HOMOLOGICAL ALGEBRA

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CONTENTS

1. Categories

Definition 1.0.1 (Category). A *category* $\mathcal{C}$ consists of the following data:

- A class of *objects* denoted $\mathrm{Ob}(\mathcal{C})$.
- For each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a class

$$\mathrm{Hom}_{\mathcal{C}}(X, Y)$$

  of *morphisms* (also called *arrows* or *homs*). If the category $\mathcal{C}$ is clear, then this *hom-class* is also denoted by $\mathrm{Hom}(X, Y)$. It may also be denoted by $\mathrm{hom}_{\mathcal{C}}(X, Y)$ or $\mathrm{hom}(X, Y)$, especially to distinguish from other types of hom's (e.g. internal hom's)
- For each triple of objects X, Y, Z, a composition law

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z),$$

  denoted $(g, f) \mapsto g \circ f$.
- For each object X, an *identity morphism*

$$\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X).$$

These data satisfy the following axioms:

- (Associativity) For all morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, and $h \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$,
$$h \circ (g \circ f) = (h \circ g) \circ f.$$
- (Identity) For all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$,
$$\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X.$$

One often writes $X \in \mathcal{C}$ synonymously with $X \in \mathrm{Ob}(\mathcal{C})$, i.e. to denote that X is an object of of $\mathcal{C}$.

We may call a category as above an *ordinary category* to distinguish this notion from the notions of *categories enriched in monoidal categories* or higher/n-categories. (♠ TODO: TODO: define n-categories)

A category as defined above may be called called a *large category* or a *class category* to emphasize that the hom-classes may be proper classes rather than sets (note, however, that the possibility that hom-classes are sets is not excluded for large categories). Accordingly, a *category* may often refer to a locally small category (Definition 1.0.5), which is a category whose hom-classes are all sets.

Definition 1.0.2 (Opposite category)**.** Let $\mathcal{C}$ be a (large) category (Definition 1.0.1). The *opposite category* of $\mathcal{C}$, denoted $\mathcal{C}^{\mathrm{op}}$, is defined as follows:

- The objects of $\mathcal{C}^{\mathrm{op}}$ are the same as those of $\mathcal{C}$.
- For any pair of objects $X, Y \in \mathcal{C}$, the morphisms from X to Y in $\mathcal{C}^{\mathrm{op}}$ are given by the morphisms from Y to X in $\mathcal{C}$:

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(Y, X).$$

- Composition in $\mathcal{C}^{\mathrm{op}}$ is defined by reversing the order of composition in $\mathcal{C}$. That is, for morphisms $f \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(Y, Z)$, their composition is

$$g \circ_{\mathcal{C}^{\mathrm{op}}} f := f \circ_{\mathcal{C}} g.$$

Intuitively, the category $\mathcal{C}^{\mathrm{op}}$ thus "reverses" the direction of all morphisms in $\mathcal{C}$.

Definition 1.0.3. Let $\mathcal{C}$ and $\mathcal{D}$ be (large) categories (Definition 1.0.1).

1. A *functor $F : \mathcal{C} \to \mathcal{D}$ (from $\mathcal{C}$ to $\mathcal{D}$)* consists of :
   - For each object X in $\mathcal{C}$, an object $F(X)$ in $\mathcal{D}$.
   - For each morphism $f : X \to Y$ in $\mathcal{C}$, a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$,
   such that:

   $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all objects X in $\mathcal{C}$,

   $F(g \circ f) = F(g) \circ F(f)$ for all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ and all $f : X \to Y, g : Y \to Z$ in $\mathcal{C}$.

   Functors as defined above are also referred to as *covariant functors* to distinguish them from contravariant functors

2. A *contravariant functor from $\mathcal{C}$ to $\mathcal{D}$* refers to a covariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. Equivalently, such a functor consists of
   - For each object X in $\mathcal{C}$, an object $F(X)$ in $\mathcal{D}$.
   - For each morphism $f : X \to Y$ in $\mathcal{C}$, a morphism $F(f) : F(Y) \to F(X)$ in $\mathcal{D}$,
   such that:

   $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all objects X in $\mathcal{C}$,

   $F(g \circ f) = F(f) \circ F(g)$ for all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ and all $f : X \to Y, g : Y \to Z$ in $\mathcal{C}$.

   A synonym for a "contravariant functor from $\mathcal{C}$ to $\mathcal{D}$" is a "presheaf on $\mathcal{C}$ with values in $\mathcal{D}$ (Definition 6.1.6)".

Note that declarations such as "Let $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ be a contravariant functor" can be common; such declarations usually mean "Let F be a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$" as opposed to "Let F be a contravariant functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$". further note that a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is equivalent to a covariant functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$.

Definition 1.0.4. Let $\mathcal{C}$ and $\mathcal{D}$ be (large) categories (Definition 1.0.1). Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors (Definition 1.0.3).

A *natural transformation η between F and G* is a family of morphisms $\eta_X : F(X) \to G(X)$ in $\mathcal{D}$, one for each object X in $\mathcal{C}$, such that for every morphism $f : X \to Y$ in $\mathcal{C}$,

$$G(f) \circ \eta_X = \eta_Y \circ F(f)$$

in $\mathcal{D}$. In other words, the following diagram commutes:

$$
\begin{array}{ccc}
F(X) & \xrightarrow{F(f)} & F(Y) \\
\eta_X \downarrow & & \downarrow \eta_Y \\
G(X) & \xrightarrow[G(f)]{} & G(Y)
\end{array}
$$

We write such a natural transformation by $\eta : F \Rightarrow G$.

If η_X is an isomorphism for all objects X of $\mathcal{C}$, then η is said to be a *natural isomorphism*.

Definition 1.0.5 (Locally small category). A (large) category (Definition 1.0.1) $\mathcal{C}$ is called a *locally small category* if for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the collection $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms between them is a (small) *set* (as opposed to a proper class). In other words, each hom-class is a set and may even be called a *hom-set*.

In some contexts, a locally small category may simply be called a *category*, especially when genuinely large categories are not considered.

A category $\mathcal{C}$ is called a *small category* if it is a locally small category and the class $\mathrm{Ob}(\mathcal{C})$ of objects is a set.

Given a universe (Definition C.0.14) U, we can define the notion of a *U-locally small category* and of a *U-small category* similarly.

Remark 1.0.6. Many "concrete" categories considered in "classical mathematics" or outside of more "abstract" category theory tend to be locally small. For example, the categories of sets, groups, R-modules, vector spaces, topological spaces, schemes, manifolds, sheaves on "small enough" sites are all locally small.

Lemma 1.0.7. Let $\mathcal{C}$ be a small category (Definition 1.0.5) (resp. U-small category where U is some universe (Definition C.0.14)) and let $\mathcal{A}$ be a locally small (Definition 1.0.5) category (resp. U-locally small category). The presheaf category $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ (Definition 6.1.6) is locally small (resp. U-locally small).

Proof. A morphism $\mathcal{F} \to \mathcal{G}$ in $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ is a natural transformation (Definition 1.0.4) of the functors $\mathcal{F}, \mathcal{G} : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$. Such a natural transformation is encoded by a family $(\eta_C)_C$ of morphisms (satisfying certain conditions) $\eta_C : \mathcal{F}(C) \to \mathcal{G}(C)$ in $\mathcal{A}$ over objects C of $\mathcal{C}^{\mathrm{op}}$. The product $\prod_{C \in \mathrm{Ob}\,\mathcal{C}^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}(C), \mathcal{G}(C))$ is a product of (U-small) sets indexed by a (U-small) set, and the collection of natural transformations is a subset of this set. Therefore, $\mathrm{Hom}_{\mathrm{PreShv}(\mathcal{C}, \mathcal{A})}(\mathcal{F}, \mathcal{G})$ is a (U-small) set. $\square$

Definition 1.0.8 (Full subcategory). Let $\mathcal{C}$ be a (large) category (Definition 1.0.1). A *full subcategory* $\mathcal{D}$ of $\mathcal{C}$ is a subcategory such that for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{D})$, the morphism classes coincide:
$$
\mathrm{Hom}_{\mathcal{D}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y).
$$
In other words, a full subcategory includes all morphisms between its objects that exist in the ambient category $\mathcal{C}$.

Definition 1.0.9. Let $\mathcal{C}$ and $\mathcal{D}$ be (large)) categories (Definition 1.0.1). Let $F : \mathcal{C} \to \mathcal{D}$ be a functor (Definition 1.0.3).

1. F is called *full* if for every pair of objects $x, y \in \mathrm{Ob}(\mathcal{C})$, the induced rule/assignment/class function

$$F_{x,y} : \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$$

on Hom-collections is "surjective", i.e. for all morphisms $g : F(x) \to F(y)$, there exists some morphism $f : x \to y$ such that $F(f) = g$.

2. F is called *faithful* if for every pair of objects $x, y \in \mathrm{Ob}(\mathcal{C})$, the induced class function (assignment)

$$F_{x,y} : \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$$

on Hom-collections is "injective", i.e., for any morphisms $f_1, f_2 \in \mathrm{Hom}_{\mathcal{C}}(x, y)$, if $F(f_1) = F(f_2)$ in $\mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$, then $f_1 = f_2$.

3. F is called *fully faithful* if it is both full and faithful.

Definition 1.0.10. An *equivalence of categories* between two (large) categories (Definition 1.0.1) $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of functors (Definition 1.0.3)

$$F : \mathcal{C} \to \mathcal{D} \quad \text{and} \quad G : \mathcal{D} \to \mathcal{C}$$

together with natural isomorphisms (Definition 1.0.4)

$$\eta : \mathrm{Id}_{\mathcal{C}} \xrightarrow{\sim} G \circ F \quad \text{and} \quad \epsilon : F \circ G \xrightarrow{\sim} \mathrm{Id}_{\mathcal{D}}.$$

Such functors F and G may be called *(natural) inverses of each other*.

When $\mathcal{C}$ and $\mathcal{D}$ are locally small categories (Definition 1.0.5), F is an equivalence of categories if and only if F is fully faithful (Definition 1.0.9) and essentially surjective

Definition 1.0.11. A category $\mathcal{C}$ is called *essentially small* if it is equivalent (Definition 1.0.10) to a small category (Definition 1.0.5), i.e., there exists a small category $\mathcal{D}$ and an equivalence of categories

$$F : \mathcal{D} \to \mathcal{C}.$$

Note that an essentially small category is necessarily locally small (Definition 1.0.5).

Definition 1.0.12 (Product Category of a Family of Categories)**.** Let $\{\mathcal{C}_i\}_{i \in I}$ be a family of (large) categories (Definition 1.0.1) indexed by a class I. The *product category of the family*, denoted

$$\prod_{i \in I} \mathcal{C}_i \, ,$$

is the very large category (🔥 TODO: define very large categories) defined as follows:

- The class of objects is

$$\mathrm{Ob}\Big(\prod_{i \in I} \mathcal{C}_i \Big) = \prod_{i \in I} \mathrm{Ob}(\mathcal{C}_i),$$

i.e., an object is a family $(A_i)_{i \in I}$ with $A_i \in \mathrm{Ob}(\mathcal{C}_i)$.

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- For two objects $(A_i)_i$ and $(B_i)_i$, the morphism class is

$$\text{Hom}_{\prod_{i \in I} \mathcal{C}_i}((A_i)_i, (B_i)_i) = \prod_{i \in I} \text{Hom}_{\mathcal{C}_i}(A_i, B_i).$$

  In other words, a morphism $(f_i)_i : (A_i)_i \to (B_i)_i$ consists of morphisms $f_i : A_i \to B_i$ in each $\mathcal{C}_i$.
- For morphisms $(f_i)_i : (A_i)_i \to (B_i)_i$ and $(g_i)_i : (B_i)_i \to (C_i)_i$, composition is defined componentwise:

$$(g_i)_i \circ (f_i)_i = (g_i \circ_i f_i)_i.$$

- For each object $(A_i)_i$, the identity morphism is given by the family

$$(\text{id}_{A_i})_i.$$

If I is a set, then $\prod_{i \in I} \mathcal{C}_i$ is a large category. If I is a set and if each $\mathcal{C}_i$ is locally small (Definition 1.0.5), then $\prod_{i \in I} \mathcal{C}_i$ is locally small.

In case that I is finite, the notation of $\times$ may be used for product categories, e.g. $\mathcal{C}_i \times \mathcal{C}_j$ denotes the product of two categories $\mathcal{C}_i \times \mathcal{C}_j$.

(♠ TODO: ordinal, U_α) If α is an ordinal such that $\mathcal{C}_i$ and I are U_α-large (i.e. they live in $U_{\alpha+1}$), then $\prod_{i \in I} \mathcal{C}_i$ is $U_{\alpha+1}$-large.

Definition 1.0.13 (n-ary (Multivariable) Functor). Let I be a finite set with $|I| = n$, and let $\{\mathcal{C}_i\}_{i \in I}$ be (large) categories (Definition 1.0.1), together with another category $\mathcal{D}$. An *n-ary functor* (also called a *multivariable functor*, a *multivariate functor*, or a *multifunctor*) from the categories $\{\mathcal{C}_i\}_{i \in I}$ to $\mathcal{D}$ is a functor (Definition 1.0.3)

$$F : \prod_{i \in I} \mathcal{C}_i \to \mathcal{D}.$$

(Definition 1.0.12) That is, F assigns:

- to each object $((A_i)_{i \in I})$ in $\prod_{i \in I} \mathcal{C}_i$, an object $F((A_i)_{i \in I})$ in $\mathcal{D}$,
- to each morphism $((f_i)_{i \in I}) : (A_i)_i \to (B_i)_i$, a morphism $F((f_i)_i) : F((A_i)_i) \to F((B_i)_i)$ in $\mathcal{D}$,

so that F preserves identities and composition componentwise. For instance, a *bifunctor* is an n-ary functor when $n = 2$, a *ternary functor/trifunctor* is an n-ary functor when $n = 3$, etc.

Lemma 1.0.14. Let $\mathcal{C}$ be a category enriched (Definition A.0.2) in a monoidal category (Definition A.0.1) $(\mathcal{V}, \otimes, 1)$ and assume that $\mathcal{V}$ is closed under finite products (Definition 2.0.3). The objects $\underline{\text{Hom}}_{\mathcal{C}}(X, Y) \in \text{Ob}(\mathcal{V})$ describe an enriched bifunctor (Definition A.0.5)

$$\underline{\text{Hom}}_{\mathcal{C}}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{V}, \quad (A, B) \mapsto \underline{\text{Hom}}_{\mathcal{C}}(A, B)$$

(Definition A.0.6) of enriched categories.

In particular, when $\mathcal{V} = \textbf{Sets}$, we have a bifunctor (Definition 1.0.13)

$$\text{Hom}_{\mathcal{C}}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \textbf{Sets}, \quad (A, B) \mapsto \text{Hom}_{\mathcal{C}}(A, B).$$

(Definition 1.0.2).

In other words, <u>Hom</u> and Hom are contravariant (Definition 1.0.3) in the first variable and covariant in the second.

1.1. **Sites and sheaves.**

Definition 1.1.1 (Ringed space)**.** A *ringed space* is a pair $(X, \mathcal{O}_X)$ where

- X is a topological space, and
- $\mathcal{O}_X$ is a sheaf of (Definition 6.1.8) commutative rings on X.

Equivalently, a ringed space is a ringed site (Definition 6.1.18) where the site is the site of opens of the topological space X The sheaf $\mathcal{O}_X$ may be suppressed from the notation and only X may be used to denote a ringed space. The sheaf $\mathcal{O}_X$, also commonly denoted by $\mathscr{O}_X$, is called the *structure sheaf of X*.

Definition 1.1.2.   1. Let $\mathcal{C}$ be a site (Definition 6.1.4), and let $\mathcal{A}$ and $\mathcal{B}$ be sheaves (Definition 6.1.7) of (not necessarily commutative) rings (Definition C.0.2) on $\mathcal{C}$.

  (a) An $(\mathcal{A}, \mathcal{B})$-*bimodule* (or a *bimodule over* $(\mathcal{A}, \mathcal{B})$) is a sheaf (Definition 6.1.7) $\mathcal{M}$ of abelian groups on $\mathcal{C}$ equipped with a left $\mathcal{A}$-module structure given by a morphism of sheaves (Definition 6.1.7) of sets

$$\lambda : \mathcal{A} \times \mathcal{M} \longrightarrow \mathcal{M},$$

  and a right $\mathcal{B}$-module structure given by a morphism of sheaves of sets

$$\rho : \mathcal{M} \times \mathcal{B} \longrightarrow \mathcal{M},$$

  such that the actions are compatible. Specifically, for every object U in $\mathcal{C}$, every section $m \in \mathcal{M}(U)$, every $a \in \mathcal{A}(U)$, and every $b \in \mathcal{B}(U)$, the equality

$$\lambda_U(a, \rho_U(m, b)) = \rho_U(\lambda_U(a, m), b)$$

  holds in $\mathcal{M}(U)$. In standard multiplicative notation where $\lambda(a, m)$ is denoted $a \cdot m$ and $\rho(m, b)$ is denoted $m \cdot b$, this condition is the associativity axiom

$$(a \cdot m) \cdot b = a \cdot (m \cdot b).$$

  In particular, for every object $U \in \mathcal{C}$, the abelian group $\mathcal{M}(U)$ has the structure of an $\mathcal{A}(U) - \mathcal{B}(U)$-bimodule (Definition C.0.4).

  (b) Let $\mathcal{M}$ and $\mathcal{N}$ be $(\mathcal{A}, \mathcal{B})$-bimodules. A *homomorphism of* $(\mathcal{A}, \mathcal{B})$-*bimodules* (or an $(\mathcal{A}, \mathcal{B})$-*linear morphism*) is a morphism of sheaves of abelian groups $f : \mathcal{M} \to \mathcal{N}$ such that for every object U of $\mathcal{C}$, every section $m \in \mathcal{M}(U)$, every $a \in \mathcal{A}(U)$, and every $b \in \mathcal{B}(U)$, the following compatibility conditions hold:

$$f_U(a \cdot m) = a \cdot f_U(m) \quad \text{and} \quad f_U(m \cdot b) = f_U(m) \cdot b.$$

  We denote the category of $(\mathcal{A}, \mathcal{B})$-bimodules, with morphisms being morphisms of sheaves of abelian groups compatible with both the left $\mathcal{A}$-action and the right $\mathcal{B}$-action, by $\mathcal{A}$-$\mathcal{B}$-Mod or sometimes by ${}_{\mathcal{A}}\mathsf{Mod}_{\mathcal{B}}$ (♠ TODO: talk about how bimodules can be identifies with left/right modules)

 2. Let $(\mathcal{C}, J)$ be a site (Definition 6.1.4). Let $\mathcal{O}$ be a sheaf of (not necessarily commutative) rings on $(\mathcal{C}, J)$ (Definition 6.1.7), i.e. $((\mathcal{C}, J), \mathcal{O})$ is a ringed site (Definition 6.1.18).

  (a) An *(left/right/two-sided) $\mathcal{O}$-module* consists of the following data:

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- A sheaf $\mathcal{F}$ of abelian groups on $(\mathcal{C}, J)$,
- for every object $U \in \mathcal{C}$, the structure of an (left/right/two-sided) $\mathcal{O}(U)$-module on $\mathcal{F}(U)$,

such that for every morphism $f : V \to U$ in $\mathcal{C}$, the restriction map

$$\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$$

is $\mathcal{O}(U)$-linear when the $\mathcal{O}(U)$-action on $\mathcal{F}(V)$ is defined via the natural ring homomorphism

$$\mathcal{O}(U) \to \mathcal{O}(V)$$

induced by f.

(b) Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}$-modules (Definition 1.1.2).
A *morphism of $\mathcal{O}$-modules* $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves (Definition 6.1.7) of abelian groups such that, for every object $U \in \mathcal{C}$, the component map

$$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

is $\mathcal{O}(U)$-linear, i.e. it satisfies

$$\varphi_U(r \cdot s) = r \cdot \varphi_U(s) \quad \text{for all } r \in \mathcal{O}(U),\ s \in \mathcal{F}(U).$$

The collection of all $\mathcal{O}$-modules together with their morphisms of $\mathcal{O}$-modules forms the *category of $\mathcal{O}$-modules*, denoted $\mathbf{Mod}(\mathcal{O})$.

In case that a sheafification functor (Definition 1.1.4)

$$\mathrm{PreShv}(\mathcal{C}, \mathbf{Rings}) \to \mathrm{Shv}(\mathcal{C}, \mathbf{Rings})$$

exists, a left, right, two-sided $\mathcal{O}$-module (and morphisms thereof) is equivalent to a $(\mathcal{O}, \mathbb{Z})$-bimodule, $(\mathbb{Z}, \mathcal{O})$-bimodule, and $(\mathcal{O}, \mathcal{O})$-bimodule (and morphisms thereof) respectively, where $\mathbb{Z}$ is the constant sheaf of the integer ring $\mathbb{Z}$.

Definition 1.1.3 (Sheaf associated to a presheaf)**.** Let X be a topological space, and let $\mathcal{D}$ be a category (Definition 1.0.1) admitting direct colimits (Definition 1.3.12) (e.g. the category of sets, groups, abelian groups, modules over rings, or vector spaces over fields). Let $\mathcal{P}$ be a presheaf on X with values in $\mathcal{D}$.

The *sheaf associated to the presheaf $\mathcal{P}$* or the *sheafification of the presheaf $\mathcal{P}$*, denoted $\mathcal{P}^+$ or sometimes by $a\mathcal{P}$, is a sheaf on X together with a morphism of presheaves

$$\eta : \mathcal{P} \to \mathcal{P}^+,$$

satisfying the following universal property: for every sheaf $\mathcal{F}$ on X (valued in $\mathcal{D}$), any morphism of presheaves

$$\varphi : \mathcal{P} \to \mathcal{F}$$

factors uniquely through η, i.e., there exists a unique morphism of sheaves

$$\widetilde{\varphi} : \mathcal{P}^+ \to \mathcal{F}$$

such that

$$\varphi = \widetilde{\varphi} \circ \eta.$$

Concretely, $\mathcal{P}^+$ can be constructed by assigning to each open set $U \subseteq X$ the set (or object in $\mathcal{D}$)

$$\mathcal{P}^+(U) := \left\{ s = (s_x)_{x \in U} \in \prod_{x \in U} \mathcal{P}_x \;\middle|\; \begin{array}{l} \forall x \in U, \\ \exists \text{ an open } V \subseteq U \text{ with } x \in V, \\ \exists t \in \mathcal{P}(V) \text{ such that} \\ \quad \forall y \in V, s_y = t_y \end{array} \right\}.$$

where $\mathcal{P}_x$ is the stalk of $\mathcal{P}$ at x, and t_y is the germ of t at y. In particular, $\mathcal{P}^+$ exists.

It is noteworthy that the assignment $\mathcal{P} \mapsto \mathcal{P}^+$ is a functor

$$\mathrm{PreShv}(X, \mathcal{D}) \to \mathrm{Shv}(X, \mathcal{D}).$$

(Definition 6.1.9) and that this functor is left adjoint to the inclusion functor

$$\mathrm{Shv}(X, \mathcal{D}) \hookrightarrow \mathrm{PreShv}(X, \mathcal{D})$$

Equivalently, the assignment $\mathcal{P} \mapsto \mathcal{P}^+$ is the sheafification functor as defined in Definition 1.1.4.

Definition 1.1.4. Let $\mathcal{C}$ be a site (Definition 6.1.4) and let $\mathcal{A}$ be a (large) category (Definition 1.0.1).

Assuming that the presheaf (Definition 6.1.6) category $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ (and hence the sheaf (Definition 6.1.7) category $\mathrm{Shv}(\mathcal{C}, \mathcal{A})$) is locally small (Definition 1.0.5) (or U-locally small if a Grothendieck universe (Definition C.0.14) U is available), a *sheafification functor* refers to a functor

$$a : \mathrm{PreShv}(\mathcal{C}, \mathcal{A}) \to \mathrm{Shv}(\mathcal{C}, \mathcal{A})$$

that is left adjoint (Definition C.0.16) to the inclusion functor

$$i : \mathrm{Shv}(\mathcal{C}, \mathcal{A}) \hookrightarrow \mathrm{PreShv}(\mathcal{C}, \mathcal{A}).$$

If such a sheafification functor exists, then it is unique up to unique natural isomorphism. Given a presheaf P, the sheafification $a(P)$ is also sometimes called the *sheaf associated to P*. See Theorem 1.1.5 for common conditions under which sheafification exists.

Theorem 1.1.5. cf. [GV72, Exposé II, Théorème 3.4]

1. Let U be a universe. Let $\mathcal{C}$ be a U-site (Definition 6.1.4). A sheafification functor (Definition 1.1.3)

$$a : \mathrm{PreShv}(\mathcal{C}, U-\mathbf{Sets}) \to \mathrm{Shv}(\mathcal{C}, U-\mathbf{Sets}).$$

exists.

2. Let $\mathcal{C}$ be a site whose underlying category is locally small (Definition 1.0.5) and which has a topologically generating family (Definition 6.1.4) that is a set (rather than a proper class). A sheafification functor

$$a : \mathrm{PreShv}(\mathcal{C}, \mathbf{Sets}) \to \mathrm{Shv}(\mathcal{C}, \mathbf{Sets})$$

exists.

3. (see e.g. [nLa25, 3]) Let $(\mathcal{C}, J)$ be a site (Definition 6.1.4) on an essentially small category (Definition 1.0.11) $\mathcal{C}$. Suppose that the category $\mathcal{A}$ is complete, cocomplete (Definition 2.0.5), that small filtered colimits (Definition 1.3.12) in $\mathcal{A}$ are exact, and that $\mathcal{A}$ satisfies the IPC-property. A sheafification functor (Definition 1.1.4)

$$a : \mathrm{PreShv}(\mathcal{C}, \mathcal{A}) \to \mathrm{Shv}(\mathcal{C}, \mathcal{A})$$

exists. (♠ TODO: IPC-property, exactess in this context.)
   (♠ TODO: state as a fact that these categories are complete, cocomplete, with small filtered colimits that are exact) This is true for instance of $\mathcal{A} = \mathbf{Set}, \mathbf{Grp}, k-\mathbf{Alg}$ for a field k, or $\mathbf{Mod}_R$ for a (not necessarily commutative unital) ring R (Definition C.0.2).

Remark 1.1.6. If the presheaf is valued in nice "algebraic category", e.g. groups, abelian groups, rings, modules over a ring, etc., then the sheafification is again valued in that category. (♠ TODO: Make this more precise.)

Lemma 1.1.7. Let C be a category (Definition 1.0.1). A final object (Definition 2.0.1), if it exists, of C is the limit (Definition 1.3.2) of the empty diagram. In particular, any category that is closed (Definition 2.0.5) under finite limits or finite products (Definition 2.0.3) has a final object.

Definition 1.1.8 (Topos)**.** There are multiple notions of a topos depending on the context (geometric vs. logical).

1. A *Grothendieck topos* (or *sheaf topos*) is a category (Definition 1.0.1) equivalent (Definition 1.0.10) to the category of sheaves (Definition 6.1.7) of sets on a *small* site (Definition 6.1.4). That is, there exists a small site $(\mathcal{C}, J)$ such that the category is equivalent to $\mathrm{Sh}(\mathcal{C}, J)$.
2. Let $\mathscr{U}$ be a universe (Definition C.0.14). A *$\mathscr{U}$-topos* is a category equivalent to the category of sheaves of sets on a $\mathscr{U}$-small site $(\mathcal{C}, J)$, where the sheaves take values in the category of $\mathscr{U}$-sets ($\mathbf{Set}_{\mathscr{U}}$). [GV72, Exposé IV Définition 1.1]
3. An *elementary topos* is a category which has all finite limits (Definition 1.3.2), is cartesian closed, and has a subobject classifier.

Remark: Every Grothendieck topos is an elementary topos, but the converse is not true (e.g., the category of finite sets is an elementary topos but not a Grothendieck topos).

Lemma 1.1.9. Let $\mathscr{U}$ be a universe and let T be a $\mathscr{U}$-site. The category $\mathrm{Shv}(T)$ of sheaves on T has a final object.

1.2. Miscellaneous categorical constructions and definitions.

Definition 1.2.1 (Monomorphism and Epimorphism in Categories)**.** Let $\mathcal{C}$ be a category (Definition 1.0.1). For objects $A, B \in \mathcal{C}$, let $f : A \to B$ be a morphism in $\mathcal{C}$.

- The morphism f is called a *monomorphism* (or a *monic morphism*) if for every object X and every pair of morphisms $g_1, g_2 : X \to A$, the equality $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.
- The morphism f is called an *epimorphism* (or an *epic morphism*) if for every object Y and every pair of morphisms $h_1, h_2 : B \to Y$, the equality $h_1 \circ f = h_2 \circ f$ implies $h_1 = h_2$.

Definition 1.2.2. Let $\mathcal{C}$ be a (large) (Definition 1.0.1) pointed category (Definition 5.9.7), i.e. a category with a zero object (Definition 2.0.1) 0. Let $X, Y \in \mathrm{Ob}(\mathcal{C})$ be an object and let $f : X \to Y$ be a morphism.

1. A morphism $i : K \to X$ is called the *kernel of f* if:
    (a) $f \circ i = 0$, where 0 is the zero morphism (Definition 5.9.8) $K \to Y$,
    (b) for any morphism $g : Z \to X$ such that $f \circ g = 0$, there exists a unique morphism $u : Z \to K$ such that $g = i \circ u$.
   The kernel, if it exists, is unique up to unique isomorphism. $\ker(f)$ denotes the object K determined (up to isomorphism) by a kernel of f.
2. a morphism $p : Y \to Q$ is called the *cokernel of f* if:
    (a) $p \circ f = 0$, where 0 is the zero morphism (Definition 2.0.1) $X \to Q$,
    (b) for any morphism $g : Y \to Z$ such that $g \circ f = 0$, there exists a unique morphism $v : Q \to Z$ such that $g = v \circ p$.
   The cokernel, if it exists, is unique up to unique isomorphism. $\mathrm{coker}(f)$ denotes the object Q determined (up to isomorphism) by a cokernel of f.

Definition 1.2.3. Let $\mathcal{C}$ be an additive category (Definition 2.0.6). Let $X \in \mathrm{Ob}(\mathcal{C})$ be an object. A *subobject of X* refers to a monomorphism (Definition 1.2.1) $i : Y \hookrightarrow X$ in $\mathcal{C}$. We regard two subobjects (Y, i) and (Y', i') of X as isomorphic if there exists an isomorphism $f : Y \to Y'$ such that $i = i' \circ f$. One often leaves the monomorphism i implicit, suppressing it from the notation.

Definition 1.2.4. Let $\mathcal{C}$ be an abelian category (Definition 2.0.10). Let $X \in \mathrm{Ob}(\mathcal{C})$ be an object. Let $i : A \hookrightarrow X$ be a subobject (Definition 1.2.3). The cokernel (Definition 1.2.2) $\pi : X \twoheadrightarrow X/A := \mathrm{coker}(i)$ is called the *quotient object of X by A*. The object X/A is determined up to canonical isomorphism.

Definition 1.2.5. Let $\mathcal{C}$ be a category (Definition 1.0.1), and let $f : A \to B$ be a morphism in $\mathcal{C}$.

1. An *image of f* consists of an object $I \in \mathrm{Ob}(\mathcal{C})$ together with a factorization of f into two morphisms

$$A \xrightarrow{e} I \xrightarrow{m} B,$$

   where e is an epimorphism (Definition 1.2.1) and m is a monomorphism (Definition 1.2.1), such that for any other factorization

$$A \xrightarrow{e'} I' \xrightarrow{m'} B$$

   with e' epi and m' mono, there exists a unique isomorphism $\varphi : I \simeq I'$ satisfying $m = m'\varphi$ and $\varphi e = e'$.

The monomorphism $m : I \to B$ (or equivalently its subobject class) is called the *image of f in $\mathcal{C}$*.

2. Let $\mathcal{C}$ be a category (Definition 1.0.1), and let $f : A \to B$ be a morphism in $\mathcal{C}$. A *coimage of f* consists of an object $C \in \mathrm{Ob}(\mathcal{C})$ together with a factorization of f into two morphisms

$$A \xrightarrow{e} C \xrightarrow{m} B,$$

where e is an epimorphism and m is a monomorphism, such that for any other factorization

$$A \xrightarrow{e'} C' \xrightarrow{m'} B$$

with e' epi and m' mono, there exists a unique isomorphism $\varphi : C \simeq C'$ satisfying $m = m'\varphi$ and $\varphi e = e'$.

$$
\begin{array}{ccc}
 & C & \\
{\scriptstyle e}\nearrow & \vdots & \searrow {\scriptstyle m} \\
A & {\scriptstyle \wr}\big\downarrow {\scriptstyle \exists!\varphi} & B \\
{\scriptstyle e'}\searrow & \downarrow & \nearrow {\scriptstyle m'} \\
 & C' &
\end{array}
$$

The epimorphism $e : A \to C$ (or equivalently its quotient class) is called the *coimage of f in $\mathcal{C}$*.

1.3. Diagrams, systems, and limits in categories.

Definition 1.3.1 (Diagram in a category and category of diagrams). Let $\mathcal{C}$ be a (large) category (Definition 1.0.1), and let I be a (large) category (Definition 1.0.1).

1. A *diagram of shape I in $\mathcal{C}$* is a functor (Definition 1.0.3) $D : I \to \mathcal{C}$. We often denote such a diagram by the family $\{D(i)\}_{i \in \mathrm{Ob}(I)}$ with transition maps given by the functorial image of morphisms in I.

   A diagram is also synonymously called a *system*. Moreover, the category I is called the *index category* or the *indexing category of the diagram D*.
2. Given two diagrams $D, E : I \to \mathcal{C}$, a *morphism of diagrams* is a simply a natural transformation (Definition 1.0.4) $D \Rightarrow E$ of the functors D and E.
3. The *category of I-shaped diagrams in $\mathcal{C}$* or simply *diagram category (of I-shaped diagrams in $\mathcal{C}$)*, often denoted $\mathcal{C}^I$, $[I, \mathcal{C}]$, or $\mathrm{Fun}(I, \mathcal{C})$, is the (large) category whose objects are functors $I \to \mathcal{C}$ (that is, diagrams of shape I in $\mathcal{C}$) and whose morphisms are natural transformations (Definition 1.0.4) between such functors. The category $\mathcal{C}^I$ is also called the *functor category of functors $I \to \mathcal{C}$*. Equivalently, the functor category $\mathcal{C}^I$ is the category $\mathrm{PreShv}(I^{\mathrm{op}}, \mathcal{C})$ of presheaves (Definition 6.1.6) on I^{op} with values in $\mathcal{C}$ and hence notations for presheaf categories are applicable as notations for functor categories.

   If $\mathcal{C}$ is locally small (Definition 1.0.5) and I is small, then $\mathcal{C}^I$ is locally small by Lemma 1.0.7.

Intuitively, a limit of a diagram in a category is a "universal" object with a morphism to (all objects in) the diagram and a colimit, the dual notion, is a "universal" object with a morphism from (all objects in) the diagram.

Definition 1.3.2 (Cones, limits and colimits in a category). Let $\mathcal{C}$ be a (large) category (Definition 1.0.1), let I be a (large) category, and let $D : I \to \mathcal{C}$ be a diagram (Definition 1.3.1) (Definition 1.3.1).

1. A *cone to the diagram D* is an object $L \in \mathcal{C}$ together with a family of morphisms

$$\{\pi_i : L \to D(i)\}_{i \in I}$$

such that for every morphism $f : i \to j$ in I, the diagram

commutes, i.e. $D(f) \circ \pi_i = \pi_j$.

2. A cone $(L, \{\pi_i\})$ is called a *limit of D* if it satisfies the following "universal property": for any cone $(C, \{f_i\})$ over D, there exists a *unique* morphism $u : C \to L$ such that

$$\pi_i \circ u = f_i \quad \text{for all } i \in I.$$

Visually, the following diagrams commute every morphism $f : i \to j$ in I:

If such a cone exists, then the object L is necessarily unique up to unique isomorphism by the universal property. In this case, L is denoted by $\lim_{i \in I} D$ or $\lim D$.

3. A *cocone from the diagram D* is an object $C \in \mathcal{C}$ together with a family of morphisms

$$\{\iota_i : D(i) \to C\}_{i \in I}$$

such that for every morphism $f : i \to j$ in I, the diagram

commutes, i.e. $\iota_j \circ D(f) = \iota_i$.

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4. A cocone $(L, \{\iota_i\})$ is called a *colimit of D* if it satisfies the following "universal property": for any cocone $(C, \{g_i\})$ under D, there exists a *unique* morphism $u : L \to C$ such that

$$u \circ \iota_i = g_i \quad \text{for all } i \in I.$$

Visually, the following diagrams commute every morphism $f : i \to j$ in I:

If such a cocone exists, then the object L is necessarily unique up to unique isomorphism by the universal property. In this case, L is denoted by $\mathrm{colim}_{i \in I} D$ or $\mathrm{colim}\, D$.

A limit/colimit is called *finite* (resp. *small*) if the index category I is finite (resp. small).

Some authors use the terms *projective limit* or *inverse limit* to refer to what is defined here as a limit, Similarly, the terms *inductive limit* or *direct limit* are sometimes used to mean a colimit. However, these phrases can have more specific meanings to other authors: a *projective* or *inverse limit* may refer to a limit over a diagram indexed by a codirected poset (Definition 1.3.7). Likewise, an *inductive* or *direct limit* may refer to a colimit over a directed poset (Definition 1.3.7) (see Definition 1.3.12) .

Thus, while the terms are sometimes used interchangeably with "limit" and "colimit," they may also emphasize particular indexing shapes and directions, distinguishing them from general limits and colimits taken over arbitrary small categories.

Theorem 1.3.3. Let $\mathcal{C}$ be a category (Definition 1.0.1), and let $\mathcal{J}$ be a category such that the limits (resp. colimits) (Definition 1.3.2) indexed by $\mathcal{J}$ exist in $\mathcal{C}$. Then the process of taking limits (resp. colimits) defines a functor

$$\lim : \mathcal{C}^{\mathcal{J}} \to \mathcal{C} \quad (\text{resp. } \mathrm{colim} : \mathcal{C}^{\mathcal{J}} \to \mathcal{C})$$

from the functor category (Definition 1.3.1) $\mathcal{C}^{\mathcal{J}}$ to $\mathcal{C}$.

More precisely:

1. If $\mathcal{C}$ has all limits indexed by the category $\mathcal{J}$, then the assignment sending each diagram $D : \mathcal{J} \to \mathcal{C}$ to its limit $\lim D$ extends to a functor

$$\lim : \mathrm{Fun}(\mathcal{J}, \mathcal{C}) \to \mathcal{C}.$$

This functor sends a natural transformation $\alpha : D \to D'$ to the unique morphism $\lim D \to \lim D'$ induced by the universal property of limits.
2. Similarly, if $\mathcal{C}$ has all colimits indexed by $\mathcal{J}$, then the assignment sending each diagram $D : \mathcal{J} \to \mathcal{C}$ to its colimit $\mathrm{colim}\, D$ extends to a functor

$$\mathrm{colim} : \mathrm{Fun}(\mathcal{J}, \mathcal{C}) \to \mathcal{C}.$$

This functor sends a natural transformation $\alpha : D \to D'$ to the unique morphism $\operatorname{colim} D \to \operatorname{colim} D'$ induced by the universal property of colimits.

For a diagram indexed by an essentially small category I, we can identify their limits/colimits with those of diagrams indexed by a small category J equivalent to I.

Theorem 1.3.4 (Limits and Colimits indexed by essentially small categories). Let $\mathcal{C}$ be a (large) category (Definition 1.0.1) and let I be an essentially small category (Definition 1.0.11). Suppose $D : I \to \mathcal{C}$ is a diagram indexed by I (Definition 1.3.1).

If J is a small category (Definition 1.0.5) that is equivalent (Definition 1.0.10) to I via an equivalence of categories
$$F : J \xrightarrow{\sim} I,$$
then the limit and colimit (Definition 1.3.2) of D indexed by I may be identified with the limit and colimit of the composed diagram
$$D \circ F : J \to \mathcal{C}.$$

In other words, if either $\lim D$ or $\operatorname{colim} D$ exists in $\mathcal{C}$, then so does $\lim(D \circ F)$ and $\operatorname{colim}(D \circ F)$, and these limits and colimits are canonically isomorphic as objects of $\mathcal{C}$.

Definition 1.3.5. Let $\mathcal{C}$ be a (large) category (Definition 1.0.1), let I be a small category (Definition 1.0.5), and let $D : I \to \mathcal{C}$ be a diagram (Definition 1.3.1).

A limit or colimit (Definition 1.3.2) is called *finite* (resp. *small*) if the indexing category I has finitely many objects and morphisms (resp. if I is a small category (Definition 1.0.5)).

Theorem 1.3.6. Let $\mathcal{C}$ be a category (Definition 1.0.1). Let $F : \mathcal{J} \to \mathcal{C}$ be a diagram (Definition 1.3.1) where $\mathcal{J}$ is a small category (Definition 1.0.5).

1. The limit (Definition 1.3.2) of F is constructed as the equalizer (Definition 2.0.12) of the pair of morphisms (s, t): assuming that the products (Definition 2.0.3) and equalizers below exist in $\mathcal{C}$, the limit $\lim F$ exists and

$$\lim F \cong \operatorname{eq}\left(\prod_{j \in \operatorname{Ob}(\mathcal{J})} F(j) \xrightarrow[t]{s} \prod_{\alpha \in \operatorname{Mor}(\mathcal{J})} F(\operatorname{cod}(\alpha)) \right)$$

where $\operatorname{cod}(\alpha)$ stands for the codomain of the morphism α, and the morphisms s and t are induced by the universal property of the product, such that for any morphism $\alpha : i \to k$ in $\mathcal{J}$, the projection to the factor indexed by α is:
   - $\pi_\alpha \circ s = F(\alpha) \circ \pi_i$
   - $\pi_\alpha \circ t = \pi_k$
2. The colimit (Definition 1.3.2) of F is constructed as the coequalizer (Definition 2.0.12) of the pair of morphisms (s, t): assuming that the coproducts (Definition 2.0.3) and coequalizers below exist in $\mathcal{C}$, the colimit $\operatorname{colim} F$ exists and

$$\operatorname{colim} F \cong \operatorname{coeq}\left(\coprod_{\alpha \in \operatorname{Mor}(\mathcal{J})} F(\operatorname{dom}(\alpha)) \xrightarrow[t]{s} \coprod_{j \in \operatorname{Ob}(\mathcal{J})} F(j) \right)$$

where $\mathrm{dom}(\alpha)$ stands for the domain of the morphism α, and the morphisms s and t are induced by the universal property of the coproduct, such that for any morphism $\alpha : i \to k$ in $\mathcal{J}$, the injection from the summand indexed by α is:

- $s \circ \iota_\alpha = \iota_k \circ F(\alpha)$
- $t \circ \iota_\alpha = \iota_i$

In particular,

1. If $\mathcal{C}$ has all nonempty finite (resp. small) products and equalizers, then $\mathcal{C}$ has all nonempty finite (resp. small) limits.
2. If $\mathcal{C}$ has all nonempty finite (resp. small) coproducts and coequalizers, then $\mathcal{C}$ has all nonempty finite (resp. small) colimits.
3. If $\mathcal{C}$ has all finite (resp. small) products and equalizers, then $\mathcal{C}$ has all finite (resp. small) limits.
4. If $\mathcal{C}$ has all finite (resp. small) coproducts and coequalizers, then $\mathcal{C}$ has all finite (resp. small) colimits.

Definition 1.3.7 (Partially ordered set).   1. A *partially ordered set* (or *poset*), or *ordered set* is a pair $(P, \leq)$ where P is a set and

$$\leq : P \times P \to \{\mathrm{true}, \mathrm{false}\}$$

is a binary relation on P satisfying the following axioms for all $a, b, c \in P$:

- *Reflexivity:* $a \leq a$,
- *Antisymmetry:* if $a \leq b$ and $b \leq a$, then $a = b$,
- *Transitivity:* if $a \leq b$ and $b \leq c$, then $a \leq c$.

The relation $\leq$ is called an *order* or a *partial order*

2. A partially ordered set $(P, \leq)$ is called a *directed partially ordered set* if for every pair $a, b \in P$, there exists $c \in P$ such that

$$a \leq c \quad \text{and} \quad b \leq c.$$

3. A partially ordered set $(P, \leq)$ is called a *codirected partially ordered set* (or *downward directed poset*) if for every pair $a, b \in P$, there exists $d \in P$ such that

$$d \leq a \quad \text{and} \quad d \leq b.$$

Definition 1.3.8 (Filtered category).   1. A *filtered category* is a (nonempty, large) category $\mathcal{I}$ satisfying the following conditions:

- For every finite collection of objects $i_1, i_2, \ldots, i_n$ in $\mathcal{I}$, there exists an object j and morphisms

$$\phi_k : i_k \to j, \quad \text{for each } k = 1, \ldots, n.$$

- For every pair of morphisms $f, g : i \to j$ in $\mathcal{I}$, there exists an object k and a morphism

$$h : j \to k$$

that satisfies

$$h \circ f = h \circ g.$$

In other words, $\mathcal{I}$ is nonempty, any finite diagram of objects admits a cocone (Definition 1.3.2), and any pair of parallel morphisms become equal after post-composition with an appropriate morphism.

i_1

ϕ_1

j

ϕ_2

i_2
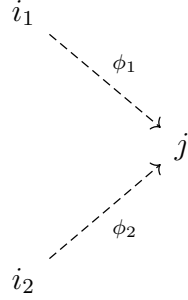
FIGURE 1. *
Condition 1: Upper Bound

$$i \xrightarrow[g]{f} j \dashrightarrow^{h} k$$

FIGURE 2. *
Condition 2: Coequalizing map

2. Dually, a *Cofiltered category* is a category whose opposite category (Definition 1.0.2) is filtered. More explicitly, A cofiltered category is a (nonempty, large) category $\mathcal{I}$ satisfying the following conditions:

- For every finite collection of objects $i_1, i_2, \ldots, i_n$ in $\mathcal{I}$, there exists an object j and morphisms
$$\phi_k : j \to i_k, \quad \text{for each } k = 1, \ldots, n.$$

- For every pair of morphisms $f, g : j \to i$ in $\mathcal{I}$, there exists an object k and a morphism
$$h : k \to j$$

that satisfies
$$f \circ h = g \circ h.$$

i_1

ϕ_1

j

ϕ_2

i_2
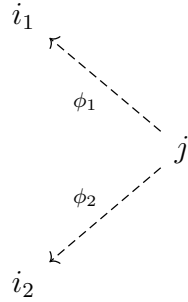
FIGURE 3. *
Condition 1: Lower Bound

$$k \dashrightarrow^{h} j \xrightarrow[g]{f} i$$

FIGURE 4. *
Condition 2: Equalizing map

In other words, $\mathcal{I}$ is nonempty, any finite diagram of objects admits a cone, and any pair of parallel morphisms become equal after pre-composition with an appropriate morphism.

**Definition 1.3.9** (Thin category). A *thin category* is a (large) category $\mathcal{C}$ such that for every pair of objects $X, Y \in \mathcal{C}$, the hom-collection

$$\mathrm{Hom}_{\mathcal{C}}(X, Y)$$

has at most one element.

Lemma 1.3.10. Let $(P, \leq)$ be a nonempty poset (Definition 1.3.7).

1. Regarding P as a category whose objects are the elements of P and such that there is a unique arrow $a \to b$ if and only if $a \leq b$, the category is filtered.
2. Every nonempty small (Definition 1.0.5) thin (Definition 1.3.9) filtered category (Definition 1.3.8) corresponds to a poset in this way.
3. Moreover, the poset P is directed (Definition 1.3.7) if and only if the category is filtered. The poset P is codirected (Definition 1.3.7) if and only if the category is cofiltered.

Definition 1.3.11 (Systems in a category)**.** Let $\mathcal{C}$ be a (large) category. Let I be a (large) category.

1. A diagram/system (Definition 1.3.1) $I \to C$ is called *filtered* (resp. *cofiltered*) if I is a filtered (Definition 1.3.8) (resp. cofiltered (Definition 1.3.8)) category.
2. A diagram/system $I \to C$ is called *directed* (resp. *codirected* if I is small and thing, i.e. is regardable/comes from (Lemma 1.3.10) a directed (resp. codirected) partially ordered set (Definition 1.3.7). A *direct system* or *inductive system* is synonymous for a directed system and a *inverse system* or *projective system* is synonymous for a codirected system.

One might also speak of a *filtered direct/inductive system* synonymously for a filtered system to emphasize that the indexing caetgory is a general filtered category, rather than a directed poset.

Definition 1.3.12 (Special cases of limits)**.** Let $\mathcal{C}$ be a (large) category. Let I be a (large) category. Let $I \to \mathcal{C}$ be a diagram/system.

- Suppose that the system is a cofiltered system (Definition 1.3.11), i.e. I is a cofiltered category. A limit (Definition 1.3.2) of this diagram is often denoted by

$$\varprojlim_{i \in I} D(i)$$

and may be called a *cofiltered (inverse/projective) limit*. In case that the system is more specifically an inverse/projective system (Definition 1.3.11), i.e. I is a cofiltered poset (Definition 1.3.7), the preferred term for such a limit is *inverse/projective limit*.
- Suppose that the system is a filtered system, i.e. I is a filtered category. A colimit of this diagram is often denoted by

$$\varinjlim_{i \in I} D(i)$$

and may be called a *filtered colimit* or a *direct/inductive/injective limit*. In case that the system is more specifically a direct/inductive system, i.e. I is a filtered poset (Definition 1.3.7), the preferred term for such a limit is *direct/inductive limit*.

Lemma 1.3.13. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be (large) categories (Definition 1.0.1) and let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ (Definition 1.0.12) be a functor (Definition 1.0.13).

1. There is a functor $A \to \mathrm{Fun}(\mathcal{B}, \mathcal{C})$ (Definition 1.3.1) given by $A \mapsto (F(A, -) : \mathcal{B} \to \mathcal{C})$.

2. Given a morphism $f : A \to A'$ in $\mathcal{A}$, there is a natural transformation (Definition 1.0.4) $F(f, -) : F(A, -) \to F(A', -)$.

Theorem 1.3.17 states the right adjoint functors commute with limits and left adjoint functors commute with colimits.

Definition 1.3.14. Let $\mathcal{C}$ and $\mathcal{D}$ be (large) categories (Definition 1.0.1), $F : \mathcal{C} \to \mathcal{D}$ a functor (Definition 1.0.3), and $D : \mathcal{J} \to \mathcal{C}$ a diagram (Definition 1.3.1) indexed by a small category $\mathcal{J}$.

1. Assume that the colimit (Definition 1.3.2) of D exists in $\mathcal{C}$, denoted by $\operatorname{colim}_{\mathcal{J}} D$, with universal cocone (Definition 1.3.2) $\eta : D \Rightarrow \Delta_{\operatorname{colim} D}$. The functor F maps this to a cocone $F(\eta) : F \circ D \Rightarrow \Delta_{F(\operatorname{colim} D)}$ in $\mathcal{D}$.
   If the colimit of the composite diagram $F \circ D$ exists in $\mathcal{D}$, denoted by $\operatorname{colim}_{\mathcal{J}}(F \circ D)$, then by the universal property of the colimit, there exists a unique morphism
   $$\phi : \operatorname{colim}_{j \in \mathcal{J}} F(D(j)) \longrightarrow F\left(\operatorname{colim}_{j \in \mathcal{J}} D(j)\right)$$
   mediating between the cocone of the composite colimit and the image cocone $F(\eta)$. This unique morphism is called the $\boxed{\textit{canonical colimit comparison morphism}}$.
2. Dually, assume that the limit of D exists in $\mathcal{C}$, denoted by $\lim_{\mathcal{J}} D$, with universal cone $\varepsilon : \Delta_{\lim D} \Rightarrow D$. The functor F maps this to a cone $F(\varepsilon) : \Delta_{F(\lim D)} \Rightarrow F \circ D$ in $\mathcal{D}$.
   If the limit of the composite diagram $F \circ D$ exists in $\mathcal{D}$, denoted by $\lim_{\mathcal{J}}(F \circ D)$, then by the universal property of the limit, there exists a unique morphism
   $$\psi : F\left(\lim_{j \in \mathcal{J}} D(j)\right) \longrightarrow \lim_{j \in \mathcal{J}} F(D(j))$$
   mediating between the image cone $F(\varepsilon)$ and the cone of the composite limit. This unique morphism is called the $\boxed{\textit{canonical limit comparison morphism}}$:

Definition 1.3.15. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor (Definition 1.0.3) between (large) categories (Definition 1.0.1) $\mathcal{C}$ and $\mathcal{D}$.

Let $D : \mathcal{J} \to \mathcal{C}$ be a diagram in $\mathcal{C}$ such that the limit $\lim_{\mathcal{J}} D$ exists in $\mathcal{C}$, with limiting cone $\lambda : \Delta_{\lim D} \Rightarrow D$. The functor F $\boxed{\textit{preserves the limit of } D}$ if the image cone
$$F(\lambda) : \Delta_{F(\lim D)} \cong F \circ \Delta_{\lim D} \Rightarrow F \circ D$$
exhibits $F(\lim_{\mathcal{J}} D)$ as a limit of the composite diagram $F \circ D$ in $\mathcal{D}$. Explicitly, the canonical comparison morphism (Definition 1.3.14)
$$F(\lim_{\mathcal{J}} D) \overset{\cong}{\Rightarrow} \lim_{\mathcal{J}}(F \circ D)$$
must be an isomorphism.

Dually, let $D : \mathcal{J} \to \mathcal{C}$ be a diagram such that the colimit $\operatorname{colim}_{\mathcal{J}} D$ exists in $\mathcal{C}$. The functor F $\boxed{\textit{preserves the colimit of } D}$ if the canonical comparison morphism (Definition 1.3.14)
$$\operatorname{colim}_{\mathcal{J}}(F \circ D) \overset{\cong}{\Rightarrow} F(\operatorname{colim}_{\mathcal{J}} D)$$
is an isomorphism.

Definition 1.3.16. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor (Definition 1.0.3) between (large) categories (Definition 1.0.1) $\mathcal{C}$ and $\mathcal{D}$.

- The functor F is called *continuous* if it preserves all small limits that exist in $\mathcal{C}$. That is, for every small category $\mathcal{J}$ and every diagram $D : \mathcal{J} \to \mathcal{C}$ having a limit in $\mathcal{C}$, F preserves the limit (Definition 1.3.15) of D.
- The functor F is called *cocontinuous* if it preserves all small colimits that exist in $\mathcal{C}$. That is, for every small category $\mathcal{J}$ and every diagram $D : \mathcal{J} \to \mathcal{C}$ having a colimit in $\mathcal{C}$, F preserves the colimit (Definition 1.3.15) of D.

Theorem 1.3.17 (Continuity of Adjoint Functors)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be (large) categories (Definition 1.0.1), and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors forming an adjunction (Definition C.0.16) $F \dashv G$, i.e. F is the left adjoint and G is the right adjoint. Let $\mathcal{J}$ be a small index category.

1. **Right Adjoints Preserve Limits:** Let $D : \mathcal{J} \to \mathcal{D}$ be a diagram (Definition 1.3.1) such that the limit (Definition 1.3.2) $\lim_{\mathcal{J}} D$ exists in $\mathcal{D}$. Then the limit of the composite diagram $G \circ D : \mathcal{J} \to \mathcal{C}$ exists in $\mathcal{C}$, and G preserves the limit (Definition 1.3.15), i.e. the comparison morphism (Definition 1.3.14) induced by the universal property of the limit,

$$\vartheta : G\left(\lim_{j \in \mathcal{J}} D(j)\right) \overset{\cong}{\to} \lim_{j \in \mathcal{J}} G(D(j)),$$

   is an isomorphism.
2. **Left Adjoints Preserve Colimits:** Let $D : \mathcal{J} \to \mathcal{C}$ be a diagram such that the colimit $\mathrm{colim}_{\mathcal{J}} D$ exists in $\mathcal{C}$. Then the colimit of the composite diagram $F \circ D : \mathcal{J} \to \mathcal{D}$ exists in $\mathcal{D}$, and F preserves the colimits (Definition 1.3.15), i.e. the comparison morphism (Definition 1.3.14) induced by the universal property of the colimit,

$$\varphi : \mathrm{colim}_{j \in \mathcal{J}} F(D(j)) \overset{\cong}{\to} F\left(\mathrm{colim}_{j \in \mathcal{J}} D(j)\right),$$

   is an isomorphism.

