TOPOLOGY

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Contents

The purpose of this document is to compile common notions in basic topology.

1. Basic topolgy

1.1. **Topological spaces.**

Definition 1.1.1 (Topology)**.** Let X be a set. A *topology on X* is a collection $\mathcal{T}$ of subsets of X such that:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
2. For any collection $\{U_i\}_{i\in I} \subseteq \mathcal{T}$ (with I arbitrary), the union $\bigcup_{i\in I} U_i \in \mathcal{T}$,
3. For any finite collection $\{U_1, \dots, U_n\} \subseteq \mathcal{T}$, the intersection $U_1 \cap \cdots \cap U_n \in \mathcal{T}$.

If $\mathcal{T}$ is a topology on X, the pair $(X, \mathcal{T})$ is called a *topological space*. Members of $\mathcal{T}$ are called *open sets*.

A subset $C \subseteq X$ is *closed* if its complement $X \setminus C$ is an open set in $\mathcal{T}$

One very often refers to X as a topological spcae, omitting the notation of the topology $\mathcal{T}$.

The collection of all topologies on a set X may be denoted by notations such as $\mathrm{Top}(X)$, $\mathbf{Top}(X)$, or $\mathsf{Top}(X)$.

1.1.1. *Basis of a topological space.*

Definition 1.1.2. Let $(X, \mathcal{T})$ be a topological space and let $x \in X$.

- An *open neighborhood of x* is any open set $U \in \mathcal{T}$ such that $x \in U$.
- A *neighborhood of x* is a set $N \subseteq X$ for which there exists an open neighborhood $U \in \mathcal{T}$ of x such that $U \subseteq N$.
- A *neighborhood basis* (or *local base*) at x is a nonempty collection $\mathcal{B}_x$ of neighborhoods of x such that for every neighborhood N of x, there exists $B \in \mathcal{B}_x$ with $B \subseteq N$.
  The elements of $\mathcal{B}_x$ are said to *form a base of neighborhoods* at x.

Definition 1.1.3. Let X be a set and let $\mathcal{B}$ be a collection of subsets of X. The collection $\mathcal{B}$ is called a *basis* (or *base*) for a topology on X if the following two conditions hold:

1. For every $x \in X$, there exists at least one $B \in \mathcal{B}$ such that $x \in B$.
2. For any $B_1, B_2 \in \mathcal{B}$ and any $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Given such a collection $\mathcal{B}$, the collection $\mathcal{T}$ of all unions of elements of $\mathcal{B}$ defines a topology on X, and it coincides with $\mathcal{T}_{\mathcal{B}}$, the topology generated by $\mathcal{B}$. In other words,

$$\mathcal{T}_{\mathcal{B}} = \{U \subseteq X : \text{for every } x \in U, \text{ there exists } B \in \mathcal{B} \text{ with } x \in B \subseteq U\}.$$

Definition 1.1.4. Let $(X, \mathcal{T})$ be a topological space.

- The space $(X, \mathcal{T})$ is said to be *first countable* if every point $x \in X$ has a countable neighborhood basis, i.e., there exists a countable collection $\{U_n\}_{n\in\mathbb{N}}$ of open neighborhoods of x such that for any open neighborhood U of x, there exists $n \in \mathbb{N}$ with $U_n \subseteq U$.
- The space $(X, \mathcal{T})$ is said to be *second countable* if there exists a countable basis $\mathcal{B} \subseteq \mathcal{T}$ for the topology $\mathcal{T}$, i.e., every open set $U \in \mathcal{T}$ can be written as a union of elements of $\mathcal{B}$.

1.1.2. *Separation axioms.*

Definition 1.1.5. Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$.

- The set A is called a *G_δ set* if there exists a countable collection of open sets $\{U_n\}_{n\in\mathbb{N}} \subseteq \mathcal{T}$ such that
$$A = \bigcap_{n\in\mathbb{N}} U_n .$$
- The set A is called an *F_σ set* if there exists a countable collection of closed sets $\{F_n\}_{n\in\mathbb{N}}$ such that
$$A = \bigcup_{n\in\mathbb{N}} F_n .$$

Definition 1.1.6 (Separation axioms)**.** Let $(X, \mathcal{T})$ be a topological space.

- $(X, \mathcal{T})$ is T_0 (*Kolmogorov*) if for every pair of distinct points $x, y \in X$, there exists an open set $U \in \mathcal{T}$ such that, without loss of generality, $x \in U$ and $y \notin U$.
- $(X, \mathcal{T})$ is T_1 (*Fréchet*) if for every pair of distinct points $x, y \in X$, there exist open sets $U, V \in \mathcal{T}$ such that $x \in U$, $y \notin U$, and $y \in V$, $x \notin V$.
- $(X, \mathcal{T})$ is T_2 or *Hausdorff* if for every pair of distinct points $x, y \in X$, there exist disjoint open sets $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$.
- $(X, \mathcal{T})$ is *regular* if it is T_1 and for each point $x \in X$ and closed set $F \subseteq X$ with $x \notin F$, there exist disjoint open sets $U, V \in \mathcal{T}$ such that $x \in U$ and $F \subseteq V$.
- $(X, \mathcal{T})$ is T_3 (regular Hausdorff) if it is T_1 and regular.
- $(X, \mathcal{T})$ is *completely regular* if for each closed set $F \subseteq X$ and $x \notin F$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f|_F = 1$.
- $(X, \mathcal{T})$ is $T_{3\frac{1}{2}}$ (completely regular Hausdorff) if it is T_1 and completely regular.
- $(X, \mathcal{T})$ is *normal* if it is T_1 and for each pair of disjoint closed sets $A, B \subseteq X$, there exist disjoint open sets $U, V \in \mathcal{T}$ such that $A \subseteq U$ and $B \subseteq V$.

- $(X, \mathcal{T})$ is T_4 (normal Hausdorff) if it is T_1 and normal.
- $(X, \mathcal{T})$ is T_5 (completely normal Hausdorff) if it is T1 and completely normal.
- $(X, \mathcal{T})$ is *perfectly normal* if every closed set is a $G\delta$ (countable intersection of open sets) and the space is normal.
- $(X, \mathcal{T})$ is T_6 (perfectly normal Hausdorff) if it is T_1 and perfectly normal.

1.1.3. *Subspaces of a topological space.*

Definition 1.1.7 (Subspace topology and subspace)**.** Let $(X, \mathcal{T})$ be a topological space and $Y \subseteq X$ a subset. The topology induced on Y by $\mathcal{T}$, called the *subspace topology*, is defined as

$$\mathcal{T}_Y := \{U \cap Y : U \in \mathcal{T}\}.$$

The pair $(Y, \mathcal{T}_Y)$ is called a *subspace of* $(X, \mathcal{T})$.

Convention 1.1.8. Given a topological space $(X, \mathcal{T})$ and a subset S of X, one almost always considers S as a subspace of X, i.e. considers S as a topological space equipped with the subspace topology.

Definition 1.1.9 (Discrete topological space)**.** Let X be a set. The *discrete topology on X* is the topology $\mathcal{T}_{\text{disc}}$ defined by

$$\mathcal{T}_{\text{disc}} := \{U : U \subseteq X\},$$

i.e., every subset of X is open. The pair $(X, \mathcal{T}_{\text{disc}})$ is called a *discrete topological space*.

Definition 1.1.10 (Indiscrete (or trivial) topological space)**.** Let X be a set. The *indiscrete topology* (also called the *trivial topology*) on X is the topology $\mathcal{T}_{\text{indisc}}$ defined by

$$\mathcal{T}_{\text{indisc}} := \{\emptyset, X\},$$

i.e., only the empty set and the entire set are open. The pair $(X, \mathcal{T}_{\text{indisc}})$ is called an *indiscrete topological space*.

Definition 1.1.11 (Closure of a subset)**.** Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$ a subset. The *closure of A in X*, denoted by $\overline{A}$, is defined as the intersection of all closed sets containing A, i.e.,

$$\overline{A} := \bigcap\{C \subseteq X : C \text{ is closed and } A \subseteq C\}.$$

Equivalently, $\overline{A}$ is the smallest closed set containing A.

1.2. **Properties of topological spaces.**

Definition 1.2.1 (Dense subset)**.** Let $(X, \mathcal{T})$ be a topological space. A subset $D \subseteq X$ is called *dense in X* if the closure of D equals X, i.e.,

$$\overline{D} = X.$$

Equivalently, D is dense in X if every non-empty open set $U \in \mathcal{T}$ intersects D, that is,

$$\forall U \in \mathcal{T},\ U \neq \emptyset \implies U \cap D \neq \emptyset.$$

Definition 1.2.2 (Connected topological space)**.** Let X be a topological space. The space X is called a *connected topological space*, or simply *connected*, if it cannot be written as a disjoint union $X = U \sqcup V$ of two nonempty open subsets $U, V \subseteq X$. Equivalently, X is connected if the only subsets of X that are both open and closed are $\varnothing$ and X itself.

1.2.1. *Compact topological spaces.*

Definition 1.2.3 (Compact topological space)**.** A topological space $(X, \mathcal{T})$ is *compact* if every open cover of X admits a finite subcover; that is, for every collection $\{U_i\}_{i \in I}$ of open sets in $\mathcal{T}$ such that $X = \bigcup_{i \in I} U_i$, there exists a finite subcollection $\{U_{i_j}\}_{j=1}^n$ such that $X = \bigcup_{j=1}^n U_{i_j}$.

Some mathematicians, e.g. algebraic geometers, would refer to this property as *quasi-compactness*.

Here is an alternate, essentially equivalent definition, of compactness:

Definition 1.2.4 (Compactness)**.** Let $(X, \mathcal{T})$ be a topological space. A subset $K \subseteq X$ is *compact* if for every collection $\{U_i\}_{i \in I}$ of open sets such that $K \subseteq \bigcup_{i \in I} U_i$, there exists a finite subcollection $\{U_{i_j}\}_{j=1}^n$ with $K \subseteq \bigcup_{j=1}^n U_{i_j}$.

