5.1.28) of sets (or sets with structure, e.g. groups, rings, abelian groups, R-modules, etc.). The sheafification (Theorem 5.2.13) $(\mathcal{F}^+, \theta)$ exists and is unique up to unique isomorphism. The sheaf $\mathcal{F}^+$ can be constructed as the sheaf of sections of the espace étalé (space of stalks) associated to $\mathcal{F}$. Specifically, for any open $U \subseteq X$, $\mathcal{F}^+(U)$ is the set of functions $s : U \to \coprod_{x \in U} \mathcal{F}_x$ such that:
   (a) For each $x \in U$, $s(x) \in \mathcal{F}_x$.
   (b) For each $x \in U$, there exists an open neighborhood $V \subseteq U$ of x and a section $t \in \mathcal{F}(V)$ such that $s(y) = t_y$ for all $y \in V$ (where t_y denotes the germ of t at y).

2. A sheafification functor (Definition 5.2.7)

$$a : \mathrm{PreShv}(X, \mathcal{A}) \to \mathrm{Shv}(X, \mathcal{A})$$

exists.

Definition 5.2.9 (cf. [**?**, Definition 3.1.10]). Let $\mathcal{A}$ be a category (Definition 1.0.1) that admits small (Definition 1.2.5) products (Definition 1.2.30) and small filtered limits (Definition 1.2.12). For any family $\{I_s\}_{s \in S}$ of small and filtered categories (Definition 1.2.8) and any family $\{\alpha_s : I_s \to \mathcal{A}\}_{s \in S}$ indexed by a small set S, there is a canonical morphism

$$colim \left(\prod_s I_s \overset{k \mapsto \prod_s \alpha_s(\pi_s(k))}{\to} A \right) \to \prod_s colim \alpha_s$$

We say that $\mathcal{A}$ satisfies the *IPC-property* (*inductive-limit-product commutation property*) if any such canonical morphism is an isomorphism.

Proposition 5.2.10 (cf. [**?**, Proposition 3.1.11(i)]). (♠ TODO: think about whether I can be made large) Let $\{\mathcal{A}_i\}_{i \in I}$ be a family of category (Definition 1.0.1) that admits small (Definition 1.2.5) products (Definition 1.2.30) and small filtered limits (Definition 1.2.12) indexed by a set I. If the categories $\mathcal{A}_i$ satisfy the IPC-property (Definition 5.2.9), then so does the product category (Definition 1.3.11) $\prod_{i \in I} \mathcal{A}_i$.

Proposition 5.2.11 ([**?**, Proposition 3.1.11(ii), Corollary 3.1.12, Corollary 3.1.13.]).   1. The category of sets (Definition 1.0.3) satisfies the IPC-property (Definition 5.2.9)
2. Given a small category (Definition 1.0.9) $\mathcal{C}$, the functor category (Definition 1.2.1) $[\mathcal{C}, \mathbf{Sets}]$ satisfies the IPC-property.
3. Let R be a (not necessarily commutative) (Definition 1.0.5) ring. The category of (left) R-modules (Definition C.0.9) satisfies the IPC-property; in particular, the category of abelian groups (Definition 1.0.4) and the category of R-S-bimodules (Definition C.0.7), given rings R and S, satisfy the IPC-property.

Proposition 5.2.12 ([**?**, Proposition 3.1.11(iii)]. (♠ TODO: think about whether local smallness can be removed) Let $\mathcal{A}$ be a locally small (Lemma 5.0.2) category (Definition 1.0.1) that admits small (Definition 1.2.5) products (Definition 1.2.30) and small filtered limits (Definition 1.2.12).

Assume that there exist a set I and a functor $\lambda : \mathcal{A} \to \mathbf{Sets}^I$ (Definition 1.0.3) (Definition 1.2.1) such that

- λ commutes with small (Definition 1.2.5) products (Definition 1.2.30)
- λ commutes with small filtered inductive limits (Definition 1.2.12)
- λ is conservative (♠ TODO: conservative)

Then $\mathcal{A}$ satisfies the IPC-property

The following expresses the idea that sheafification functors exist for small enough sites when the value category is nice enough.

Theorem 5.2.13. cf. [GV72, Exposé II, Théorème 3.4]

1. Let U be a universe. Let $\mathcal{C}$ be a U-site (Definition 5.1.9). A sheafification functor

$$a : \mathrm{PreShv}(\mathcal{C}, U-\mathbf{Sets}) \to \mathrm{Shv}(\mathcal{C}, U-\mathbf{Sets}).$$

   exists.
2. Let $\mathcal{C}$ be a site whose underlying category is locally small (Definition 1.0.9) and which has a topologically generating family (Definition 5.1.9) that is a set (rather than a proper class). A sheafification functor

$$a : \mathrm{PreShv}(\mathcal{C}, \mathbf{Sets}) \to \mathrm{Shv}(\mathcal{C}, \mathbf{Sets})$$

   exists.
   (♠ TODO: It may be the case that I need to assume $\mathcal{A}$ to be locally small; if so, recheck all the statements that use this.)
3. (see e.g. [nLa25b, 3], [?, Theorem 17.4.7]) Let $(\mathcal{C}, J)$ be a site (Definition 5.1.9) on an essentially small category (Definition 1.1.9) $\mathcal{C}$. Suppose that the category $\mathcal{A}$ is complete, cocomplete (Definition 1.3.9), that small filtered colimits (Definition 1.2.12) in $\mathcal{A}$ are exact (Definition 1.2.13), and that $\mathcal{A}$ satisfies the IPC-property (Definition 5.2.9). A sheafification functor (Definition 5.2.6)

$$a : \mathrm{PreShv}(\mathcal{C}, \mathcal{A}) \to \mathrm{Shv}(\mathcal{C}, \mathcal{A})$$

   exists.
   (♠ TODO: state as a fact that the category of groups and k-algebras are complete, cocomplete, with small filtered colimits that are exact) This is true for instance of $\mathcal{A} = \mathbf{Set}$ (Proposition 5.2.11), $\mathbf{Grp}$, $k-\mathbf{Alg}$ for a field k, or $\mathbf{Mod}_R$ (Proposition 5.2.11) for a (not necessarily commutative unital) ring R (Definition C.0.1).

Remark 5.2.14. If the presheaf is valued in nice "algebraic category", e.g. groups, abelian groups, rings, modules over a ring, etc., then the sheafification is again valued in that category. (♠ TODO: Make this more precise.)

Definition 5.2.15 (Constant sheaf on a site). Let $\mathcal{C}$ be a (large) category (Definition 1.0.1), let $\mathcal{A}$ be a (large category), and let A be an object of $\mathcal{A}$.

1. The *constant presheaf on $\mathcal{C}$ with value A* is the presheaf (Definition 5.0.1) P defined by

$$P(U) = A$$

   for every object U of $\mathcal{C}$ such that every morphism $f : V \to U$ in $\mathcal{C}$ induces the identity map $A = P(U) \to P(V) = A$.
2. Let $\mathcal{C}$ be a site (Definition 5.1.9) and assume that a sheafification functor (Definition 5.2.6)

$$a : \mathrm{Shv}(\mathcal{C}, \mathcal{A}) \to \mathrm{PreShv}(\mathcal{C}, \mathcal{A})$$

   exists (e.g. see Theorem 5.2.13) . The *constant sheaf on $\mathcal{C}$ with value A*, or the *constant sheaf on $\mathcal{C}$ associated to A* commonly denoted $\underline{A}$ or sometimes just A by abuse of notation, is the sheaf associated to (Theorem 5.2.13) the constant presheaf P with value A above.
3. Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of (not-necessarily commutative) rings on $\mathcal{C}$. Assume that the global sections ring $\Gamma(\mathcal{O})$ (Definition 5.3.11) exists. A *constant $\mathcal{O}$-module* is an $\mathcal{O}$-module (Definition 5.3.9) $\mathcal{F}$ which is isomorphic as a sheaf to the constant sheaf

on $\mathcal{C}$ with value M where M is a module of the ring $\Gamma(\mathcal{O})$. Note that sheafification functors exist for presheaves/sheaves valued in Ab (Theorem 5.2.13).

In case that $\mathcal{O}$ is the constant sheaf associated to A for some (not-necessarily commutative) ring A, a constant $\mathcal{O}$-module is simply called a *constant A-module*.

Definition 5.2.16. Let $(\mathcal{C}, J)$ be a site (Definition 5.1.9), let $\mathcal{F}$ be a sheaf (Definition 5.1.25) on $(\mathcal{C}, J)$ valued in some category, and let $U \in \text{Ob}(\mathcal{C})$ be some object. The *restriction of $\mathcal{F}$ to U* is the sheaf $\mathcal{F}|_U$ on the induced site $(\mathcal{C}/U, J/U)$ (Definition 5.1.22) defined by

$$\mathcal{F}|_U(f : V \to U) = F(V).$$

Equivalently, the forgetful functor $j_U : \mathcal{C}/U \to \mathcal{C}$ forgetful functor is a continuous functor (Definition 5.2.1) of sites and $\mathcal{F}|_U$ is given by the inverse image $j_U^* \mathcal{F}$ (Definition 5.3.2).

Definition 5.2.17 (Locally constant sheaf on a site)**.** Let $(\mathcal{C}, J)$ be a site (Definition 5.1.9).

1. Let $\mathcal{A}$ be a (large) category, and let A be an object of $\mathcal{A}$. (♠ TODO: If such a sheafification functor exist, does a sheafification functor exist when restricted to an object U?) Assume that a sheafification functor (Definition 5.2.6)

$$a : \text{PreShv}(\mathcal{C}, J, \mathcal{A}) \to \text{Shv}(\mathcal{C}, J, \mathcal{A})$$

(Definition 5.1.25) (Definition 5.0.1) exists (e.g. see Theorem 5.2.13) . Let U be an object of $\mathcal{C}$.

A sheaf $\mathcal{F}$ on C with values in $\mathcal{A}$ is said to be a *locally constant sheaf on U with value A* if there exists a covering sieve (Definition 5.1.9) $\{U_i \to U\}$ of every object U in C such that for each i, the restriction $\mathcal{F}|_{U_i}$ (Definition 5.2.16) is isomorphic to the constant sheaf (Definition 5.2.15) $\underline{A}$ on the slice site $\mathcal{C}_{/U_i}$ (Definition 5.1.22).

If $\mathcal{C}$ has a final object, then we may say that $\mathcal{F}$ is a *locally constant sheaf with value A* if it is a locally constants sheaf on the final object with value A.

2. Let $\mathcal{O}$ be a sheaf of (not-necessarily commutative) rings on $\mathcal{C}$. A *locally constant $\mathcal{O}$-module* is a $\mathcal{O}$-module (Definition 5.3.9) $\mathcal{F}$ such that there exists a covering sieve (Definition 5.1.9) $\{U_i \to U\}$ for every object U in C such that for each i, the restriction (Definition 5.2.16) $\mathcal{F}|_{U_i}$ is isomorphic, as an $\mathcal{O}|_{U_i}$-module, to a constant $\mathcal{O}|_{U_i}$-module (Definition 5.2.15)[1].

We additionally say that $\mathcal{F}$ is

   (a) *locally free of rank r over $\mathcal{O}$* if there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_i}$ is isomorphic as a $\mathcal{O}|_{U_i}$-module to $(\mathcal{O}|_{U_i})^{\oplus r}$ for each i.
   (b) *free of rank r over $\mathcal{O}$* if $\mathcal{F} \cong \mathcal{O}^{\oplus r}$ as $\mathcal{O}$-modules.
   (c) *of finite type* if there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_i}$ generated by finitely many sections over U_i as an $\mathcal{O}|_{U_i}$-module. In other words, there is an epimorphism

$$(\mathcal{O}|_{U_i})^{\oplus n_i} \to \mathcal{F}|_{U_i}$$

   of $\mathcal{O}|_{U_i}$-modules for each i.

---

[1]The global sections rings $\Gamma(\mathcal{O}|_{U_i})$ (Definition 5.3.11) of the sheaves $\mathcal{O}|_{U_i}$ on the slice sites $\mathcal{C}_{/U_i}$ (Definition 5.1.22) exist because each $\mathcal{C}_{/U_i}$ has a final object (Definition 2.0.6)(Lemma 1.3.5), so we may speak of constant $\mathcal{O}|_{U_i}$-modules.

In case that $\mathcal{O}$ is the constant sheaf on $\mathcal{C}$ associated to A (Definition 5.2.15) for some (not-necessarily commutative) ring A, a locally constant $\mathcal{O}$-module is simply called a locally constant A-module.

5.3. Inverse image and direct image of sheaves.

Definition 5.3.1 (Relative Slice Category). Let $\mathcal{C}$, $\mathcal{D}$ be categories (Definition 1.0.1), let $u : \mathcal{D} \to \mathcal{C}$ be a functor (Definition 1.1.1) and let U be an object of $\mathcal{C}$. The relative slice category (also called the relative over-category or category of objects over U relative to u), denoted $(u \downarrow U)$ or u/U, is defined as follows:

- **Objects:** Pairs (V, φ), where V is an object of $\mathcal{D}$ and $\varphi : u(V) \to U$ is a morphism in $\mathcal{C}$.
- **Morphisms:** A morphism from (V, φ) to (V', φ') is a morphism $f : V \to V'$ in $\mathcal{D}$ such that the following triangle in $\mathcal{C}$ commutes:

$$
\begin{array}{ccc}
u(V) & \xrightarrow{\;\;u(f)\;\;} & u(V') \\
& \varphi \searrow \quad \swarrow \varphi' & \\
& U &
\end{array}
$$

That is, $\varphi' \circ u(f) = \varphi$.
- **Composition:** Inherited from $\mathcal{D}$.

Slice categories are special cases of comma categories (Definition 1.3.12).

Definition 5.3.2. Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be sites (Definition 5.1.9) with small topological generating families (Definition 5.1.9) (or U-small topologically generating families if a universe (Definition E.0.1) U is available), and let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor of sites (Definition 5.2.1).

For any sheaf (Definition 5.1.25)

$$\mathcal{F} \in \mathrm{Sh}(\mathcal{D}, K; \mathcal{A}),$$

Define the pushforward/direct image sheaf $u^s \mathcal{F}$ by

$$u^s \mathcal{F} := \mathcal{F} \circ u : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}.$$

Because u is continuous, $u^s \mathcal{F}$ is a sheaf on $(\mathcal{C}, J)$ valued in $\mathcal{A}$. The assignment $\mathcal{F} \mapsto u^s \mathcal{F}$ defines the direct image/pushforward functor

$$u^s : \mathrm{Sh}(\mathcal{D}, K; \mathcal{A}) \to \mathrm{Sh}(\mathcal{C}, J; \mathcal{A}).$$

If u is the functor underlying a site morphism (Definition 5.2.4) $f : (\mathcal{D}, K) \to (\mathcal{C}, J)$, we may alternatively denote $u^s \mathcal{F}$ by $f_* \mathcal{F}$ and call it the direct image/pushforward of $\mathcal{F}$ under f; the assignment $\mathcal{F} \mapsto f_* \mathcal{F}$ is then the direct image/pushforward functor.

$$f_* : \mathrm{Sh}(\mathcal{D}, K; \mathcal{A}) \to \mathrm{Sh}(\mathcal{C}, J; \mathcal{A}).$$

Note that while the continuous functor u and the site morphism f point in opposite directions, the definition $f_* := u^s$ ensures that f_* corresponds to the standard geometric pushforward used in topology and algebraic geometry.

Definition 5.3.3. Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be sites (Definition 5.1.9) with small topological generating families (Definition 5.1.9), and let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor of sites (Definition 5.2.1). Let $\mathcal{A}$ be a (large) category such that the presheaf category (Definition 5.0.1) $\mathrm{PreSh}(\mathcal{D}, K; \mathcal{A})$ has sheafification (Definition 5.2.6).

For any sheaf (Definition 5.1.25)
$$\mathcal{G} \in \mathrm{Sh}(\mathcal{C}, J; \mathcal{A}),$$
the *inverse image/pullback sheaf of $\mathcal{G}$ under u* is defined, assuming that all colimits below exist, as:
$$u_s\mathcal{G} : \mathcal{D}^{\mathrm{op}} \to \mathcal{A}, \quad V \mapsto a\left(\varinjlim_{(V \downarrow u)} \mathcal{G}(U)\right),$$
where a is the sheafification functor of presheaves and the colimit (Definition 1.2.2) is taken over the comma category $(V \downarrow u)$ (Definition 1.3.12) of pairs $(U, V \to u(U))$ with $U \in \mathcal{C}$.

The assignment $\mathcal{G} \mapsto u_s\mathcal{G}$ defines the *inverse image/pullback functor*
$$u_s : \mathrm{Sh}(\mathcal{C}, J; \mathcal{A}) \to \mathrm{Sh}(\mathcal{D}, K; \mathcal{A}).$$

If u is the functor underlying a site morphism (Definition 5.2.4) $f : (\mathcal{D}, K) \to (\mathcal{C}, J)$, we may alternatively denote $u_s\mathcal{G}$ by $f^*\mathcal{G}$ (or sometimes by $f^{-1}\mathcal{G}$) and call it the *inverse image/pullback of $\mathcal{G}$ under f*.

Note that while the continuous functor u and the site morphism f point in opposite directions, the identification $f^* := u_s$ ensures that f^* corresponds to the standard geometric pullback. In the case of topological spaces, this recovers the usual construction involving colimits over open neighborhoods to obtain stalks followed by sheafification.

Theorem 5.3.4. (♠ TODO: It is likely that some more restrictions are needed; e.g. must C and D be small and $\mathcal{A}$ locally small to ensure that we can talk about hom's between sheaves?) Let (C, J) and (D, K) be sites (Definition 5.1.9), and let $u : C \to D$ be a continuous functor (Definition 5.2.1) of sites. Let $\mathcal{A}$ be a (large) category such that the presheaf category (Definition 5.0.1) $\mathrm{PreSh}(\mathcal{D}, K; \mathcal{A})$ has sheafification (Definition 5.2.6).

Then the inverse image functor (Definition 5.3.2)
$$u_s : \mathrm{Sh}(D, K; \mathcal{A}) \to \mathrm{Sh}(C, J; \mathcal{A})$$
(assuming that the inverse images of all sheaves on (D, K) valued in $\mathcal{A}$ exist by virtue of $\mathcal{A}$ admitting enough colimits (Definition 1.2.12)) and the direct image functor (Definition 5.3.3)
$$u^s : \mathrm{Sh}(C, J; \mathcal{A}) \to \mathrm{Sh}(D, K; \mathcal{A})$$
form an adjoint pair (Definition 1.1.13), i.e., for any sheaves $\mathcal{F} \in \mathrm{Sh}(D, K; \mathcal{A})$ and $\mathcal{G} \in \mathrm{Sh}(C, J; \mathcal{A})$, there is a natural isomorphism
$$\mathrm{Hom}_{\mathrm{Sh}(C,J)}(u_s\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{Sh}(D,K)}(\mathcal{F}, u^s\mathcal{G}).$$

Definition 5.3.5 (Pushforward (direct image) of a sheaf)**.** Let $f : X \to Y$ be a continuous map between topological spaces (Definition B.0.1), and let $\mathcal{F}$ be a presheaf on X valued in a category $\mathcal{D}$ (Definition 5.1.28) with a terminal object (Lemma 2.0.7). The *pushforward* or *direct image presheaf* $f_*\mathcal{F}$ on Y is the presheaf valued in $\mathcal{D}$ on Y (Definition 5.0.1) defined as follows: For every open set $V \subseteq Y$, the value of the pushforward is given by

$$f_*\mathcal{F}(V) := \mathcal{F}(f^{-1}(V)).$$

For an inclusion of open sets $V' \subseteq V$ in Y, the restriction morphism

$$\mathrm{res}_{V,V'}^{f_*\mathcal{F}} : f_*\mathcal{F}(V) \to f_*\mathcal{F}(V')$$

is defined as the restriction morphism of $\mathcal{F}$ associated with the inclusion of preimages $f^{-1}(V') \subseteq f^{-1}(V)$ in X:

$$\mathrm{res}_{f^{-1}(V),f^{-1}(V')}^{\mathcal{F}} : \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(f^{-1}(V')).$$

If $\mathcal{F}$ is a sheaf (Definition 5.1.26), then so is $f_*\mathcal{F}$. In this case, it is equivalent to define $f_*\mathcal{F}$ as the direct image (Definition 5.3.2) $(f^{-1})^s\mathcal{F}$ of $\mathcal{F}$ under the continuous functor $f^{-1} : \mathrm{Open}\,Y \to \mathrm{Open}\,X$ (Definition 5.2.2) (Definition 5.2.2) (Definition 5.1.23), which is also equivalent to the direct image (Definition 5.3.2) of $\mathcal{F}$ under the site morphism (Definition 5.2.4) $\mathrm{Open}\,X \to \mathrm{Open}\,Y$ whose underlying continuous functor is f^{-1}.

Definition 5.3.6 (Pullback (inverse image) of a sheaf)**.** Let $f : X \to Y$ be a continuous map between topological spaces (Definition B.0.1). Let $\mathcal{D}$ be a category with a terminal object (Lemma 2.0.7).

1. Let $\mathcal{G}$ be a presheaf on Y valued in a $\mathcal{D}$ (Definition 5.1.28). The *pullback* or *inverse image presheaf* $f^{-1}\mathcal{G}$ on X is defined as the presheaf

$$U \mapsto \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$$

   where U ranges over open subsets of X and the colimit is taken over all open subsets $V \subseteq Y$ containing $f(U)$.
2. If $\mathcal{G}$ is a sheaf (Definition 5.1.26) valued in $\mathcal{D}$, then we can define the *pullback* or *inverse image sheaf* $f^*\mathcal{G}$ on X as the sheaf associated to the presheaf (Definition 5.2.6) $f^{-1}\mathcal{G}$, assuming it exists.

   Assuming that a sheafification functor (Definition 5.2.6) exists, one may equivalently define $f^*\mathcal{G}$ via Definition 5.3.3 — More concretely, $f^*\mathcal{G}$ is the following equivalent constructions:
   - The direct image (Definition 5.3.3) $(f^{-1})_s\mathcal{G}$ of $\mathcal{G}$ under the continuous functor $f^{-1} : \mathrm{Open}\,Y \to \mathrm{Open}\,X, \quad W \mapsto f^{-1}(W)$ (Definition 5.1.23) (Definition 5.2.2).
   - The inverse image (Definition 5.3.3) of $\mathcal{G}$ under the site morphism (Definition 5.2.4) $\mathrm{Open}\,X \to \mathrm{Open}\,Y$ whose underlying continuous functor (Definition 5.2.1) is f^{-1}

Theorem 5.3.7. Let $f : X \to Y$ be a continuous map between topological spaces (Definition B.0.1). Let $\mathcal{A}$ be a locally small category (Definition 1.0.9) such that $\mathcal{A}$ is cocomplete (Definition 1.3.9), i.e. admits small colimits (Definition 1.2.5), and such that $\mathrm{PreShv}(Y, \mathcal{A})$ admits sheafification (Definition 5.2.7).

The inverse image functor (Definition 5.3.6)

$$f^* : \mathrm{Sh}(Y, \mathcal{A}) \to \mathrm{Sh}(X, \mathcal{A})$$

and the direct image functor (Definition 5.3.5)

$$f_* : \mathrm{Sh}(X, \mathcal{A}) \to \mathrm{Sh}(Y, \mathcal{A})$$

form an adjoint pair (Definition 1.1.13) $f^* \dashv f_*$, i.e., for any sheaves $\mathcal{F} \in \mathrm{Sh}(Y, \mathcal{A})$ and $\mathcal{G} \in \mathrm{Sh}(X, \mathcal{A})$, there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Sh}(X,\mathcal{A})}(f^*\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{Sh}(Y,\mathcal{A})}(\mathcal{F}, f_*\mathcal{G}).$$

Definition 5.3.8. (♠ TODO: think about if assumptions on the small site and $\mathcal{A}$ are needed) Let $f : X \to Y$ be a morphism of schemes, and let $\mathbf{C}(X)$ and $\mathbf{C}(Y)$ be small sites associated to X and Y respectively, equipped with any common Grothendieck topologies such as Zariski, étale, Nisnevich, fppf, fpqc, or crystalline. Let $\mathcal{A}$ be a (large) category.

1. Given a sheaf $\mathcal{G} \in \mathrm{Sh}(\mathbf{C}(Y); \mathcal{A})$, the *inverse image* $f^*\mathcal{G}$ along f is defined to be the inverse image (Definition 5.3.3) of $\mathcal{G}$ under the continuous functor (Definition 5.2.1) $f^{-1} : \mathbf{C}(Y) \to \mathbf{C}(X)$ (Definition 5.2.3) on sites. In particular, it is an object of $\mathrm{Sh}(\mathbf{C}(X); \mathcal{A})$ and f^* yields a functor

$$f^* : \mathrm{Sh}(\mathbf{C}(Y); \mathcal{A}) \to \mathrm{Sh}(\mathbf{C}(X); \mathcal{A}),$$

2. Given a sheaf $\mathcal{F} \in \mathrm{Sh}(\mathbf{C}(X); \mathcal{A})$, the *direct image* $f_*\mathcal{F}$ along f is defined to be the direct image (Definition 5.3.2) of $\mathcal{F}$ under the continuous functor (Definition 5.2.1) $f^{-1} : \mathbf{C}(Y) \to \mathbf{C}(X)$ (Definition 5.2.3) on sites. In particular, it is an object of $\mathrm{Sh}(\mathbf{C}(Y); \mathcal{A})$ if it exists. If $f_*\mathcal{F}$ exists for all sheaves $\mathcal{F} \in \mathrm{Sh}(\mathbf{C}(X); \mathcal{A})$, then f_* yields a functor

$$f_* : \mathrm{Sh}(\mathbf{C}(X); \mathcal{A}) \to \mathrm{Sh}(\mathbf{C}(Y); \mathcal{A}).$$

5.3.1. *Sheaf of rings on a site and modules over a sheaf of rings on a site.*

Definition 5.3.9. 1. Let $\mathcal{C}$ be a site (Definition 5.1.9), and let $\mathcal{A}$ and $\mathcal{B}$ be sheaves (Definition 5.1.25) of (not necessarily commutative) rings (Definition C.0.1) on $\mathcal{C}$.

(a) An $(\mathcal{A}, \mathcal{B})$-*bimodule* (or a *bimodule over* $(\mathcal{A}, \mathcal{B})$) is a sheaf (Definition 5.1.25) $\mathcal{M}$ of abelian groups on $\mathcal{C}$ equipped with a left $\mathcal{A}$-module structure given by a morphism of sheaves (Definition 5.1.25) of sets

$$\lambda : \mathcal{A} \times \mathcal{M} \longrightarrow \mathcal{M},$$

and a right $\mathcal{B}$-module structure given by a morphism of sheaves of sets

$$\rho : \mathcal{M} \times \mathcal{B} \longrightarrow \mathcal{M},$$

such that the actions are compatible. Specifically, for every object U in $\mathcal{C}$, every section $m \in \mathcal{M}(U)$, every $a \in \mathcal{A}(U)$, and every $b \in \mathcal{B}(U)$, the equality

$$\lambda_U(a, \rho_U(m, b)) = \rho_U(\lambda_U(a, m), b)$$

holds in $\mathcal{M}(U)$. In standard multiplicative notation where $\lambda(a, m)$ is denoted $a \cdot m$ and $\rho(m, b)$ is denoted $m \cdot b$, this condition is the associativity axiom

$$(a \cdot m) \cdot b = a \cdot (m \cdot b).$$

In particular, for every object $U \in \mathcal{C}$, the abelian group $\mathcal{M}(U)$ has the structure of an $\mathcal{A}(U) - \mathcal{B}(U)$-bimodule (Definition C.0.7).

(b) Let $\mathcal{M}$ and $\mathcal{N}$ be $(\mathcal{A}, \mathcal{B})$-bimodules. A *homomorphism of $(\mathcal{A}, \mathcal{B})$-bimodules* (or an *$(\mathcal{A}, \mathcal{B})$-linear morphism*) is a morphism of sheaves of abelian groups $f : \mathcal{M} \to \mathcal{N}$ such that for every object U of $\mathcal{C}$, every section $m \in \mathcal{M}(U)$, every $a \in \mathcal{A}(U)$, and every $b \in \mathcal{B}(U)$, the following compatibility conditions hold:

$$f_U(a \cdot m) = a \cdot f_U(m) \quad \text{and} \quad f_U(m \cdot b) = f_U(m) \cdot b.$$

We denote the category of $(\mathcal{A}, \mathcal{B})$-bimodules, with morphisms being morphisms of sheaves of abelian groups compatible with both the left $\mathcal{A}$-action and the right $\mathcal{B}$-action, by $\mathcal{A}$-$\mathcal{B}$-Mod or sometimes by $_{\mathcal{A}}\mathsf{Mod}_{\mathcal{B}}$ (♠ TODO: talk about how bimodules can be identifies with left/right modules)

2. Let $(\mathcal{C}, J)$ be a site (Definition 5.1.9). Let $\mathcal{O}$ be a sheaf of (not necessarily commutative) rings on $(\mathcal{C}, J)$ (Definition 5.1.25), i.e. $((\mathcal{C}, J), \mathcal{O})$ is a ringed site (Definition 5.4.9).

(a) An *(left/right/two-sided) $\mathcal{O}$-module* consists of the following data:
- A sheaf $\mathcal{F}$ of abelian groups on $(\mathcal{C}, J)$,
- for every object $U \in \mathcal{C}$, the structure of an (left/right/two-sided) $\mathcal{O}(U)$-module on $\mathcal{F}(U)$,

such that for every morphism $f : V \to U$ in $\mathcal{C}$, the restriction map

$$\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$$

is $\mathcal{O}(U)$-linear when the $\mathcal{O}(U)$-action on $\mathcal{F}(V)$ is defined via the natural ring homomorphism

$$\mathcal{O}(U) \to \mathcal{O}(V)$$

induced by f.