Theorem 1.3.18 states the colimits commutes with colimits and limits commute with limits.

Theorem 1.3.18 (Fubini Theorem for Colimits and Limtis)**.** Let $\mathcal{C}$ be a category and let I and J be small categories. Let $D : I \times J \to \mathcal{C}$ be a bifunctor.

1. Assume that the following colimits (Definition 1.3.2) exist in $\mathcal{C}$:
   - for every object $i \in I$, $\varinjlim_{j \in J} D(i, j)$
   - $\varinjlim_{i \in I} \left(\varinjlim_{j \in J} D(i, j)\right)$
   - for every object $j \in J$, $\varinjlim_{i \in I} D(i, j)$
   - $\varinjlim_{j \in J} \left(\varinjlim_{i \in I} D(i, j)\right)$.

   Then there is a canonical isomorphism:

$$\varinjlim_{i \in I} \left(\varinjlim_{j \in J} D(i, j)\right) \cong \varinjlim_{(i,j) \in I \times J} D(i, j) \cong \varinjlim_{j \in J} \left(\varinjlim_{i \in I} D(i, j)\right).$$

2. Assume that the following limits (Definition 1.3.2) exist in $\mathcal{C}$:
   - for every object $i \in I$, $\varprojlim_{j \in J} D(i,j)$
   - $\varprojlim_{i \in I} \left(\varprojlim_{j \in J} D(i,j) \right)$
   - for every object $j \in J$, $\varprojlim_{i \in I} D(i,j)$
   - $\varprojlim_{j \in J} \left(\varprojlim_{i \in I} D(i,j) \right)$.

   Then there is a canonical isomorphism:

$$\varprojlim_{i \in I} \left(\varprojlim_{j \in J} D(i,j) \right) \cong \varprojlim_{(i,j) \in I \times J} D(i,j) \cong \varprojlim_{j \in J} \left(\varprojlim_{i \in I} D(i,j) \right).$$

Remark 1.3.19. These results assert that the colimit functor $\mathrm{colim} : \mathcal{C}^{I \times J} \to \mathcal{C}$ (assuming $\mathcal{C}$ is cocomplete) is isomorphic to the composition of partial colimit functors $\mathrm{colim}_I \circ \mathrm{colim}_J$ and $\mathrm{colim}_J \circ \mathrm{colim}_I$.

Remark 1.3.20. These results assert that the colimit functor $\mathrm{colim} : \mathcal{C}^{I \times J} \to \mathcal{C}$ (assuming $\mathcal{C}$ is cocomplete) is isomorphic to the composition of partial colimit functors $\mathrm{colim}_I \circ \mathrm{colim}_J$ and $\mathrm{colim}_J \circ \mathrm{colim}_I$.

At times, we may want to consider the limit/colimit of a system of functors. Such limits/colimits can be computed "pointwise", whenever the pointwise limits/colimits exist.

Proposition 1.3.21 (Pointwise Computation of Limits and Colimits in Functor Categories). Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Let $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ denote the category of functors from $\mathcal{C}$ to $\mathcal{D}$ (also denoted $\mathcal{D}^{\mathcal{C}}$). Let $\mathcal{J}$ be a small category and let $F : \mathcal{J} \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be a diagram of functors, denoted by $j \mapsto F_j$.

1. **Limits are computed pointwise:** Suppose that for every object $C \in \mathcal{C}$, the limit of the diagram $j \mapsto F_j(C)$ exists in $\mathcal{D}$. Then the limit of the diagram F exists in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and is computed pointwise. That is, there is an isomorphism in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$:

$$\left(\varprojlim_{j \in \mathcal{J}} F_j \right)(C) \cong \varprojlim_{j \in \mathcal{J}} (F_j(C)).$$

   The action of this limit functor on a morphism $f : C \to C'$ in $\mathcal{C}$ is the unique morphism induced by the family $\{F_j(f)\}_{j \in \mathcal{J}}$ via the universal property of limits in $\mathcal{D}$.

2. **Colimits are computed pointwise:** Suppose that for every object $C \in \mathcal{C}$, the colimit of the diagram $j \mapsto F_j(C)$ exists in $\mathcal{D}$. Then the colimit of the diagram F exists in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and is computed pointwise. That is, there is an isomorphism in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$:

$$(\mathrm{colim}_{j \in \mathcal{J}} F_j)(C) \cong \mathrm{colim}_{j \in \mathcal{J}} (F_j(C)).$$

   The action of this colimit functor on a morphism $f : C \to C'$ in $\mathcal{C}$ is the unique morphism induced by the family $\{F_j(f)\}_{j \in \mathcal{J}}$ via the universal property of colimits in $\mathcal{D}$.

Definition 1.3.22. Let $\mathcal{C}$ and $\mathcal{J}$ be categories (Definition 1.0.1). The *diagonal functor*

$$\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$$

(Definition 1.3.1) (♠ TODO: constant functor) is the functor sending an object $X \in \mathcal{C}$ to the constant functor $\Delta(X) : \mathcal{J} \to \mathcal{C}$, which maps every object in $\mathcal{J}$ to X and every morphism in $\mathcal{J}$ to the identity morphism 1_X. For a morphism $f : X \to Y$ in $\mathcal{C}$, $\Delta(f)$ is the natural transformation (Definition 1.0.4) whose component at every $j \in \mathcal{J}$ is f.

Theorem 1.3.23. Let $\mathcal{C}$ be a locally small (Definition 1.0.5) category and $\mathcal{J}$ be a small index category (Definition 1.0.5).

1. If $\mathcal{C}$ admits all colimits (Definition 1.3.2) of shape (Definition 1.3.1) $\mathcal{J}$, then the colimit functor (Theorem 1.3.3)
$$\mathrm{colim} : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$$
is left adjoint (Definition C.0.16) to the diagonal functor Δ. That is, for any functor $F : \mathcal{J} \to \mathcal{C}$ and any object $X \in \mathcal{C}$, there is a natural bijection:
$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}\, F, X) \cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{J}}}(F, \Delta(X)).$$

2. If $\mathcal{C}$ admits all limits of shape $\mathcal{J}$, then the limit functor (Theorem 1.3.3)
$$\lim : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$$
is right adjoint (Definition C.0.16) to the diagonal functor (Definition 1.3.22) Δ. That is, for any object $X \in \mathcal{C}$ and any functor $F : \mathcal{J} \to \mathcal{C}$, there is a natural bijection:
$$\mathrm{Hom}_{\mathcal{C}}(X, \lim F) \cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{J}}}(\Delta(X), F).$$

These adjunctions characterize limits and colimits via their universal properties.

2. Additive and abelian categories

Definition 2.0.1. Let $\mathcal{C}$ be a (large) category (Definition 1.0.1).

1. An object $I \in \mathcal{C}$ is called an *initial object* if for every object $X \in \mathcal{C}$ there exists a unique morphism
$$I \to X.$$
Equivalently, an initial object is a limit (Definition 1.3.2) of the empty diagram (Definition 1.3.1), if such a limit exists.

2. An object $F \in \mathcal{C}$ is called a *final object* (or *terminal object*) if for every object $X \in \mathcal{C}$ there exists a unique morphism
$$X \to F.$$
Equivalently, a final object is a colimit (Definition 1.3.2) of the empty diagram (Definition 1.3.1), if such a colimit exists.

3. An object $Z \in \mathcal{C}$ is called a *zero object* if Z is both initial and final in $\mathcal{C}$. In particular, for every object $X \in \mathcal{C}$ there exist unique morphisms
$$Z \to X \quad \text{and} \quad X \to Z.$$

In particular, if initial/final/zero objects exist in a cateogry, then they are unique up to unique isomorphism.

Lemma 2.0.2. Let $\mathcal{C}$ be a full subcategory (Definition 1.0.8) of a (large) category (Definition 1.0.1) $\mathcal{D}$. Suppose that $\mathcal{D}$ hsa an initial object (Definition 2.0.1) I (resp. a final object (Definition 2.0.1) F) and that this object also belongs to $\mathcal{C}$. The object is initial (resp. final) in $\mathcal{C}$.

Definition 2.0.3 (Product in a category)**.** Let $\mathcal{C}$ be a category and let $\{X_i\}_{i \in I}$ be a family of objects in $\mathcal{C}$ indexed by a class I.

1. A *product of the family* $\{X_i\}$ is an object P of $\mathcal{C}$ together with a "universal" family of morphisms
$$\pi_i : P \to X_i, \quad \text{for each } i \in I.$$
   More precisely, for any object Y and any family of morphisms $\{f_i : Y \to X_i\}_{i \in I}$, there exists a unique morphism
$$f : Y \to P$$
   making the following diagram commute for all $i \in I$, i.e. $\pi_i \circ f = f_i$:

   Such a product is often denoted by $\prod_{i \in I} X_i$. If $\prod_{i \in I} X_i$ exists in $\mathcal{C}$, then it is unique up to unique isomorphism by the universal property described above.
      Equivalently, the product $\prod_{i \in I} X_i$ is the limit (Definition 1.3.2) of the diagram (Definition 1.3.1) $I \to \mathcal{C}, i \mapsto X_i$, where I is made into a category whose objects are the members of I and whose morphisms are just the identity morphisms.
2. A *coproduct* (or synonymously *direct sum*) of the family $\{X_i\}$ is an object C of $\mathcal{C}$ together with a "universal" family of morphisms
$$\iota_i : X_i \to C, \quad \text{for each } i \in I.$$
   More precisely, for any object Y and any family of morphisms $\{g_i : X_i \to Y\}_{i \in I}$, there exists a unique morphism
$$g : C \to Y$$
   making the following diagram commute for all $i \in I$, i.e. $g \circ \iota_i = g_i$:

   Such a coproduct is often denoted by $\coprod_{i \in I} X_i$ or $\oplus_{i \in I} X_i$. If $\coprod_{i \in I} X_i$ exists in $\mathcal{C}$, then it is unique up to unique isomorphism by the universal property described above.
      Equivalently, the coproduct $\coprod_{i \in I} X_i$ is the colimit (Definition 1.3.2) of the diagram (Definition 1.3.1) $I \to \mathcal{C}, i \mapsto X_i$, where I is made into a category whose objects are the members of I and whose morphisms are just the identity morphisms.

Definition 2.0.4. Let $\mathcal{C}$ be a (large) category (Definition 1.0.1), and let $c_1, c_2, \ldots, c_n \in \mathrm{Ob}(\mathcal{C})$ be objects.

A *product of $c_1, c_2, \ldots, c_n$* in $\mathcal{C}$ is a tuple

$$\left(c_1 \times c_2 \times \cdots \times c_n, \ p_1 : c_1 \times \cdots \times c_n \to c_1, \ \ldots, \ p_n : c_1 \times \cdots \times c_n \to c_n \right)$$

such that for every object $d \in \mathrm{Ob}(\mathcal{C})$ and every family of morphisms

$$(f_1 : d \to c_1, \ f_2 : d \to c_2, \ \ldots, \ f_n : d \to c_n),$$

there exists a unique morphism

$$\langle f_1, f_2, \ldots, f_n \rangle : d \to c_1 \times c_2 \times \cdots \times c_n$$

satisfying

$$p_i \circ \langle f_1, f_2, \ldots, f_n \rangle = f_i \quad \text{for all } 1 \leq i \leq n.$$

The morphisms p_i are called the *projection morphisms*, and the morphism $\langle f_1, f_2, \ldots, f_n \rangle$ is called the *product morphism*. The product $c_1 \times \cdots \times c_n$ is also denoted by $\prod_{i=1}^{n} c_i$, $c_1 \oplus \cdots \oplus c_n$, or $\oplus_{i=1}^{n} c_i$.

When $n = 0$, the product is defined to be a terminal object (Definition 2.0.1) of $\mathcal{C}$, if one exists. When $n = 2$, we simply speak of the *binary product* or *biproduct* of c_1 and c_2.

Definition 2.0.5 (Complete and Cocomplete Category)**.** Let $\mathcal{C}$ be a category (Definition 1.0.1).

- The category $\mathcal{C}$ is called *complete* (resp. *finitely complete*) if all small limits (Definition 1.3.5) (resp. finite limits) exist in $\mathcal{C}$; that is, for every small diagram $D : J \to \mathcal{C}$ (with J a small category), the limit $\lim D$ exists and is an object of $\mathcal{C}$.
- The category $\mathcal{C}$ is called *cocomplete* (resp. *finitely cocomplete*) if all small colimits (Definition 1.3.5) (resp. finite colimits) exist in $\mathcal{C}$; that is, for every small diagram $D : J \to \mathcal{C}$, the colimit $\mathrm{colim}\, D$ exists and is an object of $\mathcal{C}$.

Definition 2.0.6 (Additive category)**.** Let $\mathcal{A}$ be a locally small category (Definition 1.0.5).

1. $\mathcal{A}$ is said to be a *preadditive category* if the following hold:
   - For any two objects A, B in $\mathcal{A}$, the set $\mathrm{Hom}_{\mathcal{A}}(A, B)$ is an abelian group (Definition C.0.1), and composition of morphisms is bilinear.
   - There is a zero object (Definition 2.0.1) 0 in $\mathcal{A}$.

   Equvialently, a preadditive cateogry $\mathcal{A}$ is a (necessarily locally small) category enriched in (Definition A.0.2) the monoidal category (Definition A.0.1) Ab that also possesses a zero object.

2. If $\mathcal{A}$ is preadditive, then it is called *additive* if it additionally satisfies the following:
   - For any two objects A, B in $\mathcal{A}$, there exists a product object $A \times B$ (Definition 2.0.3), often written $A \oplus B$, called the *direct sum of A and B*. In fact, $A \oplus B$ is not only a product but also a coproduct (Definition C.0.9) of A and B (Lemma 2.0.7).

   Given a finite collection $\{A_i\}_i$ of objects A_i in an additive category $\mathcal{A}$, we may more generally speak of the *direct sum* $\bigoplus_i A_i$; it has canonical injections from and projections to each A_i.

Lemma 2.0.7. Let $\mathcal{A}$ be a preadditive category (Definition 2.0.6). Finite products (Definition 2.0.3) in $\mathcal{A}$ coincide with finite coproducts (Definition 2.0.3). More precisely, if $\{A_i\}_{i=1}^n$ is a finite collection of objects of $\mathcal{A}$, then

1. if $\prod_{i=1}^n A_i$ exists, then so does $\coprod_{i=1}^n A_i$ and these are naturally isomorphic.
2. if $\coprod_{i=1}^n A_i$, then so does $\prod_{i=1}^n A_i$ and these are naturally isomorphic.

Proof. (♠ TODO:) □

Definition 2.0.8 (Additive functor)**.**   1. Let $\mathcal{A}$ and $\mathcal{B}$ be pre-additive categories. A functor

$$F : \mathcal{A} \to \mathcal{B}$$

is an *additive functor* if for every pair of objects $A, A' \in \mathcal{A}$, the induced map

$$F_{A,A'} : \mathrm{Hom}_{\mathcal{A}}(A, A') \to \mathrm{Hom}_{\mathcal{B}}(F(A), F(A'))$$

is a group homomorphism of abelian groups, or equvialently if it is enriched over the category Ab of abelian groups (Definition A.0.2).
2. Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories. A functor

$$F : \mathcal{A} \to \mathcal{B}$$

is an *additive functor* if it an additive functor of pre-additive categories and satisfies the following:
   - F sends the zero object $0_{\mathcal{A}}$ of $\mathcal{A}$ to the zero object $0_{\mathcal{B}}$ of $\mathcal{B}$, i.e.,

   $$F(0_{\mathcal{A}}) = 0_{\mathcal{B}}.$$

   - F preserves finite direct sums: For any finite family of objects $\{A_i\}_{i=1}^n$ in $\mathcal{A}$,

   $$F\left(\bigoplus_{i=1}^n A_i\right) \cong \bigoplus_{i=1}^n F(A_i)$$

   via the canonical isomorphism induced by F applied to the canonical injections and projections.

   In other words, F is a functor that is compatible with the additive structures on $\mathcal{A}$ and $\mathcal{B}$.

Definition 2.0.9. A category $\mathcal{C}$ is *pre-abelian* if it is additive (Definition 2.0.6) and every morphism hsa a kernel and a cokernel (Definition 1.2.2).

Definition 2.0.10 (Abelian category)**.** Let $\mathcal{A}$ be a category. The category $\mathcal{A}$ is an *abelian category* if:

- $\mathcal{A}$ is an additive category (Definition 2.0.6).
- Every morphism $f : A \to B$ has a kernel $\ker(f)$ and a cokernel $\mathrm{coker}(f)$ (Definition 1.2.2).
- For every morphism $f : A \to B$, the canonical morphism $\mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism, where

$$\mathrm{coim}(f) = \mathrm{coker}(\ker(f) \to A), \quad \mathrm{im}(f) = \ker(B \to \mathrm{coker}(f)).$$

(♠ TODO: I think I need to re-check this defintion) (♠ TODO: coimage)

In particular, every abelian category is Definition 2.0.9pre-abelian .

It is also worth considering Grothendieck's additional axioms for abelian categories (Definition 2.1.8).

Proposition 2.0.11. Let $\mathcal{A}$ be a preadditive (Definition 2.0.6) (resp. additive (Definition 2.0.6), abelian (Definition 2.0.10)) category and let J be a small (Definition 1.0.5) category. The diagram category (Definition 1.3.1) $\mathcal{A}^J$ is preadditive (resp. additive, abelian).

Definition 2.0.12 (Equalizer in a category)**.** Let $\mathcal{C}$ be a (large) category (Definition 1.0.1) and let $f, g : X \to Y$ be morphisms in $\mathcal{C}$.

1. An *equalizer of f and g* is an object E together with a morphism

$$e : E \to X$$

   such that

$$f \circ e = g \circ e$$

   and for any object Z with morphism $z : Z \to X$ satisfying

$$f \circ z = g \circ z,$$

   there exists a unique morphism $u : Z \to E$ making the diagram commute:

$$e \circ u = z.$$

   If such an equalizer of f and g exists, then we say that the following *equalizer diagram is exact*:

$$E \xrightarrow{\ e\ } X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

2. A *coequalizer of f and g* is an object Q together with a morphism

$$q : Y \to Q$$

   such that

$$q \circ f = q \circ g$$

   and for any object Z with morphism $w : Y \to Z$ satisfying

$$w \circ f = w \circ g,$$

   there exists a unique morphism $v : Q \to Z$ making the diagram commute:

$$v \circ q = w.$$

If such a coequalizer of f and g exists, then we say that the following coequalizer diagram is exact:

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{\ q\ } Q$$

Lemma 2.0.13. Let $f, g : X \to Y$ be morphisms in an additive category (Definition 2.0.6).

1. The equalizer (Definition 2.0.12) of f and g is given by $\ker(f - g)$.
2. The coequalizer (Definition 2.0.12) of f and g is given by $\operatorname{coker}(f - g)$.

Lemma 2.0.14. Abelian categories (Definition 2.0.10) are finitely complete and finitely cocomplete (Definition 2.0.5).

Proof. Abelian categories have finite products and finite coproducts (in the form of direct sums), empty products and coproducts (in the form of the zero object), and equalizers and coequalizers (Definition 2.0.12) (in the form of kernels and cokernels of morphisms), so Theorem 1.3.6 applies. □

Proposition 2.0.15. The following are examples of abelian categories (Definition 2.0.10):

1. The category of R-S bimodules where R,S are (not necessarily commutative) rings (Definition C.0.2) (Theorem 2.1.11).
2. The category **Ab** of abelian groups and group homomorphisms is abelian.
3. The category Vect_k of vector spaces over a field k and k-linear maps is abelian.
4. More generally, if R is a noetherian ring (Definition C.0.18), then the category of finitely generated (Definition C.0.19) R-modules is abelian.
5. For a ringed space (Definition 1.1.1) $(X, \mathcal{O}_X)$, the category of $\mathcal{O}_X$-modules (Definition 1.1.2) is abelian. (♠ TODO: a quasi-coherent sheaf on a locally ringed space)
6. If X is a scheme (Definition B.0.2) (or more generally a locally ringed space (Definition B.0.1)), the category of quasi-coherent sheaves on X (Definition B.0.3) is abelian.
7. For any essentially small category (Definition 1.0.11) $\mathcal{C}$ and any abelian category $\mathcal{A}$, the functor category $[\mathcal{C}, \mathcal{A}]$ (Definition 1.3.1) and the category $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ of presheaves (Definition 6.1.6) are abelian. (♠ TODO: apparently, the essentially smallness condition is removable, provided that the sheafification functor exists. However, the essentially small assumption is needed to show that the category of sheaves of O-modules is a Grothendieck abelian caetgory. Verify all this. Moreover, when working with a big site of a scheme, one typically fixes a unvierse or work relative to a cardinal cutoff to treat it as essentially small)
8. For any site (Definition 6.1.4) $(\mathcal{C}, J)$ on an essentially small category (Definition 1.0.11) $\mathcal{C}$ and any abelian category $\mathcal{A}$, the category $\mathrm{Shv}(\mathcal{C}, \mathcal{A})$ of sheaves (Definition 6.1.7) is abelian.
9. For any site (Definition 6.1.4) $(\mathcal{C}, J)$ on an essentially small category (Definition 1.0.11) $\mathcal{C}$ and a sheaf of rings (Definition 6.1.7) $\mathcal{O}$ on $\mathcal{C}$, the category $\mathbf{Mod}(\mathcal{O})$ of $\mathcal{O}$-modules (Definition 1.1.2) is an abelian category.

Proposition 2.0.16 (Monomorphisms and kernels, epimorphisms and cokernels in abelian categories)**.** Let $\mathcal{A}$ be an abelian category (Definition 2.0.10).

(1) A morphism $f : X \to Y$ in $\mathcal{A}$ is a monomorphism (Definition 1.2.1) if and only if f is the kernel (Definition 1.2.2) of some morphism.

(2) Dually, $f : X \to Y$ is an epimorphism (Definition 1.2.1) if and only if f is the cokernel (Definition 1.2.2) of some morphism.

Definition 2.0.17. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor (Definition 2.0.8) between abelian categories (Definition 2.0.10).

1. F is called *left exact* if it preserves all finite limits (Definition 1.3.2), or equivalently it preserves kernels (Definition 1.2.2) and any finite limit diagrams. Equivalently, for every left exact sequence in $\mathcal{A}$

$$0 \to A' \xrightarrow{f} A \xrightarrow{g} A''$$

the sequence

$$0 \to F(A') \xrightarrow{F(f)} F(A) \xrightarrow{F(g)} F(A'')$$

is exact at $F(A')$ and $F(A)$ (i.e., F preserves monomorphisms (Definition 1.2.1) and exactness at the first two terms).

2. Dually, F is called *right exact* if it preserves all finite colimits (Definition 1.3.2), or equivalently it preserves cokernels (Definition 1.2.2) and any finite colimit diagrams. Equivalently, for every right exact sequence in $\mathcal{A}$

$$A' \xrightarrow{f} A \xrightarrow{g} A'' \to 0,$$

the sequence

$$F(A') \xrightarrow{F(f)} F(A) \xrightarrow{F(g)} F(A'') \to 0$$

is exact at $F(A)$ and $F(A'')$ (i.e., F preserves epimorphisms (Definition 1.2.1) and exactness at the last two terms).

3. F is called *exact* if it is both left and right exact.

Proposition 2.0.18. Let $\mathcal{A}, \mathcal{B}$ be abelian categories (Definition 2.0.10) and let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ be adjoint (Definition C.0.16) additive functors (Definition 2.0.8), say with $F \dashv G$ (i.e. F is left adjoint to G). The left adjoint functor F is right exact (Definition 2.0.17) and and the right adjoint functor G is left exact (Definition 2.0.17)

Definition 2.0.19. Let $\mathcal{A}$ be an additive category (Definition 2.0.6). A sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

of moprhisms in $\mathcal{A}$ is called a *short exact sequence* if the morphisms satisfy:

- $f : A \to B$ is a monomorphism (Definition 1.2.1) and is the kernel of g (Definition 1.2.2),
- $g : B \to C$ is an epimorphism (Definition 1.2.1) and is the cokernel of f (Definition 1.2.2),
- the sequence is exact at (Definition 3.2.3) B, meaning $\mathrm{Im}(f) = \mathrm{Ker}(g)$ (Definition 1.2.5).

This means the sequence starts and ends with the zero object and is exact everywhere.

Proposition 2.0.20. Let $\mathcal{A}$ be an abelian category (Definition 2.0.10) and $A \in \mathrm{Ob}(\mathcal{A})$. Then the covariant Hom functor

$$\mathrm{Hom}_{\mathcal{A}}(A, -) : \mathcal{A} \to \mathbf{Ab}$$

is left exact (Definition 2.0.17), and the contravariant Hom functor

$$\mathrm{Hom}_{\mathcal{A}}(-, A) : \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$$

(Definition 1.0.2) is also left exact. Explicitly, given a left exact sequence

$$0 \to X' \to X \to X'',$$

we have a left exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(A, X') \to \mathrm{Hom}_{\mathcal{A}}(A, X) \to \mathrm{Hom}_{\mathcal{A}}(A, X''),$$

and given a right exact sequence

$$X' \to X \to X'' \to 0$$

we have a left exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(X'', A) \to \mathrm{Hom}_{\mathcal{A}}(X, A) \to \mathrm{Hom}_{\mathcal{A}}(X', A).$$

Theorem 2.0.21. Let $\mathcal{A}$ be an abelian category (Definition 2.0.10) and $\mathcal{J}$ be a small category (Definition 1.0.5). The functor category (Definition 1.3.1) $\mathcal{A}^{\mathcal{J}}$ inherits the structure of an abelian category from $\mathcal{A}$. Specifically:

- The zero object (Definition 2.0.1) in $\mathcal{A}^{\mathcal{J}}$ is the constant functor at the zero object of $\mathcal{A}$.
- Kernels (Definition 1.2.2), cokernels (Definition 1.2.2), products (Definition 2.0.3), and coproducts (Definition 2.0.3) are computed pointwise in $\mathcal{A}$. For example, if $\eta : F \to G$ is a natural transformation (Definition 1.0.4), the kernel is the functor $K : \mathcal{J} \to \mathcal{A}$ defined by $K(j) = \ker(\eta_j)$ for each $j \in \mathcal{J}$.
- A sequence of functors $0 \to F \to G \to H \to 0$ is exact (Definition 2.0.19) in $\mathcal{A}^{\mathcal{J}}$ if and only if for every object $j \in \mathcal{J}$, the sequence $0 \to F(j) \to G(j) \to H(j) \to 0$ is exact in $\mathcal{A}$.

Moreover, if $\mathcal{A}$ admits arbitrary limits (resp. colimits), then $\mathcal{A}^{\mathcal{J}}$ also admits arbitrary limits (resp. colimits), computed pointwise.

Proposition 2.0.22. Let $\mathcal{A}$ be an abelian category (Definition 2.0.10) and $\mathcal{J}$ be a small index category.

(♠ TODO:)

1. The limit functor (Theorem 1.3.3) $\lim : \mathcal{A}^{\mathcal{J}} \to \mathcal{A}$ is left exact (Definition 2.0.17). That is, if

$$0 \to F \to G \to H \to 0$$

is a short exact sequence (Definition 2.0.19) of functors in $\mathcal{A}^{\mathcal{J}}$ (equivalently (Theorem 2.0.21) it is exact object-wise), then the induced sequence

$$0 \to \lim F \to \lim G \to \lim H$$

is exact (Definition 2.0.17) in $\mathcal{A}$.

2. Limits commute with kernels (and indeed with all limits). Since the kernel is a finite limit, strictly speaking, $\lim(\ker \Phi) \cong \ker(\lim \Phi)$ for any natural transformation Φ.
3. The colimit functor (Theorem 1.3.3) $\mathrm{colim} : \mathcal{A}^{\mathcal{J}} \to \mathcal{A}$ is right exact. That is, if

$$0 \to F \to G \to H \to 0$$

   is a short exact sequence of functors, then the induced sequence

$$\mathrm{colim}\, F \to \mathrm{colim}\, G \to \mathrm{colim}\, H \to 0$$

   is exact in $\mathcal{A}$.
4. Colimits commute with cokernels (and indeed with all colimits). Strictly speaking, $\mathrm{colim}(\mathrm{coker}\, \Phi) \cong \mathrm{coker}(\mathrm{colim}\, \Phi)$.

Proof. (♠ TODO:) □

Proposition 2.0.23. Let $F : \mathcal{A} \to \mathcal{B}$ be a full and faithful (Definition 1.0.9) additive functor (Definition 2.0.8) between abelian categories (Definition 2.0.10). Then F reflects exactness (Definition C.0.15), i.e., given a sequence

$$0 \to A' \xrightarrow{f} A \xrightarrow{g} A'' \to 0$$

in $\mathcal{A}$, if the sequence

$$0 \to F(A') \xrightarrow{F(f)} F(A) \xrightarrow{F(g)} F(A'') \to 0$$

is exact in $\mathcal{B}$, then the original sequence is exact in $\mathcal{A}$.

Theorem 2.0.24 (Freyd-Mitchell Embedding Theorem)**.** Let $\mathcal{A}$ be a small (Definition 1.0.5) abelian category (Definition 2.0.10). There exists a ring (Definition C.0.2) R (which may not be commutative) and a functor $F : \mathcal{A} \to \mathrm{Mod}_R$ (Theorem 6.2.4) such that: (♠ TODO: Show thta exact functors preserve finite limits and colimits)

1. F is exact (Definition 2.0.17), meaning it preserves all finite limits and colimits (in particular, kernels, cokernels, and exact sequences).
2. F is fully faithful (Definition 1.0.9).

Consequently, any diagram-chasing argument valid for modules over a ring is also valid in any small abelian category, and by extension (using the fact that any exact diagram involves only a set of objects), in any abelian category.

2.1. **Grothendieck's additional axioms for abelian categories.**

Definition 2.1.1. Let C be a category enriched in a monidal category (Definition A.0.2) $\mathcal{V}$. Given an object X of C, the *functor of points* h_X is the functor (Definition 1.0.3)/presheaf (Definition 6.1.6) $C^{\mathrm{op}} \to \mathcal{V}$ given by $T \mapsto \mathrm{Hom}_C(T, X)$. A functor $C^{\mathrm{op}} \to \mathcal{V}$ (or equivalently, a presheaf on C valued in $\mathcal{V}$) is said to be *representable* if it is naturally isomorphic (Definition 1.0.4) to some functor h_X of points for an object X of C.

Dually, a functor $C \to \mathcal{V}$ is called *co-representable* if it is naturally isomorphic to a functor $T \mapsto \mathrm{Hom}_C(X, T)$ for an object X in C.

For instance, we may speak of these notions when $\mathcal{V}$ is the monoidal category **Sets**, i.e. C is a locally small category (Definition 1.0.5).

Theorem 2.1.2 (Yoneda Lemma). Let $\mathcal{C}$ be a locally small category (Definition 1.0.5). Let A be an object of $\mathcal{C}$, and let $F : \mathcal{C} \to \mathbf{Set}$ be a covariant functor (Definition 1.0.3) to the category of sets. Let $h^A : \mathcal{C} \to \mathbf{Set}$ denote the covariant representable functor defined by $h^A(X) = \mathrm{Hom}_{\mathcal{C}}(A, X)$.

There exists a bijection

$$y_{A,F} : \mathrm{Nat}(h^A, F) \overset{\cong}{\Rightarrow} F(A)$$

between the set of natural transformations (Definition 1.0.4) from h^A to F and the set $F(A)$. This bijection is given by the mapping

$$\alpha \mapsto \alpha_A(\mathrm{id}_A),$$

where $\alpha : h^A \to F$ is a natural transformation, $\alpha_A : h^A(A) \to F(A)$ is its component at A, and $\mathrm{id}_A \in h^A(A) = \mathrm{Hom}_{\mathcal{C}}(A, A)$ is the identity morphism.

Furthermore, this isomorphism is natural in both A and F. Explicitly:

1. For any morphism $f : A \to B$ in $\mathcal{C}$, the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{Nat}(h^B, F) & \xrightarrow{\;y_{B,F}\;} & F(B) \\
{\scriptstyle -\circ h^f}\downarrow & & \downarrow{\scriptstyle F(f)} \\
\mathrm{Nat}(h^A, F) & \xrightarrow{\;y_{A,F}\;} & F(A)
\end{array}
$$

where $h^f : h^B \to h^A$ is the natural transformation induced by pre-composition with f.

2. For any natural transformation $\eta : F \to G$, the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{Nat}(h^A, F) & \xrightarrow{\;y_{A,F}\;} & F(A) \\
{\scriptstyle \eta\circ -}\downarrow & & \downarrow{\scriptstyle \eta_A} \\
\mathrm{Nat}(h^A, G) & \xrightarrow{\;y_{A,G}\;} & G(A)
\end{array}
$$

Corollary 2.1.3 (Yoneda Embedding). Let $\mathcal{C}$ be a locally small category (Definition 1.0.5). The functor

$$h^\bullet : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathcal{C}}$$

(Definition 1.0.2) (Definition 1.3.1) defined on objects by $A \mapsto h^A = \mathrm{Hom}_{\mathcal{C}}(A, -)$ and on morphisms by $f \mapsto h^f = (- \circ f)$ is fully faithful (Definition 1.0.9). That is, for any objects A, B in $\mathcal{C}$, the map

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Nat}(h^B, h^A)$$

given by sending a morphism $f : A \to B$ to the natural transformation (Definition 1.0.4) $h^f : h^B \to h^A$ (pre-composition by f) is a bijection.

Consequently, $\mathcal{C}^{\mathrm{op}}$ embeds as a full subcategory (Definition 1.0.8) of the functor category $\mathbf{Set}^{\mathcal{C}}$.

Theorem 2.1.4 (Contravariant Yoneda Lemma). Let $\mathcal{C}$ be a locally small category (Definition 1.0.5). Let A be an object of $\mathcal{C}$, and let $G : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ be a contravariant functor

(Definition 1.0.3) (i.e. a presheaf (Definition 6.1.6)). Let $h_A : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ denote the contravariant representable functor defined by $h_A(X) = \mathrm{Hom}_{\mathcal{C}}(X, A)$.

There exists a bijection natural in A and G:

$$\mathrm{Nat}(h_A, G) \cong G(A)$$

given by $\alpha \mapsto \alpha_A(\mathrm{id}_A)$.

Corollary 2.1.5 (Contravariant Yoneda Embedding)**.** Let $\mathcal{C}$ be a locally small category (Definition 1.0.5). The functor

$$h_\bullet : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$$

defined on objects by $A \mapsto h_A = \mathrm{Hom}_{\mathcal{C}}(-, A)$ and on morphisms by $f \mapsto h_f = (f \circ -)$ is fully faithful (Definition 1.0.9). That is, for any objects A, B in $\mathcal{C}$, the map

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Nat}(h_A, h_B)$$

given by sending a morphism $f : A \to B$ to the natural transformation (Definition 1.0.4) $h_f : h_A \to h_B$ (post-composition by f) is a bijection.

Consequently, $\mathcal{C}$ embeds as a full subcategory (Definition 1.0.8) of the category of presheaves $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$.

Definition 2.1.6 (Generator of a category)**.** Let $\mathcal{C}$ be a category (Definition 1.0.1).

1. An object $G \in \mathcal{C}$ is called a *generator* (or *separator*) if for every pair of distinct morphisms $f, g : X \to Y$ in $\mathcal{C}$, there exists a morphism $h : G \to X$ such that

$$f \circ h \neq g \circ h.$$

   In case that $\mathcal{C}$ is locally small (Definition 1.0.5), this is equivalent to the condition that the representable functor (Definition 2.1.1)

$$\mathrm{Hom}_{\mathcal{C}}(G, -) : \mathcal{C} \to \mathbf{Set}$$

   is faithful (Definition 1.0.9), which in turn is equivalent to the condition that for every object $X \in \mathcal{C}$, there exists an epimorphism

$$\bigoplus_{i \in I} G \twoheadrightarrow X$$

   for some indexing set I, where $\bigoplus$ denotes the coproduct (Definition 2.0.3) in $\mathcal{C}$.
2. A family $\{G_i\}_{i \in I}$ is called a *generating family* if for every pair of distinct morphisms $f, g : X \to Y$ in $\mathcal{C}$, there exists some index $i \in I$ and a morphism $h : G_i \to X$ such that

$$f \circ h \neq g \circ h.$$

   In case $\mathcal{C}$ is locally small, this is equivalent to the condition that the collection of representable functors

$$\{\mathrm{Hom}_{\mathcal{C}}(G_i, -) : \mathcal{C} \to \mathbf{Set}\}_{i \in I}$$

is jointly faithful, which in turn is equivalent to the condition that for every object $X \in \mathcal{C}$, there exists a family of objects $\{G_i\}_{i \in J}$ from the generating set indexed by some set J, and an epimorphism

$$\bigoplus_{i \in J} G_i \twoheadrightarrow X.$$

Example 2.1.7.   1. In the category of abelian groups **Ab**, the group $\mathbb{Z}$ is a generator, as homomorphisms out of $\mathbb{Z}$ detect morphisms, and every abelian group is a quotient of a direct sum of copies of $\mathbb{Z}$.
2. In the category of sets **Set**, the one-point set $\{*\}$ is a generator.
3. In the category of left modules over a ring R, denoted R-**Mod**, the free module R is a generator.
4. In the category of sheaves of $\mathcal{O}$-modules on a ringed space or site, stalkwise generators often induce global generators, e.g., the sheaf $\mathcal{O}$ itself is a generator in $\mathrm{Mod}(\mathcal{O})$.

Definition 2.1.8 (Grothendieck's axioms for abelian categories (Ab1–Ab5))**.** Let $\mathcal{A}$ be an abelian category (Definition 2.0.10).

Grothendieck introduced the following hierarchy of additional axioms to express stronger completeness and exactness properties in $\mathcal{A}$ — we note that Ab1, Ab2, and Ab2* are already satisfied for any abelian category:

- *Ab1*: Every morphism in $\mathcal{A}$ has a kernel and a cokernel (Definition 1.2.2).
- *Ab2*: Every monic (Definition 1.2.1) in $\mathcal{A}$ is the kernel of its cokernel.
- *Ab2**: Every epi in $\mathcal{A}$ is the cokernel of its kernel.
- *AB3*: The category $\mathcal{A}$ is cocomplete (Definition 2.0.5).
  - Since $\mathcal{A}$ is abelian (and hence admits (Lemma 2.0.13) equalizers (Definition 2.0.12) as kernels (Definition C.0.7)), this is equivalent to requiring that $\mathcal{A}$ has all small coproducts (Definition 2.0.3) (direct sums).
- *AB4*: The category $\mathcal{A}$ satisfies AB3, and coproducts are *exact*.
  - That is, the coproduct of a family of short exact sequences is a short exact sequence. Explicitly, for any family of short exact sequences $0 \to A_i \to B_i \to C_i \to 0$ indexed by a set I, the sequence

    $$0 \to \bigoplus_{i \in I} A_i \to \bigoplus_{i \in I} B_i \to \bigoplus_{i \in I} C_i \to 0$$

    is exact in $\mathcal{A}$.
- *AB5*: The category $\mathcal{A}$ satisfies AB3, and filtered colimits (Definition 1.3.12) are *exact*.
  - Equivalently, for any filtered (Definition 1.3.8) index category J and any directed system (Definition 1.3.11) of short exact sequences $0 \to A_j \to B_j \to C_j \to 0$, the colimit sequence

    $$0 \to \varinjlim A_j \to \varinjlim B_j \to \varinjlim C_j \to 0$$

    is exact.
  - Note: AB5 implies AB4. An abelian category satisfying AB5 and having a generator (Definition 2.1.6) is called a *Grothendieck category*.

- *AB6*: The category $\mathcal{A}$ satisfies AB3, and for any object X and any family of filtered subobjects $\{F_i\}_{i\in I}$ of X (where each F_i is a filter of subobjects), the intersection commutes with the limit:

$$\bigcap_{i\in I}(\varinjlim_{j\in F_i} U_{i,j}) = \varinjlim_{(j_i)\in\prod F_i} (\bigcap_{i\in I} U_{i,j_i}).$$

  (This axiom is less commonly cited but appears in Grothendieck's Tohoku paper).
- *AB3**: The category $\mathcal{A}$ is complete (Definition 2.0.5) (i.e., has all small products).
- *AB4**: The category $\mathcal{A}$ satisfies AB3* and products are exact.
  - Note: This is rarely satisfied for module categories (e.g., it fails for Abelian groups), but it is satisfied for the category of sheaves on a space.
- *AB5**: The category $\mathcal{A}$ satisfies AB3* and filtered limits (inverse limits) are exact.

Notes:

- AB5 implies AB4, and AB4 implies AB3.
- AB5* implies AB4*, and AB4* implies AB3*.

Lemma 2.1.9. Let $\mathcal{A}$ be an abelian category (Definition 2.0.10) satisfying Ab5 (Definition 2.1.8) and which admits a small (Definition 1.0.5) family $\{U_i\}_{i\in I}$ of generators (Definition 2.1.6). The coproduct $\bigoplus_{i\in I} U_i$ is in fact a generator and hence $\mathcal{A}$ is a Grothendieck category (Definition 2.1.8).

Theorem 2.1.10 (Examples of Grothendieck Categories). Examples of Grothendieck categories (Definition 2.1.8) include:

- The category of abelian groups,
- The category of R-S bimodules where R,S are (not necessarily commutative) rings (Definition C.0.2) (Theorem 2.1.11)
- The category of sheaves (Definition 6.1.7) of abelian groups on a site (Definition 6.1.4) with a small topologically generating family (Definition 6.1.4),
- The category of sheaves of O_X-modules (Definition 1.1.2) for any ringed space (Definition 1.1.1) (X, O_X).
- The category of quasi-coherent sheaves on a scheme or algebraic stack. (♠ TODO: quasi-coherent sheaves) (♠ TODO: I need to figure out if for sheaves of abelian groups/sheaves of $\mathcal{O}$-modules whether essentially smallness of the site is really necessary)
- The category of sheaves (Definition 6.1.7) of abelian groups on an essentially small (Definition 1.0.11) site (Definition 6.1.4) (C, J).
- ([GV72, Exposé II, Proposition 6.7]) The category of sheaves of $\mathcal{O}$-modules on an essentially small site (or an essentially $\mathscr{U}$-small site if a universe $\mathscr{U}$ is available) (C, J).

Theorem 2.1.11. Let R, S be (not necessarily commutative) rings (Definition C.0.2). The category of (Definition C.0.8) R-S-bimodules (Definition C.0.4) is a an Grothendieck (Definition 2.1.8) category and an $AB4*$ (Definition 2.1.8) category.

Proof. We handwave details.

Given R-S-bimodules M and N, the set $\operatorname{Hom}_{R\operatorname{Mod}_S}(M, N)$ is an abelian group. Moreover, there is a 0-object, namely the zero module in $_R\operatorname{Mod}_S$. Therefore, $_R\operatorname{Mod}_S$ is preadditive (Definition 2.0.6). The direct sum of finitely many R-S-bimodules is their coproduct (Definition C.0.9). Therefore, $_R\operatorname{Mod}_S$ is additive (Definition 2.0.6).

Given a morphism (Definition C.0.6) $f : M \to N$ be R-S-bimodules, $\ker f$ and $\operatorname{coker} f$ (Definition C.0.7) are the categorical kernel and cokernel (Definition 1.2.2) of f in $_R\operatorname{Mod}_S$ (Definition C.0.8). Moreover, the monomorphisms (Definition 1.2.1) in $_R\operatorname{Mod}_S$ are the injective module homomorphisms $f : M \to N$; such an f is the kernel of its cokernel. In other words, $_R\operatorname{Mod}_S$ satisfies $AB1$ and $AB2$ and hence is an abelian category (Definition 2.0.10).

Moreover, small coproducts (Definition 2.0.3) exist in $_R\operatorname{Mod}_S$ (Definition C.0.9), and it is easy to see that they are in fact exact, so $_R\operatorname{Mod}_S$ satisfies $AB3$ and $AB4$. To show that filtered colimits in $_R\operatorname{Mod}_S$ are exact, we first note that small (Definition 1.3.5) colimits (Definition 1.3.12) are (Theorem 1.3.23) left adjoint (Definition C.0.16) and hence (Proposition 2.0.18) is right exact (Definition 2.0.17); for any small index category J and any system of short exact sequences $0 \to A_j \to B_j \to C_j \to 0$, the sequence $0 \to A \to B \to C \to 0$ of diagrams (Definition 1.3.1) is exact by Proposition 3.2.4, so applying the colimit functor (Theorem 1.3.3) yields a right exact sequence

$$\operatorname{colim}_{j \in J} A_j \to \operatorname{colim}_{j \in J} B_j \to \operatorname{colim}_{j \in J} C_j \to 0.$$

If J is additionally filtered (Definition 1.3.8) so that the system is a directed (Definition 1.3.11) one, then we show that the sequence

$$\text{(A)} \qquad \varinjlim_{j \in J} A_j \to \varinjlim_{j \in J} B_j \to \varinjlim_{j \in J} C_j \to 0.$$

is left exact as well. Take some element of the kernel of $\varinjlim_{j \in J} A_j \to \varinjlim_{j \in J} B_j$; such an element is represented by some element $a_j \in A_j$ for some $j \in J$. Since its image is zero in $\varinjlim j \in J B_j$, it must be zero as an element of B_k for some $k \in J$. Since J is filtered, there exists some $k' \in J$ so that there are arrows $j \to k'$ and $k \to k'$ and so that the image of a_j in $B_{k'}$ is 0. The image of a_j in $A_{k'}$ is then 0 due to the assumption that

$$0 \to A_{k'} \to B_{k'} \to C_{k'} \to 0$$

is exact. Therefore, a_j is 0 in $\varinjlim_{j \in J} A_j$, so (A) is left exact as claimed and $_R\operatorname{Mod}_S$ is $AB5$.

Similarly as how we argued that $_R\operatorname{Mod}_S$ is $AB3$ and $AB4$, one can argue that $_R\operatorname{Mod}_S$ is $AB3^*$ and $AB4^*$. Moreover, one can show that the R-S-bimodule $R \otimes_{\mathbb{Z}} S^{\mathrm{op}}$ (Definition C.0.12) (Definition C.0.10) is a generator for $_R\operatorname{Mod}_S$. $\qquad\square$

Definition 2.1.12 (Topological groups). (♠ TODO: Product topology) A *topological group* is a group (Definition C.0.1) $(G, \cdot)$ together with a topology $\mathcal{T}$ on G such that the maps

$$\mu : G \times G \to G, \qquad\qquad (g, h) \mapsto g \cdot h,$$
$$\iota : G \to G, \qquad\qquad g \mapsto g^{-1},$$

are continuous with respect to the product topology on $G \times G$ and the topology $\mathcal{T}$ on G.

Definition 2.1.13 (Compact topological space). A topological space $(X, \mathcal{T})$ is *compact* if every open cover of X admits a finite subcover; that is, for every collection $\{U_i\}_{i \in I}$ of open

sets in $\mathcal{T}$ such that $X = \bigcup_{i \in I} U_i$, there exists a finite subcollection $\{U_{i_j}\}_{j=1}^n$ such that $X = \bigcup_{j=1}^n U_{i_j}$.

Some mathematicians, e.g. algebraic geometers, would refer to this property as *quasi-com-pactness*.

Definition 2.1.14 (Separation axioms). Let $(X, \mathcal{T})$ be a topological space.

- $(X, \mathcal{T})$ is T_0 (*Kolmogorov*) if for every pair of distinct points $x, y \in X$, there exists an open set $U \in \mathcal{T}$ such that, without loss of generality, $x \in U$ and $y \notin U$.
- $(X, \mathcal{T})$ is T_1 (*Fréchet*) if for every pair of distinct points $x, y \in X$, there exist open sets $U, V \in \mathcal{T}$ such that $x \in U$, $y \notin U$, and $y \in V$, $x \notin V$.
- $(X, \mathcal{T})$ is T_2 or *Hausdorff* if for every pair of distinct points $x, y \in X$, there exist disjoint open sets $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$.
- $(X, \mathcal{T})$ is *regular* if it is T_1 and for each point $x \in X$ and closed set $F \subseteq X$ with $x \notin F$, there exist disjoint open sets $U, V \in \mathcal{T}$ such that $x \in U$ and $F \subseteq V$.
- $(X, \mathcal{T})$ is T_3 (regular Hausdorff) if it is T_1 and regular.
- $(X, \mathcal{T})$ is *completely regular* if for each closed set $F \subseteq X$ and $x \notin F$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f|_F = 1$.
- $(X, \mathcal{T})$ is $T_{3\frac{1}{2}}$ (completely regular Hausdorff) if it is T_1 and completely regular.
- $(X, \mathcal{T})$ is *normal* if it is T_1 and for each pair of disjoint closed sets $A, B \subseteq X$, there exist disjoint open sets $U, V \in \mathcal{T}$ such that $A \subseteq U$ and $B \subseteq V$.
- $(X, \mathcal{T})$ is T_4 (normal Hausdorff) if it is T_1 and normal.
- $(X, \mathcal{T})$ is T_5 (completely normal Hausdorff) if it is T1 and completely normal.
- $(X, \mathcal{T})$ is *perfectly normal* if every closed set is a $G\delta$ (countable intersection of open sets) and the space is normal.
- $(X, \mathcal{T})$ is T_6 (perfectly normal Hausdorff) if it is T_1 and perfectly normal.

Definition 2.1.15. A *locally compact group* is an topological group that is locally compact Hausdorff. It is called a *locally compact abelian group* if it is abelian as well.

Definition 2.1.16 (Field). A *field* is commutative division ring. In other words, a field is a commutative ring for which all nonzero elements have a multiplicative inverse.

Definition 2.1.17 (Vector space over a field). Let $(k, +, \cdot)$ be a field (Definition 2.1.16). A *vector space over k* or a *k-vector space* is a triple $(V, +, \cdot)^1$ where

1. $(V, +)$ is an abelian group, and
2. $\cdot$ is a map $k \times V \to V$, called *scalar multiplication*

such that the following axioms hold for all $a, b \in k$ and all $u, v \in V$:

1. (Compatibility with field multiplication)

$$(ab) \cdot v = a \cdot (b \cdot v).$$

---

[1]Note that $+$ and $\cdot$ are abuse of notation here as these are already used for the addition and multiplication of $\cdot$

2. (Identity scalar)
$$1 \cdot v = v.$$

3. (Distributivity over vector addition)
$$a \cdot (u + v) = a \cdot u + a \cdot v.$$

4. (Distributivity over scalar addition)
$$(a + b) \cdot v = a \cdot v + b \cdot v.$$

Theorem 2.1.18. The category of compact (Definition 2.1.13) Hausdorff (Definition 2.1.14) abelian topological groups (Definition 2.1.12) is AB5* (Definition 2.1.8).

Theorem 2.1.19. Let k be a field (Definition 2.1.16). The category of k-vector spaces (Definition 2.1.17) is AB5* (Definition 2.1.8) and AB5.

Theorem 2.1.20. Let R be a (not necessarily commutative) ring (Definition C.0.2). The category of (left or right) R-modules (Definition 2.1.17) is AB4* (Definition 2.1.8).

Remark 2.1.21. In general, the category of R-modules is not AB5*. In general, the category of $\mathcal{O}$-modules for a ringed site $(\mathcal{C}, \mathcal{O})$ is not AB5*.

Example 2.1.22. The following example shows that for general rings R, the category of R-modules is not generally AB5* (Definition 2.1.8): let $R = \mathbb{Z}$, fix a prime number p, and let
$$0 \to A_n \to B_n \to C_n \to 0$$
be the following system of short exact sequences indexed by $n \geq 0$:

- $A_n = \mathbb{Z}$ with transition maps $f : A_{n+1} \to A_n$ given by multiplication-by-p,
- $B_n = \mathbb{Z}$ with transition maps given by the identity, and
- $C_n = \mathbb{Z}/p^n\mathbb{Z}$ with transition maps given by the canonical projections.

Taking limits,

- $\varprojlim A_n = 0$,
- $\varprojlim B_n = \mathbb{Z}$, and
- $\varprojlim C_n = \mathbb{Z}_p$,

so
$$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to 0$$
fails to be exact.

Proposition 2.1.23 (Stability and constructions of Grothendieck categories).

- Any category that is *equivalent* to a Grothendieck category (Definition 2.1.8) is itself a Grothendieck category.
- Given Grothendieck categories $\mathcal{A}_1, \ldots, \mathcal{A}_n$, the product category (Definition 1.0.12)
$$\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$$
is a Grothendieck category.

- Given a small category $\mathcal{C}$ and a Grothendieck category $\mathcal{A}$, the functor category

$$\text{Funct}(\mathcal{C}, \mathcal{A})$$

consisting of all covariant functors from $\mathcal{C}$ to $\mathcal{A}$, is a Grothendieck category.
- Given a small preadditive category $\mathcal{C}$ and a Grothendieck category $\mathcal{A}$, the category of additive covariant functors

$$\text{Add}(\mathcal{C}, \mathcal{A})$$

is a Grothendieck category.
- If $\mathcal{A}$ is a Grothendieck category and $\mathcal{C}$ is a localizing subcategory of $\mathcal{A}$, then both $\mathcal{C}$ and the Serre quotient category $\mathcal{A}/\mathcal{C}$ are Grothendieck categories.

2.2. Miscellaneous definitions.

Definition 2.2.1. Let $\mathcal{A}$ be an abelian category (Definition 2.0.10). A full subcategory (Definition 1.0.8) $\mathcal{B} \subseteq \mathcal{A}$ is called a *strict abelian subcategory of $\mathcal{A}$* if the following hold:

1. $\mathcal{B}$ is itself an abelian category.
2. The inclusion functor $i : \mathcal{B} \hookrightarrow \mathcal{A}$ is exact (Definition 2.0.17).
3. For any morphism $f \in \text{Hom}_{\mathcal{B}}(X, Y)$ with $X, Y \in \text{Ob}(\mathcal{B})$, the kernel and cokernel (Definition 1.2.2) of f computed in $\mathcal{A}$ coincide with the kernel and cokernel of f computed in $\mathcal{B}$.

3. CHAIN COMPLEXES

3.1. The category of chain complexes of objects in an additive category.

Definition 3.1.1 (Chain complex in a preadditive category)**.** Let $\mathcal{A}$ be a preadditive category and let I be a totally ordered set (typically $\mathbb{Z}$, but $I \subseteq \mathbb{Z}$ is also allowed).

1. A *chain complex* $(K_\bullet, d_\bullet)$ in $\mathcal{A}$ indexed by I is the homological convention for sequences with decreasing degrees. It consists of:
   - Objects $\{K_i\}_{i \in I}$ in $\mathcal{A}$, called the *terms in degree i*,
   - Morphisms $d_i : K_i \to K_{i-1}$ in $\mathcal{A}$, called the *boundary maps* or *differentials in degree i*,

   such that for every $i \in I$, $d_{i-1} \circ d_i = 0$. That is,

$$K_\bullet : \cdots \xrightarrow{d_{i+1}} K_i \xrightarrow{d_i} K_{i-1} \xrightarrow{d_{i-1}} K_{i-2} \to \cdots$$

   with $d_{i-1}d_i = 0$ for all i. We typically use the notation $K_\bullet = (K_i, d_i)_{i \in I}$.
2. Dually, a *cochain complex* $(K^\bullet, d^\bullet)$ in $\mathcal{A}$ follows the *cohomological convention* with increasing degrees. It consists of objects $\{K^i\}_{i \in I}$ and *coboundary maps* $d^i : K^i \to K^{i+1}$ such that $d^{i+1} \circ d^i = 0$:

$$K^\bullet : \cdots \xrightarrow{d^{i-1}} K^i \xrightarrow{d^i} K^{i+1} \xrightarrow{d^{i+1}} K^{i+2} \to \cdots$$

   We typically use the notation $K^\bullet = (K^i, d^i)_{i \in I}$.