Proposition 1.2.5. Let $(X, \mathcal{T})$ be a topological space. A subspace $K \subseteq X$ is compact in the sense of **??** if and only if the space K is compact in the sense of **??**.

Definition 1.2.6 (Locally compact)**.** Let $(X, \mathcal{T})$ be a topological space. X is *locally compact* if for every $x \in X$, there exists an open set $U \in \mathcal{T}$ containing x and a compact set $K \subseteq X$ such that $x \in U \subseteq K$.

Definition 1.2.7. Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$.

- The *interior of A*, denoted by $\operatorname{int}(A)$, is the union of all open sets contained in A:
$$\operatorname{int}(A) = \bigcup \{U \in \mathcal{T} : U \subseteq A\}.$$
- The *boundary of A*, denoted by ∂A, is defined as
$$\partial A = \overline{A} \setminus \operatorname{int}(A).$$
Equivalently, a point $x \in X$ belongs to ∂A if every open neighborhood of x intersects both A and its complement $X \setminus A$.

1.3. **Continuous maps of topological spaces.**

Definition 1.3.1. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f : X \to Y$ is called *continuous* if for every open set $V \in \mathcal{T}_Y$, the preimage $f^{-1}(V)$ is an open set in X, that is,
$$\forall V \in \mathcal{T}_Y,\ f^{-1}(V) \in \mathcal{T}_X.$$
Equivalently, f is continuous if and only if for every closed set $C \subseteq Y$, the preimage $f^{-1}(C)$ is closed in X.

A *map of topological spaces* usually refers to a continuous map between the topological spaces.

The set of continuous maps from X to Y is sometimes denoted by $C(X, Y)$. Other standard notation include $\operatorname{Hom}_{\mathrm{Top}}(X, Y)$ or $\mathrm{Top}(X, Y)$ coming from more general notation for morphisms between objects in a category.

Definition 1.3.2. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A function $f : X \to Y$ is called a *homeomorphism* if it satisfies all of the following:

1. f is bijective;
2. f is continuous with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$; and
3. the inverse map $f^{-1} : Y \to X$ is also continuous.

If such a function exists, the spaces X and Y are said to be *homeomorphic*.

Definition 1.3.3 (Pointed topological space)**.** Let X be a topological space and let $x_0 \in X$ be a chosen element of X. A *pointed/based (topological) space* is a pair (X, x_0) consisting of the space X together with the distinguished point x_0, called the *base point of X*. If the base point of a pointed space (X, x_0) is understood, then it may be suppressed from notation; in particular, X may be written as a pointed space as opposed to the full notation of (X, x_0).

A *morphism of pointed spaces* (or *based map*) or *continuous map* between pointed spaces (X, x_0) and (Y, y_0) is a continuous map

$$f : X \to Y$$

such that $f(x_0) = y_0$.

The collection of pointed spaces with their morphisms form a locally small category, often called the *category of pointed spaces*. This category is often denoted by notations such as Top_*, $\mathrm{Top}_\bullet$, $\mathbf{Top}_*$, $\mathbf{Top}_\bullet$, etc. The set of continuous maps from pointed spaces X to Y may denoted by notations such as $C_*(X, Y)$, $C_\bullet(X, Y)$, $\mathrm{Top}_*(X, Y)$, $\mathrm{Top}_\bullet(X, Y)$, $\mathrm{Hom}_{\mathrm{Top}_\bullet}(X, Y)$, etc.

Definition 1.3.4 (Preimage of a subset under a map of sets)**.** Let X and Y be sets and let $f : X \to Y$ be a function. Let $B \subseteq Y$ be a subset of the codomain Y. The *preimage of B under f* (also called the *inverse image of B under f*) is the subset of X defined by

$$f^{-1}(B) := \{x \in X : f(x) \in B\} \subseteq X.$$

If B is a singleton set $\{y\}$, then we often denote $f^{-1}(B)$ by $f^{-1}(y)$.

Definition 1.3.5 (Fiber of a map of topological spaces)**.** Let X and Y be topological spaces and let $f : X \to Y$ be a continuous map. For a point $y \in Y$, the *fiber of f over y* is the inverse image $f^{-1}(y) = f^{-1}(\{y\})$ endowed with the subspace topology induced from X. The fiber is also denoted by notations such as $\mathrm{Fib}_f(y)$ or X_y.

1.4. **Compactly generated.**

Definition 1.4.1. (♠ TODO: final topology) A topological space X is said to be *compactly generated* (or a *k-space*) if a subset $U \subseteq X$ is open whenever for every compact subset $K \subseteq X$, the intersection $U \cap K$ is open in the subspace K. Equivalently, X is compactly generated if and only if the topology of X is the final topology with respect to the collection of inclusions $K \hookrightarrow X$ for compact $K \subseteq X$.

Definition 1.4.2. A topological space X is said to be *weakly Hausdorff* if for every continuous map $f : K \to X$ from a compact Hausdorff space K, the image $\mathrm{Im}(f)$ is closed in

X. Equivalently, X is weakly Hausdorff if and only if for every compact Hausdorff K, the induced map $f : K \to X$ is a closed map.

1.5. **Metrics, absolute values, and norms.**

1.5.1. *Extended metric spaces.*

Definition 1.5.1 (Extended Metric)**.** Let M be a set. An *extended metric* on M is a function

$$d : M \times M \to [0, \infty]$$

such that for all $x, y, z \in M$:

1. **Non-negativity:** $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$.
2. **Symmetry:** $d(x, y) = d(y, x)$.
3. **Triangle inequality:** $d(x, z) \leq d(x, y) + d(y, z)$,

again adopting the convention that sums involving ∞ behave so that $a + \infty = \infty$. A set equipped with an extended metric is called an *extended metric space*.

If the range of the extended metric is contained in $[0, \infty)$, then the extended metric is a *metric* in the usual sense and V may be called a *metric space*.

Definition 1.5.2 (Isometry for extended metric spaces)**.** Let (X, d_X) and (Y, d_Y) be extended metric spaces. A map $f : X \to Y$ is called an *isometry* if for all $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$

An isometry is automatically injective. If, in addition, f is surjective, it is called an *isometric isomorphism*, and X and Y are said to be *isometric*.

Definition 1.5.3 (Extended Metric Induced by an Extended Norm)**.** Let V be a vector space over a field F equipped with an absolute value

$$| \cdot | : F \to [0, \infty),$$

and let $\| \cdot \| : V \to [0, \infty]$ be an extended norm on V. Then the *extended metric induced by the extended norm* is the function

$$d : V \times V \to [0, \infty]$$

defined by

$$d(x, y) := \|x - y\|.$$

It is indeed an extended metric. If $\| \cdot \|$ is a norm, then d is a metric.

Definition 1.5.4 (Topology induced by an extended metric)**.** Let (X, d) be an extended metric space.

1. Given $\varepsilon \in (0, \infty)$ and $x \in X$, the set

   $$B(x, \varepsilon) := \{y \in X : d(x, y) < \varepsilon\}$$

   is called the *open ball oj radisu ε centered at x*.

2. The *topology induced by d on X* is the topology defined by declaring a subset $U \subseteq X$ to be open if for every $x \in U$, there exists $\varepsilon \in (0, \infty)$ such that $B(x, \varepsilon)$ is contained in U.

1.6. **Convergence in extended metric spaces.**

Definition 1.6.1 (Convergence and Limits in an Extended Metric Space)**.** Let (X, d) be an extended metric space. A sequence $(x_n)_{n=1}^{\infty}$ in X *converges to a point $x \in X$* if for every $\varepsilon > 0$, there exists an integer $N \in \mathbb{N}$ such that for all $n \geq N$,

$$d(x_n, x) < \varepsilon.$$

In that case, x is called the *limit of the sequence (x_n)*, and we write

$$\lim_{n \to \infty} x_n = x \quad \text{or} \quad x_n \to x$$

Definition 1.6.2 (Cauchy sequence in an extended metric space)**.** Let (M, d) be an extended metric space. A sequence $(x_n)_{n \in \mathbb{N}}$ in M is called a *Cauchy sequence* if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $d(x_m, x_n) < \epsilon$ (and thus, in particular, all pairwise distances $d(x_m, x_n)$ are finite for $m, n \geq N$).

Definition 1.6.3. Let (X, d) be an extended metric space. The space (X, d) is said to be *complete* if every Cauchy sequence in X converges to a point in X; that is, for every sequence $(x_n)_{n \in \mathbb{N}}$ in X such that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ with $d(x_m, x_n) < \varepsilon$ for all $m, n \geq N$, there exists $x \in X$ such that

$$\lim_{n \to \infty} d(x_n, x) = 0.$$

Definition 1.6.4. Let (X, d) be an extended metric space. A *completion of (X, d)* is a pair $(\widehat{X}, \widehat{d})$ together with an isometric embedding $i : X \to \widehat{X}$, where:

- $(\widehat{X}, \widehat{d})$ is a complete extended metric space,
- $i(X)$ is dense in $\widehat{X}$ (with respect to the topologies induced by the extended metrics),
- $d(x, y) = \widehat{d}(i(x), i(y))$ for all $x, y \in X$.

(♠ TODO: prove that completions of extended metric spaces exist) As such a space $\widehat{X}$ exist, and is unique up to unique isometry, we may refer to the space $\widehat{X}$ as called *the completion of X (with respect to d)*.

Proposition 1.6.5. Let (X, d) be an extended metric space. A completion $(\widehat{X}, \widehat{d})$ of (X, d) exists and is unique up to unique isometry. Moreover, if (X, d) is a metric space, then so is $(\widehat{X}, \widehat{d})$.