(b) Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}$-modules (Definition 5.3.9).
A *morphism of $\mathcal{O}$-modules* $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves (Definition 5.1.25) of abelian groups such that, for every object $U \in \mathcal{C}$, the component map

$$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

is $\mathcal{O}(U)$-linear, i.e. it satisfies

$$\varphi_U(r \cdot s) = r \cdot \varphi_U(s) \quad \text{for all } r \in \mathcal{O}(U),\ s \in \mathcal{F}(U).$$

The collection of all $\mathcal{O}$-modules together with their morphisms of $\mathcal{O}$-modules forms the *category of $\mathcal{O}$-modules*, denoted $\mathbf{Mod}(\mathcal{O})$.

In case that a sheafification functor (Definition 5.2.6)

$$\mathrm{PreShv}(\mathcal{C}, \mathbf{Rings}) \to \mathrm{Shv}(\mathcal{C}, \mathbf{Rings})$$

exists, a left, right, two-sided $\mathcal{O}$-module (and morphisms thereof) is equivalent to a $(\mathcal{O}, \mathbb{Z})$-bimodule, $(\mathbb{Z}, \mathcal{O})$-bimodule, and $(\mathcal{O}, \mathcal{O})$-bimodule (and morphisms thereof) respectively, where $\mathbb{Z}$ is the constant sheaf (Definition 5.2.15) of the integer ring $\mathbb{Z}$.

Lemma 5.3.10. Let $(\mathcal{C}, J)$ be a site (Definition 5.1.9) on a locally small category $\mathcal{C}$ or be a U-site for some universe U (Definition E.0.1). Let R be a not-necessarily commutative unital ring (Definition C.0.1). Let $\mathcal{O}$ be the constant sheaf (Definition 5.2.15) valued in R, which exists by Theorem 5.2.13; it is also a sheaf of rings on $(\mathcal{C}, J)$ (Definition 5.1.25).

The following categories are equivalent:

- **Mod**($\mathcal{O}$) of $\mathcal{O}$-modules (Definition 5.3.9) is equivalent (Definition 1.1.8)
- Shv($\mathcal{C}, R-\mathbf{mod}$) of sheaves (Definition 5.1.25) on $\mathcal{C}$ valued in $R-\mathbf{mod}$ (Definition 1.0.2).

In particular, the categories of modules over the constant ring $\mathbb{Z}$ on $\mathcal{C}$ is equivalent to the category Shv($\mathcal{C}, \mathbf{Ab}$) of sheaves on $\mathcal{C}$ valued in $\mathbf{Ab}$.

5.3.2. *Sections.*

Definition 5.3.11. Let $\mathcal{C}$ be a (large) category (Definition 1.0.1), and let $\mathcal{D}$ be a (large) category (Definition 1.0.1). Let $\mathcal{F} : \mathcal{C}^{op} \to \mathcal{D}$ be a presheaf valued in $\mathcal{D}$ (Definition 5.0.1).

1. For an object $U \in \mathcal{C}$, the *sections functor evaluated at U* is the functor
$$\Gamma(U, -) : \mathrm{PSh}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$$
   defined by
$$\Gamma(U, \mathcal{F}) := \mathcal{F}(U),$$
   i.e., the value of the presheaf $\mathcal{F}$ at the object U.
2. The *global sections of $\mathcal{F}$* is the object $\Gamma(\mathcal{F})$ of $\mathcal{D}$ defined as the limit (Definition 1.2.2)
$$\Gamma(\mathcal{F}) = \varprojlim_{U \in \mathcal{C}^{\mathrm{op}}} \mathcal{F}(U)$$
   assuming that such a limit exists, where the limit is taken over objects $U \in \mathcal{C}$ and the restriction morphisms $\mathcal{F}(V) \to \mathcal{F}(U)$ in $\mathcal{D}$ for morphisms $U \to V$ in $\mathcal{C}$.
   If a final object (Definition 2.0.6) $* \in \mathcal{C}$ exists, then $\Gamma(\mathcal{F})$ exists and coincides with $\Gamma(*, \mathcal{F}) = \mathcal{F}(*)$. The construction $\Gamma(\mathcal{F})$ is functorial; in particular, if $\Gamma(\mathcal{F})$ exists for all $\mathcal{F}$ in $\mathrm{PSh}(\mathcal{C}, \mathcal{D})$, e.g. if limits of (Definition 1.2.2) diagrams in $\mathcal{D}$ indexed by $\mathcal{C}$ exist, then Γ is a functor
$$\Gamma : \mathrm{PSh}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$$
   called the *global sections functor on* $\mathrm{PSh}(\mathcal{C}, \mathcal{D})$.

Proposition 5.3.12. Let $\mathcal{C}$ be an essentially small category (Definition 1.1.9) and let $\mathcal{A}$ be an abelian category (Definition 3.0.3).

1. Let $U \in \mathrm{Ob}(\mathcal{C})$ be some fixed object. The sections functor (Definition 5.3.11)
$$\Gamma(U, -) : \mathrm{PSh}(\mathcal{C}, \mathcal{A}) \to \mathcal{A}$$
   is left exact (Definition 3.0.6). (♠ TODO: state that presheaves and sheaves valued in an abelian category form abelian categories)
2. Assume that $\Gamma(\mathcal{F})$ exists for all $\mathcal{F}$ in $\mathrm{PSh}(\mathcal{C}, \mathcal{A})$ so that Γ is a functor
$$\Gamma : \mathrm{PSh}(\mathcal{C}, \mathcal{A}) \to \mathcal{A}.$$
   The functor Γ is left exact.

Proof. Recall that $\mathrm{PSh}(\mathcal{C}, \mathcal{D})$ is an abelian category (Proposition 3.0.5) since $\mathcal{C}$ is essentially small. (♠ TODO: Talk about how limits are left exact and) □

5.4. Sheaf cohomology.

Lemma 5.4.1 (cf. [Wei94, Porism 2.2.7]). Let $\mathcal{A}$ be an abelian category (Definition 3.0.3).

1. Let
$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$
be a chain complex (Definition 6.0.1) with P_i projective. For every left resolution (Definition 5.4.3) $Q_\bullet \to N$ of an object N, every map $M \to N$ lifts to a complex map (Definition 6.0.1) $P_\bullet \to Q_\bullet$ unique up to chain homotopy.

2. Let
$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$$
be a (co)chain complex (Definition 6.0.1) with I^i injective. For every right resolution (Definition 5.4.3) $N \to Q^\bullet$ of an object N, every map $N \to M$ lifts to a complex map (Definition 6.0.1) $Q^\bullet \to I^\bullet$ unique up to chain homotopy.

Proof.   1. The map $P_0 \to M \to N$ lifts to a map $P_0 \to Q_0$ because P_0 is projective and $Q_0 \to N$ is an epimorphism. Inductively suppose that there are morphisms $P_i \to Q_i$ for $0 \le i \le n$, where $n \ge 0$ that make

$$
\begin{array}{ccccccccc}
P_n & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
Q_n & \longrightarrow & Q_{n-1} & \longrightarrow & \cdots & \longrightarrow & Q_0 & \longrightarrow & N & \longrightarrow & 0
\end{array}
$$

into a commuting diagram are established. The morphism $Q_n \to Q_{n-1}$ (where we let $Q_{-1} = N$ and $P_{-1} = M$ here in case that $n = 0$) acts as 0 when restricted to $\mathfrak{I} := \mathrm{im}(P_{n+1} \to P_n \to Q_n)$ because the composition
$$P_{n+1} \to P_n \to Q_n \to Q_{n-1}$$
equals the composition
$$P_{n+1} \to P_n \to P_{n-1} \to Q_{n-1}.$$
In other words, $\mathfrak{I}$ is a subobject (Definition 3.1.1) of $\ker(Q_n \to Q_{n-1})$ (Definition 1.3.8), which is isomorphic to $\mathrm{im}(Q_{n+1} \to Q_n)$ by the acyclicity of the sequence of the Q_i's. Therefore, we have a map $P_{n+1} \twoheadrightarrow \mathfrak{I} \hookrightarrow \mathrm{im}(Q_{n+1} \to Q_n)$ along with an epimorphism $Q_{n+1} \twoheadrightarrow \mathrm{im}(Q_{n+1} \to Q_n)$. Since P_{n+1} is projective, the former map lifts to a map $P_{n+1} \to Q_{n+1}$ in a way that is compatible with the latter, i.e. the following commutes:

$$
\begin{array}{ccc}
P_{n+1} & & \\
\vdots & \searrow & \\
\downarrow & & \\
Q_{n+1} & \longrightarrow & \mathrm{im}(Q_{n+1} \to Q_n).
\end{array}
$$

By induction, this shows thta $M \to N$ lifts to a morphism $P_\bullet \to Q_\bullet$ of complexes.

We show that the morphism of complexes is unique up to chain homotopy, i.e. if $f_1, f_2 : P_\bullet \to Q_\bullet$ are two morphisms of complexes, then $h := f_1 - f_2$ is null homotopic. We construct a chain contraction $\{s_n : P_n \to Q_{n+1}\}$ of h by induction on n. If $n < 0$,

then set $s_n = 0$. If $n = 0$, note that the composition $P_0 \xrightarrow{h_0} Q_0 \to N$ equals the composition $P_0 \to M \xrightarrow{0} N$, so $\mathrm{im}(h_0)$ is a subobject of $\ker(Q_0 \to N) \cong \mathrm{im}(Q_1 \to Q_0)$. The projectivitiy of P_0 thus yields a lift $s_0 : P_0 \to Q_1$ such that h_0 equals the composition $P_0 \xrightarrow{s_0} X_1 \xrightarrow{d} Q_0$:

$$
\begin{array}{ccc}
 & & P_0 \\
 & \swarrow^{s_0} & \downarrow{h_0} \\
Q_1 & \xrightarrow{d} & Q_0
\end{array}
$$

Note moreover that $h_0 = ds_0 + s_{-1}d$ because $s_{-1} = 0$. Inductively suppose that we have maps s_i for $i \le n$ such that $h_n = ds_n + s_{n-1}d$ or equivalently that $ds_n = h_n - s_{n-1}d$. Consider the map $h_{n+1} - s_n d : P_{n+1} \to Q_{n+1}$. Compute

$$ d(h_{n+1} - s_n d) = dh_{n+1} - ds_n d = dh_{n+1} - (h_n - s_{n-1}d)d = (dh_{n+1} - h_n d) + s_{n-1}dd = 0 $$

Therefore, $\mathrm{im}(h_{n+1} - s_n d)$ is a subobject of $\ker(Q_{n+1} \to Q_n) \cong \mathrm{im}(Q_{n+2} \to Q_{n+1})$, which is in turn a quotient of Q_{n+2}. Since P_{n+1} is projective, there is a morphism $s_{n+1} : P_{n+1} \to Q_{n+2}$ such that $ds_{n+1} = h_{n+1} - s_n d$.

$$
\begin{array}{ccc}
 & & P_{n+1} \\
 & \swarrow^{s_{n+1}} & \downarrow{h_n - s_{n-1}d = ds_n} \\
Q_{n+2} & \xrightarrow{d} & \mathrm{im}(Q_{n+2} \to Q_{n+1}) \cong \ker(Q_{n+1} \to Q_n)
\end{array}
$$

The s_n thus form a chain contraction as needed.
2. This is simply dual to the previous part.

$\square$

Proposition 5.4.2 (cf.[Wei94, Lemma 2.4.1]). Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor (Definition 3.0.2) between abelian categories (Definition 3.0.3). Let A be an object of $\mathcal{A}$.

1. Suppose that F is right exact (Definition 3.0.6), and suppose that a projective resolution (Definition 5.4.3)

$$ \cdots \to P_2 \to P_1 \to P_0 \to A \to 0 $$

of A exists in $\mathcal{A}$. Let

$$ \cdots \to Q_2 \to Q_1 \to Q_0 \to A \to 0 $$

be any projective resolution of A in $\mathcal{A}$. For all n, there are natural isomorphisms

$$ H_n(F(P_\bullet)) \cong H_n(F(Q_\bullet)). $$

In other words, the left derived objects $L_n F(A)$ (Definition 5.4.8) is well defined.
2. Suppose that F is left exact (Definition 3.0.6), and suppose that a injective resolution (Definition 5.4.3)

$$ 0 \to A \to I^0 \to I^1 \to I^2 \to \cdots $$

of A exists in $\mathcal{A}$. Let

$$ 0 \to A \to Q^0 \to Q^1 \to Q^2 \to \cdots $$

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be any injective resolution of A in $\mathcal{A}$. For all n, there are natural isomorphisms
$$H_n(F(I^\bullet)) \cong H_n(F(Q^\bullet)).$$
In other words, the right derived objects $R_n F(A)$ (Definition 5.4.8) is well defined.

Proof. 1. By Lemma 5.4.1, there is a lift $f : P_\bullet \to Q_\bullet$ of the identity map $A \to A$ unique up to chain homotopy. There are then induced natural maps $H_n(F(f)) : H_n(F(P_\bullet)) \to H_n(F(Q_\bullet))$. There is also a lift $f' : Q_\bullet \to P_\bullet$ of the identity map $A \to A$ unique up to chain homotopy, and this also induces natural maps $H_n(F(f')) : H_n(F(Q_\bullet)) \to H_n(F(P_\bullet))$. The chain maps f and f' are in fact chain homotopy inverses because Lemma 5.4.1 also implies that any lifts $P_\bullet \to P_\bullet$ and $Q_\bullet \to Q_\bullet$ of the identity map $A \to A$ are chain homotopic to the identity chain maps. Therefore, $H_n(F(f))$ and $H_n(F(f'))$ are inverses of each other as morphisms in $\mathcal{B}$. (♠ TODO: prove basic facts about the fucntoriality of homology/cohomology of chain complexes)
2. This is dual to the previous part.

□

Definition 5.4.3. Let $\mathcal{A}$ be an abelian category (Definition 3.0.3) and let $\mathcal{X}$ be a class of objects in $\mathcal{A}$. Let M be an object of $\mathcal{A}$.

1. A *right resolution of M* is a cochain complex (Definition 6.0.1) $I^\bullet$ with $I^i = 0$ for $i < 0$ and a map $M \to I^0$ such that the augmented complex
$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$$
is exact.
2. A *left resolution of M* is a chain complex (Definition 6.0.1) $P_\bullet$ with $P_i = 0$ for $i < 0$ and a map $P_0 \to M$ such that the augmented complex
$$\cdots P_2 \to P_1 \to P_0 \to M \to 0$$
is exact.
3. An *$\mathcal{X}$-left resolution* of an object $M \in \mathcal{A}$ a left resolution (Definition 5.4.3) by objects of X, i.e. an exact complex
$$\cdots \to X_2 \to X_1 \to X_0 \to M \to 0$$
with each $X_i \in \mathcal{X}$.
4. An *$\mathcal{X}$-right resolution* of an object $M \in \mathcal{A}$ a right resolution (Definition 5.4.3) by objects of X, i.e. an exact complex
$$0 \to M \to X^0 \to X^1 \to X^2 \to \cdots$$
with each $X_i \in \mathcal{X}$.
5. A *projective resolution of M* is a left resolution $P^\bullet$ for which the objects P^i are all projective.
6. An *injective resolution of M* is a right resolution $I^\bullet$ for which the objects I^i are all injective.

Definition 5.4.4. Let $\mathcal{A}$ be an abelian category (Definition 3.0.3).

1. $\mathcal{A}$ is said to *have enough injectives* if for every object A in $\mathcal{A}$, there is an monomorphism (Definition 1.3.7) $A \to I$ with I an injective object of $\mathcal{A}$.
2. $\mathcal{A}$ is said to *have enough projectives* if for every object A in $\mathcal{A}$, there is a epimorphism (Definition 1.3.7) $P \to A$ with P a projective object of $\mathcal{A}$.

Theorem 5.4.5 (e.g. see [Sta25, Tag 01DU])**.** For any site (Definition 5.1.9) $(\mathcal{C}, J)$ on an essentially small category (Definition 1.1.9) $\mathcal{C}$ and a sheaf of rings (Definition 5.1.25) $\mathcal{O}$ on $\mathcal{C}$, the category $\mathbf{Mod}(\mathcal{O})$ of $\mathcal{O}$-modules (Definition 5.3.9) is an abelian category that has enough injectives (Definition 5.4.4). In fact, there is a functorial injective embedding (♠ TODO: what does this mean?)

(♠ TODO: Apparently, if an abelian category satisfies AB5 and admits a generator (whatever that means), then it has enough injectives. Find a reference for this. Moreover, apparently, the category of sheaves of abelian groups on a general U-site satisfies these criteria and hence has enough injectives.)

(♠ TODO: According to Weibel exercise 2.3.7, if $\mathcal{A}$ is complete and has enough injectives and I is (essentially) small, then the functor category $\mathcal{A}^I$ has enough injectives)

Theorem 5.4.6.
1. Examples of abelian categories with enough injectives include:
   - The category of abelian groups.
   - The category of modules over a ring.
   - The category of sheaves of abelian groups on a ringed space or on a site.
2. Examples of abelian categories with enough projectives include:
   - The category of modules over a ring with enough projectives (e.g., rings with unity and suitable properties). (♠ TODO: make this more precise)
   - The category of finitely generated modules over a semisimple ring.

Lemma 5.4.7 (cf. [Wei94, Lemma 2.2.5, Lemma 2.3.6])**.** Let $\mathcal{A}$ be an abelian category (Definition 3.0.3) and let $\mathcal{X}$ be a class of objects in $\mathcal{A}$.

1. If $\mathcal{A}$ has enough objects of class $\mathcal{X}$ on the right, then for every object $A \in \mathcal{A}$ there exists an $\mathcal{X}$-right resolution of A (Definition 5.4.3).
2. If $\mathcal{A}$ has enough objects of class $\mathcal{X}$ on the left, then for every object $A \in \mathcal{A}$ there exists an $\mathcal{X}$-left resolution of A (Definition 5.4.3).

Note that this is a special case of **??** obtained by letting the complex $M^\bullet$ be the complex such that

$$M^i = \begin{cases} A & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}.$$

In particular,

- If $\mathcal{A}$ has enough injective objects (Definition 5.4.4), then for every object $A \in \mathcal{A}$ there exists an injective resolution of A (Definition 5.4.3).
- If $\mathcal{A}$ has enough projective objects (Definition 5.4.4), then for every object $A \in \mathcal{A}$ there exists a projective resolution of A (Definition 5.4.3).

- If $F : \mathcal{A} \to \mathcal{B}$ is a left (resp. right) exact functor (Definition 3.0.6) between abelian categories and $\mathcal{A}$ has enough F-acyclic objects on the right (resp. left), then for every object $A \in \mathcal{A}$, there exists an right (resp. left) F-acyclic resolution of A.

Proof.   1. Let $A \in \mathcal{A}$ be an object. Since $\mathcal{A}$ has enough objects of class $\mathcal{X}$ of the right, there is an object X_0 of $\mathcal{X}$ and a monomorphism $\varepsilon_0 : A \to X_0$. Let $A_0 = \operatorname{coker} \varepsilon_0$ (Definition 1.3.8). Inductively, given an object A_{n-1} of $\mathcal{A}$, choose an object X_n of $\mathcal{X}$ and a monomorphism $\varepsilon_n : A_{n-1} \hookrightarrow X_n$. Let $A_n = \operatorname{coker} \varepsilon_n$. In particular, there is a surjection $X_n \twoheadrightarrow A_n$. Let d_n be the composition

$$X_{n-1} \twoheadrightarrow A_{n-1} \xrightarrow{\varepsilon_n} X_n.$$

The chain complex

$$0 \to A \xrightarrow{\varepsilon_0} X_0 \xrightarrow{d_0} X_1 \xrightarrow{d_1} \cdots$$

is thus an $\mathcal{X}$-right resolution of A.
   2. This is simply the dual statement of the next statement.

$\square$

Definition 5.4.8. (♠ TODO: I think that the definition of derived categories might be doable for more general kinds of resolutions? Perhaps it is that if I have a right exact functor F, then $L^i F$ can be computed with resolutions of F-acyclic objects?) (♠ TODO: Apparently, left/right derived functors may be defined for functors that are additive and preserve finite coproducts, and not necessarily right/left exact; the exactness condition ensures that the zeroth derived functor agrees with F.) Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories (Definition 3.0.3), and let

$$F : \mathcal{A} \to \mathcal{B}$$

be an additive functor (Definition 3.0.2).

1. Suppose that the functor F is right exact (Definition 3.0.6) and suppose that $A \in \mathcal{A}$ is an object for which a projective resolution (Definition 5.4.3)

$$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0$$

exists in $\mathcal{A}$. We define the *left derived object* $L_n F A \in \mathcal{B}$ by applying F to obtain a complex

$$\cdots \to F(P_2) \to F(P_1) \to F(P_0) \to 0$$

and letting $L_n F(A)$ be the n-th homology object of this complex in $\mathcal{B}$:

$$L_n F(A) := H_n(F(P_\bullet)).$$

The object $L_n F(A)$ is independent of the choice of projective resolution up to natural isomorphism (Proposition 5.4.2).
   By convention, set $L_n F = 0$ for $n < 0$.
   The *higher left derived objects* refer to the object $L_n F(A)$ for $n > 0$.
2. Suppose that the functor F is right exact (Definition 3.0.6) and that $\mathcal{A}$ has enough projectives (Definition 5.4.4). The *left derived functors* refer to the family of functors

$$L_n F : \mathcal{A} \to \mathcal{B}, \quad A \mapsto L_n F(A).$$

The *higher left derived functors* refer to the functors $L_n F$ for $n > 0$.

3. Suppose that the functor F is right exact (Definition 3.0.6) and suppose that $A \in \mathcal{A}$ is an object for which a injective resolution (Definition 5.4.3)

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$$

exists in $\mathcal{A}$. We define the *right derived object* $R_n F A \in \mathcal{B}$, also often denoted by $R^n F A$, by applying F to obtain a complex

$$0 \to F(I^0) \to F(I^1) \to F(I^2) \to \cdots .$$

and letting $R_n F(A)$ be the n-th cohomology object of this complex in $\mathcal{B}$:

$$R_n F(A) := H^n(F(I_\bullet)).$$

The object $R_n F(A)$ is independent of the choice of injective resolution up to natural isomorphism (Proposition 5.4.2).

By convention, set $R_n F = 0$ for $n < 0$.

The *higher right derived objects* refer to the object $R_n F(A)$ for $n > 0$.

4. Suppose that the functor F is right exact (Definition 3.0.6) and that $\mathcal{A}$ has enough injectives (Definition 5.4.4). The *right derived functors* refer to the family of functors

$$R_n F : \mathcal{A} \to \mathcal{B}, \quad A \mapsto R_n F(A).$$

The right derived functors are also often denoted by $R^n F$. The *higher right derived functors* refer to the functors $R_n F$ for $n > 0$.

Definition 5.4.9. (♠ TODO: there are places where sites and sheaves of rings on them are used, but it would be better to just have them be ringed sites.)

A *ringed site* is a site (Definition 5.1.9) $(\mathcal{C}, J)$ with a small topological generating family (Definition 5.1.9) equipped with a sheaf (Definition 5.1.25) of (not necessarily commutative) rings $\mathcal{O}$. If the Grothendieck topology J is clear in context, one may even write that $(C, \mathcal{O})$ is a ringed site.

A *morphism of ringed sites*

$$((\mathcal{C}, J), \mathcal{O}) \to ((\mathcal{C}', J'), \mathcal{O}')$$

consists of a morphism of sites (Definition 5.2.4) $f : (\mathcal{C}, J) \to (\mathcal{C}', J')$ and a morphism of sheaves (Definition 5.1.25) of rings $f^\# : \mathcal{O}' \to f_* \mathcal{O}$ (Definition 5.3.3).

Definition 5.4.10. Let $(\mathcal{C}, J)$ be a site (Definition 5.1.9) on a locally small category (Definition 1.0.9) or a U-site for some universe (Definition E.0.1) U. Let $\mathcal{O}$ be a sheaf of rings (Definition 5.1.25) on $\mathcal{C}$, so that $(\mathcal{C}, J, \mathcal{O})$ is a ringed site (Definition 5.4.9). Recall that the category $\mathbf{Mod}(\mathcal{O})$ (Definition 5.3.9) of $\mathcal{O}$-modules is abelian and has enough injectives (Definition 5.4.4) (Theorem 5.4.5).

Assume that global sections objects $\Gamma(\mathcal{G})$ (Definition 5.3.11) exist for all objects $\mathcal{G}$ of $\mathrm{Sh}(\mathcal{C}, \mathbf{Ab})^2$ so that Γ is a functor

$$\mathrm{Sh}(\mathcal{C}, \mathbf{Ab}) \to \mathbf{Ab},$$

which is a left exact functor (Definition 3.0.6) (Proposition 5.3.12). Note that Γ restricts to a left exact functor

$$\mathbf{Mod}(\mathcal{O}) \to \mathbf{Ab}.$$

---

[2]for example, this occurs when $\mathcal{C}$ is essentially small (Definition 1.1.9)

If $\mathcal{C}$ has a final object (Definition 2.0.6) $*$ as well, then recall that $\Gamma(\mathcal{F}) = \mathcal{F}(*)$.

Let $\mathcal{F}$ be an object of $\mathbf{Mod}(\mathcal{O})$.

1. For each integer $n \geq 0$, the *n-th (abelian) (global) sheaf cohomology group of $\mathcal{F}$* is

$$H^n(\mathcal{C}, J; \mathcal{F}) := R^n\Gamma(\mathcal{F}),$$

   where $R^n\Gamma$ is the n-th right derived functor (Definition 5.4.8) of the global sections functor Γ (Definition 5.3.11).
   In particular, each H^n is a functor

$$H^n : \mathbf{Mod}(\mathcal{O}) \to \mathbf{Ab}.$$

2. Given an object $U \in \mathcal{C}$ and for each integer $n \geq 0$, the *n-th (abelian) sheaf cohomology group of $\mathcal{F}$ of sections at U* is

$$H^n(U, \mathcal{F}) := (R^n\Gamma(U, -))(\mathcal{F}),$$

   where $R^n\Gamma(U, -)$ is the n-th right derived functor (Definition 5.4.8) of the sections functor $\Gamma(U, -)$ evaluated at U (Definition 5.3.11).
   In particular, $H^n(U, \mathcal{F})$ can be regarded as the nth global sheaf cohomology group of the restriction $\mathcal{F}|_U$ (Definition 5.2.16) of $\mathcal{F}$ to U.

5.4.1. Čech cohomology of a sheaf of groups.

Definition 5.4.11 (Čech Cohomology of a Sheaf of Groups)**.** Let $(\mathcal{C}, J)$ be a site (Definition 5.1.9), and $\mathcal{G}$ a sheaf (Definition 5.1.25) of groups (not necessarily abelian) on it. (♠ TODO: define cech cohomology of the covering and cech cohomology)

1. For an object $U \in \mathcal{C}$, and a covering family $\mathfrak{U} = \{U_i \to U\}_{i \in I} \in J(U)$, the *Čech complex* $C^\bullet(\mathfrak{U}, \mathcal{G})$ is defined by:

$$C^n(\mathfrak{U}, \mathcal{G}) := \prod_{(i_0, \ldots, i_n) \in I^{n+1}} \mathcal{G}(U_{i_0} \times_U \cdots \times_U U_{i_n}),$$

   with face maps defined by restriction and group operations giving a cosimplicial object.

5.4.2. Sheaf cohomology of a sheaf of groups.

Definition 5.4.12 (Torsors under a Sheaf of Groups)**.** (♠ TODO: group object, action by a group object) Let $(\mathcal{C}, J)$ be a site (Definition 5.1.9) and $\mathcal{G}$ a sheaf of groups on it (Definition 5.1.25). A *$\mathcal{G}$-torsor* (or synonymously a *principal $\mathcal{G}$-bundle* or *principal $\mathcal{G}$-homogeneous space*, etc.) is a sheaf $\mathcal{P}$ of sets on $(\mathcal{C}, J)$ equipped with a (right) action of $\mathcal{G}U$ such that (♠ TODO: This needs some genuine fixing to be definable on a general site, not just one given by a pretopology) (Definition 5.1.22)

1. For each object U of $\mathcal{C}$, the action of the group $G(U)$ on $\mathcal{P}(U)$ is free and transitive whenever $\mathcal{P}(U)$ is nonempty.

2. $\mathcal{P}$ is locally trivial. More explicitly, for each object U of $\mathcal{C}$, there exists some covering $\{V_i \to U\}_{i \in I}$ in J such that there is an isomorphism

$$\mathcal{P}|_{U_i} \mathcal{G}|_{U_i}$$

as a sheaves with $\mathcal{G}|_{U_i}$ actions for each i.

A $\mathcal{G}$-torsor is called *trivial on U* if it is isomorphic (as a sheaf with a right action of $\mathcal{G}$) to $\mathcal{G}|_U$ acting on itself by right multiplication, or equivalently, if $\mathcal{G}(U) \neq \emptyset$. In particular, any $\mathcal{G}$-torsor is *locally* trivial.

Definition 5.4.13 (Nonabelian Sheaf Cohomology: H^0 and H^1)**.** Let $(\mathcal{C}, J)$ be a site (Definition 5.1.9) and $\mathcal{G}$ a sheaf (Definition 5.1.25) of groups on $(\mathcal{C}, J)$. For $U \in \mathcal{C}$, we define the *0th and 1st (nonabelian) sheaf cohomology sets* as follows:

- The zeroth sheaf cohomology group $H^0(\mathcal{C}, J, \mathcal{G}) := \Gamma(\mathcal{G})$ is the group of global sections (Definition 5.3.11) of $\mathcal{G}$, assuming that it exists (which is always the case when $\mathcal{C}$ is essentially small (Definition 1.1.9) for example since the category of groups is closed under small projective limits (♠ TODO:)).
- The first sheaf cohomology set $H^1(\mathcal{C}, J, \mathcal{G})$ is the pointed set of isomorphism classes of $\mathcal{G}$-torsors (Definition 5.4.12) on the site $(\mathcal{C}, J)$.