3. Let $K_\bullet = (K_i, d_i^K)$ and $L_\bullet = (L_i, d_i^L)$ be chain complexes (Definition 3.1.1) in $\mathcal{A}$ indexed by the same set I. A *morphism of chain complexes* (or *chain map*)

$$f_\bullet : K_\bullet \to L_\bullet$$

consists of morphisms $f_i : K_i \to L_i$ for all $i \in I$, such that for every $i \in I$, the following diagram commutes:

$$
\begin{array}{ccc}
K_i & \xrightarrow{d_i^K} & K_{i-1} \\
\downarrow f_i & & \downarrow f_{i-1} \\
L_i & \xrightarrow{d_i^L} & L_{i-1}
\end{array}
$$

i.e., $d_i^L \circ f_i = f_{i-1} \circ d_i^K$.

A *morphism of cochain complexes* $f^\bullet : K^\bullet \to L^\bullet$ is defined similarly, satisfying the commutativity condition $d_L^i \circ f^i = f^{i+1} \circ d_K^i$.

The collection of these objects and morphisms forms a category. Notation for these categories is as follows:

- $\mathrm{Ch}(\mathcal{A})$ or $\mathbf{Ch}(\mathcal{A})$ is often used as a general term.
- To be explicit about the indexing convention, one uses $\mathrm{Ch}_\bullet(\mathcal{A})$ for chain complexes and $\mathrm{Ch}^\bullet(\mathcal{A})$ (or sometimes $\mathrm{CoCh}(\mathcal{A})$) for cochain complexes.
- The set of chain maps between two complexes is denoted by $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(K_\bullet, L_\bullet)$; it is an abelian group under pointwise addition $(f + g)_i = f_i + g_i$.

Remark 3.1.2. The convention used to define chain complexes in Definition 3.1.1 is a *cohomological one* — note that indices are written as superscripts and increase when "following the arrows". Such a chain complex may also be referred to as a *cochain complex* or a *cohomological chain complex* to emphasize an adoption of a cohomological convention.

The dual convention would be a *homological one*, in which indices are written as subscripts and decrease when "following the arrow". As such, one may speak of a *(homological) chain complex* $(K_\bullet, d_\bullet)$ indexed by I as consisting of:

- Objects $\{K_i\}_{i \in I}$ in $\mathcal{A}$, called the *terms in degree i*,
- Morphisms $d_i : K_i \to K_{i-1}$ in $\mathcal{A}$, called the *differentials in degree i*,

such that for every $i \in I$, $d_{i-1} \circ d_i = 0$. That is,

$$K_\bullet : \cdots \xrightarrow{d_{i+1}} K_i \xrightarrow{d_i} K_{i-1} \xrightarrow{d_{i-1}} K_{i-2} \xrightarrow{d_{i-2}} \cdots$$

with $d_{i-1} d_i = 0$ for all i. We might typically use notation such as $K_\bullet = (K_i, d_i)_{i \in I}$ to denote a chain complex in $\mathcal{A}$.

The differences between the conventions persist — for example, cohomological objects are usually written with superscript indicees whereas homological objects are usually written with subscript indicees.

Convention 3.1.3. When discussing homological algebra in abstract terms, we may often adopt the homological convention in some discussions and the cohomological convention in others (Remark 3.1.2).

Definition 3.1.4. A *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$, where:

- Q_0 is a collection of *vertices*.
- Q_1 is a collection of *arrows*.
- $s, t : Q_1 \to Q_0$ are functions assigning to each arrow $\alpha \in Q_1$ its *source* $s(\alpha)$ and its *target* $t(\alpha)$.

Definition 3.1.5. Let Q be a quiver (Definition 3.1.4). The *path category generated by Q*, denoted $\mathcal{F}(Q)$, is the category (Definition 1.0.1) defined as follows:

- The objects of $\mathcal{F}(Q)$ are the vertices Q_0.
- For any two objects $x, y \in Q_0$, the set of morphisms $\mathrm{Hom}_{\mathcal{F}(Q)}(x, y)$ consists of all paths from x to y — A *path of length $n \geq 1$ from x to y* is a sequence of arrows $\alpha_n \ldots \alpha_1$ such that $s(\alpha_1) = x$, $t(\alpha_n) = y$, and $s(\alpha_{i+1}) = t(\alpha_i)$ for all $1 \leq i < n$. Additionally, for each vertex x, there is a path e_x of length 0, which serves as the identity morphism.
- Composition of morphisms is defined by the concatenation of paths.

Definition 3.1.6. Let Q be a quiver (Definition 3.1.4) whose collection of arrows is small.

The *preadditive category generated by Q*, denoted $\mathbb{Z}Q$, is the preadditive category (Definition 2.0.6), i.e. the category enriched over (Definition A.0.2) the category of abelian groups defined as follows:

- The objects of $\mathbb{Z}Q$ are the vertices Q_0.
- For any objects $x, y \in Q_0$, the morphism set $\mathrm{Hom}_{\mathbb{Z}Q}(x, y)$ is the free abelian group generated by the set of all paths from x to y in Q.
- Composition is the unique bilinear extension of the path concatenation in $\mathcal{F}(Q)$. That is, for paths u, v, w where concatenation is defined, composition satisfies $(u + v) \circ w = u \circ w + v \circ w$ and $w \circ (u + v) = w \circ u + w \circ v$.

Definition 3.1.7. Let Q_{chain} be the quiver (Definition 3.1.4) with vertex set $Q_0 = \mathbb{Z}$ and arrow set $Q_1 = \{d_n : n \to n-1 \mid n \in \mathbb{Z}\}$. (♠ TODO: quotient of a category) The *walking chain complex category*, denoted $\mathcal{I}$, is the quotient of the preadditive category (Definition 2.0.6) $\mathbb{Z}Q_{\mathrm{chain}}$ by the ideal generated by the relations $d_{n-1} \circ d_n = 0$ for all $n \in \mathbb{Z}$. Explicitly:

- Objects are the integers $\mathbb{Z}$.
- Morphisms are $\mathbb{Z}$-linear combinations of paths, subject to the relation that any path containing a subsegment $d_{n-1} d_n$ is identified with the zero morphism.

Definition 3.1.8. Let $\mathcal{A}$ and $\mathcal{B}$ be preadditive categories (Definition 2.0.6) (categories enriched over (Definition A.0.2) the category of abelian groups). The *additive functor category* $\mathrm{Add}(\mathcal{A}, \mathcal{B})$ is the functor category where:

- Objects are additive functors $F : \mathcal{A} \to \mathcal{B}$. An additive functor (Definition 2.0.8) is a functor such that for any $x, y \in \mathrm{Ob}(\mathcal{A})$, the map $F : \mathrm{Hom}_{\mathcal{A}}(x, y) \to \mathrm{Hom}_{\mathcal{B}}(F(x), F(y))$ is a group homomorphism.
- Morphisms are natural transformations (Definition 1.0.4) between additive functors (Definition 2.0.8).

Proposition 3.1.9. Let $\mathcal{B}$ be a preadditive category (Definition 2.0.6). The category $\mathrm{Ch}(\mathcal{B})$ of chain complexes (Definition 3.1.1) in $\mathcal{B}$ is isomorphic to the category $\mathrm{Add}(\mathcal{I}, \mathcal{B})$ (Definition 3.1.8)(Definition 3.1.7).

An additive functor $F : \mathcal{I} \to \mathcal{B}$ corresponds to the chain complex defined by $C_n = F(n)$ and differentials $\partial_n = F(d_n)$.

Lemma 3.1.10. Let $\mathcal{A}, \mathcal{B}$ be preadditive categories (Definition 2.0.6) with $\mathcal{A}$ small (Definition 1.0.5).

1. The additive functor category (Definition 3.1.8) $\mathrm{Add}(\mathcal{A}, \mathcal{B})$ is preadditive. If $\mathcal{B}$ is additionally additive (Definition 2.0.6)/abelian (Definition 2.0.10), then so is $\mathrm{Add}(\mathcal{A}, \mathcal{B})$.
2. If $\mathcal{B}$ is an abelian category with property ABn for $n = 3, 4, 5, 6$ or ABn^* for $n = 3, 4, 5$ (Definition 2.1.8), then $\mathrm{Add}(\mathcal{A}, \mathcal{B})$ possesses the same property.

Proposition 3.1.11. Let $\mathcal{A}$ be an additive category.

1. The category $\mathrm{Ch}(\mathcal{A})$ of chain complexes is itself and additive category.
2. If $\mathcal{A}$ is an abelian category, then $\mathrm{Ch}(\mathcal{A})$ is an abelian category.
3. If $\mathcal{A}$ is an abelian category satisfying Grothendieck's axiom ABn (resp. ABn^*) (Definition 2.1.8) for $n \in \{3, 4, 5, 6\}$, then $\mathrm{Ch}(\mathcal{A})$ also satisfies ABn (resp. ABn^*). If $\mathcal{A}$ is a Grothendieck abelian category (Definition 2.1.8), then so is $\mathrm{Ch}(\mathcal{A})$

Proof. Combine Proposition 3.1.9 and Lemma 3.1.10. $\qquad\square$

Proposition 3.1.12. Let $\mathcal{A}$ be an additive category (Definition 2.0.6). Let $\mathcal{J}$ be a category (Definition 1.0.1) and let $M^\bullet : \mathcal{J} \to \mathrm{Ch}(\mathcal{A})$ be a diagram (Definition 1.3.1) of chain complexes (Definition 3.1.1), denoted by $j \mapsto M_{(j)}^\bullet$.

1. If the limit (Definition 1.3.2) of the diagram of objects $j \mapsto M_{(j)}^n$ exists in $\mathcal{A}$ for every degree $n \in \mathbb{Z}$, then the limit of the diagram $M^\bullet$ exists in $\mathrm{Ch}(\mathcal{A})$. It is computed termwise:
$$\left(\lim_{j \in \mathcal{J}} M_{(j)}^\bullet \right)^n \cong \lim_{j \in \mathcal{J}} \left(M_{(j)}^n \right).$$
The differential $d^n : (\lim M^\bullet)^n \to (\lim M^\bullet)^{n+1}$ is the unique morphism induced by the family of morphisms $\{d_{(j)}^n : M_{(j)}^n \to M_{(j)}^{n+1}\}_{j \in \mathcal{J}}$ via the universal property of limits.
2. Similarly, if the colimit (Definition 1.3.2) of the diagram of objects $j \mapsto M_{(j)}^n$ exists in $\mathcal{A}$ for every degree $n \in \mathbb{Z}$, then the colimit of the diagram $M^\bullet$ exists in $\mathrm{Ch}(\mathcal{A})$. It is computed termwise:
$$\left(\mathrm{colim}_{j \in \mathcal{J}} M_{(j)}^\bullet \right)^n \cong \mathrm{colim}_{j \in \mathcal{J}} \left(M_{(j)}^n \right).$$
The differential $\delta^n : (\mathrm{colim}\, M^\bullet)^n \to (\mathrm{colim}\, M^\bullet)^{n+1}$ is the unique morphism induced by the family of morphisms $\{d_{(j)}^n : M_{(j)}^n \to M_{(j)}^{n+1}\}_{j \in \mathcal{J}}$ via the universal property of colimits.

Proof. See Proposition C.0.20 (♠ TODO: It must be taken into account how example the categories of chain complexes are regardable as functor categories) $\qquad\square$

3.2. (Co)Homology of chain complexes.

Definition 3.2.1. Let $\mathcal{A}$ be an abelian category (Definition 2.0.10).

1. Let $C_\bullet = (\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \ldots)$ be a chain complex (Definition 3.1.1) in $\mathcal{A}$. For each integer n, we define:
   - The object of *n-cycles*, denoted $Z_n(C)$, is the kernel (Definition 1.2.2) of the differential leaving C_n:
   $$Z_n(C) := \ker(d_n : C_n \to C_{n-1}).$$
   - The object of *n-boundaries*, denoted $B_n(C)$, is the image (Definition 1.2.5) of the differential entering C_n:
   $$B_n(C) := \operatorname{im}(d_{n+1} : C_{n+1} \to C_n).$$

   Since $d_n \circ d_{n+1} = 0$, there is a canonical monomorphism (Definition 1.2.1) $B_n(C) \hookrightarrow Z_n(C)$.

2. Let $C^\bullet = (\cdots \to C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \to \ldots)$ be a *cochain complex in $\mathcal{A}$*. For each integer n, we define:
   - The object of *n-cocycles*, denoted $Z^n(C)$, is the kernel (Definition 1.2.2) of the differential leaving C^n:
   $$Z^n(C) := \ker(d^n : C^n \to C^{n+1}).$$
   - The object of *n-coboundaries*, denoted $B^n(C)$, is the image> (Definition 1.2.5) of the differential entering C^n:
   $$B^n(C) := \operatorname{im}(d^{n-1} : C^{n-1} \to C^n).$$

   Since $d^n \circ d^{n-1} = 0$, there is a canonical monomorphism (Definition 1.2.1) $B^n(C) \hookrightarrow Z^n(C)$.

Definition 3.2.2 (Chain complexes and their (co)homology objects). Let $\mathcal{A}$ be an abelian category.

- For a cochain complex $K^\bullet$, its *cohomology object in degree i* is defined as the quotient of the object of i-cocycles by the object of i-coboundaries:
$$H^i(K^\bullet) := Z^i(K)/B^i(K) = \ker(d^i)/\operatorname{im}(d^{i-1}).$$

- For a chain complex $K_\bullet$, its *homology object in degree i* is defined as the quotient of the object of i-cycles by the object of i-boundaries:
$$H_i(K_\bullet) := Z_i(K)/B_i(K). = \ker(d_i)/\operatorname{im}(d_{i+1}).$$

Definition 3.2.3 (Acyclic complex). Let $\mathcal{A}$ be an additive category (Definition 2.0.6), and let $(C_\bullet, d_\bullet)$ be a complex (Definition 3.1.1) in $\mathcal{A}$:
$$\cdots \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots .$$

The complex $(C_\bullet, d_\bullet)$ is called *acyclic at C_n* (or sometimes synonymously *exact at C_n*) if we have $\ker d_n \cong \operatorname{im} d_{n+1}$ (Definition 1.2.2) (Definition 1.2.5).

If $\mathcal{A}$ is an abelian category (Definition 2.0.10), then this is equivalent to the condition that the (co)homology objects (Definition 3.2.2) $H^n(C_\bullet) := \ker d_n / \operatorname{im} d_{n+1}$ are zero in $\mathcal{A}$.

We furthermore say that the complex $(C_\bullet, d_\bullet)$ is *acyclic* or *exact* if it is acyclic/exact everywhere.

Proposition 3.2.4. Let $\mathcal{A}$ be a abelian category (Definition 2.0.10) and let J be a small category (Definition 1.0.5).

Given a sequence $A \to B \to C$ of objects in the diagram category (Definition 1.3.1) $\mathcal{A}^J$, the sequence is exact (Definition 3.2.3) at B if and only if all the sequences $A(j) \to B(j) \to C(j)$ are exact at $B(j)$ for every $j \in \operatorname{Ob}(J)$.

Definition 3.2.5 (Boundedness conditions on chain complexes)**.** Let $\mathcal{A}$ be an additive category.

Cohomological convention: Let $K^\bullet = (K^i, d^i)_{i \in \mathbb{Z}}$ be a cohomologically indexed chain complex in $\mathcal{A}$.

- $K^\bullet$ is *bounded above* if there exists $n \in \mathbb{Z}$ such that $K^i = 0$ for all $i > n$.
- $K^\bullet$ is *bounded below* if there exists $m \in \mathbb{Z}$ such that $K^i = 0$ for all $i < m$.
- $K^\bullet$ is *bounded* if it both bounded above and below, or equivalently if $K^i = 0$ for all but finitely many $i \in \mathbb{Z}$.
- Assuming that $\mathcal{A}$ is an abelian category,
  - $K^\bullet$ is *cohomologically bounded above* if there exists $n \in \mathbb{Z}$ such that $H^i(K^\bullet) = 0$ for all $i > n$.
  - $K^\bullet$ is *cohomologically bounded below* if there exists $m \in \mathbb{Z}$ such that $H^i(K^\bullet) = 0$ for all $i < m$.
  - $K^\bullet$ is *cohomologically bounded* if it is both cohomologically bounded above and below, or equivalently if the cohomology objects $H^i(K^\bullet)$ vanish for all but finitely many $i \in \mathbb{Z}$.

Homological convention: Let $K_\bullet = (K_i, d_i)_{i \in \mathbb{Z}}$ be a homologically indexed chain complex in $\mathcal{A}$.

- $K_\bullet$ is *bounded above* if there exists $n \in \mathbb{Z}$ such that $K_i = 0$ for all $i > n$.
- $K_\bullet$ is *bounded below* if there exists $m \in \mathbb{Z}$ such that $K_i = 0$ for all $i < m$.
- $K_\bullet$ is *bounded* if it both bounded above and below, or equivalently if $K_i = 0$ for all but finitely many $i \in \mathbb{Z}$.
- Assuming that $\mathcal{A}$ is an abelian category,
  - $K_\bullet$ is *homologically bounded above* if there exists $n \in \mathbb{Z}$ such that $H_i(K_\bullet) = 0$ for all $i > n$.
  - $K_\bullet$ is *homologically bounded below* if there exists $m \in \mathbb{Z}$ such that $H_i(K_\bullet) = 0$ for all $i < m$.
  - $K_\bullet$ is *homologically bounded* if it is both homologically bounded above and below, or equivalently if the homology objects $H_i(K_\bullet)$ vanish for all but finitely many $i \in \mathbb{Z}$.

Definition 3.2.6 (Quasi-isomorphism)**.** Let $\mathcal{A}$ be an abelian category (Definition 2.0.10), and let
$$f_\bullet : (C_\bullet, d_\bullet^C) \to (D_\bullet, d_\bullet^D)$$
be a chain map between complexes (Definition 3.1.1) in $\mathcal{A}$.

The morphism $f_\bullet$ is called a *quasi-isomorphism* if it induces isomorphisms on all cohomology objects, i.e., for every integer n, the induced morphism on homology (Definition 3.2.2) (or cohomology, depending on the convention)
$$H^n(f_\bullet) : H^n(C_\bullet) \to H^n(D_\bullet)$$
is an isomorphism in $\mathcal{A}$.

Note that all of these notions are applicable to the cohomological convention as well (Remark 3.1.2).

Lemma 3.2.7. Let
$$F : \mathcal{A} \to \mathcal{B}$$
be an additive functor (Definition 2.0.8) between abelian categories (Definition 2.0.10). It is exact (Definition 2.0.17) if and only if it preserves kernels and cokernels (Definition 1.2.2).

Theorem 3.2.8 (Long Exact Sequence in Homology)**.**   1. Let $(C_\bullet, d_\bullet^C)$, $(D_\bullet, d_\bullet^D)$, and $(E_\bullet, d_\bullet^E)$ be chain complexes (Definition 3.1.1) in an abelian category (Definition 2.0.10). Recall that $\mathbf{Ch}(\mathcal{A})$ (Definition 3.1.1) is itself an abelian category (Proposition 3.1.11). Assume that
$$0 \longrightarrow C_\bullet \xrightarrow{\alpha_\bullet} D_\bullet \xrightarrow{\beta_\bullet} E_\bullet \longrightarrow 0$$
is a short exact sequence (Definition 2.0.19) of chain complexes. Equivalently, for each integer n,
$$0 \to C_n \xrightarrow{\alpha_n} D_n \xrightarrow{\beta_n} E_n \to 0$$
is an exact sequence of R-modules.
   Then there exists a natural *long exact sequence* in homology (Definition 3.2.2):
$$\cdots \longrightarrow H_{n+1}(E_\bullet) \xrightarrow{\delta_{n+1}} H_n(C_\bullet) \xrightarrow{H_n(\alpha)} H_n(D_\bullet) \xrightarrow{H_n(\beta)} H_n(E_\bullet) \xrightarrow{\delta_n} H_{n-1}(C_\bullet) \longrightarrow \cdots$$
   The homomorphisms $\delta_n : H_n(E_\bullet) \to H_{n-1}(C_\bullet)$ are called the *connecting homomorphisms* induced by the short exact sequence of chain complexes.
   Moreover, this long exact sequence is natural with respect to morphisms (Definition 3.1.1) of short exact sequences of chain complexes.
   2. Let $(C^\bullet, d_C^\bullet)$, $(D^\bullet, d_D^\bullet)$, and $(E^\bullet, d_E^\bullet)$ be cochain complexes (Definition 3.1.1) in an abelian category (Definition 2.0.10). Recall that $\mathbf{Ch}(\mathcal{A})$ (Definition 3.1.1) is itself an abelian category (Proposition 3.1.11).
   Assume that
$$0 \longrightarrow C^\bullet \xrightarrow{\alpha^\bullet} D^\bullet \xrightarrow{\beta^\bullet} E^\bullet \longrightarrow 0$$
is a short exact sequence (Definition 2.0.19) of cochain complexes. Equivalently, for each integer n,
$$0 \to C^n \xrightarrow{\alpha^n} D^n \xrightarrow{\beta^n} E^n \to 0$$
is an exact sequence of R-modules.

Then there exists a natural *long exact sequence* in cohomology (Definition 3.2.2):

$$\cdots \longrightarrow H^{n-1}(E^\bullet) \xrightarrow{\delta^{n-1}} H^n(C^\bullet) \xrightarrow{H^n(\alpha)} H^n(D^\bullet) \xrightarrow{H^n(\beta)} H^n(E^\bullet) \xrightarrow{\delta^n} H^{n+1}(C^\bullet) \longrightarrow \cdots$$

The morphisms $\delta^n : H^n(E^\bullet) \to H^{n+1}(C^\bullet)$ are called the *connecting homomorphisms* induced by the short exact sequence of cochain complexes.

Moreover, this long exact sequence is natural with respect to morphisms of short exact sequences of cochain complexes.

3.3. Homotopy between maps between chain complexes.

Definition 3.3.1. Let $\mathcal{A}$ be an additive category (Definition 2.0.6).

1. Let $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ be chain maps between complexes (Definition 3.1.1) in $\mathcal{A}$. A *chain homotopy from $f_\bullet$ to $g_\bullet$* is a collection of morphisms $\{s_n : C_n \to D_{n+1}\}$ such that for all n,
$$f_n - g_n = d^D_{n+1} \circ s_n + s_{n-1} \circ d^C_n.$$
If such an $s_\bullet$ exists, we say that $f_\bullet$ and $g_\bullet$ are *chain homotopic* and write $f_\bullet \simeq g_\bullet$.
2. Let $f_\bullet : C_\bullet \to D_\bullet$ be a chain map between complexes (Definition 3.1.1) in $\mathcal{A}$. A *chain contraction* is a chain homotopy from $f_\bullet$ to the zero complex. The chain map $f_\bullet$ is said to be *null homotopic* if a chain contraction of $f_\bullet$ exists, i.e. $f_\bullet$ is chain homotopic to the 0 chain complex.
3. Let $f_\bullet : C_\bullet \to D_\bullet$ be a chain map between complexes. We say that $f_\bullet$ is a *chain homotopy equivalence* if there exists a chain map and $h_\bullet : D_\bullet \to C_\bullet$ such that
$$fg \simeq \mathrm{id}_{D_\bullet} \quad \text{and} \quad gf \simeq \mathrm{id}_{C_\bullet}.$$
In this case, it is appropriate to call f and g *chain homotopy inverses of each other*.

One similarly defines the above notions for cochain complexes and their morphisms.

Lemma 3.3.2. Let $\mathcal{A}$ be an additive category (Definition 2.0.6). Being chain homotopic (Definition 3.3.1) is an equivalence relation on chain complexes (Definition 3.1.1) of objects in $\mathcal{A}$. (♠ TODO: equivalence relation)

3.4. Truncations of chain complexes.

Definition 3.4.1. Let $\mathcal{A}$ be an additive category (Definition 2.0.6).

1. Let $A_\bullet$ be a chain complex (Definition 3.1.1) in $\mathbf{Ch}(\mathcal{A})$ (Definition 3.1.1), and let $n \in \mathbb{Z}$. The *stupid/brutal truncations of $A_\bullet$* are defined as follows.
$$(\sigma_{\leq n} A)_i = \begin{cases} 0, & i > n, \\ A_i, & i \leq n, \end{cases} \qquad (\sigma_{\geq n} A)_i = \begin{cases} A_i, & i \geq n, \\ 0, & i < n. \end{cases}$$
Both are complexes with differentials induced from that of $A_\bullet$. If $\mathcal{A}$ is an abelian category (Definition 2.0.10), then they satisfy canonical isomorphisms
$$\sigma_{\leq n} A_\bullet / \sigma_{\leq n-1} A_\bullet \cong A_n[-n] \quad \text{and} \quad \sigma_{\geq n} A_\bullet / \sigma_{\geq n+1} A_\bullet \cong A_n[-n].$$

2. Let $A^\bullet$ be a cochain complex in $\mathbf{Ch}(\mathcal{A})$, i.e.

$$\cdots \xrightarrow{d^{n-2}} A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \xrightarrow{d^{n+1}} \cdots$$

with $d^{n+1} \circ d^n = 0$ for all $n \in \mathbb{Z}$. The *stupid truncations of $A^\bullet$* are defined by the rules

$$(\sigma_{\geq n} A)^i = \begin{cases} 0, & i < n, \\ A^i, & i \geq n, \end{cases} \qquad (\sigma_{\leq n} A)^i = \begin{cases} A^i, & i \leq n, \\ 0, & i > n. \end{cases}$$

These are cochain complexes (Definition 3.1.1) with differentials inherited from $A^\bullet$. If $\mathcal{A}$ is abelian, then they fit canonically into short exact sequences of cochain complexes

$$0 \longrightarrow \sigma_{\geq n+1} A^\bullet \longrightarrow \sigma_{\geq n} A^\bullet \longrightarrow A^n[-n] \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow A^n[-n] \longrightarrow \sigma_{\leq n} A^\bullet \longrightarrow \sigma_{\leq n-1} A^\bullet \longrightarrow 0.$$

Definition 3.4.2. Let $\mathcal{A}$ be an abelian category (Definition 2.0.10).

1. The *canonical truncations* of a chain complex (Definition 3.1.1) $A_\bullet$ in $\mathbf{Ch}(\mathcal{A})$ (Definition 3.1.1) are defined by

$$(\tau_{\geq n} A)_i = \begin{cases} A_i, & i > n, \\ \ker(d_n : A_n \to A_{n-1}), & i = n, \\ 0, & i < n, \end{cases} \quad (\tau_{\leq n} A)_i = \begin{cases} 0, & i > n, \\ \mathrm{coker}(d_{n+1} : A_{n+1} \to A_n), & i = n, \\ A_i, & i < n. \end{cases}$$

The differentials are the restrictions and/or quotient maps induced from $A_\bullet$. In particular,

$$H_i(\tau_{\geq n} A_\bullet) = \begin{cases} H_i(A_\bullet), & i \geq n, \\ 0, & i < n, \end{cases} \quad \text{and} \quad H_i(\tau_{\leq n} A_\bullet) = \begin{cases} H_i(A_\bullet), & i \leq n, \\ 0, & i > n. \end{cases}$$

The assignments $A_\bullet \mapsto \tau_{\geq n} A_\bullet$ and $A_\bullet \mapsto \tau_{\leq n} A_\bullet$ extend to endofunctors

$$\tau_{\geq n}, \tau_{\leq n} : \mathbf{Ch}(\mathcal{A}) \to \mathbf{Ch}(\mathcal{A}),$$

called the *truncation functors*. They are natural in both $A_\bullet$ and n, and fit into canonical morphisms of complexes

$$\tau_{\geq n} A_\bullet \longrightarrow A_\bullet \longrightarrow \tau_{\leq n} A_\bullet.$$

2. Similarly, let $A^\bullet$ be a cochain complex in $\mathbf{Ch}(\mathcal{A})$, i.e.

$$\cdots \xrightarrow{d^{n-2}} A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \xrightarrow{d^{n+1}} \cdots$$

with $d^{n+1} \circ d^n = 0$ for all $n \in \mathbb{Z}$. The *canonical truncations of $A^\bullet$* are defined by

$$(\tau_{\leq n} A)^i = \begin{cases} A^i, & i < n, \\ \ker(d^n : A^n \to A^{n+1}), & i = n, \\ 0, & i > n, \end{cases} \quad (\tau_{\geq n} A)^i = \begin{cases} 0, & i < n, \\ \mathrm{coker}(d^{n-1} : A^{n-1} \to A^n), & i = n, \\ A^i, & i > n. \end{cases}$$

The differentials are the restrictions or quotient maps induced by those of $A^\bullet$. These truncations satisfy

$$H^i(\tau_{\leq n}A^\bullet) = \begin{cases} H^i(A^\bullet), & i \leq n, \\ 0, & i > n, \end{cases} \quad \text{and} \quad H^i(\tau_{\geq n}A^\bullet) = \begin{cases} 0, & i < n, \\ H^i(A^\bullet), & i \geq n. \end{cases}$$

They also extend to endofunctors

$$\tau_{\leq n},\ \tau_{\geq n} : \mathbf{Ch}(\mathcal{A}) \to \mathbf{Ch}(\mathcal{A}),$$

natural in both $A^\bullet$ and n, fitting into canonical morphisms of cochain complexes

$$\tau_{\leq n}A^\bullet \longrightarrow A^\bullet \longrightarrow \tau_{\geq n}A^\bullet.$$

Proposition 3.4.3 (Exhaustion of Complexes by Brutal Truncations). Let $\mathcal{A}$ be an additive category and let X be a chain complex in $\mathrm{Ch}(\mathcal{A})$.

1. For any integer $n \in \mathbb{Z}$, let $\sigma_{\leq n}X$ and $\sigma_{\geq n}X$ denote the brutal truncations (Definition 3.4.1) defined componentwise by:

$$(\sigma_{\leq n}X)_k = \begin{cases} X_k & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases} \quad \text{and} \quad (\sigma_{\geq n}X)_k = \begin{cases} X_k & \text{if } k \geq n \\ 0 & \text{if } k < n. \end{cases}$$

   The canonical inclusions $\iota_n : \sigma_{\leq n}X \to \sigma_{\leq n+1}X$ define a directed system (Definition 1.3.11), and the canonical projections $\pi_n : \sigma_{\geq n}X \to \sigma_{\geq n+1}X$ define an inverse system.

   The complex X is canonically isomorphic to both the colimit (Definition 1.3.2) of its "bounded above" truncations and the limit (Definition 1.3.2) of its "bounded below" truncations:

   $$X \cong \varinjlim_{n \to +\infty} \sigma_{\leq n}X \quad \text{and} \quad X \cong \varprojlim_{n \to -\infty} \sigma_{\geq n}X.$$

2. For any integer $n \in \mathbb{Z}$, let $\tau_{\leq n}X$ and $\tau_{\geq n}X$ denote the canonical truncations (Definition 3.4.2) (or Postnikov sections) defined by:

   $$(\tau_{\leq n}X)_k = \begin{cases} X_k & \text{if } k < n \\ \mathrm{Ker}(d_n) & \text{if } k = n \\ 0 & \text{if } k > n \end{cases}$$

   $$(\tau_{\geq n}X)_k = \begin{cases} X_k & \text{if } k > n \\ \mathrm{Coker}(d_{n+1}) & \text{if } k = n \\ 0 & \text{if } k < n. \end{cases}$$

   The canonical morphisms $\iota_n : \tau_{\leq n}X \to \tau_{\leq n+1}X$ define a directed system (Definition 1.3.11), and the canonical projections $\pi_n : \tau_{\geq n}X \to \tau_{\geq n+1}X$ define an inverse system (where the limit is taken as n decreases).

   The complex X is canonically isomorphic to both the colimit (Definition 1.3.2) of its "homologically bounded above" truncations and the limit (Definition 1.3.2) of its "homologically bounded below" truncations:

   $$X \cong \varinjlim_{n \to +\infty} \tau_{\leq n}X \quad \text{and} \quad X \cong \varprojlim_{n \to -\infty} \tau_{\geq n}X.$$

3.5. Mapping cone of maps of chain complexes.

Definition 3.5.1.   1. Let $f : (C_\bullet, d_\bullet^C) \to (D_\bullet, d_\bullet^D)$ be a morphism of chain complexes (Definition 3.1.1) in an additive category $\mathcal{A}$ (Definition 2.0.6).

The *mapping cone of f*, denoted $\boxed{\text{Cone}(f)}$, is the chain complex defined by:

- Objects: For each n,

$$\text{Cone}(f)_n = D_n \oplus C_{n-1}.$$

- Differential: For each n, define

$$d_n^{\text{Cone}(f)} : \text{Cone}(f)_n \to \text{Cone}(f)_{n-1}$$

by the matrix morphism

$$d_n^{\text{Cone}(f)} = \begin{pmatrix} d_n^D & f_{n-1} \\ 0 & -d_{n-1}^C \end{pmatrix} : D_n \oplus C_{n-1} \to D_{n-1} \oplus C_{n-2}.$$

This construction defines a chain complex, i.e., $d_{n-1}^{\text{Cone}(f)} \circ d_n^{\text{Cone}(f)} = 0$.

2. Dually, let $g : (C^\bullet, d_C^\bullet) \to (D^\bullet, d_D^\bullet)$ be a morphism of cochain complexes (Definition 3.1.1) in $\mathcal{A}$.

The *mapping cone of g*, denoted $\boxed{\text{Cone}(g)}$, is the cochain complex (Definition 3.1.1) defined by:

- Objects: For each n,

$$\text{Cone}(g)^n = D^n \oplus C^{n+1}.$$

- Differential: For each n, define

$$d_{\text{Cone}(g)}^n : \text{Cone}(g)^n \to \text{Cone}(g)^{n+1}$$

by the matrix morphism

$$d_{\text{Cone}(g)}^n = \begin{pmatrix} d_D^n & g^{n+1} \\ 0 & -d_C^{n+1} \end{pmatrix} : D^n \oplus C^{n+1} \to D^{n+1} \oplus C^{n+2}.$$

This construction defines a cochain complex, i.e., $d_{\text{Cone}(g)}^{n+1} \circ d_{\text{Cone}(g)}^n = 0$.

4. RIGHT/LEFT DERIVED FUNCTORS OF LEFT/RIGHT EXACT ADDITIVE FUNCTORS BETWEEN ABELIAN CATEGORIES

4.1. Injective/projective objects.

Definition 4.1.1 (Injective and Projective objects in a general category)**.** Let $\mathcal{C}$ be a category (Definition 1.0.1)

- An object $I \in \mathcal{C}$ is called *injective* if for every monomorphism (Definition 1.2.1) $m : A \to B$ in $\mathcal{C}$ and every morphism $f : A \to I$, there exists a morphism $\tilde{f} : B \to I$ such that the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & I \\
{\scriptstyle m}\downarrow & \nearrow {\scriptstyle \tilde{f}} & \\
B & &
\end{array}
$$

commutes, i.e., $\tilde{f} \circ m = f$.

- Dually, an object $P \in \mathcal{C}$ is called *projective* if for every epimorphism (Definition 1.2.1) $e : X \to Y$ in $\mathcal{C}$ and every morphism $g : P \to Y$, there exists a morphism $\tilde{g} : P \to X$ such that the diagram

$$
\begin{array}{ccc}
 & & P \\
 & {\scriptstyle \tilde{g}} \swarrow & \downarrow {\scriptstyle g} \\
X & \xrightarrow{\ e\ } & Y
\end{array}
$$

  commutes, i.e., $e \circ \tilde{g} = g$.

Lemma 4.1.2 (cf. e.g. [Wei94, Lemma 2.3.4]). Let $\mathcal{C}$ be a (large) category (Definition 1.0.1). (♠ TODO: Does this generalize to a category enriched in a symmetric monoidal category) The following are equivalent for an object $I \in \mathcal{C}$:

1. I is injective (Definition 4.1.1) in $\mathcal{C}$
2. I is projective (Definition 4.1.1) in $\mathcal{C}^{\mathrm{op}}$ (Definition 1.0.2)

Proof. The notions of injectivity and projectivity of objects in a category are dual objects, so it is clear that I is injective in $\mathcal{C}$ if and only if it is projective in $\mathcal{C}^{\mathrm{op}}$. $\qquad\square$

Lemma 4.1.3.   1. Let R be a (not necessarily commutative) ring (Definition C.0.2).
   - (a) In the category $R\text{-}\mathbf{Mod}$ of left R-modules, projective objects are precisely direct summands of free modules.
   - (b) For $R = \mathbb{Z}$, projective $\mathbb{Z}$-modules are the free abelian groups, and injective ones are the divisible abelian groups, e.g. $\mathbb{Q}$ or $\mathbb{Q}/\mathbb{Z}$.
2. In the category $\mathbf{Set}$ of sets, every object is both projective and injective. Indeed, epimorphisms and monomorphisms in $\mathbf{Set}$ are surjective and injective maps, respectively, both of which split.
3. In the category $\mathbf{Grp}$ of groups, projective objects are precisely the free groups, since every homomorphism from a free group lifts along surjective homomorphisms.
4. In $\mathbf{Grp}$, the only injective object is the trivial group, since injectivity requires extensions along all inclusions, which only the terminal object satisfies.

Theorem 4.1.4 (Baer's Criterion). Let R be a (not necessarily commutative) ring (Definition C.0.2) with unit and let I be an R-module (Definition C.0.4). Then I is an injective (Definition 4.1.1) R-module if and only if for every left ideal (Definition C.0.5) $J \subseteq R$, every R-module homomorphism (Definition C.0.6)

$$ f : J \to I $$

extends to an R-module homomorphism

$$ \tilde{f} : R \to I. $$

Intuitively, Baer's criterion reduces the problem of verifying injectivity of a module to checking extension properties from the simplest possible submodules of the ring itself — its left ideals. This characterization is fundamental in homological algebra and module theory, and is widely used to identify and construct injective modules.

Lemma 4.1.5. Let $\mathcal{C}$ be a locally small category (Definition 1.0.5). (♠ TODO: Does this generalize to a category enriched in a symmetric monoidal category)

1. The following are equivalent for an object $I \in \mathcal{C}$:
   - I is injective (Definition 4.1.1) in $\mathcal{C}$.
   - The contravariant functor

   $$\mathrm{Hom}_{\mathcal{C}}(-, I) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}$$

   takes monomorphisms (Definition 1.2.1) to surjections, i.e. for any monomorphism $A \hookrightarrow B$, the induced map

   $$\mathrm{Hom}_{\mathcal{C}}(B, I) \to \mathrm{Hom}_{\mathcal{C}}(A, I)$$

   of sets is a surjection.
2. The following are equivalent for an object $P \in \mathcal{C}$:
   - P is projective (Definition 4.1.1) in $\mathcal{C}$.
   - The covariant functor

   $$\mathrm{Hom}_{\mathcal{C}}(P, -) : \mathcal{C} \to \mathbf{Sets}$$

   takes epimorphisms (Definition 1.2.1) to surjections, i.e. for any epimorphism $A \hookrightarrow B$, the induced map

   $$\mathrm{Hom}_{\mathcal{C}}(P, A) \to \mathrm{Hom}_{\mathcal{C}}(A, B)$$

   of sets is a surjection.

Proof. These follow immediately from the definitions of injective and projective objects. □

Lemma 4.1.6 (cf. e.g. [Wei94, Lemma 2.3.4]). Let $\mathcal{A}$ be a abelian category (Definition 2.0.10). (♠ TODO: Does this generalize to a category enriched in a symmetric monoidal category)

1. The following are equivalent for an object $I \in \mathcal{A}$:
   - I is injective (Definition 4.1.1) in $\mathcal{A}$.
   - The contravariant functor

   $$\mathrm{Hom}_{\mathcal{A}}(-, I) : \mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$$

   is exact (Definition 2.0.17).
2. The following are equivalent for an object $P \in \mathcal{A}$:
   - P is projective (Definition 4.1.1) in $\mathcal{A}$.
   - The covariant functor

   $$\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to \mathrm{Ab}$$

   is exact (Definition 2.0.17).

Proof. (♠ TODO:) □

4.2. **Resolutions of objects by chain complexes.** In this section, we list out some basic facts about resolutions (Definition 4.2.4), particular of injective or projective (Definition 4.1.1) objects. These will be needed to define derived functors (Definition 4.3.1)

Definition 4.2.1. Let $\mathcal{A}$ be an abelian category (Definition 2.0.10) and let $\mathcal{X}$ be a class of objects in $\mathcal{A}$.

We say that $\mathcal{A}$ *has enough objects of class $\mathcal{X}$ on the left (resp. on the right)* if for any object $M \in \mathcal{A}$, there exists an object X of the class $\mathcal{X}$ and an epimorphism (Definition 1.2.1) $X \twoheadrightarrow M$ (resp. a monomorphism $M \hookrightarrow X$).

Definition 4.2.2. Let $\mathcal{A}$ be an abelian category (Definition 2.0.10).

1. $\mathcal{A}$ is said to *have enough injectives* if for every object A in $\mathcal{A}$, there is an monomorphism (Definition 1.2.1) $A \to I$ with I an injective object (Definition 4.1.1) of $\mathcal{A}$. Equivalently, $\mathcal{A}$ has enough injectives if it has enough objects of the class of injectives on the right (Definition 4.2.1)
2. $\mathcal{A}$ is said to *have enough projectives* if for every object A in $\mathcal{A}$, there is a epimorphism (Definition 1.2.1) $P \to A$ with P a projective object (Definition 4.1.1) of $\mathcal{A}$. Equivalently, $\mathcal{A}$ has enough projectives if it has enough objects of the class of projectives on the left (Definition 4.2.1)

Theorem 4.2.3.   1. Examples of abelian categories with enough injectives include:
   - The category of abelian groups.
   - The category of modules over a ring.
   - The category of sheaves of abelian groups on a ringed space or on a site.
2. Examples of abelian categories with enough projectives include:
   - The category of modules over a ring with enough projectives (e.g., rings with unity and suitable properties). (♠ TODO: make this more precise)
   - The category of finitely generated modules over a semisimple ring.

Definition 4.2.4. Let $\mathcal{A}$ be an abelian category (Definition 2.0.10) and let $\mathcal{X}$ be a class of objects in $\mathcal{A}$. Let M be an object of $\mathcal{A}$.

1. A *right resolution of M* is a cochain complex (Definition 3.1.1) $I^\bullet$ with $I^i = 0$ for $i < 0$ and a map $M \to I^0$ such that the augmented complex
$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$$
is exact (Definition 3.2.3).
2. A *left resolution of M* is a chain complex (Definition 3.1.1) $P_\bullet$ with $P_i = 0$ for $i < 0$ and a map $P_0 \to M$ such that the augmented complex
$$\cdots P_2 \to P_1 \to P_0 \to M \to 0$$
is exact (Definition 3.2.3).
3. An *$\mathcal{X}$-left resolution* of an object $M \in \mathcal{A}$ a left resolution (Definition 4.2.4) by objects of X, i.e. an exact complex
$$\cdots \to X_2 \to X_1 \to X_0 \to M \to 0$$
with each $X_i \in \mathcal{X}$.

4. An $\mathcal{X}$-*right resolution* of an object $M \in \mathcal{A}$ a right resolution (Definition 4.2.4) by objects of X, i.e. an exact complex

$$0 \to M \to X^0 \to X^1 \to X^2 \to \cdots$$

with each $X_i \in \mathcal{X}$.

5. A *projective resolution of M* is a left resolution $P^\bullet$ for which the objects P^i are all projective (Definition 4.1.1).

6. An *injective resolution of M* is a right resolution $I^\bullet$ for which the objects I^i are all injective (Definition 4.1.1).

Lemma 4.2.5. Let $\mathcal{A}$ be an abelian category (Definition 2.0.10).

1. A projective object (Definition 4.1.1) $\mathcal{P}$ always has a projective resolution (Definition 4.2.4) given by

$$\cdots \to 0 \to \mathcal{P} \xrightarrow{\mathrm{id}} \mathcal{P} \to 0.$$

2. A injective object (Definition 4.1.1) $\mathcal{I}$ always has a injective resolution (Definition 4.2.4) given by

$$0 \to \mathcal{I} \xrightarrow{\mathrm{id}} \mathcal{I} \to 0 \to \cdots.$$

Proof. This is clear. $\square$

Lemma 4.2.6 (cf. [Wei94, Lemma 2.2.5, Lemma 2.3.6]). Let $\mathcal{A}$ be an abelian category (Definition 2.0.10) and let $\mathcal{X}$ be a class of objects in $\mathcal{A}$.

1. If $\mathcal{A}$ has enough objects of class $\mathcal{X}$ on the right (Definition 4.2.1), then for every object $A \in \mathcal{A}$ there exists an $\mathcal{X}$-right resolution of A (Definition 4.2.4).

2. If $\mathcal{A}$ has enough objects of class $\mathcal{X}$ on the left (Definition 4.2.1), then for every object $A \in \mathcal{A}$ there exists an $\mathcal{X}$-left resolution of A (Definition 4.2.4).

Note that this is a special case of Proposition 4.2.7 obtained by letting the complex $M^\bullet$ be the complex such that

$$M^i = \begin{cases} A & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}.$$

In particular,

- If $\mathcal{A}$ has enough injective objects (Definition 4.2.2), then for every object $A \in \mathcal{A}$ there exists an injective resolution of A (Definition 4.2.4).
- If $\mathcal{A}$ has enough projective objects (Definition 4.2.2), then for every object $A \in \mathcal{A}$ there exists a projective resolution of A (Definition 4.2.4).
- If $F : \mathcal{A} \to \mathcal{B}$ is a left (resp. right) exact functor (Definition 2.0.17) between abelian categories and $\mathcal{A}$ has enough F-acyclic objects on the right (resp. left), then for every object $A \in \mathcal{A}$, there exists an right (resp. left) F-acyclic resolution (Definition 4.6.2) of A.

Proof. 1. Let $A \in \mathcal{A}$ be an object. Since $\mathcal{A}$ has enough objects of class $\mathcal{X}$ of the right, there is an object X_0 of $\mathcal{X}$ and a monomorphism $\varepsilon_0 : A \to X_0$. Let $A_0 = \operatorname{coker} \varepsilon_0$ (Definition 1.2.2). Inductively, given an object A_{n-1} of $\mathcal{A}$, choose an object X_n of $\mathcal{X}$ and a monomorphism $\varepsilon_n : A_{n-1} \hookrightarrow X_n$. Let $A_n = \operatorname{coker} \varepsilon_n$. In particular, there is a surjection $X_n \twoheadrightarrow A_n$. Let d_n be the composition

$$X_{n-1} \twoheadrightarrow A_{n-1} \xrightarrow{\varepsilon_n} X_n.$$

The chain complex

$$0 \to A \xrightarrow{\varepsilon_0} X_0 \xrightarrow{d_0} X_1 \xrightarrow{d_1} \cdots$$

is thus an $\mathcal{X}$-right resolution of A.
2. This is simply the dual statement of the next statement.

$\square$

In fact, a generalization is possible: if the abelian category $\mathcal{A}$ has enough objects in a class $\mathcal{X}$ (on the right/left) *complex*, then the complex has a "resolution" by objects of $\mathcal{X}$.

Proposition 4.2.7. Let $\mathcal{A}$ be an abelian category (Definition 2.0.10) and let $\mathcal{X}$ be a class of objects in $\mathcal{A}$.

1. If $\mathcal{A}$ has enough objects of class $\mathcal{X}$ on the right (Definition 4.2.1), then for every bounded below (Definition 3.2.5) complex $M^\bullet$ of objects in $\mathcal{A}$, there exists a bounded below complex $I^\bullet$ of objects in $\mathcal{X}$ and a quasi-isomorphism (Definition 3.2.6) $M^\bullet \to I^\bullet$.
2. If $\mathcal{A}$ has enough objects of class $\mathcal{X}$ on the left (Definition 4.2.1), then for every bounded above (Definition 3.2.5) complex $M^\bullet$ of objects in $\mathcal{A}$, there exists a bounded above complex $P^\bullet$ of objects in $\mathcal{X}$ and a quasi-isomorphism (Definition 3.2.6) $P^\bullet \to M^\bullet$.

Proof. We prove that if $\mathcal{A}$ has enough objects of class $\mathcal{X}$ on the left, then there exists a complex $P^\bullet$ of objects in $\mathcal{X}$ and a quasi-isomorphism $P^\bullet \to M^\bullet$. The other statement can be proven basically symmetrically.

First suppose that $M^\bullet$ is bounded above (Definition 3.2.5); say that $M^i = 0$ for all $i > n$. We inductively construct $P^\bullet$ and the quasi-isomoprhism to $M^\bullet$. Choose an object P^n from $\mathcal{X}$ and a surjective morphism $\epsilon_n : P^n \twoheadrightarrow M^n$, and let $d : P^n \to P^{n+1}$ be the zero map. Assume inductively that we have constructed the complex $P^\bullet$ and maps $\epsilon_i : P^i \to M^i$ for $i = k + 1, k + 2, \ldots, n$. We want to construct P^k, the differential $d : P^k \to P^{k+1}$ and the map $\epsilon_k : P^k \to M^k$.

Let $L_k = Z^{k+1}(P) \times_{Z^{k+1}(M)} M^k$

$$
\begin{array}{ccc}
L_k & \longrightarrow & Z^{k+1}(P) \\
\downarrow & & \downarrow{\scriptstyle \epsilon_{k+1}} \\
M^k & \xrightarrow{\ d\ } & Z^{k+1}(M)
\end{array}
$$

where Z^i denotes the ith cycle (Definition 3.2.1) of a complex (Definition 3.2.2); recall that abelian categories have finite limits by Lemma 2.0.14, so fiber products (Definition C.0.21) exist. Choose an object P^k from $\mathcal{X}$ and a surjective moprhism $\pi : P^k \twoheadrightarrow L_k$. Set the differential $d : P^k \to P^{k+1}$ to be $\mathrm{proj}_{Z^{k+1}(P)} \circ \pi$ and the map $\epsilon_k : P^k \to M^k$ to be $\mathrm{proj}_{M^k} \circ \pi$.

We verify that the square

$$
\begin{array}{ccc}
P^k & \xrightarrow{\ d\ } & P^{k+1} \\
\downarrow{\scriptstyle \epsilon_k} & & \downarrow{\scriptstyle \epsilon_{k+1}} \\
M^k & \xrightarrow{\ d\ } & M^{k+1}
\end{array}
$$

commutes, i.e. that $\epsilon_{k+1} \circ d = d \circ \varepsilon_k$. The left is $\epsilon_{k+1} \circ \mathrm{proj}_{Z^{k+1}(P)} \circ \pi$ and the right is $d \circ \mathrm{proj}_{M^k} \circ \pi$, which do indeed coincide.

We verify that the chain map ϵ induces isomorphisms $H^*(P) \xrightarrow{\sim} H^*(M)$ on cohomology objects. We first show that the induced maps on cohomology are epimorphisms. Let $u : Z^k(M) \to L_k = Z^{k+1}(P) \times_{Z^{k+1}(M)} M^k$ be the unique morphism corresponding to the morphisms $0 : Z^k(M) \to Z^{k+1}(P)$ and $Z^k(M) \hookrightarrow M^k$. Let Y be the pullback of π along u:

$$
\begin{array}{ccc}
Y & \xrightarrow{\ \tilde{\pi}\ } & Z^k(M) \\
\downarrow{\scriptstyle \tilde{u}} & & \downarrow{\scriptstyle u} \\
P^k & \xrightarrow{\ \pi\ } & L_k.
\end{array}
$$

Note that $\mathrm{im}\,\tilde{u}$ is a subobject of $Z^k(P) = \ker(d : P^k \to P^{k+1})$ because

$$
d \circ \tilde{u} = \mathrm{proj}_{Z^{k+1}(P)} \circ \pi \circ \tilde{u} = \mathrm{proj}_{Z^{k+1}(P)} \circ u \circ \tilde{\pi} = 0.
$$

Therefore, $\tilde{u}$ factors through $Z^k(P)$. Writing $[\tilde{u}]$ for the composition $Y \xrightarrow{\tilde{u}} Z^k(P) \twoheadrightarrow H^k(P)$, note that

$$
H^k(\epsilon) \circ [\tilde{u}] = [\epsilon_k \circ \tilde{u}] = [\mathrm{proj}_{M^k} \circ \pi \circ \tilde{u}] = [\mathrm{proj}_{M^k} \circ u \circ \tilde{\pi}] = [(\mathrm{id} : Z^k(M) \to Z^k(M)) \circ \tilde{\pi}].
$$

The right most expression is the composition

$$
Y \xrightarrow{\tilde{\pi}} Z^k(M) \twoheadrightarrow H^k(M).
$$

Since π is an epimorphism, $\tilde{\pi}$ is an epimorphism, so the above composition is an epimorphism. We have thus shown that $H^k(\epsilon) \circ [\tilde{u}]$ is an epimorphism, so $H^k(\epsilon)$ is an epimorphism.

We now show that $H^{k+1}(\epsilon) : H^{k+1}(P) \to H^{k+1}(M)$ is a monomorphism. Let K be the kernel of $Z^{k+1}(P) \xrightarrow{\epsilon_{k+1}} Z^{k+1}(M) \twoheadrightarrow H^{k+1}(M)$; this kernel coincides with "the (k+1)-cycles of P mapping to (k+1)-boundaries of M". More precisely, K can be regarded as the fiber product

$$\begin{array}{ccc} K & \longrightarrow & Z^{k+1}(P) \\ \downarrow & & \downarrow{\scriptstyle \epsilon_{k+1}} \\ B^{k+1}(M) & \lhook\joinrel\longrightarrow & Z^{k+1}(M), \end{array}$$

and note that this Cartesian diagram displays K as a subobject of $Z^{k+1}(P)$. Furthe note that the morphism $d : M^k \to B^{k+1}(M)$ naturally induces a morphism $L_k \to K$; in fact, K is then the image of the projection map $\mathrm{proj}_{Z^{k+1}(P)} : L_k \to Z^{k+1}(P)$. On the other hand, by definition,

$$B^{k+1}(P) = \mathrm{im}(d : P^k \to Z^{k+1}(P)) = \mathrm{im}(\mathrm{proj}_{Z^{k+1}(P)} \circ \pi).$$

Since π is an epimorphism, this image in turn equals $\mathrm{im}(\mathrm{proj}_{Z^{k+1}(P)})$, which equals K as we have seen. Therefore, K coincides with $B^{k+1}(P)$, which means that the map $Z^{k+1}(P) \to H^{k+1}(M)$, whose kernel is K by definition, naturally induces a monomorphism $H^{k+1}(P) \to H^{k+1}(M)$ as desired.

$\square$

Lemma 4.2.8 (cf. [Wei94, Porism 2.2.7]). Let $\mathcal{A}$ be an abelian category (Definition 2.0.10).

1. Let
$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$
be a chain complex (Definition 3.1.1) with P_i projective (Definition 4.1.1). For every left resolution (Definition 4.2.4) $Q_\bullet \to N$ of an object N, every map $M \to N$ lifts to a complex map (Definition 3.1.1) $P_\bullet \to Q_\bullet$ unique up to chain homotopy (Definition 3.3.1).

2. Let
$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$$
be a (co)chain complex (Definition 3.1.1) with I^i injective (Definition 4.1.1). For every right resolution (Definition 4.2.4) $N \to Q^\bullet$ of an object N, every map $N \to M$ lifts to a complex map (Definition 3.1.1) $Q^\bullet \to I^\bullet$ unique up to chain homotopy (Definition 3.3.1).

Proof. 1. The map $P_0 \to M \to N$ lifts to a map $P_0 \to Q_0$ because P_0 is projective and $Q_0 \to N$ is an epimorphism. Inductively suppose that there are morphisms $P_i \to Q_i$ for $0 \le i \le n$, where $n \ge 0$ that make

$$\begin{array}{ccccccccc} P_n & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ Q_n & \longrightarrow & Q_{n-1} & \longrightarrow & \cdots & \longrightarrow & Q_0 & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

into a commuting diagram are established. The morphism $Q_n \to Q_{n-1}$ (where we let $Q_{-1} = N$ and $P_{-1} = M$ here in case that $n = 0$) acts as 0 when restricted to $\mathfrak{I} := \mathrm{im}(P_{n+1} \to P_n \to Q_n)$ (Definition 1.2.5) because the composition

$$P_{n+1} \to P_n \to Q_n \to Q_{n-1}$$

equals the composition

$$P_{n+1} \to P_n \to P_{n-1} \to Q_{n-1}.$$

In other words, $\mathfrak{I}$ is a subobject (Definition 1.2.3) of $\ker(Q_n \to Q_{n-1})$ (Definition 1.2.2), which is isomorphic to $\operatorname{im}(Q_{n+1} \to Q_n)$ by the acyclicity of the sequence of the Q_i's. Therefore, we have a map $P_{n+1} \twoheadrightarrow \mathfrak{I} \hookrightarrow \operatorname{im}(Q_{n+1} \to Q_n)$ along with an epimorphism $Q_{n+1} \twoheadrightarrow \operatorname{im}(Q_{n+1} \to Q_n)$. Since P_{n+1} is projective, the former map lifts to a map $P_{n+1} \to Q_{n+1}$ in a way that is compatible with the latter, i.e. the following commutes:

$$
\begin{array}{c}
P_{n+1} \\
\Big\downarrow \qquad \searrow \\
Q_{n+1} \longrightarrow \operatorname{im}(Q_{n+1} \to Q_n).
\end{array}
$$

By induction, this shows thta $M \to N$ lifts to a morphism $P_\bullet \to Q_\bullet$ of complexes.