Proof. (♠ TODO: :) actually complete this proof Let $\mathcal{C}$ be the set of all Cauchy sequences in (X, d):

$$\mathcal{C} = \{(x_n)_{n \in \mathbb{N}} : (x_n) \text{ is a Cauchy sequence in } (X, d)\},$$

and define a relation $\sim$ on $\mathcal{C}$ by

$$(x_n) \sim (y_n) \iff \lim_{n \to \infty} d(x_n, y_n) = 0.$$

This relation is well-defined and in fact an equivalence relation. (♠ TODO: Show that the relation is well-defined and an equivalence relation) Let $\widehat{X} = \mathcal{C}/\sim$ and let $\widehat{d} : \widehat{X} \times \widehat{X} \to [0, \infty]$ be defined by

$$\widehat{d}([(x_n)], [(y_n)]) := \lim_{n\to\infty} d(x_n, y_n).$$

This limit exists as a value in $[0, \infty]$ and is independent of representative sequences chosen. (♠ TODO: Demonstrate why the limit exists and is well defined)

It can be verified that $\widehat{d}$ is an extended metric on $\widehat{X}$ and that $(\widehat{X}, \widehat{d})$ is complete by construction.

Define an embedding

$$i : X \to \widehat{X}, \quad i(x) := [(x, x, x, \ldots)],$$

which is an isometry:

$$\widehat{d}(i(x), i(y)) = \lim_{n\to\infty} d(x, y) = d(x, y).$$

Moreover, $i(X)$ is dense in $\widehat{X}$ because any equivalence class $[(x_n)]$ of a Cauchy sequence can be approximated arbitrarily closely by points $i(x_n)$ in the image of X.

Uniqueness:

Suppose $(\widehat{X}', \widehat{d}')$ is another completion of (X, d) with isometric embedding $i' : X \to \widehat{X}'$.

By the universal property of completions, there exists a unique isometry

$$\Phi : \widehat{X} \to \widehat{X}'$$

such that $\Phi \circ i = i'$.

The map Φ is onto since $i'(X)$ is dense in $\widehat{X}'$ and Φ preserves limits of Cauchy sequences.

Therefore, Φ is a unique isometry between the completions, and the completion is unique up to unique isometry. $\square$

1.6.1. *Absolute values on fields.*

Definition 1.6.6 (Absolute Value on a Field)**.** Let F be a field. An *absolute value on F* is a function

$$|\cdot| : F \to \mathbb{R}_{\geq 0}$$

that satisfies the following properties for all $a, b \in F$:

1. Non-negativity: $|a| \geq 0$,
2. Positive-definiteness: $|a| = 0 \iff a = 0$,
3. Multiplicativity: $|ab| = |a| \cdot |b|$,
4. Triangle inequality: $|a + b| \leq |a| + |b|$.

Here, 0 denotes the additive identity of the field F, and the codomain $\mathbb{R}_{\geq 0}$ consists of non-negative real numbers.

1.6.2. *Norms on vector spaces over fields with absolute values.*

Definition 1.6.7 (Extended Norm)**.** Let V be a vector space over a field F equipped with an absolute value

$$|\cdot| : F \to [0, \infty).$$

An *extended norm on V* is a function

$$\|\cdot\| : V \to [0, \infty]$$

satisfying for all $x, y \in V$ and all scalars $\alpha \in F$:

1. **Positive definiteness:** $\|x\| = 0$ if and only if $x = 0$.
2. **Homogeneity:** $\|\alpha x\| = |\alpha| \cdot \|x\|$.
3. **Triangle inequality:** $\|x + y\| \leq \|x\| + \|y\|$,

where arithmetic is extended to allow sums involving ∞ with the convention that $a+\infty = \infty$ for any $a \in [0, \infty]$. A vector space with an extended norm over a field with an absolute value is called an *extended normed (vector) space*.

If the range of the extended norm is contained in $[0, \infty)$, then the extended norm is a *norm* in the usual sense and V may be called a *normed (vector) space*.

Definition 1.6.8 (Topology induced by a norm)**.** Let V be a vector space over a field K with absolute value $|\cdot|$, and let $\|\cdot\|$ be an extended norm on V. The *topology induced by the extended norm $\|\cdot\|$ on V* is defined by declaring a subset $U \subseteq V$ to be open if for every $x \in U$, there exists $\varepsilon > 0$ such that

$$B(x, \varepsilon) := \{y \in V : \|y - x\| < \varepsilon\}$$

is contained in U. The set $B(x, \varepsilon)$ is called the *open ball of radius ε around x*. The collection of all such open sets forms a topology on V.

1.6.3. *Inner product on a vector space over the reals or complex numbers.*

Definition 1.6.9. An *inner product on a vector space V over a field $\mathbb{F}$* (either $\mathbb{R}$ or $\mathbb{C}$) is a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$$

satisfying, for all $u, v, w \in V$ and all scalars $\alpha \in \mathbb{F}$:

1. **Conjugate symmetry:** $\langle u, v \rangle = \overline{\langle v, u \rangle}$
2. **Linearity in the first argument:** $\langle \alpha u + v, w \rangle = \alpha \langle u, w \rangle + \langle v, w \rangle$
3. **Positive-definiteness:** $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ if and only if $v = 0$

A vector space over $\mathbb{F}$ equipped with such an inner product is called an *inner product space*.

Definition 1.6.10. Let V be a vector space over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $\langle \cdot, \cdot \rangle$ be an inner product. The *norm induced by $\langle \cdot, \cdot \rangle$* is the norm

$$\|\cdot\| : V \to \mathbb{R}_{\geq 0}$$

defined by

$$\|v\| = \sqrt{\langle v, v \rangle}$$

for any $v \in V$.

1.7. **Euclidean space.**

Definition 1.7.1. For a positive integer n, let $\mathbb{R}^n$ denote the n-fold Cartesian product of the real line $\mathbb{R}$ with itself:

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R} \text{ for all } i = 1, \ldots, n\}.$$

The set $\mathbb{R}^n$ is called Euclidean n-space. A point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is associated with the Euclidean norm

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

The corresponding metric $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ is given by

$$d(x, y) = \|x - y\|.$$

This metric induces the standard topology on $\mathbb{R}^n$, called the *Euclidean topology*.

2. Homotopy

The notions of homotopy have the following variants:

1. Absolute, on unpointed spaces
2. Relative, on unpointed spaces
3. Absolute, on pointed spaces
4. Relative, on pointed spaces.

Definition 2.0.1 (Homotopy of maps of topological spaces)**.** Let X and Y be topological spaces and let $K \subseteq X$ be a subset. Let $C(X, Y)$ denote the set of all continuous maps $f : X \to Y$.

1. A *homotopy between two maps $f, g \in C(X, Y)$ relative to K* is a continuous map

$$H : X \times [0, 1] \to Y$$

such that for all $x \in X$,

$$H(x, 0) = f(x), \quad H(x, 1) = g(x),$$

and for all $x \in K$ and $t \in [0, 1]$,

$$H(x, t) = f(x) = g(x).$$

If such an H exists, we say f and g are *homotopic relative to K*, and we write $f \simeq g$ rel K; this is an equivalence relation.

A *homotopy between two maps $f, g \in C(X, Y)$* is simply a homotopy relative to $\emptyset$. We write we write $f \simeq g$ if a homotopy between them exists.

2. Let (X, x_0) and (Y, y_0) be pointed topological spaces and let $K \subseteq X$ be a subset with $x_0 \in K$. Let $C_*(X, Y)$ denote the set of all continuous based maps $f : X \to Y$ satisfying $f(x_0) = y_0$.

   A *homotopy of based maps $f, g \in C_*(X, Y)$ relative to K* is a continuous map

$$H : X \times [0, 1] \to Y$$

   such that for all $x \in X$,

$$H(x, 0) = f(x), \quad H(x, 1) = g(x),$$

   and for all $k \in K$ and $t \in [0, 1]$,

$$H(k, t) = f(k) = g(k),$$

   in particular fixing the basepoint throughout,

$$H(x_0, t) = y_0 \quad \text{for all } t \in [0, 1].$$

   If such an H exists, we say f and g are *based homotopic relative to K*, and we write $f \simeq g$ rel K. This is an equivalence relation.

   A *homotopy of based maps $f, g \in C_*(X, Y)$* without relative condition is the special case $K = \{x_0\}$ and is called a *homotopy of based maps* or *based homotopy*. We write $f \simeq g$ if such a homotopy exists.

Definition 2.0.2 (Homotopy equivalence)**.** Let X and Y be topological spaces and let $K \subseteq X$.

1. A continuous map $f : X \to Y$ is a *homotopy equivalence relative to K* if there exists a continuous map $g : Y \to X$ such that

$$g \circ f \simeq \mathrm{id}_X \ \text{rel } K \quad \text{and} \quad f \circ g \simeq \mathrm{id}_Y \ \text{rel } f(K).$$

   (**??**) That is, there exist homotopies $H : X \times [0, 1] \to X$ and $G : Y \times [0, 1] \to Y$ satisfying

$$\begin{aligned} &H(x,0) = (g \circ f)(x), && H(x,1) = x, && H(k,t) = k \text{ for all } k \in K, t \in [0,1], \\ &G(y,0) = (f \circ g)(y), && G(y,1) = y, && G(f(k),t) = f(k) \text{ for all } k \in K, t \in [0,1]. \end{aligned}$$

   In this case, the map g is called a *homotopy inverse relative to K of f*, and the pairs (X, K) and $(Y, f(K))$ are said to be *homotopy equivalent relative to K*, denoted $(X, K) \simeq (Y, f(K))$ rel K. This is an eqiuvalence relation.

