SCHEMES

December 5, 2025

Contents

1. Formalities of locally ringed spaces and definition of schemes

1.1. Sheaves on topological spaces.

Definition 1.1.1 (Category of opens of a topological space)**.** Let X be a topological space (Definition B.0.1). The *category of opens of X*, sometimes denoted $\mathbf{Open}(X)$ (or $\mathrm{Open}(X)$ or $\mathrm{Ouv}(X)$ (for the French word "ouvert", meaning open), etc.), is the small (Definition C.0.2) category (Definition C.0.1) defined as follows:

- The objects are the open subsets $U \subseteq X$.
- For two open sets $U, V \subseteq X$, the morphism set is
$$\mathrm{Hom}_{\mathbf{Open}(X)}(U, V) = \begin{cases} \{\iota_{U,V}\}, & \text{if } U \subseteq V, \\ \varnothing, & \text{otherwise,} \end{cases}$$
where $\iota_{U,V}$ denotes the inclusion morphism $U \hookrightarrow V$.
- Composition of morphisms is given by composition of set-theoretic inclusions, i.e.
$$\iota_{V,W} \circ \iota_{U,V} = \iota_{U,W} \quad \text{whenever } U \subseteq V \subseteq W.$$
- The identity morphism on an object U is the inclusion $\iota_{U,U} = \mathrm{id}_U$.

Definition 1.1.2. Let (X, τ_X) be a topological space. The *small site assciated to X* or *the site of open covers of X* or *the canonical site on* $\mathrm{Open}\, X$ is the category $\mathrm{Open}(X)$ of open subsets (Definition 1.1.1) of X with inclusion morphisms, equipped with the canonical Grothendieck topology (Definition D.0.2) defined by open coverings.

Definition 1.1.3 (Presheaf on a topological space)**.** Let X be a topological space (Definition B.0.1). Let $\mathcal{D}$ be a category.

A *presheaf (of objects of $\mathcal{D}$/valued in $\mathcal{D}$) on X* is a rule $\mathcal{F}$ that assigns:

- to each open set $U \subseteq X$, an object $\mathcal{F}(U) \in \mathrm{Ob}\,\mathcal{D}$, called the *sections of $\mathcal{F}$ over U*,
- to each inclusion of open sets $V \subseteq U$, a morphism
$$\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V), \quad s \mapsto s|_V,$$

in the category $\mathcal{D}$ called the *restriction map* such that the following conditions hold:

- (Identity) For every open set $U \subseteq X$, the restriction map ρ_U^U is the identity on $\mathcal{F}(U)$.
- (Transitivity) For inclusions $W \subseteq V \subseteq U$ of open sets, one has
$$\rho_W^U = \rho_W^V \circ \rho_V^U.$$

For instance, we may speak of a *presheaf of sets/groups/rings/etc. on the topological space* X.

Equivalently, a presheaf on X (of objects in a category $\mathcal{D}$) is a functor (Definition C.0.4)
$$\mathbf{Open}(X)^{\mathrm{op}} \to \mathcal{D}$$
from the opposite of the category $\mathbf{Open}(X)$ (Definition 1.1.1) of open subsets of X (see also Definition D.0.1) .

Equivalently, a presheaf on X is a presheaf on the category $\mathbf{Open}(X)$ in the sense of Definition D.0.1 .

The sections object $\mathcal{F}(U)$ is also denoted by $\Gamma(U, \mathcal{F})$ (see Definition 1.1.4) . Moreover, the object $\mathcal{F}(X) = \Gamma(X, \mathcal{F})$ is called the *global sections object of* $\mathcal{F}$. This agrees with the notion of global sections as discussed in Definition 1.1.4.

Definition 1.1.4. Let $\mathcal{C}$ be a (large) category (Definition C.0.1), and let $\mathcal{D}$ be a (large) category (Definition C.0.1). Let $\mathcal{F} : \mathcal{C}^{op} \to \mathcal{D}$ be a presheaf valued in $\mathcal{D}$ (Definition D.0.1).