We show that the morphism of complexes is unique up to chain homotopy, i.e. if $f_1, f_2 : P_\bullet \to Q_\bullet$ are two morphisms of complexes, then $h := f_1 - f_2$ is null homotopic. We construct a chain contraction (Definition 3.3.1) $\{s_n : P_n \to Q_{n+1}\}$ of h by induction on n. If $n < 0$, then set $s_n = 0$. If $n = 0$, note that the composition $P_0 \xrightarrow{h_0} Q_0 \to N$ equals the composition $P_0 \to M \xrightarrow{0} N$, so $\operatorname{im}(h_0)$ is a subobject of $\ker(Q_0 \to N) \cong \operatorname{im}(Q_1 \to Q_0)$. The projectivitiy of P_0 thus yields a lift $s_0 : P_0 \to Q_1$ such that h_0 equals the composition $P_0 \xrightarrow{s_0} X_1 \xrightarrow{d} Q_0$:

$$
\begin{array}{c}
P_0 \\
{\scriptstyle s_0}\swarrow \quad \Big\downarrow {\scriptstyle h_0} \\
Q_1 \xrightarrow{\ d\ } Q_0
\end{array}
$$

Note moreover that $h_0 = ds_0 + s_{-1}d$ because $s_{-1} = 0$. Inductively suppose that we have maps s_i for $i \le n$ such that $h_n = ds_n + s_{n-1}d$ or equivalently that $ds_n = h_n - s_{n-1}d$. Consider the map $h_{n+1} - s_n d : P_{n+1} \to Q_{n+1}$. Compute

$$d(h_{n+1} - s_n d) = dh_{n+1} - ds_n d = dh_{n+1} - (h_n - s_{n-1}d)d = (dh_{n+1} - h_n d) + s_{n-1}dd = 0$$

Therefore, $\operatorname{im}(h_{n+1} - s_n d)$ is a subobject of $\ker(Q_{n+1} \to Q_n) \cong \operatorname{im}(Q_{n+2} \to Q_{n+1})$, which is in turn a quotient of Q_{n+2}. Since P_{n+1} is projective, there is a morphism $s_{n+1} : P_{n+1} \to Q_{n+2}$ such that $ds_{n+1} = h_{n+1} - s_n d$.

$$
\begin{array}{c}
\qquad\qquad P_{n+1} \\
{\scriptstyle s_{n+1}}\nearrow \qquad \Big\downarrow {\scriptstyle h_n - s_{n-1}d = ds_n} \\
Q_{n+2} \xrightarrow{\ d\ } \operatorname{im}(Q_{n+2} \to Q_{n+1}) \cong \ker(Q_{n+1} \to Q_n)
\end{array}
$$

The s_n thus form a chain contraction as needed.

2. This is simply dual to the previous part.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

4.3. Derived functors of right or left exact functors between abelian categories where the source category has enough projectives or injectives.

Definition 4.3.1. (♠ TODO: I think that the definition of derived categories might be doable for more general kinds of resolutions? Perhaps it is that if I have a right exact functor F, then $L^i F$ can be computed with resolutions of F-acyclic objects? (Definition 4.6.1)) (♠ TODO: Apparently, left/right derived functors may be defined for functors that are additive and preserve finite coproducts, and not necessarily right/left exact; the exactness condition ensures that the zeroth derived functor agrees with F.) Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories (Definition 2.0.10), and let

$$F : \mathcal{A} \to \mathcal{B}$$

be an additive functor (Definition 2.0.8).

1. Suppose that the functor F is right exact (Definition 2.0.17) and suppose that $A \in \mathcal{A}$ is an object for which a projective resolution (Definition 4.2.4)

$$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0$$

exists in $\mathcal{A}$. We define the *left derived object* $L_n F A \in \mathcal{B}$ by applying F to obtain a complex

$$\cdots \to F(P_2) \to F(P_1) \to F(P_0) \to 0$$

and letting $L_n F(A)$ be the n-th homology object (Definition 3.2.2) of this complex in $\mathcal{B}$:

$$L_n F(A) := H_n(F(P_\bullet)).$$

The object $L_n F(A)$ is independent of the choice of projective resolution up to natural isomorphism (Proposition 4.3.3).

By convention, set $L_n F = 0$ for $n < 0$.

The *higher left derived objects* refer to the object $L_n F(A)$ for $n > 0$.

2. Suppose that the functor F is right exact (Definition 2.0.17) and that $\mathcal{A}$ has enough projectives (Definition 4.2.2). The *left derived functors* refer to the family of functors

$$L_n F : \mathcal{A} \to \mathcal{B}, \quad A \mapsto L_n F(A).$$

The *higher left derived functors* refer to the functors $L_n F$ for $n > 0$.

3. Suppose that the functor F is right exact (Definition 2.0.17) and suppose that $A \in \mathcal{A}$ is an object for which a injective resolution (Definition 4.2.4)

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$$

exists in $\mathcal{A}$. We define the *right derived object* $R_n F A \in \mathcal{B}$, also often denoted by $R^n F A$, by applying F to obtain a complex

$$0 \to F(I^0) \to F(I^1) \to F(I^2) \to \cdots .$$

and letting $R_n F(A)$ be the n-th cohomology object (Definition 3.2.2) of this complex in $\mathcal{B}$:

$$R_n F(A) := H^n(F(I_\bullet)).$$

The object $R_n F(A)$ is independent of the choice of injective resolution up to natural isomorphism (Proposition 4.3.3).

By convention, set $R_n F = 0$ for $n < 0$.

The *higher right derived objects* refer to the object $R_n F(A)$ for $n > 0$.

4. Suppose that the functor F is right exact (Definition 2.0.17) and that $\mathcal{A}$ has enough injectives (Definition 4.2.2). The *right derived functors* refer to the family of functors

$$R_n F : \mathcal{A} \to \mathcal{B}, \quad A \mapsto R_n F(A).$$

The right derived functors are also often denoted by $R^n F$. The *higher right derived functors* refer to the functors $R_n F$ for $n > 0$.

Lemma 4.3.2 (Horseshoe lemma, cf. [Wei94, Horsehoe Lemma 2.2.8]). Let $\mathcal{A}$ be an abelian category (Definition 2.0.10).

1. Suppose that

$$0 \to A' \xrightarrow{i_A} A \xrightarrow{\pi_A} A'' \to 0$$

is a short exact sequence in $\mathcal{A}$, and that $\varepsilon' : P'_\bullet \to A'$ and $\varepsilon'' : P''_\bullet \to A''$ are respectively projective resolutions (Lemma 4.10.5).

Let $P_\bullet = P'_\bullet \oplus P''_\bullet$. The complex $P_\bullet$ is a projective resolution of A, and the short exact sequence lifts to an exact esquence of complexes

$$0 \to P' \xrightarrow{i} P \xrightarrow{\pi} P'' \to 0$$

where $i_n : P'_n \to P_n$ and $\pi_n : P_n \to P''_n$ are the natural inclusion and projection respectively.
2. Suppose that

$$0 \to A' \xrightarrow{i_A} A \xrightarrow{\pi_A} A'' \to 0$$

is a short exact sequence in $\mathcal{A}$, and that $\eta' : A' \to I'^\bullet$ and $\eta'' : A'' \to I''^\bullet$ are respectively injective resolutions.

$$
\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & A' & \xrightarrow{\eta'} & I'^0 & \longrightarrow & I'^1 & \longrightarrow & I'^2 \cdots \\
 & & \downarrow{\scriptstyle i_A} & & & & & & \\
 & & A & & & & & & \\
 & & \downarrow{\scriptstyle \pi_A} & & & & & & \\
0 & \longrightarrow & A' & \xrightarrow{\eta''} & I''^0 & \longrightarrow & I''^1 & \longrightarrow & I''^2 \cdots \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

Let $I^\bullet = I'^\bullet \oplus I''^\bullet$. The complex $I^\bullet$ is an injective resolution of A, and the short exact sequence lifts to a short exact sequence of complexes

$$
0 \to I'^\bullet \xrightarrow{i} I^\bullet \xrightarrow{\pi} I''^\bullet \to 0,
$$

where $i^n : I'^n \to I^n$ and $\pi^n : I^n \to I''^n$ are the natural inclusion and projection at each degree n.

Proof. (♠ TODO:) □

Proposition 4.3.3 (cf.[Wei94, Lemma 2.4.1]). Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor (Definition 2.0.8) between abelian categories (Definition 2.0.10). Let A be an object of $\mathcal{A}$.

1. Suppose that F is right exact (Definition 2.0.17), and suppose that a projective resolution (Definition 4.2.4)

$$
\cdots \to P_2 \to P_1 \to P_0 \to A \to 0
$$

of A exists in $\mathcal{A}$. Let

$$
\cdots \to Q_2 \to Q_1 \to Q_0 \to A \to 0
$$

be any projective resolution of A in $\mathcal{A}$. For all n, there are natural isomorphisms

$$
H_n(F(P_\bullet)) \cong H_n(F(Q_\bullet)).
$$

In other words, the left derived objects $L_n F(A)$ (Definition 4.3.1) is well defined.

2. Suppose that F is left exact (Definition 2.0.17), and suppose that a injective resolution (Definition 4.2.4)

$$
0 \to A \to I^0 \to I^1 \to I^2 \to \cdots
$$

of A exists in $\mathcal{A}$. Let

$$
0 \to A \to Q^0 \to Q^1 \to Q^2 \to \cdots
$$

be any injective resolution of A in $\mathcal{A}$. For all n, there are natural isomorphisms

$$
H_n(F(I^\bullet)) \cong H_n(F(Q^\bullet)).
$$

In other words, the right derived objects $R_n F(A)$ (Definition 4.3.1) is well defined.

Proof. 1. By Lemma 4.2.8, there is a lift $f : P_\bullet \to Q_\bullet$ of the identity map $A \to A$ unique up to chain homotopy. There are then induced natural maps $H_n(F(f)) : H_n(F(P_\bullet)) \to H_n(F(Q_\bullet))$. There is also a lift $f' : Q_\bullet \to P_\bullet$ of the identity map $A \to A$ unique up to chain homotopy, and this also induces natural maps $H_n(F(f')) : H_n(F(Q_\bullet)) \to H_n(F(P_\bullet))$. The chain maps f and f' are in fact chain homotopy inverses (Definition 3.3.1) because Lemma 4.2.8 also implies that any lifts $P_\bullet \to P_\bullet$ and $Q_\bullet \to Q_\bullet$ of the identity map $A \to A$ are chain homotopic to the identity chain maps. Therefore, $H_n(F(f))$ and $H_n(F(f'))$ are inverses of each other as morphisms in $\mathcal{B}$. (♠ TODO: prove basic facts about the fucntoriality of homology/cohomology of chain complexes)

2. This is dual to the previous part.

□

4.4. **Homological and cohomological δ functors.** Derived functors (Definition 4.3.1) are the main example of δ-functors (Definition 4.4.1), which associate long exact sequences to short exact sequences.

Definition 4.4.1. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories (Definition 2.0.10).

1. A *homological δ functor from $\mathcal{A}$ to $\mathcal{B}$* is a pair $(T_n, \delta_n)_{n \geq 0}$ consisting of:
   - a sequence of additive functors (Definition 2.0.8) $T_n : \mathcal{A} \to \mathcal{B}$ for each integer $n \geq 0$, and
   - for every short exact sequence (Definition 2.0.19) $0 \to A' \xrightarrow{u} A \xrightarrow{v} A'' \to 0$ in $\mathcal{A}$, a *connecting morphism*

   $$\delta_n(A', A, A'') : T_n(A'') \to T_{n-1}(A')$$

   in $\mathcal{B}$ for each $n > 0$

   that make the induced sequence

   $$\cdots \to T_{n+1}(A'') \xrightarrow{\delta_{n+1}} T_n(A') \to T_n(A) \to T_n(A'') \xrightarrow{\delta_n} T_{n-1}(A') \to \cdots$$

   exact (Definition 3.2.3) in $\mathcal{B}$, and are natural in short exact sequences. That is, for any morphism of short exact sequences, the induced morphisms between these long exact sequences commute.

2. A *cohomological δ-functor from $\mathcal{A}$ to $\mathcal{B}$* is defined dually: it consists of additive functors (Definition 2.0.8) $T^n : \mathcal{A} \to \mathcal{B}$ for $n \geq 0$ and *connecting morphisms*

   $$\delta^n(A', A, A'') : T^n(A') \to T^{n+1}(A'')$$

   such that for each short exact sequence (Definition 2.0.19) $0 \to A' \to A \to A'' \to 0$ the resulting sequence

   $$0 \to T^0(A') \to T^0(A) \to T^0(A'') \xrightarrow{\delta^0} T^1(A') \to T^1(A) \to T^1(A'') \xrightarrow{\delta^1} \cdots$$

   is exact (Definition 3.2.3) and the construction is natural with respect to morphisms of short exact sequences.

3. Let (T_n, δ_n) and (S_n, ∂_n) be homological δ-functors from $\mathcal{A}$ to $\mathcal{B}$. A *morphism of (homological) δ-functors*

$$\eta : T_\bullet \to S_\bullet$$

is a family of natural transformations (Definition 1.0.4) $\eta_n : T_n \Rightarrow S_n$ for each $n \geq 0$ such that for every short exact sequence $0 \to A' \xrightarrow{u} A \xrightarrow{v} A'' \to 0$, the following diagram in $\mathcal{B}$ commutes for all $n > 0$:

$$
\begin{array}{ccc}
T_n(A'') & \xrightarrow{\delta_n} & T_{n-1}(A') \\
\downarrow \eta_n & & \downarrow \eta_{n-1} \\
S_n(A'') & \xrightarrow{\partial_n} & S_{n-1}(A')
\end{array}
$$

The dual notion (for cohomological δ-functors) is defined analogously, reversing the direction of the connecting morphisms.

Theorem 4.4.2 (cf. [Wei94, Theorem 2.4.6]). Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor (Definition 2.0.8) between abelian categories (Definition 2.0.10).

1. Suppose that F is right exact (Definition 2.0.17).
   (a) Let
   $$0 \to A' \to A \to A'' \to 0$$
   be a short exact sequence (Definition 2.0.19) in $\mathcal{A}$. Suppose that there exist projective resolutions (Definition 4.2.4) $P' \to A'$ and $P'' \to A''$. There exists a long exact sequence

$$\cdots \xrightarrow{\partial} L_i F(A') \to L_i F(A) \to L_i F(A'') \xrightarrow{\partial} L_{i-1} F(A') \to L_{i-1} F(A) \to L_{i-1} F(A'') \xrightarrow{\partial} \cdots$$

   of derived objects (Definition 4.3.1) and this long exact sequence is natural.
   (b) If $\mathcal{A}$ has enough projectives (Definition 4.2.2), then the derived functor $L_* F$ form a homological δ-functor (Definition 4.4.1).
2. Suppose that G is left exact (Definition 2.0.17).
   (a) Let
   $$0 \to A' \to A \to A'' \to 0$$
   be a short exact sequence (Definition 2.0.19) in $\mathcal{A}$. Suppose that there exist injective resolutions (Definition 4.2.4) $I' \to A'$, $I \to A$, and $I'' \to A''$. There exists a long exact sequence

$$\cdots \xrightarrow{\partial} R^i G(A') \to R^i G(A) \to R^i G(A'') \xrightarrow{\partial} R^{i+1} G(A') \to R^{i+1} G(A) \to R^{i+1} G(A'') \xrightarrow{\partial} \cdots$$

   of derived objects (Definition 4.3.1), and this long exact sequence is natural.
   (b) If $\mathcal{A}$ has enough injectives (Definition 4.2.2), then the derived functors $R^* G$ form a cohomological δ-functor (Definition 4.4.1).

Proof. 1. (a) By the Horseshoe lemma (Lemma 4.3.2), there is a projective resolution $P \to A$ fitting into a short exact sequence $0 \to P' \to P \to P'' \to 0$. Since the P''_n are projective, each sequence $0 \to P'_n \to P_n \to P''_n \to 0$ is split exact. (♠ TODO: show that SES's ending in projective objects are split). Since F is additive, each sequence

$$0 \to F(P'_n) \to F(P_n) \to F(P''_n) \to 0$$

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is split exact in $\mathcal{B}$ (♠ TODO: show why). Therefore,

$$0 \to F(P') \to F(P) \to F(P'') \to 0$$

is a short exact sequence of chain complex. Its associated long exact sequence in homology (Theorem 4.4.2) is the desired long exact sequence.

We now show that the long exact sequence is natural. (♠ TODO: continue)

$\square$

(♠ TODO: define universal delta functors) (♠ TODO: Put a staetment about how derived functors are universal delta functors)

4.5. Homological and cohomological dimension of an additive functor.

Definition 4.5.1 (Homological dimension). Let $\mathcal{A}$ be an abelian category (Definition 2.0.10) and let $M \in \mathcal{A}$ be an object.

1. Suppose that M has a projective resolution (Definition 4.2.4).
   The *homological/projective dimension of M*, denoted by notations such as $\mathrm{hdim}(M)$ and $\mathrm{hd}(M)$, is the smallest integer $n \geq 0$ such that there exists a projective resolution of M of length n, i.e., an exact sequence (Definition 3.2.3)

   $$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0,$$

   with each P_i projective (Definition 4.1.1).
   If no such finite n exists, we say $\mathrm{hdim}(M) = \infty$.
2. Suppose that M has an injective resolution (Definition 4.2.4).
   The *cohomological/injective dimension of M*, denoted by notations such as $\mathrm{cdim}(M)$ or $\mathrm{cd}(M)$, is the smallest integer $n \geq 0$ such that there exists a injective resolution of M of length n, i.e., an exact sequence (Definition 3.2.3)

   $$0 \to M \to I^1 \to \cdots \to I^{n-1} \to I^n \to 0,$$

   with each I^i injective (Definition 4.1.1).
   If no such finite n exists, we say $\mathrm{cdim}(M) = \infty$.

Definition 4.5.2 (Cohomological/homological dimension of a functor). Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor (Definition 2.0.8) between abelian categories (Definition 2.0.10).

1. Assume that F is left exact (Definition 2.0.17) and that $\mathcal{A}$ has enough projectives (Definition 4.2.2). The *cohomological dimension of F*, denoted $\mathrm{cdim}(F)$ or $\mathrm{cd}(F)$, is the smallest integer $n \geq 0$ such that the right derived functors $R^i F$ (Definition 4.3.1) vanish for all $i > n$, i.e.,

   $$R^i F = 0 \quad \text{for all } i > n.$$

   If no such finite n exists, then $\mathrm{cdim}(F) = \infty$.
2. Assume that F is right exact (Definition 2.0.17) and that $\mathcal{A}$ has enough injectives (Definition 4.2.2). The *homological dimension of F*, denoted $\mathrm{hdim}(F)$ or $\mathrm{hd}(F)$, is the

smallest integer $n \geq 0$ such that the left derived functors $L_i F$ (Definition 4.3.1) vanish for all $i > n$, i.e.,

$$L_i F = 0 \quad \text{for all } i > n.$$

If no such finite n exists, then $\text{hdim}(F) = \infty$.

4.6. F-acyclic objects.

Recall by definition that derived functors (Definition 4.3.1) are computed using projective/injective (Lemma 4.2.5) resolutions (Definition 4.2.4). In Proposition 4.6.5, we see that derived functors can be more generally computed with resolutions of F-acyclic (Definition 4.6.1).

Definition 4.6.1 (F-acyclic object for a functor of categories)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories (Definition 2.0.10).

1. Let
$$F : \mathcal{A} \to \mathcal{B}$$
be a right exact functor (Definition 2.0.17).

   An object $A \in \mathcal{A}$ for which a projective resolution (Definition 4.2.4) exists is called *F-acyclic* if for all integers $n > 0$, its higher left derived functors (Definition 4.3.1) vanish:
$$L_n F(A) = 0 \text{ for all } n > 0$$

2. Let
$$F : \mathcal{A} \to \mathcal{B}$$
be a left exact functor (Definition 2.0.17).

   An object $A \in \mathcal{A}$ for which a injective resolution (Definition 4.2.4) exists is called *F-acyclic* if for all integers $n > 0$, its higher right derived functors (Definition 4.3.1) vanish:
$$R_n F(A) = 0 \text{ for all } n > 0$$

Definition 4.6.2 (F-acyclic resolution for a right/left exact functor)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories (Definition 2.0.10) and let
$$F : \mathcal{A} \to \mathcal{B}$$
be an additive functor (Definition 2.0.8).

1. Suppose that F is a right exact functor (Definition 2.0.17). *An (left) F-acyclic resolution* of an object $A \in \mathcal{A}$ is a left resolution of A for (Definition 4.2.4) the class of F-acyclic objects (Definition 4.6.1).
2. Suppose that F is a left exact functor (Definition 2.0.17). *An (right) F-acyclic resolution* of an object $A \in \mathcal{A}$ is a right resolution of A for (Definition 4.2.4) the class of F-acyclic objects (Definition 4.6.1).

Lemma 4.6.3. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor (Definition 2.0.8) between abelian categories (Definition 2.0.10).

1. If F is right exact (Definition 2.0.17), then any projective object (Definition 4.1.1) $\mathcal{P}$ is F-acyclic (Definition 4.6.1).

2. If F is left exact (Definition 2.0.17), then any injective object (Definition 4.1.1) $\mathcal{I}$ is F-acyclic (Definition 4.6.1).

Proof. Note that any projective/injective object has a projective/injective resolution (Definition 4.2.4) (Lemma 4.2.5), so we may talk about the derived objects $L_n F(\mathcal{P})$ and $R^n F(\mathcal{I})$ (Definition 4.3.1) of projective objects $\mathcal{P}$ and injective objects $\mathcal{I}$ of $\mathcal{A}$. In fact, the projective/injective resolution is simply given by

$$\cdots \to 0 \to \mathcal{P} \xrightarrow{\mathrm{id}} \mathcal{P} \to 0$$

and

$$0 \to \mathcal{I} \xrightarrow{\mathrm{id}} \mathcal{I} \to 0 \to \cdots$$

so we compute $L_n F(\mathcal{P}) = 0$ and $R^n F(\mathcal{I}) = 0$. $\qquad \square$

Lemma 4.6.4 (Dimension shifting, cf. [Wei94, Exercise 2.4.3]). Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor (Definition 2.0.8) between abelian categories (Definition 2.0.10). Let A be an object of $\mathcal{A}$.

1. Suppose that F is right exact (Definition 2.0.17).
   (a) Suppose that $0 \to M \to C \to A \to 0$ is an exact sequence in $\mathcal{A}$ where C is F-acyclic, and that A and M have projective resolutions (Definition 4.2.4). We have $L_i F(A) \cong L_{i-1} F(M)$ for $i \geq 2$ and $L_1 F(A) \cong \ker(F(M) \to F(C))$.
   (b) Suppose that $\mathcal{A}$ has enough projectives (Definition 4.2.2). Let

   $$0 \to M_m \to C_m \to C_{m-1} \to \cdots \to C_0 \to A \to 0$$

   be an acyclic complex (Definition 3.2.3) with the C_i F-acyclic (Definition 4.6.1). We have
      (i) $L_i F(A) \cong L_{i-m-1} F(M_m)$ for $i \geq m+2$ and
      (ii) $L_{m+1} F(A) \cong \ker(F(M_m) \to F(C_m))$.
2. (♠ TODO: dual statement)

Proof.   1. Suppose that F is right exact
   (a) Since C is F-acyclic, the long exact sequence of derived functors associated to (Theorem 4.4.2) $0 \to M \to C \to A \to 0$ yields an exact sequence

   $$0 \to L_1 F(A) \to F(M) \to F(C) \to F(A) \to 0$$

   along with isomorphisms $L_i F(A) \cong L_{i-1} F(M)$ for $i \geq 2$. In particular, $L_1 F(A) = \ker(F(M) \to F(C))$.
   (b) We now proceed by induction. The above proves the base case of $m = 0$. Associated to the acyclic complex

   $$0 \to M_m \to C_m \to C_{m-1} \to \cdots \to C_0 \to A \to 0$$

   is a short exact sequence

   $$0 \to M_m \to C_m \to M_{m-1} \to 0$$

   where $M_{m-1} = \mathrm{coker}(M_m \to C_m)$ and an acyclic complex

   $$0 \to M_{m-1} \to C_{m-1} \to \cdots \to C_0 \to A \to 0.$$

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The derived functor long exact sequence of the short exact sequence yields isomorphisms
$$L_i F(M_{m-1}) \cong L_{i-1} F(M_m)$$
for all $i \geq 1$. By induction, we also have $L_i F(A) \cong L_{i-(m-1)-1} F(M_{m-1}) = L_{i-m} F(M_{m-1})$ for $i \geq m+1$, so we in fact have
$$L_i F(A) \cong L_{i-m} F(M_{m-1}) \cong L_{i-m-1} F(M_m)$$
for all $i \geq m+1$. Moreover, the derived functor long exact sequence also includes
$$0 \to L_1 F(M_{m-1}) \to F(M_m) \to F(C_m) \to F(M_{m-1}) \to 0,$$
so $L_1 F(M_{m-1}) \cong \ker(F(M_m) \to F(C_m))$. We found that $L_1 F(M_{m-1}) \cong L_{m+1} F(A)$, so $L_{m+1} F(A)$ is the kernel of $F(M_m) \to F(C_m)$ as desired.

2. These hold dually.

$\square$

Proposition 4.6.5 (cf.[Wei94, Exercise 2.4.3]). Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor (Definition 2.0.8) between abelian categories (Definition 2.0.10). Let A be an object of $\mathcal{A}$.

1. Suppose that F is right exact (Definition 2.0.17), and suppose that a projective resolution (Definition 4.2.4)
$$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0$$
of A exists in $\mathcal{A}$. Let
$$\cdots \to Q_2 \to Q_1 \to Q_0 \to A \to 0$$
be any F-acyclic (left) resolution (Definition 4.6.2) of A in $\mathcal{A}$. For all n, there are natural isomorphisms
$$H_n(F(P_\bullet)) \cong H_n(F(Q_\bullet)).$$
In particular, the left derived objects $L_n F(A)$ may be computed using any F-acyclic (left) resolution of A and are independent of the choice of left resolution. In particular, since all projective objects are F-acyclic (Lemma 4.6.3), $L_n F(A)$ is well defined.

2. Suppose that F is left exact (Definition 2.0.17), and suppose that a injective resolution (Definition 4.2.4)
$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$$
of A exists in $\mathcal{A}$. Let
$$0 \to A \to Q^0 \to Q^1 \to Q^2 \to \cdots$$
be any F-acyclic (right) resolution (Definition 4.6.2) of A in $\mathcal{A}$. For all n, there are natural isomorphisms
$$H_n(F(I^\bullet)) \cong H_n(F(Q^\bullet)).$$
In particular, the right derived objects $R_n F(A)$ may be computed using any F-acyclic (right) resolution of A and are independent of the choice of right resolution. In particular, since all injectives objects are F-acyclic (Lemma 4.6.3), $R_n F(A)$ is well defined.

Proof. (♠ TODO: There is a "dimension shifting" fact that goes into proving this kind of thing.) $\square$

4.7. Double complexes. Just as we define and compute derived functors (Definition 4.3.1) of (left/right exact (Definition 2.0.17)) single-variate additive functors (Definition 2.0.8) by (projective/injective/F-acyclic) resolutions (Definition 4.2.4) of objects, we can define and compute derived functors (Definition 4.8.4) of bi-additive (Definition 4.7.5) functors via flat resolutions (Definition 4.8.3). To show that such derived functors of bi-additive functors are well defined under mild conditions (see Theorem 4.8.6), it is useful (see Lemma 4.8.5) to use obtain (Definition 4.7.6) double complexes (Definition 4.7.1) from the biadditive functor.

Definition 4.7.1. Let $\mathcal{A}$ be an additive category (Definition 2.0.6). A *double complex* (also called a *bicomplex*) in $\mathcal{A}$ is a collection of objects

$$\{A^{p,q}\}_{(p,q)\in\mathbb{Z}^2}$$

together with morphisms

$$d_A'^{\,p,q} : A^{p,q} \to A^{p+1,q}, \quad d_A''^{\,p,q} : A^{p,q} \to A^{p,q+1},$$

satisfying the identities

$$(d_A')^{p+1,q} \circ (d_A')^{p,q} = 0, \qquad (d_A'')^{p,q+1} \circ (d_A'')^{p,q} = 0,$$

and the *anti-commutativity relation*

$$(d_A'')^{p+1,q} \circ (d_A')^{p,q} + (d_A')^{p,q+1} \circ (d_A'')^{p,q} = 0.$$

An alternative (equivalent) convention defines a double complex $(A^{p,q}, d_A', d_A'')$ so that

$$(d_A'')^{p+1,q} \circ (d_A')^{p,q} - (d_A')^{p,q+1} \circ (d_A'')^{p,q} = 0.$$

This convention differs by a sign and corresponds to replacing d_A'' by $-d_A''$. Thus, both conventions are equivalent up to the natural isomorphism $A^{p,q} \mapsto A^{p,q}$, $d_A' \mapsto d_A'$, $d_A'' \mapsto -d_A''$.

Definition 4.7.2. Let $\mathcal{A}$ be an additive category (Definition 2.0.6), and let

$$(A^{p,q}, d_A', d_A'')_{(p,q)\in\mathbb{Z}^2}, \quad (B^{p,q}, d_B', d_B'')_{(p,q)\in\mathbb{Z}^2}$$

be double complexes (Definition 4.7.1) in $\mathcal{A}$.

A *morphism of double complexes* $f : A \to B$ is a collection of morphisms

$$f^{p,q} : A^{p,q} \to B^{p,q},$$

for all $(p, q) \in \mathbb{Z}^2$, such that the following diagrams commute:

$$d_B'^{\,p,q} \circ f^{p,q} = f^{p+1,q} \circ d_A'^{\,p,q},$$
$$d_B''^{\,p,q} \circ f^{p,q} = f^{p,q+1} \circ d_A''^{\,p,q}.$$

In other words, f respects both horizontal and vertical differentials of the double complexes.

The double complexes and their morphisms form a category, sometimes denoted by $\mathbf{DC}(\mathcal{A})$.

Theorem 4.7.3. Let $\mathcal{A}$ be an additive category (Definition 2.0.6). Consider the category $\mathbf{DC}(\mathcal{A})$ (Definition 4.7.2) of double complexes (Definition 4.7.1) in $\mathcal{A}$ with morphisms (Definition 4.7.2), and the category $\mathbf{Ch}(\mathbf{Ch}(\mathcal{A}))$ of chain complexes (Definition 3.1.1) in the category of chain complexes over $\mathcal{A}$.

(♠ TODO: natural isomorphism of categories) There exists a natural isomorphism of categories
$$\mathbf{DC}(\mathcal{A}) \cong \mathbf{Ch}(\mathbf{Ch}(\mathcal{A})),$$
given by the sign trick: for a double complex $(A^{p,q}, d'_A, d''_A)$, define the associated chain complex of chain complexes by adjusting the vertical differentials as
$$d''^{p,q}_{\text{new}} := (-1)^p d''^{p,q}_A,$$
while keeping the horizontal differentials d'_A unchanged.

This identification respects morphisms of double complexes and chain complexes of chain complexes, making the two categories canonically equivalent.

Given two chain complexes and a biadditive functor of their underlying additive categories, we can obtain a double complex.

Lemma 4.7.4 (Product of Additive Categories is Additive). Let I be any set and let $\{\mathcal{A}_i\}_{i\in I}$ be a family of additive categories (Definition 2.0.6).

1. The product category (Definition 1.0.12) $\prod_{i\in I} \mathcal{A}_i$ is additive. Explicitly:
    - Each hom-set
    $$\operatorname{Hom}_{\prod_i \mathcal{A}_i}((A_i)_i, (B_i)_i) = \prod_{i\in I} \operatorname{Hom}_{\mathcal{A}_i}(A_i, B_i)$$

    is an abelian group under componentwise addition.
    - Composition is bilinear with respect to this group structure.
    - The family $(0_i)_{i\in I}$ of zero objects in each $\mathcal{A}_i$ is a zero object in $\prod_i \mathcal{A}_i$.
    - Finite direct sums exist and are given componentwise.
2. If each $\mathcal{A}_i$ is abelian (Definition 2.0.10), then so is $\prod_{i\in I} \mathcal{A}_i$. (♠ TODO: If each $\mathcal{A}_i$ satisfies Ab1-Ab5, does the product also?) Explicitly,
    - Kernels and cokernels exist and are computed componentwise:
    $$\ker((f_i)_i) = (\ker(f_i))_i, \quad \operatorname{coker}((f_i)_i) = (\operatorname{coker}(f_i))_i.$$

    - Every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel.

Definition 4.7.5 (n-ary Additive Functor). Let I be a finite set with $|I| = n$. Let $\{\mathcal{A}_i\}_{i\in I}$ be additive categories (Definition 2.0.6) and let $\mathcal{B}$ be an additive category. An *n-ary additive functor* (or *multilinear functor*)
$$F : \prod_{i\in I} \mathcal{A}_i \to \mathcal{B}$$
(Definition 1.0.12) is a functor such that for each fixed collection of all but one variable, the resulting functor in the remaining variable is additive (Definition 2.0.8). Equivalently, for every $j \in I$ and objects $(A_i)_{i\in I}$ and morphisms $f_1, f_2 : A_j \to A'_j$ in $\mathcal{A}_j$, we have
$$F(A_1, \ldots, A_{j-1}, f_1 + f_2, A_{j+1}, \ldots, A_n)$$
$$= F(A_1, \ldots, A_{j-1}, f_1, A_{j+1}, \ldots, A_n)$$
$$+ F(A_1, \ldots, A_{j-1}, f_2, A_{j+1}, \ldots, A_n),$$

and F preserves zero morphisms componentwise:

$$F(A_1, \ldots, 0_{A_j, A_j'}, \ldots, A_n) = 0_{F(A_1, \ldots), F(A_1', \ldots)}.$$

A bifunctor that satisfies this property for $n = 2$ is simply called a *biadditive functor*.

Definition 4.7.6 (Double Complex associated to biadditive functor and chain complexes). Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be additive categories (Definition 2.0.6), and suppose that

$$F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$$

is a biadditive functor (Definition 4.7.5). Let $X_\bullet$ and $Y_\bullet$ be chain complexes (Definition 3.1.1) of objects in $\mathcal{A}$ and $\mathcal{B}$ respectively.

Construct the double complex (Definition 4.7.1) $Z_{\bullet,\bullet} = (Z_{n,m}, d_{n,m}^h, d_{n,m}^v)$ in C associated to F, $X_\bullet$, and $Y_\bullet$ as follows:

$$Z_{n,m} := F(X_n, Y_m),$$

with horizontal differentials

$$d_{n,m}^h := F(d_n^X, \mathrm{id}_{Y_m}) : Z_{n,m} \to Z_{n-1,m},$$

and vertical differentials

$$d_{n,m}^v := (-1)^n F(\mathrm{id}_{X_n}, d_m^Y) : Z_{n,m} \to Z_{n,m-1}.$$

These differentials indeed satisfy the double complex conditions:

$$d^h \circ d^h = 0, \quad d^v \circ d^v = 0, \quad \text{and} \quad d^h \circ d^v + d^v \circ d^h = 0.$$

We may often denote the double complex $Z_{\bullet,\bullet}$ by $F(X_\bullet, Y_\bullet)$. In particular, F induces a bifunctor

$$F : \mathbf{Ch}(\mathcal{A}) \times \mathbf{Ch}(\mathcal{B}) \to \mathbf{DC}(\mathcal{C})$$

(Definition 4.7.2) that is in fact a biadditive functor of additive categories (see Proposition 3.1.11) (♠ TODO: verify that we indeed get a biadditive functor)

In particular, we may speak of the total complexes (Definition 4.7.8) $\mathrm{Tot}^\oplus(F(X_\bullet, Y_\bullet))$ and $\mathrm{Tot}^\Pi(F(X_\bullet, Y_\bullet))$, and these specify biadditive functors

$$\mathrm{Tot}^\oplus(F(-,-)), \mathrm{Tot}^\Pi(F(-,-)) : \mathbf{Ch}(\mathcal{A}) \times \mathbf{Ch}(\mathcal{B}) \to \mathbf{Ch}(\mathcal{C}).$$

Definition 4.7.7. Let $(A^{p,q}, d_A', d_A'')$ be a double complex in an additive category $\mathcal{A}$.

- The double complex is called *bounded above* if there exist integers p_0, q_0 such that $A^{p,q} = 0$ whenever $p > p_0$ or $q > q_0$.
- The double complex is called *bounded below* if there exist integers p_0, q_0 such that $A^{p,q} = 0$ whenever $p < p_0$ or $q < q_0$.
- The double complex is called *bounded* if it is both bounded above and below.
- The double complex is said to be in the *first quadrant* (also called *first-quadrant double complex*) if $A^{p,q} = 0$ whenever $p < 0$ or $q < 0$. In particular, any first quadrant double complex is bounded below.
- The double complex is said to be in the *third quadrant* (also called *third-quadrant double complex*) if $A^{p,q} = 0$ whenever $p > 0$ or $q > 0$. In particular, any third quadrant double complex is bounded above.

- Let us say that the double complex is *locally finite along diagonals* or *locally bounded along diagonals*[2] if for each integer n, there exist at most finitely many pairs (p, q) with $p + q = n$ such thta $A^{p,q} \neq 0$.
- Let us say that the double complex is *bounded in total degree*[3] if there exist integers m and M such that $A^{p,q} = 0$ whenever $m \leq p + q \leq M$.

Definition 4.7.8. Let $(A^{p,q}, d'_A, d''_A)$ be a double complex (Definition 4.7.1) in an additive category (Definition 2.0.6) $\mathcal{A}$. For each integer n, define:

$$\mathrm{Tot}^n_\oplus(A) = \bigoplus_{p+q=n} A^{p,q},$$

$$\mathrm{Tot}^n_\Pi(A) = \prod_{p+q=n} A^{p,q}.$$

assuming that the direct sum $\bigoplus_{p+q=n} A^{p,q}$ and the product $\prod_{p+q=n} A^{p,q}$ respectively exist. These are called the *direct-sum total complex* and the *product total complex*, respectively.

Define the differentials categorically as follows:

- For the direct sum total complex, the differential $d^n : \mathrm{Tot}^n_\oplus(A) \to \mathrm{Tot}^{n+1}_\oplus(A)$ is the unique morphism such that, for each (p, q) with $p + q = n$, we have

$$d^n \circ \iota_{p,q} = \iota_{p+1,q} \circ d'_A{}^{p,q} + (-1)^p \iota_{p,q+1} \circ d''_A{}^{p,q},$$

where $\iota_{p,q} : A^{p,q} \to \mathrm{Tot}^n_\oplus(A)$ is the canonical inclusion.
- For the product total complex, the differential $d^n : \mathrm{Tot}^n_\Pi(A) \to \mathrm{Tot}^{n+1}_\Pi(A)$ is the unique morphism such that, for each (p, q) with $p + q = n + 1$, we have

$$\pi_{p,q} \circ d^n = d'_A{}^{p-1,q} \circ \pi_{p-1,q} + (-1)^{p-1} d''_A{}^{p,q-1} \circ \pi_{p,q-1},$$

where $\pi_{p,q} : \mathrm{Tot}^{n+1}_\Pi(A) \to A^{p,q}$ is the canonical projection.

Then $(\mathrm{Tot}^\bullet_\oplus(A), d)$ and $(\mathrm{Tot}^\bullet_\Pi(A), d)$ are chain complexes (Definition 3.1.1) in $\mathcal{A}$, whenever the corresponding sums or products exist. The complexes $\mathrm{Tot}^\bullet_\oplus(A)$ and $\mathrm{Tot}^\bullet_\Pi(A)$ are also denoted by $\mathrm{Tot}^\oplus(A)$ and $\mathrm{Tot}^\Pi(A)$.

Lemma 4.7.9. Let C be a double complex (Definition 4.7.1) of objects in an additive category (Definition 2.0.6) $\mathcal{A}$. If C is locally bounded along diagonals (Definition 4.7.7), then the complexes $\mathrm{Tot}^\oplus(A)$ and $\mathrm{Tot}^\Pi(A)$ (Definition 4.7.8) are naturally isomorphic.

Proof. Since C is locally bounded along diagonals, the degreee n components

$$\left(\mathrm{Tot}^\oplus\right)^n(A) = \bigoplus_{p+q=n} A^{p,q},$$

$$\left(\mathrm{Tot}^\Pi\right)^n(A) = \prod_{p+q=n} A^{p,q}.$$

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[2]These do not seem to be standard teminology.
[3]This does not seem to be standard teminology.

are finite direct sums and finite products respectively and hence are naturally isomorphic (Lemma 2.0.7). The differential maps of the two total complexes also naturally coincide. $\square$

Lemma 4.7.10 (cf. [Wei94, Acyclic Assembly Lemma 2.7.3]). (♠ TODO: It may be the case that this is generalizable beyond first quadrant double complexes, but I don't have a slick way to show this. See the commented out code for the statements; also, it may be necessary to assume something like AB4∗ for such statements) Let $\mathcal{A}$ be an abelian category (Definition 2.0.10) for which (small) filtered colimits (Definition 1.3.12) which exist are exact (e.g. which holds if $\mathcal{A}$ satisfies Ab5 (Definition 2.1.8)). Let C be a double complex (Definition 4.7.1) in $\mathcal{A}$.

If C has exact columns or has exact rows and C is a bounded below or bounded above double complex (Definition 4.7.7), then $\mathrm{Tot}^{\Pi}(C)$ is an acyclic chain complex (Definition 3.2.3).

Proof. We show that if C has exact columns and C is bounded below, then $\mathrm{Tot}^{\Pi}(C)$ is an acyclic chain complex (Definition 3.2.3); it can then be argued symmetrically that if C has exact columns and C is bounded above, then $\mathrm{Tot}^{\Pi}(C)$ is acyclic. Moreover, the case of exact rows can be deduced by reflecting the rows and columns of double complexes.

Note that since C is assumed to be bounded below and hence is locally bounded along diagonals (Definition 4.7.7), $\mathrm{Tot}^{\Pi}(C)$ and $\mathrm{Tot}^{\oplus}(C)$ exist, are constructed by finite products (which are also finite coproducts), and are naturally isomorphic by Lemma 4.7.9. Further recall that finite coproducts in an abelian category are exact.

Define the sub-double complexes $F^k C$ of C by

$$(F^k C)^{p,q} = \begin{cases} C^{p,q} & \text{if } p \leq k \\ 0 & \text{otherwise} \end{cases}.$$

This yields a filtration

$$\cdots \subseteq F^{k-1}C \subseteq F^k C \subseteq F^{k+1} \subseteq \cdots \subseteq C.$$

Moreover, for each n,

$$\left(\mathrm{Tot}^{\Pi}(F^k C)\right)^n = \prod_{p \leq k, p+q=n} C^{p,q}.$$

For each n, the above stabilizes as $k \to \infty$ to $\mathrm{Tot}^{\Pi}(C)^n$. Now let

$$D^k = F^k C / F^{k-1}C = \begin{cases} C^{p,q} & \text{if } p = k \\ 0 & \text{otherwise.} \end{cases}$$

Since each column $C^{k,*}$ is exact by assumption, the total complex $\mathrm{Tot}^{\Pi}(D^k)$ is acyclic. Note that we have short exact sequences

$$0 \to F^{k-1}C \to F^k C \to D^k \to 0$$

of double complexes. The totalization functor $\mathrm{Tot}^{\Pi}(-)$ in this case is exact because all of the double complexes are locally bounded along diagonals (Definition 4.7.7). We hence have a short exact sequence

$$0 \to \mathrm{Tot}^{\Pi}(F^{k-1}C) \to \mathrm{Tot}^{\Pi}(F^k C) \to \mathrm{Tot}^{\Pi}(D^k) \to 0.$$

Since $\mathrm{Tot}^{\Pi}(D^k)$ is acyclic (Definition 3.2.3), the long exact cohomology sequences (Theorem 4.4.2) yield isomorphisms

$$H^n(\mathrm{Tot}^{\Pi}(F^{k-1}C)) \cong H^n(\mathrm{Tot}^{\Pi}(F^k C)).$$

Since C is assumed to be bounded, the subcomplex $F^k C$ is zero for sufficiently negative k, in which case $\mathrm{Tot}^{\Pi}(F^k C)$ is acyclic. By induction on k, $\mathrm{Tot}^{\Pi}(F^k C)$ remains acyclic for all k. Moroever, the filtered colimit $\varinjlim_k \mathrm{Tot}^{\Pi}(F^k C)$ is $\mathrm{Tot}^{\Pi}(C)$. The assumed exactness of filtered colimits in $\mathcal{A}$ concludes that $\mathrm{Tot}^{\Pi}(C)$ is acyclic.

By symmetry, if C instead has exact rows, then $\mathrm{Tot}^{\Pi}(C)$ is an acyclic chain complex. $\square$

4.8. **Derived functors of biadditive functors.** Given a biadditive functor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ of abelian categories, we consider "derived" functors (in a broad sense) obtained by fixing an object A or B either of $\mathcal{A}$ or $\mathcal{B}$ respectively and applying $F(A, -)$ or $F(-, B)$ to a suitable resolution. Such derived functors are not a priori independent of the choice of resolution, so we state sufficient conditions for such derived functors to be well defined.

4.8.1. *Flat objects with respect to biadditive functors.*

Definition 4.8.1 (Flat object with respect to a tensor (monoidal) product)**.** Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories (Definition 2.0.10), and let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a biadditive functor (Definition 4.7.5) (in practice, the following definitions are usually considered when F is some kind of "tensor product" $\otimes$ and is right exact (Definition 2.0.17) in each variable).

1. An object $X \in \mathcal{A}$ is called *flat (with respect to F on the left)* if the functor $F(X, -) : \mathcal{B} \to \mathcal{C}$ if exact (Definition 2.0.17).
2. An object $Y \in \mathcal{B}$ is called *flat (with respect to F on the right)* if the functor $F(-, Y) : \mathcal{A} \to \mathcal{C}$ if exact (Definition 2.0.17).

If $\mathcal{A} = \mathcal{B} = \mathcal{C}$ and F makes $\mathcal{A}$ into a symmetric monoidal category (Definition C.0.24), then certainly $F(X, -)$ is exact if and only if $F(-, Y)$ is exact, i.e. flatness is equivalent on the two sides of $\otimes$.

Definition 4.8.2 (Having enough flat objects in an abelian category)**.** Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories (Definition 2.0.10), and let $\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a biadditive functor (Definition 4.7.5)

1. We say that *$\mathcal{A}$ has enough flat objects with respect to $\otimes$* if for every object $A \in \mathcal{A}$ there exists an epimorphism

$$F \twoheadrightarrow A$$

in $\mathcal{A}$ where F is flat with respect to $\otimes$ (Definition 4.8.1), i.e. $F \otimes - : \mathcal{B} \to \mathcal{C}$ is exact. Equivalently, $\mathcal{A}$ has enough flat objects with respect to $\otimes$ if it has enough objects of the class of objects of $\mathcal{A}$ with are flat with respect to $\otimes$ (Definition 4.2.1) on the left.
2. Similarly, we say that *$\mathcal{B}$ has enough flat objects with respect to $\otimes$* if for every $B \in \mathcal{B}$ there exists an epimorphism

$$G \twoheadrightarrow B$$

in $\mathcal{B}$ with G flat with respect to $\otimes$, i.e. $- \otimes G : \mathcal{A} \to \mathcal{C}$ is exact. Equivalently, $\mathcal{B}$ has enough flat objects with respect to $\otimes$ if it has enough objects of the class of objects of $\mathcal{B}$ with are flat with respect to $\otimes$ (Definition 4.2.1) on the left.

Definition 4.8.3. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories (Definition 2.0.10), and let $\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a bifunctor that is additive (Definition 2.0.8) and right exact (Definition 2.0.17) in each variable.

1. A *flat (left-)resolution of an object $F \in \mathcal{A}$* is a left resolution of F (Definition 4.2.4) consisting of flat objects (Definition 4.8.1) in $\mathcal{A}$ with respect to $\otimes$.
2. A *flat (left-)resolution of an object $G \in \mathcal{B}$* is a left resolution of G (Definition 4.2.4) consisting of flat objects (Definition 4.8.1) in $\mathcal{B}$ with respect to $\otimes$.

Note that we are reserving these default term "flat resolution" for "left resolutions with flat objects". We may still use *flat right-resolution* to be a right resolution (Definition 4.2.4) consisting of flat objects (Definition 4.8.1).

4.8.2. *Derived functors for biadditive functors computed via resolutions of flat objects.*

Definition 4.8.4 (Derived functors for biadditive functors of abelian categories via resolutions by flat objects). Let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a biadditive functor (Definition 4.7.5) of abelian categories (Definition 2.0.10).

1. Fix an object A of $\mathcal{A}$. Let B be an object of $\mathcal{B}$.
   (a) Let $L_n^{II} F(A, B) = L_n^{II}(F(A, -))(B)$ be an object of $\mathcal{C}$ obtained as follows: take a left resolution (Definition 4.2.4)
   $$P_\bullet \to B$$
   of B of objects P_i for which the functors $F(-, P_i) : \mathcal{A} \to \mathcal{C}$ are exact (Definition 2.0.17), i.e. $P_\bullet$ is a flat resolution of B (Definition 4.8.3). Let
   $$L_n^{II}(F(A, -))(B) := H_n(F(A, P_\bullet))$$
   (Definition 3.2.2).
   (b) Let $R_{II}^n F(A, B) = R_{II}^n(F(A, -))(B)$ be an object of $\mathcal{C}$ obtained as follows: take a right resolution (Definition 4.2.4)
   $$B \to I^\bullet$$
   of B of objects I^i for which the functors $F(-, I^i) : \mathcal{A} \to \mathcal{C}$ are exact (Definition 2.0.17), i.e. $I^\bullet$ is a flat right-resolution of B (Definition 4.8.3). Let
   $$R_{II}^n(F(A, -))(B) := H^n(F(A, I^\bullet))$$
   (Definition 3.2.2).
2. Fix an object B of $\mathcal{B}$. Let A be an object of $\mathcal{A}$.
   (a) Let $L_n^I F(A, B) = L_n^I(F(-, B))(A)$ be an object of $\mathcal{C}$ obtained as follows: take a left resolution (Definition 4.2.4)
   $$P_\bullet \to A$$

of A of objects P_i for which the functors $F(P_i, -) : \mathcal{B} \to \mathcal{C}$ are exact (Definition 2.0.17), i.e. $P_\bullet$ is a flat resolution (Definition 4.8.3) of A. Let

$$L_n^I(F(-, B))(A) := H_n(F(P_\bullet, B))$$

(Definition 3.2.2).

(b) Let $R_I^n F(A, B) = R_I^n(F(-, B))(A)$ be an object of $\mathcal{C}$ obtained as follows: take a right resolution (Definition 4.2.4)

$$A \to I^\bullet$$

of A of objects I^i for which the functors $F(I^i, -) : \mathcal{B} \to \mathcal{C}$ are exact (Definition 2.0.17), i.e. $I^\bullet$ is a flat right-resolution of A (Definition 4.8.3). Let

$$R_I^n(F(-, B))(A) := H^n(F(I^\bullet, B))$$

(Definition 3.2.2).

A priori, the objects $L_n^{II} F(A, B)$, $R_{II}^n F(A, B)$, $L_n^I F(A, B)$, and $R_I^n F(A, B)$ are not well defined, i.e. they may possibly depend on the choice of resolution. See Theorem 4.8.6, which asserts that the pair $L_n^{II} F(A, B)$ and $L_n^I F(A, B)$ (resp. $R_{II}^n F(A, B)$ and $R_I^n F(A, B)$) are in agreement and are well defined under mild conditions.

In case that F is a functor thought of as some kind of "tensor product" and denoted by $\otimes$ (and usually, but not necessarily, such that $\otimes$ is right exact in each variable), it is customary to denote $L_n^{II} F(A, B)$ and $L_n^I F(A, B)$ by $\mathrm{Tor}_n^{II}(A, B)$ and $\mathrm{Tor}_n^I(A, B)$ respectively and to call these objects *Tor objects*. In other words, Tor objects are obtained by taking flat (Definition 4.8.1) resolutions with respect to $\otimes$

In case that F is a functor thought of as some kind of "Hom" and denoted as (some variation of) Hom : $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathcal{C}$ (most usually, $\mathcal{A} = \mathcal{B}$ for such Hom functors), it is customary to denote $R_n^{II} F(A, B)$ and $R_n^I F(A, B)$ by $\mathrm{Ext}_n^{II}(A, B)$ and $\mathrm{Ext}_n^I(A, B)$ respectivley and to call these objects *Ext objects*.

Lemma 4.8.5. Let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a biadditive functor (Definition 4.7.5) of abelian categories (Definition 2.0.10). Assume that (small) filtered colimits which exist in $\mathcal{C}$ are exact (e.g. which holds if $\mathcal{C}$ satisfies Ab5 (Definition 2.1.8)).

Let $A \in \mathcal{A}$ and $B \in \mathcal{B}$ be objects.

1. Suppose that left resolutions (Definition 4.2.4) $P_{A,\bullet} \to A$ and $P_{B,\bullet} \to B$ exist such that $P_{A,i}$ and $P_{B,i}$ are flat (Definition 4.8.1) with respect to F on the left and right respectively, i.e. $F(P_{A,i}, -) : \mathcal{B} \to \mathcal{C}$ and $F(-, P_{B,i}) : \mathcal{A} \to \mathcal{C}$ are exact for all i.
   The complexes $F(P_{A,\bullet}, B)$ and $F(A, P_{B,\bullet})$ are quasi-isomorphic (Definition 3.2.6) to the complex $\mathrm{Tot}(F(P_{A,\bullet}, P_{B,\bullet}))$ (Definition 4.7.8) (Definition 4.7.6).
2. Suppose that right resolutions (Definition 4.2.4) $A \to I^{A,\bullet}$ and $B \to I^{B,\bullet}$ exist such that $I^{A,i}$ and $I^{B,i}$ are flat (Definition 4.8.1) with respect to F on the left and right respectively, $F(I^{A,i}, -) : \mathcal{B} \to \mathcal{C}$ and $F(-, I^{B,i}) : \mathcal{A} \to \mathcal{C}$ are exact for all i.
   The complexes $F(I^{A,\bullet}, B)$ and $F(A, I^{B,\bullet})$ are quasi-isomorphic (Definition 3.2.6) to the complex $\mathrm{Tot}(F(I^{A,\bullet}, I^{B,\bullet}))$ (Definition 4.7.8) (Definition 4.7.6).

Proof. We prove 1. The other part is the dual statement.

Choose resolutions $P_{A,\bullet} \xrightarrow{\varepsilon} A$ and $P_{B,\bullet} \xrightarrow{\eta} B$ such that $F(P_{A,i}, -) : \mathcal{B} \to \mathcal{C}$ and $F(-, P_{B,i}) : \mathcal{A} \to \mathcal{C}$ are exact for all i. Identifying A and B with complexes concentrated in degree 0, we can form (Definition 4.7.6) the three double complexes (Definition 4.7.1) $F(P_{A,\bullet}, P_{B,\bullet})$, $F(A, P_{B,\bullet})$, and $F(P_{A,\bullet}, B)$. Note that the augmentation morphisms ε and η induce morphisms $P_{A,\bullet} \otimes P_{B,\bullet} \to A \otimes P_{B,\bullet}, P_{A,\bullet} \otimes B$.