   A continuous map $f : X \to Y$ is a *homotopy eqiuvalence* if it is a homotopy equivalence relative to $\emptyset$. A *homotopy inverse of f* is then simply a homotopy inverse of f relative to $\emptyset$. In this case, and the spaces X and Y are said to be *homotopy equivalent*, denoted $X \simeq Y$. This is an equivalence relation.

2. Let (X, x_0) and (Y, y_0) be pointed topological spaces and let $K \subseteq X$ be a subset with $x_0 \in K$.

   A continuous map $f : (X, x_0) \to (Y, y_0)$ of pointed spaces is a *homotopy equivalence relative to K* if there exists a continuous map $g : (Y, y_0) \to (X, x_0)$ such that

$$g \circ f \simeq \mathrm{id}_X \ \text{rel } K \quad \text{and} \quad f \circ g \simeq \mathrm{id}_Y \ \text{rel } f(K),$$

   where the homotopies are homotopies of based maps relative to K and $f(K)$, respectively.

That is, there exist homotopies of based maps $H : X\times[0,1] \to X$ and $G : Y\times[0,1] \to Y$ satisfying

$$H(x,0) = (g \circ f)(x), \qquad H(x,1) = x, \qquad H(k,t) = k \text{ for all } k \in K, t \in [0,1],$$
$$G(y,0) = (f \circ g)(y), \qquad G(y,1) = y, \qquad G(f(k),t) = f(k) \text{ for all } k \in K, t \in [0,1],$$

with $H(x_0,t) = x_0$ and $G(y_0,t) = y_0$ for all t.

The map g is called a *homotopy inverse relative to K of f*, and the pairs (X,K) and $(Y,f(K))$ are said to be *homotopy equivalent relative to K*, denoted $(X,K) \simeq (Y,f(K))$ rel K. This is an equivalence relation.

A continuous map $f : (X,x_0) \to (Y,y_0)$ of pointed spaces is a *homotopy equivalence* if it is a homotopy equivalence relative to $\{x_0\}$. A *homotopy inverse of f* is then simply a homotopy inverse relative to $\{x_0\}$. In this case, the pointed spaces (X,x_0) and (Y,y_0) are said to be *homotopy equivalent*, denoted $(X,x_0) \simeq (Y,y_0)$.

Definition 2.0.3 (Homotopy class of maps relative to a subset)**.** Let X and Y be topological spaces and let $K \subseteq X$. Let $C(X,Y)$ denote the set of all continuous maps $f : X \to Y$.

1. Two maps $f,g \in C(X,Y)$ are said to be in the same *homotopy class relative to K* if there exists a homotopy relative to K

   $$H : X \times [0,1] \to Y$$

   such that

   $$H(x,0) = f(x), \quad H(x,1) = g(x),$$

   and

   $$H(k,t) = f(k) = g(k) \quad \text{for all } k \in K, t \in [0,1].$$

   The *homotopy class of maps relative to K* containing a map $f : X \to Y$ is denoted by $[f]_K$.

   Two maps $f,g \in C(X,Y)$ are said to be in the same *homotopy class* if they are in the same homotopy class relative to $\emptyset$.

   The *homotopy class of maps* containing a map $f : X \to Y$ is denoted by $[f]$.

   The set of homotopy classes of maps may often be denoted by $[X,Y]$.
2. Let (X,x_0) and (Y,y_0) be pointed topological spaces and let $K \subseteq X$ be a subset containing x_0. Let $C_*(X,Y)$ denote the set of all continuous based maps $f : X \to Y$ with $f(x_0) = y_0$.

   Two based maps $f,g \in C_*(X,Y)$ are said to be in the same *homotopy class relative to K* if there exists a homotopy of based maps relative to K

   $$H : X \times [0,1] \to Y$$

   such that for all $x \in X$,

   $$H(x,0) = f(x), \quad H(x,1) = g(x),$$

   and for all $k \in K$ and $t \in [0,1]$,

   $$H(k,t) = f(k) = g(k),$$

   particularly ensuring the basepoint is fixed throughout,

   $$H(x_0,t) = y_0 \quad \text{for all } t \in [0,1].$$

The *homotopy class relative to K* containing $f : (X, x_0) \to (Y, y_0)$ is denoted by $[f]_K$.

Two based maps $f, g \in C_*(X, Y)$ are said to be in the same *homotopy class* if they are in the same homotopy class relative to $\{x_0\}$.

The *homotopy class* containing a map $f : (X, x_0) \to (Y, y_0)$ is denoted by $[f]$.

The set of homotopy classes of pointed maps $(X, x_0) \to (Y, y_0)$ may often be denoted by $[(X, x_0), (Y, y_0)]$ or by $[X, Y]$ if the base points are clear.

Definition 2.0.4. The *homotopy category of topological spaces*, denoted hTop, is the category whose objects are topological spaces and whose morphisms are homotopy classes of continuous maps. In other words, for objects X and Y, the set of morphisms is defined as $\mathrm{Hom}_{\mathrm{hTop}}(X, Y) = [X, Y] = C(X, Y)/\simeq$.

Proposition 2.0.5. Composition in the homotopy category of topological spaces is well-defined. If $f_1, f_2 : X \to Y$ are homotopic and $g_1, g_2 : Y \to Z$ are homotopic, then the compositions $g_1 \circ f_1$ and $g_2 \circ f_2$ are homotopic as maps from X to Z. That is, $[g] \circ [f] = [g \circ f]$ is independent of the choice of representatives.

Theorem 2.0.6. There exists a canonical functor $Q : \mathrm{Top} \to \mathrm{hTop}$ which is the identity on objects and maps each continuous map f to its homotopy class $[f]$. This functor is full and essentially surjective.

Definition 2.0.7 (Path in a topological space)**.** Let X be a topological space, let $x_0, x_1 \in X$, and let $[0, 1] \subset \mathbb{R}$ be the closed unit interval equipped with the subspace topology from $\mathbb{R}$.

1. A *path in X from x_0 to x_1* is a continuous map $\gamma : [0, 1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. More generally, a *path in X* is a path from some point $x_0 \in X$ to some point $x_1 \in X$ in this sense.
2. A *loop in X based at $x_0 \in X$* is a path $\gamma : [0, 1] \to X$ in X from x_0 to x_0, that is, a continuous map γ such that $\gamma(0) = \gamma(1) = x_0$. When a basepoint x_0 is fixed and understood from the context, a *loop in X* means a loop in X based at this chosen basepoint.

Definition 2.0.8 (Homotopy groups)**.** For any pointed topological space (X, x_0) and integer $n \geq 0$, the *n-th homotopy group of X at x_0*, denoted $\pi_n(X, x_0)$, is defined as the set of all homotopy classes (rel. ∂I^n) of based maps

$$f : (I^n, \partial I^n) \to (X, x_0),$$

where $I^n = [0, 1]^n$. For $n \geq 1$, $\pi_n(X, x_0)$ is a group under concatenation of based maps, and for $n \geq 2$, it is abelian.

The *fundamental group of (X, x_0)* refers to $\pi_1(X, x_0)$. Equivalently, it is the group of homotopy classes (rel. endpoints) of loops $\gamma : [0, 1] \to X$ satisfying $\gamma(0) = \gamma(1) = x_0$.

Definition 2.0.9 (Weak homotopy equivalence)**.** Let $f : X \to Y$ be a continuous map between topological spaces. The map f is a *weak homotopy equivalence* if, for every choice of basepoint $x_0 \in X$ with $y_0 = f(x_0)$, the induced maps on homotopy groups

$$f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$$

are isomorphisms for all integers $n \geq 0$.

Definition 3.0.1. Let V be a real vector space of finite dimension. A *k-simplex in topology* (or a *geometric k-simplex*) is the convex hull of $k+1$ affinely independent points $v_0, v_1, \dots, v_k \in V$, and is denoted by

$$[v_0, v_1, \dots, v_k] := \left\{ \sum_{i=0}^{k} t_i v_i \;\middle|\; t_i \geq 0,\ \sum_{i=0}^{k} t_i = 1 \right\}.$$

It is also standard to talk of the *standard topological n-simplex* — the topological space $|\Delta^n|$ defined as the subset of Euclidean space $\mathbb{R}^{n+1}$ given by

$$|\Delta^n| = \left\{ (t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} t_i = 1,\ \text{and } t_i \geq 0 \text{ for all } i \right\}$$

equipped with the induced topology from the usual Euclidean topology on $\mathbb{R}^{n+1}$.

(♠ TODO: comment on how $|\Delta^n|$ makes sense via a geometric realization)

Definition 3.0.2 (Singular simplices)**.** Let X be a topological space. For each integer $n \geq 0$, A *singular n-simplex in X* is a continuous map

$$\sigma : \Delta^n \to X.$$

where Δ^n is the standard topological n-simplex. The set of all singular n-simplices in X is denoted by $S_n(X)$.

Definition 3.0.3 (Singular chain group with coefficients)**.** Let X be a topological space, let $S_n(X)$ be the set of singular n-simplices in X, and let R be a commutative ring with unity.

1. The *singular n-chain group of X with coefficients in R* is the free R-module $C_n(X; R)$ whose elements are finite formal linear combinations

$$\sum_i r_i \sigma_i, \quad \text{with } \sigma_i \in S_n(X),\ r_i \in R.$$

   Elements of $C_n(X; R)$ are called *singular n-chains in X with coefficients in R*.
2. If $A \subseteq X$ is a subspace, the quotient groups

$$C_n(X, A; R) = C_n(X; R)/C_n(A; R)$$

   may be referred to as the *relative singular n-chain groups of X in A with coefficients in R* and elements of this group may be referred to as *relative singular n-chains in X relative to A with coefficients in R*.