1. For an object $U \in \mathcal{C}$, the *sections functor evaluated at* U is the functor
$$\Gamma(U, -) : \mathrm{PSh}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$$
defined by
$$\Gamma(U, \mathcal{F}) := \mathcal{F}(U),$$
i.e., the value of the presheaf $\mathcal{F}$ at the object U.
2. The *global sections of* $\mathcal{F}$ is the object $\Gamma(\mathcal{F})$ of $\mathcal{D}$ defined as the limit (Definition C.1.1)
$$\Gamma(\mathcal{F}) = \varprojlim_{U \in \mathcal{C}^{\mathrm{op}}} \mathcal{F}(U)$$
assuming that such a limit exists, where the limit is taken over objects $U \in \mathcal{C}$ and the restriction morphisms $\mathcal{F}(V) \to \mathcal{F}(U)$ in $\mathcal{D}$ for morphisms $U \to V$ in $\mathcal{C}$.
   If a final object (Definition E.0.28) $* \in \mathcal{C}$ exists, then $\Gamma(\mathcal{F})$ exists and coincides with $\Gamma(*, \mathcal{F}) = \mathcal{F}(*)$. The construction $\Gamma(\mathcal{F})$ is functorial; in particular, if $\Gamma(\mathcal{F})$ exists for all $\mathcal{F}$ in $\mathrm{PSh}(\mathcal{C}, \mathcal{D})$, e.g. if limits of (Definition C.1.1) diagrams in $\mathcal{D}$ indexed by $\mathcal{C}$ exist, then Γ is a functor
$$\Gamma : \mathrm{PSh}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$$
called the *global sections functor on* $\mathrm{PSh}(\mathcal{C}, \mathcal{D})$.

Definition 1.1.5 (Sheaf on a topological space)**.** Let X be a topological space (Definition B.0.1), let $\mathcal{D}$ be a category (Definition C.0.1), and let $\mathcal{F}$ be a presheaf valued in $\mathcal{D}$ on X (Definition 1.1.3). Then $\mathcal{F}$ is a *sheaf* if it satisfies the following two additional conditions (known as the *sheaf axioms*):

For every open set $U \subseteq X$ and every open cover $\{U_i\}_{i \in I}$ of U,

- (Locality) If $s, t \in \mathcal{F}(U)$ are such that $s|_{U_i} = t|_{U_i}$ for all i, then $s = t$.
- (Gluing) If for each i there is $s_i \in \mathcal{F}(U_i)$ such that for all i, j one has $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all i.

Equivalently, a sheaf on a topological space X may be defined as a sheaf on (Definition D.0.3) the site (Definition D.0.2) of opens on X (Definition 1.1.2).

Definition 1.1.6 (Morphism of presheaves on a topological space)**.** Let X be a topological space (Definition B.0.1), let $\mathcal{D}$ be a category (Definition C.0.1) and let $\mathcal{F}$, and $\mathcal{G}$ be presheaves valued in $\mathcal{D}$ on X (Definition 1.1.3).

A *morphism of presheaves* $\varphi : \mathcal{F} \to \mathcal{G}$ is a collection of maps

$$\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$$

in $\mathcal{D}$ defined for every open set $U \subseteq X$, such that the maps are compatible with restriction: for every inclusion $V \subseteq U$ of open sets, the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \downarrow \rho_V^U & & \downarrow \rho_V^U \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

commutes.

Equivalently, φ is a natural transformation (Definition C.0.5) of $\mathcal{F}, \mathcal{G}$, regarded as (Definition 1.1.3) functors

$$\mathcal{F}, \mathcal{G} : \mathbf{Open}(X)^{\mathrm{op}} \to \mathcal{D}.$$

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then a *morphism of sheaves* is a morphism between $\mathcal{F}$ and $\mathcal{G}$ as presheaves.

Definition 1.1.7. (♠ TODO: Move these notations to the defintions of presheaves and sheaves on topological spaces) Let X be a topological space (Definition B.0.1), and let $\mathcal{D}$ be a category (Definition C.0.1).

The presheaves on X valued in $\mathcal{D}$ (Definition 1.1.3), along with the morphisms (Definition 1.1.6) thereof, form a (in general large) category (Definition C.0.1) often denoted by notations such as $\mathrm{PreShv}(X, \mathcal{D})$ (♠ TODO: include more notations) (or $\mathrm{PreShv}(X)$ if the category $\mathcal{D}$ is clear). If $\mathcal{D}$ is locally small (Definition C.0.2), then so is $\mathrm{PreShv}(X, \mathcal{D})$.