Let C be the double complex of objects in $\mathcal{C}$ obtained from $F(P_{A,\bullet}, P_{B,\bullet})$ by adding $F(A, P_{B,\bullet}[-1])$ in the column $p = -1$. One can show that the translate $\mathrm{Tot}(C)[1]$ is the mapping cone (Definition 3.5.1) of the map

$$\mathrm{Tot}(F(P_{A,\bullet}, P_{B,\bullet})) \xrightarrow{\varepsilon \otimes \mathrm{id}} \mathrm{Tot}(F(A, P_{B,\bullet})) = F(A, P_{B,\bullet}).$$

Moreover, since each $F(-, P_{B,i})$ is an exact functor, every row of C is exact, so $\mathrm{Tot}(C)$ is exact by Lemma 4.7.10. Therefore, $F(\varepsilon, \mathrm{id})$ is a quasi-isomorphism and hence

$$H_*(\mathrm{Tot}(F(P_{A,\bullet}, P_{B,\bullet}))) \xrightarrow{H_*(F(\varepsilon, P_{B,\bullet}))} H_*(F(A, P_{B,\bullet}))$$

is a natural isomorphism.

By symmetry, there is a natural isomorphism $H_*(\mathrm{Tot}(F(P_{A,\bullet} P_{B,\bullet}))) \to H_*(F(P_{A,\bullet}, B))$. $\square$

Theorem 4.8.6 (Balancing generalized derived functors of a biadditive functor of abelian categories computed via flat resolutions). Let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a biadditive functor (Definition 4.7.5) of abelian categories (Definition 2.0.10). Assume that (small) filtered colimits which exist in $\mathcal{C}$ are exact (e.g. which holds if $\mathcal{C}$ satisfies Ab5 (Definition 2.1.8)).

Let $A \in \mathcal{A}$ and $B \in \mathcal{B}$ be objects.

1. Suppose that left resolutions (Definition 4.2.4) $P_{A,\bullet} \to A$ and $P_{B,\bullet} \to B$ exist such that $F(P_{A,i}, -) : \mathcal{B} \to \mathcal{C}$ and $F(-, P_{B,i}) : \mathcal{A} \to \mathcal{C}$ are exact for all i, i.e. $P_{A,\bullet}$ and $P_{B,\bullet}$ are flat resolutions (Definition 4.8.3) of A and B respectively.
   (a) The objects $L_n^I F(A, B)$ (Definition 4.8.4) and $L_n^{II} F(A, B)$ are naturally isomorphic.
   (b) The objects $L_n^I F(A, B)$ and $L_n^{II} F(A, B)$ are well defined (up to natural isomorphism), i.e. do not depend on the choice of left resolutions of A and B respectively.
2. Suppose that right resolutions (Definition 4.2.4) $A \to I^{A,\bullet}$ and $B \to I^{B,\bullet}$ exist such that $F(I^{A,i}, -) : \mathcal{B} \to \mathcal{C}$ and $F(-, I^{B,i}) : \mathcal{A} \to \mathcal{C}$ are exact for all i.
   (a) The objects $R_I^n F(A, B)$ (Definition 4.8.4) and $R_{II}^n F(A, B)$ are naturally isomorphic.
   (b) The objects $R_I^n F(A, B)$ and $R_{II}^n F(A, B)$ are well defined (up to natural isomorphism), i.e. do not depend on the choice of left resolutions of A and B respectively.

Proof. We prove 1. The other part is the dual statement.

Choose resolutions $P_{A,\bullet} \xrightarrow{\varepsilon} A$ and $P_{B,\bullet} \xrightarrow{\eta} B$ such that $F(P_{A,i}, -) : \mathcal{B} \to \mathcal{C}$ and $F(-, P_{B,i}) : \mathcal{A} \to \mathcal{C}$ are exact for all i. As per Lemma 4.8.5, $F(\varepsilon, \mathrm{id})$ is a quasi-isomorphism and hence

$$H_*(\mathrm{Tot}(F(P_{A,\bullet}, P_{B,\bullet}))) \xrightarrow{H_*(F(\varepsilon, P_{B,\bullet}))} H_*(F(A, P_{B,\bullet}))$$

is a natural isomorphism. By symmetry, there is a natural isomorphism $H_*(\text{Tot}(F(P_{A,\bullet}P_{B,\bullet}))) \to$ $H_*(F(P_{A,\bullet}, B))$. Therefore, $L_n^I(A, B)$ and $L_n^{II}(A, B)$ are naturally isomorphic as claimed. In particular, $L_n^I(A, B)$ and $L_n^{II}(A, B)$ are independent of the choice of resolution of A and B respectively.

$\square$

4.9. **Homological and Cohomological amplitude.**

Definition 4.9.1 (Homological and cohomological amplitude of a functor). (♠ TODO:) Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and let $F \colon D(\mathcal{A}) \to D(\mathcal{B})$ be a triangulated functor between their derived categories. Suppose F is *bounded* on cohomological degrees of objects in $D(\mathcal{A})$ in the sense explained below.

We say that F has *homological amplitude contained in the interval* $[a, b] \subset \mathbb{Z}$ if for every complex $X^\bullet \in D(\mathcal{A})$ whose cohomology vanishes outside degrees $m \leq n$, the complex $F(X^\bullet)$ has vanishing cohomology outside degrees $m + a$ through $n + b$. Equivalently, F shifts the cohomology of any complex by at least a and at most b degrees.

Dually, F has *cohomological amplitude contained in the interval* $[c, d] \subset \mathbb{Z}$ if its right or left derived functor (whichever is defined in context) satisfies the analogous bounds on cohomological degrees.

More explicitly, for integers $a \leq b$, F has homological amplitude in $[a, b]$ if and only if for all $X^\bullet$ in $D(\mathcal{A})$,

$$F(X^\bullet) \in D^{[m+a,n+b]}(\mathcal{B}) \quad \text{whenever} \quad X^\bullet \in D^{[m,n]}(\mathcal{A}),$$

where $D^{[m,n]}(\mathcal{A})$ denotes complexes with cohomology supported between degrees m and n.

(♠ TODO: try defining flat dimensions via derived tensor products)

Definition 4.9.2. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories (Definition 2.0.10), and let $\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a biadditive functor (Definition 4.7.5) that is right exact (Definition 2.0.17) in each variable.

1. Suppose that $\mathcal{B}$ has enough projectives (Definition 4.2.2) or has enough flats (Definition 4.8.2).
   Let M be an object of $\mathcal{A}$. The *Tor dimension of M*, denoted by $\text{tdim}(M)$ or $\text{td}(M)$, is the smallest integer $n \geq 0$ such that
   $$\text{Tor}_i(M, -) = 0$$
   for all $i > n$, where $\text{Tor}_i(M, -) : \mathcal{B} \to \mathcal{C}$ (Definition 4.10.1) is computed via projective or flat resolutions of objects of $\mathcal{B}$; by Lemma 4.10.5, it does not matter whether a projective resolution or a flat resolution is used.
2. Symmetrically, suppose that $\mathcal{A}$ has enough projectives (Definition 4.2.2) or has enough flats (Definition 4.8.2).

Let N be an object of $\mathcal{B}$. The *Tor dimension of N*, denoted by $\mathrm{tdim}(N)$ or $\mathrm{td}(N)$, is the smallest integer $n \geq 0$ such that

$$\mathrm{Tor}_i(-, N) = 0$$

for all $i > n$, where $\mathrm{Tor}_i(-, N) : \mathcal{A} \to \mathcal{C}$ (Definition 4.10.1) is computed via projective or flat resolutions of objects of $\mathcal{A}$; by Lemma 4.10.5, it does not matter whether a projective resolution or a flat resolution is used.

4.10. **Tor/derived functor of a (right exact) biadditive functor via projective resolutions or flat resolutions.** Recall that the derived functor (Definition 4.8.4) of a biadditive functor (Definition 4.7.5) may be computed via flat resolutions (Definition 4.8.3); this seems to be a departure from the notion of a left derived functor (Definition 4.3.1) of a single-variate right exact (Definition 2.0.17) additive functor, which a priori uses projective (Definition 4.1.1) resolutions (Definition 4.2.4). The motivation for this is that certain common abelian categories, such as those of sheaves (Definition 6.1.7) of abelian groups or modules (Definition 1.1.2), may have enough flats (Definition 4.8.2) (with respect to the biadditive functor of interest, which is a right exact "tensor product" $\otimes$ in common contexts) but not enough projectives (Definition 4.2.2), so projective resolutions have not be available to compute the derived functors. Nevertheless, in the case that the abelian categories involved may both have enough flats and have enough projectives (e.g. when the abelian categories are the categories of abelian groups or modules over rings), then using either flat resolutions or projective resolutions to compute the derived functors of the bi-additive functor (which again is usually some kind of right exact "tensor product" $\otimes$) will yield the same result, see Corollary 4.10.6.

Definition 4.10.1 (General circumstances for the existence of Tor functors)**.** Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories (Definition 2.0.10), and let $\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a biadditive functor (Definition 4.7.5) that is right exact (Definition 2.0.17) in each variable.

(♠ TODO: Do notation for tor in two categories, one category, modules over rings)

For objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$, the *Tor objects*

$$\mathrm{Tor}_n(A, B) = \mathrm{Tor}_n^{\mathcal{A}, \mathcal{B}}(A, B)$$

may be defined in one of several not-necessarily-equivalent ways:

1. as the left derived functors (Definition 4.3.1) of $A \otimes -$ computed via projective resolutions (Definition 4.2.4) of B in $\mathcal{B}$, assuming that such a projective resolution exists.
2. as the left derived functors (Definition 4.3.1) of $- \otimes B$ computed via projective resolutions (Definition 4.2.4) of B in $\mathcal{A}$, assuming that such a projective resolution exists.
3. as the "left derived functors" of $A \otimes -$ computed via flat resolutions (Definition 4.8.3) of B in $\mathcal{B}$, assuming that such a flat resolution exists. More precisely, $\mathrm{Tor}_n(A, B) = H_n(A \otimes F_\bullet)$ in this case where $F_\bullet \to B$ is a flat resolution. Equivalently, $\mathrm{Tor}_n(A, B) = \mathrm{Tor}_n^{II}(A, B)$ as in Definition 4.8.4. A priori, this might depend on the choice of flat resolution.
4. as the "left derived functors" of $- \otimes B$ computed via flat resolutions (Definition 4.8.3) of A in $\mathcal{A}$. More precisely, $\mathrm{Tor}_n(A, B) = H_n(F_\bullet \otimes B)$ in this case where $F_\bullet \to A$ is a

flat resolution. Equivalently, $\mathrm{Tor}_n(A, B) = \mathrm{Tor}_n^I(A, B)$ as in Definition 4.8.4. A priori, this might depend on the choice of flat resolution.

See Theorem 4.10.2 and Lemma 4.10.5, which describe sufficient conditions under which the Tor functors defined via flat resolutions are independent of the choice of flat resolution. In particular Corollary 4.10.6 shows that all of the above notions are in natural agreement if $\mathcal{A}$ and $\mathcal{B}$ have enough projectives (Definition 4.2.2) and have enough flats (Definition 4.8.2) and filtered direct limits that exist in $\mathcal{C}$ are exact.

For any of the above notions of Tor_n, note that

1. for fixed $A \in \mathcal{A}$, if $\mathrm{Tor}_n(A, B)$ exists and is well defined for any $B \in \mathcal{B}$, then $\mathrm{Tor}_n(A, -)$ is an additive functor $\mathcal{B} \to \mathcal{C}$.
2. for fixed $B \in \mathcal{B}$, if $\mathrm{Tor}_n(A, B)$ exists and is well defined for any $A \in \mathcal{A}$, then $\mathrm{Tor}_n(A, -)$ is an additive functor $\mathcal{A} \to \mathcal{C}$.
3. if $\mathrm{Tor}_n(A, B)$ exists and is well defined for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then $\mathrm{Tor}_n(-, -)$ is an biadditive $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$.

In the case that $\mathcal{A}$ is the category of $R - S$-modules, $\mathcal{B}$ is the category of $S - T$-modules, $\mathcal{C}$ is the category of $R - T$-modules for some (not necessarily commutative) rings R, S, T (Definition C.0.2), and $\otimes$ is the usual tensor product between $R - S$-modules and $S - T$-modules producing $R - T$-modules, then the Tor functors may be denoted by

$$\boxed{\mathrm{Tor}_n^S(A, B)}$$

for $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Theorem 4.10.2 (Balancing of Tor, cf. [Wei94, Theorem 2.7.2]). Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories (Definition 2.0.10), and let $\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a biadditive functor (Definition 4.7.5) that is right exact (Definition 2.0.17) in each variable. Assume that (small) filtered colimits which exist in $\mathcal{C}$ are exact (e.g. which holds if $\mathcal{C}$ satisfies Ab5 (Definition 2.1.8)).

Given $A \in \mathcal{A}$ and $B \in \mathcal{B}$ for which flat resolutions exist (Definition 4.8.3), let $\mathrm{Tor}_n^I(A, B)$ and $\mathrm{Tor}_n^{II}(A, B)$ respectively be the Tor objects $\mathrm{Tor}_n^{\mathcal{A}}(A, B)$ (Definition 4.10.1) (Definition 4.8.4) computed via flat resolutions (Definition 4.8.3) of A in $\mathcal{A}$ and of B in $\mathcal{B}$.

1. $\mathrm{Tor}_n^I(A, B)$ and $\mathrm{Tor}_n^{II}(A, B)$ are naturally isomorphic.
2. $\mathrm{Tor}_n^I(A, B)$ and $\mathrm{Tor}_n^{II}(A, B)$ are independent of the choice of flat resolution of A and B respectively.

In particular, we may identify the objects $\mathrm{Tor}_n^I(A, B)$ and $\mathrm{Tor}_n^{II}(A, B)$ and simply write $\mathrm{Tor}_n(A, B)$ for either.

Proof. This follows from Theorem 4.8.6.

$\square$

Remark 4.10.3. Some common abelian categories do not have enough projectives (Definition 4.2.2), but nevertheless have enough flats (Definition 4.8.2) and hence Tor objects are

well defined as per Theorem 4.10.2. One such abelian category is the category of torsion abelian groups. Similarly, some categories of sheaves of torsion abelian groups on some sites may also have enought flats, but not enough projectives.

Lemma 4.10.4. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories (Definition 2.0.10), and let $\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a biadditive functor (Definition 4.7.5) that is right exact (Definition 2.0.17) in each variable.

1. Say that $\mathcal{B}$ has enough projectives (Definition 4.2.2) and let $\mathrm{Tor}_n(A, B)$ be the Tor object computed as the left derived functor (Definition 4.3.1) of $A \otimes -$ applied to B. If A is flat (Definition 4.8.1), then $\mathrm{Tor}_n(A, B) = 0$ for all $n \neq 0$ and all B.
2. Say that $\mathcal{A}$ has enough projectives (Definition 4.2.2) and let $\mathrm{Tor}_n(A, B)$ be the Tor object computed as the left derived functor of $- \otimes B$ applied to A. If B is flat (Definition 4.8.1), then $\mathrm{Tor}_n(A, B) = 0$ for all $n \neq 0$ and all A.
3. Say that $\mathcal{B}$ has enough flats (Definition 4.8.2) $F_\bullet \to B$ and let $\mathrm{Tor}_n(A, B)$ be the Tor object computed as $H_n(A \otimes F_\bullet)$. If A is flat (Definition 4.8.1), then $\mathrm{Tor}_n(A, B) = 0$ for all $n \neq 0$ and all B.
4. Say that $\mathcal{A}$ has enough flats (Definition 4.8.2) $F_\bullet \to A$ and let $\mathrm{Tor}_n(A, B)$ be the Tor object computed as $H_n(F_\bullet \otimes B)$. If B is flat (Definition 4.8.1), then $\mathrm{Tor}_n(A, B) = 0$ for all $n \neq 0$ and all A.

Proof. We prove the first part. The other parts hold similarly.

If A is flat, then for any $B \in \mathcal{B}$, letting $P_\bullet \to B$ be some projective resolution (Definition 4.2.4), the exactness of $A \otimes -$ implies that $\mathrm{Tor}_n(A, B) = H_n(A \otimes P_\bullet) = 0$ for all $n \neq 0$. $\square$

Lemma 4.10.5 (cf. [Wei94, Flat Resolution Lemma 3.2.8]). Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories (Definition 2.0.10), and let $\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a biadditive functor (Definition 4.7.5) that is right exact (Definition 2.0.17) in each variable. Let $A \in \mathcal{A}$ and $B \in \mathcal{B}$ be objects.

1. Suppose that $\mathcal{A}$ has enough projective objects (Definition 4.2.2). Suppose that A has some flat resolution (Definition 4.8.3). The Tor objects (Definition 4.10.1)
$$\mathrm{Tor}_n(A, B) \in \mathcal{C}$$
obtained via a projective resolution of A (Definition 4.2.4) and via a flat resolution of A (Definition 4.8.3) are naturally isomorphic. In particular, either notion of Tor_n is well defined.
2. Suppose that $\mathcal{B}$ has enough projective objects (Definition 4.2.2). Suppose that B has some flat resolution (Definition 4.8.3). The Tor objects
$$\mathrm{Tor}_n(A, B) \in \mathcal{C}$$
obtained via a projective resolution of B (Definition 4.2.4) and via a flat resolution of B (Definition 4.8.3) are naturally isomorphic. In particular, either notion of Tor_n is well defined.

Proof. We show 1. Write $\mathrm{Tor}_n(A, B)$ for the tor object obtained via a projective resolution of A. Write $H_n(F_\bullet \otimes B)$ for the tor object obtained via a flat resolution $F_\bullet \to A$ of A. We

argue by induction. For $n = 0$, we have $\mathrm{Tor}_0(A, B) \cong H_0(F_\bullet \otimes B)$ because $- \otimes B$ is right exact by assumption.

Let K be such that $0 \to K \to F_0 \to A$ is exact. write $E_\bullet = (\cdots \to F_2 \to F_1)$ so that $E_\bullet \to K$ is a resolution of K by flat objects (Lemma 4.2.6). Since $\mathcal{A}$ has enough projectives, further note that all flat objects are acyclic (Definition 4.6.1) for the functor $- \otimes B$ by Lemma 4.10.4.

By Lemma 4.6.4, we have

$$\mathrm{Tor}_1(A, B) = \ker(K \otimes B \to F_0 \otimes B) = \ker\left\{ \frac{F_1 \otimes B}{\mathrm{im}(F_2 \otimes B) \to F_0 \otimes B} = H_1(F \otimes B) \right\},$$

thus establishing the desired isomorphism for $n = 1$. For $n \geq 2$, use induction to see that

$$\mathrm{Tor}_n(A, B) \cong \mathrm{Tor}_{n-1}(K, B) \cong H_{n-1}(E_\bullet \otimes B) = H_n(F \otimes B).$$

$\square$

Corollary 4.10.6. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories (Definition 2.0.10), and let $\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a biadditive functor (Definition 4.7.5) that is right exact (Definition 2.0.17) in each variable. Suppose that $\mathcal{A}$ and $\mathcal{B}$ have enough projectives (Definition 4.2.2) and flats (Definition 4.8.2). Further suppose that (small) filtered colimits which exist in $\mathcal{C}$ are exact (e.g. which holds if $\mathcal{C}$ satisfies Ab5 (Definition 2.1.8)).

For any objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$, all notions of $\mathrm{Tor}_n(A, B)$ as defined in Definition 4.10.1 naturally agree with one another.

Proof. This follows from Theorem 4.10.2 and Lemma 4.10.5. $\square$

5. Derived categories

5.1. **Homotopy category of chain complexes of an additive category.** We will define homotopy categories and derived categories using the cohomological convention (Convention 3.1.3)

Definition 5.1.1 (Homotopy category $K(\mathcal{A})$)**.** Let $\mathcal{A}$ be an additive category (Definition 2.0.6). (♠ TODO: describe the distinguished triangles of this category)

1. The *homotopy category* $K(\mathcal{A})$ has as objects the (cochain) complexes over $\mathcal{A}$, and as morphisms the cochain maps (Definition 3.1.1) modulo cochain homotopy equivalence (Definition 3.3.1). More explicitly,

$$\mathrm{Hom}_{K(\mathcal{A})}(C^\bullet, D^\bullet) := \frac{\{\text{cochain maps } C^\bullet \to D^\bullet\}}{\{\text{cochain homotopies}\}},$$

where morphisms differing by a cochain homotopy are identified.
2. Let

$$F : \mathcal{A} \to \mathcal{B}$$

be an additive functor (Definition 2.0.8) of additive categories.

The functor F naturally induces a functor between the homotopy categories of complexes,
$$K(F) : K(\mathcal{A}) \to K(\mathcal{B}),$$
by applying F degreewise to complexes and cochain maps. That is, for a complex $(C^\bullet, d^\bullet)$ in $\mathcal{A}$, define
$$K(F)(C^\bullet) := (F(C^\bullet), F(d^\bullet)),$$
and for a cochain map $f^\bullet : C^\bullet \to D^\bullet$, define
$$K(F)(f^\bullet) := (F(f^\bullet)).$$
The functor $K(F)$ respects cochain homotopies, thus it is well-defined on the homotopy category.

3. (See [DBG$^+$77, Catégories dérivées Chapitre 1 2-3] for notation) We write $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ for the full subcategories of bounded below, bounded abaove, and bounded (Definition 3.2.5) complexes respectively.

4. (See [DBG$^+$77, Catégories dérivées Chapitre 1 2-3] for notation) If $\mathcal{A}$ is abelian, we may also write $K^{\infty,+}(\mathcal{A})$, $K^{\infty,-}(\mathcal{A})$, $K^{\infty,b}(\mathcal{A})$, and $K^{\infty,\emptyset}(\mathcal{A})$ for the full subcategories of cohomologically bounded below, cohomologically bounded abaove, cohomologically bounded (Definition 3.2.5), and acyclic complexes (Definition 3.2.3) respectively. We may further write $K^{?,??}(\mathcal{A})$ for the full subcategory where $? \in \{+, -, b\}$ indicates the boundedness of the complex and $?? \in \{+, -, b, (\text{blank})\}$ indicates the cohomological boundedness of the complex.

The various categories here are (Proposition 5.1.6) triangulated categories.

Definition 5.1.2 (Triangulated category). A *triangulated category* is a triple $(\mathcal{T}, [1], \Delta)$ where:

- $\mathcal{T}$ is an additive category,
- $[1] : \mathcal{T} \to \mathcal{T}$ (also denoted by notations such as Σ) is an additive auto-equivalence called the *shift* or *suspension functor*,
- Δ is a class of distinguished triangles of the form
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$
with objects $X, Y, Z \in \mathcal{T}$ and morphisms f, g, h in $\mathcal{T}$,

satisfying the following axioms:

(TR1) The class Δ is closed under isomorphisms of triangles and contains the triangles isomorphic to
$$A \xrightarrow{\text{id}} A \to 0 \to A[1].$$
Moreover, for every morphism $f : X \to Y$ in $\mathcal{T}$, there exists a distinguished triangle
$$X \xrightarrow{f} Y \to Z \to X[1].$$

(TR2) (Rotation) If
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is a distinguished triangle, then so are

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

and

$$Z[-1] \xrightarrow{-h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z.$$

(TR3) (Octahedral axiom) Given morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ in $\mathcal{T}$, there exist distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{u} C(f) \xrightarrow{v} X[1],$$
$$Y \xrightarrow{g} Z \xrightarrow{w} C(g) \xrightarrow{x} Y[1],$$
$$X \xrightarrow{g \circ f} Z \xrightarrow{y} C(g \circ f) \xrightarrow{z} X[1],$$

and morphisms

$$s : C(f) \to C(g \circ f), \quad t : C(g \circ f) \to C(g)$$

such that the following diagram commutes and its rows and columns are distinguished triangles:

$$
\begin{array}{ccccccc}
X & \xrightarrow{\ f\ } & Y & \xrightarrow{\ u\ } & C(f) & \xrightarrow{\ v\ } & X[1] \\
\| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle s} & & \| \\
X & \xrightarrow{\ g\circ f\ } & Z & \xrightarrow{\ y\ } & C(g \circ f) & \xrightarrow{\ z\ } & X[1] \\
& & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle t} & & \\
& & C(g) & =\!=\!= & C(g) & & \\
& & \downarrow{\scriptstyle x} & & \downarrow & & \\
& & Y[1] & \xrightarrow{\ f[1]\ } & X[1][1] & &
\end{array}
$$

In particular,
  − The compositions $s \circ u = y$, $t \circ s = w$, and $x \circ t = v[1]$ hold,
  − All rows and columns form distinguished triangles.

(TR4) The class Δ is closed under the shift functor $[1]$: if

$$X \to Y \to Z \to X[1]$$

is distinguished, then so is

$$X[1] \to Y[1] \to Z[1] \to X[2].$$

Definition 5.1.3. A functor $T : D_1 \to D_2$ between triangulated categories is called *exact* if it is additive, is translation preserving, and transforms distinguished triangles to distinguished triangles. Oftentimes, a *morphism between triangulated categories* refers to an exact functor between triangulated categories.

Definition 5.1.4. In the context of a triangulated category (Definition 5.1.2) $\mathscr{D}$, a multiplicative system (Definition 5.2.1) S is *compatible with the triangulation* if:

1. For any $s \in S$, $T(s) \in S$ where T is the translation functor.
2. For any commutative square between the first two steps of two distinguished triangles where the vertical arrows are in S, there exists a morphism in S between the third objects that makes the whole diagram of triangles commute.

Definition 5.1.5. Let I be a finite set and $\varepsilon : I \to \{1, -1\}$ a variance function. Let $(\mathscr{A}_i)_{i \in I}$ and $\mathscr{A}$ be triangulated categories (Definition 5.1.2). A multifunctor (Definition 1.0.13) $F : \prod_{i \in I} \mathscr{A}_i \to \mathscr{A}$ is an *exact multifunctor of variance ε* if for every index $k \in I$, the partial functor

$$F_k : \mathscr{A}_k \to \mathscr{A}$$

(obtained by fixing all objects A_j for $j \neq k$) satisfies the following:

1. If $\varepsilon(k) = 1$, F_k is a covariant exact functor: it maps any distinguished triangle $X \to Y \to Z \to T_k X$ in $\mathscr{A}_k$ to a distinguished triangle

$$F_k(X) \to F_k(Y) \to F_k(Z) \to TF_k(X)$$

   in $\mathscr{A}$.
2. If $\varepsilon(k) = -1$, F_k is a contravariant exact functor: it maps any distinguished triangle $X \to Y \to Z \to T_k X$ in $\mathscr{A}_k$ to a distinguished triangle

$$F_k(Z) \to F_k(Y) \to F_k(X) \to F_k(T_k^{-1}X)$$

   in $\mathscr{A}$, using the isomorphism $F_k(T_k^{-1}X) \cong TF_k(X)$.

Furthermore, F must be a graded functor equipped with natural isomorphisms $\varphi_i : T \circ F \xrightarrow{\sim} F \circ T_i^{\varepsilon_i}$ for each $i \in I$. These must satisfy the anticommutativity condition: for any pair of distinct indices $i, j \in I$ ($i \neq j$), the following diagram of natural transformations commutes with a factor of -1:

(B)
$$
\begin{array}{ccc}
T^2 F & \xrightarrow{T\varphi_i} & TFT_i^{\varepsilon_i} \\
\downarrow{T\varphi_j} & & \downarrow{\varphi_j T_i^{\varepsilon_i}} \\
TFT_j^{\varepsilon_j} & \xrightarrow{-\varphi_i T_j^{\varepsilon_j}} & FT_i^{\varepsilon_i} T_j^{\varepsilon_j}
\end{array}
$$

Equivalently, the following square is anticommutative:

(C)
$$(\varphi_j T_i^{\varepsilon_i}) \circ (T\varphi_i) = -(\varphi_i T_j^{\varepsilon_j}) \circ (T\varphi_j)$$

as morphisms of functors $T^2 F \to FT_i^{\varepsilon_i} T_j^{\varepsilon_j}$.

In particular in the case that $\epsilon(i) = 1$ for all i, we are reduced to the following definition: Let $(\mathscr{A}_i)_{1 \leq i \leq n}$ and $\mathscr{A}$ be triangulated categories (Definition 5.1.2). A multifunctor (Definition 1.0.13) $F : \prod \mathscr{A}_i \to \mathscr{A}$ is called an *exact covariant multifunctor* if it is covariant in each argument and if for every index $i \in \{1, \ldots, n\}$ and every fixed collection of objects $(A_j)_{j \neq i}$, the partial functor $F(\ldots, A_{i-1}, -, A_{i+1}, \ldots) : \mathscr{A}_i \to \mathscr{A}$ is an exact functor (Definition 5.1.3) of triangulated categories.

Proposition 5.1.6. Let $\mathcal{A}$ be an additive category (Definition 2.0.6).

1. The homotopy category $K(\mathcal{A})$ (Definition 5.1.1) is a triangulated category (Definition 5.1.2).(♠ TODO: shift of chain complexes, describe what the distinguished triangles are)
2. The subcategory $K^?(\mathcal{A})$ is a triangulated subcategory of $K(\mathcal{A})$ for $? \in \{+, -, b\}$.
3. Let $\mathcal{A}$ be an abelian category. The subcategory $K^{?,??}(\mathcal{A})$ is a triangulated subcategory of $K(\mathcal{A})$ for $? \in \{\infty, +, -, b\}$ and $?? \in \{+, -, b, (\text{blank})\}$.

(♠ TODO: How homotopy categories of chain complexes relate to topological homotopy categories)

Theorem 5.1.7 (Exact functor induces a functor on homotopy categories, cf. [DA73, Exposé XVII, Proposition 1.1.7.]). Let $F : \prod_{i=1}^{n} \mathscr{A}_i \to \mathscr{B}$ be an additive multifunctor (Definition 4.7.5) of abelian categories (Definition 2.0.10). F induces exact multifunctors (Definition 5.1.5)[4] between the corresponding homotopy categories (Definition 5.1.1):

$$K(\mathscr{A}) \to K(\mathscr{B})$$

$$K^?(\prod \mathscr{A}_i) \to K^?(\mathscr{B})$$

$$K^{?,??}(\prod \mathscr{A}_i) \to K^{?,??}(\mathscr{B})$$

(Definition 5.1.1). We write these functors by $K(F)$.

5.2. **Multiplicative systems in categories.**

Definition 5.2.1 (Multiplicative system). (♠ TODO: Write a precise definition) Let $\mathcal{C}$ be a category. A *multiplicative system S in $\mathcal{C}$* consists of a collection of morphisms in $\mathcal{C}$ satisfying:

- S contains all identity morphisms,
- S is closed under composition,
- (Ore conditions) For any morphism f in $\mathcal{C}$ and any morphism $s \in S$ with suitable domain and codomain, there exist morphisms to form commutative squares allowing localization.

The precise Ore condition varies depending on context but guarantees the localization exists.

Definition 5.2.2 (Localization by a multiplicative system). Given a category $\mathcal{C}$ and a multiplicative system (Definition 5.2.1) $S \subseteq \mathrm{Mor}(\mathcal{C})$, the *localization $\mathcal{C}[S^{-1}]$* is a category equipped with a functor

$$Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$$

that sends every morphism in S to an isomorphism satisfying the universal property: any functor from $\mathcal{C}$ sending the morphisms in S to isomorphisms factors uniquely through Q.

Definition 5.2.3. A multiplicative system (Definition 5.2.1) S in a category (Definition 1.0.1) $\mathscr{C}$ is called *saturated* if it satisfies the following condition: any morphism f in $\mathscr{C}$ whose image $Q(f)$ in the localized category (Definition 5.2.2) $\mathscr{C}(S^{-1})$ is an isomorphism belongs to S.

---

[4]wlog covariant in each variable, but this can be made multi-variant by taking opposite categories (Definition 1.0.2)

If F is contravariant in the j-th variable, the definitions are adjusted by replacing $\mathscr{A}_j$ with its opposite category $\mathscr{A}_j^{\mathrm{op}}$, thereby interchanging K^+ with K^- and RF with LF for that specific variable.

Proposition 5.2.4. Let $\mathscr{C}$ be a category (Definition 1.0.1) and S a multiplicative system (Definition 5.2.1). There exists a unique saturated multiplicative system (Definition 5.2.3) $\bar{S}$ containing S such that the canonical functors $\mathscr{C}(S^{-1}) \to \mathscr{C}(\bar{S}^{-1})$ define an equivalence of categories (Definition 1.0.10). $\bar{S}$ consists of all morphisms f such that there exist morphisms g, h with $g \circ f \in S$ and $f \circ h \in S$.

5.3. **Derived categories of abelian categories.** (♠ TODO: describe the objects and morphisms in a derived category)

Definition 5.3.1 (Derived category). Let $\mathcal{A}$ be an abelian category. Consider the class S of quasi-isomorphisms (Definition 3.2.6) in the homotopy category $K(\mathcal{A})$ (Definition 5.1.1).

The *derived category* $D(\mathcal{A})$ is the localization (Definition 5.2.2) of the homotopy category at the multiplicative system (Definition 5.2.1) S:

$$D(\mathcal{A}) := K(\mathcal{A})[S^{-1}],$$

We often write the localization functor (Definition 5.2.2) $K(\mathcal{A}) \to D(\mathcal{A})$ by q or Q.

- We often write $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, and $D^b(\mathcal{A})$ for the full subcategories of cohomologically bounded below, cohomologically bounded above, and cohomologically bounded (Definition 3.2.5) complexes respectively.

In general, $D(\mathcal{A})$ (and the aforementioned subcategories) may only exist as a large category (Definition 1.0.1), rather than a locally small category. When $\mathcal{A}$ is a Grothendieck abelian category (Definition 2.1.8), however, $D(\mathcal{A})$ is a locally small category (see [Sta25, Tag 09PA]).

Convention 5.3.2. Given an abelian category (Definition 5.3.1) A, any object M of A may be identified with an object of the homotopy category $K(A)$ (Definition 5.1.1) and the derived category $D(A)$ (Definition 5.3.1) — simply take the chain complex concentrated at degree 0 with the object M. In fact, this derived object can often be regarded as an object of "nice" enough subcategories of $D(A)$, such as the bounded derived categories $D^+(A)$, $D^-(A)$, $D^b(A)$.

(♠ TODO: State how a derived category is a triangulated category)

Definition 5.3.3. Let $\mathcal{A}$ be an additive category (Definition 2.0.6). For $? \in \{+, -, b, \text{(blank)}\}$, write $K^?(\mathcal{I})$ (resp. $K^?(\mathcal{P})$) for the full subcategory of $K^?(\mathcal{A})$ (Definition 5.1.1) whose objects are appropriately bounded (Definition 3.2.5) complexes of injectives (resp. projectives) (Definition 4.1.1).

Even though derived categories of abelian categories may not be guaranteed to be locally small, the following gives sufficient conditions for the appropriately (cohomologically) bounded derived categories to be locally small. The following also shows that appropriately bounded complexes under these circumstances can be represented in the derived category by a complex of injective or projective objects; we might think of such complexes of injectives or projectives as acting like an injective/projective resolution.

Theorem 5.3.4 (see e.g. [Wei94, Theorem 10.4.8]). Let $\mathcal{A}$ be an abelian category.

1. If $\mathcal{A}$ has enough injectives (Definition 4.2.2), then the category $D^+(\mathcal{A})$ (Definition 5.3.1) is equivalent to the category $K^+(\mathcal{I})$ (Definition 5.3.3), which is locally small (Definition 1.0.5). In fact, an equivalence is given by

$$K^+(\mathcal{I}) \xrightarrow{q} D^+(\mathcal{I}) \hookrightarrow D^+(\mathcal{A}).$$

2. If $\mathcal{A}$ has enough projectives (Definition 4.2.2), then the category $D^-(\mathcal{A})$ (Definition 5.3.1) is equivalent to the category $K^-(\mathcal{P})$ (Definition 5.3.3), which is locally small (Definition 1.0.5).

$$K^-(\mathcal{P}) \xrightarrow{q} D^-(\mathcal{P}) \hookrightarrow D^-(\mathcal{A}).$$

5.4. **Total derived functors.** The following subsection mostly takes the point of view of [Wei94, Chapter 10]

Definition 5.4.1 (cf. [Wei94, Definition 10.5.1]). Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $F : K(\mathcal{A}) \to K(\mathcal{B})$ (Definition 5.1.1) be an exact functor (Definition 5.1.3) of triangulated categories. Write $q_{\mathcal{A}} : K(\mathcal{A}) \to D(\mathcal{A})$ and $q_{\mathcal{B}} : K(\mathcal{B}) \to D(\mathcal{B})$ for the localization functors (Definition 5.3.1),

1. A *(total) right derived functor on F on $K(\mathcal{A})$* is an exact functor $RF : D(\mathcal{A}) \to D(\mathcal{B})$ together with a natural transformation

$$\xi : q_{\mathcal{B}}F \Rightarrow (RF)q_{\mathcal{A}},$$

which is universal in the sense that if $G : D(\mathcal{A}) \to D(\mathcal{B})$ is another morphism equipped with a natural transformation $\zeta : q_{\mathcal{B}}F \Rightarrow Gq_{\mathcal{A}}$, then there exists a unique natural transformation $\eta : RF \Rightarrow G$ so that $\zeta_A = \eta_{q_{\mathcal{A}}A} \circ \xi_A$ for all A in $K(\mathcal{A})$.

2. Dually, a *(total) left derived functor on F on $K(\mathcal{A})$* is an exact functor $LF : D(\mathcal{A}) \to D(\mathcal{B})$ together with a natural transformation

$$\xi : (LF)q_{\mathcal{A}} \Rightarrow q_{\mathcal{B}}F,$$

which is universal in the sense that if $G : D(\mathcal{A}) \to D(\mathcal{B})$ is another exact functor equipped with a natural transformation $\zeta : Gq_{\mathcal{A}} \Rightarrow q_{\mathcal{B}}F$, then there exists a unique natural transformation $\eta : G \Rightarrow LF$ such that

$$\zeta_A = \xi_A \circ \eta_{q_{\mathcal{A}}A}$$

for all A in $K(\mathcal{A})$.

The universal properties guarantee that if the total derived functors exist, then they are unique up to natural isomorphism, and that

These definitions/ideas may be extended to exact functors $K^?(\mathcal{A}) \to K(\mathcal{B})$ for $? \in \{+, -, b\}$ and to exact functors $K^{?,??}(\mathcal{A}) \to K(\mathcal{B})$ for $? \in \{\infty, +, -, b\}$ and $?? \in \{+, -, b, (\text{blank})\}$ (Definition 5.1.1)

Proposition 5.4.2. See e.g. [Wei94, 10.5.2] Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor (Definition 2.0.17) of abelian categories (Definition 2.0.10). There is an induced functor

$$D(\mathcal{A}) \to D(\mathcal{B})$$

(Definition 5.3.1) also denoted by F by abuse of notation.

This functor is constructible in the "simple" way —

- For an object M of $D(\mathcal{A})$, take a chain complex K representing it. The object FM of $D(\mathcal{B})$ is the object represented by the chain complex FK.
- For a morphism $f : M \to N$ in $D(\mathcal{A})$, take a morphism $\tilde{f} : K \to L$ representing it. The morphism $Ff : FM \to FN$ is the morphism represented by $F\tilde{f} : FK \to FL$.

Due to the exactness of F, these do not depend on the choices of representatives.

The below shows that right and left total derived functors exist for nice enough abelian categories on sufficiently (cohomologically) boundeded objects and can be computed using injective/projective resolutions.

Theorem 5.4.3 (See e.g. [Wei94, 10.5.6]). Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $F : K^+(\mathcal{A}) \to K(\mathcal{B})$ (Definition 5.1.1) be a morphism of triangulated categories (Definition 5.1.3).

1. If $\mathcal{A}$ has enough injectives (Theorem 5.1.7), then the right derived functor RF (Definition 5.4.1) exists (as a functor $D^+(\mathcal{A}) \to D(\mathcal{B})$ (Definition 5.3.1)), and if I is a bounded below (Definition 3.2.5) complex of injectives (Definition 4.1.1), then

$$RF(I) \cong qF(I)$$

   where $q : K(\mathcal{B}) \to D(\mathcal{B})$ is the localization functor (Definition 5.3.1).
2. Dually, if $\mathcal{A}$ has enough projectives, then the left derived functor LF exists (as a functor $D^-(\mathcal{A}) \to D(\mathcal{B})$), and if I is a bounded above complex of projectives, then

$$LF(P) \cong qF(P)$$

Corollary 5.4.4. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor (Definition 2.0.8) between abelian categories (Definition 2.0.10). Write $q : K(\mathcal{B}) \to D(\mathcal{B})$ (Definition 5.1.1 Definition 5.3.1) for the localization functor (Definition 5.2.2).

1. Assuming that $\mathcal{A}$ has enough injectives (Definition 4.2.2), there exists a right derived functor (Definition 5.4.1) $RF : D^+(\mathcal{A}) \to D(\mathcal{B})$ and if $I \in K^+(\mathcal{I})$ (Definition 5.3.3) is a bounded below complex of injectives (Definition 4.1.1), then

$$RF(I) \cong q(K(F))(I).$$

   where $K(F) : K^+(\mathcal{A}) \to K(\mathcal{B})$ is the exact functor (Definition 5.1.3) induced by F (Theorem 5.1.7).
   If F is additionally left exact (Definition 2.0.17), then for all objects A of $\mathcal{A}$ we have

$$H^i(RF(A)) \cong R^i F(A)$$

   (Definition 4.3.1) for all integers i where we regard A as the chain complex (Definition 3.1.1) object of $D^+(\mathcal{A})$ with A concentrated in degree 0 (Convention 5.3.2).
2. Assuming that $\mathcal{A}$ has enough projectives (Definition 4.2.2), there exists a left derived functor (Definition 5.4.1) $LF : D^-(\mathcal{A}) \to D(\mathcal{B})$ and if $P \in K^-(\mathcal{P})$ (Definition 5.3.3) is a bounded above complex of projectives (Definition 4.1.1), then

$$LF(P) \cong q(K(F))(P).$$

where $K(F) : K^-(\mathcal{A}) \to K(\mathcal{B})$ is the exact functor (Definition 5.1.3) induced by F (Theorem 5.1.7).

If F is additionally right exact (Definition 2.0.17), then for all objects A of $\mathcal{A}$ we have

$$H^{-i}(LF(A)) \cong L_i F(A)$$

(Definition 4.3.1) for all integers i where we regard A as the chain complex (Definition 3.1.1) object of $D^+(\mathcal{A})$ with A concentrated in degree 0 (Convention 5.3.2).

Proof. Apply Theorem 5.4.3 to the exact functor $K(F)$ to obtain the isomorphisms

$$RF(I) \cong q(K(F))(I)$$

$$LF(P) \cong q(K(F))(P).$$

If F is left exact and A is an object of $\mathcal{A}$, then the right derived functors $R^i F(A)$ (Definition 4.3.1) are defined. Recall that they are defined first by taking an injective resolution (Definition 4.2.4)

$$0 \to A \to I^0 \to I^1 \to \cdots .$$

Since the injective resolution is acyclic (Definition 3.2.3), The object A and the complex $0 \to I^0 \to I^1 \to \cdots$ are equivalent as objects of $D^+(\mathcal{A})$ as an object of $D^+(\mathcal{A})$ are equivalent objects. Moreover, we have the isomorphism

$$RF(I) \cong q(K(F))(I).$$

In fact, the ith cohomology of the right hand side object precisely describes $R^i F(A)$ by definition. Therefore,

$$H^i(RF(A)) \cong R^i F(A)$$

as desired.

The analogous statement for $L_i F(A)$ holds similarly. $\square$

(♠ TODO: define hyper-derived functors)

Corollary 5.4.5 (See e.g. [Wei94, Corollary 10.5.7, Remark 10.5.8])**.** Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor (Definition 2.0.8) between abelian categories (Definition 2.0.10).

1. If $\mathcal{A}$ has enough injectives (Definition 4.2.2), the hyper-derived functors $\mathbb{R}^i F(X)$ are the cohomology of $\mathbf{R}F(X)$: $\quad \mathbb{R}^i F(X) \cong H^i \mathbf{R}F(X)$ for all i.
2. If $\mathcal{A}$ has enough projectives (Definition 4.2.2), the hyper-derived functors $\mathbb{L}_i F(X)$ are the cohomology of $\mathbf{L}F(X)$: $\quad \mathbb{L}_i F(X) \cong H^{-i} \mathbf{L}F(X)$ for all i.

5.5. **General derived functors for exact multifunctors between triangulated categories.** The following subsection mostly takes the point of view of [DA73, Exposé XVII, 1.2]

Definition 5.5.1 (Ind-category)**.** Let $\mathcal{C}$ be a locally small category (Definition 1.0.5).

1. The *Ind-category of* $\mathcal{C}$, denoted Ind($\mathcal{C}$), is defined as follows:

- Objects of Ind($\mathcal{C}$) are formal filtered colimits (Definition 1.3.12) of objects in $\mathcal{C}$. More precisely, an object is given by a filtered (Definition 1.3.8) small category I and a functor
$$X : I \to \mathcal{C}.$$
- Morphisms between objects $X : I \to \mathcal{C}$ and $Y : J \to \mathcal{C}$ are defined by
$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(X, Y) := \varprojlim_{i \in I} \varinjlim_{j \in J} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j),$$
(Definition 1.3.12) where X_i and Y_j denote the images of $i \in I$ and $j \in J$ under X and Y, respectively.

The composition of morphisms is induced naturally from composition in $\mathcal{C}$. Hence, Ind($\mathcal{C}$) is the completion of $\mathcal{C}$ under filtered colimits. Objects of Ind($\mathcal{C}$) are called *Ind-objects of $\mathcal{C}$*.

2. The *Pro-category of $\mathcal{C}$*, denoted Pro($\mathcal{C}$), is defined as follows:
- Objects of Pro($\mathcal{C}$) are formal cofiltered limits (Definition 1.3.12) of objects in $\mathcal{C}$. More precisely, an object is given by a cofiltered small category I and a functor
$$X : I \to \mathcal{C}.$$
- Morphisms between objects $X : I \to \mathcal{C}$ and $Y : J \to \mathcal{C}$ are defined by
$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X, Y) := \varinjlim_{j \in J} \varprojlim_{i \in I} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j),$$
where X_i and Y_j denote the images of $i \in I$ and $j \in J$ under X and Y, respectively.

The composition of morphisms is induced naturally from composition in $\mathcal{C}$.

Hence, Pro($\mathcal{C}$) is the completion of $\mathcal{C}$ under cofiltered limits. Objects of Pro($\mathcal{C}$) are called *Pro-objects of $\mathcal{C}$*.

Since **Sets** has all limits and colimits (♠ TODO:) and hence has all projective and inductive limits and since $\mathcal{C}$ is locally small, Ind($\mathcal{C}$) and Pro($\mathcal{C}$) are locally small.

Definition 5.5.2. Let $\mathscr{C}$ be a locally small category (Lemma 1.0.7).

1. An Ind-object (Definition 5.5.1) $X = \varinjlim X_i$ is said to be *essentially constant* if it is isomorphic in Ind($\mathscr{C}$) to an object in the image of the canonical fully faithful functor $h : \mathscr{C} \to \mathrm{Ind}(\mathscr{C})$, which maps an object $A \in \mathrm{ob}(\mathscr{C})$ to the Ind-object represented by the constant diagram.
2. A Pro-object (Definition 5.5.1) $X = \varprojlim X_i$ is said to be *essentially constant* if it is isomorphic in Pro($\mathscr{C}$) to an object in the image of the canonical fully faithful functor $h : \mathscr{C} \to \mathrm{Pro}(\mathscr{C})$, which maps an object $A \in \mathrm{ob}(\mathscr{C})$ to the Pro-object represented by the constant diagram.

In either definition, the object A may be referred to as the *value/limit/constant value of the Ind/Pro object* or the *object representing the Ind/Pro object*.

Proposition 5.5.3. Let $\mathscr{C}$ be a locally small category (Lemma 1.0.7).

1. An Ind-object (Definition 5.5.1) $X = \varinjlim_{i \in I} X_i$ (where I is a filtered (Definition 1.3.8) index category) is essentially constant (Definition 5.5.2) and isomorphic to $A \in \mathrm{ob}(\mathscr{C})$

if and only if there exists a family of morphisms $(f_i : X_i \to A)_{i \in I}$ such that for every morphism $u : i \to j$ in I, the diagram

$$X_i \xrightarrow{X(u)} X_j$$
$$\searrow_{f_i} \quad \downarrow_{f_j}$$
$$A$$

commutes, and the induced morphism $\varinjlim X_i \to A$ (Definition 1.3.12) is an isomorphism in $\mathrm{Ind}(\mathscr{C})$.

2. A Pro-object (Definition 5.5.1) $X = \varprojlim_{i \in I} X_i$ (where I is a filtered (Definition 1.3.8) index category) is essentially constant (Definition 5.5.2) and isomorphic to $A \in \mathrm{ob}(\mathscr{C})$ if and only if there exists a family of morphisms $(g_i : A \to X_i)_{i \in I}$ such that for every morphism $u : i \to j$ in I, the diagram

$$A \xrightarrow{g_i} X_i$$
$$\searrow_{g_j} \quad \downarrow^{X(u)}$$
$$X_j$$

commutes, and the induced morphism $A \to \varprojlim X_i$ (Definition 1.3.12) is an isomorphism in $\mathrm{Pro}(\mathscr{C})$.

Definition 5.5.4 (Mixed-Variance Derived Multifunctors). [cf. [DA73, Exposé XVII, Définition 1.2.1., 1.2.5.]] Let $(\mathscr{A}_i)_{1 \leq i \leq n}$ and $(\mathscr{B}_j)_{1 \leq j \leq m}$ be families of triangulated categories (Definition 5.1.2), and let $\mathscr{C}$ be a triangulated category. Let

$$F : \prod_{i=1}^{n} \mathscr{A}_i \times \prod_{j=1}^{m} \mathscr{B}_j \to \mathscr{C}$$

be an exact multifunctor (Definition 5.1.5), covariant in A_i and contravariant in B_j. Let S_i, Σ_j, T be saturated multiplicative systems (Definition 5.2.3) in $\mathscr{A}_i, \mathscr{B}_j, \mathscr{C}$ respectively.

1. The *right derived functor RF of F relative to (S_i, Σ_j) and T* is a functor

$$RF : \prod_{i=1}^{n} \mathscr{A}_i(S_i^{-1}) \times \prod_{j=1}^{m} \mathscr{B}_j(\Sigma_j^{-1}) \to \mathrm{Ind}(\mathscr{C}(T^{-1}))$$

which makes the following diagram commutative:

$$\begin{array}{ccc}
\prod_{i=1}^{n} \mathscr{A}_i \times \prod_{j=1}^{m} \mathscr{B}_j & \xrightarrow{F} & \mathscr{C} \\
\downarrow{\prod Q_{A_i} \times \prod Q_{B_j}} & & \downarrow{Q_C} \\
\prod_{i=1}^{n} \mathscr{A}_i(S_i^{-1}) \times \prod_{j=1}^{m} \mathscr{B}_j(\Sigma_j^{-1}) & \xrightarrow{RF} & \mathrm{Ind}(\mathscr{C}(T^{-1}))
\end{array}$$

2. The *left derived functor LF of F relative to (S_i, Σ_j) and T* is a functor

$$LF : \prod_{i=1}^{n} \mathscr{A}_i(S_i^{-1}) \times \prod_{j=1}^{m} \mathscr{B}_j(\Sigma_j^{-1}) \to \mathrm{Pro}(\mathscr{C}(T^{-1}))$$

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which makes the following diagram commutative:

$$
\begin{array}{ccc}
\prod_{i=1}^{n}\mathscr{A}_i \times \prod_{j=1}^{m}\mathscr{B}_j & \xrightarrow{F} & \mathscr{C} \\
\downarrow{\scriptstyle \prod Q_{A_i}\times\prod Q_{B_j}} & & \downarrow{\scriptstyle Q_C} \\
\prod_{i=1}^{n}\mathscr{A}_i(S_i^{-1}) \times \prod_{j=1}^{m}\mathscr{B}_j(\Sigma_j^{-1}) & \xrightarrow{LF} & \mathrm{Pro}(\mathscr{C}(T^{-1}))
\end{array}
$$

Definition 5.5.5 (Derivability and Adapted Families). Let $(\mathscr{A}_i)_{1\leq i\leq n}$ and $(\mathscr{B}_j)_{1\leq j\leq m}$ be families of triangulated categories (Definition 5.1.2), and let $\mathscr{C}$ be a triangulated category. Let

$$
F : \prod_{i=1}^{n}\mathscr{A}_i \times \prod_{j=1}^{m}\mathscr{B}_j \to \mathscr{C}
$$

be an exact multifunctor (Definition 5.1.5), covariant in the variables $A_i \in \mathscr{A}_i$ and contravariant in the variables $B_j \in \mathscr{B}_j$. Let S_i, Σ_j, T be saturated multiplicative systems (Definition 5.2.3) in $\mathscr{A}_i, \mathscr{B}_j, \mathscr{C}$ respectively.

1. Let RF be the right derived functor (Definition 5.5.4) of F relative to the multiplicative systems.
   (a) RF is *defined at a family* (A_i, B_j) if the Ind-object (Definition 5.5.1) $RF(A_i, B_j)$ is essentially constant. This object is the *value of RF at* (A_i, B_j).
   (b) F is *right derivable* if RF is defined at all objects. In this case, we view it as a functor into $\mathscr{C}(T^{-1})$.
   (c) A family (A_i, B_j) is *adapted for RF* (or *deployed for RF* (*déployée pour RF* in French)) if the natural morphism $Q_C(F(A_i, B_j)) \to RF(Q_{A_i}A_i, Q_{B_j}B_j)$ is an isomorphism in $\mathrm{Ind}(\mathscr{C}(T^{-1}))$, i.e. the Ind-object $RF(A_i)$ of $A(S^{-1})$ is essentially constant (Definition 5.5.2) with value $F((A_i), (B_j))$.
2. Let LF be the left derived functor (Definition 5.5.4) of F relative to the multiplicative systems.
   (a) LF is *defined at a family* (A_i, B_j) if the Pro-object (Definition 5.5.1) $LF(A_i, B_j)$ is essentially constant.
   (b) F is *left derivable* if LF is defined at all objects.
   (c) A family (A_i, B_j) is *adapted for LF* (or *deployed for LF* (*déployée pour LF* in French)) if the natural morphism $LF(Q_{A_i}A_i, Q_{B_j}B_j) \to Q_C(F(A_i, B_j))$ is an isomorphism in $\mathrm{Pro}(\mathscr{C}(T^{-1}))$, i.e. the Pro-object $LF(A_i)$ of $A(S^{-1})$ is essentially constant (Definition 5.5.2) with value $F((A_i), (B_j))$.

5.5.1. *Derived multifunctors of additive multifunctors between abelian categories.*

Definition 5.5.6. Let $F : \prod_{i=1}^{n}\mathscr{A}_i \to \mathscr{B}$ be an additive multifunctor (Definition 4.7.5) of abelian categories (Definition 2.0.10).

The *right (resp. left) derived multifunctors of F* are the derived functors (Definition 5.5.4) RF (resp. LF) of the induced multifunctors (Theorem 5.1.7)

- $K(\prod \mathscr{A}_i) \to K(\mathscr{B})$
- $K^{+}(\prod \mathscr{A}_i) \to K^{+}(\mathscr{B})$
- $K^{-}(\prod \mathscr{A}_i) \to K^{-}(\mathscr{B})$

(Recall Proposition 5.1.6) relative to the systems (Definition 5.2.3) of quasi-isomorphisms (Definition 3.2.6) in the respective homotopy categories (Definition 5.1.1). In particular, the functor RF refers to one of the functors

$$D^+(\prod \mathscr{A}_i) \to \operatorname{Ind} D^+(\mathscr{B})D(\prod \mathscr{A}_i) \to \operatorname{Ind} D(\mathscr{B})$$

and the functor LF refers to one of the functors

$$D^-(\prod \mathscr{A}_i) \to \operatorname{Pro} D^-(\mathscr{B})D(\prod \mathscr{A}_i) \to \operatorname{Pro} D(\mathscr{B})$$

(Definition 5.3.1) (Definition 5.5.1).

Definition 5.5.7 ([DA73, Definition 1.2.3]). Let $F : \prod_{i=1}^{n} \mathscr{A}_i \to \mathscr{B}$ be an additive multifunctor (Definition 4.7.5) of abelian categories (Definition 2.0.10).