Proposition 5.4.14. See [Sta25, Tag 03AJ] for a statement[3] Let $(\mathcal{C}, J)$ be a site (Definition 5.1.9). Assume that global sections objects $\Gamma(\mathcal{G})$ (Definition 5.3.11) exist for all objects $\mathcal{G}$ of $\mathrm{Sh}(\mathcal{C}, \mathbf{Ab})$[4] so that Γ is a functor

$$\mathrm{Sh}(\mathcal{C}, \mathbf{Ab}) \to \mathbf{Ab}.$$

Let $\mathcal{G}$ be a sheaf (Definition 5.1.25) of abelian groups on $(\mathcal{C}, J)$. For $i = 0, 1$, there is a canonical bijection between the set of isomorphism classes of $\mathcal{G}$-torsors (Definition 5.4.12) and the (abelian) sheaf cohomology group $H^i(\mathcal{C}, J; \mathcal{G})$ (Definition 5.4.10). In other words, for $i = 0, 1$, there is a canonical bijection between the ith nonabelian sheaf cohomology (Definition 5.4.13) and the ith abelian sheaf cohomology (Definition 5.4.10) of $\mathcal{G}$.

Proof. In the case of $i = 0$, both are identifiable with the groups of global sections. In the case of $i = 1$, see [Sta25, Tag 03AJ]. $\square$

6. Homological Algebra

Definition 6.0.1 (Chain complex in a preadditive category)**.** Let $\mathcal{A}$ be a preadditive category and let I be a totally ordered set (typically $\mathbb{Z}$, but $I \subseteq \mathbb{Z}$ is also allowed).

1. A *chain complex* $(K_\bullet, d_\bullet)$ in $\mathcal{A}$ indexed by I is the homological convention for sequences with decreasing degrees. It consists of:
    - Objects $\{K_i\}_{i \in I}$ in $\mathcal{A}$, called the *terms in degree i*,

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[3]While the Stacks Project defines a site as a category eqiupped with a Grothendieck *pre*topology, its proof of this statement should be applicable for more general sites

[4]for example, this occurs when $\mathcal{C}$ is essentially small (Definition 1.1.9)

- Morphisms $d_i : K_i \to K_{i-1}$ in $\mathcal{A}$, called the boundary maps or differentials in degree i,

such that for every $i \in I$, $d_{i-1} \circ d_i = 0$. That is,

$$K_\bullet : \cdots \xrightarrow{d_{i+1}} K_i \xrightarrow{d_i} K_{i-1} \xrightarrow{d_{i-1}} K_{i-2} \to \cdots$$

with $d_{i-1} d_i = 0$ for all i. We typically use the notation $K_\bullet = (K_i, d_i)_{i \in I}$.

2. Dually, a cochain complex $(K^\bullet, d^\bullet)$ in $\mathcal{A}$ follows the cohomological convention with increasing degrees. It consists of objects $\{K^i\}_{i \in I}$ and coboundary maps $d^i : K^i \to K^{i+1}$ such that $d^{i+1} \circ d^i = 0$:

$$K^\bullet : \cdots \xrightarrow{d^{i-1}} K^i \xrightarrow{d^i} K^{i+1} \xrightarrow{d^{i+1}} K^{i+2} \to \cdots$$

We typically use the notation $K^\bullet = (K^i, d^i)_{i \in I}$.

3. Let $K_\bullet = (K_i, d_i^K)$ and $L_\bullet = (L_i, d_i^L)$ be chain complexes (Definition 6.0.1) in $\mathcal{A}$ indexed by the same set I. A morphism of chain complexes (or chain map)

$$f_\bullet : K_\bullet \to L_\bullet$$

consists of morphisms $f_i : K_i \to L_i$ for all $i \in I$, such that for every $i \in I$, the following diagram commutes:

$$\begin{array}{ccc} K_i & \xrightarrow{d_i^K} & K_{i-1} \\ \downarrow f_i & & \downarrow f_{i-1} \\ L_i & \xrightarrow{d_i^L} & L_{i-1} \end{array}$$

i.e., $d_i^L \circ f_i = f_{i-1} \circ d_i^K$.

A morphism of cochain complexes $f^\bullet : K^\bullet \to L^\bullet$ is defined similarly, satisfying the commutativity condition $d_L^i \circ f^i = f^{i+1} \circ d_K^i$.

The collection of these objects and morphisms forms a category. Notation for these categories is as follows:

- $\mathrm{Ch}(\mathcal{A})$ or $\mathbf{Ch}(\mathcal{A})$ is often used as a general term.
- To be explicit about the indexing convention, one uses $\mathrm{Ch}_\bullet(\mathcal{A})$ for chain complexes and $\mathrm{Ch}^\bullet(\mathcal{A})$ (or sometimes $\mathrm{CoCh}(\mathcal{A})$) for cochain complexes.
- The set of chain maps between two complexes is denoted by $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(K_\bullet, L_\bullet)$; it is an abelian group under pointwise addition $(f + g)_i = f_i + g_i$.

If k is a commutative ring (Definition C.0.3) such that $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is enriched in (Definition 4.0.1) the category of k-modules (Definition C.0.7), then $\mathrm{Ch}(\mathcal{A})$ can be equipped with (Definition 9.4.3) the structure of a dg-category over k (Definition 9.4.2).

Definition 6.0.2. A quiver is a quadruple $Q = (Q_0, Q_1, s, t)$, where:

- Q_0 is a collection of vertices.
- Q_1 is a collection of arrows.
- $s, t : Q_1 \to Q_0$ are functions assigning to each arrow $\alpha \in Q_1$ its source $s(\alpha)$ and its target $t(\alpha)$.

Definition 6.0.3. Let Q be a quiver (Definition 6.0.2). The *path category generated by Q*, denoted $\mathcal{F}(Q)$, is the category (Definition 1.0.1) defined as follows:

- The objects of $\mathcal{F}(Q)$ are the vertices Q_0.
- For any two objects $x, y \in Q_0$, the set of morphisms $\mathrm{Hom}_{\mathcal{F}(Q)}(x, y)$ consists of all paths from x to y — A *path of length $n \geq 1$ from x to y* is a sequence of arrows $\alpha_n \ldots \alpha_1$ such that $s(\alpha_1) = x$, $t(\alpha_n) = y$, and $s(\alpha_{i+1}) = t(\alpha_i)$ for all $1 \leq i < n$. Additionally, for each vertex x, there is a path e_x of length 0, which serves as the identity morphism.
- Composition of morphisms is defined by the concatenation of paths.

Definition 6.0.4. Let Q be a quiver (Definition 6.0.2) whose collection of arrows is small.

The *preadditive category generated by Q*, denoted $\mathbb{Z}Q$, is the preadditive category (Definition 3.0.1), i.e. the category enriched over (Definition 4.0.1) the category of abelian groups (Definition 1.0.4) defined as follows:

- The objects of $\mathbb{Z}Q$ are the vertices Q_0.
- For any objects $x, y \in Q_0$, the morphism set $\mathrm{Hom}_{\mathbb{Z}Q}(x, y)$ is the free abelian group generated by the set of all paths from x to y in Q.
- Composition is the unique bilinear extension of the path concatenation in $\mathcal{F}(Q)$. That is, for paths u, v, w where concatenation is defined, composition satisfies $(u + v) \circ w = u \circ w + v \circ w$ and $w \circ (u + v) = w \circ u + w \circ v$.

Definition 6.0.5. Let Q_{chain} be the quiver (Definition 6.0.2) with vertex set $Q_0 = \mathbb{Z}$ and arrow set $Q_1 = \{d_n : n \to n-1 \mid n \in \mathbb{Z}\}$. (♠ TODO: quotient of a category) The *walking chain complex category*, denoted $\mathcal{I}$, is the quotient of the preadditive category (Definition 3.0.1) $\mathbb{Z}Q_{\mathrm{chain}}$ by the ideal generated by the relations $d_{n-1} \circ d_n = 0$ for all $n \in \mathbb{Z}$. Explicitly:

- Objects are the integers $\mathbb{Z}$.
- Morphisms are $\mathbb{Z}$-linear combinations of paths, subject to the relation that any path containing a subsegment $d_{n-1}d_n$ is identified with the zero morphism.

Definition 6.0.6. Let $\mathcal{A}$ and $\mathcal{B}$ be preadditive categories (Definition 3.0.1) (categories enriched over (Definition 4.0.1) the category of abelian groups (Definition 1.0.4)). The *additive functor category* $\mathrm{Add}(\mathcal{A}, \mathcal{B})$ is the functor category where:

- Objects are additive functors $F : \mathcal{A} \to \mathcal{B}$. An additive functor (Definition 3.0.2) is a functor such that for any $x, y \in \mathrm{Ob}(\mathcal{A})$, the map $F : \mathrm{Hom}_{\mathcal{A}}(x, y) \to \mathrm{Hom}_{\mathcal{B}}(F(x), F(y))$ is a group homomorphism.
- Morphisms are natural transformations (Definition 1.1.3) between additive functors (Definition 3.0.2).

Proposition 6.0.7. Let $\mathcal{B}$ be a preadditive category (Definition 3.0.1). The category $\mathrm{Ch}(\mathcal{B})$ of chain complexes (Definition 6.0.1) in $\mathcal{B}$ is isomorphic to the category $\mathrm{Add}(\mathcal{I}, \mathcal{B})$ (Definition 6.0.6)(Definition 6.0.5).

An additive functor $F : \mathcal{I} \to \mathcal{B}$ corresponds to the chain complex defined by $C_n = F(n)$ and differentials $\partial_n = F(d_n)$.

Lemma 6.0.8. Let $\mathcal{A}, \mathcal{B}$ be preadditive categories (Definition 3.0.1) with $\mathcal{A}$ small (Definition 1.0.9).

1. The additive functor category (Definition 6.0.6) $\text{Add}(\mathcal{A}, \mathcal{B})$ is preadditive. If $\mathcal{B}$ is additionally additive (Definition 3.0.1)/abelian (Definition 3.0.3), then so is $\text{Add}(\mathcal{A}, \mathcal{B})$.
2. If $\mathcal{B}$ is an abelian category with property ABn for $n = 3, 4, 5, 6$ or ABn^* for $n = 3, 4, 5$ (Definition 3.0.4), then $\text{Add}(\mathcal{A}, \mathcal{B})$ possesses the same property.

Proposition 6.0.9. Let $\mathcal{A}$ be an additive category.

1. The category $\text{Ch}(\mathcal{A})$ of chain complexes is itself and additive category.
2. If $\mathcal{A}$ is an abelian category, then $\text{Ch}(\mathcal{A})$ is an abelian category.
3. If $\mathcal{A}$ is an abelian category satisfying Grothendieck's axiom ABn (resp. ABn^*) (Definition 3.0.4) for $n \in \{3, 4, 5, 6\}$, then $\text{Ch}(\mathcal{A})$ also satisfies ABn (resp. ABn^*). If $\mathcal{A}$ is a Grothendieck abelian category (Definition 3.0.4), then so is $\text{Ch}(\mathcal{A})$

Proof. Combine Proposition 6.0.7 and Lemma 6.0.8. $\qquad\qquad\square$

7. Derived categories

7.1. **Homotopy category of chain complexes of an additive category.** We will define homotopy categories and derived categories using the cohomological convention

Definition 7.1.1 (Homotopy category $K(\mathcal{A})$). Let $\mathcal{A}$ be an additive category (Definition 3.0.1). (♠ TODO: describe the distinguished triangles of this category)

1. The *homotopy category* $K(\mathcal{A})$ has as objects the (cochain) complexes over $\mathcal{A}$, and as morphisms the cochain maps (Definition 6.0.1) modulo cochain homotopy equivalence. More explicitly,
$$\text{Hom}_{K(\mathcal{A})}(C^\bullet, D^\bullet) := \frac{\{\text{cochain maps } C^\bullet \to D^\bullet\}}{\{\text{cochain homotopies}\}},$$
where morphisms differing by a cochain homotopy are identified.
2. Let
$$F : \mathcal{A} \to \mathcal{B}$$
be an additive functor (Definition 3.0.2) of additive categories.

The functor F naturally induces a functor between the homotopy categories of complexes,
$$K(F) : K(\mathcal{A}) \to K(\mathcal{B}),$$
by applying F degreewise to complexes and cochain maps. That is, for a complex $(C^\bullet, d^\bullet)$ in $\mathcal{A}$, define
$$K(F)(C^\bullet) := (F(C^\bullet), F(d^\bullet)),$$
and for a cochain map $f^\bullet : C^\bullet \to D^\bullet$, define
$$K(F)(f^\bullet) := (F(f^\bullet)).$$

The functor $K(F)$ respects cochain homotopies, thus it is well-defined on the homotopy category.

3. (See [DBG$^+$77, Catégories dérivées Chapitre 1 2-3] for notation) We write $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ for the full subcategories of bounded below, bounded abaove, and bounded complexes respectively.
4. (See [DBG$^+$77, Catégories dérivées Chapitre 1 2-3] for notation) If $\mathcal{A}$ is abelian, we may also write $K^{\infty,+}(\mathcal{A})$, $K^{\infty,-}(\mathcal{A})$, $K^{\infty,b}(\mathcal{A})$, and $K^{\infty,\emptyset}(\mathcal{A})$ for the full subcategories of cohomologically bounded below, cohomologically bounded abaove, cohomologically bounded, and acyclic complexes respectively. We may further write $K^{?,??}(\mathcal{A})$ for the full subcategory where $? \in \{+, -, b\}$ indicates the boundedness of the complex and $?? \in \{+, -, b, \text{(blank)}\}$ indicates the cohomological boundedness of the complex.

The various categories here are (Proposition 7.1.6) triangulated categories.

Definition 7.1.2 (Triangulated category). A *triangulated category* is a triple $(\mathcal{T}, [1], \Delta)$ where:

- $\mathcal{T}$ is an additive category,
- $[1] : \mathcal{T} \to \mathcal{T}$ (also denoted by notations such as Σ) is an additive auto-equivalence called the *shift* or *suspension functor*,
- Δ is a class of distinguished triangles of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

  with objects $X, Y, Z \in \mathcal{T}$ and morphisms f, g, h in $\mathcal{T}$,

satisfying the following axioms:

(TR1) The class Δ is closed under isomorphisms of triangles and contains the triangles isomorphic to

$$A \xrightarrow{\text{id}} A \to 0 \to A[1].$$

Moreover, for every morphism $f : X \to Y$ in $\mathcal{T}$, there exists a distinguished triangle

$$X \xrightarrow{f} Y \to Z \to X[1].$$

(TR2) (Rotation) If

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is a distinguished triangle, then so are

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

and

$$Z[-1] \xrightarrow{-h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z.$$

(TR3) (Octahedral axiom) Given morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ in $\mathcal{T}$, there exist distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{u} C(f) \xrightarrow{v} X[1],$$
$$Y \xrightarrow{g} Z \xrightarrow{w} C(g) \xrightarrow{x} Y[1],$$
$$X \xrightarrow{g \circ f} Z \xrightarrow{y} C(g \circ f) \xrightarrow{z} X[1],$$

and morphisms

$$s : C(f) \to C(g \circ f), \quad t : C(g \circ f) \to C(g)$$

such that the following diagram commutes and its rows and columns are distinguished triangles:

$$
\begin{array}{ccccccc}
X & \xrightarrow{\;f\;} & Y & \xrightarrow{\;u\;} & C(f) & \xrightarrow{\;v\;} & X[1] \\
\| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle s} & & \| \\
X & \xrightarrow{\;g\circ f\;} & Z & \xrightarrow{\;y\;} & C(g\circ f) & \xrightarrow{\;z\;} & X[1] \\
& & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle t} & & \\
& & C(g) & = & C(g) & & \\
& & \downarrow{\scriptstyle x} & & \downarrow & & \\
& & Y[1] & \xrightarrow{\;f[1]\;} & X[1][1] & &
\end{array}
$$

In particular,
- The compositions $s \circ u = y$, $t \circ s = w$, and $x \circ t = v[1]$ hold,
- All rows and columns form distinguished triangles.

(TR4) The class Δ is closed under the shift functor $[1]$: if

$$X \to Y \to Z \to X[1]$$

is distinguished, then so is

$$X[1] \to Y[1] \to Z[1] \to X[2].$$

Definition 7.1.3. A functor $T : D_1 \to D_2$ between triangulated categories is called *exact* if it is additive, is translation preserving, and transforms distinguished triangles to distinguished triangles. Oftentimes, a *morphism between triangulated categories* refers to an exact functor between triangulated categories.

Definition 7.1.4. In the context of a triangulated category (Definition 7.1.2) $\mathscr{D}$, a multiplicative system (Definition 7.2.1) S is *compatible with the triangulation* if:

1. For any $s \in S$, $T(s) \in S$ where T is the translation functor.
2. For any commutative square between the first two steps of two distinguished triangles where the vertical arrows are in S, there exists a morphism in S between the third objects that makes the whole diagram of triangles commute.

Definition 7.1.5. Let I be a finite set and $\varepsilon : I \to \{1, -1\}$ a variance function. Let $(\mathscr{A}_i)_{i\in I}$ and $\mathscr{A}$ be triangulated categories (Definition 7.1.2). A multifunctor (Definition 1.3.16) $F : \prod_{i\in I} \mathscr{A}_i \to \mathscr{A}$ is an *exact multifunctor of variance ε* if for every index $k \in I$, the partial functor

$$F_k : \mathscr{A}_k \to \mathscr{A}$$

(obtained by fixing all objects A_j for $j \neq k$) satisfies the following:

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1. If $\varepsilon(k) = 1$, F_k is a covariant exact functor: it maps any distinguished triangle $X \to Y \to Z \to T_k X$ in $\mathscr{A}_k$ to a distinguished triangle

$$F_k(X) \to F_k(Y) \to F_k(Z) \to TF_k(X)$$

in $\mathscr{A}$.

2. If $\varepsilon(k) = -1$, F_k is a contravariant exact functor: it maps any distinguished triangle $X \to Y \to Z \to T_k X$ in $\mathscr{A}_k$ to a distinguished triangle

$$F_k(Z) \to F_k(Y) \to F_k(X) \to F_k(T_k^{-1}X)$$

in $\mathscr{A}$, using the isomorphism $F_k(T_k^{-1}X) \cong TF_k(X)$.

Furthermore, F must be a graded functor equipped with natural isomorphisms $\varphi_i : T \circ F \xrightarrow{\sim} F \circ T_i^{\varepsilon_i}$ for each $i \in I$. These must satisfy the anticommutativity condition: for any pair of distinct indices $i, j \in I$ $(i \neq j)$, the following diagram of natural transformations commutes with a factor of -1:

(A)
$$
\begin{array}{ccc}
T^2 F & \xrightarrow{T\varphi_i} & TFT_i^{\varepsilon_i} \\
\downarrow{T\varphi_j} & & \downarrow{\varphi_j T_i^{\varepsilon_i}} \\
TFT_j^{\varepsilon_j} & \xrightarrow{-\varphi_i T_j^{\varepsilon_j}} & FT_i^{\varepsilon_i} T_j^{\varepsilon_j}
\end{array}
$$

Equivalently, the following square is anticommutative:

(B)
$$(\varphi_j T_i^{\varepsilon_i}) \circ (T\varphi_i) = -(\varphi_i T_j^{\varepsilon_j}) \circ (T\varphi_j)$$

as morphisms of functors $T^2 F \to FT_i^{\varepsilon_i} T_j^{\varepsilon_j}$.

In particular in the case that $\epsilon(i) = 1$ for all i, we are reduced to the following definition: Let $(\mathscr{A}_i)_{1 \leq i \leq n}$ and $\mathscr{A}$ be triangulated categories (Definition 7.1.2). A multifunctor (Definition 1.3.16) $F : \prod \mathscr{A}_i \to \mathscr{A}$ is called an *exact covariant multifunctor* if it is covariant in each argument and if for every index $i \in \{1, \ldots, n\}$ and every fixed collection of objects $(A_j)_{j \neq i}$, the partial functor $F(\ldots, A_{i-1}, -, A_{i+1}, \ldots) : \mathscr{A}_i \to \mathscr{A}$ is an exact functor (Definition 7.1.3) of triangulated categories.

Proposition 7.1.6. Let $\mathcal{A}$ be an additive category (Definition 3.0.1).

1. The homotopy category $K(\mathcal{A})$ (Definition 7.1.1) is a triangulated category (Definition 7.1.2).(♠ TODO: shift of chain complexes, describe what the distinguished triangles are)

2. The subcategory $K^?(\mathcal{A})$ is a triangulated subcategory of $K(\mathcal{A})$ for $? \in \{+, -, b\}$.

3. Let $\mathcal{A}$ be an abelian category. The subcategory $K^{?,??}(\mathcal{A})$ is a triangulated subcategory of $K(\mathcal{A})$ for $? \in \{\infty, +, -, b\}$ and $?? \in \{+, -, b, (\text{blank})\}$.

(♠ TODO: How homotopy categories of chain complexes relate to topological homotopy categories)

Theorem 7.1.7 (Exact functor induces a functor on homotopy categories, cf. [?, Exposé XVII, Proposition 1.1.7.]). Let $F : \prod_{i=1}^{n} \mathscr{A}_i \to \mathscr{B}$ be an additive multifunctor of abelian

categories (Definition 3.0.3). F induces exact multifunctors (Definition 7.1.5)[5] between the corresponding homotopy categories (Definition 7.1.1):

$$K(\mathscr{A}) \to K(\mathscr{B})$$
$$K^?(\prod \mathscr{A}_i) \to K^?(\mathscr{B})$$
$$K^{?,??}(\prod \mathscr{A}_i) \to K^{?,??}(\mathscr{B})$$

(Definition 7.1.1). We write these functors by $K(F)$.

7.2. Multiplicative systems in categories.

Definition 7.2.1 (Multiplicative system). (♠ TODO: Write a precise definition) Let $\mathcal{C}$ be a category. A *multiplicative system S in $\mathcal{C}$* consists of a collection of morphisms in $\mathcal{C}$ satisfying:

- S contains all identity morphisms,
- S is closed under composition,
- (Ore conditions) For any morphism f in $\mathcal{C}$ and any morphism $s \in S$ with suitable domain and codomain, there exist morphisms to form commutative squares allowing localization.

The precise Ore condition varies depending on context but guarantees the localization exists.

Definition 7.2.2 (Localization by a multiplicative system). Given a category $\mathcal{C}$ and a multiplicative system (Definition 7.2.1) $S \subseteq \mathrm{Mor}(\mathcal{C})$, the *localization $\mathcal{C}[S^{-1}]$* is a category equipped with a functor

$$Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$$

that sends every morphism in S to an isomorphism satisfying the universal property: any functor from $\mathcal{C}$ sending the morphisms in S to isomorphisms factors uniquely through Q.

Definition 7.2.3. A multiplicative system (Definition 7.2.1) S in a category (Definition 1.0.1) $\mathscr{C}$ is called *saturated* if it satisfies the following condition: any morphism f in $\mathscr{C}$ whose image $Q(f)$ in the localized category (Definition 7.2.2) $\mathscr{C}(S^{-1})$ is an isomorphism belongs to S.

Proposition 7.2.4. Let $\mathscr{C}$ be a category (Definition 1.0.1) and S a multiplicative system (Definition 7.2.1). There exists a unique saturated multiplicative system (Definition 7.2.3) $\bar{S}$ containing S such that the canonical functors $\mathscr{C}(S^{-1}) \to \mathscr{C}(\bar{S}^{-1})$ define an equivalence of categories (Definition 1.1.8). $\bar{S}$ consists of all morphisms f such that there exist morphisms g, h with $g \circ f \in S$ and $f \circ h \in S$.

7.3. Derived categories of abelian categories. (♠ TODO: describe the objects and morphisms in a derived category)

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[5]wlog covariant in each variable, but this can be made multi-variant by taking opposite categories (Definition 1.0.12)

If F is contravariant in the j-th variable, the definitions are adjusted by replacing $\mathscr{A}_j$ with its opposite category $\mathscr{A}_j^{\mathrm{op}}$, thereby interchanging K^+ with K^- and RF with LF for that specific variable.

Definition 7.3.1 (Derived category)**.** Let $\mathcal{A}$ be an abelian category. Consider the class S of quasi-isomorphisms in the homotopy category $K(\mathcal{A})$ (Definition 7.1.1).

The *derived category* $D(\mathcal{A})$ is the localization (Definition 7.2.2) of the homotopy category at the multiplicative system (Definition 7.2.1) S:

$$D(\mathcal{A}) := K(\mathcal{A})[S^{-1}],$$

We often write the localization functor (Definition 7.2.2) $K(\mathcal{A}) \to D(\mathcal{A})$ by q or Q.

- We often write $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, and $D^b(\mathcal{A})$ for the full subcategories of cohomologically bounded below, cohomologically bounded above, and cohomologically bounded complexes respectively.

In general, $D(\mathcal{A})$ (and the aforementioned subcategories) may only exist as a large category (Definition 1.0.1), rather than a locally small category. When $\mathcal{A}$ is a Grothendieck abelian category (Definition 3.0.4), however, $D(\mathcal{A})$ is a locally small category (see [Sta25, Tag 09PA]).

Convention 7.3.2. Given an abelian category (Definition 7.3.1) A, any object M of A may be identified with an object of the homotopy category $K(A)$ (Definition 7.1.1) and the derived category $D(A)$ (Definition 7.3.1) — simply take the chain complex concentrated at degree 0 with the object M. In fact, this derived object can often be regarded as an object of "nice" enough subcategories of $D(A)$, such as the bounded derived categories $D^+(A)$, $D^-(A)$, $D^b(A)$.

(♠ TODO: State how a derived category is a triangulated category)

Definition 7.3.3. Let $\mathcal{A}$ be an additive category (Definition 3.0.1). For $? \in \{+, -, b, (\text{blank})\}$, write $K^?(\mathcal{I})$ (resp. $K^?(\mathcal{P})$) for the full subcategory of $K^?(\mathcal{A})$ (Definition 7.1.1) whose objects are appropriately bounded complexes of injectives (resp. projectives).

Even though derived categories of abelian categories may not be guaranteed to be locally small, the following gives sufficient conditions for the appropriately (cohomologically) bounded derived categories to be locally small. The following also shows that appropriately bounded complexes under these circumstances can be represented in the derived category by a complex of injective or projective objects; we might think of such complexes of injectives or projectives as acting like an injective/projective resolution.

Theorem 7.3.4 (see e.g. [Wei94, Theorem 10.4.8])**.** Let $\mathcal{A}$ be an abelian category.

1. If $\mathcal{A}$ has enough injectives (Definition 5.4.4), then the category $D^+(\mathcal{A})$ (Definition 7.3.1) is equivalent to the category $K^+(\mathcal{I})$ (Definition 7.3.3), which is locally small (Definition 1.0.9). In fact, an equivalence is given by

$$K^+(\mathcal{I}) \xrightarrow{q} D^+(\mathcal{I}) \hookrightarrow D^+(\mathcal{A}).$$

2. If $\mathcal{A}$ has enough projectives (Definition 5.4.4), then the category $D^-(\mathcal{A})$ (Definition 7.3.1) is equivalent to the category $K^-(\mathcal{P})$ (Definition 7.3.3), which is locally small (Definition 1.0.9).

$$K^-(\mathcal{P}) \xrightarrow{q} D^-(\mathcal{P}) \hookrightarrow D^-(\mathcal{A}).$$

7.4. **Total derived functors.** The following subsection mostly takes the point of view of [Wei94, Chapter 10]

Definition 7.4.1 (cf. [Wei94, Definition 10.5.1]). Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $F : K(\mathcal{A}) \to K(\mathcal{B})$ (Definition 7.1.1) be an exact functor (Definition 7.1.3) of triangulated categories. Write $q_{\mathcal{A}} : K(\mathcal{A}) \to D(\mathcal{A})$ and $q_{\mathcal{B}} : K(\mathcal{B}) \to D(\mathcal{B})$ for the localization functors (Definition 7.3.1),

1. A *(total) right derived functor on F on $K(\mathcal{A})$* is an exact functor $RF : D(\mathcal{A}) \to D(\mathcal{B})$ together with a natural transformation

$$\xi : q_{\mathcal{B}}F \Rightarrow (RF)q_{\mathcal{A}},$$

which is universal in the sense that if $G : D(\mathcal{A}) \to D(\mathcal{B})$ is another morphism equipped with a natural transformation $\zeta : q_{\mathcal{B}}F \Rightarrow Gq_{\mathcal{A}}$, then there exists a unique natural transformation $\eta : RF \Rightarrow G$ so that $\zeta_A = \eta_{q_{\mathcal{A}}A} \circ \xi_A$ for all A in $K(\mathcal{A})$.
2. Dually, a *(total) left derived functor on F on $K(\mathcal{A})$* is an exact functor $LF : D(\mathcal{A}) \to D(\mathcal{B})$ together with a natural transformation

$$\xi : (LF)q_{\mathcal{A}} \Rightarrow q_{\mathcal{B}}F,$$

which is universal in the sense that if $G : D(\mathcal{A}) \to D(\mathcal{B})$ is another exact functor equipped with a natural transformation $\zeta : Gq_{\mathcal{A}} \Rightarrow q_{\mathcal{B}}F$, then there exists a unique natural transformation $\eta : G \Rightarrow LF$ such that

$$\zeta_A = \xi_A \circ \eta_{q_{\mathcal{A}}A}$$

for all A in $K(\mathcal{A})$.

The universal properties guarantee that if the total derived functors exist, then they are unique up to natural isomorphism, and that

These definitions/ideas may be extended to exact functors $K^?(\mathcal{A}) \to K(\mathcal{B})$ for $? \in \{+, -, b\}$ and to exact functors $K^{?,??}(\mathcal{A}) \to K(\mathcal{B})$ for $? \in \{\infty, +, -, b\}$ and $?? \in \{+, -, b, (\text{blank})\}$ (Definition 7.1.1)

Proposition 7.4.2. See e.g. [Wei94, 10.5.2] Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor (Definition 3.0.6) of abelian categories (Definition 3.0.3). There is an induced functor

$$D(\mathcal{A}) \to D(\mathcal{B})$$

(Definition 7.3.1) also denoted by F by abuse of notation.

This functor is constructible in the "simple" way —

- For an object M of $D(\mathcal{A})$, take a chain complex K representing it. The object FM of $D(\mathcal{B})$ is the object represented by the chain complex FK.
- For a morphism $f : M \to N$ in $D(\mathcal{A})$, take a morphism $\tilde{f} : K \to L$ representing it. The morphism $Ff : FM \to FN$ is the morphism represented by $F\tilde{f} : FK \to FL$.

Due to the exactness of F, these do not depend on the choices of representatives.

The below shows that right and left total derived functors exist for nice enough abelian categories on sufficiently (cohomologically) boundeded objects and can be computed using injective/projective resolutions.