Similarly, the sheaves on X valued in $\mathcal{D}$ (Definition 1.1.5), along with the morphisms (Definition 1.1.6) thereof, form a (in general large) category (Definition C.0.1) often denoted by notations such as $\mathrm{Shv}(X, \mathcal{D})$ (♠ TODO: include more notations) (or $\mathrm{Shv}(X)$ if the category $\mathcal{D}$ is clear). The category $\mathrm{Shv}(X, \mathcal{D})$ is a full subcategory (Definition C.0.6) of $\mathrm{PreShv}(X, \mathcal{D})$.

Equivalently, the categories of presheaves and sheaves are the categories $\mathrm{PreShv}(\mathbf{Open}(X), \mathcal{D})$ and $\mathrm{Shv}(\mathbf{Open}(X), \mathcal{D})$ of presheaves (Definition D.0.1) and sheaves (Definition D.0.3) where $\mathbf{Open}(X)$ (Definition 1.1.1) is the category of open subsets of X equipped with its usual (Definition 1.1.2) Grothendieck topology (Definition D.0.2).

Definition 1.1.8 (Stalk of a sheaf)**.** Let X be a topological space, and let $\mathcal{D}$ be a category (Definition C.0.1) admitting direct colimits (Definition C.1.4) (e.g. the category of sets, groups, abelian groups, modules over rings, or vector spaces over fields). Let $\mathcal{F}$ be a sheaf on X valued in $\mathcal{D}$ (Definition 1.1.5). For a point $x \in X$, the *stalk of $\mathcal{F}$* at x, denoted $\mathcal{F}_x$, is defined as the direct limit (Definition C.1.4)

$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U),$$

where the limit ranges over all open neighborhoods U of x in X, ordered by inclusion.

An element of $\mathcal{F}_x$ is called a *germ of a section at x*. Concretely, a germ at x is given by a pair (U, s) with U an open neighborhood of x and $s \in \mathcal{F}(U)$, modulo the equivalence relation: $(U, s) \sim (V, t)$ if there exists an open neighborhood $W \subseteq U \cap V$ of x such that $s|_W = t|_W$.

If $\mathcal{F}$ is a sheaf of groups, rings, or modules, then each stalk $\mathcal{F}_x$ inherits the corresponding algebraic structure.

Definition 1.1.9 (Morphism induced on stalks by a morphism of presheaves)**.** Let X be a topological space, and let $\mathcal{D}$ be a category (Definition C.0.1) admitting direct colimits (Definition C.1.4) (e.g. the category of sets, groups, abelian groups, modules over rings, or vector spaces over fields). Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on X valued in $\mathcal{D}$, and let

$$\varphi : \mathcal{F} \to \mathcal{G}$$

be a morphism of presheaves (Definition 1.1.6).

For each point $x \in X$, the morphism φ induces a morphism $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ on stalks (Definition 1.1.8) defined as follows:

Recall that the stalk $\mathcal{F}_x$ is the colimit

$$\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U).$$

The morphism φ_x is the canonical morphism induced by the universal property of the colimit: for each open neighborhood U of x, the map

$$\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$$

induces compatible morphisms on the directed system defining stalks, so that

$$\varphi_x = \varinjlim_{x \in U} \varphi(U).$$

In other words, the following diagram commutes for each $U \ni x$:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_x & \xrightarrow{\varphi_x} & \mathcal{G}_x \end{array}$$

where the vertical arrows are the canonical maps to the stalks.

1.2. **Sheaf associated to a presheaf on a topological space.**

Definition 1.2.1 (Sheaf associated to a presheaf)**.** Let X be a topological space, and let $\mathcal{D}$ be a category (Definition C.0.1) admitting direct colimits (Definition C.1.4) (e.g. the category of sets, groups, abelian groups, modules over rings, or vector spaces over fields). Let $\mathcal{P}$ be a presheaf on X with values in $\mathcal{D}$ (Definition 1.1.3).