1. An object $(A_1, \ldots, A_n)$ with $A_i \in \operatorname{ob}(\mathscr{A}_i)$ is *right acyclic for F* if the complex concentrated in degree 0 associated to (A_i) is adapted for (Definition 5.5.5) RF .
2. Similarly, (A_i) is *left acyclic for F* if the complex concentrated in degree 0 is adapted for (Definition 5.5.5) LF.

In the case that $n = 1$ and $A_1 \in \mathcal{A}_1$ has a projective (Definition 4.1.1) resolution (Definition 4.2.4) (resp. injective resolution), the notion of left acyclic for F (resp. right acyclic for F) is equivalent to the notion of F-acyclicity (Definition 4.6.1).

Proposition 5.5.8 (cf. [DA73, Exposé XVII, Proposition 1.2.4.]). Let $F : \prod_{i=1}^{n} \mathscr{A}_i \to \mathscr{B}$ be an additive multifunctor (Definition 4.7.5) of abelian categories (Definition 2.0.10).

1. The following diagram of categories is commutative for the right derived multifunctor:

$$\begin{array}{ccc} D^+(\prod \mathscr{A}_i) & \longrightarrow & D(\prod \mathscr{A}_i) \\ \downarrow {\scriptstyle RF} & & \downarrow {\scriptstyle RF} \\ \operatorname{Ind} D^+(\mathscr{B}) & \longrightarrow & \operatorname{Ind} D(\mathscr{B}) \end{array}$$

(Definition 5.3.1) (Definition 4.8.1) (Definition 5.5.6) where the horizontal arrows are the natural functors induced by inclusions. If RF is everywhere defined on $D(\prod \mathscr{A}_i)$, its restriction to $D^+(\prod \mathscr{A}_i)$ factors through $D^+(\mathscr{B})$ [DA73, Proposition 1.2.4].

2. Dually, the following diagram is commutative for the left derived multifunctor:

$$\begin{array}{ccc} D^-(\prod \mathscr{A}_i) & \longrightarrow & D(\prod \mathscr{A}_i) \\ \downarrow {\scriptstyle LF} & & \downarrow {\scriptstyle LF} \\ \operatorname{Pro} D^-(\mathscr{B}) & \longrightarrow & \operatorname{Pro} D(\mathscr{B}) \end{array}$$

If LF is everywhere defined on $D(\prod \mathscr{A}_i)$, its restriction to $D^-(\prod \mathscr{A}_i)$ factors through $D^-(\mathscr{B})$.

Theorem 5.5.9 ([DA73, Exposé XVII, Proposition 1.2.7.]). (♠ TODO:) Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor (Definition 2.0.8) between abelian categories (Definition 2.0.10).

If $\mathcal{A}$ has enough objects (Definition 4.2.1) that are acyclic for (Definition 5.5.7) LF (resp. for RF), the derived functor (Definition 5.5.4) $LF : D^-(\mathcal{A}) \to D^-(\mathcal{B})$ (resp. $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$) is left (resp. right) derivable (Definition 5.5.5) and every homologically upper bounded (Definition 3.2.5) (resp. homologically lower bounded) complex of acyclics for LF (resp. for RF) is adapted for LF (resp. for RF) (Definition 5.5.5).

(♠ TODO:

- **Address the Biadditive/Multifunctor Gap**
  - Add a Lemma or Remark following Definition 5.5.5 clarifying that for a biadditive functor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$, the condition of a pair of complexes $(M^\bullet, N^\bullet)$ being adapted (deployed) is what allows the representation of LF via the Total Complex construction.
  - Explicitly define the Total Complex functor $\text{Tot} : \text{K}(\mathcal{A}) \times \text{K}(\mathcal{B}) \to \text{K}(\mathcal{C})$ used to induce the multifunctor in Definition 5.5.6. Specify the sign convention for the total differential: $d(x \otimes y) = d_x(x) \otimes y + (-1)^p x \otimes d_y(y)$ for x of degree p.
- **Formulate the Balancing Results**
  - Add a Proposition (The Balancing Lemma) stating that if $\mathcal{P} \subset \mathcal{A}$ is a class of objects adapted for $F(-, N)$ and $\mathcal{Q} \subset \mathcal{B}$ is a class adapted for $F(M, -)$, there exist canonical isomorphisms in $D(\mathcal{C})$:
  $$\text{Tot}(F(P^\bullet, N^\bullet)) \cong \text{Tot}(F(M^\bullet, Q^\bullet)) \cong \text{Tot}(F(P^\bullet, Q^\bullet))$$
  - Cite [SGA 4, XVII, 1.2.7] to justify that these isomorphisms are well-defined because complexes of adapted objects are "deployed".
- **Incorporate the "Everywhere Defined" Logic**
  - Include the "Finite Dimension" case from [SGA 4, XVII, Prop 1.2.10] to explain the conditions under which LF and RF can be extended to the unbounded derived category $D(\mathcal{A})$.
- **Technical Refinements**
  - Provide a precise proof or reference for the property: "If F is bi-right-exact, then projective objects are left-acyclic (adapted) for F."
  - Define the term "Deployed in the first variable": A complex $P^\bullet$ is deployed in the first variable if the family $(P^\bullet, N^\bullet)$ is deployed for the multifunctor F for every $N^\bullet \in \text{K}(\mathcal{A}_2)$.
  - Resolve the ♠ TODO regarding the existence of limits/colimits in `Sets` to ensure Ind/Pro categories are locally small.
- **Categorical Equivalence**
  - Add a Remark confirming the equivalence of categories $D(\mathcal{A}_1) \times D(\mathcal{A}_2) \simeq D(\mathcal{A}_1 \times \mathcal{A}_2)$, noting that quasi-isomorphisms are defined component-wise.
- **derived tensor product**
  - Do proposition 4.1.7. expose XVII

)

5.6. **Total derived functors of biadditive functors.** We extend the ideas from Section 4.8.2

Definition 5.6.1. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories (Definition 2.0.10), and let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a biadditive functor (Definition 4.7.5).

There are notions of derived functors that we may consider depending on whether $\mathcal{A}$ or $\mathcal{B}$ has enough projectives or injectives or flats (Definition 4.8.1) with respect to F. (♠ TODO: define for flats)

(♠ TODO: I really should be letting M and N be complxes, not just objects.)

1.

1. (a) Let $M \in \mathcal{A}$. Assume that $\mathcal{B}$ has enough projectives (Definition 4.2.2). By Corollary 5.4.4, the functor $F(M, -) : \mathcal{B} \to \mathcal{C}$ has a left derived functor (Definition 5.4.1) $L(F(M, -)) : D^-(\mathcal{B}) \to D(\mathcal{C})$ (Definition 5.3.1), which we may write as $LF(M, -)$.
   (b) Symmetrically, let $N \in \mathcal{B}$. Assume that $\mathcal{A}$ has enough projectives (Definition 4.2.2). By Corollary 5.4.4, the functor $F(-, N) : \mathcal{A} \to \mathcal{C}$ has a left derived functor (Definition 5.4.1) $L(F(-, N)) : D^-(\mathcal{A}) \to D(\mathcal{C})$ (Definition 5.3.1), which we may write as $LF(-, N)$.
   Assuming that $\mathcal{A}$ and $\mathcal{B}$ have enough projectives, the notations $LF(M, N)$ above are in agreement (Proposition 5.6.2)
2. (a) Let $M \in \mathcal{A}$. Assume that $\mathcal{B}$ has enough injectives (Definition 4.2.2). By Corollary 5.4.4, the functor $F(M, -) : \mathcal{B} \to \mathcal{C}$ has a right derived functor (Definition 5.4.1) $R(F(M, -)) : D^+(\mathcal{B}) \to D(\mathcal{C})$ (Definition 5.3.1), which we may write as $RF(M, -)$.
   (b) Symmetrically, let $N \in \mathcal{B}$. Assume that $\mathcal{A}$ has enough injectives (Definition 4.2.2). By Corollary 5.4.4, the functor $F(-, N) : \mathcal{A} \to \mathcal{C}$ has a right derived functor (Definition 5.4.1) $R(F(-, N)) : D^+(\mathcal{A}) \to D(\mathcal{C})$ (Definition 5.3.1), which we may write as $RF(-, N)$.
   Assuming that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives, the notations $RF(M, N)$ above are in agreement (Proposition 5.6.2)
3. Let $M \in \mathcal{A}$. Assume that $\mathcal{B}$ has enough flats (Definition 4.8.2) with respect to $\otimes$. Define $LF(M, -) : D^-(\mathcal{B}) \to D(\mathcal{C})$ as follows:

See Definition 5.6.3 for notation used in the case that F is written as a tensor product.

Proposition 5.6.2. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories (Definition 2.0.10), and let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a biadditive functor (Definition 4.7.5).

1. Suppose that $\mathcal{A}$ and $\mathcal{B}$ both have enough projectives (Definition 4.2.2). Given objects $M \in D^-(\mathcal{A})$ and $N \in D^-(\mathcal{A})$ (Definition 5.3.1), the objects $LF(M, N)$ obtained as $(LF(M, -))(N)$ and $(LF(-, N))(M)$ are naturally isomorphic. Thus, the two definitions of $LF(M, N)$ in Definition 5.6.1 are in agreement.
2. Dually, suppose that $\mathcal{A}$ and $\mathcal{B}$ both have enough injectives (Definition 4.2.2). Given objects $M \in D^+(\mathcal{A})$ and $N \in D^+(\mathcal{A})$ (Definition 5.3.1), the objects $RF(M, N)$ obtained as $(RF(M, -))(N)$ and $(RF(-, N))(M)$ are naturally isomorphic. Thus, the two definitions of $RF(M, N)$ in Definition 5.6.1 are in agreement.

Proof. We prove 1. The other part is dual. By Theorem 5.3.4, note that $D^-(\mathcal{A})$ and $D^-(\mathcal{B})$ are respectively equivalent to the categories $K^-(\mathcal{P}_\mathcal{A})$ and $K^-(\mathcal{P}_\mathcal{B})$ (Definition 5.3.3) of cohomologically bounded above complexes of projectives in $\mathcal{A}$ and $\mathcal{B}$ respectively. Let

$$P_\bullet \to M$$

and

$$Q_\bullet \to N$$

be projective resolutions. By Corollary 5.4.4,

$$(LF(M,-)(N) \cong q(K(F(M,-)))(N)$$
$$(LF(-,N)(M) \cong q(K(F(-,N)))(M)$$

The former is represented by the complex $F(M, Q_\bullet)$ and the latter is represented by the complex $F(P_\bullet, N)$. These are quasi-isomorphic by Lemma 4.8.5. □

5.6.1. *Derived tensor product.*

Definition 5.6.3. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories (Definition 2.0.10), and let $\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a biadditive functor (Definition 4.7.5) written as tensor product.

(♠ TODO: need a definition with enough flats)

1. Let $M \in \mathcal{A}$. Assume that $\mathcal{B}$ has enough projectives (Definition 4.2.2). We write $M \otimes^L -$ for $LF(M,-) : D^-(\mathcal{B}) \to D(\mathcal{C})$ (Definition 5.6.1) in the case that $F = M \otimes - : \mathcal{B} \to \mathcal{C}$.
2. Symmetrically, let $N \in \mathcal{B}$. Assume that $\mathcal{A}$ has enough projectives (Definition 4.2.2). We write $- \otimes^L N$ for $LF(-,N) : D^+(\mathcal{A}) \to D(\mathcal{C})$ (Definition 5.6.1) in the case that $F = - \otimes N : \mathcal{A} \to \mathcal{C}$.
   (♠ TODO: show that projectives vs. flats yield the same thing) (♠ TODO: show that flats in each variable yield the same thing)
3. Let $M \in \mathcal{A}$. Assume that $\mathcal{B}$ has enough flats (Definition 4.8.2). We alternatively define $M \otimes^L - : D^-(\mathcal{B}) \to D(\mathcal{C})$ as follows — given an object $N \in D^-(\mathcal{B})$, say that $Q^\bullet$ is a complex of flat objects in $\mathcal{B}$ representing N (♠ TODO: show that such a thing exists), and let $M \otimes^L N$ be the object of $D(\mathcal{C})$ represented by the complex $M \otimes Q^\bullet$. (♠ TODO: show that this is well defined, i.e. does not depend on the choice of flat resolution)
4. Symmetrically, let $N \in \mathcal{B}$. Assume that $\mathcal{A}$ has enough flats (Definition 4.8.2). We alternatively define $- \otimes^L N : D^-(\mathcal{A}) \to D(\mathcal{C})$ as follows — given an object $M \in D^-(\mathcal{A})$, say that $P^\bullet$ is a complex of flat objects in $\mathcal{A}$ representing M (♠ TODO: show that such a thing exists), and let $M \otimes^L N$ be the object of $D(\mathcal{C})$ represented by the complex $P^\bullet \otimes N$. (♠ TODO: show that this is well defined, i.e. does not depend on the choice of flat resolution)

The first two notions agree assuming that $\mathcal{A}$ and $\mathcal{B}$ have enough projectives. (♠ TODO: comment on the next two notions agreeing)

Definition 5.6.4. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories (Definition 2.0.10), and let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a bi-additive functor (Definition 4.7.5). Let us say that a complex (Definition 3.1.1) $K^\bullet$ of $\mathcal{A}$ (resp. of $\mathcal{B}$) is K-flat (with respect to F on the left, resp. right) if for every acyclic complex (Definition 3.2.3) $M^\bullet$ of $\mathcal{B}$ (resp. of $\mathcal{A}$), the direct sum total complex (Definition 4.7.8) $\mathrm{Tot}^\oplus(F(K^\bullet, M^\bullet))$ (Definition 4.7.6) (resp. $\mathrm{Tot}^\oplus(F(M^\bullet, K^\bullet)))$) is acyclic.

Lemma 5.6.5. Let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a bi-additive functor (Definition 4.7.5) where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are additive categories (Definition 2.0.6). Let J be a category (Definition 1.0.1).

1. Assume that $F(X, -) : \mathcal{B} \to \mathcal{C}$ preserves colimits (Definition 1.3.2) of diagrams of shape J (Definition 1.3.1) for all objects $X \in \mathcal{A}$. Then the induced functor $F(X^\bullet, -) : \mathbf{Ch}(\mathcal{B}) \to \mathbf{DC}(\mathcal{C})$ (Definition 4.7.6) (Definition 3.1.1) (Definition 4.7.2) also preserves colimits of diagrams of shape J for all objects $X^\bullet \in \mathbf{Ch}(\mathcal{A})$.
2. Dually, assume that $F(-, Y) : \mathcal{A} \to \mathcal{C}$ preserves colimits (Definition 1.3.2) of diagrams of shape J (Definition 1.3.1) for all objects $Y \in \mathcal{B}$. Then the induced functor $F(-, Y^\bullet) : \mathbf{Ch}(\mathcal{A}) \to \mathbf{DC}(\mathcal{C})$ (Definition 4.7.6) (Definition 3.1.1) (Definition 4.7.2) also preserves colimits of diagrams of shape J for all objects $Y^\bullet \in \mathbf{Ch}(\mathcal{B})$.
3. Assume that $F(X, -) : \mathcal{B} \to \mathcal{C}$ preserves limits (Definition 1.3.2) of diagrams of shape J (Definition 1.3.1) for all objects $X \in \mathcal{A}$. Then the induced functor $F(X^\bullet, -) : \mathbf{Ch}(\mathcal{B}) \to \mathbf{DC}(\mathcal{C})$ (Definition 4.7.6) (Definition 3.1.1) (Definition 4.7.2) also preserves limits of diagrams of shape J for all objects $X^\bullet \in \mathbf{Ch}(\mathcal{A})$.
4. Dually, assume that $F(-, Y) : \mathcal{A} \to \mathcal{C}$ preserves limits (Definition 1.3.2) of diagrams of shape J (Definition 1.3.1) for all objects $Y \in \mathcal{B}$. Then the induced functor $F(-, Y^\bullet) : \mathbf{Ch}(\mathcal{A}) \to \mathbf{DC}(\mathcal{C})$ (Definition 4.7.6) (Definition 3.1.1) (Definition 4.7.2) also preserves limits of diagrams of shape J for all objects $Y^\bullet \in \mathbf{Ch}(\mathcal{B})$.

Proof. We prove that if $F(X, -) : \mathcal{B} \to \mathcal{C}$ preserves colimtis of diagrams of shape J for all $X \in \mathcal{A}$, then so does $F(X^\bullet, -) : \mathbf{Ch}(\mathcal{B}) \to \mathbf{DC}(\mathcal{C})$ for all $X^\bullet \in \mathbf{Ch}(\mathcal{A})$. Given $X^\bullet \in \mathbf{Ch}(\mathcal{A})$ and $Y^\bullet \in \mathbf{Ch}(\mathcal{B})$, the term of $F(X^\bullet, Y^\bullet)$ at position (p, q) is $F(X^p, Y^q)$. Let $J \to \mathbf{Ch}(\mathcal{B}) : j \mapsto Y^\bullet_{(j)}$ be some diagram (Definition 1.3.1) whose colimit is $Y^\bullet$. In fact, since colimits of chain complexes are computed termwise (Proposition 3.1.12), there are canonical isomorphisms

$$Y^q \cong \mathrm{colim}_{j \in J} Y^q_{(j)}$$

in $\mathcal{B}$.

We wish to show that $F(X^\bullet, Y^\bullet)$ is the colimit of the diagram $j \mapsto F(X^\bullet, Y^\bullet_{(j)})$ in $\mathbf{DC}(\mathcal{C})$. The colimit in $\mathbf{DC}(\mathcal{C})$ is also computed termwise (note that $\mathbf{DC}(\mathcal{C})$ is identifiable (Theorem 4.7.3) as $\mathbf{Ch}(\mathbf{Ch}(\mathcal{C}))$), it suffices to verify the isomorphism at each position (p, q) and ensure it respects the differentials.

Consider the term at position (p, q). The functor $F(X^\bullet, -)$ maps the colimit diagram in $\mathbf{Ch}(\mathcal{B})$ to a cocone (Definition 1.3.2) in $\mathbf{DC}(\mathcal{C})$. This induces a canonical comparison morphism:

$$\phi : \mathrm{colim}_{j \in J} F(X^\bullet, Y^\bullet_{(j)}) \longrightarrow F(X^\bullet, \mathrm{colim}_{j \in J} Y^\bullet_{(j)}) = F(X^\bullet, Y^\bullet).$$

At the position (p, q), this morphism is the map:

$$\phi^{p,q} : \mathrm{colim}_{j \in J} F(X^p, Y^q_{(j)}) \longrightarrow F(X^p, Y^q).$$

By hypothesis, $F(X^p, -) : \mathcal{B} \to \mathcal{C}$ preserves colimits of shape J. Therefore, $\phi^{p,q}$ is an isomorphism for all $p, q \in \mathbb{Z}$.

Since a morphism of double complexes is an isomorphism if and only if it is an isomorphism at every term (p, q), we conclude that ϕ is an isomorphism. Thus, $F(X^\bullet, -)$ preserves colimits of shape J.

$\square$

Lemma 5.6.6. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories (Definition 2.0.10), and let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a bi-additive functor (Definition 4.7.5). Assume that $\mathcal{C}$ is Ab5 (Definition 2.1.8).

1. Suppose that $F(X, -) : \mathcal{B} \to \mathcal{C}$ preserves filtered colimits (Definition 1.3.12) for all objects $X \in \mathcal{A}$. Let $P^\bullet$ be a bounded above (Definition 3.2.5) complex of objects of $\mathcal{A}$ that are flat (Definition 4.8.1) with respect to F on the left. The complex $P^\bullet$ is K-flat (Definition 5.6.4) with respect to F on the left.
2. Suppose that $F(-, Y) : \mathcal{A} \to \mathcal{C}$ preserves filtered colimits (Definition 1.3.12) for all objects $Y \in \mathcal{B}$. Let $P^\bullet$ be a bounded above complex of objects in $\mathcal{B}$ that are flat with respect to F on the right. The complex $P^\bullet$ is K-flat with respect to F on the right.

Proof. We show the claimed statement in the case that $P^\bullet$ is a complex of flat objects of $\mathcal{A}$ with respect to F on the left; the other case can be argued symmetrically. Let $M^\bullet$ be an acyclic complex (Definition 3.2.3) of objects in $\mathcal{A}$. Writing $\tau_{\leq k} M^\bullet$ for the brutal truncation (Definition 3.4.1) of $M^\bullet$, note that $M^\bullet$ is the filtered colimit of these truncations (Proposition 3.4.3):

$$M^\bullet = \varinjlim_k \tau_{\leq k} M^\bullet.$$

Since $F(X, -) : \mathcal{B} \to \mathcal{C}$ preserves filtered colimits, so does $F(X^\bullet, -) : \mathbf{Ch}(\mathcal{B}) \to \mathbf{DC}(\mathcal{C})$ (Definition 4.7.6) (Definition 3.1.1) (Definition 4.7.2) by Lemma 5.6.5. Therefore,

$$\operatorname{Tot}^\oplus F(P^\bullet, M^\bullet) \cong \operatorname{Tot}^\oplus F(P^\bullet, \varinjlim_k \tau_{\leq k} M^\bullet) \cong \varinjlim_k \operatorname{Tot}^\oplus F(P^\bullet, \tau_{\leq k} M^\bullet).$$

Since $\tau_{\leq k} M^\bullet$ is bounded above and is of flat objects, the double complex $F(P^\bullet, \tau_{\leq k} M^\bullet)$ (Definition 4.7.6) is bounded above (Definition 4.7.7), and each row is acyclic. Since $\mathcal{C}$ satisfies Ab5, $\operatorname{Tot}^\oplus F(P^\bullet, \tau_{\leq k} M^\bullet)$ is acyclic by Lemma 4.7.10. That $\mathcal{C}$ satisfies AB5 further implies that

$$H^n(\operatorname{Tot}^\oplus F(P^\bullet, M^\bullet)) \cong \varinjlim_k H^n \operatorname{Tot}^\oplus F(P^\bullet, \tau_{\leq k} M^\bullet) = 0,$$

i.e. $\operatorname{Tot}^\oplus F(P^\bullet, M^\bullet)$ is acyclic. Therefore, $P^\bullet$ is K-flat as desired. $\qquad\square$

5.7. Ext functors.

5.8. Serre subcategories.

Definition 5.8.1 (Serre subcategory). Let $\mathcal{A}$ be an abelian category (Definition 2.0.10). A full subcategory (Definition 1.0.8) $\mathcal{S} \subseteq \mathcal{A}$ is called a *Serre subcategory* (or sometimes a *thick subcategory*) if it satisfies the following conditions:

1. For any short exact sequence (Definition 2.0.19)

$$0 \to A' \to A \to A'' \to 0$$

in $\mathcal{A}$, the object A lies in $\mathcal{S}$ if and only if both A' and A'' lie in $\mathcal{S}$.
   (♠ TODO: extension)
2. Equivalently, $\mathcal{S}$ is closed under taking subobjects (Definition 1.2.3), quotients (Definition 1.2.4), and extensions in $\mathcal{A}$.

In other words, $\mathcal{S}$ is a Serre subcategory if for every exact sequence in $\mathcal{A}$, the presence of any two of the objects in $\mathcal{S}$ forces the third to be in $\mathcal{S}$.

The notion of a thick subcategory of an abelian category should not be confused for the notion of a thick subcategory (Definition 5.8.2) of a triangulated category.

Definition 5.8.2 (Thick subcategory). Let $\mathcal{T}$ be a triangulated category (Definition 5.1.2). A full (Definition 1.0.8) triangulated subcategory $\mathcal{S} \subseteq \mathcal{T}$ is called *thick* (also *epaisse*) if it is closed under direct summands (Definition 2.0.6). More explictly, for any object $X \in \mathcal{T}$, if X is isomorphic to a direct sum (Definition 2.0.6)

$$X \cong Y \oplus Z,$$

and X lies in $\mathcal{S}$, then both Y and Z lie in $\mathcal{S}$.

The notion of a thick subcategory of a triangulated category should not be confused for the notion of a thick subcategory (Definition 5.8.1) of an abelian category.

Proposition 5.8.3. Let $\mathcal{A}$ be an abelian category and let $\mathcal{A}'$ be a Serre subcategory (Definition 5.8.1) of $\mathcal{A}$. The full subcategories of $K(\mathcal{A})$, $K^?(\mathcal{A})$ for $? \in \{+, -, b\}$, and $K^{?,??}(\mathcal{A})$ for $? \in \{\infty, +, -, b\}$ and $?? \in \{+, -, b, \emptyset\}$ (Definition 5.1.1) consisting of complexes $X^\bullet$ such that $H^i(X^\bullet)$ (Definition 3.2.2) are objects of $\mathcal{A}'$ for all i are a triangulated subcategories (Definition 5.1.2) of their respective triangulated parent categories.

Proof. (♠ TODO: cf. [Fu15, Page 272 and 6.1.2]) □

Notation 5.8.4. Let $\mathcal{A}$ be an abelian category and let $\mathcal{A}'$ be a Serre subcategory (Definition 5.8.1) of $\mathcal{A}$. We may write $K_{\mathcal{A}'}(\mathcal{A})$ for the triangulated subcategory of $K(\mathcal{A})$ (Definition 5.1.1) consisting of complexes $X^\bullet$ such that $H^i(X^\bullet)$ (Definition 3.2.2) are objects of $\mathcal{A}'$ for all i. Similarly, we may write $K^?_{\mathcal{A}'}(\mathcal{A})$ for $? \in \{+, -, b\}$, and $K^{?,??}_{\mathcal{A}'}(\mathcal{A})$ for $? \in \{\infty, +, -, b\}$ and $?? \in \{+, -, b, \emptyset\}$ for the corresponding subcategories of $K^?(\mathcal{A})$ and $K^{?,??}(\mathcal{A})$ (Definition 5.1.1) respectively.

Notation 5.8.5. Let $\mathcal{A}$ be an abelian category and let $\mathcal{A}'$ be a Serre subcategory (Definition 5.8.1) of $\mathcal{A}$. We may write $D^?_{\mathcal{A}'}(\mathcal{A})$ for $? \in \{(\text{blank}), +, -, b\}$ for the full subcategory of $D^?(\mathcal{A})$ (Definition 5.3.1) of objects from $K_{\mathcal{A}'}(\mathcal{A})$ (Notation 5.8.4). These are triangulated subcategories of their respective parent categories.

5.9. **Derived functors of the inverse limit.**

Definition 5.9.1. Let $\mathcal{C}$ be a category (Definition 1.0.1) and let I be the poset (Definition 1.3.7) of non-negative integers $\mathbb{N}$ with the standard ordering $\geq$ (i.e. I has arrows $\cdots \to 2 \to 1 \to 0$). A *tower in $\mathcal{C}$* is a functor (Definition 1.0.3) $X : I \to \mathcal{C}$. Explicitly, a tower consists of a sequence of objects $\{A_i\}_{i \in \mathbb{N}}$ and morphisms $f_i : A_i \to A_{i-1}$ for each $i \geq 1$:

$$\cdots \xrightarrow{f_3} A_2 \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0.$$

Moreover a *morphism of towers* is a morphism of towers as objects of the functor category (Definition 1.3.1) $\mathcal{C}^I$.

Definition 5.9.2. Let $\mathcal{C}$ be a category (Definition 1.0.1) and let I be the poset (Definition 1.3.7) of non-negative integers $\mathbb{N}$ with the standard ordering $\geq$ (i.e. I has arrows $\cdots \to 2 \to 1 \to 0$) (or more generally let I be the poset of integers $\mathbb{Z}$ with similar ordering).

Given a tower (Definition 5.9.1) $A = (A_i, u_i)_{i \in I}$ where $u_i : A_i \to A_{i-1}$ (for $i \geq 1$ if $\mathbb{N}$ is the collection of objects of I) and $r \geq 0$, the *shifted tower* $A[r]$ is the tower $A[r] = (A[r]_i, u[r]_i)$ given by $A[r]_i = A_{r+i}$ and whose transition maps are given by $u[r]_i : A[r]_i = Ar + i \xrightarrow{u_{r+i}} A_{r+i-1} = A[r]_{i-1}$.

Note that there is a morphism of towers (Definition 5.9.1) $A[r] \to A$ where $A[r]_i \to A_i$ is given by the transition morphism $A_{r+i} \to A_i$.

Definition 5.9.3. Let $\mathcal{A}$ be an abelian category (Definition 2.0.10) with enough injectives (Definition 4.2.2). Let I be a small category (Definition 1.0.5). Assume that $\mathcal{A}$ admits all limits indexed by I. (♠ TODO: comment on how $\mathcal{A}^I$ has enough injectives)

For $n \geq 0$, the *nth derived inverse limit functor* is the right derived functor $R^n \lim : \mathcal{A}^I \to \mathcal{A}$, also denoted $\lim^n A_i$, of the inverse limit functor (Definition 1.3.12) $\lim : \mathcal{A}^I \to \mathcal{A}$. Note that a limit functor (Theorem 1.3.23) (of diagrams indexed by small categories on $\mathcal{A}$) is left exact (Definition 2.0.17) because it is right adjoint (Definition C.0.16) to a diagonal functor (Definition 1.3.22) (Proposition 2.0.18) (Theorem 1.3.23).

Theorem 5.9.4 ([Roo06], cf. [Wei94, Corollary 3.5.4. and the following remark]). Let $\mathcal{A}$ be an abelian category (Definition 2.0.10) satisfying axiom (AB4*) (Definition 2.1.8) that is a Grothendieck category (Definition 2.1.8). For any tower (Definition 5.9.1) $\{A_i\}$ of objects in $\mathcal{A}$, the higher derived functors (Definition 4.3.1) of the inverse limit (Definition 1.3.12) vanish for $n \geq 2$, i.e., $R^n \lim A_i = 0$ (Definition 1.3.12) for all $n \geq 2$.

Remark 5.9.5. Roos [Roo61, Corollaire á Proposition 5] had originally stated Theorem 5.9.4 for general abelian categories satisying $(AB4^*)$ (Definition 2.1.8) (which are not necessarily Grothendieck (Definition 2.1.8)). Neeman [Nee02] later produced a counterexample to Roos' statement. Roos later provided a corrected treatment [Roo06], establishing Theorem 5.9.4. In particular, [Wei94, Corollary 3.5.4. and the following remark] is a formulation of Roos' original erred statement.

Proposition 5.9.6. Let $\mathcal{A}$ be an abelian category (Definition 2.0.10) satisfying (AB4*) (Definition 2.1.8). For a tower (Definition 5.9.1) $A = \{A_i\}$ of objects in $\mathcal{A}$, the first derived functor $R^1 \lim A_i$ (Definition 1.3.12) can be computed via the following exact sequence:

$$0 \to \lim A_i \to \prod_{i \in \mathbb{N}} A_i \xrightarrow{d} \prod_{i \in \mathbb{N}} A_i \to \lim{}^1 A_i \to 0$$

where the map d is defined on the components by the difference of the identity and the transition maps: $d(\{a_i\}) = \{a_i - f_{i+1}(a_{i+1})\}$.

Definition 5.9.7. A *pointed category* is a category (Definition 1.0.1) that has a zero object (Definition 2.0.1).

Definition 5.9.8 (Zero morphism). Let $\mathcal{C}$ be a pointed category (Definition 5.9.7). Let X and Y be objects in $\mathcal{C}$, and let 0 be a zero object (Definition 2.0.1) of $\mathcal{C}$. The *zero morphism*

from X to Y is the unique composite morphism

$$0_{XY} : X \to 0 \to Y,$$

where $X \to 0$ is the unique morphism into the terminal object 0 and $0 \to Y$ is the unique morphism out of the initial object 0. This morphism is independent of the choice of the zero object 0.

Definition 5.9.9 (cf. [Wei94, Definition 3.5.6]). Let I be a directed set (Definition 1.3.8) and let $\{A_i, \phi_{ji}\}_{i \in I}$ be an inverse system (Definition 1.3.11) of objects in a category $\mathcal{C}$ where images (Definition 1.2.5) are well-defined (such as the category of sets, abelian groups, or modules).

The system is said to satisfy the *Mittag-Leffler condition* if for every index $i \in I$, there exists an index $j \geq i$ such that for all $k \geq j$, the image of the transition map $\phi_{ki} : A_k \to A_i$ is equal to the image of $\phi_{ji} : A_j \to A_i$.

For a fixed i, let $I_{k,i} = \operatorname{im}(\phi_{ki}) \subseteq A_i$ for all $k \geq i$. The condition states that the decreasing family of subobjects

$$A_i \supseteq I_{i,i} \supseteq I_{i+1,i} \supseteq I_{i+2,i} \supseteq \cdots$$

becomes stationary.

Definition 5.9.10 (cf. [Wei94, Definition 3.5.6]). Let I be a directed set (Definition 1.3.8) and let $\{A_i, \phi_{ji}\}_{i \in I}$ be an inverse system (Definition 1.3.11) of objects in a pointed category (Definition 5.9.7) $\mathcal{C}$.

The system is said to satisfy the *trivial Mittag-Leffler condition* if for every index $i \in I$, there exists an index $j > i$ such that the map $A_j \to A_i$ is a zero morphism (Definition 5.9.8).

Definition 5.9.11. Let I be the poset (Definition 1.3.7) of non-negative integers $\mathbb{N}$ with the standard ordering $\geq$ (i.e. I has arrows $\cdots \to 2 \to 1 \to 0$) and let $\{A_i, \phi_{ji}\}_{i \in I}$ be an inverse system (Definition 1.3.11) of objects (so a tower (Definition 5.9.1)) in a pointed category (Definition 5.9.7) $\mathcal{C}$ where images (Definition 1.2.5) are well-defined (such as the category of sets, abelian groups, or modules).

The system is said to be a *null system* if there exists $r \geq 0$ such that for every $i \in I$, the transition map $\phi_{(i+r)i} : A_{i+r} \to A_i$ is the zero morphism (Definition 5.9.8).

Equivalently, the tower is a null system if there exists an integer $r \geq 0$ such that the morphism of towers (Definition 5.9.1) $A[r] \to A$ (Definition 5.9.2) is the zero morphism (Definition 5.9.8). (♠ TODO: The category of towers $\mathcal{C}^I$ is pointed as $\mathcal{C}$ is pointed; the zero morphism is defined component-wise.)

Definition 5.9.12 (See e.g. [Gro77, Exposé V, VI], [Del80, 1.1] [Fu15, 10.1]). Let $A = \{A_i, \varphi_{ji}\}$ be a tower (Definition 5.9.1) in a category $\mathcal{C}$ where images (Definition 1.2.5) are well-defined.

The tower is said to satisfy the *Artin-Rees-Mittag-Leffler (ARML) condition* if there exists an integer $r \geq 0$ such that for every index $i \in \mathbb{N}$ and every $k \geq i + r$, the image of the transition map $\phi_{ki} : A_k \to A_i$ is equal to the image of $\phi_{(i+r)i} : A_{i+r} \to A_i$.

Equivalently, the tower satisfies the ARML condition if and only if for any integer n, there exists an integer $r \geq 0$ such that

$$\operatorname{im}(A[r] \to A) = \operatorname{im}(A[t] \to A)$$

Definition 5.9.2 for all $t \geq r$.

Lemma 5.9.13. Let I be a directed set (Definition 1.3.8) and let $\{A_i, \phi_{ji}\}_{i \in I}$ be an inverse system (Definition 1.3.11) of objects in a pointed category (Definition 5.9.7) $\mathcal{C}$ where images (Definition 1.2.5) are well-defined (such as the category of sets, abelian groups, or modules).

If $\{A_i, \phi_{ji}\}_{i \in I}$ satisfies the trivial Mittag-Leffler condition (Definition 5.9.10), then it satisfies the Mittag-Leffler condition (Definition 5.9.9).

Lemma 5.9.14. Let I be the poset (Definition 1.3.7) of non-negative integers $\mathbb{N}$ with the standard ordering $\geq$ (i.e. I has arrows $\cdots \to 2 \to 1 \to 0$) and let $\{A_i, \phi_{ji}\}_{i \in I}$ be an inverse system (Definition 1.3.11) (so a tower (Definition 5.9.1)) of objects in a pointed category (Definition 5.9.7) $\mathcal{C}$ where images (Definition 1.2.5) are well-defined (such as the category of sets, abelian groups, or modules).

If $\{A_i, \phi_{ji}\}_{i \in I}$ is a null system (Definition 5.9.11), then it satisfies the trivial Mittag-Leffler condition (Definition 5.9.10).

Lemma 5.9.15. Let I be the poset (Definition 1.3.7) of non-negative integers $\mathbb{N}$ with the standard ordering $\geq$ (i.e. I has arrows $\cdots \to 2 \to 1 \to 0$) and let $\{A_i, \phi_{ji}\}_{i \in I}$ be an inverse system (Definition 1.3.11) (so a tower (Definition 5.9.1)) of objects in a pointed category (Definition 5.9.7) $\mathcal{C}$ where images (Definition 1.2.5) are well-defined (such as the category of sets, abelian groups, or modules).

If $\{A_i, \phi_{ji}\}_{i \in I}$ satisfies the ARML condition (Definition 5.9.12), then it satisfies the ML condition (Definition 5.9.9).

Proposition 5.9.16. (♠ TODO: say how lim 1 vanishes for ML systems, cf. webiel 3.5.7)

6. SHEAVES AND SHEAF COHOMOLOGY

6.1. **Sites and sheaves.**

Definition 6.1.1 ([GV72, Exposé I Définition 4.1]). Let C be a (large) category (Definition 1.0.1).

1. A *sieve S on the category C* is a full subcategory (Definition 1.0.8) D of C such that for any object U of C there exists an object V of (♠ TODO: correctly parse the definiton)
2. A *sieve S on an object $U \in \operatorname{Ob}(C)$* is a collection of morphisms in C with codomain U that is closed under precomposition by any compatible morphism in C. In other words, S is a sieve if for every $f : V \to U$ in S and morphism $g : W \to V$ in C, the composition $f \circ g : W \to U$ is also in S.
   Given a morphism $f : V \to U$ in a sieve S, we also say that *f factors through U*.

Definition 6.1.2. Let $\mathcal{C}$ be a category (Definition 1.0.1) and $U \in \mathcal{C}$ an object. Let $\mathcal{S} = \{f_i : U_i \to U\}_{i \in I}$ be a family of morphisms with codomain U.

The *sieve generated by* $\mathcal{S}$, denoted $(\mathcal{S})$ or $\langle \mathcal{S} \rangle$, is the smallest sieve on U (Definition 6.1.1) containing all the morphisms in $\mathcal{S}$.

Explicitly, a morphism $h : V \to U$ belongs to the generated sieve if and only if h factors through some morphism in $\mathcal{S}$. That is, there exists an index $i \in I$ and a morphism $g : V \to U_i$ such that

$$h = f_i \circ g.$$

Definition 6.1.3. Let C be a category, let $U \in \mathrm{Ob}(C)$, and let S be a sieve on U (Definition 6.1.1). For a morphism $f : V \to U$ in C, the *pullback sieve* f^*S (or *basechange sieve* $S \times_U V$) on V is defined by

$$f^*S = \{g : W \to V \mid f \circ g \in S\}.$$

In other words, f^*S consists of all morphisms into V whose composite with f belongs to the sieve S on U.

Definition 6.1.4 (Grothendieck topology)**.** Let $\mathscr{U}$ be a universe (Definition C.0.14).

1. (See [GV72, Exposé II, Définition 1.1]) Let $\mathcal{C}$ be a category (Definition 1.0.1). A *Grothendieck topology on* $\mathcal{C}$ assigns to each object U of $\mathcal{C}$ a collection $J(U)$ of sieves (Definition 6.1.1) $\{U_i \to U\}_{i \in I}$, each called a *covering sieve of* U, satisfying:
   (a) (Stability under "base change"): If $S \in J(U)$ is a covering sieve of an object U, and $f : V \to U$ is any morphism in $\mathcal{C}$, then the pullback sieve (Definition 6.1.3) f^*S is a covering sieve of U.
   (b) (Local character condition) If S is a sieve on U, and if there exists a covering sieve $R \in J(U)$ such that for all $f : V \to U$ in R the pullback sieve (Definition 6.1.3) f^*S is in $J(V)$, then $S \in J(U)$.
   (c) The maximal sieve is a covering sieve.
   Some will refer to a Grothendieck topology as simply a *topology*, not to be confused with the related, but less general, notion of a topology on a set.
2. (See [GV72, Exposé II, 1.1.5]) A *site* is a category $\mathcal{C}$ equipped with a Grothendieck topology.
   When we are working with a Grothendieck pretopology (Definition C.0.22) K on a category $\mathcal{C}$, we may regard $\mathcal{C}$ as a site by equipping it with the Grothendieck topology generated by (Definition C.0.23) K.
3. (See [GV72, Exposé II, Définition 1.2]) Let $(\mathcal{C}, J)$ be a site. A family of morphisms $(U_i \to U)_{i \in I}$ is called a *covering family of* U *(with respect to the site/topology)* or a *cover of* U *(with respect to the site/topology)* if the sieve generated by (Definition 6.1.2) the family is a covering sieve of U.
4. (See [GV72, Exposé II, Définition 3.0.1]) Let $(\mathcal{C}, J)$ be a site (Definition 6.1.4), where J is a Grothendieck topology on $\mathcal{C}$.
   A family G of objects $\mathcal{C}$ is called a *topologically generating family of the site/topology* or a *generating family/collection of the site/topology* if for every object $X \in \mathcal{C}$, there is a covering family $\{X_\alpha \to X\}_{\alpha \in A}$ of X such that every X_α is a member of G.
   Equivalently, the Grothendieck topology J is the smallest Grothendieck topology containing all covers of the U_i. Also equivalently, for any $S \in J(X)$, the sieve S contains a covering family $\{V_i \to X\}$ such that each morphism $V_i \to X$ factors through

some member of G. (♠ TODO: Verify that these claimed equivalences are indeed equivalences)

5. (See [GV72, Exposé II, Définition 3.0.2]) A $\mathscr{U}$-*site* is a site whose underlying category $\mathcal{C}$ is $\mathscr{U}$-locally small (Definition 1.0.5) and which has a $\mathscr{U}$-small topologically generating family. A $\mathscr{U}$-site is called $\mathscr{U}$-*small* if its underlying category is $\mathscr{U}$-small. Similarly, a *small site* is a site whose underlying category is a set and a *locally small site* is a site whose underlying category is locally small (Definition 1.0.5).

Definition 6.1.5. An *essentially small site* is a site (Definition 6.1.4) whose underlying category is essentially small (Definition 1.0.11).

Definition 6.1.6 (Presheaf on a category)**.** Let C and $\mathcal{A}$ be (large) categories (Definition 1.0.1).

1. A *presheaf $\mathcal{F}$ on C with values in $\mathcal{A}$* is a functor
$$\mathcal{F} : C^{\mathrm{op}} \to \mathcal{A}.$$

   In other words, a presheaf $\mathcal{F}$ on C with values in $\mathcal{A}$ is simply a contravariant functor (Definition 1.0.3) from C to $\mathcal{A}$. Explicitly, for every object U in C, one has an object $\mathcal{F}(U)$ in $\mathcal{A}$ (called the *U-valued sections/sections evaluated at U of $\mathcal{F}$*, cf. Definition 6.2.1), and for every morphism $f : V \to U$ in C, one has a morphism (called the *restriction map*)
$$\mathcal{F}(f) : \mathcal{F}(U) \to \mathcal{F}(V)$$

   in $\mathcal{A}$, such that for all composable morphisms $W \xrightarrow{g} V \xrightarrow{f} U$ in C, the following diagram in $\mathcal{A}$ commutes:

$$\mathcal{F}(U) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(V) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(W)$$

with $\mathcal{F}(f \circ g)$ the outer arrow from $\mathcal{F}(U)$ to $\mathcal{F}(W)$.

   That is,
$$\mathcal{F}(g) \circ \mathcal{F}(f) = \mathcal{F}(f \circ g),$$
   and for every object U in C, $\mathcal{F}(\mathrm{id}_U) = \mathrm{id}_{\mathcal{F}(U)}$.

2. Let $\mathcal{F}, \mathcal{G} : C^{\mathrm{op}} \to \mathcal{A}$ be two presheaves on C with values in $\mathcal{A}$. A *morphism of presheaves*
$$\varphi : \mathcal{F} \to \mathcal{G}$$

   is a natural transformation of functors (Definition 1.0.4): for each object U of C, one has a morphism
$$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

   in $\mathcal{A}$, such that for every morphism $f : V \to U$ in C, the diagram

$$
\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(V) \\
\downarrow{\varphi_U} & & \downarrow{\varphi_V} \\
\mathcal{G}(U) & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}(V)
\end{array}
$$

commutes, i.e.,
$$\varphi_V \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \varphi_U$$
for all objects and morphisms in C.

3. Given a universe (Definition C.0.14) U, a U-*presheaf on $\mathcal{C}$* typically refers to a presheaf of U-sets on C.

4. The *presheaf category/category of $\mathcal{A}$-valued presheaves on $\mathcal{C}$* is the (large) category whose objects are the presheaves on C with values in $\mathcal{A}$ and whose morphisms are the presheaf morphisms. Common notations for the presheaf category include, but are not limited to: $\mathcal{A}^{\mathcal{C}^{\mathrm{op}}}$, $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$, $[\mathcal{C}^{\mathrm{op}}, \mathcal{A}]$. If the value category $\mathcal{A}$ is clear from context, then notations such as $\mathrm{PreShv}(\mathcal{C})$ are also common. Note that the presheaf category $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ is equivalent to the category of functors (Definition 1.3.1) $\mathcal{C}^{\mathrm{op}} \to \mathcal{A}$ and hence notations for the functor categories are applicable as notations for presheaf categories.

Definition 6.1.7 (Sheaf on a site)**.** Let $(\mathcal{C}, J)$ be a site (Definition 6.1.4). Let $\mathcal{A}$ be a (large) category (Definition 1.0.1).

1. A presheaf (Definition 6.1.6) $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$ (Definition 1.0.2) is called a *sheaf on the site $(\mathcal{C}, J)$ valued in $\mathcal{A}$* if, for every object U of $\mathcal{C}$ and every covering sieve (Definition 6.1.4) $S \in J(U)$, the limit (Definition 1.3.2)
$$\varprojlim_{(V \to U) \in (\mathcal{D}_S)^{\mathrm{op}}} \mathcal{F}|_{\mathcal{D}_S}(V),$$
exists and the canonical natural morphism
$$\mathcal{F}(U) \to \varprojlim_{(V \to U) \in (\mathcal{D}_S)^{\mathrm{op}}} \mathcal{F}|_{\mathcal{D}_S}(V)$$
is an isomorphism. Here, $\mathcal{D}_S \hookrightarrow \mathcal{C}/U$ (Definition 6.2.5) is the full downward-closed subcategory such that $\mathrm{Ob}(\mathcal{D}_S) = \{(f : V \to U) : f \in S(V)\}$,

   In particular, when we are working with a Grothendieck pretopology (Definition C.0.22) K on a category $\mathcal{C}$, we may speak of sheaves on the site whose Grothendieck topology is the one generated by (Definition C.0.23) K.

2. Given sheaves $\mathcal{F}, \mathcal{G} : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$ on the site $(\mathcal{C}, J)$, a *morphism between the sheaves* is a morphism (Definition 6.1.6) between $\mathcal{F}$ and $\mathcal{G}$ as presheaves.

3. Let U be a universe (Definition C.0.14). A U-*sheaf* typically refers to a U-presheaf that is a sheaf for a U-site. In other words, a U-sheaf is a sheaf on a site whose underlying category is U-locally small (Definition 1.0.5) and which has a U-small topologically generating family such that the sheaf is valued in U-sets.

4. The *sheaf category/category of $\mathcal{A}$-valued sheaves on $\mathcal{C}$* is the (large) category defined as the full subcategory of $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ whose objects are the sheaves on $\mathcal{C}$ with values in $\mathcal{A}$. Common notations for the sheaf category include $\mathrm{Shv}(\mathcal{C}, \mathcal{A})$, $\mathrm{Shv}(\mathcal{C}, J, \mathcal{A})$, $\mathrm{Sh}(\mathcal{C}, \mathcal{A})$, $\mathrm{Sh}(\mathcal{C}, J, \mathcal{A})$. If the value category $\mathcal{A}$ is clear from context, then notations such as $\mathrm{Shv}(\mathcal{C})$, $\mathrm{Shv}(\mathcal{C}, J)$, $\mathrm{Sh}(\mathcal{C})$, $\mathrm{Sh}(\mathcal{C}, J)$ are also common.

Definition 6.1.8 (Sheaf on a topological space)**.** Let X be a topological space, let $\mathcal{D}$ be a category (Definition 1.0.1) with a terminal object (Lemma 2.0.2), and let $\mathcal{F}$ be a presheaf valued in $\mathcal{D}$ on X. Then $\mathcal{F}$ is a *sheaf* if it satisfies the following additional condition (known as the *sheaf axioms*):

For every open set $U \subseteq X$ and every open cover $\{U_i\}_{i \in I}$ of U, let $\mathcal{J}$ be the diagram (Definition 1.3.1) in the category of opens of U consisting of the inclusions $U_i \cap U_j \hookrightarrow U_i$ for all $i, j \in I$. Then $\mathcal{F}$ is a sheaf if the limit (Definition 1.3.2) of the diagram $\mathcal{F} \circ \mathcal{J}$ exists in $\mathcal{D}$ and the natural morphism

$$\mathcal{F}(U) \to \lim_{j \in \mathcal{J}} \mathcal{F}(j)$$

is an isomorphism. More precisely, $\mathcal{J} : J \to \text{Open}(U)$ should be the functor whose index category J consists of

1. An object i for every $i \in I$ and an object (i, j) for every pair $i, j \in I$,
2. Morphisms $p_1 : (i, j) \to i$ and $p_2 : (i, j) \to j$ for every pair $i, j \in I$

and which sends the objects and morphisms as follows:

1. $\mathcal{J}(i) = U_i$
2. $\mathcal{J}(i, j) = U_i \cap U_j$
3. $\mathcal{J}(p_1) : U_i \cap U_j \hookrightarrow U_i$
4. $\mathcal{J}(p_2) : U_i \cap U_j \hookrightarrow U_j$.

In particular, taking $U = \emptyset$ and taking the empty open cover of the empty set, $\mathcal{F}(\emptyset)$ must be the terminal object (Lemma 2.0.2) of $\mathcal{D}$

In the case that $\mathcal{D}$ admits all small limits (Definition 1.3.5), the sheaf condition is equivalent to the following: For every open set $U \subset X$ and every open cover $\{U_i\}_{i \in I}$ of U, the following equalizer diagram is exact (Definition 2.0.12):

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j).$$

Here, the morphism $\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i)$ and the two morphisms $\prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$ are induced by the restriction maps $\mathcal{F}(U) \to \mathcal{F}(U_i)$ and $\mathcal{F}(U_i) \to \mathcal{F}(U_i \cap U_j)$.

In the case that $\mathcal{D}$ is some subcategory of the category of sets, the sheaf condition is equivalent to the following: For every open set $U \subseteq X$ and every open cover $\{U_i\}_{i \in I}$ of U,

- (Locality) If $s, t \in \mathcal{F}(U)$ are such that $s|_{U_i} = t|_{U_i}$ for all i, then $s = t$.
- (Gluing) If for each i there is $s_i \in \mathcal{F}(U_i)$ such that for all i, j one has $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all i.

Equivalently, a sheaf on a topological space X may be defined as a sheaf on (Definition 6.1.7) the site (Definition 6.1.4) of opens on X.

Definition 6.1.9. (♠ TODO: Move these notations to the defintions of presheaves and sheaves on topological spaces) Let X be a topological space, and let $\mathcal{D}$ be a category (Definition 1.0.1) with a terminal object (Lemma 2.0.2).

The presheaves on X valued in $\mathcal{D}$, along with the morphisms thereof, form a (in general large) category (Definition 1.0.1) often denoted by notations such as $\text{PreShv}(X, \mathcal{D})$ (♠ TODO: include more notations) (or $\text{PreShv}(X)$ if the category $\mathcal{D}$ is clear). If $\mathcal{D}$ is locally small (Definition 1.0.5), then so is $\text{PreShv}(X, \mathcal{D})$.

Similarly, the sheaves on X valued in $\mathcal{D}$ (Definition 6.1.8), along with the morphisms thereof, form a (in general large) category (Definition 1.0.1) often denoted by notations such as $\mathrm{Shv}(X, \mathcal{D})$ (♠ TODO: include more notations) (or $\mathrm{Shv}(X)$ if the category $\mathcal{D}$ is clear). The category $\mathrm{Shv}(X, \mathcal{D})$ is a full subcategory (Definition 1.0.8) of $\mathrm{PreShv}(X, \mathcal{D})$.

Equivalently, the categories of presheaves and sheaves are the categories $\mathrm{PreShv}(\mathbf{Open}(X), \mathcal{D})$ and $\mathrm{Shv}(\mathbf{Open}(X), \mathcal{D})$ of presheaves (Definition 6.1.6) and sheaves (Definition 6.1.7) where $\mathbf{Open}(X)$ is the category of open subsets of X equipped with its usual Grothendieck pretopology (Definition C.0.22).

Definition 6.1.10. Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be sites (Definition 6.1.4).

A functor $u : \mathcal{C} \to \mathcal{D}$ is said to be a *continuous functor of sites* if, for every object $U \in \mathrm{Ob}(\mathcal{D})$ and every covering sieve (Definition 6.1.4) $S \in K(U)$, the pullback sieve u^*S (Definition 6.1.3) belongs to $J(V)$ for all $V \in \mathcal{C}$ with a morphism $u(V) \to U$ in $\mathcal{D}$.

Equivalently, u is continuous if for every sheaf (Definition 6.1.7) of sets F on $\mathcal{D}$, the presheaf (Definition 6.1.6) $\mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}, X \mapsto F(u(X))$ is a sheaf on $\mathcal{C}$. (♠ TODO: show these are equivalent) (♠ TODO: define morphism of sites and recheck ref's to this definition)

Definition 6.1.11. Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be sites (Definition 6.1.4) with small topological generating families (Definition 6.1.4), and let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor of sites (Definition 6.1.10). Let $\mathcal{A}$ be a (large) category such that the presheaf category (Definition 6.1.6) $\mathrm{PreSh}(\mathcal{D}, K; \mathcal{A})$ has sheafification (Definition 1.1.4).

For any sheaf (Definition 6.1.7)

$$\mathcal{G} \in \mathrm{Sh}(\mathcal{C}, J; \mathcal{A}),$$

the *inverse image/pullback sheaf of $\mathcal{G}$ under u* is defined, assuming that all colimits below exist, as:

$$u_s \mathcal{G} : \mathcal{D}^{\mathrm{op}} \to \mathcal{A}, \quad V \mapsto a \left(\varinjlim_{(V \downarrow u)} \mathcal{G}(U) \right),$$

where a is the sheafification functor of presheaves and the colimit (Definition 1.3.2) is taken over the comma category $(V \downarrow u)$ of pairs $(U, V \to u(U))$ with $U \in \mathcal{C}$.

The assignment $\mathcal{G} \mapsto u_s \mathcal{G}$ defines the *inverse image/pullback functor*

$$u_s : \mathrm{Sh}(\mathcal{C}, J; \mathcal{A}) \to \mathrm{Sh}(\mathcal{D}, K; \mathcal{A}).$$

If u is the functor underlying a site morphism $f : (\mathcal{D}, K) \to (\mathcal{C}, J)$, we may alternatively denote $u_s \mathcal{G}$ by $f^* \mathcal{G}$ (or sometimes by $f^{-1} \mathcal{G}$) and call it the *inverse image/pullback of $\mathcal{G}$ under f*.

Note that while the continuous functor u and the site morphism f point in opposite directions, the identification $f^* := u_s$ ensures that f^* corresponds to the standard geometric pullback. In the case of topological spaces, this recovers the usual construction involving colimits over open neighborhoods to obtain stalks followed by sheafification.

Definition 6.1.12. Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be sites (Definition 6.1.4) with small topological generating families (Definition 6.1.4) (or U-small topologically generating families if a universe (Definition C.0.14) U is available), and let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor of sites (Definition 6.1.10).