Theorem 7.4.3 (See e.g. [Wei94, 10.5.6]). Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $F : K^+(\mathcal{A}) \to K(\mathcal{B})$ (Definition 7.1.1) be a morphism of triangulated categories (Definition 7.1.3).

1. If $\mathcal{A}$ has enough injectives (Theorem 7.1.7), then the right derived functor RF (Definition 7.4.1) exists (as a functor $D^+(\mathcal{A}) \to D(\mathcal{B})$ (Definition 7.3.1)), and if I is a bounded below complex of injectives, then

$$RF(I) \cong qF(I)$$

where $q : K(\mathcal{B}) \to D(\mathcal{B})$ is the localization functor (Definition 7.3.1).
2. Dually, if $\mathcal{A}$ has enough projectives, then the left derived functor LF exists (as a functor $D^-(\mathcal{A}) \to D(\mathcal{B})$), and if I is a bounded above complex of projectives, then

$$LF(P) \cong qF(P)$$

Corollary 7.4.4. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor (Definition 3.0.2) between abelian categories (Definition 3.0.3). Write $q : K(\mathcal{B}) \to D(\mathcal{B})$ (Definition 7.1.1 Definition 7.3.1) for the localization functor (Definition 7.2.2).

1. Assuming that $\mathcal{A}$ has enough injectives (Definition 5.4.4), there exists a right derived functor (Definition 7.4.1) $RF : D^+(\mathcal{A}) \to D(\mathcal{B})$ and if $I \in K^+(\mathcal{I})$ (Definition 7.3.3) is a bounded below complex of injectives, then

$$RF(I) \cong q(K(F))(I).$$

where $K(F) : K^+(\mathcal{A}) \to K(\mathcal{B})$ is the exact functor (Definition 7.1.3) induced by F (Theorem 7.1.7).
   If F is additionally left exact (Definition 3.0.6), then for all objects A of $\mathcal{A}$ we have

$$H^i(RF(A)) \cong R^i F(A)$$

(Definition 5.4.8) for all integers i where we regard A as the chain complex (Definition 6.0.1) object of $D^+(\mathcal{A})$ with A concentrated in degree 0 (Convention 7.3.2).
2. Assuming that $\mathcal{A}$ has enough projectives (Definition 5.4.4), there exists a left derived functor (Definition 7.4.1) $LF : D^-(\mathcal{A}) \to D(\mathcal{B})$ and if $P \in K^-(\mathcal{P})$ (Definition 7.3.3) is a bounded above complex of projectives, then

$$LF(P) \cong q(K(F))(P).$$

where $K(F) : K^-(\mathcal{A}) \to K(\mathcal{B})$ is the exact functor (Definition 7.1.3) induced by F (Theorem 7.1.7).
   If F is additionally right exact (Definition 3.0.6), then for all objects A of $\mathcal{A}$ we have

$$H^{-i}(LF(A)) \cong L_i F(A)$$

(Definition 5.4.8) for all integers i where we regard A as the chain complex (Definition 6.0.1) object of $D^+(\mathcal{A})$ with A concentrated in degree 0 (Convention 7.3.2).

Proof. Apply Theorem 7.4.3 to the exact functor $K(F)$ to obtain the isomorphisms

$$RF(I) \cong q(K(F))(I)$$

$$LF(P) \cong q(K(F))(P).$$

If F is left exact and A is an object of $\mathcal{A}$, then the right derived functors $R^i F(A)$ (Definition 5.4.8) are defined. Recall that they are defined first by taking an injective resolution (Definition 5.4.3)

$$0 \to A \to I^0 \to I^1 \to \cdots .$$

Since the injective resolution is acyclic, The object A and the complex $0 \to I^0 \to I^1 \to \cdots$ are equivalent as objects of $D^+(\mathcal{A})$ as an object of $D^+(\mathcal{A})$ are equivalent objects. Moreover, we have the isomorphism

$$RF(I) \cong q(K(F))(I).$$

In fact, the ith cohomology of the right hand side object precisely describes $R^i F(A)$ by definition. Therefore,

$$H^i(RF(A)) \cong R^i F(A)$$

as desired.

The analogous statement for $L_i F(A)$ holds similarly. □

(♠ TODO: define hyper-derived functors)

Corollary 7.4.5 (See e.g. [Wei94, Corollary 10.5.7, Remark 10.5.8])**.** Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor (Definition 3.0.2) between abelian categories (Definition 3.0.3).

1. If $\mathcal{A}$ has enough injectives (Definition 5.4.4), the hyper-derived functors $\mathbb{R}^i F(X)$ are the cohomology of $\mathbf{R}F(X)$: $\quad \mathbb{R}^i F(X) \cong H^i \mathbf{R}F(X)$ for all i.
2. If $\mathcal{A}$ has enough projectives (Definition 5.4.4), the hyper-derived functors $\mathbb{L}_i F(X)$ are the cohomology of $\mathbf{L}F(X)$: $\quad \mathbb{L}_i F(X) \cong H^{-i} \mathbf{L}F(X)$ for all i.

7.5. **General derived functors for exact multifunctors between triangulated categories.** The following subsection mostly takes the point of view of [?, Exposé XVII, 1.2]

Definition 7.5.1 (Ind-category)**.** Let $\mathcal{C}$ be a locally small category (Definition 1.0.9).

1. The *Ind-category of $\mathcal{C}$*, denoted $\mathrm{Ind}(\mathcal{C})$, is defined as follows:
    - Objects of $\mathrm{Ind}(\mathcal{C})$ are formal filtered colimits (Definition 1.2.12) of objects in $\mathcal{C}$. More precisely, an object is given by a filtered (Definition 1.2.8) small category I and a functor

    $$X : I \to \mathcal{C}.$$

    - Morphisms between objects $X : I \to \mathcal{C}$ and $Y : J \to \mathcal{C}$ are defined by

    $$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(X, Y) \ := \ \varprojlim_{i \in I} \varinjlim_{j \in J} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j),$$

    (Definition 1.2.12) where X_i and Y_j denote the images of $i \in I$ and $j \in J$ under X and Y, respectively.

The composition of morphisms is induced naturally from composition in $\mathcal{C}$. Hence, $\mathrm{Ind}(\mathcal{C})$ is the completion of $\mathcal{C}$ under filtered colimits. Objects of $\mathrm{Ind}(\mathcal{C})$ are called *Ind-objects of $\mathcal{C}$*.

2. The *Pro-category of $\mathcal{C}$*, denoted $\mathrm{Pro}(\mathcal{C})$, is defined as follows:
   - Objects of $\mathrm{Pro}(\mathcal{C})$ are formal cofiltered limits (Definition 1.2.12) of objects in $\mathcal{C}$. More precisely, an object is given by a cofiltered small category I and a functor

$$X : I \to \mathcal{C}.$$

   - Morphisms between objects $X : I \to \mathcal{C}$ and $Y : J \to \mathcal{C}$ are defined by

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X, Y) := \varinjlim_{j \in J} \varprojlim_{i \in I} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j),$$

   where X_i and Y_j denote the images of $i \in I$ and $j \in J$ under X and Y, respectively. The composition of morphisms is induced naturally from composition in $\mathcal{C}$.

   Hence, $\mathrm{Pro}(\mathcal{C})$ is the completion of $\mathcal{C}$ under cofiltered limits. Objects of $\mathrm{Pro}(\mathcal{C})$ are called *Pro-objects of $\mathcal{C}$*.

Since **Sets** has all limits and colimits (♠ TODO:) and hence has all projective and inductive limits and since $\mathcal{C}$ is locally small, $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C})$ are locally small.

Definition 7.5.2. Let $\mathscr{C}$ be a locally small category (Lemma 5.0.2).

1. An Ind-object (Definition 7.5.1) $X = \varinjlim X_i$ is said to be *essentially constant* if it is isomorphic in $\mathrm{Ind}(\mathscr{C})$ to an object in the image of the canonical fully faithful functor $h : \mathscr{C} \to \mathrm{Ind}(\mathscr{C})$, which maps an object $A \in \mathrm{ob}(\mathscr{C})$ to the Ind-object represented by the constant diagram.
2. A Pro-object (Definition 7.5.1) $X = \varprojlim X_i$ is said to be *essentially constant* if it is isomorphic in $\mathrm{Pro}(\mathscr{C})$ to an object in the image of the canonical fully faithful functor $h : \mathscr{C} \to \mathrm{Pro}(\mathscr{C})$, which maps an object $A \in \mathrm{ob}(\mathscr{C})$ to the Pro-object represented by the constant diagram.

In either definition, the object A may be referred to as the *value/limit/constant value of the Ind/Pro object* or the *object representing the Ind/Pro object*.

Proposition 7.5.3. Let $\mathscr{C}$ be a locally small category (Lemma 5.0.2).

1. An Ind-object (Definition 7.5.1) $X = \varinjlim_{i \in I} X_i$ (where I is a filtered (Definition 1.2.8) index category) is essentially constant (Definition 7.5.2) and isomorphic to $A \in \mathrm{ob}(\mathscr{C})$ if and only if there exists a family of morphisms $(f_i : X_i \to A)_{i \in I}$ such that for every morphism $u : i \to j$ in I, the diagram

$$
\begin{array}{ccc}
X_i & \xrightarrow{X(u)} & X_j \\
& \searrow_{f_i} & \downarrow^{f_j} \\
& & A
\end{array}
$$

commutes, and the induced morphism $\varinjlim X_i \to A$ (Definition 1.2.12) is an isomorphism in $\mathrm{Ind}(\mathscr{C})$.

2. A Pro-object (Definition 7.5.1) $X = \varprojlim_{i \in I} X_i$ (where I is a filtered (Definition 1.2.8) index category) is essentially constant (Definition 7.5.2) and isomorphic to $A \in \mathrm{ob}(\mathscr{C})$ if and only if there exists a family of morphisms $(g_i : A \to X_i)_{i \in I}$ such that for every morphism $u : i \to j$ in I, the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{g_i} & X_i \\
& \searrow{\scriptstyle g_j} & \downarrow{\scriptstyle X(u)} \\
& & X_j
\end{array}
$$

commutes, and the induced morphism $A \to \varprojlim X_i$ (Definition 1.2.12) is an isomorphism in $\mathrm{Pro}(\mathscr{C})$.

Definition 7.5.4 (Mixed-Variance Derived Multifunctors). [cf. [?, Exposé XVII, Définition 1.2.1., 1.2.5.]]   Let $(\mathscr{A}_i)_{1 \leq i \leq n}$ and $(\mathscr{B}_j)_{1 \leq j \leq m}$ be families of triangulated categories (Definition 7.1.2), and let $\mathscr{C}$ be a triangulated category. Let

$$
F : \prod_{i=1}^{n} \mathscr{A}_i \times \prod_{j=1}^{m} \mathscr{B}_j \to \mathscr{C}
$$

be an exact multifunctor (Definition 7.1.5), covariant in A_i and contravariant in B_j. Let S_i, Σ_j, T be saturated multiplicative systems (Definition 7.2.3) in $\mathscr{A}_i, \mathscr{B}_j, \mathscr{C}$ respectively.

1. The *right derived functor RF of F relative to (S_i, Σ_j) and T* is a functor

$$
RF : \prod_{i=1}^{n} \mathscr{A}_i(S_i^{-1}) \times \prod_{j=1}^{m} \mathscr{B}_j(\Sigma_j^{-1}) \to \mathrm{Ind}(\mathscr{C}(T^{-1}))
$$

which makes the following diagram commutative:

$$
\begin{array}{ccc}
\prod_{i=1}^{n} \mathscr{A}_i \times \prod_{j=1}^{m} \mathscr{B}_j & \xrightarrow{F} & \mathscr{C} \\
\downarrow{\scriptstyle \prod Q_{A_i} \times \prod Q_{B_j}} & & \downarrow{\scriptstyle Q_C} \\
\prod_{i=1}^{n} \mathscr{A}_i(S_i^{-1}) \times \prod_{j=1}^{m} \mathscr{B}_j(\Sigma_j^{-1}) & \xrightarrow{RF} & \mathrm{Ind}(\mathscr{C}(T^{-1}))
\end{array}
$$

2. The *left derived functor LF of F relative to (S_i, Σ_j) and T* is a functor

$$
LF : \prod_{i=1}^{n} \mathscr{A}_i(S_i^{-1}) \times \prod_{j=1}^{m} \mathscr{B}_j(\Sigma_j^{-1}) \to \mathrm{Pro}(\mathscr{C}(T^{-1}))
$$

which makes the following diagram commutative:

$$
\begin{array}{ccc}
\prod_{i=1}^{n} \mathscr{A}_i \times \prod_{j=1}^{m} \mathscr{B}_j & \xrightarrow{F} & \mathscr{C} \\
\downarrow{\scriptstyle \prod Q_{A_i} \times \prod Q_{B_j}} & & \downarrow{\scriptstyle Q_C} \\
\prod_{i=1}^{n} \mathscr{A}_i(S_i^{-1}) \times \prod_{j=1}^{m} \mathscr{B}_j(\Sigma_j^{-1}) & \xrightarrow{LF} & \mathrm{Pro}(\mathscr{C}(T^{-1}))
\end{array}
$$

Definition 7.5.5 (Derivability and Adapted Families). Let $(\mathscr{A}_i)_{1 \leq i \leq n}$ and $(\mathscr{B}_j)_{1 \leq j \leq m}$ be families of triangulated categories (Definition 7.1.2), and let $\mathscr{C}$ be a triangulated category. Let

$$
F : \prod_{i=1}^{n} \mathscr{A}_i \times \prod_{j=1}^{m} \mathscr{B}_j \to \mathscr{C}
$$

be an exact multifunctor (Definition 7.1.5), covariant in the variables $A_i \in \mathscr{A}_i$ and contravariant in the variables $B_j \in \mathscr{B}_j$. Let S_i, Σ_j, T be saturated multiplicative systems (Definition 7.2.3) in $\mathscr{A}_i, \mathscr{B}_j, \mathscr{C}$ respectively.

1. Let RF be the right derived functor (Definition 7.5.4) of F relative to the multiplicative systems.
   (a) RF is *defined at a family (A_i, B_j)* if the Ind-object (Definition 7.5.1) $RF(A_i, B_j)$ is essentially constant. This object is the *value of RF at (A_i, B_j)*.
   (b) F is *right derivable* if RF is defined at all objects. In this case, we view it as a functor into $\mathscr{C}(T^{-1})$.
   (c) A family (A_i, B_j) is *adapted for RF* (or *deployed for RF* (*déployée pour RF* in French)) if the natural morphism $Q_C(F(A_i, B_j)) \to RF(Q_{A_i} A_i, Q_{B_j} B_j)$ is an isomorphism in $\mathrm{Ind}(\mathscr{C}(T^{-1}))$, i.e. the Ind-object $RF(A_i)$ of $A(S^{-1})$ is essentially constant (Definition 7.5.2) with value $F((A_i), (B_j))$.
2. Let LF be the left derived functor (Definition 7.5.4) of F relative to the multiplicative systems.
   (a) LF is *defined at a family (A_i, B_j)* if the Pro-object (Definition 7.5.1) $LF(A_i, B_j)$ is essentially constant.
   (b) F is *left derivable* if LF is defined at all objects.
   (c) A family (A_i, B_j) is *adapted for LF* (or *deployed for LF* (*déployée pour LF* in French)) if the natural morphism $LF(Q_{A_i} A_i, Q_{B_j} B_j) \to Q_C(F(A_i, B_j))$ is an isomorphism in $\mathrm{Pro}(\mathscr{C}(T^{-1}))$, i.e. the Pro-object $LF(A_i)$ of $A(S^{-1})$ is essentially constant (Definition 7.5.2) with value $F((A_i), (B_j))$.

7.5.1. *Derived multifunctors of additive multifunctors between abelian categories.*

Definition 7.5.6. Let $F : \prod_{i=1}^{n} \mathscr{A}_i \to \mathscr{B}$ be an additive multifunctor of abelian categories (Definition 3.0.3).

The *right (resp. left) derived multifunctors of F* are the derived functors (Definition 7.5.4) RF (resp. LF) of the induced multifunctors (Theorem 7.1.7)

- $K(\prod \mathscr{A}_i) \to K(\mathscr{B})$
- $K^+(\prod \mathscr{A}_i) \to K^+(\mathscr{B})$
- $K^-(\prod \mathscr{A}_i) \to K^-(\mathscr{B})$

(Recall Proposition 7.1.6) relative to the systems (Definition 7.2.3) of quasi-isomorphisms in the respective homotopy categories (Definition 7.1.1). In particular, the functor RF refers to one of the functors

$$D^+(\prod \mathscr{A}_i) \to \mathrm{Ind}\, D^+(\mathscr{B}) D(\prod \mathscr{A}_i) \to \mathrm{Ind}\, D(\mathscr{B})$$

and the functor LF refers to one of the functors

$$D^-(\prod \mathscr{A}_i) \to \mathrm{Pro}\, D^-(\mathscr{B}) D(\prod \mathscr{A}_i) \to \mathrm{Pro}\, D(\mathscr{B})$$

(Definition 7.3.1) (Definition 7.5.1).

Definition 7.5.7 ([?, Definition 1.2.3]). Let $F : \prod_{i=1}^{n} \mathscr{A}_i \to \mathscr{B}$ be an additive multifunctor of abelian categories (Definition 3.0.3).

1. An object $(A_1, \ldots, A_n)$ with $A_i \in \mathrm{ob}(\mathscr{A}_i)$ is *right acyclic for F* if the complex concentrated in degree 0 associated to (A_i) is adapted for (Definition 7.5.5) RF .
2. Similarly, (A_i) is *left acyclic for F* if the complex concentrated in degree 0 is adapted for (Definition 7.5.5) LF.

In the case that $n = 1$ and $A_1 \in \mathcal{A}_1$ has a projective resolution (Definition 5.4.3) (resp. injective resolution), the notion of left acyclic for F (resp. right acyclic for F) is equivalent to the notion of F-acyclicity.

Proposition 7.5.8 (cf. [?, Exposé XVII, Proposition 1.2.4.]). Let $F : \prod_{i=1}^n \mathscr{A}_i \to \mathscr{B}$ be an additive multifunctor of abelian categories (Definition 3.0.3).

1. The following diagram of categories is commutative for the right derived multifunctor:

$$
\begin{array}{ccc}
D^+(\prod \mathscr{A}_i) & \longrightarrow & D(\prod \mathscr{A}_i) \\
\downarrow{\scriptstyle RF} & & \downarrow{\scriptstyle RF} \\
\mathrm{Ind}\ D^+(\mathscr{B}) & \longrightarrow & \mathrm{Ind}\ D(\mathscr{B})
\end{array}
$$

(Definition 7.3.1) (**??**) (Definition 7.5.6) where the horizontal arrows are the natural functors induced by inclusions. If RF is everywhere defined on $D(\prod \mathscr{A}_i)$, its restriction to $D^+(\prod \mathscr{A}_i)$ factors through $D^+(\mathscr{B})$ [?, Proposition 1.2.4].
2. Dually, the following diagram is commutative for the left derived multifunctor:

$$
\begin{array}{ccc}
D^-(\prod \mathscr{A}_i) & \longrightarrow & D(\prod \mathscr{A}_i) \\
\downarrow{\scriptstyle LF} & & \downarrow{\scriptstyle LF} \\
\mathrm{Pro}\ D^-(\mathscr{B}) & \longrightarrow & \mathrm{Pro}\ D(\mathscr{B})
\end{array}
$$

If LF is everywhere defined on $D(\prod \mathscr{A}_i)$, its restriction to $D^-(\prod \mathscr{A}_i)$ factors through $D^-(\mathscr{B})$.

Theorem 7.5.9 ([?, Exposé XVII, Proposition 1.2.7.]). (♠ TODO:) Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor (Definition 3.0.2) between abelian categories (Definition 3.0.3).

If $\mathcal{A}$ has enough objects that are acyclic for (Definition 7.5.7) LF (resp. for RF), the derived functor (Definition 7.5.4) $LF : D^-(\mathcal{A}) \to D^-(\mathcal{B})$ (resp. $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$) is left (resp. right) derivable (Definition 7.5.5) and every homologically upper bounded (resp. homologically lower bounded) complex of acyclics for LF (resp. for RF) is adapted for LF (resp. for RF) (Definition 7.5.5).

(♠ TODO:

- **Address the Biadditive/Multifunctor Gap**
  - Add a Lemma or Remark following Definition 5.5.5 clarifying that for a biadditive functor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$, the condition of a pair of complexes $(M^\bullet, N^\bullet)$ being adapted (deployed) is what allows the representation of LF via the Total Complex construction.
  - Explicitly define the Total Complex functor $\mathrm{Tot} : \mathrm{K}(\mathcal{A}) \times \mathrm{K}(\mathcal{B}) \to \mathrm{K}(\mathcal{C})$ used to induce the multifunctor in Definition 5.5.6. Specify the sign convention for the total differential: $d(x \otimes y) = d_x(x) \otimes y + (-1)^p x \otimes d_y(y)$ for x of degree p.
- **Formulate the Balancing Results**

- Add a Proposition (The Balancing Lemma) stating that if $\mathcal{P} \subset \mathcal{A}$ is a class of objects adapted for $F(-, N)$ and $\mathcal{Q} \subset \mathcal{B}$ is a class adapted for $F(M, -)$, there exist canonical isomorphisms in $D(\mathcal{C})$:

$$\mathrm{Tot}(F(P^\bullet, N^\bullet)) \cong \mathrm{Tot}(F(M^\bullet, Q^\bullet)) \cong \mathrm{Tot}(F(P^\bullet, Q^\bullet))$$

- Cite [SGA 4, XVII, 1.2.7] to justify that these isomorphisms are well-defined because complexes of adapted objects are "deployed".
- **Incorporate the "Everywhere Defined" Logic**
  - Include the "Finite Dimension" case from [SGA 4, XVII, Prop 1.2.10] to explain the conditions under which LF and RF can be extended to the unbounded derived category $D(\mathcal{A})$.
- **Technical Refinements**
  - Provide a precise proof or reference for the property: "If F is bi-right-exact, then projective objects are left-acyclic (adapted) for F."
  - Define the term "Deployed in the first variable": A complex $P^\bullet$ is deployed in the first variable if the family $(P^\bullet, N^\bullet)$ is deployed for the multifunctor F for every $N^\bullet \in \mathrm{K}(\mathcal{A}_2)$.
  - Resolve the ♠ TODO regarding the existence of limits/colimits in `Sets` to ensure Ind/Pro categories are locally small.
- **Categorical Equivalence**
  - Add a Remark confirming the equivalence of categories $D(\mathcal{A}_1) \times D(\mathcal{A}_2) \simeq D(\mathcal{A}_1 \times \mathcal{A}_2)$, noting that quasi-isomorphisms are defined component-wise.
- **derived tensor product**
  - Do proposition 4.1.7. expose XVII

)

8. SIMPLICIAL CATEGORIES

(♠ TODO:)

Notation 8.0.1. Let **Cat** denote the category of (small) categories. Let Δ denote the simplex category, whose objects are finite nonempty totally ordered sets $[n] := \{0, 1, \ldots, n\}$ for $n \geq 0$, and whose morphisms are all order-preserving functions $\theta : [m] \to [n]$. Let Δ^{op} denote its opposite category.

Definition 8.0.2. Let V be a real vector space of finite dimension. A *k-simplex in topology* (or a *geometric k-simplex*) is the convex hull of $k + 1$ affinely independent points $v_0, v_1, \ldots, v_k \in V$, and is denoted by

$$[v_0, v_1, \ldots, v_k] := \left\{ \sum_{i=0}^{k} t_i v_i \ \middle| \ t_i \geq 0, \ \sum_{i=0}^{k} t_i = 1 \right\}.$$

It is also standard to talk of the *standard topological n-simplex* — the topological space $|\Delta^n|$ defined as the subset of Euclidean space $\mathbb{R}^{n+1}$ given by

$$|\Delta^n| = \left\{ (t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} t_i = 1, \text{ and } t_i \geq 0 \text{ for all } i \right\}$$

equipped with the induced topology from the usual Euclidean topology on $\mathbb{R}^{n+1}$.

(♠ TODO: comment on how $|\Delta^n|$ makes sense via a geometric realization)

Definition 8.0.3. The *simplex category*, or *nonempty finite ordinal category*, denoted by Δ, is the locally small (Definition 1.0.9) category (Definition 1.0.1) whose

- objects are the finite nonempty totally ordered sets $[n] := \{0, 1, 2, \ldots, n\}$ for each integer $n \geq 0$;
- morphisms are all order-preserving (non-decreasing) functions (Definition A.0.2) $\theta : [m] \to [n]$.

Composition in Δ is given by composition of functions.

Definition 8.0.4. Let $\mathcal{C}$ be a category (Definition 1.0.1).

1. The *category of simplicial objects in $\mathcal{C}$*, commonly denoted by notations such as $\mathbf{s}\mathcal{C}$, $\text{Simp}\,\mathcal{C}$, $\mathcal{C}_\Delta$, or $(\Delta^{\text{op}})^{\mathcal{C}}$ (cf. Definition 1.2.1), or $\Delta^{\text{op}}\mathcal{C}$, is the functor category (Definition 1.2.1)

$$\mathbf{s}\mathcal{C} := \mathbf{Fun}(\Delta^{\text{op}}, \mathcal{C}).$$

(Definition 1.0.12) (Definition 8.0.3) In particular, a morphism between objects $X, Y : \Delta^{\text{op}} \to \mathcal{C}$ in this category is a natural transform $X \Rightarrow Y$ from X to Y as functors.

An object X of $\mathbf{s}\mathcal{C}$ is called a *simplicial object of $\mathcal{C}$*, and, by definition (Definition 1.1.1), consists of a family of objects $\{X_n\}_{n \geq 0}$ in $\mathcal{C}$ together with morphisms

$$X(\theta) : X_n \to X_m, \quad \text{for each } \theta : [m] \to [n] \text{ in } \Delta,$$

satisfying the functoriality conditions

$$X(\text{id}_{[n]}) = \text{id}_{X_n}, \qquad X(\theta \circ \psi) = X(\psi) \circ X(\theta).$$

2. Dually, the *category of cosimplicial objects in $\mathcal{C}$*, commonly denoted by notatoins such as $\mathbf{c}\mathcal{C}$, $\text{Cosimp}\,\mathcal{C}$, or $\Delta^{\mathcal{C}}$ (cf. Definition 1.2.1) is the functor category

$$\mathbf{c}\mathcal{C} := \mathbf{Fun}(\Delta, \mathcal{C}).$$

In particular, a morphism between objects $X, Y : \Delta \to \mathcal{C}$ in this category is a natural transform $X \Rightarrow Y$ from X to Y as functors.

An object Y of $\mathbf{c}\mathcal{C}$ is called a *cosimplicial object of $\mathcal{C}$*, and consists of a family of objects $\{Y^n\}_{n \geq 0}$ in $\mathcal{C}$ together with morphisms

$$Y(\theta) : Y^m \to Y^n, \quad \text{for each } \theta : [m] \to [n] \text{ in } \Delta,$$

satisfying the functoriality conditions

$$Y(\text{id}_{[n]}) = \text{id}_{Y^n}, \qquad Y(\theta \circ \psi) = Y(\theta) \circ Y(\psi).$$

For instance, a *(co)simplicial set, group, topological space, ring, etc.* refers to a (co)simplicial object in the category of sets, of groups, of topological spaces, of rings, etc. and such categories are denoted by notations such as $\mathbf{Sets}_\Delta$, $\mathbf{Grps}_\Delta$, $\mathbf{Top}_\Delta$, $\mathbf{Rings}_\Delta$, etc. Accordingly, a (co)simplicial map between such (co)simplicial objects refers to a morphism in the appropriate (co)simplicial category.

If $\mathcal{C}$ is locally small (Definition 1.0.9), then both $\mathbf{s}\mathcal{C}$ and $\mathbf{c}\mathcal{C}$ are locally small as well.

8.0.1. *Geometric realization and singular complex adjunction.*

Definition 8.0.5. Let S be a simplicial set (Definition 8.0.4). The *geometric realization of S*, denoted $|S|$, is the compactly generated (Definition 9.3.1) Hausdorff (Definition E.0.18) topological space obtained as the coend (♠ TODO: coend)

$$|S| \;=\; \int^{[n]\in\Delta} S_n \times |\Delta^n|,$$

(Definition 8.0.11) where $|\Delta^n|$ is the standard topological n-simplex (Definition 8.0.2), and the equivalence relation is generated by the simplicial structure maps in S and the face and degeneracy maps of the simplices $|\Delta^n|$. Concretely, $|S|$ is given by the quotient space

$$\left(\coprod_{n\geq 0} S_n \times |\Delta^n| \right) \Big/ \sim,$$

where for every morphism $\varphi : [m] \to [n]$ in the simplex category Δ, the identification

$$(s, \varphi^*(x)) \sim (\varphi_*(s), x)$$

holds for all $s \in S_n$ and $x \in |\Delta^m|$, with $\varphi^* : |\Delta^m| \to |\Delta^n|$ induced by φ, and $\varphi_* : S_n \to S_m$ the simplicial map induced by φ. This construction endows $|S|$ with the quotient topology from the disjoint union.

Definition 8.0.6. Let $\mathcal{C}$ be a topological space (Definition B.0.1). The *singular complex of $\mathcal{C}$*, denoted $\mathrm{Sing}(\mathcal{C})$, is the simplicial set (Definition 8.0.4) defined as follows:

- For each integer $n \geq 0$, the set of n-simplices (Definition 8.0.11) is the set of continuous maps (Definition B.0.3) from the standard topological n-simplex $|\Delta^n|$ (Definition 8.0.2) to $\mathcal{C}$:
$$\mathrm{Sing}(\mathcal{C})_n = \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, \mathcal{C}).$$

- The face and degeneracy maps (Definition 8.0.11) of $\mathrm{Sing}(\mathcal{C})$ are induced by the standard coface and codegeneracy maps of the simplices $|\Delta^n|$ in the usual way, making $\mathrm{Sing}(\mathcal{C})$ a simplicial set.