The *sheaf associated to the presheaf* $\mathcal{P}$ or the *sheaffification of the presheaf* $\mathcal{P}$, denoted $\mathcal{P}^+$ or sometimes by $a\mathcal{P}$, is a sheaf on X together with a morphism of presheaves

$$\eta : \mathcal{P} \to \mathcal{P}^+,$$

satisfying the following universal property: for every sheaf $\mathcal{F}$ on X (valued in $\mathcal{D}$), any morphism of presheaves

$$\varphi : \mathcal{P} \to \mathcal{F}$$

factors uniquely through η, i.e., there exists a unique morphism of sheaves

$$\widetilde{\varphi} : \mathcal{P}^+ \to \mathcal{F}$$

such that

$$\varphi = \widetilde{\varphi} \circ \eta.$$

Concretely, $\mathcal{P}^+$ can be constructed by assigning to each open set $U \subseteq X$ the set (or object in $\mathcal{D}$)

$$\mathcal{P}^+(U) := \left\{ s = (s_x)_{x \in U} \in \prod_{x \in U} \mathcal{P}_x \;\middle|\; \begin{array}{l} \forall x \in U, \\ \exists \text{ an open } V \subseteq U \text{ with } x \in V, \\ \exists t \in \mathcal{P}(V) \text{ such that} \\ \quad \forall y \in V, s_y = t_y \end{array} \right\}.$$

where $\mathcal{P}_x$ is the stalk (Definition 1.1.8) of $\mathcal{P}$ at x, and t_y is the germ (Definition 1.1.8) of t at y. In particular, $\mathcal{P}^+$ exists.

It is noteworthy that the assignment $\mathcal{P} \mapsto \mathcal{P}^+$ is a functor

$$\mathrm{PreShv}(X, \mathcal{D}) \to \mathrm{Shv}(X, \mathcal{D}).$$

(Definition 1.1.7) and that this functor is left adjoint to the inclusion functor

$$\mathrm{Shv}(X, \mathcal{D}) \hookrightarrow \mathrm{PreShv}(X, \mathcal{D})$$

Equivalently, the assignment $\mathcal{P} \mapsto \mathcal{P}^+$ is the sheafification functor as defined in Definition D.0.4.

1.3. **Pushforward/direct image and pullback/inverse image of sheaves on topological spaces under continuous maps.**

Definition 1.3.1 (Pushforward (direct image) of a sheaf)**.** Let $f : X \to Y$ be a continuous map (Definition B.0.2) between topological spaces (Definition B.0.1), and let $\mathcal{F}$ be a presheaf on X valued in a category $\mathcal{D}$ (Definition 1.1.3). The *pushforward* or *direct image presheaf* $f_*\mathcal{F}$ on Y is the sheaf valued in $\mathcal{D}$ on Y (Definition 1.1.5) defined by

$$f_*\mathcal{F}(V) := \mathcal{F}(f^{-1}(V))$$

for every open set $V \subseteq Y$, with restriction maps induced from those of $\mathcal{F}$ via preimages.

If $\mathcal{F}$ is a sheaf (Definition 1.1.5), then so is $f_*\mathcal{F}$.

Definition 1.3.2 (Pullback (inverse image) of a sheaf)**.** Let $f : X \to Y$ be a continuous map (Definition B.0.2) between topological spaces (Definition B.0.1)

1. Let $\mathcal{G}$ be a presheaf on Y valued in a category $\mathcal{D}$ (Definition 1.1.3). The *pullback* or *inverse image presheaf* $f^{-1}\mathcal{G}$ on X is defined as the presheaf
$$U \mapsto \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$$
where U ranges over open subsets of X and the colimit is taken over all open subsets $V \subseteq Y$ containing $f(U)$. This construction admits a natural morphism of sheaves
$$f^{-1}\mathcal{G} \to \mathcal{G}_{f(x)}$$
on stalks (Definition 1.1.8), inducing the identification $(f^{-1}\mathcal{G})_x \cong \mathcal{G}_{f(x)}$.
2. If $\mathcal{G}$ is a sheaf (Definition 1.1.5), then we can define the *pullback* or *inverse image sheaf* $f^*\mathcal{G}$ on X as the sheaf associated to the presheaf (Definition 1.2.1) $f^{-1}\mathcal{G}$, assuming it exists.

1.4. **Ringed spaces, locally ringed spaces, and scheme.**

Definition 1.4.1 (Ringed space)**.** A *ringed space* is a pair $(X, \mathcal{O}_X)$ where

- X is a topological space (Definition B.0.1), and
- $\mathcal{O}_X$ is a sheaf of (Definition 1.1.5) commutative rings (Definition A.0.7) on X.