In either case, when $R = \mathbb{Z}$, the ring R may be suppressed from notation, so we may write $C_n(X)$ and $C_n(X, A)$ for $C_n(X, \mathbb{Z})$ and $C_n(X, A, \mathbb{Z})$ respectively.

Definition 3.0.4 (Singular chain complex with coefficients)**.** Let X be a topological space, and let R be a commutative ring with unity.

For each $n \geq 1$, define the R-linear *boundary operator*

$$\partial_n : C_n(X;R) \to C_{n-1}(X;R)$$

by

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \, \sigma \circ \delta_i,$$

where $\delta_i : \Delta^{n-1} \to \Delta^n$ is the i-th face inclusion. Extend ∂_n to $C_n(X;R)$ by R-linearity. Then $(C_n(X;R), \partial_n)$ forms a chain complex in the abelian category of R-modules. This chain complex is called the *singular chain complex of X with coefficients in R*.

If $A \subseteq X$ is a subspace, then the boundary maps above induce maps

$$C_n(X, A; R) \to C_{n-1}(X, A; R)$$

on the relative chain groups $C_n(X, A; R)$, yielding a chain complex of R-modules; this chain complex may be called the *relative singular chain complex of the pair (X, A) with coefficients in R*.

Definition 3.0.5 (Singular homology with coefficients)**.** Let X be a topological space and R a commutative ring with 1. The *n-th singular homology group of X with coefficients in R* is the homology group

$$H_n(X;R) = H_n(C_*(X;R))$$

where $C_*(X;R)$ is the singular chain complex of X with coefficients in R.

Given a subspace $A \subseteq X$, the *n-th relative singular homology group of (X, A) with coefficients in R* is defined as

$$H_n(X, A; R) = H_n(C_*(X, A; R))$$

where $C_*(X, A; R)$ is the relative singular chain complex of (X, A) with coefficients in R.

We may denote $H_n(X;\mathbb{Z})$ and $H_n(X, A;\mathbb{Z})$ by $H_n(X)$ and $H_n(X, A)$ respectively.

Definition 3.0.6 (Singular cochain complex of a topological space)**.** Let X be a topological space and R a commutative ring with 1. The *singular n-cochain group of X with coefficients in R* is the R-module

$$C^n(X;R) = \mathrm{Hom}_R(C_n(X;R), R) .$$

We define *coboundary operators*

$$\delta^n : C^n(X;R) \to C^{n+1}(X;R), \quad \delta^n(\varphi) = \varphi \circ \partial_{n+1}.$$

The pair $(C^n(X;R), \delta^n)_n$ form a cochain complex called the *singular cochain complex of X with coefficients in R*. Given a subspace $A \subseteq X$, the *relative singular n-cochain group of (X, A) with coefficients in R* is the R-module

$$C^n(X, A; R) = \mathrm{Hom}_R(C_n(X, A; R), R) .$$

The coboundary operators δ^n induce maps $C^n(X, A; R) \to C^{n+1}(X, A; R)$ for which the $C^n(X, A; R)$ form a cochain complex, called the *relative singular cochain complex of (X, A) with coefficients in R*

Definition 3.0.7. Let X be a topological space and R a commutative ring with 1. The *n-th singular cohomology group of X with coefficients in R* is

$$H^n(X;R) = \ker(\delta^n)/\operatorname{im}(\delta^{n-1}).$$

For a subspace $A \subseteq X$ the *relative cohomology groups of* (X,A) are

$$H^n(X,A;R) = H^n(C^*(X,A;R)).$$

In other words, these singular cohomology groups are the cohomology R-modules of the singular cochain complex $C^*(X;R)$ and the relative singular cochain complex $C^*(X,A;R)$ respectively.

Appendix A. Set theory

Definition A.0.1. Let A be a set.

- The set A is said to be *countably infinite* (or simply *countable*) if there exists a bijection $f : \mathbb{N} \to A$.
- The set A is said to be *finite* if there exists some $n \in \mathbb{N}$ and a bijection $g : \{1, 2, \ldots, n\} \to A$.
- The set A is said to be *at most countable* if it is either finite or countably infinite.
- The set A is said to be *uncountable* if it is not at most countable.

Definition A.0.2. Let X and Y be sets and let $f : X \to Y$ be a function.

- The function f is said to be *injective* (or *one-to-one*) if for all $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
- The function f is said to be *surjective* (or *onto*) if for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$.
- The map f is *bijective* if it is both injective and surjective. In this case, there exists a unique *inverse map* $f^{-1} : Y \to X$ such that for all $x \in X$ and $y \in Y$,

$$f^{-1}(f(x)) = x \text{ and } f(f^{-1}(y)) = y.$$

Appendix B. Category Theory

Definition B.0.1 (Category)**.** A category name=Category, description=A nice enough collection of objects and morphisms (**??**), *category*[format=textbf]category *category* $\mathcal{C}$ consists of the following data:

- A class of object$_o f_{ac} ategoryname = Objectofacategory, description =$ **??**,$[format = textbf]object_o f_{ac} ategorydenotedclass_o f_o bjects_o f_{ac} ategoryname = \mathrm{Ob}(\mathcal{C})$, description=Class of objects of a category $\mathcal{C}$ **??**, sort=Ob, $\mathrm{Ob}(\mathcal{C})$[format=textbf]class$_o f_o bjects_o f_{ac} ategory.$

- For each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a class class$_of_morphisms_between_two_objects_of_a_category name =$ description=Class of morphisms between objects X and Y of the category $\mathcal{C}$ (**??**), sort=Hom,

$$\mathrm{Hom}_{\mathcal{C}}(X, Y)$$

[format=textbf]class$_of_morphisms_between_two_objects_of_a_category of morphism_between_objects_of_a_categ$ $textbf]morphism_between_objects_of_a_category(also called arrow_between_objects_of_a_category name = Arr$ $textbf]arrow_between_objects_of_a_category or hom_between_objects_of_a_category name = Hom between objec$ $textbf]hom_between_objects_of_a_category). If the category$ C $is clear, then this$ is also $denoted by$. It may also b

- For each triple of objects X, Y, Z, a composition law

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z),$$

denoted $(g, f) \mapsto g \circ f$.
- For each object X, an *identity morphism*

$$\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X).$$

These data satisfy the following axioms:

- (Associativity) For all morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, and $h \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

- (Identity) For all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$,

$$\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X .$$

One often writes $X \in \mathcal{C}$ synonymously with $X \in \mathrm{Ob}(\mathcal{C})$, i.e. to denote that X is an object of of $\mathcal{C}$.

We may call a category as above an *ordinary category* to distinguish this notion from the notions of *categories enriched in monoidal categories* or higher/n-categories. (♠ TODO: TODO: define n-categories)

A category as defined above may be called called a *large category* or a *class category* to emphasize that the hom-classes may be proper classes rather than sets (note, however, that the possibility that hom-classes are sets is not excluded for large categories). Accordingly, a *category* may often refer to a locally small category, which is a category whose hom-classes are all sets.

Definition B.0.2 (Locally small category)**.** A (large) category $\mathcal{C}$ is called a *locally small category* if for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the collection $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms between them is a (small) *set* (as opposed to a proper class). In other words, each hom-class is a set and may even be called a *hom-set*.

In some contexts, a locally small category may simply be called a *category*, especially when genuinely large categories are not considered.

A category $\mathcal{C}$ is called a *small category* if it is a locally small category and the class $\mathrm{Ob}(\mathcal{C})$ of objects is a set.

Remark B.0.3. Many "concrete" categories considered in "classical mathematics" or outside of more "abstract" category theory tend to be locally small. For example, the categories of sets, groups, R-modules, vector spaces, topological spaces, schemes, manifolds, sheaves on "small enough" sites are all locally small.

Appendix C. Group Theory

Definition C.0.1 (Groups)**.** A *group* is a pair $(G, \cdot)$ where G is a set and $\cdot : G \times G \to G$ is a binary operation, subject to the following conditions:

1. (Associativity) For all $g, h, k \in G$ one has

$$(g \cdot h) \cdot k = g \cdot (h \cdot k).$$

2. (Identity element) There exists an element $e \in G$ such that for all $g \in G$,

$$e \cdot g = g \cdot e = g.$$

3. (Inverse element) For all $g \in G$ there exists an element $g^{-1} \in G$ such that

$$g \cdot g^{-1} = g^{-1} \cdot g = e.$$

The element e is called the *identity element of G*, and g^{-1} is called the *inverse of g*.

Equivalently, a group is a monoid with inverse elements.

A group $(G, \cdot)$ is often simply written as G, when the notation for the binary operation $\cdot$ is clear.

An *abelian group* or synonymously, a *commutative group*, is a group $(G, \cdot)$ whose binary operation $\cdot$ is *abelian* or *commutative*, i.e. satisfies

$$g \cdot h = h \cdot g$$

for all $g, h \in G$.

Definition C.0.2 (Group homomorphism)**.** Let $(G, \cdot)$ and $(H, *)$ be groups. A map $f : G \to H$ is called a *group homomorphism* if for all $g_1, g_2 \in G$ one has

$$f(g_1 \cdot g_2) = f(g_1) * f(g_2).$$

The collection of all groups with the group homomorphisms forms a locally small category, called the *category of groups*.

If f is bijective, then f is called a *group isomorphism*. .

Definition D.0.1. A *commutative (unital) ring* is a ring $(R, +, \cdot)$ such that $\cdot$ is a commutative operation, i.e. $a \cdot b = b \cdot a$.

For many writers (e.g. "commutative" algebraists or number theorists), a *ring* refers to a commutative ring as above.

Definition D.0.2. Let R be a not-necessarily commutative ring.