For any sheaf (Definition 6.1.7)

$$\mathcal{F} \in \mathrm{Sh}(\mathcal{D}, K; \mathcal{A}),$$

Define the *pushforward/direct image sheaf* $u^s \mathcal{F}$ by

$$u^s \mathcal{F} := \mathcal{F} \circ u : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}.$$

Because u is continuous, $u^s \mathcal{F}$ is a sheaf on $(\mathcal{C}, J)$ valued in $\mathcal{A}$. The assignment $\mathcal{F} \mapsto u^s \mathcal{F}$ defines the *direct image/pushforward functor*

$$u^s : \mathrm{Sh}(\mathcal{D}, K; \mathcal{A}) \to \mathrm{Sh}(\mathcal{C}, J; \mathcal{A}).$$

If u is the functor underlying a site morphism $f : (\mathcal{D}, K) \to (\mathcal{C}, J)$, we may alternatively denote $u^s \mathcal{F}$ by $f_* \mathcal{F}$ and call it the *direct image/pushforward of $\mathcal{F}$ under f*; the assignment $\mathcal{F} \mapsto f_* \mathcal{F}$ is then the *direct image/pushforward functor*.

$$f_* : \mathrm{Sh}(\mathcal{D}, K; \mathcal{A}) \to \mathrm{Sh}(\mathcal{C}, J; \mathcal{A}).$$

Note that while the continuous functor u and the site morphism f point in opposite directions, the definition $f_* := u^s$ ensures that f_* corresponds to the standard geometric pushforward used in topology and algebraic geometry.

Theorem 6.1.13. (♠ TODO: It is likely that some more restrictions are needed; e.g. must C and D be small and $\mathcal{A}$ locally small to ensure that we can talk about hom's between sheaves?) Let (C, J) and (D, K) be sites (Definition 6.1.4), and let $u : C \to D$ be a continuous functor (Definition 6.1.10) of sites. Let $\mathcal{A}$ be a (large) category such that the presheaf category (Definition 6.1.6) $\mathrm{PreSh}(\mathcal{D}, K; \mathcal{A})$ has sheafification (Definition 1.1.4).

Then the inverse image functor (Definition 6.1.12)

$$u_s : \mathrm{Sh}(D, K; \mathcal{A}) \to \mathrm{Sh}(C, J; \mathcal{A})$$

(assuming that the inverse images of all sheaves on (D, K) valued in $\mathcal{A}$ exist by virtue of $\mathcal{A}$ admitting enough colimits (Definition 1.3.12)) and the direct image functor (Definition 6.1.11)

$$u^s : \mathrm{Sh}(C, J; \mathcal{A}) \to \mathrm{Sh}(D, K; \mathcal{A})$$

form an adjoint pair (Definition C.0.16), i.e., for any sheaves $\mathcal{F} \in \mathrm{Sh}(D, K; \mathcal{A})$ and $\mathcal{G} \in \mathrm{Sh}(C, J; \mathcal{A})$, there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Sh}(C,J)}(u_s \mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{Sh}(D,K)}(\mathcal{F}, u^s \mathcal{G}).$$

Definition 6.1.14 (Pushforward (direct image) of a sheaf)**.** Let $f : X \to Y$ be a continuous map between topological spaces, and let $\mathcal{F}$ be a presheaf on X valued in a category $\mathcal{D}$ with a terminal object (Lemma 2.0.2). The *pushforward* or *direct image presheaf* $f_* \mathcal{F}$ on Y is the

presheaf valued in $\mathcal{D}$ on Y (Definition 6.1.6) defined as follows: For every open set $V \subseteq Y$, the value of the pushforward is given by

$$f_* \mathcal{F}(V) := \mathcal{F}(f^{-1}(V)).$$

For an inclusion of open sets $V' \subseteq V$ in Y, the restriction morphism

$$\mathrm{res}_{V,V'}^{f_* \mathcal{F}} : f_* \mathcal{F}(V) \to f_* \mathcal{F}(V')$$

is defined as the restriction morphism of $\mathcal{F}$ associated with the inclusion of preimages $f^{-1}(V') \subseteq f^{-1}(V)$ in X:

$$\mathrm{res}_{f^{-1}(V), f^{-1}(V')}^{\mathcal{F}} : \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(f^{-1}(V')).$$

If $\mathcal{F}$ is a sheaf (Definition 6.1.8), then so is $f_* \mathcal{F}$. In this case, it is equivalent to define $f_* \mathcal{F}$ as the direct image (Definition 6.1.12) $(f^{-1})^s \mathcal{F}$ of $\mathcal{F}$ under the continuous functor $f^{-1} : \mathrm{Open}\, Y \to \mathrm{Open}\, X$, which is also equivalent to the direct image (Definition 6.1.12) of $\mathcal{F}$ under the site morphism $\mathrm{Open}\, X \to \mathrm{Open}\, Y$ whose underlying continuous functor is f^{-1}.

Definition 6.1.15 (Pullback (inverse image) of a sheaf). Let $f : X \to Y$ be a continuous map between topological spaces. Let $\mathcal{D}$ be a category with a terminal object (Lemma 2.0.2).

1. Let $\mathcal{G}$ be a presheaf on Y valued in a $\mathcal{D}$. The *pullback* or *inverse image presheaf* $f^{-1}\mathcal{G}$ on X is defined as the presheaf

$$U \mapsto \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$$

   where U ranges over open subsets of X and the colimit is taken over all open subsets $V \subseteq Y$ containing $f(U)$.

2. If $\mathcal{G}$ is a sheaf (Definition 6.1.8) valued in $\mathcal{D}$, then we can define the *pullback* or *inverse image sheaf* $f^* \mathcal{G}$ on X as the sheaf associated to the presheaf (Definition 1.1.4) $f^{-1}\mathcal{G}$, assuming it exists.

   Assuming that a sheafification functor (Definition 1.1.4) exists, one may equivalently define $f^* \mathcal{G}$ via Definition 6.1.11 — More concretely, $f^* \mathcal{G}$ is the following equivalent constructions:

   - The direct image (Definition 6.1.11) $(f^{-1})_s \mathcal{G}$ of $\mathcal{G}$ under the continuous functor $f^{-1} : \mathrm{Open}\, Y \to \mathrm{Open}\, X$, $W \mapsto f^{-1}(W)$.
   - The inverse image (Definition 6.1.11) of $\mathcal{G}$ under the site morphism $\mathrm{Open}\, X \to \mathrm{Open}\, Y$ whose underlying continuous functor (Definition 6.1.10) is f^{-1}

Theorem 6.1.16. Let $f : X \to Y$ be a continuous map between topological spaces. Let $\mathcal{A}$ be a locally small category (Definition 1.0.5) such that $\mathcal{A}$ is cocomplete (Definition 2.0.5), i.e. admits small colimits (Definition 1.3.5), and such that $\mathrm{PreShv}(Y, \mathcal{A})$ admits sheafification.

The inverse image functor (Definition 6.1.15)

$$f^* : \mathrm{Sh}(Y, \mathcal{A}) \to \mathrm{Sh}(X, \mathcal{A})$$

and the direct image functor (Definition 6.1.14)

$$f_* : \mathrm{Sh}(X, \mathcal{A}) \to \mathrm{Sh}(Y, \mathcal{A})$$

form an adjoint pair (Definition C.0.16) $f^* \dashv f_*$, i.e., for any sheaves $\mathcal{F} \in \mathrm{Sh}(Y, \mathcal{A})$ and $\mathcal{G} \in \mathrm{Sh}(X, \mathcal{A})$, there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Sh}(X,\mathcal{A})}(f^*\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{Sh}(Y,\mathcal{A})}(\mathcal{F}, f_*\mathcal{G}).$$

Theorem 6.1.17 (Comparison Lemma, cf. [GV72, Exposé III, Théorème 4.1]). Let $\mathcal{C}$ be a small category (Definition 1.0.5), let $(\mathcal{D}, K)$ be a site (Definition 6.1.4), let $u : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor (Definition 1.0.9), and equip $\mathcal{C}$ with the Grothendieck topology (Definition 6.1.4) J induced by u. Let $\mathcal{A}$ be a locally small (Definition 1.0.5) complete category (Definition 2.0.5), i.e. a category which admits all small limits (Definition 1.3.5).

If the objects of $\mathcal{C}$ form a topologically generating family (Definition 6.1.4)(♠ TODO: I need to check that the condition of covering in SGA really matches the notion of topologically generating family) of $\mathcal{D}$, then the direct image (Definition 6.1.12)/restriction functor

$$u_s : \mathrm{Sh}(\mathcal{D}, K; \mathcal{A}) \to \mathrm{Sh}(\mathcal{C}, J; \mathcal{A})$$

is an equivalence of categories (Definition 1.0.10). In particular, $\mathrm{Sh}(\mathcal{D}, K; \mathcal{A})$ is a locally small category (Definition 1.0.5).

Proof. We explicitly construct the inverse functor and prove the equivalence without assuming *a priori* that $\mathrm{Sh}(\mathcal{D}, K)$ is locally small.

Step 1: Construction of the Inverse Functor u_*

We define the functor $u_* : \mathrm{Sh}(\mathcal{C}, J) \to \mathrm{Sh}(\mathcal{D}, K)$ as the **Right Kan Extension** of a sheaf F along u.

For a sheaf $F \in \mathrm{Sh}(\mathcal{C}, J)$ and an object $d \in \mathcal{D}$, define:

$$u_*F(d) = \lim_{(c,f) \in (u \downarrow d)^{op}} F(c)$$

Here, the limit is taken over the opposite of the comma category $(u \downarrow d)$, whose objects are pairs $(c, f : u(c) \to d)$.

Verification of Well-Definedness (Size): Since $\mathcal{C}$ is a small category and $\mathcal{D}$ is locally small, the collection of objects in $(u \downarrow d)$ is a set (indexed by objects of $\mathcal{C}$ and hom-sets of $\mathcal{D}$). Therefore, the limit defining $u_*F(d)$ is a **small limit of sets**, which exists in **Set**. Thus, u_*F takes values in **Set** rather than proper classes.

Verification that u_*F is a Sheaf: Since limits commute with limits, and the sheaf condition is a limit condition, the Right Kan Extension of a sheaf along a continuous functor is a sheaf. The density condition ensures that covers in $\mathcal{D}$ are "seen" by $\mathcal{C}$, ensuring the sheaf condition is preserved.

Step 2: The Unit of Adjunction ($u^* u_* \cong \mathrm{Id}$)

We examine $u^* u_* F$ for $F \in \mathrm{Sh}(\mathcal{C}, J)$. Evaluating at an object $c_0 \in \mathcal{C}$:

$$(u^* u_* F)(c_0) = u_*F(u(c_0)) = \lim_{(c,f) \in (u \downarrow u(c_0))^{op}} F(c)$$

Since u is **fully faithful**, the comma category $(u \downarrow u(c_0))$ has a terminal object: $(c_0, \mathrm{id}_{u(c_0)})$. The limit over a category with a terminal object is isomorphic to the value at that object. Thus, $(u^* u_* F)(c_0) \cong F(c_0)$, concluding $u^* u_* \cong \mathrm{Id}_{\mathrm{Sh}(\mathcal{C})}$.

Step 3: The Counit of Adjunction ($H \cong u_* u^* H$)

This is the critical step that uses the **Density Condition** to control the size of sheaves on $\mathcal{D}$. Let $H \in \mathrm{Sh}(\mathcal{D}, K)$. We construct a map $\eta_d : H(d) \to u_*(u^* H)(d)$. By definition:

$$u_*(u^* H)(d) = \lim_{u(c) \to d} H(u(c))$$

There is a canonical map η_d induced by the morphisms $H(f) : H(d) \to H(u(c))$ for each $f : u(c) \to d$.

To see that η_d is an isomorphism:

1. **Density as a Cover:** By hypothesis, the family of morphisms $\mathcal{S} = \{f : u(c) \to d \mid c \in \mathcal{C}\}$ generates a covering sieve S on d.
2. **Sheaf Property:** Since H is a sheaf on $(\mathcal{D}, K)$, $H(d) \cong \mathrm{Match}(S, H)$.
3. **Matching Families:** The limit over the comma category $(u \downarrow d)$ is exactly the set of compatible families indexed by the generators of the sieve. Because $\mathcal{S}$ generates S, the data of a matching family on $\mathcal{S}$ extends uniquely to the whole sieve S.

The canonical map $H(d) \to \lim_{u(c) \to d} H(u(c))$ is therefore an isomorphism, so $H \cong u_*(u^* H)$.

Step 4: Conclusion of Equivalence and Local Smallness

We have established natural isomorphisms $u^* u_* \cong \mathrm{Id}$ and $\mathrm{Id} \cong u_* u^*$, establishing an equivalence. Since $\mathcal{C}$ is small, $\mathrm{Sh}(\mathcal{C}, J)$ is a locally small category. Since $\mathrm{Sh}(\mathcal{D}, K)$ is equivalent to it, $\mathrm{Sh}(\mathcal{D}, K)$ is itself locally small. Specifically:

$$\mathrm{Hom}_{\mathcal{D}}(H, G) \cong \mathrm{Hom}_{\mathcal{C}}(u^* H, u^* G)$$

where the latter is a set.

Since $\mathcal{C}$ is small and $\mathcal{A}$ is locally small, the category of presheaves on $\mathcal{C}$ valued in $\mathcal{A}$ is locally small by Lemma 1.0.7. Therefore, the category of sheaves is locally small. □

Definition 6.1.18. (♠ TODO: there are places where sites and sheaves of rings on them are used, but it would be better to just have them be ringed sites.)

A *ringed site* is a site (Definition 6.1.4) $(\mathcal{C}, J)$ with a small topological generating family (Definition 6.1.4) equipped with a sheaf (Definition 6.1.7) of (not necessarily commutative) rings $\mathcal{O}$. If the Grothendieck topology J is clear in context, one may even write that $(C, \mathcal{O})$ is a ringed site.

A *morphism of ringed sites*

$$((\mathcal{C}, J), \mathcal{O}) \to ((\mathcal{C}', J'), \mathcal{O}')$$

consists of a morphism of sites $f : (\mathcal{C}, J) \to (\mathcal{C}', J')$ and a morphism of sheaves (Definition 6.1.7) of rings $f^{\#} : \mathcal{O}' \to f_* \mathcal{O}$ (Definition 6.1.11).

6.2. Sheaf cohomology.

Definition 6.2.1. Let $\mathcal{C}$ be a (large) category (Definition 1.0.1), and let $\mathcal{D}$ be a (large) category (Definition 1.0.1). Let $\mathcal{F} : \mathcal{C}^{op} \to \mathcal{D}$ be a presheaf valued in $\mathcal{D}$ (Definition 6.1.6).

1. For an object $U \in \mathcal{C}$, the *sections functor evaluated at U* is the functor
$$\Gamma(U, -) : \mathrm{PSh}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$$
   defined by
$$\Gamma(U, \mathcal{F}) := \mathcal{F}(U),$$
   i.e., the value of the presheaf $\mathcal{F}$ at the object U.

2. The *global sections of $\mathcal{F}$* is the object $\Gamma(\mathcal{F})$ of $\mathcal{D}$ defined as the limit (Definition 1.3.2)
$$\Gamma(\mathcal{F}) = \varprojlim_{U \in \mathcal{C}^{op}} \mathcal{F}(U)$$
   assuming that such a limit exists, where the limit is taken over objects $U \in \mathcal{C}$ and the restriction morphisms $\mathcal{F}(V) \to \mathcal{F}(U)$ in $\mathcal{D}$ for morphisms $U \to V$ in $\mathcal{C}$.

   If a final object (Definition 2.0.1) $* \in \mathcal{C}$ exists, then $\Gamma(\mathcal{F})$ exists and coincides with $\Gamma(*, \mathcal{F}) = \mathcal{F}(*)$. The construction $\Gamma(\mathcal{F})$ is functorial; in particular, if $\Gamma(\mathcal{F})$ exists for all $\mathcal{F}$ in $\mathrm{PSh}(\mathcal{C}, \mathcal{D})$, e.g. if limits of (Definition 1.3.2) diagrams in $\mathcal{D}$ indexed by $\mathcal{C}$ exist, then Γ is a functor
$$\Gamma : \mathrm{PSh}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$$
   called the *global sections functor on $\mathrm{PSh}(\mathcal{C}, \mathcal{D})$*.

Proposition 6.2.2. Let $\mathcal{C}$ be a locally small category (Definition 1.0.5) such that $\mathcal{C}$ has a final object (Definition 2.0.1) $*$. The presheaf (Definition 6.1.6) represented by (Definition 2.1.1) $*$ is a final object in the category $\mathrm{PSh}(\mathcal{C}, \mathbf{Sets})$ of set-valued presheaves.

Proposition 6.2.3. Let $\mathcal{C}$ be an essentially small category (Definition 1.0.11) and let $\mathcal{A}$ be an abelian category (Definition 2.0.10).

1. Let $U \in \mathrm{Ob}(\mathcal{C})$ be some fixed object. The sections functor (Definition 6.2.1)
$$\Gamma(U, -) : \mathrm{PSh}(\mathcal{C}, \mathcal{A}) \to \mathcal{A}$$
   is left exact (Definition 2.0.17). (♠ TODO: state that presheaves and sheaves valued in an abelian category form abelian categories)

2. Assume that $\Gamma(\mathcal{F})$ exists for all $\mathcal{F}$ in $\mathrm{PSh}(\mathcal{C}, \mathcal{A})$ so that Γ is a functor
$$\Gamma : \mathrm{PSh}(\mathcal{C}, \mathcal{A}) \to \mathcal{A}.$$
   The functor Γ is left exact.

Proof. Recall that $\mathrm{PSh}(\mathcal{C}, \mathcal{D})$ is an abelian category (Proposition 2.0.15) since $\mathcal{C}$ is essentially small. (♠ TODO: Talk about how limits are left exact and) □

Theorem 6.2.4 (e.g. see [Sta25, Tag 01DU])**.** For any site (Definition 6.1.4) $(\mathcal{C}, J)$ on an essentially small category (Definition 1.0.11) $\mathcal{C}$ and a sheaf of rings (Definition 6.1.7) $\mathcal{O}$ on $\mathcal{C}$, the category $\mathbf{Mod}(\mathcal{O})$ of $\mathcal{O}$-modules (Definition 1.1.2) is an abelian category that has enough injectives (Definition 4.2.2). In fact, there is a functorial injective embedding (♠ TODO: what does this mean?)

Definition 6.2.5 (Category of objects over a fixed object). Let $\mathcal{C}$ be a category (Definition 1.0.1) and let $X \in \mathrm{Ob}(\mathcal{C})$ be a fixed object.

1. The *category of objects over X* (or synonymously the *slice category of X in $\mathcal{C}$* or the *over category of X in $\mathcal{C}$*), commonly denoted $\mathcal{C}/X$, $\mathcal{C}_{/X}$, or $(\mathcal{C} \downarrow X)$ is the category defined as follows:
   - An object of $\mathcal{C}/X$ is a morphism $f \colon A \to X$ in $\mathcal{C}$, where $A \in \mathrm{Ob}(\mathcal{C})$.
   - A morphism from $f \colon A \to X$ to $g \colon B \to X$ in $\mathcal{C}/X$ is a morphism $h \colon A \to B$ in $\mathcal{C}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
A & \overset{h}{\longrightarrow} & B \\
 & {\scriptstyle f}\searrow & \downarrow {\scriptstyle g} \\
 & & X
\end{array}
$$

   i.e. such that $g \circ h = f$.
   - The identity morphisms and composition in $\mathcal{C}/X$ are inherited from $\mathcal{C}$.
2. The *category of objects under X* (or synonymously the *coslice category of X in $\mathcal{C}$* or the *under category of X in $\mathcal{C}$*), commonly denoted $X/\mathcal{C}$, $X\backslash\mathcal{C}$, $\mathcal{C}_{X/}$, or $(X \downarrow \mathcal{C})$, is the category defined as follows:
   - An object of $X/\mathcal{C}$ is a morphism $f \colon X \to A$ in $\mathcal{C}$, where $A \in \mathrm{Ob}(\mathcal{C})$.
   - A morphism from $f \colon X \to A$ to $g \colon X \to B$ in $X/\mathcal{C}$ is a morphism $h \colon A \to B$ in $\mathcal{C}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
X & \overset{f}{\longrightarrow} & A \\
 & {\scriptstyle g}\searrow & \downarrow {\scriptstyle h} \\
 & & B
\end{array}
$$

   i.e. such that $h \circ f = g$.
   - The identity morphisms and composition in $X/\mathcal{C}$ are inherited from $\mathcal{C}$.

Definition 6.2.6 (Slice site). Let $(\mathcal{C}, \tau)$ be a site (Definition 6.1.4), where τ is a Grothendieck topology on the (locally small or U-locally small (Definition 1.0.5), if a universe (Definition C.0.14) U is available) category $\mathcal{C}$. For a fixed object X in $\mathcal{C}$, the *slice site* (or the *over site*, the *site on the slice category $\mathcal{C}_{/X}$*, the *site induced on the over category $\mathcal{C}_{/X}$*, the *localization of the site $\mathcal{C}$ at the object X*, etc.) $(\mathcal{C}_{/X}, \tau_{/X})$ is the site whose underlying category is the slice category $\mathcal{C}_{/X}$ (Definition 6.2.5), and whose Grothendieck topology $\tau_{/X}$ (also denoted by notations such as $\tau|_X$ or τ/X) is defined by declaring a family of morphisms $\{f_i : Y_i \to Y\}$ in $\mathcal{C}_{/X}$ to be a covering if and only if the family $\{f_i : Y_i \to Y\}$ is a covering in $(\mathcal{C}, \tau)$.

Definition 6.2.7. Let $(\mathcal{C}, J)$ be a site (Definition 6.1.4) on a locally small category (Definition 1.0.5) or a U-site for some universe (Definition C.0.14) U. Let $\mathcal{O}$ be a sheaf of rings (Definition 6.1.7) on $\mathcal{C}$, so that $(\mathcal{C}, J, \mathcal{O})$ is a ringed site (Definition 6.1.18). Recall that the category $\mathbf{Mod}(\mathcal{O})$ (Definition 1.1.2) of $\mathcal{O}$-modules is abelian and has enough injectives (Definition 4.2.2) (Theorem 6.2.4).

Assume that global sections objects $\Gamma(\mathcal{G})$ (Definition 6.2.1) exist for all objects $\mathcal{G}$ of $\mathrm{Sh}(\mathcal{C}, \mathbf{Ab})$[5] so that Γ is a functor

$$\mathrm{Sh}(\mathcal{C}, \mathbf{Ab}) \to \mathbf{Ab},$$

which is a left exact functor (Definition 2.0.17) (Proposition 6.2.3). Note that Γ restricts to a left exact functor

$$\mathbf{Mod}(\mathcal{O}) \to \mathbf{Ab}.$$

If $\mathcal{C}$ has a final object (Definition 2.0.1) $*$ as well, then recall that $\Gamma(\mathcal{F}) = \mathcal{F}(*)$.

Let $\mathcal{F}$ be an object of $\mathbf{Mod}(\mathcal{O})$.

1. For each integer $n \geq 0$, the *n-th (abelian) (global) sheaf cohomology group of $\mathcal{F}$* is

$$H^n(\mathcal{C}, J; \mathcal{F}) := R^n\Gamma(\mathcal{F}),$$

where $R^n\Gamma$ is the n-th right derived functor (Definition 4.3.1) of the global sections functor Γ (Definition 6.2.1).

In particular, each H^n is a functor

$$H^n : \mathbf{Mod}(\mathcal{O}) \to \mathbf{Ab}.$$

2. Given an object $U \in \mathcal{C}$ and for each integer $n \geq 0$, the *n-th (abelian) sheaf cohomology group of $\mathcal{F}$ of sections at U* is

$$H^n(U, \mathcal{F}) := (R^n\Gamma(U, -))(\mathcal{F}),$$

where $R^n\Gamma(U, -)$ is the n-th right derived functor (Definition 4.3.1) of the sections functor $\Gamma(U, -)$ evaluated at U (Definition 6.2.1).

In particular, $H^n(U, \mathcal{F})$ can be regarded as the nth global sheaf cohomology group of the restriction $\mathcal{F}|_U$ of $\mathcal{F}$ to U.

We show that sheaf cohomology commutes with filtered colimits.

Definition 6.2.8. A site (Definition 6.1.4) $(\mathcal{C}, J)$ is called a *coherent site* if it satisfies the following conditions:

1. The category $\mathcal{C}$ is locally small with a small topologically generating family (Definition 6.1.4).
2. The category $\mathcal{C}$ admits all finite limits (Definition 1.3.2).
3. Every object in $\mathcal{C}$ is quasi-compact relative to the topology J.
4. The collection of quasi-compact objects relative to J is closed under fiber products (Definition C.0.21). Explicitly, if $X \to Z$ and $Y \to Z$ are morphisms where X, Y, Z are quasi-compact, then $X \times_Z Y$ is also quasi-compact.

Definition 6.2.9. A Grothendieck topos (Definition 1.1.8) $\mathcal{E}$ is called a *coherent topos* if there exists a coherent site (Definition 6.2.8) $(\mathcal{C}, J)$ such that $\mathcal{E}$ is equivalent to the category of sheaves (Definition 6.1.7) $\mathcal{E} \simeq \mathrm{Sh}(\mathcal{C}, J)$ on that site.

Alternatively, an intrinsic characterization [GSDV72, Exposé VI] is that $\mathcal{E}$ is coherent if there exists a generating family (Definition 2.1.6) $\mathcal{G}$ of objects in $\mathcal{E}$ such that:

---

[5]for example, this occurs when $\mathcal{C}$ is essentially small (Definition 1.0.11)

1. Every object in $\mathcal{G}$ is coherent (quasi-compact and quasi-separated).
2. The full subcategory generated by $\mathcal{G}$ is closed under fiber products (Definition C.0.21).

It is a theorem that the full subcategory of *all* coherent objects in a coherent topos is closed under finite limits and finite colimits (forming a so-called *pretopos*).

Theorem 6.2.10 (Standard Examples of Coherent Sites)**.** The following are standard examples of coherent sites. In each case, the topos of sheaves on the site is a coherent topos.

1. **The Zariski Site of a Coherent Scheme**: Let X be a *coherent scheme* (♠ TODO: define coherent scheme) (e.g., any *Noetherian scheme*). Let $\mathcal{C}$ be the category of open subsets of X that are *quasi-compact* (i.e., finite unions of affine opens). Let J be the standard finite open cover topology.
   Then $(\mathcal{C}, J)$ is a coherent site. The sheaves on this site are equivalent to the category of sheaves on the full Zariski site of X.
2. **The Étale Site of a Coherent Scheme**: Let X be a coherent scheme. Let $\mathcal{C}$ be the category of schemes *étale* over X which are themselves coherent (quasi-compact and quasi-separated over X) (♠ TODO: define étale morphism). Let J be the topology generated by finite surjective families of étale maps.
   Then $(\mathcal{C}, J)$ is a coherent site.
3. **The Nisnevich Site of a Coherent Scheme**: Let X be a coherent scheme. The Nisnevich site over X is a coherent site.
4. **The Site of Finite Sets**: Let $\mathcal{C} = \mathbf{FinSet}$ be the category of finite sets. Let J be the *canonical topology* (where covering families are jointly surjective finite families).
   This is a coherent site. The corresponding topos is **Set** itself.
5. **The Syntactic Site of a Coherent Theory**: Let $\mathbb{T}$ be a *coherent theory* in first-order logic (♠ TODO: define coherent theory). Let $\mathcal{C}_{\mathbb{T}}$ be the *syntactic category* of $\mathbb{T}$, whose objects are formulas modulo equivalence and morphisms are provable functional relations.
   Equipped with the topology where covers correspond to finite disjunctions ($\phi \leftrightarrow \bigvee_{i=1}^{n} \psi_i$), this forms a coherent site. The sheaf topos is the *classifying topos* of $\mathbb{T}$.

Theorem 6.2.11 (Sheaf cohomology of modules on a coherent site commutes with filtered colimits [GSDV72, Exposé VI Corollaire 5.2])**.** Let $\mathcal{E}$ be a coherent topos (Definition 6.2.9). In particular, the site (Definition 6.1.4) underlying $\mathcal{E}$ has a final object and so we may speak of the sheaf cohomology functors (Definition 6.2.7) $H^q(\mathcal{E}, -)$ as the derived functors of the global sections functor.

The functors $H^q(\mathcal{E}, -)$ commute with filtered colimits (Definition 1.3.12).

Explicitly, let $\{\mathcal{F}_i\}_{i \in I}$ be a filtered system (Definition 1.3.11) of abelian sheaves in $\mathcal{E}$. Then the canonical map

$$\operatorname{colim}_{i \in I} H^q(\mathcal{E}, \mathcal{F}_i) \xrightarrow{\cong} H^q(\mathcal{E}, \operatorname{colim}_{i \in I} \mathcal{F}_i)$$

is an isomorphism for all $q \geq 0$. In particular, the global sections functor $\Gamma(\mathcal{E}, -) = H^0(\mathcal{E}, -)$ commutes with filtered colimits.

6.3. **Sheafy Tor and Ext, and derived tensor product and Hom on sheaves.**

Definition 6.3.1. Let $\mathcal{A}$, $\mathcal{B}$ be categories (Definition 1.0.1), let $F : \mathcal{A} \to \mathcal{B}$ be a functor (Definition 1.0.3), and let $(\mathcal{D}, K)$ be a site (Definition 6.1.4).

Assume at least one of the following:

1. A sheafification functor (Definition 1.1.4)
$$a : \mathrm{PreSh}(\mathcal{D}, K; \mathcal{B}) \to \mathrm{Sh}(\mathcal{D}, K; \mathcal{B})$$
exists.
2. F is a continuous functor (Definition C.0.17) and $\mathcal{B}$ admits limits (Theorem 1.3.23) indexed by covering sieves (Definition 6.1.4) of K.

Define the $\boxed{\textit{sheafy functor } \mathscr{F}}$ of F under each of the above assumptions as follows:

1. As the functor
$$\mathscr{F} : \mathrm{PreSh}(\mathcal{D}, \mathcal{A}) \to \mathrm{Sh}(\mathcal{D}, K; \mathcal{B})$$
given by $S \mapsto a(F \circ S)$. By restriction, there is an induced functor
$$\mathrm{Sh}(\mathcal{D}, K; \mathcal{A}) \to \mathrm{Sh}(\mathcal{D}, K; \mathcal{B})$$
on the category of sheaves on $\mathcal{D}$ valued in $\mathcal{A}$; we usually denote this restricted functor by $\boxed{\mathscr{F}}$ as well.
2. As the functor
$$\mathscr{F} : \mathrm{Sh}(\mathcal{D}, K; \mathcal{A}) \to \mathrm{Sh}(\mathcal{D}, K; \mathcal{B})$$
given by $S \mapsto F \circ S$. Due to the assumptions on F and $\mathcal{B}$, $F \circ S$ is a sheaf.

Note that if both assumptions hold, then the two functors
$$\mathrm{Sh}(\mathcal{D}, K; \mathcal{A}) \to \mathrm{Sh}(\mathcal{D}, K; \mathcal{B})$$
are naturally isomorphic.

Sheafy functors are often notated with scripted letters.

Proposition 6.3.2. Let $(\mathcal{D}, K)$ be a site (Definition 6.1.4). Let $\mathcal{A}, \mathcal{B}$ be categories (Definition 1.0.1). There is an equivalence (Definition 1.0.10)
$$\mathrm{Sh}(\mathcal{D}, K; \mathcal{A} \times \mathcal{B}) \cong \mathrm{Sh}(\mathcal{D}, K; \mathcal{A}) \times \mathrm{Sh}(\mathcal{D}, K; \mathcal{B})$$
(Definition 6.1.7) (Definition 1.0.12).

Proof. (♠ TODO:) □

Theorem 6.3.3. Let $(\mathcal{D}, K)$ be a site (Definition 6.1.4). Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be categories (Definition 1.0.1). Let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ and $G : \mathcal{A}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{B}$ be functors such that for any fixed $A \in \mathcal{A}$, there is an adjunction (Definition C.0.16) $F(A, -) \dashv G(A, -)$. Assume the following:

- there is a sheafification functor (Definition 1.1.4)
$$a : \mathrm{PreSh}(\mathcal{D}, \mathcal{C}) \to \mathrm{Sh}(\mathcal{D}, K; \mathcal{C}),$$
and

- $\mathcal{B}$ admits limits (Definition 1.3.2) indexed by the covering sieves in the Grothendieck topology (Definition 6.1.4) K.

Let $\mathscr{F}$ be the composition

$$\mathrm{Sh}(D, K; \mathcal{A}) \times \mathrm{Sh}(D, K; \mathcal{B}) \cong \mathrm{Sh}(D, K; \mathcal{A} \times \mathcal{B}) \to \mathrm{Sh}(D, K; \mathcal{C})$$

of the equivalence of Proposition 6.3.2 with the sheafy functor (Definition 6.3.1) of F.

For a fixed sheaf $\mathcal{R} \in \mathrm{Sh}(\mathcal{D}, K; \mathcal{A})$, let $\mathscr{G}_{\mathcal{R}} : \mathrm{Sh}(\mathcal{D}, K; \mathcal{C}) \to \mathrm{Sh}(\mathcal{D}, K; \mathcal{B})$ be the functor defined section-wise by

$$\mathcal{T} \mapsto (U \mapsto G(\mathcal{R}(U), \mathcal{T}(U))).$$

Note that since $G(\mathcal{R}(U), -)$ is a right adjoint (Definition C.0.16), it preserves (Theorem 1.3.17) the limits (Definition 1.3.2) indexed by the covering sieves of K; thus, $\mathscr{G}_{\mathcal{R}}(\mathcal{T})$ is a sheaf whenever $\mathcal{T}$ is a sheaf.

Furthermore, the assignment $(\mathcal{R}, \mathcal{T}) \mapsto \mathscr{G}_{\mathcal{R}}(\mathcal{T})$ defines a functor

$$\mathscr{G} : \mathrm{Sh}(\mathcal{D}, K; \mathcal{A})^{\mathrm{op}} \times \mathrm{Sh}(\mathcal{D}, K; \mathcal{C}) \to \mathrm{Sh}(\mathcal{D}, K; \mathcal{B}).$$

Then for all $\mathcal{R} \in \mathrm{Sh}(\mathcal{D}, K; \mathcal{A})$, there is an adjunction

$$\mathscr{F}(\mathcal{R}, -) \dashv \mathscr{G}_{\mathcal{R}}(-)$$

induced by the pointwise adjunctions $F(\mathcal{R}(U), -) \dashv G(\mathcal{R}(U), -)$ and the universal property of sheafification.

Proof. (♠ TODO:) $\qquad \qquad \square$

6.3.2. *Tensor product and Hom's of sheaves.*

Definition 6.3.4 (Sheafified Tensor Product)**.** Let $(\mathcal{D}, K)$ be a site (Definition 6.1.4) that admits a small topologically generating family (Definition 6.1.4) (e.g. by being essentially small (Definition 6.1.5)) and let $\mathcal{R}, \mathcal{S}, \mathcal{T}$ be sheaves of rings (Definition 6.1.7) on $(\mathcal{D}, K)$. Let $\mathcal{M}$ be a sheaf (Definition 6.1.7) of $\mathcal{R}$-$\mathcal{S}$ bimodules (Definition 1.1.2) and $\mathcal{N}$ be a sheaf of $\mathcal{S}$-$\mathcal{T}$ bimodules. The *sheafified tensor product*, denoted $\mathcal{M} \otimes_{\mathcal{S}} \mathcal{N}$, is the sheaf of $\mathcal{R}$-$\mathcal{T}$ bimodules defined as the sheafification (Definition 1.1.4) of the section-wise tensor product presheaf:

$$U \mapsto \mathcal{M}(U) \otimes_{\mathcal{S}(U)} \mathcal{N}(U).$$

(Definition C.0.11) Explicitly, $\mathcal{M} \otimes_{\mathcal{S}} \mathcal{N} = a(U \mapsto \mathcal{M}(U) \otimes_{\mathcal{S}(U)} \mathcal{N}(U))$, where a is the sheafification functor, which exists by Theorem 1.1.5. (♠ TODO: to apply the theorem, it seems necessary to show that the category of sheaves of $\mathcal{R}$-$\mathcal{T}$ bimodules is complete, cocomplete (probably need sheafification of pointwise construction), has exact filtered colimits, and the IPC property)

In particular, we have a bi-additive functor (Definition 4.7.5)

$$\otimes : {}_{\mathcal{R}}\mathsf{Mod}_{\mathcal{S}} \times {}_{\mathcal{S}}\mathsf{Mod}_{\mathcal{T}} \to {}_{\mathcal{R}}\mathsf{Mod}_{\mathcal{T}}$$

In the framework of Definition 6.3.1, the sheafified tensor product is the evaluation of the sheafy functor $\mathscr{F}$ induced by the tensor product functor $F = \otimes : \mathcal{B}imod \times_{\mathcal{R}ing} \mathcal{B}imod \to$

$\mathcal{B}imod$ (**??**) (**??**); (♠ TODO: must show that the category $\mathcal{B}imod$ is complete, cocomplete, has exact essentially small filtered limits, and satisfies the IPC property to argue that there is a sheafification functor by Theorem 1.1.5) (♠ TODO: show that the "composable" subcategory of sheaves of bimodules over all sheaves of rings on the site is equivalence to the category of sheaves valued in the caetgory of all composable bimodules) Specifically, if $\mathbb{M} := (\mathcal{R}, \mathcal{S}, \mathcal{M}) \in \mathrm{Sh}(\mathcal{D}, K; \mathcal{B}imod)$ and $\mathbb{N} := (\mathcal{S}, \mathcal{T}, \mathcal{N}) \in \mathrm{Sh}(\mathcal{D}, K; \mathcal{B}imod)$, then the sheafified tensor product is the module component of the sheaf $\mathscr{F}(\mathbb{M}, \mathbb{N}) \in \mathrm{Sh}(\mathcal{D}, K; \mathcal{B}imod)$, which is the triple $(\mathcal{R}, \mathcal{T}, \mathcal{M} \otimes_{\mathcal{S}} \mathcal{N})$.

Definition 6.3.5 (Sheafified Hom). Let $(\mathcal{D}, K)$ be a site (Definition 6.1.4) that admits a small topologically generating family (Definition 6.1.4) (e.g. by being essentially small (Definition 6.1.5)) and let $\mathcal{R}, \mathcal{S}, \mathcal{T}$ be sheaves of rings on $(\mathcal{D}, K)$. Let $\mathcal{M}$ be a sheaf (Definition 6.1.7) of $\mathcal{R}$-$\mathcal{S}$ bimodules (Definition 1.1.2) and $\mathcal{N}$ be a sheaf of $\mathcal{R}$-$\mathcal{T}$ bimodules.

The *sheafified Hom*, denoted by notations such as $\mathcal{H}om_{\mathcal{R}}(\mathcal{M}, \mathcal{N})$, $\mathscr{H}om_{\mathcal{R}}(\mathcal{M}, \mathcal{N})$, or $\underline{\mathrm{Hom}}_{\mathcal{R}}(\mathcal{M}, \mathcal{N})$, is the sheaf of $\mathcal{S}$-$\mathcal{T}$ bimodules defined section-wise by:

$$\mathcal{H}om_{\mathcal{R}}(\mathcal{M}, \mathcal{N})(U) = \mathrm{Hom}_{\mathrm{Sh}(\mathcal{D}/U, K|_U; \mathrm{Mod}(\mathcal{R}|_U))}(\mathcal{M}|_U, \mathcal{N}|_U).$$

(Definition 6.2.5) (Definition 6.2.6) (**??**)

Equivalently, it is the unique sheaf which satisfies the right adjoint property relative to the sheafified tensor product (Theorem 6.3.6).

In the framework of Definition 6.3.1, the sheafified Hom corresponds to the right adjoint $\mathscr{G}$ to the sheafified tensor product $\mathscr{F}$. While the general framework defines sheafy functors via section-wise application, the internal Hom for sheaves of modules requires the restriction-based definition (using $\mathcal{D}/U$) to correctly represent the right adjoint in the category of sheaves, ensuring compatibility with the varying ring structure $\mathcal{R}$.

(♠ TODO: the following would actually need to be modified to have a notion of "Hom-composable" category of functors) Also in the framework of Definition 6.3.1, the sheafified Hom is the evaluation of the sheafy functor $\mathscr{G}$ induced by the algebraic Hom functor $G : \mathcal{A}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{B}$ under assumption (2). Given input sheaves $\mathbb{M} := (\mathcal{R}, \mathcal{S}, \mathcal{M})$ and $\mathbb{N} := (\mathcal{R}, \mathcal{T}, \mathcal{N})$, the result is the sheaf $\mathscr{G}(\mathbb{M}, \mathbb{N}) = (\mathcal{S}, \mathcal{T}, \mathcal{H}om_{\mathcal{R}}(\mathcal{M}, \mathcal{N}))$. Because G is a right adjoint in the base categories, it is a continuous functor (i.e., it preserves limits); thus, $\mathscr{G}$ can be defined section-wise as in Case (2) of the framework. This section-wise construction $U \mapsto \mathrm{Hom}_{\mathcal{R}(U)}(\mathcal{M}(U), \mathcal{N}(U))$ is naturally isomorphic to the restriction-based internal Hom defined above, as both satisfy the same local-to-global universal property.

Theorem 6.3.6 (Tensor-Hom Adjunction for Sheaves of Bimodules). Let $(\mathcal{D}, K)$ be a site (Definition 6.1.4) that admits a small topologically generating family (Definition 6.1.4) (e.g. by being essentially small (Definition 6.1.5)). Let $\mathcal{R}, \mathcal{S}, \mathcal{T}$ be sheaves (Definition 6.1.7) of rings on $(\mathcal{D}, K)$. Let $\mathcal{M}$ be a sheaf of $\mathcal{R}$-$\mathcal{S}$ bimodules (Definition 1.1.2), $\mathcal{N}$ be a sheaf of $\mathcal{S}$-$\mathcal{T}$ bimodules, and $\mathcal{P}$ be a sheaf of $\mathcal{R}$-$\mathcal{T}$ bimodules.

Then there is a natural isomorphism:

$$\mathrm{Hom}_{\mathrm{Sh}(\mathcal{R}\text{-}\mathcal{T})}(\mathcal{M} \otimes_{\mathcal{S}} \mathcal{N}, \mathcal{P}) \cong \mathrm{Hom}_{\mathrm{Sh}(\mathcal{S}\text{-}\mathcal{T})}(\mathcal{N}, \mathcal{H}om_{\mathcal{R}}(\mathcal{M}, \mathcal{P}))$$

where $\mathcal{M} \otimes_{\mathcal{S}} \mathcal{N}$ is the sheafified tensor product and $\mathcal{H}om_{\mathcal{R}}(\mathcal{M}, \mathcal{P})$ is the sheaf of local $\mathcal{R}$-linear homomorphisms.

Proof. (♠ TODO: Verify this proof We apply Theorem 6.3.3 by specifying the categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and the functors F, G as follows:

1. **Categories:** Let $\mathcal{A}$ be the category whose objects are triples (R, S, M) where M is an R-S bimodule. Define $\mathcal{B}$ and $\mathcal{C}$ similarly for S-T and R-T bimodules.
2. **Base Functors:** Define $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ by $(M, N) \mapsto M \otimes_S N$ and $G : \mathcal{A}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{B}$ by $(M, P) \mapsto \mathrm{Hom}_R(M, P)$. At each component, $F(A, -) \dashv G(A, -)$ is the standard algebraic tensor-hom adjunction for bimodules.
3. **Sheaf Interpretation:** The sheaves $\mathcal{M}, \mathcal{N}, \mathcal{P}$ are viewed as objects in $\mathrm{Sh}(\mathcal{D}, K; \mathcal{A})$, $\mathrm{Sh}(\mathcal{D}, K; \mathcal{B})$, and $\mathrm{Sh}(\mathcal{D}, K; \mathcal{C})$ respectively, where the ring components of these sheaves are fixed to be the sheaves $\mathcal{R}, \mathcal{S}, \mathcal{T}$.

By Theorem 6.3.3, the sheafy functor $\mathscr{F}$ associated to the tensor product is:

$$\mathscr{F}(\mathcal{M}, \mathcal{N}) = a(U \mapsto \mathcal{M}(U) \otimes_{\mathcal{S}(U)} \mathcal{N}(U))$$

which is precisely the definition of the sheaf of $\mathcal{R}$-$\mathcal{T}$ bimodules $\mathcal{M} \otimes_{\mathcal{S}} \mathcal{N}$.

The right adjoint functor $\mathscr{G}_{\mathcal{M}}$ is defined section-wise by:

$$\mathscr{G}_{\mathcal{M}}(\mathcal{P}) : U \mapsto \mathrm{Hom}_{\mathcal{R}(U)}(\mathcal{M}(U), \mathcal{P}(U))$$

Per the theorem, since G is a right adjoint, $\mathscr{G}_{\mathcal{M}}(\mathcal{P})$ is a sheaf. In the category of modules, this section-wise Hom sheaf is naturally isomorphic to the internal sheaf $\mathcal{H}om_{\mathcal{R}}(\mathcal{M}, \mathcal{P})$ because any morphism of sheaves $\mathcal{M}|_U \to \mathcal{P}|_U$ is uniquely determined by its action on sections when the sheafification of the section-wise Hom agrees with the internal Hom.

Thus, the general adjunction $\mathscr{F}(\mathcal{M}, -) \dashv \mathscr{G}_{\mathcal{M}}(-)$ specializes to the required isomorphism of Hom-sets in the category of sheaves of bimodules.) □

6.3.3. *Sheafy Tor and Ext.* (♠ TODO:)

6.3.4. *Derived tensor product and Hom on sheaves.*

Definition 6.3.7 (cf. [DA73, Exposé XVII, Proposition 4.1.7.])**.** Let $(\mathcal{C}, J)$ be a site (Definition 6.1.4) that admits a small topologically generating family (Definition 6.1.4) (e.g. by being essentially small (Definition 6.1.5)). Let $\mathcal{R}, \mathcal{S}, \mathcal{T}$ be sheaves of rings on $(\mathcal{C}, J)$.

The tensor product functor (Definition 6.3.4)

$$\otimes : {}_{\mathcal{R}}\mathsf{Mod}_{\mathcal{S}} \times {}_{\mathcal{S}}\mathsf{Mod}_{\mathcal{T}} \to {}_{\mathcal{R}}\mathsf{Mod}_{\mathcal{T}}$$

(Definition 1.1.2) induces (Theorem 5.1.7) an exact multifunctor (Definition 5.1.5)

$$\otimes = K(\otimes) : K^-({}_{\mathcal{R}}\mathsf{Mod}_{\mathcal{S}}) \times K^-({}_{\mathcal{S}}\mathsf{Mod}_{\mathcal{T}}) \to K^-({}_{\mathcal{R}}\mathsf{Mod}_{\mathcal{T}})$$

(Definition 5.1.1). The functor $\otimes = K(\otimes)$ is left differentiable (Definition 5.5.5) into a *(total derived) tensor product functor*

$$\overset{\mathbf{L}}{\otimes} \colon D^-({}_{\mathcal{R}}\mathsf{Mod}_{\mathcal{S}}) \times D^-({}_{\mathcal{S}}\mathsf{Mod}_{\mathcal{T}}) \to D^-({}_{\mathcal{R}}\mathsf{Mod}_{\mathcal{T}})$$

(Definition 5.3.1) (♠ TODO: prove that the functor is indeed left differentiable, say by appealing to Theorem 5.5.9 along with the fact that enough flats exist in the category of sheaves of modules, and that the homotopy and derived categories of products of abelian categories are equivalent to the products of homotopy and derived categories of the individual abelian categories)

7. Spectral Sequences

Definition 7.0.1. Let $\mathcal{A}$ be an abelian category (Definition 2.0.10) . A *graded object in $\mathcal{A}$* is a collection of objects $M = \{M_n\}_{n\in\mathbb{Z}}$ in $\mathcal{A}$ indexed by the integers.

Definition 7.0.2. Let $\mathcal{A}$ be an abelian category (Definition 2.0.10). Let $M = \{M_n\}_{n\in\mathbb{Z}}$ and $N = \{N_n\}_{n\in\mathbb{Z}}$ be graded objects (Definition 7.0.1) in $\mathcal{A}$. A *morphism of graded objects* $f : M \to N$ is a family of morphisms in $\mathcal{A}$

$$f = \{f_n : M_n \to N_n\}_{n\in\mathbb{Z}}.$$

Composition of such morphisms is defined component-wise: $(g \circ f)_n = g_n \circ f_n$.

Definition 7.0.3. Let $\mathcal{A}$ be an abelian category (Definition 2.0.10). The *category of graded objects in $\mathcal{A}$*, denoted $\mathrm{Gr}(\mathcal{A})$, is the category whose objects are graded objects in $\mathcal{A}$ and whose morphisms are morphisms of graded objects.

Proposition 7.0.4. Let $\mathcal{A}$ be an abelian category (Definition 2.0.10). The category $\mathrm{Gr}(\mathcal{A})$ (Definition 7.0.3) is an abelian category. Limits, colimits (Definition 1.3.2), kernels, and cokernels (Definition 1.2.2) in $\mathrm{Gr}(\mathcal{A})$ are formed component-wise in $\mathcal{A}$. Explicitly, a sequence of graded objects

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

is exact (Definition 2.0.19) in $\mathrm{Gr}(\mathcal{A})$ if and only if for every $n \in \mathbb{Z}$, the sequence of components

$$0 \to L_n \xrightarrow{f_n} M_n \xrightarrow{g_n} N_n \to 0$$

is exact in $\mathcal{A}$.

Definition 7.0.5. Let $\mathcal{A}$ be an abelian category (Definition 2.0.10). Let $M = \{M_n\}_{n\in\mathbb{Z}}$ be a graded object (Definition 7.0.1) in $\mathcal{A}$. A *graded subobject of M* is a graded object $N = \{N_n\}_{n\in\mathbb{Z}}$ in $\mathcal{A}$ together with a family of monomorphisms (Definition 1.2.1)

$$\iota_n : N_n \hookrightarrow M_n$$

for each $n \in \mathbb{Z}$. If we view M as an object in the category of graded objects $\mathrm{Gr}(\mathcal{A})$ (Definition 7.0.3), this is equivalent to saying that N is a subobject (Definition 1.2.3) of M in $\mathrm{Gr}(\mathcal{A})$.

Definition 7.0.6. Let R, S be (not necessarily commutative) rings (Definition C.0.2). A *graded R-S-bimodule* is a collection of R-S-bimodules (Definition C.0.4) $M = \{M_n\}_{n \in \mathbb{Z}}$ indexed by the integers. Equivalently, a graded R-S-bimodule is a graded object (Definition 7.0.1) in the abelian category of R-S-bimodules (Definition C.0.8). An element $x \in M_n$ is called a *homogeneous element of degree n*.

Definition 7.0.7. Let $\mathcal{A}$ be an abelian category (Definition 2.0.10). A *bigraded object in $\mathcal{A}$* is a collection of objects $E = \{E_{p,q}\}_{p,q \in \mathbb{Z}}$ in $\mathcal{A}$ indexed by pairs of integers.

Definition 7.0.8. Let E be a bigraded object (Definition 7.0.7) in an abelian category (Definition 2.0.10) $\mathcal{A}$.

A *differential of bidegree (δ_p, δ_q) on E* is a family of morphisms in $\mathcal{A}$

$$d : E_{p,q} \to E_{p+\delta_p, q+\delta_q}$$

(or $d : E^{p,q} \to E^{p+\delta_p, q+\delta_q}$ depending on notation) defined for all $p, q \in \mathbb{Z}$, such that $d \circ d = 0$. The pair (E, d) is called a *differential bigraded object*.

Two specific conventions are standard in spectral sequence theory (where $r \geq 0$ is the page index):

1. **Homological convention:** The objects are denoted $E_{p,q}$. The differential usually has bidegree $(-r, r-1)$. Thus, d **decreases** the total degree $p + q$ by 1.
2. **Cohomological convention:** The objects are denoted $E^{p,q}$. The differential usually has bidegree $(r, 1-r)$. Thus, d **increases** the total degree $p + q$ by 1.

Definition 7.0.9. Let (E, d) be a differential bigraded object (Definition 7.0.8) in an abelian category (Definition 2.0.10) $\mathcal{A}$. Since $\mathcal{A}$ is abelian, the condition $d \circ d = 0$ implies that for every (p, q), the image (Definition 1.2.5) of the incoming differential is a subobject of the kernel (Definition 1.2.2) of the outgoing differential. We define the *homology* (or *cohomology*) $H(E, d)$ as the bigraded object of quotients $\ker(d)/\mathrm{im}(d)$. Explicitly:

1. If d has *homological bidegree* $(-r, r-1)$, then:

$$H_{p,q}(E, d) = \frac{\ker(d : E_{p,q} \to E_{p-r, q+r-1})}{\mathrm{im}(d : E_{p+r, q-r+1} \to E_{p,q})}.$$

2. If d has *cohomological bidegree* $(r, 1-r)$, then:

$$H^{p,q}(E, d) = \frac{\ker(d : E^{p,q} \to E^{p+r, q-r+1})}{\mathrm{im}(d : E^{p-r, q+r-1} \to E^{p,q})}.$$

Definition 7.0.10. Let M be an object in an abelian category (Definition 2.0.10) $\mathcal{A}$. A *filtration F on M* is a family of subobjects of M indexed by $\mathbb{Z}$. There are two standard conventions:

1. An *increasing filtration* (or *ascending filtration*) is denoted by subscripts $\{F_p M\}_{p \in \mathbb{Z}}$ such that:

$$\cdots \hookrightarrow F_{p-1}M \hookrightarrow F_p M \hookrightarrow F_{p+1}M \hookrightarrow \cdots \hookrightarrow M.$$

2. A *decreasing filtration* (or *descending filtration*) is denoted by superscripts $\{F^p M\}_{p\in\mathbb{Z}}$ such that:
$$\cdots \hookrightarrow F^{p+1}M \hookrightarrow F^p M \hookrightarrow F^{p-1}M \hookrightarrow \cdots \hookrightarrow M.$$

The pair (M, F) is called a *filtered object*.

Definition 7.0.11. Let $\mathcal{A}$ be an abelian category (Definition 2.0.10). Let (M, F) be a filtered object (Definition 7.0.10) in $\mathcal{A}$.

1. Let F be an **increasing** filtration $\{F_p M\}_{p\in\mathbb{Z}}$.
   - We say F is *exhaustive* if the colimit (Definition 1.3.2) of the diagram $\cdots \hookrightarrow F_p M \hookrightarrow \ldots$ exists in $\mathcal{A}$ and the canonical morphism induced by the inclusions $F_p M \hookrightarrow M$ is an isomorphism:
     $$\operatorname{colim}_{p\to\infty} F_p M \xrightarrow{\cong} M.$$
   - We say F is *separated* (or Hausdorff) if the limit of the diagram $\cdots \hookrightarrow F_p M \hookrightarrow \ldots$ exists in $\mathcal{A}$ and is the zero object:
     $$\lim_{p\to-\infty} F_p M \cong 0.$$

2. Let F be a **decreasing** filtration $\{F^p M\}_{p\in\mathbb{Z}}$.
   - We say F is *exhaustive* if the colimit of the diagram $\cdots \hookrightarrow F^p M \hookrightarrow \ldots$ exists in $\mathcal{A}$ and the canonical morphism is an isomorphism:
     $$\operatorname{colim}_{p\to-\infty} F^p M \xrightarrow{\cong} M.$$
   - We say F is *separated* if the limit of the diagram $\cdots \hookrightarrow F^p M \hookrightarrow \ldots$ exists in $\mathcal{A}$ and is zero:
     $$\lim_{p\to\infty} F^p M \cong 0.$$

When $\mathcal{A}$ is the category of R-modules, these conditions correspond to $\bigcup F_p M = M$ and $\bigcap F_p M = 0$ (increasing case).

Definition 7.0.12. Let (M, F) be a (increasing or decreasing) filtered object (Definition 7.0.10) in an abelian category (Definition 2.0.10) $\mathcal{A}$. If M is a graded object (Definition 7.0.1), then the filtration is said to be *compatible with the grading* if each $F_p M$ (or $F^p M$) is a graded subobject (Definition 7.0.5) of M. A graded object equipped with a filtration compatible with the grading is called a *filtered graded object*.