Equivalently, a ringed space is a ringed site (Definition 4.0.2) where the site is the site of opens (Definition 1.1.2) of the topological space X The sheaf $\mathcal{O}_X$ may be suppressed from the notation and only X may be used to denote a ringed space. The sheaf $\mathcal{O}_X$, also commonly denoted by $\mathscr{O}_X$, is called the *structure sheaf of X*.

Definition 1.4.2 (Locally ringed space)**.** A *locally ringed space* is a ringed space (Definition 1.4.1) $(X, \mathcal{O}_X)$ such that for every point $x \in X$, the stalk $\mathcal{O}_{X,x}$ (Definition 1.1.8) is a local ring (Definition 1.5.4). The unique maximal ideal (Definition A.0.12) of $\mathcal{O}_{X,x}$ is often denoted $\mathfrak{m}_x$ and is called the *maximal ideal at x*.

Definition 1.4.3 (Affine scheme)**.** Let A be a commutative ring with unity (Definition A.0.7). Define the set $\mathrm{Spec}(A)$ to be the set of all prime ideals (Definition A.0.12) of A. Equip it with the *Zariski topology*, which is the topology (Definition B.0.1) whose closed sets are given by *vanishing loci*
$$V(I) = \{\mathfrak{p} \in \mathrm{Spec}(A) : I \subseteq \mathfrak{p}\}$$
for ideals $I \subseteq A$. Define the sheaf $\mathcal{O}_{\mathrm{Spec}(A)}$, called the *structure sheaf of* $\mathrm{Spec}\, A$, by
$$\mathcal{O}_{\mathrm{Spec}(A)}(U) = \{ \text{ locally defined fractions of elements of } A \text{ on } U \},$$
for each open set $U \subseteq \mathrm{Spec}(A)$. It is the case that the stalk at $\mathfrak{p} \in \mathrm{Spec}(A)$ is canonically the localization $A_\mathfrak{p}$ (Definition A.0.15). Then $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ is a locally ringed space (Definition 1.4.2), called the *affine scheme associated to A*.

Moreover, given $f \in A$, we define the locus $D(f)$ by

$$D(f) = \operatorname{Spec} A \setminus V((f)) = \{\mathfrak{p} \in \operatorname{Spec} A : f \notin \mathfrak{p}\}$$

Definition 1.4.4 (Morphism of ringed spaces)**.** Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces (Definition 1.4.2). A *morphism of ringed spaces*

$$(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

consists of:

- a continuous map (Definition B.0.2) $f : X \to Y$, and
- a morphism of (Definition 1.1.6) sheaves of rings (Definition 1.1.5)

  $$f^{\#} : \mathcal{O}_Y \to f_* \mathcal{O}_X,$$

  where $f_* \mathcal{O}_X$ is the direct image sheaf (Definition 1.3.1) of $\mathcal{O}_X$ on Y. (♠ TODO: state the adjunction between direct image and inverse image)

Equivalently, for each open set $V \subseteq Y$, there is a ring homomorphism

$$f^{\#}(V) : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V)),$$

compatible with restriction maps of sections.

Definition 1.4.5. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces (Definition 1.4.1). Suppose that $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces (Definition 1.4.4). By Definition 1.1.9 there is a ring homomorphism (Definition A.0.3)

$$f_x^{\#} : \mathcal{O}_{Y, f(x)} \to \mathcal{O}_{X,x}$$

of the stalks (Definition 1.1.8). This may be referred to the *ring homomorphism induced by f on the stalks*.

Definition 1.4.6 (Morphism of locally ringed spaces)**.** Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be locally ringed spaces (Definition 1.4.2). A *morphism of locally ringed spaces* is a morphism of ringed spaces (Definition 1.4.4) $(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that for each $x \in X$, the induced ring homomorphism (Definition 1.4.5) of stalks (Definition 1.1.8)

$$f_x^{\#} : \mathcal{O}_{Y, f(x)} \to \mathcal{O}_{X,x}$$

is a local homomorphism of local rings (Definition A.0.13); that is, $(f_x^{\#})^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}$ where $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ and $\mathfrak{m}_{f(x)} \subseteq \mathcal{O}_{Y,f(x)}$ are the maximal ideals.