1. A *left R-module* is an abelian group $(M, +)$ together with an operation $R \times M \to M$, denoted $(r, m) \mapsto rm$, such that for all $r, s \in R$ and $m, n \in M$:
   - $r(m + n) = rm + rn$,
   - $(r + s)m = rm + sm$,
   - $(rs)m = r(sm)$,
   - $1_R m = m$ where 1_R is the multiplicative identity of R.
2. A *right R-module* is defined similarly as an abelian group $(M, +)$ with an operation $M \times R \to M$, denoted $(m, r) \mapsto mr$, such that for all $r, s \in R$ and $m, n \in M$:
   - $(m + n)r = mr + nr$,
   - $m(r + s) = mr + ms$,
   - $m(rs) = (mr)s$,
   - $m1_R = m$.
3. Let R and S be (not necessarily commutative) rings.
   An *R-S-bimodule* (or an *R-S-module* or an (R, S)-module, etc.)is an abelian group $(M, +)$ equipped with
   (a) a left action of R:
   $$R \times M \to M, \quad (r, m) \mapsto r \cdot m,$$
   making M a left R-module,
   (b) a right action of S:
   $$M \times S \to M, \quad (m, s) \mapsto m \cdot s,$$
   making M a right S-module,
   such that the left and right actions commute; that is, for all $r \in R$, $s \in S$, and $m \in M$,
   $$r \cdot (m \cdot s) = (r \cdot m) \cdot s.$$
4. A *two-sided R-module* (or *R-bimodule*) is an R-R-bimodule.

If R is a commutative ring, then a left/right R-module can automatically be regarded as a two-sided R-module. As such, we simply talk about *R-modules* in this case.

Any abelian group is equivalent to a two-sided $\mathbb{Z}$-module. Moreover, any left R-module is equivalent to an $R - \mathbb{Z}$-bimodule and any right R-module is equivalent to an $\mathbb{Z} - R$-bimodule. Given a left/right/two-sided R-module, its *natural bimodule structure* will refer to its structure as a R-$\mathbb{Z}$/$\mathbb{Z}$-R/R-R bimodule. In this way, many definitions associated with the notions of left/right/two-sided R-modules can be defined as special cases for definitions for R-S-bimodules.

Definition D.0.3 (Hom of left/right/bi-modules)**.** Let R, S, T be (not necessarily commutative) rings.

1. Let M and N be left R-modules. The *homomorphism group of left R-modules from M to N* is the abelian group

$$\mathrm{Hom}(M,N) = \mathrm{Hom}_R(M,N) := \{f : M \to N \mid f \text{ is a left } R\text{-module homomorphism}\}.$$

2. Let M and N be right R-modules. The *homomorphism group of right R-modules from M to N* is the abelian group

$$\mathrm{Hom}(M,N) = \mathrm{Hom}_R(M,N) := \{f : M \to N \mid f \text{ is a right } R\text{-module homomorphism}\}.$$

3. Let S be a (not necessarily commutative ring) and let M and N be $R-S$-bimodules. The *homomorphism group of R-S-bimodules from M to N* is the abelian group

$$\mathrm{Hom}(M,N) = \mathrm{Hom}_{R-S}(M,N) \coloneqq \{f : M \to N | f \text{ is a } R-S\text{-bimodule homomorphism}\}$$

In each case, $\mathrm{Hom}(M,N)$ has a natural structure of an *abelian group* given by *pointwise addition*: for $f, g \in \mathrm{Hom}(M,N)$,

$$(f+g)(m) := f(m) + g(m),$$

and the zero morphism 0 given by $0(m) := 0_N$ acts as the identity element. The additive inverse $-f$ is defined by $(-f)(m) := -f(m)$. Moreover, depending on bi-module structures that M and N may be carrying, $\mathrm{Hom}(M,N)$ may itself carry additional module structures:

- In case that M is a $R-S$-bimodule and N is a $R-T$-bimodule, $\mathrm{Hom}_R(M,N)$, the group of left R-module homomorphisms, is an $S-T$-bimodule as follows:

$$(s \cdot f \cdot t)(m) = f(m \cdot s) \cdot t \quad f \in \mathrm{Hom}_R(M,N), s \in S, t \in T.$$

- Dually, in case that M is a $S-R$-bimodule and N is a $T-R$-bimodule, $\mathrm{Hom}_R(M,N)$, the group of right R-module homomorphisms, is an $S-T$-bimodule as follows:

$$(s \cdot f \cdot t)(m) = f(s \cdot m) \cdot t \quad f \in \mathrm{Hom}_R(M,N), s \in S, t \in T.$$

Some cases of interest may be when R, S, or T is in fact $\mathbb{Z}$ — these allow us to see module structures on $\mathrm{Hom}(M,N)$ even when M and N are one-sided modules.

(♠ TODO: state this as a theorem) We furthermore note that $\mathrm{Hom}_R(-,-)$ yields biadditive functors

$$\mathrm{Hom}_R(-,-) : {}_R\mathbf{Mod}_S^{\mathrm{op}} \times {}_R\mathbf{Mod}_T \to {}_S\mathbf{Mod}_T$$
$$\mathrm{Hom}_R(-,-) : {}_S\mathbf{Mod}_R^{\mathrm{op}} \times {}_T\mathbf{Mod}_R \to {}_S\mathbf{Mod}_T.$$

Appendix E. Homological algebra

Definition E.0.1 (Chain complex in a preadditive category)**.** Let $\mathcal{A}$ be a preadditive category and let I be a totally ordered set (typically $\mathbb{Z}$, but $I \subseteq \mathbb{Z}$ is also allowed).

1. A *chain complex* $(K_\bullet, d_\bullet)$ in $\mathcal{A}$ indexed by I is the homological convention for sequences with decreasing degrees. It consists of:
   - Objects $\{K_i\}_{i\in I}$ in $\mathcal{A}$, called the *terms in degree i*,
   - Morphisms $d_i : K_i \to K_{i-1}$ in $\mathcal{A}$, called the *boundary maps* or *differentials in degree i*,

   such that for every $i \in I$, $d_{i-1} \circ d_i = 0$. That is,
$$K_\bullet : \cdots \xrightarrow{d_{i+1}} K_i \xrightarrow{d_i} K_{i-1} \xrightarrow{d_{i-1}} K_{i-2} \to \cdots$$
   with $d_{i-1}d_i = 0$ for all i. We typically use the notation $K_\bullet = (K_i, d_i)_{i\in I}$.
2. Dually, a *cochain complex* $(K^\bullet, d^\bullet)$ in $\mathcal{A}$ follows the *cohomological convention* with increasing degrees. It consists of objects $\{K^i\}_{i\in I}$ and *coboundary maps* $d^i : K^i \to K^{i+1}$ such that $d^{i+1} \circ d^i = 0$:
$$K^\bullet : \cdots \xrightarrow{d^{i-1}} K^i \xrightarrow{d^i} K^{i+1} \xrightarrow{d^{i+1}} K^{i+2} \to \cdots$$
   We typically use the notation $K^\bullet = (K^i, d^i)_{i\in I}$.
3. Let $K_\bullet = (K_i, d_i^K)$ and $L_\bullet = (L_i, d_i^L)$ be chain complexes in $\mathcal{A}$ indexed by the same set I. A *morphism of chain complexes* (or *chain map*)
$$f_\bullet : K_\bullet \to L_\bullet$$
   consists of morphisms $f_i : K_i \to L_i$ for all $i \in I$, such that for every $i \in I$, the following diagram commutes:
$$\begin{array}{ccc} K_i & \xrightarrow{d_i^K} & K_{i-1} \\ \downarrow f_i & & \downarrow f_{i-1} \\ L_i & \xrightarrow{d_i^L} & L_{i-1} \end{array}$$
   i.e., $d_i^L \circ f_i = f_{i-1} \circ d_i^K$.

   A *morphism of cochain complexes* $f^\bullet : K^\bullet \to L^\bullet$ is defined similarly, satisfying the commutativity condition $d_L^i \circ f^i = f^{i+1} \circ d_K^i$.

The collection of these objects and morphisms forms a category. Notation for these categories is as follows:

- $\mathrm{Ch}(\mathcal{A})$ or $\mathbf{Ch}(\mathcal{A})$ is often used as a general term.
- To be explicit about the indexing convention, one uses $\mathrm{Ch}_\bullet(\mathcal{A})$ for chain complexes and $\mathrm{Ch}^\bullet(\mathcal{A})$ (or sometimes $\mathrm{CoCh}(\mathcal{A})$) for cochain complexes.
- The set of chain maps between two complexes is denoted by $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(K_\bullet, L_\bullet)$; it is an abelian group under pointwise addition $(f + g)_i = f_i + g_i$.

Definition E.0.2 (Chain complexes and their (co)homology objects)**.** Let $\mathcal{A}$ be an abelian category.

- For a cochain complex $K^\bullet$, its *cohomology object in degree i* is defined as the quotient of the object of i-cocycles by the object of i-coboundaries:
$$H^i(K^\bullet) := Z^i(K)/B^i(K) = \ker(d^i)/\operatorname{im}(d^{i-1}).$$

- For a chain complex $K_\bullet$, its *homology object in degree i* is defined as the quotient of the object of i-cycles by the object of i-boundaries:

$$H_i(K_\bullet) := Z_i(K)/B_i(K). = \ker(d_i)/\operatorname{im}(d_{i+1}).$$

References

[AP07] Jeffrey D. Achter and Rachel Pries. The integral monodromy of hyperelliptic and trielliptic curves. *Mathematische Annalen*, 338(1):187–206, 2007.

[Arm92] Brumer Armand. The average rank of elliptic curves i. *Inventiones mathematicae*, 109(1):445–472, 1992.

[Ayo23] Joseph Ayoub. Counterexamples to F. Morel's conjecture on $\pi_0^{\mathbb{A}^1}$. *Comptes Rendus. Mathématique*, 361:1087–1090, 2023.