Proposition 8.0.7.   1. The geometric realization functor (Definition 8.0.5) and the singular complex functor (Definition 8.0.6) form an adjoint pair (Definition 1.1.13)

$$|\cdot| \dashv \mathrm{Sing}$$

$$|\cdot| : \mathbf{sSets} \rightleftarrows \mathrm{Top} : \mathrm{Sing}.$$

(Definition 8.0.4) which infact restrict to an adjoint pair

$$|\cdot| : \mathbf{sSets} \rightleftarrows \mathcal{CG}\mathrm{Haus} : \mathrm{Sing}.$$

where $\mathcal{CG}\mathrm{Haus}$ is the category of compactly generated (Definition 9.3.1) Hausdorff (Definition E.0.18) topological spaces (Definition B.0.1).

2. Moreover, for every topological space X, the counit map (Definition 1.1.13)

$$|\operatorname{Sing} X| \to X$$

is a weak homotopy equivalence (Definition B.0.7) of topological spaces.

Theorem 8.0.8 (Quillen)**.** The unit and counit morphisms (Definition 1.1.13)

$$S \to \operatorname{Sing} |S|$$
$$|\operatorname{Sing} X| \to X$$

for simplicial sets (Definition 8.0.4) S and topological spaces X are weak homotopy equivalences (of simplicial sets (Definition 8.1.5) and of topological spaces (Definition B.0.7) respectively).

8.0.2. *Homotopy groups of a Kan Simplicial Set.*

Definition 8.0.9 (Basepoint of a Simplicial Set)**.** Let X be a simplicial set (Definition 8.0.4). A *(base)point* or *basepoint vertex* of X is a chosen element

$$x \in X_0.$$

A *pointed simplicial set* is a simplicial set equipped with a choice of base point.

Definition 8.0.10 (Homotopy Groups of a Kan Simplicial Set)**.** Let X be a Kan simplicial set (Definition 8.2.6), and fix a basepoint (Definition 8.0.9) $x \in X_0$.

For each integer $n \geq 1$, define the *n-th homotopy group of X at x*, denoted $\pi_n(X, x)$, as the set of homotopy classes of maps (♠ TODO: homotopy, boundary of Δ^n)

$$f : \partial\Delta^{n+1} \to X$$

that restrict to the constant map at x on the basepoint simplex, modulo homotopies relative to the boundary.

Equivalently, $\pi_n(X, x)$ can be described as the set of equivalence classes of n-simplices (Definition 8.0.4) whose faces are degenerate at x, with composition induced by the combinatorial structure of simplices.

These sets carry natural group structures for $n \geq 1$, with $\pi_1(X, x)$ being the fundamental group and $\pi_n(X, x)$ for $n \geq 2$ being abelian groups.

8.0.3.

Definition 8.0.11. Let $\mathcal{C}$ be a category (Definition 1.0.1).

1. Let X be a simplicial object (Definition 8.0.4) in $\mathcal{C}$. An object $X_n := X([n])$ of $\mathcal{C}$ is called the *n-simplices of X*. In case that $\mathcal{C}$ is some kind of category of sets, an element of X_n is called an *n-simplex of X*, so X_n is the *set of n-simplices of X*. In this case, a *vertex of X* moreover refers to a 0-simplex of X and a *edge of X* refers to a 1-simplex of X.

For each morphism $\theta : [m] \to [n]$ in Δ, the induced morphism

$$X(\theta) : X_n \to X_m$$

in $\mathcal{C}$ is called a *simplicial morphism*.

For each $0 \le j \le n$, the *jth face map of the n-simplicies* refers to the map

$$d_j = X(p_j) : X_n \to X_{n-1}, \quad p_j : [n-1] \to [n], \quad p_j(i) = \begin{cases} i & \text{if } i < j \\ i+1 & \text{if } i \ge j \end{cases}.$$

d_j is also denoted by ∂_j.

For each $0 \le j \le n$, the *jth degeneracy map of the n-simplicies* refers to the map

$$s_j = X(q_j) : X_n \to X_{n+1}, \quad q_j : [n+1] \to [n], \quad q_j(i) = \begin{cases} i & \text{if } i \le j \\ i-1 & \text{if } i > j \end{cases}.$$

2. Let $Y : \Delta \to \mathcal{C}$ be a cosimplicial object (Definition 8.0.4) of $\mathcal{C}$. An object $Y^n := Y([n])$ of $\mathcal{C}$ is called the *n-cosimplicies of Y*. In case that $\mathcal{C}$ is some kind of category of sets, an element of Y_n is called an *n-cosimplex of Y*, so Y^n is the *set of n-cosimplices of Y*.

.

For each morphism $\theta : [m] \to [n]$ in Δ, the induced morphism

$$Y(\theta) : Y^m \to Y^n$$

is called a *cosimplicial morphism*.

For each $0 \le j \le n$, the *jth coface map of the n-cosimplicies* refers to the map

$$d^j = Y(p_j) : Y^n \to Y^{n+1}, \quad p_j : [n] \to [n+1], \quad p_j(i) = \begin{cases} i & \text{if } i < j \\ i+1 & \text{if } i \ge j \end{cases}$$

d^j is also denoted by ∂^j.

For each $0 \le j \le n$, the *jth codegeneracy map of the n-cosimplicies* refers to the map

$$s^j = Y(q_j) : Y^n \to Y^{n-1}, \quad q_j : [n] \to [n-1], \quad q_j(i) = \begin{cases} i & \text{if } i \le j \\ i-1 & \text{if } i > j \end{cases}$$

A *(co)face/degeneracy of a the n-(co)simplicies of a (co)simplicial object* may also refer to the images of the (co)face/degeneracy maps.

Definition 8.0.12. Let $n \ge 0$ be an integer. The representable functor (Definition 4.0.6)

$$\mathrm{Hom}_\Delta(-, [n]) : \Delta^{\mathrm{op}} \to \mathbf{Sets}$$

(Definition 8.0.3) (Definition 1.0.12) (Definition 1.0.3) is a simplicial set (Definition 8.0.4) often denoted by Δ^n, $\Delta^{[n]}$, or $\Delta[n]$, and is called the *standard n-simplex*. More generally, if J is a finite nonempty linearly ordered set, then we may speak of the simplicial set Δ^J given by the representable functor $\mathrm{Hom}_\Delta(-, J)$.

The *Standard simplicial n-simplex functor* refers to the functor

$$\Delta^\bullet : \Delta \to \mathbf{sSets}$$

given by $[n] \mapsto \Delta^n$. By construct, note that $\Delta^\bullet$ is a cosimplicial object (Definition 8.0.4) in the category of simplicial sets.

Dually, the functor

$$\mathrm{Hom}_\Delta([n], -) : \Delta \to \mathbf{Sets}$$

is a cosimplicial set (Definition 8.0.4) called the *standard n-cosimplex*.

0-simplices of a simplicial set may be regarded like objects in a category and 1-simplices between such objects may be regarded like morphisms between the objects — however, unless the simplicial set satisfies more conditions, composition of morphisms may not exist and may not be unique.

Definition 8.0.13. Let X be a simplicial set (Definition 8.0.4).

1. An *object of X* refers to a vertex of X (Definition 8.0.11), i.e. a 0-simplex of X (Definition 8.0.11). We may write $x \in X$ or $x \in \mathrm{Ob}\, X$ to denote that x is an object of X.
2. A *morphism* $x \to y$ where $x, y : \Delta^0 \to X$ (Definition 8.0.12) (Lemma 8.0.17) are objects of X is an edge (Definition 8.0.11) $f : \Delta^1 \to S$ such that $d_1(f) = x$ and $d_0(f) = y$, or equivalently $f(0) = x$ and $f(1) = y$. We may write such a morphism by $f : x \to y$.
3. Given an object x of X, the *identity morphism of x* is the edge $\mathrm{id}_X = s_0(X) : \Delta^1 \to X$.
4. Let $f : x \to y$ and $g : y \to z$ be morphisms in X. A *composition of f and g* (which may not exist and may not be unique) is a 2-simplex (Definition 8.0.11) $F : \Delta^2 \to S$ such that
$$d_2(F) = f, \quad \text{and} \quad d_0(F) = g.$$
(Definition 8.0.11). The composite of f and g with respect to the composition F is then the morphism $d_1(F) : x \to z$.

   In particular, compositions of morphisms exist whenever X is a quasi-category (Definition 8.2.2) and are unique whenever X satisfies the conditions of Proposition 8.2.3, i.e. when X is isomorphic to the nerve (Definition 8.0.20) of a small category. (♠ TODO: comment on how different composites of f and g become equivalent in the homotopy category of X in case that X is a quasi-category.)

Definition 8.0.14 (Functor between quasi-categories)**.** Let C and D be ∞-categories/quasi-categories (Definition 8.2.2). A *functor of quasi-categories from C to D* is a simplicial map (Definition 8.0.4) $F \colon C \to D$.

Equivalently, a functor $C \to D$ is a 0-simplex (Definition 8.0.11) of the simplicial mapping object D^C (Definition 8.0.22).

Definition 8.0.15 (Simplicial category of quasi-categories)**.** (♠ TODO: read this and make it more precise) Let $\mathbf{QCat}_\mathcal{U}^\Delta$ be the simplicial category defined as follows. Its objects are the $\mathcal{U}$-small quasi-categories. For objects C, D, the mapping space is the Kan complex

$$\mathrm{Map}_{\mathbf{QCat}_\mathcal{U}^\Delta}(C, D) \coloneqq (D^C)^\simeq,$$

where D^C is the exponential simplicial set (the functor simplicial set) and $(-)^\simeq$ denotes the largest Kan subcomplex. Composition

$$\mathrm{Map}(C, D) \times \mathrm{Map}(D, E) \longrightarrow \mathrm{Map}(C, E)$$

is induced by the exponential composition $E^D \times D^C \to E^C$, and identities by the units $C \to C$ under the exponential adjunction; these maps restrict to Kan subcomplexes and equip $\mathbf{QCat}_{\mathcal{U}}^\Delta$ with a simplicial category structure.

Definition 8.0.16 (Infinity category of infinity categories)**.** (♠ TODO: reformulate this definition) Let $\mathcal{U}$ be a Grothendieck universe. The ∞-*category of* ∞-*categories* (of $\mathcal{U}$-small quasi-categories) is the quasi-category (Definition 8.2.2)

$$\mathrm{Cat}_\infty^{\mathcal{U}} := N_\Delta\big(\mathbf{QCat}_{\mathcal{U}}^\Delta\big).$$

(Definition 8.0.15) Concretely, the objects of $\mathrm{Cat}_\infty^{\mathcal{U}}$ are $\mathcal{U}$-small quasi-categories, and for C, D the mapping space in $\mathrm{Cat}_\infty^{\mathcal{U}}$ is equivalent to $(D^C)^\simeq$, the largest Kan (Definition 8.2.6) subcomplex of the exponential simplicial set D^C (Definition 8.0.22), so that 0-simplices correspond to functors $C \to D$ and higher simplices encode their coherent homotopies.

Lemma 8.0.17. 1. Let X be a simplicial set (Definition 8.0.4). The n-simplices (Definition 8.0.11) X^n is naturally isomorphic to $\mathrm{Hom}_{\mathrm{Simp\,\mathbf{Sets}}}(\Delta^n, X)$ (Definition 8.0.12).
2. Let Y be a cosimplicial set (Definition 8.0.4). The n-cosimplicies (Definition 8.0.11) Y^n is naturally isomorphic to $\mathrm{Hom}_{\mathrm{Cosimp\,\mathbf{Sets}}}(\mathrm{Hom}_\Delta([n], -), Y)$ (Definition 8.0.12).

Proof. This is a consequence of the Yoneda lemma (Theorem 4.0.7). □

Definition 8.0.18. 1. Let X and Y be simplicial sets (Definition 8.0.4), that is, functors $X, Y : \Delta^{\mathrm{op}} \to \mathbf{Set}$. A *simplicial subset of X* is a simplicial set Y together with a family of injective (Definition A.0.3) maps

$$\iota_n : Y_n \hookrightarrow X_n \quad (n \geq 0)$$

that form a natural transformation (Definition 1.1.3) $\iota : Y \Rightarrow X$. Equivalently, Y is a simplicial subset of X if and only if for every morphism $\theta : [m] \to [n]$ in Δ, the following diagram commutes:

$$
\begin{array}{ccc}
Y_n & \xrightarrow{Y(\theta)} & Y_m \\
\downarrow \iota_n & & \downarrow \iota_m \\
X_n & \xrightarrow{X(\theta)} & X_m
\end{array}
$$

and each ι_n is injective.
2. Let X and Y be cosimplicial sets, that is, functors $X, Y : \Delta \to \mathbf{Set}$. A *cosimplicial subset* of X is a cosimplicial set Y together with a family of injective maps

$$\iota^n : Y^n \hookrightarrow X^n \quad (n \geq 0)$$

that form a natural transformation $\iota : Y \Rightarrow X$. Equivalently, Y is a cosimplicial subset of X if and only if for every morphism $\theta : [m] \to [n]$ in Δ, the following

diagram commutes:

$$
\begin{array}{ccc}
Y^m & \xrightarrow{Y(\theta)} & Y^n \\
\downarrow \iota^m & & \downarrow \iota^n \\
X^m & \xrightarrow{X(\theta)} & X^n
\end{array}
$$

and each ι^n is injective.

Definition 8.0.19. Let $n \geq 1$ be an integer and $\Delta[n]$ be the standard n-simplex. For each i with $0 \leq i \leq n$, the *ith horn of the nth standard simplex*, denoted by $\Lambda^i[n]$, is the simplicial subset of $\Delta[n]$ (Definition 8.0.12) defined as follows: the set of k-simplices (Definition 8.0.11) of $\Lambda^i[n]$ is

$$\Lambda^i[n]_k := \{\, \theta \in \mathrm{Hom}_\Delta([k],[n]) \mid \mathrm{Im}(\theta) \text{ does not contain all of } [n] \setminus \{i\} \,\}.$$

Equivalently,

$$\Lambda^i[n] = \bigcup_{j \neq i} \mathrm{Im}(d^j : \Delta[n-1] \to \Delta[n]),$$

where each d^j is the coface map (Definition 8.0.11) in the simplex category corresponding to the injective order-preserving inclusion $[n-1] \hookrightarrow [n]$ that omits j.

Definition 8.0.20. Let $\mathcal{C}$ be a (large) category (Definition 1.0.1). The *nerve of $\mathcal{C}$* is the simplicial "class" (Definition 8.0.4) $N(\mathcal{C})$ given as follows: for $n \geq 0$, the n-simplicies (Definition 8.0.11) $N(\mathcal{C})_n$ is the class of all functors $[n] \to \mathcal{C}$ (Definition 8.0.3). In other words, $N(\mathcal{C})_n$ is the class of all composable sequences

$$C_0 \xrightarrow{f_1} C_1 \ldots \xrightarrow{f_n} C_n$$

of morphisms of length n. The face map (Definition 8.0.11) $d_i : N(\mathcal{C})_n \to N(\mathcal{C})_{n-1}$ carries the above sequence to

$$C_0 \xrightarrow{f_1} C_1 \ldots \xrightarrow{f_{i-1}} C_{i-1} \xrightarrow{f_{i+1} \circ f_i} C_{i+1} \xrightarrow{f_{i+2}} \ldots \xrightarrow{f_n} C_n$$

while the degeneracy s_i carries it to

$$C_0 \xrightarrow{f_1} C_1 \ldots \xrightarrow{f_i} C_i \xrightarrow{id_{C_i}} C_i \xrightarrow{f_{i+1}} C_{i+1} \xrightarrow{f_{i+2}} \ldots \xrightarrow{f_n} C_n.$$

If $\mathcal{C}$ is a small category (Definition 1.0.9), then $N(\mathcal{C})$ is a simplicial set (Definition 8.0.4).

Note that the category $\mathcal{C}$ can be recovered, roughly up to isomoprhism, from its nerve $N(\mathcal{C})$. The objects of $\mathcal{C}$ are simply the objects of $N(\mathcal{C})_0$. The morphisms $C_0 \to C_1$ are given by objects $\phi \in N(\mathcal{C})_1$ such that $d_1(\phi) = C_0$ and $d_0(\phi) = C_1$. The identity morphism $C \to C$ is given by the degenerate simplex $s_0(C)$. Moreover, given a diagram $C_0 \xrightarrow{\phi} C_1 \xrightarrow{\psi} C_2$, the object $N(\mathcal{C})_1$ corresponding to $\psi \circ \phi : C_0 \to C_2$ is uniquely characterized by the 2-simplex $\sigma \in N(\mathcal{C})_2$ such that $d_2(\sigma) = \phi$, $d_0(\sigma) = \psi$, and $d_1(\sigma) = \psi \circ \phi$.

Simplicial categories provide one framework for ∞-categories.

Definition 8.0.21. A *simplicial category* is a category enriched over (Definition 4.0.1) the category of simplicial sets (Definition 8.0.4). Common notation for the category of simplicial categories include $\mathbf{Cat}_\Delta$ or $\mathbf{sCat}$. However, note that these would overlap with notation

for the category of simplicial objects of the category **Cat** (Definition 8.0.4). Further note that all simplicial categories are regardable as simplicial objects in **Cat**, but not vice versa. Unless otherwise specified, the notations here will take precedence; in other words, $\mathbf{Cat}_\Delta$ or **sCat** will mean the caetgory of simplicial categories unless otherwise stated.

Definition 8.0.22. The category **sSets** of simplicial sets (Definition 8.0.4) can be enriched with the structure of a simplicial category (Definition 8.0.21) (i.e. **sSets** can be regarded as a category enriched (Definition 4.0.1) in **sSets**) as follows: given simplicial sets X and Y, we let $\mathrm{Mor}(X, Y)$, which is also often denoted by notations such as Y^X and referred to as the *exponential simplicial set from X to Y* or the *simplicial mapping object from X to Y*

$$\mathrm{Mor}(X, Y)_n = \mathrm{Hom}_{\mathbf{sSets}}(X \times \Delta^n, Y)$$

(Definition 8.0.11) (Definition 8.0.12) (♠ TODO: discuss what the face and degeneracy maps do) and we let composition be given by the simplicial set morphisms (Definition 8.0.4)

$$\circ : \mathrm{Mor}(Y, Z) \times \mathrm{Mor}(X, Y) \to \mathrm{Mor}(X, Z)$$

where X, Y, Z are simplicial sets as follows:

1. The n-simplices of the product simplicial set $\mathrm{Mor}(Y, Z) \times \mathrm{Mor}(X, Y)$ is the product set $\mathrm{Mor}(Y, Z)_n \times \mathrm{Mor}(X, Y)_n$
2. Since the n-simplices of $\mathrm{Mor}(Y, Z)$ and the n-simplicies of $\mathrm{Mor}(X, Y)$ are respectively given by simplicial set morphisms

$$f : Y \times \Delta^n \to Z$$

$$g : X \times \Delta^n \to Y$$

respectively, the n-simplices of the product simplicial set are given by pairs (f, g) of such morphisms.
3. The composition $\circ(f, g)$ is the simplicial set morphism

$$X \times \Delta^n \xrightarrow{\mathrm{id}_X \times \Delta} X \times (\Delta^n \times \Delta^n) \cong (X \times \Delta^n) \times \Delta^n \xrightarrow{g, \mathrm{id}_{\Delta^n}} Y \times \Delta^n \xrightarrow{f} Z$$

where $\Delta : \Delta^n \to \Delta^n \times \Delta^n$ is the diagonal morphism of the standard n-simplex Δ^n.

This simplicial category may be called the *simplicial category of simplicial sets* and the same notations, such as **sSets** or $\mathbf{Sets}_\Delta$, used to denote the (ordinary) category of simplicial sets may be used to denote this simplicial category.

When the simplicial set Y is ∞-categories (Definition 8.2.2), it is also customary to denote $\mathrm{Mor}(X, Y)$ by $\mathrm{Fun}(X, Y)$, and to call this the *functor ∞-category from X to Y*, especially when both X and Y are ∞-categories; it is itself an ∞-category. See also Definition 8.0.14.

Definition 8.0.23. Let X be a simplicial set (Definition 8.0.4). Let $x, y \in X$ be objects (Definition 8.0.13). The *mapping space between x and y* is the simplicial set $\mathrm{Mor}(x, y)$ whose n-simplicies (Definition 8.0.11) are given by

$$\mathrm{Mor}(x, y)_n = \{F : \Delta^n \times \Delta^1 \to X : F|_{\Delta^n \times \{0\}} = x, F|_{\Delta^n \times \{1\}} = y\}.$$

(Definition 8.0.12) (♠ TODO: comment on how composition of morphisms becomes well defined up to homotopy; it seems reasonable that compositions to be well defined on the homotopy categories h Mor of the mapping spaces)

Definition 8.0.24. Let J be a finite nonempty linearly ordered set. Define the *simplicial path category* $\mathfrak{C}[\Delta^J]$ of Δ^J (Definition 8.0.12) as the following simplicial category (Definition 8.0.21):

- The objects of $\mathfrak{C}[\Delta^J]$ are the elements of J.
- If $i, j \in J$, then

$$\text{Hom}_{\mathfrak{C}[\Delta^J]}(i,j) = \begin{cases} \emptyset & \text{if } j < i \\ N(P_{i,j}) & \text{if } i \leq j \end{cases}$$

(Definition 8.0.20) where $P_{i,j}$ is the partially ordered set

$$P_{i,j} = \{I \subseteq J : (i, j \in I) \wedge (\forall k \in I)[i \leq k \leq j])\}.$$

- If $i_0 \leq i_1 \cdots i_n$, then the composition

$$\text{Hom}_{\mathfrak{C}[\Delta^J]}(i_0, i_1) \times \cdots \times \text{Hom}_{\mathfrak{C}[\Delta^J]}(i_{n-1}, i_n) \to \text{Hom}_{\mathfrak{C}[\Delta^J]}(i_0, i_n)$$

is induced by the map of partially ordered sets

$$P_{i_0, i_1} \times \ldots \times P_{i_{n-1}, i_n} \to P_{i_0, i_n}$$
$$(I_1, \ldots, I_n) \mapsto I_1 \cup \ldots \cup I_n.$$

The functor $\mathfrak{C} : \Delta \to \mathbf{Cat}_\Delta$ (Definition 8.0.3, Definition 8.0.21) extends uniquely (up to unique isomorphism) to a (small) colimit-preserving functor $\mathfrak{C} : \mathbf{Sets}_\Delta \to \mathbf{Cat}_\Delta$ (Definition 8.0.4, Lemma 9.3.8)

Definition 8.0.25. Let $\mathcal{C}$ be a simplicial category (Definition 8.0.21). The *(simplicial) nerve* $N(\mathcal{C})$ of $\mathcal{C}$ is the simplicial set (Definition 8.0.4) described by the formula

$$N(\mathcal{C})_n \cong \text{Hom}_{\mathbf{Sets}_\Delta}(\Delta^n, N(\mathcal{C})) = \text{Hom}_{\mathcal{C}at_\Delta}(\mathfrak{C}[\Delta^n], \mathcal{C}).$$

(Lemma 8.0.17, Definition 8.0.4, Definition 8.0.11, Definition 8.0.21, Definition 8.0.24) It is synonymously called the *homotopy coherent nerve of $\mathcal{C}$*.

Definition 8.0.26. Let Kan be the full subcategory (Definition 1.3.2) of **sSets** (Definition 8.0.4) spanned by the collection of Kan complexes (Definition 8.2.6). Regard Kan as a simplicial category (Definition 8.0.21). (♠ TODO: how exactly is Kan regarded as a simplicial category?) In the context of ∞-category theory, the simplicial nerve (Definition 8.0.25) of Kan is called the *∞-category of spaces*.

8.1. Model categories.

Definition 8.1.1 (Model Category)**.** (♠ TODO: I don't like the axioms as stated here) A *model category*, or synonymously a *closed model category*, is a complete and cocomplete category (Definition 1.3.9) $\mathcal{M}$ equipped with three distinguished classes of morphisms:

- *Weak equivalences* $\mathcal{W}$,
- *Fibrations* $\mathcal{F}$,
- *Cofibrations* $\mathcal{C}$,

1. (**Two-out-of-three**): For any composable morphisms $f : X \to Y$, $g : Y \to Z$, if any two of f, g, or $g \circ f$ lie in $\mathcal{W}$, then so does the third.

2. **(Retracts)**: Each of the classes $\mathcal{W}, \mathcal{F}, \mathcal{C}$ is closed under retracts in the arrow category $\mathcal{M}^2$. That is, if f is a retract of g and g belongs to one of these classes, then f also belongs to that class.

3. **(Lifting)**: Given any commutative square

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\
B & \longrightarrow & Y
\end{array}
$$

where $i \in \mathcal{C}$ and $p \in \mathcal{F}$, a diagonal filler (lift) exists making both triangles commute provided either

- i is also a weak equivalence (called an acyclic cofibration), or
- p is also a weak equivalence (called an acyclic fibration).

Formally, acyclic cofibrations have the left lifting property with respect to all fibrations, and cofibrations have the left lifting property with respect to all acyclic fibrations.

4. **(Factorization)**: Every morphism $f : X \to Y$ in $\mathcal{M}$ admits two functorial factorizations:

- $f = p \circ i$, where $i \in \mathcal{C}$ is a cofibration and $p \in \mathcal{F} \cap \mathcal{W}$ is an acyclic fibration.
- $f = q \circ j$, where $j \in \mathcal{C} \cap \mathcal{W}$ is an acyclic cofibration and $q \in \mathcal{F}$ is a fibration.

Here, an *acyclic fibration* (or *trivial fibration*) is a morphism in $\mathcal{F} \cap \mathcal{W}$, and an *acyclic cofibration* (or *trivial cofibration*) is a morphism in $\mathcal{C} \cap \mathcal{W}$.

Definition 8.1.2 (Proper Model Category). Let $\mathcal{M}$ be a model category (Definition 8.1.1) with classes of weak equivalences $\mathcal{W}$, cofibrations $\mathcal{C}of$, and fibrations $\mathcal{F}ib$. (♠ TODO: pushout, pullback)

1. Then $\mathcal{M}$ is called *left proper* if weak equivalences are preserved under pushouts along cofibrations, i.e., if for every pushout diagram

$$
\begin{array}{ccc}
A & \xrightarrow{i} & B \\
{\scriptstyle f \in \mathcal{W}}\downarrow & & \downarrow \\
C & \to & D
\end{array}
$$

with $i \in \mathcal{C}of$, the induced map $C \to D$ lies in $\mathcal{W}$.

2. Dually, $\mathcal{M}$ is called *right proper* if weak equivalences are preserved under pullbacks along fibrations, i.e., if for every pullback diagram

$$
\begin{array}{ccc}
P & \to & X \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle p \in \mathcal{F}ib} \\
Y & \xrightarrow{h \in \mathcal{W}} & Z
\end{array}
$$

the induced map $P \to Y$ is in $\mathcal{W}$.

3. A model category $\mathcal{M}$ is *proper* if it is both left proper and right proper.

Lemma 8.1.3. A model category (Definition 8.1.1) always has a initial object and a final object (Definition 2.0.6).

Proof. This is because a model category, by definition, is complete and cocomplete (Definition 1.3.9). □

Definition 8.1.4 (Fibrant Object)**.** Let $(\mathcal{C}, \mathcal{W}, \mathcal{C}of, \mathcal{F}ib)$ be a model category (Definition 8.1.1).

1. An object $X \in \mathcal{C}$ is called *fibrant* if the unique morphism
$$X \longrightarrow *$$
to the terminal object (Definition 2.0.6) (Lemma 8.1.3) $*$ of $\mathcal{C}$ is a fibration (i.e., belongs to $\mathcal{F}ib$).
2. An object $X \in \mathcal{C}$ is called *cofibrant* if the unique morphism
$$\emptyset \longrightarrow X$$
from the initial object (Definition 2.0.6) (Lemma 8.1.3) $\emptyset$ of $\mathcal{C}$ is a cofibration (i.e., belongs to $\mathcal{C}of$).

Definition 8.1.5 (Standard model category structure on simplicial sets)**.** Let sSet denote the category of simplicial sets (Definition 8.0.4). The *standard model category structure on* sSet, also known as the *Kan model structure* or the *Quillen model structure* or the *Kan-Quillen model structure*, is the model category (Definition 8.1.1) given by specifying three classes of morphisms:

- **Weak equivalences:** A map $f : X \to Y$ in sSet is a weak equivalence, also called a *weak homotopy equivalence between the simplicial sets X and Y*, if the induced map of geometric realizations (Definition 8.0.5) $|f| : |X| \to |Y|$ is a weak homotopy equivalence (Definition B.0.7) of topological spaces (i.e., induces isomorphisms on all homotopy groups for all choices of basepoint).
- **Fibrations:** A map $f : X \to Y$ is a *(Kan) fibration* if it has the right lifting property with respect to the horn inclusions $\Lambda_k^n \hookrightarrow \Delta^n$ for all $n \geq 1$ and $0 \leq k \leq n$.
- **Cofibrations:** A map $f : X \to Y$ is a cofibration if it is a monomorphism (Definition 1.3.7) (i.e., injective at each level).

For this model category structure, all objects are cofibrant (Definition 8.1.4), and the fibrant objects (Definition 8.1.4) are exactly the Kan complexes (Definition 8.2.6).

Definition 8.1.6 (Model Structure on $\mathcal{T}op$)**.** The category **Top** of topological spaces has a model structure (Definition 8.1.1), referred to such names as the *Quillen model structure (on the category of topological spaces)*, *classical model structure on the category of topological spaces*, *Serre model structure*, or *Quillen-Serre model structure*, given by the following three classes of morphisms
$$\mathcal{W}, \quad \mathcal{C}of, \quad \mathcal{F}ib$$
such that:

- $\mathcal{W}$ is the class of weak homotopy equivalences (Definition B.0.7). (♠ TODO: cofibrations, Serre fibrations of topological spaces)
- $\mathcal{C}of$ is the class of cofibrations: maps that are retracts of relative cell complexes (e.g., inclusions satisfying the homotopy extension property).
- $\mathcal{F}ib$ is the class of Serre fibrations: maps having the right lifting property with respect to all inclusions of disks into disks times the interval.