Definition 7.0.13. Let (M, F) be a filtered graded object (Definition 7.0.12) in an abelian category (Definition 2.0.10) $\mathcal{A}$. The *associated graded object*, denoted $\operatorname{Gr}^F(M)$, is the bi-graded object (Definition 7.0.7) defined by the short exact sequences (Definition 2.0.19):
$$0 \to F_{p-1}M_{p+q} \to F_p M_{p+q} \to \operatorname{Gr}^F_{p,q}(M) \to 0.$$

In other words, $\operatorname{Gr}^F_{p,q}(M)$ is the cokernel of the inclusion $F_{p-1}M_{p+q} \hookrightarrow F_p M_{p+q}$.

Definition 7.0.14. Let $\mathcal{A}$ be an abelian category (Definition 2.0.10). Let r_0 be an integer (typically $0, 1,$ or 2).

1. A *homological spectral sequence (starting at page r_0) in $\mathcal{A}$* is a sequence of bigraded objects (Definition 7.0.7) $E = \{E^r, d^r\}_{r \geq r_0}$, where for each $r \geq r_0$:
   (a) $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ is a differential (Definition 7.0.8) on E^r of bidegree $(-r, r-1)$;
   (b) There is an isomorphism of bigraded objects $E^{r+1} \cong H(E^r, d^r)$, i.e. for all indices (p, q), we have isomorphisms $E^{r+1}_{p,q} \cong H_{p,q}(E^r, d^r)$ (Definition 7.0.9).
   The object $E^r_{p,q}$ is called the term of bidegree (p, q) on the *r-th page*.
2. A *cohomological spectral sequence (starting at page r_0) in $\mathcal{A}$* is a sequence of bigraded objects (Definition 7.0.7) $E = \{E_r, d_r\}_{r \geq r_0}$, where for each $r \geq r_0$:
   (a) $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ is a differential on E_r of bidegree $(r, 1-r)$;
   (b) There is an isomorphism of bigraded objects $E_{r+1} \cong H(E_r, d_r)$, i.e. for all indices (p, q), we have isomorphisms $E_{r+1}^{p,q} \cong H^{p,q}(E_r, d_r)$ (Definition 7.0.9).

Definition 7.0.15. Let $\mathcal{A}$ be an abelian category (Definition 2.0.10).

1. Let $E = \{E^r, d^r\}_{r \geq r_0}$ be a homological spectral sequence (Definition 7.0.14) in $\mathcal{A}$. A term $E^r_{p,q}$ is said to *stabilize* if there exists an integer $r(p, q) \geq r_0$ such that for all $r \geq r(p, q)$, the differentials entering and leaving $E^r_{p,q}$ are zero:
$$d^r : E^r_{p,q} \to E^r_{p-r,q+r-1} = 0 \quad \text{and} \quad d^r : E^r_{p+r,q-r+1} \to E^r_{p,q} = 0.$$
In this case, we have isomorphisms $E^{r+1}_{p,q} \cong E^r_{p,q}$ for all $r \geq r(p, q)$. The *limit term* (or *infinity page*), denoted $E^\infty_{p,q}$, is defined as this stable object:
$$E^\infty_{p,q} = E^{r(p,q)}_{p,q}.$$

2. Let $E = \{E_r, d_r\}_{r \geq r_0}$ be a cohomological spectral sequence (Definition 7.0.14) in $\mathcal{A}$. A term $E_r^{p,q}$ is said to *stabilize* if there exists an integer $r(p, q) \geq r_0$ such that for all $r \geq r(p, q)$, the differentials entering and leaving $E_r^{p,q}$ are zero:
$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1} = 0 \quad \text{and} \quad d_r : E_r^{p-r,q+r-1} \to E_r^{p,q} = 0.$$
In this case, we have isomorphisms $E_{r+1}^{p,q} \cong E_r^{p,q}$ for all $r \geq r(p, q)$. The *limit term* (or *infinity page*), denoted $E_\infty^{p,q}$, is defined as this stable object:
$$E_\infty^{p,q} = E_{r(p,q)}^{p,q}.$$

Definition 7.0.16. Let $\mathcal{A}$ be an abelian category (Definition 2.0.10) and let $H = \{H_n\}_{n \in \mathbb{Z}}$ be a graded object (Definition 7.0.1) in $\mathcal{A}$.

1. Let $E = \{E^r, d^r\}_{r \geq r_0}$ be a homological spectral sequence (Definition 7.0.14). We say that E *converges to H*, denoted $E^r_{p,q} \Rightarrow H_{p+q}$, if:
   (a) For every pair (p, q), the term $E^r_{p,q}$ stabilizes (Definition 7.0.15) to a limit $E^\infty_{p,q}$.
   (b) Each object H_n is equipped with a finite, exhaustive, and separated (Definition 7.0.11) increasing filtration (Definition 7.0.10) F compatible with the grading (Definition 7.0.12), such that there is an isomorphism:
$$E^\infty_{p,q} \cong \mathrm{Gr}^F_{p,q}(H) := \frac{F_p H_{p+q}}{F_{p-1} H_{p+q}}.$$

2. Let $E = \{E_r, d_r\}_{r \geq r_0}$ be a cohomological spectral sequence (Definition 7.0.14). We say that E *converges to H*, denoted $E_r^{p,q} \Rightarrow H^{p+q}$, if:

(a) For every pair (p,q), the term $E_r^{p,q}$ stabilizes (Definition 7.0.15) to a limit $E_\infty^{p,q}$.

(b) Each object H^n (where H is viewed as co-graded, $H^n = H_{-n}$) is equipped with a finite, exhaustive, and separated decreasing filtration (Definition 7.0.10) F compatible with the grading (Definition 7.0.12), such that there is an isomorphism:

$$E_\infty^{p,q} \cong \mathrm{Gr}_F^{p,q}(H) := \frac{F^p H^{p+q}}{F^{p+1} H^{p+q}}.$$

In either case, the object H is called the *abutment of the spectral sequence*. Usually, H is explicitly realized as the homology $H_*(C)$ or cohomology $H^*(C)$ of a filtered complex C.

Definition 7.0.17. Let $E_{p,q}^r \Rightarrow H_{p+q}$ be a spectral sequence (Definition 7.0.14) converging (Definition 7.0.16) to a filtered graded object (Definition 7.0.12) (H, F). The convergence is called *strong convergence* if the filtration F on the abutment (Definition 7.0.16) H is exhaustive and separated (Definition 7.0.11).

Remark 7.0.18. Strong convergence ensures that the abutment H is "fully" determined by the spectral sequence (up to extension problems). Without these conditions, non-zero elements in H could be "invisible" to the spectral sequence (e.g., elements in $\cap F_p H$ or elements not captured by $\cup F_p H$). Explicitly, for an increasing filtration, strong convergence implies that H_n is built from the factors $E_{p,n-p}^\infty$ via a sequence of short exact sequences:

$$0 \to F_{p-1}H_n \to F_p H_n \to E_{p,n-p}^\infty \to 0.$$

Definition 7.0.19. Let $\mathcal{A}$ be an abelian category.

1. A homological spectral sequence (Definition 7.0.14) $E = \{E^r\}_{r \geq r_0}$ is called a *first quadrant spectral sequence* if the terms $E_{p,q}^{r_0}$ vanish unless $p \geq 0$ and $q \geq 0$. In this case, for any fixed (p,q), the differentials d^r entering or leaving $E_{p,q}^r$ eventually vanish because the indices $(p-r, q+r-1)$ and $(p+r, q-r+1)$ eventually land outside the first quadrant for large r. Thus, every term stabilizes (Definition 7.0.15).

2. A cohomological spectral sequence (Definition 7.0.14) $E = \{E_r\}_{r \geq r_0}$ is called a *first quadrant spectral sequence* if the terms $E_{r_0}^{p,q}$ vanish unless $p \geq 0$ and $q \geq 0$. Similarly, the differentials d_r eventually vanish as the indices $(p+r, q-r+1)$ and $(p-r, q+r-1)$ leave the first quadrant, guaranteeing stabilization (Definition 7.0.15).

Definition 7.0.20. Let $\mathcal{A}$ be an abelian category (Definition 2.0.10).

1. A homological spectral sequence (Definition 7.0.14) E is called *bounded* if for every integer n, there are only finitely many pairs (p,q) with $p + q = n$ such that $E_{p,q}^{r_0} \neq 0$. Boundedness ensures that the filtration (Definition 7.0.10) on the abutment (Definition 7.0.16) H_n is finite, which is a key condition for strong convergence (Definition 7.0.17).

2. A cohomological spectral sequence (Definition 7.0.14) E is called *bounded* if for every integer n, there are only finitely many pairs (p,q) with $p + q = n$ such that $E_{r_0}^{p,q} \neq 0$. Boundedness ensures that the filtration on the abutment H^n is finite.

(♠ TODO:

[STRUCTURAL] Exact Couple (D, E, i, j, k) - A pair of bigraded modules D, E and morphisms $i : D \to D$, $j : D \to E$, $k : E \to D$ forming an exact triangle - [Purpose: the algebraic machine that generates spectral sequences] - [Dependencies: Exact sequence, bigraded module]

[LEMMA] Derivation of a Spectral Sequence from an Exact Couple - Given an exact couple, $d = j \circ k$ is a differential on E, and a derived couple (D', E') exists - [Purpose: shows how to iterate the process to get pages $E^1, E^2, \ldots$] - [Dependencies: Exact couple]

[SPECIALIZED] Filtered Chain Complex (C_*, d, F) - A chain complex C_* equipped with a filtration $F_p C_*$ preserved by the differential d - [Purpose: the most common source of spectral sequences in nature] - [Dependencies: Chain complex, filtered module]

[THEOREM] Spectral Sequence of a Filtered Complex - For a filtered complex (C_*, F), there exists a spectral sequence with $E^0_{p,q} = F_p C_{p+q}/F_{p-1} C_{p+q}$ and $E^1_{p,q} = H_{p+q}(E^0_{p,q})$ converging to $H_*(C)$ - [Purpose: the fundamental existence theorem used in applications] - [Dependencies: Filtered chain complex, convergence]

[SPECIALIZED] Double Complex (Bicomplex) $C_{p,q}$ - A bigraded module with two commuting (or anticommuting) differentials $d_h : C_{p,q} \to C_{p-1,q}$ and $d_v : C_{p,q} \to C_{p,q-1}$ - [Purpose: a specific type of filtered complex that yields two spectral sequences] - [Dependencies: Chain complex]

[STRUCTURAL] Total Complex of a Double Complex $\mathrm{Tot}(C)_n$ - The chain complex defined by $\mathrm{Tot}(C)_n = \bigoplus_{p+q=n} C_{p,q}$ with differential $D = d_h + d_v$ - [Purpose: the object to which the double complex spectral sequences converge] - [Dependencies: Double complex]

[THEOREM] Spectral Sequences of a Double Complex - There are two spectral sequences associated to $C_{p,q}$, one filtering by rows ($^I E$) and one by columns ($^{II} E$), both converging to $H_*(\mathrm{Tot}(C))$ - [Purpose: allows computation of unknown homology by comparing two different filtrations] - [Dependencies: Double complex, total complex, convergence]

[SPECIALIZED] Edge Homomorphisms - Natural maps $H_n \to E^\infty_{n,0} \hookrightarrow E^2_{n,0}$ and $E^2_{0,n} \twoheadrightarrow E^\infty_{0,n} \to H_n$ (for first quadrant) - [Purpose: relates the spectral sequence terms directly to the abutment] - [Dependencies: Convergence, first quadrant spectral sequence]

[STRUCTURAL] Five-Term Exact Sequence - For a first quadrant spectral sequence converging to H_*, the sequence $H_2 \to E^2_{2,0} \xrightarrow{d^2} E^2_{0,1} \to H_1 \to E^2_{1,0} \to 0$ is exact - [Purpose: extracts concrete low-dimensional data without analyzing the full sequence] - [Dependencies: Edge homomorphisms, convergence]

[SPECIALIZED] Transgression τ - The differential $d^n : E^n_{n,0} \to E^n_{0,n-1}$ in a first quadrant spectral sequence - [Purpose: the longest possible non-zero differential connecting the axes] - [Dependencies: First quadrant spectral sequence]

[SPECIALIZED] Collapse of a Spectral Sequence - A situation where $E^r_{p,q} = E^\infty_{p,q}$ for some finite r (often $r = 2$) - [Purpose: simplifying condition where the calculation terminates early] - [Dependencies: Limit page]

APPENDIX A. CATEGORIES ENRICHED IN MONOIDAL CATEGORIES

Definition A.0.1. A *monoidal category* is a (large) category (Definition 1.0.1) $\mathcal{C}$ equipped with:

- a bifunctor (Definition 1.0.13) $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (Definition 1.0.12) (called the *tensor product*);
- an object $\mathbb{I} \in \mathrm{Ob}(\mathcal{C})$ (often called the *unit object*); common notations for the unit object include $\mathbb{I}$ and $\mathbb{1}$.
- natural isomorphisms (*associator*) $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ for all $X, Y, Z \in \mathcal{C}$;
- natural isomorphisms (*left and right unitors*) $\lambda_X : \mathbb{I} \otimes X \to X$, $\rho_X : X \otimes \mathbb{I} \to X$ for all $X \in \mathcal{C}$;

such that the following coherence diagrams commute:

Pentagon coherence: For all $W, X, Y, Z \in \mathcal{C}$, the diagram

$$
((W \otimes X) \otimes Y) \otimes Z \xrightarrow{\alpha_{W \otimes X, Y, Z}} (W \otimes X) \otimes (Y \otimes Z) \xrightarrow{\alpha_{W, X, Y \otimes Z}} W \otimes (X \otimes (Y \otimes Z))
$$

with $\alpha_{W,X,Y} \otimes \mathrm{id}_Z$ down the left, $\mathrm{id}_W \otimes \alpha_{X,Y,Z}$ up the right,

$$
(W \otimes (X \otimes Y)) \otimes Z \xrightarrow{\alpha_{W, X \otimes Y, Z}} W \otimes ((X \otimes Y) \otimes Z).
$$

Triangle coherence: For all $X, Y \in \mathcal{C}$, the diagram

$$
(X \otimes \mathbb{I}) \otimes Y \xrightarrow{\alpha_{X, \mathbb{I}, Y}} X \otimes (\mathbb{I} \otimes Y)
$$

with $\rho_X \otimes \mathrm{id}_Y$ and $\mathrm{id}_X \otimes \lambda_Y$ to

$$
X \otimes Y.
$$

Definition A.0.2 (Category enriched in a monoidal category)**.** Let $(\mathcal{V}, \otimes, \mathbf{1})$ be a monoidal category (Definition A.0.1). A *category enriched in $\mathcal{V}$* (or a *$\mathcal{V}$-enriched category* or a *$\mathcal{V}$-category*) $\mathcal{C}$ consists of the following data:

- A class $\mathrm{Ob}(\mathcal{C})$ of *objects*. As with regular categories, we may write $X \in \mathrm{Ob}(\mathcal{C})$ or $X \in \mathcal{C}$ to mean that X is an object of $\mathcal{C}$.
- For each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, an object $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \in \mathrm{Ob}(\mathcal{V})$ of *morphisms*; it is an object of the monoidal category $\mathcal{V}$. It is also often denoted by notations such as $\mathcal{C}(X, Y)$, $\mathrm{Hom}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$, or $\mathrm{Mor}(X, Y) = \mathrm{Mor}_{\mathcal{C}}(X, Y)$.

- For each triple $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a composition morphism

$$\mu_{X,Y,Z} : \underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Z).$$

  It is a morphism in $\mathcal{V}$.
- For each object X, a unit morphism $\eta_X : \mathbf{1} \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, X)$ in $\mathcal{V}$.

These data satisfy the following axioms:

- (Associativity) For all $W, X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, the following diagram in $\mathcal{V}$ commutes:

$$\big(\underline{\mathrm{Hom}}_{\mathcal{C}}(Z, W) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z)\big) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \xrightarrow{\ \alpha\ } \underline{\mathrm{Hom}}_{\mathcal{C}}(Z, W) \otimes \big(\underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)\big)$$

$$\downarrow \mu \otimes \mathrm{id} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \mathrm{id} \otimes \mu$$

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(Y, W) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \qquad\qquad\qquad\qquad \underline{\mathrm{Hom}}_{\mathcal{C}}(Z, W) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Z)$$

$$\downarrow \mu \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \mu$$

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X, W) = \!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!= \underline{\mathrm{Hom}}_{\mathcal{C}}(X, W)$$

  where α is the associativity constraint in $\mathcal{V}$.
- (Unit) For all $X, Y \in \mathrm{Ob}(\mathcal{C})$, the following diagrams commute:

$$\mathbf{1} \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \xrightarrow{\ \eta_Y \otimes \mathrm{id}\ } \underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Y) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$$

$$\searrow \lambda \qquad\qquad \downarrow \mu$$

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$$

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \otimes \mathbf{1} \xrightarrow{\ \mathrm{id} \otimes \eta_X\ } \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X, X)$$

$$\searrow \rho \qquad\qquad \downarrow \mu$$

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$$

  where λ and ρ are the left and right unit constraints in $\mathcal{V}$.

Definition A.0.3. Let $\mathcal{V}$ be a monoidal category (Definition A.0.1) and let $\mathcal{C}$ and $\mathcal{D}$ be categories enriched in $\mathcal{V}$ (Definition A.0.2). A $\mathcal{V}$-functor between $\mathcal{C}$ and $\mathcal{D}$ or an enriched functor between $\mathcal{C}$ and $\mathcal{D}$, written

$$F : \mathcal{C} \to \mathcal{D},$$

consists of:

- a function on objects $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$,
- for all $A, B \in \mathrm{Ob}(\mathcal{C})$, morphisms in $\mathcal{V}$,

$$F_{A,B} : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB),$$

such that the following diagrams in $\mathcal{V}$ commute:

$$\begin{array}{c}
\mathcal{C}(B,C) \otimes \mathcal{C}(A,B) \xrightarrow{\circ_{\mathcal{C}}} \mathcal{C}(A,C) \\
\downarrow F_{B,C} \otimes F_{A,B} \qquad \downarrow F_{A,C} \\
\mathcal{D}(FB,FC) \otimes \mathcal{D}(FA,FB) \xrightarrow{\circ_{\mathcal{D}}} \mathcal{D}(FA,FC)
\end{array}$$

and for all $A \in \mathrm{Ob}(\mathcal{C})$, the unit compatibility condition holds:

$$\iota_{FA} = F_{A,A} \circ \iota_A.$$

Definition A.0.4. More generally, let $\mathcal{M} = (\mathcal{M}, \otimes, \mathbb{1})$ be a monoidal category (Definition A.0.1), and let $\{\mathcal{C}_i\}_{i \in I}$ be a family of $\mathcal{M}$-enriched categories (Definition A.0.2) indexed by a class I such that $\mathcal{M}$ is closed under all products (Definition 2.0.3) indexed by I. The *product $\mathcal{M}$-enriched category of the family*, denoted

$$\prod_{i \in I} \mathcal{C}_i,$$

is defined as follows:

- The class of objects is

$$\mathrm{Ob}\Big(\prod_{i \in I} \mathcal{C}_i \Big) = \prod_{i \in I} \mathrm{Ob}(\mathcal{C}_i),$$

  i.e., an object is a family $(A_i)_{i \in I}$ with $A_i \in \mathrm{Ob}(\mathcal{C}_i)$.
- For two objects $(A_i)_i$ and $(B_i)_i$, the hom-object in $\mathcal{M}$ is

$$\Big(\prod_{i \in I} \mathcal{C}_i \Big) \big((A_i)_i, (B_i)_i \big) := \prod_{i \in I} \mathcal{C}_i(A_i, B_i).$$

  Here, the product on the right is taken inside $\mathcal{M}$.
- Composition morphisms in $\mathcal{M}$ are defined componentwise using the composition morphisms of each $\mathcal{C}_i$:

$$\circ_{(A_i),(B_i),(C_i)} : \Big(\prod_i \mathcal{C}_i(B_i, C_i) \Big) \otimes \Big(\prod_i \mathcal{C}_i(A_i, B_i) \Big) \to \prod_i \mathcal{C}_i(A_i, C_i),$$

  given by the universal property of products and the componentwise compositions

$$\circ_i : \mathcal{C}_i(B_i, C_i) \otimes \mathcal{C}_i(A_i, B_i) \to \mathcal{C}_i(A_i, C_i).$$

- For each object $(A_i)_i$, the identity morphism is given by the family

$$(\mathrm{id}_{A_i})_i : \mathbb{1} \to \prod_i \mathcal{C}_i(A_i, A_i),$$

  where $\mathbb{1}$ is the unit object of $\mathcal{M}$.

In case that I is finite, the notation $\times$ may be used for product $\mathcal{M}$-enriched categories, e.g. $\mathcal{C}_i \times \mathcal{C}_j$ denotes the product of two $\mathcal{M}$-enriched categories $\mathcal{C}_i \times \mathcal{C}_j$.

Definition A.0.5 (n-ary Functor of $\mathcal{V}$-Enriched Categories). Let $\mathcal{V}$ be a monoidal category (Definition A.0.1) that is closed under finite products (Definition 2.0.3), and let I be a finite set with $|I| = n$. Suppose $\{\mathcal{C}_i\}_{i\in I}$ are $\mathcal{V}$-enriched categories (Definition A.0.2), and $\mathcal{D}$ is another $\mathcal{V}$-enriched category. An *n-ary enriched functor* (or *multivariable enriched functor*) from $\{\mathcal{C}_i\}_{i\in I}$ to $\mathcal{D}$ is a $\mathcal{V}$-enriched functor (Definition A.0.3)

$$F : \prod_{i\in I} \mathcal{C}_i \to \mathcal{D},$$

where $\prod_{i\in I}\mathcal{C}_i$ is the $\mathcal{V}$-enriched product category (Definition A.0.4).

That is, F assigns:

- to each object $((A_i)_{i\in I})$ in $\prod_{i\in I}\mathcal{C}_i$, an object $F((A_i)_{i\in I})$ in $\mathcal{D}$,
- to each family of objects $((A_i)_i, (B_i)_i)$, a morphism in $\mathcal{V}$

$$F_{(A_i),(B_i)} : \prod_{i\in I} \mathcal{C}_i(A_i, B_i) \to \mathcal{D}(F((A_i)_i), F((B_i)_i)),$$

respecting the enriched composition and unit axioms.

When $n = 2$, an n-ary enriched functor is called a *bifunctor* enriched over $\mathcal{V}$, etc.

Definition A.0.6. Let $\mathcal{V}$ be a monoidal category (Definition A.0.1) and $\mathcal{C}$ a $\mathcal{V}$-category (Definition A.0.2). The *opposite $\mathcal{V}$-category of $\mathcal{C}$*, denoted by $\mathcal{C}^{\mathrm{op}}$, is defined as follows:

- Objects: $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$.
- Hom-objects: For all $A, B \in \mathrm{Ob}(\mathcal{C})$, set

$$\mathcal{C}^{\mathrm{op}}(A, B) = \mathcal{C}(B, A).$$

- Composition: For all $A, B, C \in \mathrm{Ob}(\mathcal{C})$, the composition morphism

$$\circ^{\mathrm{op}}_{A,B,C} : \mathcal{C}^{\mathrm{op}}(B, C) \otimes \mathcal{C}^{\mathrm{op}}(A, B) \to \mathcal{C}^{\mathrm{op}}(A, C)$$

is defined by the composite in $\mathcal{V}$

$$\mathcal{C}(C, B) \otimes \mathcal{C}(B, A) \xrightarrow{s_{\mathcal{C}(C,B),\mathcal{C}(B,A)}} \mathcal{C}(B, A) \otimes \mathcal{C}(C, B) \xrightarrow{\circ_{C,B,A}} \mathcal{C}(C, A).$$

- Units: For each $A \in \mathrm{Ob}(\mathcal{C})$, the unit morphism in $\mathcal{C}^{\mathrm{op}}$ is the same as that in $\mathcal{C}$,

$$\iota_A^{\mathrm{op}} = \iota_A : I \to \mathcal{C}(A, A) = \mathcal{C}^{\mathrm{op}}(A, A).$$

The associativity and unit axioms for $\mathcal{C}^{\mathrm{op}}$ follow from those for $\mathcal{C}$ and the naturality and symmetry properties of $s_{X,Y}$.

Appendix B. Schemes

Definition B.0.1 (Locally ringed space). A *locally ringed space* is a ringed space (Definition 1.1.1) $(X, \mathcal{O}_X)$ such that for every point $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring. The unique maximal ideal of $\mathcal{O}_{X,x}$ is often denoted $\mathfrak{m}_x$ and is called the *maximal ideal at x*.

Definition B.0.2 (Scheme). A *scheme* is a locally ringed space (Definition B.0.1) $(X, \mathcal{O}_X)$ that admits an open cover $\{U_i\}_{i \in I}$ such that each $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic (as a locally ringed space) to an affine scheme $(\mathrm{Spec}(A_i), \mathcal{O}_{\mathrm{Spec}(A_i)})$ for some commutative ring A_i. In other words, a scheme is a locally ringed space locally isomorphic to affine schemes.

Definition B.0.3. Let X be a scheme (Definition B.0.2) and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules (Definition 1.1.2). The sheaf $\mathcal{F}$ is called *quasi-coherent* if for every open affine subset $U = \mathrm{Spec}\, A$ of X, there exists an A-module M_U such that the restriction $\mathcal{F}|_U$ is isomorphic to the sheaf $\widetilde{M_U}$ associated to M_U.

For a scheme X, we denote by

$$\mathrm{QCoh}(X)$$

the full subcategory of $\mathrm{Mod}(\mathcal{O}_X)$ consisting of all quasi-coherent $\mathcal{O}_X$-modules.

Appendix C. Miscellaneous definitions

Definition C.0.1 (Groups). A *group* is a pair $(G, \cdot)$ where G is a set and $\cdot : G \times G \to G$ is a binary operation, subject to the following conditions:

1. (Associativity) For all $g, h, k \in G$ one has
$$(g \cdot h) \cdot k = g \cdot (h \cdot k).$$

2. (Identity element) There exists an element $e \in G$ such that for all $g \in G$,
$$e \cdot g = g \cdot e = g.$$

3. (Inverse element) For all $g \in G$ there exists an element $g^{-1} \in G$ such that
$$g \cdot g^{-1} = g^{-1} \cdot g = e.$$

The element e is called the *identity element of G*, and g^{-1} is called the *inverse of g*.

Equivalently, a group is a monoid with inverse elements.

A group $(G, \cdot)$ is often simply written as G, when the notation for the binary operation $\cdot$ is clear.

An *abelian group* or synonymously, a *commutative group*, is a group $(G, \cdot)$ whose binary operation $\cdot$ is *abelian* or *commutative*, i.e. satisfies
$$g \cdot h = h \cdot g$$
for all $g, h \in G$.

An abelian group is equivalent to a $\mathbb{Z}$-module.

Definition C.0.2. A *ring* is a triple $(R, +, \cdot)$ where

1. $(R, +)$ is a commutative group (Definition C.0.1), and
2. $(R, \cdot)$ is a monoid.

3. $\cdot$ is distributive over $+$, i.e. for all $a, b, c \in R$, we have
$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and} \quad (a + b) \cdot c = a \cdot c + b \cdot c.$$

Equivalently, a ring is a triple $(R, +, \cdot)$ where $+, \cdot : R \times R \to R$ are binary operations satisfying

1. $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$ for all $a, b, c \in R$
2. There exists an element $0 \in R$ such that $a + 0 = a = 0 + a$ for all $a \in R$.
3. For every $a \in R$, there exists an element $-a \in R$ such that $a + (-a) = 0 = (-a) + a$ for all $a \in R$.
4. There exists an element $1 \in R$ such that $a \cdot 1 = a = 1 \cdot a$ for all $a \in R$.
5. For all $a, b, c \in R$, we have
$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and} \quad (a + b) \cdot c = a \cdot c + b \cdot c.$$

The operation $+$ is often called *addition* and the operation $\cdot$ is often called *multiplication*. Accordingly, the identity element 0 of $+$ is often called the *additive identity* and the identity element 1 of $\cdot$ is often called the *multiplicative identity*.

Remark C.0.3. Some writers might not require a ring to have a multiplicative identity element, i.e. would define a ring so that $(R, +)$ is a commutative group, $(R, \cdot)$ is a semigroup, and $\cdot$ is distributive over $+$. Such writers would call the notion of ring in Definition C.0.2 a *unitary ring* to emphasize the existence of the multiplicative identity 1.

Definition C.0.4. Let R be a not-necessarily commutative ring (Definition C.0.2).

1. A *left R-module* is an abelian group $(M, +)$ together with an operation $R \times M \to M$, denoted $(r, m) \mapsto rm$, such that for all $r, s \in R$ and $m, n \in M$:
   - $r(m + n) = rm + rn$,
   - $(r + s)m = rm + sm$,
   - $(rs)m = r(sm)$,
   - $1_R m = m$ where 1_R is the multiplicative identity of R.
2. A *right R-module* is defined similarly as an abelian group $(M, +)$ with an operation $M \times R \to M$, denoted $(m, r) \mapsto mr$, such that for all $r, s \in R$ and $m, n \in M$:
   - $(m + n)r = mr + nr$,
   - $m(r + s) = mr + ms$,
   - $m(rs) = (mr)s$,
   - $m1_R = m$.
3. Let R and S be (not necessarily commutative) rings (Definition C.0.2).
   An *R-S-bimodule* (or an *R-S-module* or an (R, S)-module, etc.) is an abelian group (Definition C.0.1) $(M, +)$ equipped with
   (a) a left action of R:
   $$R \times M \to M, \quad (r, m) \mapsto r \cdot m,$$
   making M a left R-module (Definition C.0.4),
   (b) a right action of S:
   $$M \times S \to M, \quad (m, s) \mapsto m \cdot s,$$

making M a right S-module,
such that the left and right actions commute; that is, for all $r \in R$, $s \in S$, and $m \in M$,

$$r \cdot (m \cdot s) = (r \cdot m) \cdot s.$$

4. A *two-sided R-module* (or *R-bimodule*) is an R-R-bimodule.

If R is a commutative ring, then a left/right R-module can automatically be regarded as a two-sided R-module. As such, we simply talk about *R-modules* in this case.

Any abelian group is equivalent to a two-sided $\mathbb{Z}$-module. Moreover, any left R-module is equivalent to an $R - \mathbb{Z}$-bimodule (Definition C.0.4) and any right R-module is equivalent to an $\mathbb{Z} - R$-bimodule (Definition C.0.4). Given a left/right/two-sided R-module, its *natural bimodule structure* will refer to its structure as a R-$\mathbb{Z}$/$\mathbb{Z}$-R/R-R bimodule. In this way, many definitions associated with the notions of left/right/two-sided R-modules can be defined as special cases for definitions for R-S-bimodules.

Definition C.0.5. Let R be a (not necessarily commutative, possibly nonunital) ring (Definition C.0.2). A *left ideal of R* is a subset $I \subseteq R$ such that

- $(I, +)$ is an additive subgroup of $(R, +)$,
- $RI \subseteq I$, i.e., for all $r \in R$ and $x \in I$, one has $rx \in I$.

Similarly, a *right ideal of R* is a subset $I \subseteq R$ such that

- $(I, +)$ is an additive subgroup of $(R, +)$,
- $IR \subseteq I$, i.e., for all $r \in R$ and $x \in I$, one has $xr \in I$.

A *two-sided ideal* (or simply an *ideal*) of R is a subset $I \subseteq R$ which is both a left ideal and a right ideal of R. We denote by $I \trianglelefteq R$ the relation expressing that I is a two-sided ideal of R.

Equivalently, an left/right/two-sided ideal of R is a submodule of R as an R-module (Definition C.0.4).

A left/right/two-sided ideal is said to be *proper* if it is strictly contained in R.

Note that every left or right ideal of a commutative ring is a two-sided ideal.

Definition C.0.6. Let R, S be (not-necessarily commutative) rings (Definition C.0.2).

1. Let M and N be R-S-bimodules (Definition C.0.4). A function $\varphi : M \to N$ is called an *R-S-bimodule homomorphism* or *R-S-linear* if it is a group homomorphism of the underlying abelian groups of M and N and respects the scalar actions as follows: for all $m_1, m_2 \in M$, $r \in R$, and $s \in S$,

$$\varphi(r \cdot m_1) = r \cdot \varphi(m_1),$$
$$\varphi(m_1 \cdot s) = \varphi(m_1) \cdot s.$$

2. Let M and N be left/right/two-sided R-modules (Definition C.0.4). A function $\varphi : M \to N$ is called a *left/right/two-sided R-module homomorphism* if it is an bimodule

homomorphism on the natural bimodule structures (Definition C.0.4) of M and N. Such a function is also called *R-linear*.

Modules and homomorphisms of a fixed type (i.e. R-S-bimodules or left/righ/two-sided R-modules) form a locally small (Definition 1.0.5) category (Definition 1.0.1).

Definition C.0.7. Let R, S be (not-necessarily commutative) rings with unity (Definition C.0.2), and let M, N be R-S-bimodules (Definition C.0.4). Let

$$\varphi : M \to N$$

be a homomorphism of R-S-bimodules (Definition C.0.6). We define:

1. The *kernel of φ* is the submodule of M given by

$$\ker(\varphi) := \{ m \in M \mid \varphi(m) = 0 \} \subseteq M.$$

2. The *image of φ* is the submodule of N given by

$$\mathrm{im}(\varphi) := \{ \varphi(m) \mid m \in M \} \subseteq N.$$

3. The *cokernel of φ* is the quotient module of N defined by

$$\mathrm{coker}(\varphi) := N/\mathrm{im}(\varphi).$$

4. The *coimage of φ* is the quotient module of M defined by

$$\mathrm{coim}(\varphi) := M/\ker(\varphi).$$

It is not difficult to see that each of these are indeed R-S bimodules. In case M and N are left/right/two-sided R-modules, the *kernel, image, cokernel, and coimage* of a module homomorphism $\varphi : M \to N$ are respectively defined to be the kernel, image, cokernel, and coimage for the natural bimodule structures (Definition C.0.4) of M and N.

The kerel, cokernel, image, and coimage of f are respectively the categorical kernel, cokernel (Definition 1.2.2), image, and coimage (Definition 1.2.5) (**??**).

Definition C.0.8. Let R and S be (not necessarily commutative) rings (Definition C.0.2).

1. The *category of (R, S)-bimodules* (or R-S-bimodules), denoted by notations such as $_R\mathsf{Mod}_S$, is the category whose objects are (R, S)-bimodules (Definition C.0.4) and whose R-S-bimodule homomorphisms (Definition C.0.6).
2. The *category of left R-modules*, denoted by notations such as $_R\mathsf{Mod}$ or $R - \mathbf{Mod}$, is the category $_R\mathsf{Mod}_{\mathbb{Z}}$, i.e. the category whose objects are left R-modules (Definition C.0.4) and whose morphisms are left R-linear maps (Definition C.0.6).
3. The *category of right R-modules*, denoted by notations such as Mod_R or $\mathbf{Mod} - R$, is the category $_{\mathbb{Z}}\mathsf{Mod}_R$, i.e. the category whose objects are right R-modules (Definition C.0.4) and whose morphisms are right R-linear maps (Definition C.0.6).

The category of bimodules can be canonically identified with module categories over tensor product rings (Definition C.0.12):

- $_R\mathsf{Mod}_S$ is isomorphic to the category of left modules over the ring $R \otimes_\mathbb{Z} S^{\mathrm{op}}$.
- $_R\mathsf{Mod}_S$ is isomorphic to the category of right modules over the ring $R^{\mathrm{op}} \otimes_\mathbb{Z} S$.

Consequently, standard module-theoretic concepts (such as projective objects, injective objects, and flat objects) in $_R\mathsf{Mod}_S$ correspond exactly to the respective concepts in $_{R\otimes S^{\mathrm{op}}}\mathsf{Mod}$.

Note that there are canonical isomorphisms of categories:
$$_R\mathsf{Mod} \cong {}_R\mathsf{Mod}_\mathbb{Z} \quad \text{and} \quad \mathsf{Mod}_S \cong {}_\mathbb{Z}\mathsf{Mod}_S.$$

That is, left R-modules are exactly $(R, \mathbb{Z})$-bimodules, and right S-modules are exactly $(\mathbb{Z}, S)$-bimodules.

Definition C.0.9 (Coproduct of Modules)**.** Let R and S be (not necessarily commutative) rings (Definition C.0.2), and let $\{M_i\}_{i \in I}$ be a (possibly infinite but small) family of (R, S)-bimodules.

The *coproduct (direct sum) of the family $\{M_i\}_{i\in I}$*, denoted by $\bigoplus_{i \in I} M_i$, is constructed as
$$\bigoplus_{i\in I} M_i := \left\{ (m_i)_{i\in I} \in \prod_{i\in I} M_i \mid m_i = 0 \text{ for all but finitely many } i \in I \right\}$$
consisting of all tuples with only finitely many nonzero entries.

Addition and scalar multiplication in $\bigoplus_{i\in I} M_i$ are defined componentwise as in the direct product:
$$(m_i)_{i\in I} + (n_i)_{i\in I} := (m_i + n_i)_{i\in I}, \qquad r \cdot (m_i)_{i\in I} \cdot s := (r \cdot m_i \cdot s)_{i\in I}, \qquad r \in R,\ s \in S.$$

In all cases, the zero element is $(0)_{i\in I}$, and additive inverses are given by $-(m_i)_{i\in I} := (-m_i)_{i\in I}$.

Note that we can define the coproduct of a family $\{M_i\}_{i\in I}$ of left/right/two-sided R-modules by taking the natural bimodule structure (Definition C.0.4) of each module.

(♠ TODO: submodule) Note that $\bigoplus_{i\in I} M_i$ is a submodule of $\prod_{i\in I} M_i$. Moreover, $\bigoplus_{i\in I} M_i$ is the coproduct (Definition 2.0.3) in the appropriate category of modules (Definition C.0.8).

For finitely many modules $M_1, \ldots, M_n$, the direct sum $\bigoplus_{j=1}^n M_j$, which may also be written as $M_1 \oplus \cdots \oplus M_n$, is simply the usual Cartesian product $\prod_{j=1}^n M_j$ of the modules, as every tuple automatically has only finitely many nonzero entries.

Definition C.0.10 (Opposite ring)**.** Let $R = (R, +, \cdot, 0, 1)$ be a ring (Definition C.0.2) with addition $+$, multiplication $\cdot$, additive identity 0, and multiplicative identity 1 (not necessarily commutative).

The *opposite ring of R*, denoted R^{op}, is the ring with the same underlying set R and the same addition $+$ and additive identity 0, but with multiplication defined by
$$r \star s := s \cdot r$$
for all $r, s \in R$.

That is, multiplication in R^{op} is the multiplication of R reversed in order.

If R is commutative, then R and R^{op} are naturally isomorphic to each other.

Definition C.0.11 (Tensor product of bimodules). Let R, S, T be (not necessarily commutative) rings (Definition C.0.2), let M be an R-S bimodule (Definition C.0.4), and let N be an S-T bimodule. In the free abelian group $\mathbb{Z}[M \times N]$ generated by the Cartesian product $M \times N$, let U be the subgroup generated by elements of the form (♠ TODO: subgroup generated)

$$(m + m', n) - (m, n) - (m', n),$$
$$(m, n + n') - (m, n) - (m, n'),$$
$$(m \cdot s, n) - (m, s \cdot n),$$

for all $m, m' \in M$, $n, n' \in N$, and $s \in S$. The *tensor product of M and N over S* is the quotient abelian group

$$M \otimes_S N := \mathbb{Z}[M \times N]/U.$$

The image of an element of the form $(m, n) \in M \times N$ in $M \otimes_S N$ is denoted $m \otimes n$ and called a *pure tensor*. In general, the elements of $M \otimes_S N$ are finite sums

$$\sum_{i=1}^{n} m_i \otimes n_i \quad m_i \in M, n_i \in N$$

of pure tensors. Thus, the pure tensors satisfy the following relations:

$$(m + m') \otimes n = m \otimes n + m' \otimes n$$
$$m \otimes (n + n') = m \otimes n + m \otimes n'$$
$$(m \cdot s) \otimes n = m \otimes (s \cdot n)$$

This tensor product becomes naturally an R-T bimodule with left action and right action defined by

$$r \cdot (m \otimes n) = (r \cdot m) \otimes n,$$
$$(m \otimes n) \cdot t = m \otimes (n \cdot t),$$

for all $r \in R$, $t \in T$, $m \in M$, and $n \in N$.

Inductively, given rings $R_0, \ldots, R_k$ and $R_{i-1} - R_i$-bimodules M_i for $i = 1, \ldots, k$, we may speak of the tensor product

$$M_0 \otimes_{R_1} M_1 \otimes_{R_2} \cdots \otimes_{R_{k-1}} M_k;$$

tensor products are associative(♠ TODO:), so parentheses are not strictly needed to notate them. Its *pure tensors* are elements of the form $m_0 \otimes m_1 \otimes \cdots \otimes m_k$ for $m_i \in M_i$, and its general elements are finite sums

$$\sum_{j=1}^{n} m_{0j} \otimes m_{1j} \otimes \cdots m_{kj} \quad m_{ij} \in M_i.$$

of pure tensors. It also has a natural $R_0 - R_k$-bimodule structure.

In general, $(M_0, \ldots, M_k) \mapsto M_0 \otimes_{R_1} M_1 \otimes_{R_2} \cdots \otimes_{R_{k-1}} M_k$ defines a $(k+1)$-ary additive functor (Definition 4.7.5)

$$_{R_0}\mathbf{Mod}_{R_1} \times \cdots \times {_{R_{k-1}}}\mathbf{Mod}_{R_k} \to {_{R_0}}\mathbf{Mod}_{R_k}$$

(Theorem 2.1.11).

Given a ring R and a two-sided R-module M, we may also speak of the *n-fold tensor product* $M^{\otimes n} = M^{\otimes_R n}$

Definition C.0.12. Let k be a not necessarily commutative ring (Definition C.0.2). Let R and S be k-rings (not necessarily commutative). Assume that at least one of R or S is a k-algebra. The *tensor product ring* $R \otimes_k S$ is the k-module $R \otimes_k S$ (Definition C.0.11) equipped with a multiplication defined on simple tensors by

$$(r_1 \otimes s_1) \cdot (r_2 \otimes s_2) = (r_1 r_2) \otimes (s_1 s_2)$$

and extended by linearity. This multiplication is well-defined and makes $R \otimes_k S$ into a k-ring under the ring homomorphism

$$k \to R \otimes_k S, \quad a \mapsto a \otimes 1 = 1 \otimes a.$$

The unit element is $1_R \otimes 1_S$.

In this ring, the subrings $R \otimes 1$ and $1 \otimes S$ commute with each other; that is, for all $r \in R$ and $s \in S$,

$$(r \otimes 1) \cdot (1 \otimes s) = r \otimes s = (1 \otimes s) \cdot (r \otimes 1).$$

If R and S are both k-algebras, then $R \otimes_k S$ is also a k-algebra.

Theorem C.0.13. Let R and S be (not necessarily commutative) rings (Definition C.0.2). The category of (Definition C.0.8) R-S-bimodules (Definition C.0.4) is equivalent (Definition 1.0.10) to the category of left $R \otimes_{\mathbb{Z}} S^{\mathrm{op}}$ (Definition C.0.12) (Definition C.0.10) modules.

Definition C.0.14 (Grothendieck Universe)**.** Let U be a set. We say U is a *Grothendieck universe* (or just a *universe*) if the following conditions hold:

1. If $x \in U$ and $y \in x$, then $y \in U$ (transitivity).
2. If $x, y \in U$, then $\{x, y\} \in U$ (closed under pair formation).
3. If $x \in U$, then the power set $\mathcal{P}(x) \in U$.
4. If $I \in U$ and $(x_\alpha)_{\alpha \in I}$ is a family with each $x_\alpha \in U$, then $\bigcup_{\alpha \in I} x_\alpha \in U$.

A set X is called *U-small* or a *U-set* if $X \in U$.

Definition C.0.15 (Reflecting a type of morphism)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between (large) categories (Definition 1.0.3), and let $\mathcal{P}$ be a property of morphisms (or more generally a property of sequences or families of morphisms) that is stable under isomorphism (e.g. monomorphism, epimorphism (Definition 1.2.1), isomorphism, etc.). We say that F *reflects $\mathcal{P}$-morphisms* if for every morphism $f : x \to y$ in $\mathcal{C}$, whenever $F(f)$ has property $\mathcal{P}$ in $\mathcal{D}$, it follows that f has property $\mathcal{P}$ in $\mathcal{C}$.

Definition C.0.16. Let $\mathcal{C}$ and $\mathcal{D}$ be categories (Definition 1.0.1). Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors.

An *adjunction between F and G* consists of two natural transformations (Definition 1.0.4): $\eta : \mathrm{Id}_{\mathcal{C}} \implies GF$ (the *unit*), and $\varepsilon : FG \implies \mathrm{Id}_{\mathcal{D}}$ (the *counit*)

These must satisfy the triangle identities: For every object $X \in \mathcal{C}$ and $Y \in \mathcal{D}$,

$$\varepsilon_{FX} \circ F(\eta_X) = \mathrm{id}_{FX}$$

$$G(\varepsilon_Y) \circ \eta_{GY} = \mathrm{id}_{GY}.$$

In diagrammatic form, the triangle identities assert that the following are commutative diagrams:

$$F(X) \xrightarrow{F(\eta_X)} FGF(X) \qquad G(Y) \xrightarrow{\eta_{G(Y)}} GFG(Y)$$

with $\mathrm{id}_{F(X)}$ and $\varepsilon_{F(X)}$ to $F(X)$, and $\mathrm{id}_{G(Y)}$ and $G(\varepsilon_Y)$ to $G(Y)$.

We say that F is a *left adjoint to G* and G is a *right adjoint to F* (written $F \dashv G$).

In the case that $\mathcal{C}$ and $\mathcal{D}$ are locally small (Definition 1.0.5) categories (or U-locally small categories if a universe (Definition C.0.14) U is available), we have an adjunction $F \dashv G$ if and only if for every object X in $\mathcal{C}$ and Y in $\mathcal{D}$ there is a natural isomorphism (Definition 1.0.4)

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

that is natural in both X and Y. In this case, the *unit of the adjunction* is the natural transformation $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ such that,

1. for every $X \in \mathcal{C}$, the morphism $\eta_X : X \to GF(X)$ (each called a *unit morphism*) in $\mathcal{C}$ is obtained as the image of $\mathrm{id}_{F(X)}$ via the adjoint isomorphism

   $$\mathrm{Hom}_{\mathcal{D}}(F(X), F(X)) \cong \mathrm{Hom}_{\mathcal{C}}(X, GF(X)).$$

2. for every $Y \in \mathcal{D}$, the morphism $\epsilon_Y : FG(Y) \to Y$ (each called a *counit morphism*) in $\mathcal{D}$ is obtained as the image of $\mathrm{id}_{G(Y)}$ via the adjoint isomorphism

   $$\mathrm{Hom}_{\mathcal{C}}(G(Y), G(Y)) \cong \mathrm{Hom}_{\mathcal{D}}(FG(Y), Y).$$

Definition C.0.17. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor (Definition 1.0.3) between (large) categories (Definition 1.0.1) $\mathcal{C}$ and $\mathcal{D}$.

- The functor F is called *continuous* if it preserves all small limits that exist in $\mathcal{C}$. That is, for every small category $\mathcal{J}$ and every diagram $D : \mathcal{J} \to \mathcal{C}$ having a limit in $\mathcal{C}$, F preserves the limit (Definition 1.3.15) of D.
- The functor F is called *cocontinuous* if it preserves all small colimits that exist in $\mathcal{C}$. That is, for every small category $\mathcal{J}$ and every diagram $D : \mathcal{J} \to \mathcal{C}$ having a colimit in $\mathcal{C}$, F preserves the colimit (Definition 1.3.15) of D.

Definition C.0.18 (Noetherian conditions for a ring)**.** Let R be a ring (Definition C.0.2). We say:

- R is *left-Noetherian* if every ascending chain of left ideals (Definition C.0.5) of R stabilizes, i.e., if for any chain

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

  of left ideals, there exists n such that $I_m = I_n$ for all $m \geq n$.
- R is *right-Noetherian* if every ascending chain of right ideals of R stabilizes.
- R is *Noetherian* if it is both left-Noetherian and right-Noetherian.

(♠ TODO: finitely generated ideal) If R is commutative, then R is Noetherian if and only if every ideal is finitely generated.

Definition C.0.19 (Finitely generated modules and bimodules)**.** Let R and S be (not necessarily commutative) rings (Definition C.0.2).

1. An R-S-bimodule M is *finitely generated* if it has a finite spanning set.
2. A left/right/two-sided R-module is *finitely generated* if has a finite spanning set, or equivalently if its natural bimodule structure (Definition C.0.4) is finitely generated.

Proposition C.0.20 (Pointwise Computation of Limits and Colimits in Functor Categories)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Let $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ denote the category of functors from $\mathcal{C}$ to $\mathcal{D}$ (also denoted $\mathcal{D}^{\mathcal{C}}$). Let $\mathcal{J}$ be a small category and let $F : \mathcal{J} \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be a diagram of functors, denoted by $j \mapsto F_j$.

1. **Limits are computed pointwise:** Suppose that for every object $C \in \mathcal{C}$, the limit of the diagram $j \mapsto F_j(C)$ exists in $\mathcal{D}$. Then the limit of the diagram F exists in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and is computed pointwise. That is, there is an isomorphism in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$:

$$\left(\lim_{j \in \mathcal{J}} F_j \right)(C) \cong \lim_{j \in \mathcal{J}} (F_j(C)).$$

   The action of this limit functor on a morphism $f : C \to C'$ in $\mathcal{C}$ is the unique morphism induced by the family $\{F_j(f)\}_{j \in \mathcal{J}}$ via the universal property of limits in $\mathcal{D}$.
2. **Colimits are computed pointwise:** Suppose that for every object $C \in \mathcal{C}$, the colimit of the diagram $j \mapsto F_j(C)$ exists in $\mathcal{D}$. Then the colimit of the diagram F exists in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and is computed pointwise. That is, there is an isomorphism in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$:

$$(\mathrm{colim}_{j \in \mathcal{J}} F_j)(C) \cong \mathrm{colim}_{j \in \mathcal{J}} (F_j(C)).$$

   The action of this colimit functor on a morphism $f : C \to C'$ in $\mathcal{C}$ is the unique morphism induced by the family $\{F_j(f)\}_{j \in \mathcal{J}}$ via the universal property of colimits in $\mathcal{D}$.

Definition C.0.21. Let $\mathcal{C}$ be a category (Definition 1.0.1), let Z be an object, and let X, Y be objects of $\mathcal{C}$ over (Definition 6.2.5) Z, i.e. morphisms $X \to Z$ and $Y \to Z$ are fixed. A *cartesian product of X and Y over Z in $\mathcal{C}$* (or *fiber product* or *pullback diagram*) is an object, often denoted by $X \times_Z Y$, with *projection morphisms* $X \times_Z Y \to X$ and $X \times_Z Y \to Y$ that are universal. More precisely, for any object T of $\mathcal{C}$ and morphisms $f_X : T \to X$, $f_Y : T \to Y$, there exists a unique morphism $u : T \to X \times_Z Y$ such that the following diagram commutes:
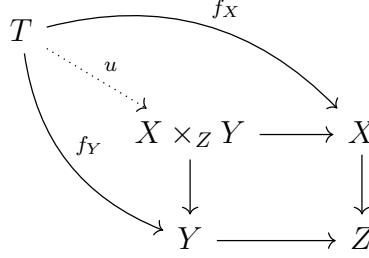
Equivalently, $X \times_Z Y$ is the limit (Definition 1.3.2) of the diagram (Definition 1.3.1)

$$
\begin{array}{c}
X \\
\downarrow \\
Y \longrightarrow Z
\end{array}
$$

in $\mathcal{C}$.

The commutative diagram

$$
\begin{array}{ccc}
X \times_Z Y & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Z
\end{array}
$$

may be referred to as a *cartesian square*.

Definition C.0.22. Let $\mathcal{C}$ be a category (Definition 1.0.1). A *basis for a Grothendieck topology* (also called a *Grothendieck pretopology* or simply a *pretopology*) on $\mathcal{C}$ is a collection of families $K(U)$ of morphisms for each object $U \in \mathcal{C}$, called *coverings* or *covering families*, satisfying the following axioms:

1. **(Identity)** For every isomorphism $U' \to U$, the singleton family $\{U' \to U\}$ is in $K(U)$.
2. **(Base Change)** If $\{U_i \to U\}_{i \in I}$ is a covering family in $K(U)$ and $V \to U$ is any morphism in $\mathcal{C}$, then the fiber products (Definition C.0.21) $U_i \times_U V$ exist, and the family of projections $\{U_i \times_U V \to V\}_{i \in I}$ is in $K(V)$.
3. **(Composition)** If $\{U_i \to U\}_{i \in I}$ is in $K(U)$ and for each $i \in I$, $\{V_{ij} \to U_i\}_{j \in J_i}$ is in $K(U_i)$, then the composite family $\{V_{ij} \to U_i \to U\}_{i \in I, j \in J_i}$ is in $K(U)$.

Definition C.0.23. Let $\mathcal{C}$ be a category equipped with a Grothendieck pretopology (Definition C.0.22) K. The *Grothendieck topology generated by K*, denoted J_K, is the smallest Grothendieck topology (Definition 6.1.4) on $\mathcal{C}$ such that every family in $K(U)$ is a covering family for J_K.

Explicitly, a sieve (Definition 6.1.1) S on an object U belongs to $J_K(U)$ if and only if there exists a covering family (Definition C.0.22) $\{U_i \to U\}_{i \in I} \in K(U)$ such that for every $i \in I$, the morphism $U_i \to U$ belongs to S.

The condition that S contains the family $\{U_i \to U\}$ is equivalent to saying that the sieve generated by this family is a sub-sieve of S.

Definition C.0.24. A *symmetric monoidal category* is a monoidal category $(\mathcal{C}, \otimes, \mathbb{I})$ together with a natural isomorphism (symmetry)

$$\gamma_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$$

for all $X, Y \in \mathcal{C}$, such that for all $X, Y, Z \in \mathcal{C}$ the following holds:

- $\gamma_{Y,X} \circ \gamma_{X,Y} = \mathrm{id}_{X \otimes Y}$ (involutivity);
- the **hexagon coherence diagrams** commute:

$$
\begin{array}{ccccc}
(X \otimes Y) \otimes Z & \xrightarrow{\alpha_{X,Y,Z}} & X \otimes (Y \otimes Z) & \xrightarrow{\gamma_{X,Y \otimes Z}} & (Y \otimes Z) \otimes X \\
{\scriptstyle \gamma_{X,Y} \otimes \mathrm{id}_Z} \downarrow & & & & \uparrow {\scriptstyle \mathrm{id}_Y \otimes \gamma_{X,Z}} \\
(Y \otimes X) \otimes Z & & \xrightarrow{\hspace{3cm} \alpha_{Y,X,Z} \hspace{3cm}} & & Y \otimes (X \otimes Z)
\end{array}
$$

and the analogous hexagon with inverse braiding:

$$
\begin{array}{ccccc}
X \otimes (Y \otimes Z) & \xrightarrow{\alpha^{-1}_{X,Y,Z}} & (X \otimes Y) \otimes Z & \xrightarrow{\gamma_{X \otimes Y, Z}} & Z \otimes (X \otimes Y) \\
{\scriptstyle \mathrm{id}_X \otimes \gamma_{Y,Z}} \downarrow & & & & \uparrow {\scriptstyle \gamma_{X,Z} \otimes \mathrm{id}_Y} \\
X \otimes (Z \otimes Y) & & \xrightarrow{\hspace{3cm} \alpha^{-1}_{X,Z,Y} \hspace{3cm}} & & (X \otimes Z) \otimes Y
\end{array}
$$

- the **symmetry coherence diagram** commutes:

$$
\begin{array}{ccc}
X \otimes Y & \xrightarrow{\gamma_{X,Y}} & Y \otimes X \\
& {\scriptstyle \mathrm{id}_{X \otimes Y}} \searrow & \downarrow {\scriptstyle \gamma_{Y,X}} \\
& & X \otimes Y
\end{array}
$$

A *closed symmetric monoidal category* usually refers to a symmetric monoidal category that is closed as a monoidal category.

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