[BBD82] Alexander A. Beilinson, Joseph Berstein, and Pierre Deligne. Analyse et topologie sur les espaces singuliers (i). *Astérisque*, 100, 1982.

[BBDG18] Alexander A. Beilinson, Joseph Berstein, Pierre Deligne, and Ofer Gabber. *Faisceaux pervers*, volume 4. Société mathématique de France Paris, 2018.

[BBK+23] Barinder S. Banwait, Armand Brumer, Hyun Jong Kim, Zev Klagsbrun, Jacob Mayle, Padmavathi Srinivasan, and Isabel Vogt. Computing nonsurjective primes associated to galois representations of genus 2 curves. *LuCaNT: LMFDB, Computation, and Number Theory*, 796:129, 2023.

[BC19] Tilman Bauer and Magnus Carlson. Tensor products of affine and formal abelian groups. *Documenta Mathematica*, 24:2525–2582, 2019.

[BFK+17] Valentin Blomer, Étienne Fouvry, Emmanuel Kowalski, Philippe Michel, and Djordje Milićević. Some applications of smooth bilinear forms with kloosterman sums. *Proceedings of the Steklov Institute of Mathematics*, 296:18–29, 2017.

[BGI71] Pierre Berthelot, Alexander Grothendieck, and Luc Illusie. *Théorie des Intersections et Théorème de Riemann-Roch (SGA6)*, volume 225 of *Lecture Notes in Mathematics*. Springer-Verlag, 1971.

[BH12] Salman Baig and Chris Hall. Experimental data for goldfeld's conjecture over function fields. *Experimental Mathematics*, 21(4):362–374, 2012.

[BLGHT11] Tom Barnet-Lamb, David Geraghty, Michael Harris, and Richard Taylor. A family of Calabi–Yau varieties and potential automorphy ii. *Publications of the Research Institute for Mathematical Sciences*, 47(1):29–98, 2011.

[Bor12] Armand Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer New York, 2012.

[BS15a] Manjul Bhargava and Arul Shankar. Binary quartic forms having bounded invariants, and the boundedness of the average rank of elliptic curves. *Annals of Mathematics*, pages 191–242, 2015.

[BS15b] Manjul Bhargava and Arul Shankar. Ternary cubic forms having bounded invariants, and the existence of a positive proportion of elliptic curves having rank 0. *Annals of Mathematics*, pages 587–621, 2015.

[BS15c] Bhargav Bhatt and Peter Scholze. The pro-étale topology for schemes. *Astérisque*, 360:99–201, 2015.

[BSD65] Bryan John Birch and Peter Francis Swinnerton-Dyer. Notes on elliptic curves. ii. *Journal für die reine und angewandte Mathematik*, 218:79–108, 1965.

[BSS18] Bhargav Bhatt, Christian Schnell, and Peter Scholze. Vanishing theorems for perverse sheaves on abelian varieties, revisited. *Selecta Mathematica*, 24:63–84, 2018.

[Cho08] Utsav Choudhury. Homotopy theory of schemes and a^1-fundamental groups. Master's thesis, Università degli Studi di Padova, 2008.

[CHT08] Laurent Clozel, Michael Harris, and Richard Taylor. Automorphy for some l-adic lifts of automorphic mod l galois representations. *Publications mathématiques*, 108:1–181, 2008.

[DA73] Pierre Deligne and Michael Artin. *Théorie des Topos et Cohomologies Étale des Schémas. Séminaire de Géométrie Algébrique due Bois-Marie 1963-1964 (SGA 4)*. Lecture Notes in Mathematics. Springer Berlin, 1973.

[DBG⁺77] Pierre Deligne, Jean-François Boutot, Alexander Grothendieck, Luc Illusie, and Jean-Louis Verdier. *Étale Cohomology. Séminaire de Géométrie Algébrique due Bois-Marie 1963-1964 (SGA 4 1/2)*. Lecture Notes in Mathematics. Springer-Verlag, 1977.

[Del80] Pierre Deligne. La conjecture de Weil : II. *Publications Mathématiques de l'IHÉS*, 52:137–252, 1980.

[Del89] Pierre Deligne. Le groupe fondamental de la droite projective moins trois points. In *Galois Groups over Q: Proceedings of a Workshop Held March 23–27, 1987*, pages 79–297. Springer, 1989.

[Die02] Luis V. Dieulefait. Explicit determination of the images of the Galois representations attached to abelian surfaces with $\mathrm{End}(A) = \mathbb{Z}$. *Experiment. Math.*, 11(4):503–512 (2003), 2002.

[DR04] Luis V. Dieulefait and Victor Rotger. The arithmetic of qm-abelian surfaces through their galois representations. *Journal of Algebra*, 281:124–143, 2004.

[Dri89] Vladimir Gershonovich Drinfeld. Cohomology of compactified manifolds of modules of f-sheaves. *Journal of Soviet Mathematics*, 46(2):1789–1821, 1989.

[DZ19] Alexander Dunn and Alexandru Zaharescu. Sums of Kloosterman sums over primes in an arithmetic progression. *The Quaterly Journal of Mathematics*, 70(1):319–342, 2019.

[Eke07] Torsten Ekedahl. *On The Adic Formalism*, pages 197–218. Birkhäuser Boston, Boston, MA, 2007.

[ELS20] Jordan S. Ellenberg, Wanlin Li, and Mark Shusterman. Nonvanishing of hyperelliptic zeta functions over finite fields. *Algebra & Number Theory*, 14(7):1895–1909, 8 2020.

[EVW16] Jordan S. Ellenberg, Akshay Venkatesh, and Craig Westerland. Homology stability for Hurwitz spaces and the Cohen-Lenstra conjecture over function fields. *Annals of Mathematics*, 183:729–786, 2016.

[FFK25] Arthur Forey, Javier Fresán, and Emmanuel Kowalski. Arithmetic fourier transforms over finite fields: generic vanishing, convolution, and equidistribution, 2025.

[FK12] Sergey Finashin and Viatcheslav Kharlamov. Abundance of Real Lines on Real Projective Hypersurfaces. *International Mathematics Research Notices*, 2013(16):3639–3646, 06 2012.

[FLR23] Tony Feng, Aaron Landesman, and Eric M. Rains. The geometric distribution of Selmer groups of elliptic curves over function fields. *Mathematische Annalen*, 387:615–687, 2023.

[Fu15] Lei Fu. *Etale Cohomology Theory*, volume 14 of *Nankai Tracts in Mathematics*. World Scientific, 2015.

[FvdG04] Carel Faber and Gerard van der Geer. Complete subvarieties of moduli spaces and the Prym map. *Journal für die reine und angewandte Mathematik*, 2004(573):117–137, 2004.

[GL96] Ofer Gabber and François Loeser. Faisceaux pervers ℓ-adiques sur un tore. *Duke Math J.*, 83(3):501–606, 1996.

[Gol06] Dorian Goldfeld. Conjectures on elliptic curves over quadratic fields. In *Number Theory Carbondale 1979: Proceedings of the Southern Illinois Number Theory Conference Carbondale, March 30 and 31, 1979*, pages 108–118. Springer, 2006.

[GR04] Alexander Grothendieck and Michèle Raynaud. Revêtements étales et groupe fondamental (SGA 1). eprint arXiv matyh/0206203, 2004. Updated edition of the book of the same title published by Springer-Verlag in 1971 as volume 224 of the series Lecture Notes in Mathematics.

[Gro77] Alexander Grothendieck. *Cohomologie l-adique et fonctions L Séminaire de Géométrie Algébrique due Bois-Marie 1965-1966 (SGA 5)*, volume 589 of *Springer Lecture Notes*. Springer-Verlag, 1977. Avec la collaboration de I. Bucur, C. Houzel, L. Illusie, J.-P. Jouanolou, et J.-P. Serre.

[GV72] Alexander Grothendieck and Jean-Louis Verdier. *Theorie des Topos et Cohomologie Etale des Schemas. Seminaire de Geometrie Algebrique du Bois-Marie 1963-1964 (SGA 4)*. Lecture Notes in mathematics. Springer-Verlag Berlin Heidelberg, 1 edition, 1972.

[HB04] David Heath-Brown. The average analytic rank of elliptic curves. *Duke Math. J.*, 122:591–623, 2004.

[HK25] Chris Hall and Hyun Jong Kim. Independence of ℓ (title to be determined). In progress, 2025.

[HM73] Dale Husemoller and John Milnor. *Symmetric Bilinear Forms*, volume 73 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 2. Folge*. Springer Berlin Heidelberg, 1973.

[HSBT05] Michael Harris, Nick Shepherd-Barron, and Richard Taylor. Ihara's lemma and potential automorphy, 2005.

[Hub97] Annette Huber. Mixed perverse sheaves for schemes over number fields. *Compositio Mathematica*, 108:107–121, 1997.

[Ibu22] Tomoyoshi Ibukiyama. Supersingular loci of low dimensions and parahoric subgroups. *Osaka Journal of Mathematics*, 59:703–726, 2022.

[Jor05] Andrei Jorza. The birch and swinnerton-dyer conjecture for abelian varieties over number fields, 2005.

[Kat90] Nicholas M. Katz. *Exponential sums and differential equations*, volume 124 of *Annals of Mathematics Studies*. Princeton University Press, 1990.

[Kat96] Nicholas M. Katz. *Rigid Local Systems*, volume 139 of *annals of Mathematics Studies*. Princeton University Press, 1996.

[Kat98] Nicholas M. Katz. *Gauss Sums, Kloosterman Sums, and Monodromy Groups*, volume 116 of *Annals of Mathematics Studies*. Princeton University Press, 1998.

[Kat12] Nicholas M. Katz. *Convolution and Equidistribution Sato-Tate Theorems for Finite-Field Mellin Transforms*, volume 180 of *Annals of Mathematics Studies*. Princeton University Press, 2012.