8.1.1. *Homotopy categories of model categories.*

Definition 8.1.7 (Homotopy category of a model category)**.** Let $\mathcal{M}$ be a model category (Definition 8.1.1). The *homotopy category of $\mathcal{M}$*, denoted by notations such as Ho($\mathcal{M}$), h($\mathcal{M}$), etc. , is the category (Definition 1.0.1) $\mathcal{M}[\mathcal{W}^{-1}]$ whose objects are those of $\mathcal{M}$, and whose morphisms are equivalence classes of morphisms in $\mathcal{M}$ under the relation of left and right homotopy, localized (Definition 7.2.2) at the weak equivalences. Explicitly,

- The objects of Ho($\mathcal{M}$) are the same as those in $\mathcal{M}$.
- For objects X, Y in $\mathcal{M}$, the morphism set $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X, Y)$ consists of maps in $\mathcal{M}$ modulo homotopy, with weak equivalences formally inverted.

Definition 8.1.8 (Homotopy category of the category of simplicial sets)**.** Let **sSets** denote the category of simplicial sets (Definition 8.0.4), equipped with its standard model structure (Definition 8.1.5). The *homotopy category of (the category of) simplicial sets*, denoted by notations such as Ho(**sSets**), h($\mathbf{Sets}_\Delta$), etc., is the homotopy category (Definition 8.1.7) of the standard model category of simplicial sets (Definition 8.1.5). In other words, It is the category whose objects are simplicial sets, and whose morphisms are homotopy classes of maps between simplicial sets, with weak equivalences inverted. That is,

- Objects: simplicial sets X.
- Morphisms: equivalence classes of maps $f : X \to Y$ under simplicial homotopy, where weak equivalences (maps inducing isomorphism on all homotopy groups) are made invertible.

The homotopy category of the category of simplicial sets is equivalent (Definition 1.1.8) to the category obtained from the category $\mathcal{CG}$ of compactly generated (Definition 9.3.1), weakly Hausdorff (Definition 9.3.2) topological spaces by inverting (Definition 7.2.2) weak homotopy equivalences (Definition B.0.7) of spaces (Corollary 8.1.9).

The homotopy category of simplicial sets should not be confused with the homotopy category of a simplicial set (Definition 8.1.11).

Corollary 8.1.9. The category obtained from the category $\mathcal{CG}$ of compactly generated (Definition 9.3.1), weakly Hausdorff (Definition 9.3.2) topological spaces by inverting (Definition 7.2.2) weak homotopy equivalences (Definition B.0.7) of spaces is equivalent (Definition 1.1.8) to the homotopy category of the category of simplicial sets (Definition 8.1.8).

Proof. This follows from Theorem 8.0.8. $\square$

Definition 8.1.10 ([Lur09, Before Definition 1.1.4.4])**.** Let $\mathcal{C}$ be a simplicial category (Definition 8.0.21). Let $\mathcal{H}$ be the homotopy category of the category of simplicial sets (Definition 8.1.8). The *homotopy category of $\mathcal{C}$* is the $\mathcal{H}$-enriched category (Definition 4.0.1) h$\mathcal{C}$ obtained by applying the localization functor h : $\mathbf{Sets}_\Delta \to \mathcal{H}$ (Definition 8.0.4) to the Hom's of $\mathcal{C}$. In other words, h$\mathcal{C}$ is the $\mathcal{H}$-enriched category category whose

1. Objects are the objects of $\mathcal{C}$
2. Hom's are given for $X, Y \in \mathrm{Ob}\, \mathrm{h}\mathcal{C}$ by

$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y) = \mathrm{h}\,\mathrm{Hom}_{\mathcal{C}}(X, Y).$$

(♠ TODO: homotopy category of the cateogry of simplicial sets, equivalently the homotopy category of the category of compactly generated spaces)

Definition 8.1.11. Let S be a simplicial set (Definition 8.0.4). Its *homotopy category* hS is defined to be the homotopy category (Definition 8.1.10) $h\mathfrak{C}[S]$ of the simplicial category (Definition 8.0.21) $\mathfrak{C}[S]$ (Definition 8.0.24).

The homotopy category of a simplicial set should not be confused with the homotopy category of the category of simplicial sets (Definition 8.1.8).

Proposition 8.1.12 ([Lur09, Proposition 1.2.3.1]). (♠ TODO:) There is an adjunction (Definition 1.1.13) $N \dashv h$ between the functor of nerves of a small categories (Definition 8.0.20) and the functor of homotopy categories (Definition 8.1.11) of simplicial sets (Definition 8.0.4)

$$N : \mathbf{Cat} \leftrightarrows \mathbf{Sets}_\Delta : h$$

(Definition 8.0.4) (where we ignore that enriched category structure (Definition 4.0.1) of hS for simplicial sets (Definition 8.0.4) S)

Definition 8.1.13 (Joyal Model Structure, [Lur09, Theorem 2.2.5.1]). (♠ TODO: This description seems a bit incomplete; it should be necessary to say that the model structure is the left proper combinatorial model structure with these properties.) The *Joyal model structure* on the category $s\mathbf{Set}$ of simplicial sets (Definition 8.0.4) is characterized by: (♠ TODO: fibrant/cofibrant objects)

- Fibrant objects (Definition 8.1.4) are exactly the quasi-categories (Definition 8.2.2),
- Cofibrations are the monomorphisms,
- Weak equivalences are categorical equivalences, i.e., maps inducing equivalences of ∞-categories.

This model category satisfies the axioms of a model category (Definition 8.1.1) and presents the homotopy theory of $(\infty, 1)$-categories.

Definition 8.1.14 (Model for the Homotopy Theory of $(\infty, 1)$-Categories). A model category (Definition 8.1.1) $\mathcal{M}$ is called a *model for the homotopy theory of $(\infty, 1)$-categories* if:

- The fibrant objects of $\mathcal{M}$ are equivalent (under suitable equivalences) to $(\infty, 1)$-categories,
- The weak equivalences correspond to equivalences of $(\infty, 1)$-categories,
- $\mathcal{M}$ is Quillen equivalent to other established models of ∞-categories (e.g., simplicial categories (Definition 8.0.21), Segal categories, complete Segal spaces). (♠ TODO: Quillen equivalent)

8.2. **Quasi-categories.** Quasi-categories provide one framework for ∞-categories.

Definition 8.2.1. Let K be a simplicial set (Definition 8.0.4). We may say that K *satisfies the Kan condition/extension condition* for the horn (Definition 8.0.19) $\Lambda_i^n \subset \Delta^n$ ((Definition 8.0.12)) if any morphism $\lambda_i^n \to K$ of simplicial sets admits an extension $f : \Delta^n \to K$.

Definition 8.2.2. A *quasi-category* is a simplicial set (Definition 8.0.4) K satisfying the Kan extension condition (Definition 8.2.1) for the horns $\Lambda_i^n \subset \Delta^n$ for all n and all $0 < i < n$. More explicitly, for any $0 < i < n$, any map $f_0 : \Lambda_i^n \to K$ (Definition 8.0.19) admits an extension $f : \Delta^n \to K$ (Definition 8.0.12).

In ∞-category theory, a quasi-category is synonymously referred to as an *$(\infty, 1)$-category*. Moreover, an *∞-category*, without further qualifications, often refers to a quasi-category.

A basic class of quasi-categories (Definition 8.2.2) would be nerves of categories (Definition 8.0.20) (Proposition 8.2.3).

Proposition 8.2.3 (see [Lur09, Proposition 1.1.2.2]). Let K be a simplicial set (Definition 8.0.4). Then the following conditions are equivalent:

- There exists a small category (Definition 1.0.9) $\mathcal{C}$ and an isomorphism $K \simeq N(\mathcal{C})$ (Definition 8.0.20).
- For each $0 < i < n$ and each diagram

$$
\begin{array}{ccc}
\Lambda_i^n & \longrightarrow & K \\
\downarrow & \nearrow & \\
\Delta^n & &
\end{array}
$$

there exists a *unique* dotted arrow rendering the diagram commutative.

Corollary 8.2.4. The nerve $N(\mathcal{C})$ (Definition 8.0.20) of any small category (Definition 1.0.9) is a quasi-category (Definition 8.2.2).

Proof. This is immediate by the definition of a quasi-category and Proposition 8.2.3. $\qquad\square$

Definition 8.2.5. An *∞-groupoid* is an ∞-category (Definition 8.2.2) whose homotopy category (Definition 8.1.11) $h\mathcal{C}$ is a groupoid, i.e. every morphism in $\mathcal{C}$ is an equivalence. (♠ TODO: objects and morphisms, equivalence in an ∞-category, which is a simplicial set) An ∞-groupoid is also often referred to as an *$(\infty, 0)$-category*.

∞-groupoids are equivalent to Kan-complexes (Definition 8.2.6) (Proposition 8.2.8).

An ∞-groupoid may be referred to as an *anima* (in the Clausen-Scholze termionology), especially if the ∞-groupoid is of the form $\mathrm{Mor}_{\mathcal{C}}(x, y)$ (Definition 8.0.23) for an ∞-category (Definition 8.2.2) $\mathcal{C}$ and objects (Definition 8.0.13)/0-simplicies (Definition 8.0.11) x, y of $\mathcal{C}$, cf. Proposition 8.2.9.

Definition 8.2.6. A *Kan complex* or a *Kan simplicial set* is a simplicial set (Definition 8.0.4) K satisfying the Kan extension condition (Definition 8.2.1) for the horns $\Lambda_i^n \subset \Delta^n$ for all n and all $0 \leq i < n$. More explicitly, for any $0 \leq i \leq n$, any map $f_0 : \Lambda_i^n \to K$ (Definition 8.0.19) admits an extension $f : \Delta^n \to K$ (Definition 8.0.12).

Kan-complexes are equivalent to ∞-groupoids (Definition 8.2.5) (Proposition 8.2.8).

Lemma 8.2.7. Every Kan complex (Definition 8.2.6) is a quasi-category (Definition 8.2.2).

Proof. This is immediate by definition. □

Proposition 8.2.8 ([Joy02, Corollary 1.4], cf. [Lur09, Proposition 1.2.5.1]). Let $\mathcal{C}$ be a simplicial set (Definition 8.0.4). The following are equivalent:

1. $\mathcal{C}$ is an ∞-groupoid (Definition 8.2.5).
2. $\mathcal{C}$ satisfies by Kan extension condition (Definition 8.2.1) for all horns (Definition 8.0.19) $\Lambda_i^n \subseteq \Delta^n$ (Definition 8.0.12) for all $0 \leq i < n$.
3. $\mathcal{C}$ satisfies by Kan extension condition (Definition 8.2.1) for all horns (Definition 8.0.19) $\Lambda_i^n \subseteq \Delta^n$ (Definition 8.0.12) for all $0 < i \leq n$.
4. $\mathcal{C}$ is a Kan complex (Definition 8.2.6).

Proposition 8.2.9. Let X be a quasi-category (Definition 8.2.2) and let $x, y \in X$ be objects (Definition 8.0.13). The mapping space $\mathrm{Mor}(x, y)$ (Definition 8.0.23) is an ∞-groupoid (Definition 8.2.5).

9. HIGHER CATEGORY THEORY

9.1. 2-categories.

Definition 9.1.1. A *strict 2-category* consists of:

1. A collection $\mathrm{Ob}(\mathcal{C})$ of objects, referred to as the *0-cells*.
2. For each pair of objects $x, y \in \mathrm{Ob}(\mathcal{C})$, a category (Definition 1.0.1) $\mathrm{Hom}(x, y) = \mathrm{Hom}_{\mathcal{C}}(x, y)$.
   (a) The objects in this category are called the *1-morphisms* or *1-cells of* $\mathcal{C}$.
   (b) The morphisms in this category are called the *2-morphisms* or *2-cells of* $\mathcal{C}$.
3. For each object $x \in \mathrm{Ob}(\mathcal{C})$, an object
$$\mathrm{id}_x \in \mathrm{Hom}_{\mathcal{C}}(x, x)$$
   called the *identity object*,
4. For each triple of objects $x, y, z \in \mathrm{Ob}(\mathcal{C})$, A functor
$$\circ_{x,y,z} : \mathrm{Hom}_{\mathcal{C}}(x, y) \times \mathrm{Hom}_{\mathcal{C}}(y, z) \to \mathrm{Hom}_{\mathcal{C}}(x, z)$$
   (Definition 1.3.11) called *composition*

that satisfy the following:

1. **(Strict associativity)** For any quadruple of objects $w, x, y, z \in \mathrm{Ob}(\mathcal{C})$, the two compositions
$$\circ_{w,y,z} \circ (1 \times \circ_{w,x,y}), \qquad \circ_{w,x,z} \circ (\circ_{x,y,z} \times 1)$$
   are *equal* as functors (♠ TODO: what is 1?)
$$\mathrm{Hom}_{\mathcal{C}}(y, z) \times \mathrm{Hom}_{\mathcal{C}}(x, y) \times \mathrm{Hom}_{\mathcal{C}}(w, x) \to \mathrm{Hom}_{\mathcal{C}}(w, z).$$

2. **(Strict unit laws)** for all $x, y \in \mathrm{Ob}(\mathcal{C})$,
$$\circ_{x,x,y}(\mathrm{id}_x, -) = 1_{\mathrm{Hom}_{\mathcal{C}}(x,y)}, \qquad \circ_{x,y,y}(-, \mathrm{id}_y) = 1_{\mathrm{Hom}_{\mathcal{C}}(x,y)}.$$
   That is, composing with an identity morphism acts as the identity functor.

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Roughly speaking, a strict 2-category is a strict n-category (Definition 9.2.2) for $n = 2$, at least when a strict 0-category is understood to be a set.

Definition 9.1.2. A *bicategory* or a *weak 2-category* $\mathcal{B}$ consists of the following data:

1. A collection of *0-cells* (or *objects*), denoted by $A, B, C, \ldots$;
2. For each pair of 0-cells A, B, a category (Definition 1.0.1) $\mathcal{B}(A, B)$, whose objects are called *1-cells* (denoted $f : A \to B$) and whose morphisms are called *2-cells* (denoted $\alpha : f \Rightarrow g$);
3. For each triple of 0-cells A, B, C, a composition functor

$$\odot_{ABC} : \mathcal{B}(B, C) \times \mathcal{B}(A, B) \to \mathcal{B}(A, C),$$

   denoted by $(g, f) \mapsto g \odot f$;
4. For each 0-cell A, an identity 1-cell $I_A \in \mathcal{B}(A, A)$;
5. For each quadruple of 0-cells A, B, C, D and 1-cells $f : A \to B$, $g : B \to C$, $h : C \to D$, a natural isomorphism called the *associator*,

$$a_{h,g,f} : (h \odot g) \odot f \xrightarrow{\cong} h \odot (g \odot f);$$

6. For each pair of 0-cells A, B and 1-cell $f : A \to B$, natural isomorphisms called the *left and right unitors*,

$$\lambda_f : I_B \odot f \xrightarrow{\cong} f \quad \text{and} \quad \rho_f : f \odot I_A \xrightarrow{\cong} f.$$

These must satisfy the following two coherence axioms:

1. **The Pentagon Identity:** For 1-cells f, g, h, k, the following diagram of 2-cells commutes:

2. **The Triangle Identity:** For 1-cells $f : A \to B$ and $g : B \to C$, the following diagram of 2-cells commutes:

Lemma 9.1.3. Any strict 2-category (Definition 9.1.1) is a weak 2-category (Definition 9.1.2).

Proof. In a strict 2-category, the strict associativity and the strict unit laws in particular yield associators and left/right unitors. $\qquad\square$

Definition 9.1.4. The *category of small categories* is the category defined by the following data:

- The objects are all small categories (Definition 1.0.9).
- The morphisms between two small categories $\mathcal{C}$ and $\mathcal{D}$ are the functors (Definition 1.1.1) $F : \mathcal{C} \to \mathcal{D}$.
- The composition of morphisms is the standard composition of functors.
- The identity morphism for each object $\mathcal{C}$ is the identity functor $1_{\mathcal{C}}$.

This category is a large category (Definition 1.0.1) and is denoted by **Cat**

Definition 9.1.5. Let $\mathcal{U}$ be a fixed Grothendieck universe (Definition E.0.1). The *category of categories (relative to $\mathcal{U}$)* or the *category of U-small categories* is the category defined by:

- The objects are all U-small (Definition 1.0.9) categories (Definition 1.0.1) $\mathcal{C}$.
- The morphisms are functors (Definition 1.1.1) between such categories.

This category is denoted by $U - \mathbf{Cat}$ or by $\mathbf{Cat}$, if U is understood.

In the case that Tarski-Grothendieck (Definition E.0.15) set theory is assumed (so in particular, the axiom of universes is assumed), one often adopts the convention of denoting $U_n - \mathbf{Cat}$ (Definition E.0.16) by $\mathbf{Cat}$ and $U_{n+1} - \mathbf{Cat}$ (Definition E.0.16) by $\mathbf{CAT}$. Thus, for $n = 1$, $\mathbf{Cat}$ would be the category of small categories, whereas $\mathbf{CAT}$ would serve as the "category of large categories" which consists of many common categories that are too large to be U_1-categories such as **Set**, **Grp**, **Top** or even **Cat**.

Definition 9.1.6. Let $\mathcal{U}$ be a fixed Grothendieck universe (Definition E.0.1). The category $U - \mathbf{Cat}$ (Definition 9.1.5) can be enriched with the structure of a strict 2-category (Definition 9.1.1) as follows:

- The objects/0-cells are all U-small categories (Definition 1.0.9).
- Given U-small categories $\mathcal{C}, \mathcal{D}$, the category $\mathrm{Hom}(\mathcal{C}, \mathcal{D}) = \mathrm{Hom}_{U-\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$ is the functor category (Definition 1.2.1) $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$; Recall that
  - the objects of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ are functors (Definition 1.1.1) $\mathcal{C} \to \mathcal{D}$
  - the morphisms between functors $F, G : \mathcal{C} \to \mathcal{D}$ are natural transformations (Definition 1.1.3) $F \Rightarrow G$.

  note that $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is a U-small category.
- The identity objects $\mathrm{id}_{\mathcal{C}} \in \mathrm{Ob}(\mathrm{Fun}(\mathcal{C}, \mathcal{C}))$ are given by the standard identity functors
- For every triple $\mathcal{C}, \mathcal{D}, \mathcal{E}$ of U-small categories, the composition functors

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \times \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

  are given by the standard composition of functors

Definition 9.1.7. A *pseudofunctor* $F : \mathcal{C} \to \mathcal{D}$ between weak 2-categories (Definition 9.1.2) $\mathcal{C}$ and $\mathcal{D}$ consists of:

- An assignment for each object $x \in \mathcal{C}$ to an object $Fx \in \mathcal{D}$.
- For each pair of objects $x, y \in \mathcal{C}$, a functor $F_{x,y} : \mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$ between the mapping categories.

- For each object $x \in \mathcal{C}$, a 2-isomorphism $F_{id_x} : id_{Fx} \Rightarrow F(id_x)$ called the *identity constraint*.
- For each pair of composable 1-morphisms $f : x \to y$ and $g : y \to z$ in $\mathcal{C}$, a 2-isomorphism $F_{g,f} : Fg \circ Ff \Rightarrow F(g \circ f)$ called the *composition constraint*, which is natural in f and g.

The identity 2-isomorphism is often denoted by $\phi_x : id_{Fx} \cong F(id_x)$ and the composition 2-isomorphism by $\phi_{g,f} : Fg \circ Ff \cong F(g \circ f)$.

Theorem 9.1.8. Let $\mathcal{C}$ and $\mathcal{D}$ be weak 2-categories (Definition 9.1.2). For a pseudofunctor (Definition 9.1.7) $F : \mathcal{C} \to \mathcal{D}$ to be well-defined, the constraints must satisfy the following coherence conditions:

1. Associativity Coherence: For any three composable 1-morphisms f, g, h, the following diagram of 2-isomorphisms commutes:

$$
\begin{array}{ccc}
(Fh \circ Fg) \circ Ff & \xrightarrow{\alpha_{Fx,Fy,Fz}} & Fh \circ (Fg \circ Ff) \\
\downarrow \phi_{h,g} \circ 1 & & \downarrow 1 \circ \phi_{g,f} \\
F(h \circ g) \circ Ff & & Fh \circ F(g \circ f) \\
\downarrow \phi_{h \circ g, f} & & \downarrow \phi_{h, g \circ f} \\
F((h \circ g) \circ f) & \xrightarrow{F(\alpha_{x,y,z})} & F(h \circ (g \circ f))
\end{array}
$$

   where α denotes the associator in the respective bicategories.

2. Unit Coherence: For any 1-morphism $f : x \to y$, the following diagrams commute:

$$
\begin{array}{ccc}
id_{Fy} \circ Ff & \xrightarrow{\ell_{Ff}} & Ff \\
\downarrow \phi_y \circ 1 & & \uparrow F(\ell_f) \\
F(id_y) \circ Ff & \xrightarrow{\phi_{id_y,f}} & F(id_y \circ f)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
Ff \circ id_{Fx} & \xrightarrow{r_{Ff}} & Ff \\
\downarrow 1 \circ \phi_x & & \uparrow F(r_f) \\
Ff \circ F(id_x) & \xrightarrow{\phi_{f,id_x}} & F(f \circ id_x)
\end{array}
$$

   where ℓ and r denote the left and right unitors.

Definition 9.1.9. Let $F, G : \mathcal{C} \to \mathcal{D}$ be pseudofunctors (Definition 9.1.7) between weak 2-categories (bicategories) (Definition 9.1.2). A *pseudonatural transformation* $\alpha : F \Rightarrow G$ consists of:

- For each object $x \in \mathcal{C}$, a 1-morphism $\alpha_x : Fx \to Gx$ in $\mathcal{D}$.
- For each 1-morphism $f : x \to y$ in $\mathcal{C}$, an invertible 2-morphism (a 2-isomorphism) $\alpha_f : Gf \circ \alpha_x \Rightarrow \alpha_y \circ Ff$ in $\mathcal{D}$.

These data must satisfy two coherence conditions: (♠ TODO: make diagrams of these conditions)

1. Naturality with respect to 2-morphisms $\sigma : f \Rightarrow g$: The 2-morphisms α_f and α_g must be compatible with $F\sigma$ and $G\sigma$.
2. Compatibility with composition and identities: The 2-isomorphisms α_f must respect the composition constraints $\phi_{g,f}$ and identity constraints ϕ_x of the pseudofunctors.

The 2-isomorphism α_f in a pseudonatural transformation is often called the *filler of the naturality square*.

Definition 9.1.10. Let $F, G : \mathcal{C} \to \mathcal{D}$ be pseudofunctors (Definition 9.1.7) between weak 2-categories (bicategories) (Definition 9.1.2). A *modification* $\Gamma : \alpha \Rrightarrow \beta$ *between pseudonatural transformations* $\alpha, \beta : F \Rightarrow G$ is a collection of 2-morphisms $\Gamma_x : \alpha_x \Rightarrow \beta_x$ in $\mathcal{D}$ for each $x \in \mathcal{C}$, such that for every 1-morphism $f : x \to y$ in $\mathcal{C}$, the following square of 2-morphisms commutes:

$$
\begin{array}{ccc}
Gf \circ \alpha_x & \xrightarrow{\alpha_f} & \alpha_y \circ Ff \\
\downarrow 1 \circ \Gamma_x & & \downarrow \Gamma_y \circ 1 \\
Gf \circ \beta_x & \xrightarrow{\beta_f} & \beta_y \circ Ff
\end{array}
$$

Definition 9.1.11. Let $F, G : \mathcal{C} \to \mathcal{D}$ be pseudofunctors (Definition 9.1.7) between weak 2-categories (bicategories) (Definition 9.1.2). A pseudofunctor (Definition 9.1.7) $F : \mathcal{C} \to \mathcal{D}$ is a *biequivalence* if there exists a pseudofunctor $G : \mathcal{D} \to \mathcal{C}$ and pseudonatural transformations (Definition 9.1.9) $\eta : id_{\mathcal{C}} \Rightarrow G \circ F$ and $\varepsilon : F \circ G \Rightarrow id_{\mathcal{D}}$ that are internal equivalences in the respective bicategories of functors. (♠ TODO: what does internal equivalences mean here?)

Theorem 9.1.12. Let $F : \mathcal{C} \to \mathcal{D}$ be pseudofunctors (Definition 9.1.7) between weak 2-categories (bicategories) (Definition 9.1.2). The pseudofunctor (Definition 9.1.7) $F : \mathcal{C} \to \mathcal{D}$ is a biequivalence (Definition 9.1.11) if and only if the following two conditions hold:

1. F is locally an equivalence: For every pair of objects $x, y \in \mathcal{C}$, the functor $F_{x,y} : \mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$ is an equivalence of categories. (♠ TODO: biessentially surjective)
2. F is *biessentially* surjective on objects: For every object $d \in \mathcal{D}$, there exists an object $x \in \mathcal{C}$ and an equivalence $Fx \simeq d$ in the bicategory $\mathcal{D}$.

Theorem 9.1.13 (Coherence theorem/strictness theorem for bicategories)**.** Every weak 2-category (Definition 9.1.2) is biequivalent (Definition 9.1.11) to a strict 2-category (Definition 9.1.1). (♠ TODO: a citation for this)

9.2. n-categories.

Notation 9.2.1. Let **Cat** denote the category of (small) categories and functors between them. For integers $n \geq 0$, we define inductively the category of n-categories.

In a strict n-category, the associativity and unit axioms are specified "strictly" —- i.e. they are expressed in terms of literal equalities as opposed to looser isomorphisms.

Definition 9.2.2. A *strict 0-category* is a set. A *strict $(n + 1)$-category* is a category enriched in (Definition 4.0.1) the category of strict n-categories. Explicitly, a strict $(n + 1)$-category $\mathcal{C}$ consists of:

- a collection of objects $\text{Ob}(\mathcal{C})$; such an object is called a *0-cell* or synonymously a *0-morphism*,
- for each pair of objects $x, y \in \text{Ob}(\mathcal{C})$, a strict n-category $\text{Hom}_{\mathcal{C}}(x, y)$; inductively, this strict n-category itself contains *k-cells*, or synonymously *k-morphisms*, for $0 \leq k \leq n$; the k-cells of $\text{Hom}_{\mathcal{C}}(x, y)$ are exactly the $k + 1$-cells of $\mathcal{C}$.
  - For each object/0-cell x of $\mathcal{C}$, there is an *identity* $id_x \in \text{Hom}_{\mathcal{C}}(x, x)$, which is an $n + 1$ cell of $\mathcal{C}$ and or equivalently an n cell of $\text{Hom}_{\mathcal{C}}(x, x)$.

- composition functors

$$\circ_{x,y,z} : \mathrm{Hom}_{\mathcal{C}}(y, z) \times \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{C}}(x, z)$$

that satisfy the following:

1. **(Strict associativity)** For any quadruple of objects w, x, y, z, the two possible compositions

$$\circ_{w,y,z} \circ (1 \times \circ_{w,x,y}), \qquad \circ_{w,x,z} \circ (\circ_{x,y,z} \times 1)$$

are *equal* as functors (♠ TODO: what is 1?)

$$\mathrm{Hom}_{\mathcal{C}}(y, z) \times \mathrm{Hom}_{\mathcal{C}}(x, y) \times \mathrm{Hom}_{\mathcal{C}}(w, x) \to \mathrm{Hom}_{\mathcal{C}}(w, z).$$

2. **(Strict unit laws)** For each object $x \in \mathrm{Ob}(\mathcal{C})$, there exists a distinguished identity element

$$\mathrm{id}_x \in \mathrm{Hom}_{\mathcal{C}}(x, x),$$

such that for all $x, y \in \mathrm{Ob}(\mathcal{C})$,

$$\circ_{x,x,y}(\mathrm{id}_x, -) = 1_{\mathrm{Hom}_{\mathcal{C}}(x,y)}, \qquad \circ_{x,y,y}(-, \mathrm{id}_y) = 1_{\mathrm{Hom}_{\mathcal{C}}(x,y)}.$$

That is, composing with an identity morphism acts as the identity functor.

3. **(Functoriality)** (♠ TODO: strict n-functor) Each composition map $\circ_{x,y,z}$ is a strict n-functor; it preserves all k-cell compositions and identities in the strict n-category structure of each $\mathrm{Hom}_{\mathcal{C}}(x, y)$ strictly (i.e., on the nose).

In particular, collecting all the strict 0-categories yields the category of sets (Definition 1.0.3), i.e. the 1-category whose objects are sets and morphisms are functions between these sets. Moreover, a strict 1-category is equivalent to a locally small category (Definition 1.0.9).

Morphisms between strict $(n + 1)$-categories are functors that preserve all levels of composition strictly.

Definition 9.2.3. A *weak n-category* (in the sense of Tamsamani and Simpson) is defined inductively on $n \geq 0$.

Base Case ($n = 0$): A weak 0-category is a set.

Inductive Step: For $n \geq 1$, a *weak n-category* is a simplicial object $X_\bullet$ in the category of weak $(n-1)$-categories, denoted $X : \Delta^{op} \to \mathrm{Weak}\text{-}(n-1)\text{-Cat}$, satisfying the Segal condition (composition) and the discreteness condition (identities).

Explicitly, X consists of:

- A sequence of weak $(n - 1)$-categories X_k for each integer $k \geq 0$ (where X_0 represents the objects, X_1 the morphisms, X_2 composable pairs, etc.);
- Face maps $d_i : X_k \to X_{k-1}$ and degeneracy maps $s_i : X_k \to X_{k+1}$ in the category of weak $(n - 1)$-categories, satisfying the simplicial identities.

These data must satisfy the following properties:

1. **Discreteness (Units):** The object of 0-simplices X_0 is "discrete" in the sense that the underlying simplicial set of X_0 is constant (homotopically equivalent to a set).

2. **Segal Condition (Composition):** For every $k \geq 2$, the map induced by the simplicial segment inclusions

$$\mu_k : X_k \xrightarrow{\approx} X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1 \quad (k \text{ times})$$

is an *equivalence* of weak $(n-1)$-categories.

Here, the fiber product is formed using the source and target maps $(d_1, d_0 : X_1 \to X_0)$. The "equivalence" is the notion of equivalence appropriate for $(n-1)$-categories (defined inductively as maps inducing weak homotopy equivalences on all hom-spaces).