[Kim23] Hyun Jong Kim. `trouver`, 2023. GitHub repository: https://github.com/hyunjongkimmath/trouver.

[Kim24] Hyun Jong Kim. *Cohen-Lenstra heuristics and vanishing of zeta functions for superelliptic curves over finite fields*. PhD thesis, University of Wisconsin-Madison, 2024.

[KL85] Nicholas M. Katz and Gérard Laumon. Transformation de fourier et majoration de sommes exponentielles. *Publications Mathématiques de l'IHÉS*, 62:145–202, 1985.

[KLSW23] Jesse Leo Kass, Marc Levine, Jake P. Solomon, and Kirsten Wickelgren. A quadratically enriched count of rational curves. arXiv 2307.01936, 2023.

[KM23] Seoyoung Kim and M. Ram Murty. From the Birch and Swinnerton-Dyer conjecture to Nagao's conjecture. *Mathematics of Computation*, 92(339):385–408, 2023.

[KMS17] Emmanuel Kowalski, Philippe Michel, and Will Sawin. Bilinear forms with Kloosterman sums and applciations. *Annals of Mathematics*, 186:413–500, 2017.

[KP24] Hyun Jong Kim and Sun Woo Park. Global $\mathbb{A}^1$ degrees of covering maps between modular curves, 2024.

[Krä14] Thomas Krämer. Perverse sheaves on semiabelian varieties. *Rendiconti del Seminario Matematico della Università di Padova*, 132:83–102, 2014.

[KS99] Nicholas M. Katz and Peter Sarnak. *Random matrices, Frobenius eigenvalues, and monodromy*, volume 45. American Mathematical Society, 1999.

[KS22] Timo Keller and Michael Stoll. Exact verification of the strong bsd conjecture for some absolutely simple abelian surfaces. *Comptes Rendus Mathématique*, 360:483–489, 2022.

[KW13] Reinhardt Kiehl and Rainer Weissauer. *Weil Conjectures, Perverse Sheaves and ℓ'adic Fourier Transform*, volume 42 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics*. Springer Berlin, Heidelberg, 2013.

[KW15] Thomas Krämer and Rainer Weissauer. Vanishing theorems for constructible sheaves on abelian varieties. *J. Algebraic Geometry*, 24:531–568, 2015.

[KW19] Jesse Leo Kass and Kirsten Wickelgren. The class of Eisenbud-Khimshiashvili-Levine is the local $\mathbb{A}^1$ − Brouwer degree. *Duke Mathematical Journal*, 168(3):429–469, 2019.

[KW21] Jesse Leo Kass and Kirsten Wickelgren. An arithmetic count of the lines on a smooth cubic surface. *Compositio Mathematica*, 157(4):677–709, 2021.

[Laf02] Laurent Lafforgue. Chtoucas de Drinfeld et correspondance de Langlands. *Inventiones mathematicae*, 147:1–241, 2002.

[Lom17] Davide Lombardo. Galois representations attached to abelian varieties of cm type. *Bulletin de la Société mathématique de France*, 145(3):469–501, 2017.

[May99] Jon Peter May. *A Concise Course in Algebraic Topology*. Chicago Lectures in Mathematics. University of Chicago Press, 1999.

[Mil80] James S. Milne. *Etale cohomology*. Number 33 in Princeton Mathematical Series. Princeton university press, 1980.

[Mil07] James S. Milne. Quotients of Tannakian categories. *Theory and Applications of Categories*, 18(21):654–664, 2007.

[Mil13] James S. Milne. Lie algebras, algebraic groups, and lie groups, 2013. Available at www.jmilne.org/math/.

[Mil17] James S. Milne. *Algebraic Groups*, volume 170 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2017.

[Mit14] Howard H. Mitchell. The subgroups of the quaternary abelian linear group. *Trans. Amer. Math. Soc.*, 15(4):379–396, 1914.

[Mor06] Fabien Morel. A1-algebraic topology. In *International Congress of Mathematicians*, volume 2, pages 1035–1059, 2006.

[Mor12] Fabien Morel. *A1-Algebraic Topology over a field*. Lecture Notes in Mathematics. Springer Berlin, Heidelberg, 2012.

[MV99] Fabien Morel and Vladimir Voevodsky. A1-homotopy theory of schemes. *Publications Mathématiques de l'IHÉS*, 90:45–143, 1999.

[Nag97] Koh-ichi Nagao. Q(t)-rank of elliptic curves and certain limit coming from the local points. *Manuscripta mathematica*, 92(1):13–32, 1997.

[nLa25a] nLab authors. geometric morphism. `https://ncatlab.org/nlab/show/geometric+morphism`, July 2025. Revision 61.

[nLa25b] nLab authors. homotopy group of a spectrum. `https://ncatlab.org/nlab/show/homotopy+group+of+a+spectrum`, June 2025. Revision 7.

[nLa25c] nLab authors. Introduction to Stable homotopy theory – 1-1. `https://ncatlab.org/nlab/show/Introduction+to+Stable+homotopy+theory+--+1-1`, June 2025. Revision 43.

[nLa25d] nLab authors. model structure on topological sequential spectra. `https://ncatlab.org/nlab/show/model+structure+on+topological+sequential+spectra`, June 2025. Revision 61.

[nLa25e] nLab authors. point of a topos. `https://ncatlab.org/nlab/show/point+of+a+topos`, July 2025. Revision 53.

[nLa25f] nLab authors. sheafification. `https://ncatlab.org/nlab/show/sheafification`, September 2025. Revision 40.

[nLa25g] nLab authors. stable homotopy category. `https://ncatlab.org/nlab/show/stable+homotopy+category`, June 2025. Revision 31.

[OT14] Christian Okonek and Andrei Teleman. Intrinsic signs and lower bounds in real algebraic geometry. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2014(688):219–241, 2014.

[Poo18] Bjorn Poonen. Heuristics for the arithmetic of elliptic curves. In *Proceedings of the International Congress of Mathematicians: Rio de Janeiro 2018*, pages 399–414. World Scientific, 2018.

[PR12] BJORN POONEN and ERIC RAINS. Random maximal isotropic subspaces and selmer groups. *Journal of the American Mathematical Society*, 25(1):245–269, 2012.

[Pri24] Rachel Pries. The torelli locus and newton polygons, 2 2024. Lecture Notes for the 2024 Arizona Winter School.

[Pri25] Rachel Pries. Some cases of oort's conjecture about newton polygons of curves. *Nagoya Mathematical Journal*, 257:93–103, 2025.

[PW21] Sabrina Pauli and Kirsten Wickelgren. Applications to $\mathbb{A}^1$-enumerative geometry of the $\mathbb{A}^1$-degree. *Research in the Mathematical Sciences*, 8(24):24–29, 2021.

[Ros02] Michael Rosen. *Number Theory in Function Fields*, volume 210 of *Graduate Texts in Mathematics*. Springer-Verlag New York, 2002.

[RS98] Michael Rosen and Joseph H. Silverman. On the rank of an elliptic surface. *Inventiones mathematicae*, 133:43–67, 1998.

[Rud87] Walter Rudin. *Real and Complex Analysis*. Mathematics Series. McGraw-Hill Book Company, 3 edition, 1987.

[Saw24] Will Sawin. General multiple dirichlet series from perverse sheaves. *Journal of Number Theory*, 262:408–453, 2024.

[Ser72] Jean-Pierre Serre. Propriétés galoisiennes des points d'ordre fini des courbes elliptiques. *Invent. math*, 15:259–331, 1972.

[Ser00] Jean-Pierre Serre. Lettre à marie-france vignéras du 10/2/1986. *Oeuvres–Collected Papers*, 4:38–55, 2000.

[SFFK23] Will Sawin, Arthur Forey, Javier Fresán, and Emmanuel Kowalski. Quantitative sheaf theory. *Journal of the American Mathematical Society*, 36(3):653–726, 2023.

[Sil89] Joseph H. Silverman. Elliptic curves of bounded degree and height. *Proceedings of the American Mathematical Society*, 105(3):540–545, 1989.

[Sil09] Joseph H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer-Verlag, 2 edition, 2009.

[ST21] Will Sawin and Jacob Tsimerman. Bounds for the stalks of perverse sheaves in characteristic p and a conjecture of shende and tsimerman. *Inventiones mathematicae*, 224(1):1–32, 2021.

[Sta25] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2025.

[Tat65] John T. Tate. Algebraic cycles and poles of zeta functions. In *Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963)*, pages 93–110. Harper & Row, 1965. Also in Collected works of John Tate (2 vols.), Amer. Math. Soc. (2016), vol. 2.

[Tat66] John Tate. On the conjectures of Birch and Swinnerton-Dyer and a geometric analog. In *Séminaire Bourbaki : années 1964/65 1965/66, exposés 277-312*, number 9 in Astérisque, pages 415–440. Société mathématique de France, 1966. talk:306.

[Tay08] Richard Taylor. Automorphy for some l-adic lifts of automorphic mod l galois representations. ii. *Publications mathématiques*, 108:183–239, 2008.

[Voe98] Vladimir Voevodsky. A1-homotopy theory. In *Proceedings of the international congress of mathematicians*, volume 1, pages 579–604. Berlin, 1998.

[Wei94] Charles A. Weibel. *An Introduction to Homological Algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1994. First paperback edition 1995 Reprinted 1997.

[Wik25] Wikipedia contributors. Frobenius endomorphisms#frobenius for schemes — Wikipedia, the free encyclopedia, 2025. [Online; accessed 08-July-2025].

[You06] Matthew Young. Low-lying zeros of families of elliptic curves. *Journal of the American Mathematical Society*, 19(1):205–250, 2006.

[Yu97] Jiu-Kang Yu. Toward a proof of the Cohen-Lenstra conjecture in the function field case. preprint, 1997.