Definition 9.2.4. A *weak 3-category* (Definition 9.2.3) (or *tricategory* in the sense of Gordon-Power-Street) consists of:

- A collection of 0-cells (objects);
- For every pair of 0-cells A, B, a bicategory $\mathcal{C}(A, B)$;
- **Composition:** Pseudofunctors $\otimes : \mathcal{C}(B, C) \times \mathcal{C}(A, B) \to \mathcal{C}(A, C)$;
- **Units:** Identity 1-cells I_A (objects in $\mathcal{C}(A, A)$);

Equipped with the following coherent equivalence data (constraints):

- **Associator Adjoint Equivalence:** A pseudonatural equivalence in the bicategory of bicategories:
$$\mathbf{a} : \otimes \circ (\otimes \times 1) \xrightarrow{\approx} \otimes \circ (1 \times \otimes);$$
- **Unitor Adjoint Equivalences:** Pseudonatural equivalences $\mathbf{l}, \mathbf{r}$ for left and right units.
- **Modifications (The 3-dimensional Axioms):** Invertible 3-cells satisfying specific axioms:
    - The *pentagon modification* π (relating the various 3-cell associators);
    - The *middle triangle modification* μ;
    - The *left/right triangle modifications* λ, ρ.

These modifications must satisfy the non-abelian 4-cocycle conditions, specifically the *non-abelian pentagon equation* (often called the associahedron equation K_5).

Definition 9.2.5. Let $\mathcal{C}$ and $\mathcal{D}$ be strict n-categories (Definition 9.2.2). For each pair of objects $x, y \in \mathrm{Ob}(\mathcal{C})$, write

$$\mathrm{Hom}_{\mathcal{C}}(x, y), \quad \mathrm{Hom}_{\mathcal{D}}(Fx, Fy)$$

for the corresponding hom $((n-1))$-categories.

A *functor between n-categories* (or *n-functor*)

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

consists of the following data:

- a function on objects
$$F_0 : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D}),$$
- for each pair of objects $x, y \in \mathrm{Ob}(\mathcal{C})$, a strict $(n-1)$-functor
$$F_{x,y} : \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(Fx, Fy),$$

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- these assignments must satisfy the following axioms.

1. **(Functoriality of composition)** For all $x, y, z \in \mathrm{Ob}(\mathcal{C})$, the following square of strict $(n-1)$-functors commutes strictly:

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(y,z) \times \mathrm{Hom}_{\mathcal{C}}(x,y) & \xrightarrow{\circ_{\mathcal{C}}} & \mathrm{Hom}_{\mathcal{C}}(x,z) \\
\downarrow F_{y,z} \times F_{x,y} & & \downarrow F_{x,z} \\
\mathrm{Hom}_{\mathcal{D}}(Fy,Fz) \times \mathrm{Hom}_{\mathcal{D}}(Fx,Fy) & \xrightarrow{\circ_{\mathcal{D}}} & \mathrm{Hom}_{\mathcal{D}}(Fx,Fz)
\end{array}
$$

That is,

$$
F_{x,z} \circ \circ_{\mathcal{C}} \;=\; \circ_{\mathcal{D}} \circ (F_{y,z} \times F_{x,y}).
$$

2. **(Preservation of identities)** For every object $x \in \mathrm{Ob}(\mathcal{C})$, the identity n-cell is preserved strictly:

$$
F_{x,x}(\mathrm{id}_x) = \mathrm{id}_{Fx}.
$$

Morphisms between functors of n-categories are $(n+1)$-natural transformations, defined inductively in the same manner.

Definition 9.2.6. (♠ TODO: this definition is not very precise) A *weak n-category* is a structure that generalizes the notion of a strict n-category by relaxing the associativity and unit laws to hold only up to specified coherent equivalences. It can be defined inductively or through several equivalent formal models.

Formally, a weak n-category $\mathcal{C}$ consists of:

- for each $0 \leq k \leq n$, a collection of k-morphisms;
- source and target operations assigning to each k-morphism its $(k-1)$-boundary morphisms;
- composition operations of k-morphisms along $(k-1)$-boundaries;
- identity k-morphisms for each $(k-1)$-morphism;
- and coherent associativity, unit, and interchange equivalences expressing that composition is associative and unital up to higher morphisms, with these higher coherence conditions themselves satisfying higher coherence laws.

Several equivalent frameworks make this notion precise:

- In the **algebraic approach** (Batanin, Leinster), a weak n-category is encoded as a system of operations on n-cells governed by a globular operad, whose algebras satisfy the necessary coherence conditions.
- In the **simplicial or topological approach**, weak n-categories are modeled by higher Segal-type objects, such as Tamsamani–Simpson n-categories or Rezk's $(n+k, n)$–Θ–spaces, which are presheaves of simplicial sets on the category Θ_n satisfying Segal and completeness conditions.
- In the **truncated ∞-categorical approach** (Baez–Dolan), a weak n-category is a weak ∞-category with trivial cells above dimension n.

Thus, a weak n-category can be viewed as a category-like structure containing morphisms of all orders up to n, where composition is associative and unital only up to a coherent family of higher isomorphisms organized by the same structure.

Definition 9.2.7. (♠ TODO: this definition is not very precise) A *weak n-category* (or *weak higher category*) is a structure similar to a strict n-category, except that the associativity and unit axioms hold only up to coherent higher equivalences. That is, an n-category $\mathcal{C}$ consists of

- objects, 1-morphisms, 2-morphisms, ..., n-morphisms,
- composition operations defined at each level,
- associativity, unit, and interchange data holding up to coherent $(k+1)$-morphisms for $k < n$.

Different models exist for weak n-categories (e.g. bicategories for $n = 2$, tricategories, quasicategories, Segal spaces, etc.), which satisfy equivalent homotopy-coherent conditions.

Definition 9.2.8. (♠ TODO: this definition is not very precise) An *∞-category* (or *weak ∞-category*) is a category-like structure that encodes morphisms of all dimensions, where composition is associative and unital only up to coherent higher homotopies. More precisely, one usually works with the notion of an *$(\infty, 1)$-category*, in which all k-morphisms for $k > 1$ are invertible up to higher homotopy.

Formally, an ∞-category can be defined in several equivalent ways, each specifying a precise model of this higher structure. Each of the standard models forms an *∞-cosmos* — a simplicially enriched category whose hom-objects are themselves ∞-categories — and all such models are equivalent up to categorical equivalence. Common frameworks include:

- **Quasicategories:** Simplicial sets C satisfying the *inner horn-filling condition*: for every $n \geq 2$ and $0 < i < n$, every map $\Lambda_i[n] \to C$ (where $\Lambda_i[n]$ is the ith inner horn of the standard simplex $\Delta[n]$) extends to a map $\Delta[n] \to C$. Such objects are the fibrant objects in the Joyal model structure on simplicial sets.
- **Complete Segal spaces:** Simplicial spaces $W_\bullet : \Delta^{\mathrm{op}} \to \mathbf{sSet}$ satisfying the Segal condition
$$W_n \simeq W_1 \times_{W_0} \cdots \times_{W_0} W_1$$
(encoding associative composition up to homotopy) together with the completeness condition ensuring that equivalences are detected correctly. These are the fibrant objects in the Rezk model structure on simplicial spaces.
- **Simplicial categories:** Categories enriched in simplicial sets whose hom-objects are Kan complexes and satisfy homotopy-coherent associativity laws. These are related to quasicategories via the homotopy coherent nerve construction.
- **Other equivalent models:** Segal categories, naturally marked simplicial sets, and Θ-spaces, each giving a distinct but equivalent encoding of higher categorical coherence.

Morphisms between ∞-categories, called *∞-functors*, are maps in the chosen model that preserve the higher compositional structure up to coherent homotopy. The totality of ∞-categories and ∞-functors forms an $(\infty, 2)$-category commonly denoted by ∞-**Cat**.

In summary, an ∞-category formalizes the idea of a category enriched in homotopy types (or spaces) — a "category up to higher coherent homotopies." All standard models of ∞-categories (quasicategories, complete Segal spaces, simplicial categories, etc.) are equivalent in the sense of ∞-cosmoi theory.

9.3. **Topological categories.** Topological categories (Definition 9.3.3) provide one framework for ∞-categories.

Definition 9.3.1. (♠ TODO: final topology) A topological space (Definition B.0.1) X is said to be *compactly generated* (or a *k-space*) if a subset $U \subseteq X$ is open whenever for every compact (Definition E.0.17) subset $K \subseteq X$, the intersection $U \cap K$ is open in the subspace K. Equivalently, X is compactly generated if and only if the topology of X is the final topology with respect to the collection of inclusions $K \hookrightarrow X$ for compact $K \subseteq X$.

Definition 9.3.2. A topological space (Definition B.0.1) X is said to be *weakly Hausdorff* if for every continuous map (Definition B.0.3) $f : K \to X$ from a compact (Definition E.0.17) Hausdorff space K, the image $\mathrm{Im}(f)$ is closed in X. Equivalently, X is weakly Hausdorff if and only if for every compact Hausdorff K, the induced map $f : K \to X$ is a closed map.

Definition 9.3.3. A *topological category* is a category (Definition 1.0.1) which is enriched over (Definition 4.0.1) $\mathcal{CG}$, the category of compactly generated (Definition 9.3.1) (and weakly Hausdorff (Definition 9.3.2)) topological spaces. The category of topological categories will often be denoted by $\mathcal{C}at_{\mathrm{top}}$.

More explicitly, a topological category $\mathcal{C}$ consists of a collection of objects, together with a (compactly generated) topological space $Map_{\mathcal{C}}(X, Y)$ for any pair of objects $X, Y \in \mathcal{C}$. These mapping spaces must be equipped with an associative composition law, given by continuous maps

$$Map_{\mathcal{C}}(X_0, X_1) \times Map_{\mathcal{C}}(X_1, X_2) \times \ldots Map_{\mathcal{C}}(X_{n-1}, X_n) \to Map_{\mathcal{C}}(X_0, X_n)$$

(defined for all $n \geq 0$). Here the product is taken in the category of compactly generated topological spaces.

There are analogues to the geometric realization functor (Definition 8.0.5) of simplicial sets and the singular complex functor (Definition 8.0.6) of topological spaces:

Definition 9.3.4. Let $\mathcal{C}$ be a simplicial category (Definition 8.0.21). The *geometric realization of $\mathcal{C}$*, denoted $|\mathcal{C}|$, is the topological category (Definition 9.3.3) defined as follows:

- The objects of $|\mathcal{C}|$ are the same as the objects of $\mathcal{C}$:

$$\mathrm{Ob}(|\mathcal{C}|) = \mathrm{Ob}(\mathcal{C}).$$

- For any objects $X, Y \in \mathcal{C}$, the morphism space is the geometric realization (Definition 8.0.5) of the simplicial mapping space:

$$\mathrm{Hom}_{|\mathcal{C}|}(X, Y) = |\mathrm{Hom}_{\mathcal{C}}(X, Y)|.$$

(Definition 4.0.1)

- The composition maps

$$\mathrm{Hom}_{|\mathcal{C}|}(Y, Z) \times \mathrm{Hom}_{|\mathcal{C}|}(X, Y) \to \mathrm{Hom}_{|\mathcal{C}|}(X, Z)$$

  are induced by applying the geometric realization functor to the composition maps of $\mathcal{C}$, using that geometric realization commutes with finite products.

Definition 9.3.5. Let $\mathcal{C}$ be a topological category (Definition 9.3.3). Its *singular simplicial category* $\mathrm{Sing}\,\mathcal{C}$ is the simplicial category (Definition 8.0.21) defined as follows:

- The objects of $\mathrm{Sing}\,\mathcal{C}$ are the same as those of $\mathcal{C}$:

$$\mathrm{Ob}(\mathrm{Sing}\,\mathcal{C}) = \mathrm{Ob}(\mathcal{C}).$$

- For any objects $X, Y \in \mathcal{C}$, the morphism simplicial set is given by the singular complex (Definition 8.0.6) of the topological morphism space:

$$\mathrm{Hom}_{\mathrm{Sing}\,\mathcal{C}}(X, Y) = \mathrm{Sing}(\mathrm{Hom}_{\mathcal{C}}(X, Y)).$$

  Explicitly, this singular complex is given by

$$\mathrm{Sing}_n(\mathrm{Hom}_{\mathcal{C}}(X, Y)) = \mathrm{Top}(|\Delta^n|, \mathrm{Hom}_{\mathcal{C}}(X, Y)),$$

  where $|\Delta^n|$ is the standard topological n-simplex (Definition 8.0.2).
- The composition maps in $\mathrm{Sing}\,\mathcal{C}$ are induced by the composition maps in $\mathcal{C}$ via functoriality of the singular complex:

$$\mathrm{Sing}(\mathrm{Hom}_{\mathcal{C}}(Y, Z)) \times \mathrm{Sing}(\mathrm{Hom}_{\mathcal{C}}(X, Y)) \to \mathrm{Sing}(\mathrm{Hom}_{\mathcal{C}}(X, Z)).$$

There is an adjunction analogous to that of Proposition 8.0.7:

Proposition 9.3.6. The geometric realization functor (Definition 9.3.4) and the singular simplicial category functor (Definition 9.3.5) form an adjoint pair (Definition 1.1.13)

$$|\cdot| \dashv \mathrm{Sing}$$

$$|\cdot| : \mathbf{Cat}_\Delta \rightleftarrows \mathcal{C}at_{\mathrm{top}} : \mathrm{Sing}$$

(Definition 8.0.21, Definition 9.3.3).

Definition 9.3.7. If $\mathcal{C}$ is a topological category (Definition 9.3.3), we define the *(topological) nerve* $N(\mathcal{C})$ of $\mathcal{C}$ to be the simplicial nerve (Definition 8.0.25) of $\mathrm{Sing}\,\mathcal{C}$ (Definition 9.3.5).

Lemma 9.3.8 ([Lur09, between Examples 1.1.5.8 and 1.1.5.9]). The functor $\mathfrak{C} : \Delta \to \mathbf{Cat}_\Delta$ (Definition 8.0.24, Definition 8.0.3, Definition 8.0.21) extends uniquely (up to unique isomorphism) to a (small) colimit-preserving functor $\mathfrak{C} : \mathbf{Sets}_\Delta \to \mathbf{Cat}_\Delta$ (Definition 8.0.4).

Theorem 9.3.9 ([Lur09, Theorem 1.1.5.13]). Let $\mathcal{C}$ be a topological category (Definition 9.3.3), and let $X, Y \in \mathcal{C}$ be objects. The counit map (Definition 1.1.13) (obtained via the adjunction (Definition 1.1.13) of (♠ TODO: the adjunction should come from $\mathfrak{C}$ and N))

$$|\mathrm{Hom}_{\mathfrak{C}[N(\mathcal{C})]}(X, Y)| \to \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

(Definition 8.0.5, Lemma 9.3.8, Definition 9.3.7) is a weak homotopy equivalence (Definition B.0.7) of topological spaces.

9.4. Derived categories via ∞-categories.

Definition 9.4.1 (∞-category localization)**.** (♠ TODO: ∞-category in what sense) Let $\mathcal{C}$ be an ∞-category and let $W \subseteq \mathrm{Mor}(\mathcal{C})$ be a collection of morphisms in $\mathcal{C}$, called *weak equivalences* or *localizing morphisms*.

A *localization of the ∞-category $\mathcal{C}$ at W* is an ∞-category $\mathcal{C}[W^{-1}]$ together with an ∞-functor

$$L : \mathcal{C} \to \mathcal{C}[W^{-1}]$$

satisfying the following universal property:

- For every morphism $w \in W$, the image $L(w)$ is an equivalence in $\mathcal{C}[W^{-1}]$.
- For any ∞-category $\mathcal{D}$, composition with L induces a fully faithful embedding of ∞-categories
$$\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$
  whose essential image consists precisely of the ∞-functors that send every morphism in W to an equivalence in $\mathcal{D}$.

Definition 9.4.2 (dg-category)**.** Let k be a commutative ring. A *dg-category* (differential graded category) $\mathcal{C}$ over k is a category enriched in (Definition 4.0.1) the category of chain complexes (Definition 6.0.1) of k-modules (Definition C.0.7). Explicitly, a dg-category $\mathcal{C}$ consists of the following data:

- A class of objects $\mathrm{Ob}(\mathcal{C})$.
- For each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a complex (Definition 6.0.1) of k-modules $\mathrm{Hom}_{\mathcal{C}}(X, Y) = (\mathrm{Hom}_{\mathcal{C}}^n(X, Y), d)$; that is, a $\mathbb{Z}$-graded k-module
$$\mathrm{Hom}_{\mathcal{C}}(X, Y) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}^n(X, Y)$$
  equipped with a differential $d : \mathrm{Hom}_{\mathcal{C}}^n(X, Y) \to \mathrm{Hom}_{\mathcal{C}}^{n+1}(X, Y)$ satisfying $d^2 = 0$.
- For each triple of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a morphism of complexes (composition)
$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \otimes_k \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$
  which is associative and satisfies the graded Leibniz rule with respect to the differential d:
$$d(f \circ g) = d(f) \circ g + (-1)^{|f|} f \circ d(g),$$
  for homogeneous f, g with $|f|$ denoting the degree of f.
- For each object X, an identity morphism $1_X \in \mathrm{Hom}_{\mathcal{C}}^0(X, X)$ such that for any $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$,
$$1_Y \circ f = f, \quad f \circ 1_X = f.$$

Definition 9.4.3 (category of chain complexes as a dg-category)**.** Let $\mathcal{A}$ be an additive category (Definition 3.0.1) and suppose that $\mathcal{A}$ is enriched over (Definition 4.0.1) a commutative ring k. The category $\mathrm{Ch}(\mathcal{A})$ (Definition 6.0.1) can be given the structure of a dg-category over k (Definition 9.4.2) as follows:

- An object in $\mathrm{Ch}(\mathcal{A})$ is, as usual, a chain complex (Definition 6.0.1) of objects in $\mathcal{A}$.

- For a pair of chain complexes $X^\bullet, Y^\bullet \in \mathrm{Ch}(\mathcal{A})$, the hom-complex $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(X^\bullet, Y^\bullet) \in \mathrm{Ob}\,\mathrm{Ch}(\mathcal{A})$ is the complex of k-modules with

$$\mathrm{Hom}^n_{\mathrm{Ch}(\mathcal{A})}(X^\bullet, Y^\bullet) := \prod_{m \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}\big(X^m, Y^{m+n}\big).$$

  The differential on this complex, $\delta : \mathrm{Hom}^n_{\mathrm{Ch}(\mathcal{A})}(X^\bullet, Y^\bullet) \to \mathrm{Hom}^{n+1}_{\mathrm{Ch}(\mathcal{A})}(X^\bullet, Y^\bullet)$, is given by

$$\delta(f)^m := d_Y^{m+n} \circ f^m - (-1)^n f^{m+1} \circ d_X^m,$$

  for $f = (f^m)_{m \in \mathbb{Z}}$.
- Composition of morphisms

$$\circ : \mathrm{Hom}^p_{\mathrm{Ch}(\mathcal{A})}(Y^\bullet, Z^\bullet) \otimes \mathrm{Hom}^q_{\mathrm{Ch}(\mathcal{A})}(X^\bullet, Y^\bullet) \to \mathrm{Hom}^{p+q}_{\mathrm{Ch}(\mathcal{A})}(X^\bullet, Z^\bullet)$$

  is given by component-wise composition in $\mathcal{A}$ with the Koszul sign rule:

$$(g \circ f)^m := g^{m+q} \circ f^m.$$

- For each $X^\bullet$, the identity morphism is the family $(1_{X^m})_{m \in \mathbb{Z}}$ in degree zero.

Definition 9.4.4 (Moore complex of a simplicial object)**.** Let $\mathcal{A}$ be an abelian category (Definition 3.0.3) and let $X_\bullet$ be a simplicial object (Definition 8.0.4) in $\mathcal{A}$, i.e., a functor

$$X_\bullet : \Delta^{op} \to \mathcal{A}$$

where Δ is the simplex category.

The *Moore complex*, synonymously called the *normalized chain complex* or *complex of normalized chains*, $M(X_\bullet) \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$ associated to $X_\bullet$ is the chain complex (Definition 6.0.1) given by:

- For each integer $n \geq 0$, the degree n component is

$$M_n(X_\bullet) := \bigcap_{i=1}^{n} \ker(d_i : X_n \to X_{n-1}),$$

  where d_i are the face maps (Definition 8.0.11) of the simplicial object.
- The differential $d_n : M_n(X_\bullet) \to M_{n-1}(X_\bullet)$ is given by the restriction of the face map d_0:

$$d_n := d_0|_{M_n(X_\bullet)}.$$

Theorem 9.4.5 (Dold-Kan correspondence)**.** Let $\mathcal{A}$ be an abelian category with enough projectives. There exists an equivalence of categories, called the *Dold-Kan equivalence*, (♠ TODO: normalized chain complex functor, moore complex functor)

$$N : \mathbf{s}\,\mathrm{Ab}(\mathcal{A}) \leftrightarrows \mathrm{Ch}_{\geq 0}(\mathcal{A}) : \Gamma,$$

(Definition 9.4.4) between the category of simplicial objects (Definition 8.0.4) in Ab, and the category of connective (non-negatively graded) chain complexes in $\mathcal{A}$.

This equivalence is exact, functorial, and compatible with the homotopy structures on both categories:

- The *normalized chain complex functor* N sends a simplicial object to its normalized chain complex.
- The inverse functor Γ takes a chain complex supported in degrees ≥ 0 to a simplicial object.
- The adjunction (N, Γ) restricts to an equivalence of categories.

Theorem 9.4.6 (Stable enhancement of the Dold-Kan correspondence)**.** (♠ TODO: verify that the following is true) Let $\mathcal{A}$ be an abelian category with enough projectives. There exists an equivalence of stable ∞-categories

$$\mathcal{D}(\mathcal{A}) \simeq \mathrm{Sp}(\mathrm{S}_{\geq 0}(\mathcal{A})),$$

where

- $\mathcal{D}(\mathcal{A})$ denotes the derived ∞-category of unbounded chain complexes in $\mathcal{A}$, obtained by inverting quasi-isomorphisms in $\mathrm{Ch}(\mathcal{A})$ and passing to the homotopy-coherent enhancement.
- $\mathrm{S}_{\geq 0}(\mathcal{A})$ is the ∞-category of connective (nonnegatively graded) simplicial objects in $\mathcal{A}$.
- $\mathrm{Sp}(\mathrm{S}_{\geq 0}(\mathcal{A}))$ denotes the stable ∞-category of spectrum objects in $\mathrm{S}_{\geq 0}(\mathcal{A})$, i.e., the stabilization of the connective simplicial objects.

This equivalence *categorifies and stabilizes* the classical Dold-Kan correspondence and extends it from connective chain complexes to all unbounded chain complexes:

- The classical Dold-Kan equivalence identifies nonnegatively graded chain complexes with connective simplicial objects.
- The stable enhancement takes spectra in the simplicial world to model all chain complexes, thus producing a stable ∞-category representing $\mathrm{Ch}(\mathcal{A})$ internally as an ∞-category.

10. 6-FUNCTOR FORMALISM

(♠ TODO: read through these)

Notation 10.0.1 (Ambient data, sizes, pullbacks, spans)**.** Fix a symmetric monoidal ∞-category $(\mathcal{E}, \otimes, \mathbb{1}_{\mathcal{E}})$ and a class of morphisms $\mathcal{P}$ in an ∞-category $\mathcal{C}$ such that:

- $\mathcal{C}$ admits pullbacks along morphisms in $\mathcal{P}$, and $\mathcal{P}$ is stable under pullback and composition.
- There is a functor $\omega \colon \mathcal{C} \to \mathcal{E}$ (often a coefficient or "sheaf/transfer" functor) that sends $\mathcal{P}$-pullback squares in $\mathcal{C}$ to $\otimes$-Beck–Chevalley squares in $\mathcal{E}$ (so that base-change along $\mathcal{P}$ exists and is coherent).

Write objects of $\mathcal{C}$ by X, Y, Z, and spans by diagrams $X \xleftarrow{p} U \xrightarrow{q} Y$. For a morphism $f \colon X \to Y$ in $\mathcal{C}$, its graph span is $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{f} Y$. We use the following notation:

- $(\mathcal{E}, \otimes, \mathbb{1}_{\mathcal{E}})$

- $\mathcal{C}$
- $\mathcal{P} \subseteq \mathrm{Mor}(\mathcal{C})$
- $\omega \colon \mathcal{C} \to \mathcal{E}$
- $X \xleftarrow{p} U \xrightarrow{q} Y$

Definition 10.0.2 (Category of $\mathcal{P}$-correspondences)**.** Given $\mathcal{C}$ and a class $\mathcal{P}$ as in the notation above, the *∞-category of $\mathcal{P}$-correspondences* $\mathrm{Corr}_{\mathcal{P}}(\mathcal{C})$ is defined as follows. Its objects are those of $\mathcal{C}$. For $X, Y \in \mathcal{C}$, the mapping anima $\mathrm{Map}_{\mathrm{Corr}_{\mathcal{P}}(\mathcal{C})}(X, Y)$ is the space of spans $X \xleftarrow{p} U \xrightarrow{q} Y$ with $p \in \mathcal{P}$, and morphisms given by equivalences of such spans. Composition is given by pullback: the composite of $X \xleftarrow{p} U \xrightarrow{q} Y$ and $Y \xleftarrow{p'} V \xrightarrow{q'} Z$ is

$$X \xleftarrow{\mathrm{pr}_1} U \times_Y V \xrightarrow{q' \circ \mathrm{pr}_2} Z,$$

where $\mathrm{pr}_1 \in \mathcal{P}$ by stability of $\mathcal{P}$ under pullback, and associativity/unitality hold up to coherent homotopy. The unit at X is the identity span $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} X$.

Definition 10.0.3 (Coefficient systems on correspondences)**.** A *coefficient system* (with values in $\mathcal{E}$ along $\mathcal{P}$) is the data of $(\omega, \mathcal{P})$ as in the notation, together with:

- for every span $X \xleftarrow{p \in \mathcal{P}} U \xrightarrow{q} Y$, a morphism in $\mathcal{E}$ written $\omega_!(p) \circ \omega^*(q) \colon \omega(X) \to \omega(Y)$, where $\omega^*(q)$ is pullback along q and $\omega_!(p)$ is pushforward along p;
- for every pullback square with left arrow in $\mathcal{P}$, a canonical base-change equivalence exhibiting Beck–Chevalley for $\omega_!$ and ω^*;
- coherences making these assignments functorial for composition of spans.

We denote such a system by $(\omega, \mathcal{P})$.

Definition 10.0.4 (Symmetric monoidal structure on correspondences)**.** Assume $\mathcal{C}$ admits finite products $\times$ preserved by pullbacks along $\mathcal{P}$, and that $(\mathcal{E}, \otimes, \mathbb{1}_{\mathcal{E}})$ is symmetric monoidal with ω lax symmetric monoidal. The *symmetric monoidal structure* on $\mathrm{Corr}_{\mathcal{P}}(\mathcal{C})$ is given on objects by $X \otimes Y \coloneqq X \times Y$, unit $\mathbb{1} \coloneqq *$ (a terminal object of $\mathcal{C}$), and on morphisms by external product of spans:

$$\left(X \xleftarrow{p} U \xrightarrow{q} Y \right) \otimes \left(X' \xleftarrow{p'} U' \xrightarrow{q'} Y' \right) \coloneqq X \times X' \xleftarrow{p \times p'} U \times U' \xrightarrow{q \times q'} Y \times Y',$$

which is well-defined since $p \times p' \in \mathcal{P}$ by closure of $\mathcal{P}$ under products and pullbacks. Symmetry, associativity, and unit constraints are induced from those of $(\mathcal{C}, \times, *)$ and satisfy the ∞-categorical coherences.

Definition 10.0.5 (Symmetric monoidal ∞-category of correspondences with coefficients)**.** Given $(\omega, \mathcal{P})$ and assumptions as above, the *symmetric monoidal ∞-category of correspondences* is the symmetric monoidal ∞-category

$$\mathrm{Corr}(\mathcal{C}, \mathcal{E}) \coloneqq \left(\mathrm{Corr}_{\mathcal{P}}(\mathcal{C}),\ \otimes = \times,\ \mathbb{1} = * \right),$$

equipped with the symmetric monoidal functor $\Omega \colon \mathrm{Corr}_{\mathcal{P}}(\mathcal{C}) \to \mathcal{E}$ determined on objects by $\Omega(X) = \omega(X)$ and on a span $X \xleftarrow{p \in \mathcal{P}} U \xrightarrow{q} Y$ by

$$\Omega\left(X \xleftarrow{p} U \xrightarrow{q} Y \right) \coloneqq \omega_!(p) \circ \omega^*(q) \colon \omega(X) \to \omega(Y),$$

with monoidal structure maps induced by the external product and the lax symmetric monoidal structure on ω together with Beck–Chevalley for base-change. This realizes $\mathrm{Corr}(\mathcal{C}, \mathcal{E})$ as a symmetric monoidal ∞-category encoding spans in $\mathcal{C}$ and their functorial "push–pull" action on coefficients in $\mathcal{E}$.

APPENDIX A. SET THEORY

Definition A.0.1. (♠ TODO: first order language) The *axioms of Zermelo–Fraenkel set theory (ZF)* are the following statements, formulated in the first-order language with symbol $\in$:

1. **Axiom of Extensionality:** Two sets are equal if and only if they have the same elements.
$$\forall A \, \forall B \, (\forall x \, (x \in A \leftrightarrow x \in B) \to A = B).$$

2. **Axiom of Pairing:** For any sets x, y, there exists a set containing exactly x and y as elements.
$$\forall x \, \forall y \, \exists A \, \forall z \, [z \in A \leftrightarrow (z = x \lor z = y)].$$

3. **Axiom of Union:** For any set A, there exists a set that is the union of the elements of A.
$$\forall A \, \exists U \, \forall x \, [x \in U \leftrightarrow \exists B \, (x \in B \land B \in A)].$$

4. **Axiom of Power Set:** For any set A, there exists a set $\mathcal{P}(A)$ containing all subsets of A.
$$\forall A \, \exists P \, \forall B \, [B \in P \leftrightarrow B \subseteq A].$$

5. **Axiom of Infinity:** There exists an inductive set, that is, a set containing the empty set and closed under the successor operation.
$$\exists I \, [\varnothing \in I \land \forall x \, (x \in I \to x \cup \{x\} \in I)].$$

6. **Axiom Schema of Separation:** For any property $\varphi(x)$ expressible in the language of set theory and any set A, there is a subset of A consisting of the elements of A satisfying $\varphi(x)$.
$$\forall A \, \exists B \, \forall x \, [x \in B \leftrightarrow (x \in A \land \varphi(x))].$$

7. **Axiom Schema of Replacement:** For any definable function given by a formula $\varphi(x, y)$ ensuring that for each x there exists a unique y with $\varphi(x, y)$, the image of any set under this function is also a set.
$$\forall A \, [\forall x \in A \, \exists! y \, \varphi(x, y)] \to \exists B \, \forall y \, (y \in B \leftrightarrow \exists x \in A \, \varphi(x, y)).$$

8. **Axiom of Foundation (Regularity):** Every nonempty set A contains an element x that is disjoint from A.
$$\forall A \, [A \neq \varnothing \to \exists x \, (x \in A \land x \cap A = \varnothing)].$$

The *Axiom of Choice (AC)* asserts that for every set X of nonempty sets, there exists a function $f : X \to \bigcup X$ such that $f(A) \in A$ for all $A \in X$. Formally:
$$\forall X \, [(\forall A \in X \, A \neq \varnothing) \to \exists f \, (\mathrm{dom}(f) = X \land \forall A \in X \, f(A) \in A)].$$

When ZF is augmented by the Axiom of Choice, the resulting system is called *Zermelo–Fraenkel set theory with Choice (ZFC)*.

Definition A.0.2. Let X and Y be sets. A *map* (or *function*) from X to Y is a rule f assigning to each element $x \in X$ exactly one element $f(x) \in Y$. We write $f : X \to Y$.

We say that X is the *domain* and that Y is the *codomain of f*.

Definition A.0.3. Let X and Y be sets and let $f : X \to Y$ be a function.

- The function f is said to be *injective* (or *one-to-one*) if for all $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
- The function f is said to be *surjective* (or *onto*) if for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$.
- The map f is *bijective* if it is both injective and surjective. In this case, there exists a unique *inverse map* $f^{-1} : Y \to X$ such that for all $x \in X$ and $y \in Y$,

$$f^{-1}(f(x)) = x \text{ and } f(f^{-1}(y)) = y.$$

Appendix B. Topology

Definition B.0.1 (Topology)**.** Let X be a set. A *topology on X* is a collection $\mathcal{T}$ of subsets of X such that:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
2. For any collection $\{U_i\}_{i \in I} \subseteq \mathcal{T}$ (with I arbitrary), the union $\bigcup_{i \in I} U_i \in \mathcal{T}$,
3. For any finite collection $\{U_1, \ldots, U_n\} \subseteq \mathcal{T}$, the intersection $U_1 \cap \cdots \cap U_n \in \mathcal{T}$.

If $\mathcal{T}$ is a topology on X, the pair $(X, \mathcal{T})$ is called a *topological space*. Members of $\mathcal{T}$ are called *open sets*.

A subset $C \subseteq X$ is *closed* if its complement $X \setminus C$ is an open set in $\mathcal{T}$

One very often refers to X as a topological spcae, omitting the notation of the topology $\mathcal{T}$.

The collection of all topologies on a set X may be denoted by notations such as $\text{Top}(X)$, $\mathbf{Top}(X)$, or $\mathsf{Top}(X)$.

Definition B.0.2. Let (X, τ) be a topological space (Definition B.0.1). An *open covering of X* is a family of open sets

$$\mathcal{U} = \{U_i\}_{i \in I}$$

such that

$$\bigcup_{i \in I} U_i = X.$$

Here, each $U_i \in \tau$ is an open subset of X indexed by a set I, which can be finite or infinite.

Definition B.0.3. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces (Definition B.0.1). A map $f : X \to Y$ is called *continuous* if for every open set $V \in \mathcal{T}_Y$, the preimage $f^{-1}(V)$ is an open set in X, that is,

$$\forall V \in \mathcal{T}_Y, \ f^{-1}(V) \in \mathcal{T}_X.$$

Equivalently, f is continuous if and only if for every closed set $C \subseteq Y$, the preimage $f^{-1}(C)$ is closed in X.

A *map of topological spaces* usually refers to a continuous map between the topological spaces.

The set of continuous maps from X to Y is sometimes denoted by $C(X, Y)$. Other standard notation include $\mathrm{Hom}_{\mathrm{Top}}(X, Y)$ or $\mathrm{Top}(X, Y)$ coming from more general notation for morphisms between objects in a category (Definition 1.0.1).

Definition B.0.4 (Separation axioms)**.** Let $(X, \mathcal{T})$ be a topological space (Definition B.0.1).

- $(X, \mathcal{T})$ is T_0 (*Kolmogorov*) if for every pair of distinct points $x, y \in X$, there exists an open set $U \in \mathcal{T}$ such that, without loss of generality, $x \in U$ and $y \notin U$.
- $(X, \mathcal{T})$ is T_1 (*Fréchet*) if for every pair of distinct points $x, y \in X$, there exist open sets $U, V \in \mathcal{T}$ such that $x \in U$, $y \notin U$, and $y \in V$, $x \notin V$.
- $(X, \mathcal{T})$ is T_2 or *Hausdorff* if for every pair of distinct points $x, y \in X$, there exist disjoint open sets $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$.
- $(X, \mathcal{T})$ is *regular* if it is T_1 and for each point $x \in X$ and closed set $F \subseteq X$ with $x \notin F$, there exist disjoint open sets $U, V \in \mathcal{T}$ such that $x \in U$ and $F \subseteq V$.
- $(X, \mathcal{T})$ is T_3 (regular Hausdorff) if it is T_1 and regular.
- $(X, \mathcal{T})$ is *completely regular* if for each closed set $F \subseteq X$ and $x \notin F$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f|_F = 1$.
- $(X, \mathcal{T})$ is $T_{3\frac{1}{2}}$ (completely regular Hausdorff) if it is T_1 and completely regular.
- $(X, \mathcal{T})$ is *normal* if it is T_1 and for each pair of disjoint closed sets $A, B \subseteq X$, there exist disjoint open sets $U, V \in \mathcal{T}$ such that $A \subseteq U$ and $B \subseteq V$.
- $(X, \mathcal{T})$ is T_4 (normal Hausdorff) if it is T_1 and normal.
- $(X, \mathcal{T})$ is T_5 (completely normal Hausdorff) if it is T1 and completely normal.
- $(X, \mathcal{T})$ is *perfectly normal* if every closed set is a $G\delta$ (countable intersection of open sets) and the space is normal.
- $(X, \mathcal{T})$ is T_6 (perfectly normal Hausdorff) if it is T_1 and perfectly normal.

Definition B.0.5 (Homotopy of maps of topological spaces)**.** Let X and Y be topological spaces and let $K \subseteq X$ be a subset. Let $C(X, Y)$ denote the set of all continuous maps $f : X \to Y$.

1. A *homotopy between two maps $f, g \in C(X, Y)$ relative to K* is a continuous map

$$H : X \times [0, 1] \to Y$$

such that for all $x \in X$,

$$H(x, 0) = f(x), \quad H(x, 1) = g(x),$$

and for all $x \in K$ and $t \in [0, 1]$,

$$H(x, t) = f(x) = g(x).$$

If such an H exists, we say f and g are *homotopic relative to K*, and we write $f \simeq g$ rel K; this is an equivalence relation.

A *homotopy between two maps $f, g \in C(X, Y)$* is simply a homotopy relative to $\emptyset$. We write we write $f \simeq g$ if a homotopy between them exists.

2. Let (X, x_0) and (Y, y_0) be pointed topological spaces and let $K \subseteq X$ be a subset with $x_0 \in K$. Let $C_*(X, Y)$ denote the set of all continuous based maps $f : X \to Y$ satisfying $f(x_0) = y_0$.

   A *homotopy of based maps $f, g \in C_*(X, Y)$ relative to K* is a continuous map

   $$H : X \times [0, 1] \to Y$$

   such that for all $x \in X$,

   $$H(x, 0) = f(x), \quad H(x, 1) = g(x),$$

   and for all $k \in K$ and $t \in [0, 1]$,

   $$H(k, t) = f(k) = g(k),$$

   in particular fixing the basepoint throughout,

   $$H(x_0, t) = y_0 \quad \text{for all } t \in [0, 1].$$

   If such an H exists, we say f and g are *based homotopic relative to K*, and we write $f \simeq g$ rel K. This is an equivalence relation.

   A *homotopy of based maps $f, g \in C_*(X, Y)$* without relative condition is the special case $K = \{x_0\}$ and is called a *homotopy of based maps* or *based homotopy*. We write $f \simeq g$ if such a homotopy exists.

Definition B.0.6 (Homotopy groups)**.** For any pointed topological space (X, x_0) and integer $n \geq 0$, the *n-th homotopy group of X at x_0*, denoted $\pi_n(X, x_0)$, is defined as the set of all homotopy classes (rel. ∂I^n) of based maps

$$f : (I^n, \partial I^n) \to (X, x_0),$$

where $I^n = [0, 1]^n$. For $n \geq 1$, $\pi_n(X, x_0)$ is a group under concatenation of based maps, and for $n \geq 2$, it is abelian.

The *fundamental group of (X, x_0)* refers to $\pi_1(X, x_0)$. Equivalently, it is the group of homotopy classes (rel. endpoints) of loops $\gamma : [0, 1] \to X$ satisfying $\gamma(0) = \gamma(1) = x_0$.

Definition B.0.7 (Weak homotopy equivalence)**.** Let $f : X \to Y$ be a continuous map (Definition B.0.3) between topological spaces. The map f is a *weak homotopy equivalence* if, for every choice of basepoint $x_0 \in X$ with $y_0 = f(x_0)$, the induced maps on homotopy groups (Definition B.0.6)

$$f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$$

are isomorphisms (Definition C.0.6) for all integers $n \geq 0$.

Appendix C. Algebra

Definition C.0.1. A *ring* is a triple $(R, +, \cdot)$ where

1. $(R, +)$ is a commutative group (Definition C.0.5), and
2. $(R, \cdot)$ is a monoid.
3. $\cdot$ is distributive over $+$, i.e. for all $a, b, c \in R$, we have

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and} \quad (a + b) \cdot c = a \cdot c + b \cdot c.$$

Equivalently, a ring is a triple $(R, +, \cdot)$ where $+, \cdot : R \times R \to R$ are binary operations satisfying

1. $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$ for all $a, b, c \in R$
2. There exists an element $0 \in R$ such that $a + 0 = a = 0 + a$ for all $a \in R$.
3. For every $a \in R$, there exists an element $-a \in R$ such that $a + (-a) = 0 = (-a) + a$ for all $a \in R$.
4. There exists an element $1 \in R$ such that $a \cdot 1 = a = 1 \cdot a$ for all $a \in R$.
5. For all $a, b, c \in R$, we have

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and} \quad (a + b) \cdot c = a \cdot c + b \cdot c.$$

The operation $+$ is often called *addition* and the operation $\cdot$ is often called *multiplication*. Accordingly, the identity element 0 of $+$ is often called the *additive identity* and the identity element 1 of $\cdot$ is often called the *multiplicative identity*.

Remark C.0.2. Some writers might not require a ring to have a multiplicative identity element, i.e. would define a ring so that $(R, +)$ is a commutative group, $(R, \cdot)$ is a semigroup, and $\cdot$ is distributive over $+$. Such writers would call the notion of ring in Definition C.0.1 a *unitary ring* to emphasize the existence of the multiplicative identity 1.

Definition C.0.3. A *commutative (unital) ring* is a ring (Definition C.0.1) $(R, +, \cdot)$ such that $\cdot$ is a commutative operation, i.e. $a \cdot b = b \cdot a$.

For many writers (e.g. "commutative" algebraists or number theorists), a *ring* refers to a commutative ring as above.

Definition C.0.4. Let $(R, +, \cdot)$ and $(S, +, \cdot)$ be rings (Definition C.0.1), not assumed to be commutative. A function $f : R \to S$ is called a *ring homomorphism* if for all $r_1, r_2 \in R$ the following properties hold:

1. $f(r_1 + r_2) = f(r_1) + f(r_2)$,
2. $f(r_1 r_2) = f(r_1) f(r_2)$,
3. $f(1_R) = 1_S$ where 1_R and 1_S denote the multiplicative identities in R and S, respectively.

A ring homomorphism is said to be a *ring isomorphism* if it is invertible as a map of sets.

An *R-ring* refers to a ring S equipped with a ring homomorphism $f : R \to S$.

We note that a ring homomorphism $f : R \to S$ yields a natural left R-module (Definition C.0.7) structure on S and a natural right R-module structure on S respectively as follows for $r \in R$ and $s \in S$:

$$r \cdot s = f(r) \cdot s$$
$$s \cdot r = s \cdot f(r).$$

However, these left and right module structures need not yield a two-sided R–module structure.

Definition C.0.5 (Groups). A *group* is a pair $(G, \cdot)$ where G is a set and $\cdot : G \times G \to G$ is a binary operation, subject to the following conditions:

1. (Associativity) For all $g, h, k \in G$ one has
$$(g \cdot h) \cdot k = g \cdot (h \cdot k).$$

2. (Identity element) There exists an element $e \in G$ such that for all $g \in G$,
$$e \cdot g = g \cdot e = g.$$

3. (Inverse element) For all $g \in G$ there exists an element $g^{-1} \in G$ such that
$$g \cdot g^{-1} = g^{-1} \cdot g = e.$$

The element e is called the *identity element of G*, and g^{-1} is called the *inverse of g*.

Equivalently, a group is a monoid with inverse elements.

Equivalently, a group is a group object (Definition 1.3.19) in the category of sets (Definition 1.0.3).

A group $(G, \cdot)$ is often simply written as G, when the notation for the binary operation $\cdot$ is clear.

An *abelian group* or synonymously, a *commutative group*, is a group $(G, \cdot)$ whose binary operation $\cdot$ is *abelian* or *commutative*, i.e. satisfies
$$g \cdot h = h \cdot g$$
for all $g, h \in G$.

An abelian group is equivalent to a $\mathbb{Z}$-module.

Definition C.0.6 (Group homomorphism). Let $(G, \cdot)$ and $(H, *)$ be groups (Definition C.0.5). A map $f : G \to H$ is called a *group homomorphism* if for all $g_1, g_2 \in G$ one has
$$f(g_1 \cdot g_2) = f(g_1) * f(g_2).$$

The collection of all groups with the group homomorphisms forms a locally small (Definition 1.0.9) category (Definition 1.0.1), called the *category of groups*.

If f is bijective (Definition A.0.3), then f is called a *group isomorphism*. Equivalently, a group isomorphism is an isomorphism (Definition 1.0.8) in the category of groups .

Definition C.0.7. Let R be a not-necessarily commutative ring (Definition C.0.1).

1. A *left R-module* is an abelian group $(M, +)$ together with an operation $R \times M \to M$, denoted $(r, m) \mapsto rm$, such that for all $r, s \in R$ and $m, n \in M$:
   - $r(m + n) = rm + rn$,
   - $(r + s)m = rm + sm$,
   - $(rs)m = r(sm)$,
   - $1_R m = m$ where 1_R is the multiplicative identity of R.
2. A *right R-module* is defined similarly as an abelian group $(M, +)$ with an operation $M \times R \to M$, denoted $(m, r) \mapsto mr$, such that for all $r, s \in R$ and $m, n \in M$:
   - $(m + n)r = mr + nr$,

- $m(r + s) = mr + ms$,
- $m(rs) = (mr)s$,
- $m1_R = m$.

3. Let R and S be (not necessarily commutative) rings (Definition C.0.1).

   An *R-S-bimodule* (or an *R-S-module* or an (R, S)-module, etc.) is an abelian group (Definition C.0.5) $(M, +)$ equipped with

   (a) a left action of R:

$$R \times M \to M, \quad (r, m) \mapsto r \cdot m \,,$$

   making M a left R-module (Definition C.0.7),

   (b) a right action of S:

$$M \times S \to M, \quad (m, s) \mapsto m \cdot s \,,$$

   making M a right S-module,

   such that the left and right actions commute; that is, for all $r \in R$, $s \in S$, and $m \in M$,

$$r \cdot (m \cdot s) = (r \cdot m) \cdot s.$$

4. A *two-sided R-module* (or *R-bimodule*) is an R-R-bimodule.

If R is a commutative ring (Definition C.0.3), then a left/right R-module can automatically be regarded as a two-sided R-module. As such, we simply talk about *R-modules* in this case.

Any abelian group is equivalent to a two-sided $\mathbb{Z}$-module. Moreover, any left R-module is equivalent to an $R - \mathbb{Z}$-bimodule (Definition C.0.7) and any right R-module is equivalent to an $\mathbb{Z} - R$-bimodule (Definition C.0.7). Given a left/right/two-sided R-module, its *natural bimodule structure* will refer to its structure as a R-$\mathbb{Z}$/$\mathbb{Z}$-R/R-R bimodule. In this way, many definitions associated with the notions of left/right/two-sided R-modules can be defined as special cases for definitions for R-S-bimodules.

Definition C.0.8. Let R, S be (not-necessarily commutative) rings (Definition C.0.1).

1. Let M and N be R-S-bimodules (Definition C.0.7). A function $\varphi : M \to N$ is called an *R-S-bimodule homomorphism* or *R-S-linear* if it is a group homomorphism (Definition C.0.6) of the underlying abelian groups of M and N and respects the scalar actions as follows: for all $m_1, m_2 \in M$, $r \in R$, and $s \in S$,

$$\varphi(r \cdot m_1) = r \cdot \varphi(m_1),$$
$$\varphi(m_1 \cdot s) = \varphi(m_1) \cdot s.$$

2. Let M and N be left/right/two-sided R-modules (Definition C.0.7). A function $\varphi : M \to N$ is called a *left/right/two-sided R-module homomorphism* if it is an bimodule homomorphism on the natural bimodule structures (Definition C.0.7) of M and N. Such a function is also called *R-linear*.

Modules and homomorphisms of a fixed type (i.e. R-S-bimodules or left/righ/two-sided R-modules) form a locally small (Definition 1.0.9) category (Definition 1.0.1).

Definition C.0.9. Let R and S be (not necessarily commutative) rings (Definition C.0.1).

1. The *category of (R, S)-bimodules* (or R-S-bimodules), denoted by notations such as $_R\mathsf{Mod}_S$, is the category whose objects are (R, S)-bimodules (Definition C.0.7) and whose R-S-bimodule homomorphisms (Definition C.0.8).

2. The *category of left R-modules*, denoted by notations such as $_R\mathsf{Mod}$ or $R - \mathbf{Mod}$, is the category $_R\mathsf{Mod}_{\mathbb{Z}}$, i.e. the category whose objects are left R-modules (Definition C.0.7) and whose morphisms are left R-linear maps (Definition C.0.8).

3. The *category of right R-modules*, denoted by notations such as Mod_R or $\mathbf{Mod} - R$, is the category $_{\mathbb{Z}}\mathsf{Mod}_R$, i.e. the category whose objects are right R-modules (Definition C.0.7) and whose morphisms are right R-linear maps (Definition C.0.8).

The category of bimodules can be canonically identified with module categories over tensor product rings:

- $_R\mathsf{Mod}_S$ is isomorphic to the category of left modules over the ring $R \otimes_{\mathbb{Z}} S^{\mathrm{op}}$.
- $_R\mathsf{Mod}_S$ is isomorphic to the category of right modules over the ring $R^{\mathrm{op}} \otimes_{\mathbb{Z}} S$.

Consequently, standard module-theoretic concepts (such as projective objects, injective objects, and flat objects) in $_R\mathsf{Mod}_S$ correspond exactly to the respective concepts in $_{R \otimes S^{\mathrm{op}}}\mathsf{Mod}$.

Note that there are canonical isomorphisms of categories:

$$_R\mathsf{Mod} \cong {}_R\mathsf{Mod}_{\mathbb{Z}} \quad \text{and} \quad \mathsf{Mod}_S \cong {}_{\mathbb{Z}}\mathsf{Mod}_S.$$

That is, left R-modules are exactly $(R, \mathbb{Z})$-bimodules, and right S-modules are exactly $(\mathbb{Z}, S)$-bimodules.

Appendix D. Schemes

Definition D.0.1. A morphism of schemes $f : X \to Y$ is called *étale* if it satisfies the following conditions: (♠ TODO: sheaf of relative differentials)

- f is locally of finite presentation,
- f is flat,
- f is unramified, i.e., the sheaf of relative differentials $\Omega_{X/Y}$ equals 0.

(♠ TODO: relative dimension) Equivalently, a morphism of schemes is étale if and only if it is smooth of relative dimension 0. A finite étale morphism is synonymously called a *finite étale cover*.

Appendix E. Grothendieck universes

Recall that the collection of all sets within Zermelo-Fraenkel set theory is a proper class and not a set. We often use Grothendieck universes in categorical discussions as a way to restrict the sets considered so as to guarantee that only sets result in the categorical constructions that matter (e.g. to guarantee that the hom's in a newly constructed category form a set).

Definition E.0.1 (Grothendieck Universe)**.** Let U be a set. We say U is a *Grothendieck universe* (or just a *universe*) if the following conditions hold:

1. If $x \in U$ and $y \in x$, then $y \in U$ (transitivity).
2. If $x, y \in U$, then $\{x, y\} \in U$ (closed under pair formation).
3. If $x \in U$, then the power set $\mathcal{P}(x) \in U$.
4. If $I \in U$ and $(x_\alpha)_{\alpha \in I}$ is a family with each $x_\alpha \in U$, then $\bigcup_{\alpha \in I} x_\alpha \in U$.

A set X is called U-*small* or a U-*set* if $X \in U$.

Remark E.0.2. Given a Grothendieck universe U, the collection of all U-small sets is U itself, which contrasts against the fact that the collection of all sets (for the ZF (and ZFC) axioms) is a proper class.

Remark E.0.3. Within ZF/ZFC alone, there is no guarantee that every given set S satisfies $S \in U$ for some universe S.

Remark E.0.4. While universes are a powerful tool for studying category theory, they are not universally used or necessary for doing so. Usage depends on the context and foundational preferences. For instance, Grothendieck universes can be used to simplify foundational technicalities when one needs to work with such things as "categories of all categories" or "functor categories" of large size.

Definition E.0.5. A set x is a *transitive set* if every element of x is also a subset of x. Formally, x is transitive if for all $y \in x$, the condition $y \subseteq x$ holds, which is equivalent to the statement that for all $z \in y$, it follows that $z \in x$.

Proposition E.0.6. For any set x, the following conditions are equivalent:

1. x is a transitive set (Definition E.0.5).
2. $\cup x \subseteq x$.
3. $x \subseteq \mathcal{P}(x)$, where $\mathcal{P}(x)$ denotes the power set of x.

Definition E.0.7. The *transitive closure of a set* x denoted $\mathrm{tc}(x)$, is the unique minimal transitive set such that $x \subseteq \mathrm{tc}(x)$. It can be constructed as the union $\bigcup_{n \in \mathbb{N}} \cup^n x$, where $\cup^0 x = x$ and $\cup^{n+1} x = \bigcup(\cup^n x)$.

Definition E.0.8. A set α is an *ordinal number* (or simply an *ordinal*) if it satisfies the following two conditions:

1. α is *transitive*, meaning that every element of α is also a subset of α. That is, if $x \in \alpha$, then $x \subseteq \alpha$.
2. α is *strictly well-ordered* by the membership relation $\in$. That is, the relation $<$ on α defined by $x < y \iff x \in y$ is a strict total ordering, and every non-empty subset of α has a least element under this ordering.

The class of all ordinal numbers is denoted by Ord or **ON**. For ordinals α and β, we write $\alpha < \beta$ when $\alpha \in \beta$, and we write and $\alpha \leq \beta$ when $\alpha \in \beta$ or $\alpha = \beta$.

Theorem E.0.9. An ordinal (Definition E.0.8) is a transitive set (Definition E.0.5) that is strictly well-ordered by the membership relation $\in$. (♠ TODO: strictly well-ordered)

Definition E.0.10. A set x is a *hereditarily finite set* if its transitive closure (Definition E.0.7) $\mathrm{tc}(x)$ is a finite set.

The collection of all hereditarily finite sets is denoted by V_ω. It is a set (Proposition E.0.11).

Proposition E.0.11. The collection V_ω of all hereditarily finite sets (Definition E.0.10) is a set and it can be constructed via the cumulative hierarchy by defining $V_0 = \emptyset$, $V_{n+1} = \mathcal{P}(V_n)$ for $n \in \mathbb{N}$, and taking the union $V_\omega = \bigcup_{n<\omega} V_n$.

Theorem E.0.12. A set x is hereditarily finite (Definition E.0.10) if and only if x is finite and every element $y \in x$ is also a hereditarily finite set.

Proposition E.0.13. The structure $(V_\omega, \in)$ satisfies all the axioms of Zermelo-Fraenkel set theory (Definition A.0.1) excluding the Axiom of Infinity.

Theorem E.0.14. The set V_ω of all hereditarily finite sets (Definition E.0.10) is a Grothendieck universe (Definition E.0.1). Furthermore, V_ω is initial in the sense that it is contained in every Grothendieck universe (that may exist with respect to the chosen system of axioms containing ZF (Definition A.0.1)); more precisely, it is the unique minimal Grothendieck universe under the inclusion relation $\subseteq$.

Definition E.0.15. *Tarski–Grothendieck set theory*, denoted by TG, is the theory consisting of the axioms of ZFC (Definition A.0.1) together with *Tarski's Axiom of Universes*, which asserts that for every set x there exists a universe (Definition E.0.1) U such that $x \in U$:

$$\forall x\, \exists U\, (U \text{ is a Grothendieck universe} \wedge x \in U)$$

Definition E.0.16. Given an axiom system containing the Tarski-Grothendieck set theory (Definition E.0.15) axioms, the *hierarchy of Grothendieck universes* consists of the Grothendieck universes (Definition E.0.1) U_n for $n \in \mathbb{N} \cup \{0\}$ constructed as follows:

1. Base case: $U_0 := V_\omega$, the set of all hereditarily finite sets (Definition E.0.10).
2. Successor step: For each $n \geq 0$, U_{n+1} is the unique minimal Grothendieck universe such that $U_n \in U_{n+1}$.

In particular, $U_n \in U_{n+1}$ and $U_n \subset U_{n+1}$ for all n.

Definition E.0.17 (Compact topological space)**.** A topological space $(X, \mathcal{T})$ is *compact* if every open cover of X admits a finite subcover; that is, for every collection $\{U_i\}_{i \in I}$ of open sets in $\mathcal{T}$ such that $X = \bigcup_{i \in I} U_i$, there exists a finite subcollection $\{U_{i_j}\}_{j=1}^n$ such that $X = \bigcup_{j=1}^n U_{i_j}$.

Some mathematicians, e.g. algebraic geometers, would refer to this property as *quasi-compactness*.

Definition E.0.18 (Separation axioms)**.** Let $(X, \mathcal{T})$ be a topological space (Definition B.0.1).

- $(X, \mathcal{T})$ is T_0 (*Kolmogorov*) if for every pair of distinct points $x, y \in X$, there exists an open set $U \in \mathcal{T}$ such that, without loss of generality, $x \in U$ and $y \notin U$.
- $(X, \mathcal{T})$ is T_1 (*Fréchet*) if for every pair of distinct points $x, y \in X$, there exist open sets $U, V \in \mathcal{T}$ such that $x \in U$, $y \notin U$, and $y \in V$, $x \notin V$.
- $(X, \mathcal{T})$ is T_2 or *Hausdorff* if for every pair of distinct points $x, y \in X$, there exist disjoint open sets $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$.

- $(X, \mathcal{T})$ is *regular* if it is T$_1$ and for each point $x \in X$ and closed set $F \subseteq X$ with $x \notin F$, there exist disjoint open sets $U, V \in \mathcal{T}$ such that $x \in U$ and $F \subseteq V$.
- $(X, \mathcal{T})$ is T_3 (regular Hausdorff) if it is T$_1$ and regular.
- $(X, \mathcal{T})$ is *completely regular* if for each closed set $F \subseteq X$ and $x \notin F$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f|_F = 1$.
- $(X, \mathcal{T})$ is $T_{3\frac{1}{2}}$ (completely regular Hausdorff) if it is T$_1$ and completely regular.
- $(X, \mathcal{T})$ is *normal* if it is T$_1$ and for each pair of disjoint closed sets $A, B \subseteq X$, there exist disjoint open sets $U, V \in \mathcal{T}$ such that $A \subseteq U$ and $B \subseteq V$.
- $(X, \mathcal{T})$ is T_4 (normal Hausdorff) if it is T$_1$ and normal.
- $(X, \mathcal{T})$ is T_5 (completely normal Hausdorff) if it is T1 and completely normal.
- $(X, \mathcal{T})$ is *perfectly normal* if every closed set is a $G\delta$ (countable intersection of open sets) and the space is normal.
- $(X, \mathcal{T})$ is T_6 (perfectly normal Hausdorff) if it is T$_1$ and perfectly normal.

Definition E.0.19. Let $\mathcal{C}$ be a category (Definition 1.0.1). A *category fibred in groupoids over $\mathcal{C}$* is a category $\mathcal{X}$ equipped with a functor (Definition 1.1.1) $p : \mathcal{X} \to \mathcal{C}$ such that:

1. For any morphism $f : V \to U$ in $\mathcal{C}$ and any object $x \in \mathcal{X}$ with $p(x) = U$, there exists a morphism $\phi : y \to x$ in $\mathcal{X}$ with $p(\phi) = f$.
2. For any pair of morphisms $\phi : y \to x$ and $\psi : z \to x$ in $\mathcal{X}$, and any morphism $g : p(z) \to p(y)$ in $\mathcal{C}$ such that $p(\phi) \circ g = p(\psi)$, there exists a unique morphism $\chi : z \to y$ in $\mathcal{X}$ over g (i.e., $p(\chi) = g$) such that $\phi \circ \chi = \psi$.

For each object U in $\mathcal{C}$, the *fiber category $\mathcal{X}_U$* is the category

- whose objects are those $x \in \mathcal{X}$ with $p(x) = U$, and
- whose morphisms α satisfy $p(\alpha) = \mathrm{id}_U$)

The fiber category $\mathcal{X}_U$ is a groupoid.

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