Definition 1.4.7 (Scheme)**.** A *scheme* is a locally ringed space (Definition 1.4.2) $(X, \mathcal{O}_X)$ that admits an open cover $\{U_i\}_{i \in I}$ such that each $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic (as a locally ringed space) (Definition 1.4.6) to an affine scheme $(\operatorname{Spec}(A_i), \mathcal{O}_{\operatorname{Spec}(A_i)})$ (Definition 1.4.3) for some commutative ring A_i. In other words, a scheme is a locally ringed space locally isomorphic to affine schemes.

Definition 1.4.8 (Morphism of schemes)**.** Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be schemes (Definition 1.4.7). A *morphism of schemes* is a morphism as locally ringed spaces (Definition 1.4.6).

In particular, there is a category (Definition C.0.1), often denoted by Sch, **Sch** etc., whose objects are schemes and whose morphisms are morphisms of schemes.

Definition 1.4.9. Let X be a scheme (Definition 1.4.7). A subset $U \subseteq X$ is called an *affine open subset/subscheme of X* if U is open in the underlying topological space of X and the restricted scheme $(U, \mathcal{O}_X|_U)$ is isomorphic, as a scheme, to $\operatorname{Spec} A$ (Definition 1.4.3) for some commutative ring A.

Definition 1.4.10 (Scheme over a scheme)**.** Let $(S, \mathcal{O}_S)$ be a scheme. A *scheme over S* (or an *S-scheme*) is a scheme $(X, \mathcal{O}_X)$ together with a morphism of schemes

$$\pi : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S).$$

This morphism π is called the *structure morphism of the scheme X over S*.

If $S = \operatorname{Spec}(R)$ is an affine scheme for a commutative ring R, then an S-scheme is synonymously called an *R-scheme* or a *scheme over R*.

Let $(S, \mathcal{O}_S)$ be a scheme, and let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be schemes over S with structure morphisms

$$\pi_X : X \to S, \quad \pi_Y : Y \to S.$$

A *morphism of S-schemes* (or synonymously a *S-scheme morphism*) is a morphism of schemes (Definition 1.4.8)

$$(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \pi_X \downarrow & & \downarrow \pi_Y \\ S & = & S \end{array}$$

In other words,

$$\pi_Y \circ f = \pi_X.$$

Given a fixed scheme S, there is a category, often denoted by Sch_S, $\mathrm{Sch}_{/S}$, Sch/S, $\mathbf{Sch}_S$, $\mathbf{Sch}_{/S}$, $\mathbf{Sch}/S$ etc. whose objects are schemes T over S and whose morphisms $T_1 \to T_2$ are morphisms of schemes over S. If $S = \operatorname{Spec} R$ for some commutative ring R, then we may instead write Sch_R to denote $\mathrm{Sch}_{\operatorname{Spec} R}$, etc. It is noteworthy that $\mathrm{Sch}_{\mathbb{Z}}$ coincides with the category Sch (Definition 1.4.8) of all schemes. In other words, a $\mathbb{Z}$-scheme can be identified simply with a scheme.

Equivalently, the category $\mathrm{Sch}_{/S}$ is the category of schemes over S in the sense of Definition C.0.7.

1.5. **Preimages.**

Definition 1.5.1. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces (Definition 1.4.1), and let

$$f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

be a morphism of ringed spaces (Definition 1.4.4). For any open subset $V \subseteq Y$, we define the *preimage of V under f* to be the ringed space whose underlying topological space is the open subset

$$f^{-1}(V) := \{x \in X \mid f(x) \in V\} \subseteq X$$

and whose sheaf of rings is the restriction of $\mathcal{O}_X$ to $f^{-1}(V)$.

Definition 1.5.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be locally ringed spaces (Definition 1.4.2), and let

$$f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

be a morphism of locally ringed spaces (Definition 1.4.6). The *preimage of an open subset* $V \subseteq Y$ is defined as the preimage of V under f (Definition 1.5.1) as a morphism of ringed spaces (Definition 1.4.4):

$$f^{-1}(V) := \{x \in X \mid f(x) \in V\} \subseteq X.$$

Definition 1.5.3. Let X and Y be schemes (Definition 1.4.7), and let

$$f : X \to Y$$

be a morphism of schemes (Definition 1.4.8). For an open subset $V \subseteq Y$, the *preimage of V under f* is preimage of V under f (Definition 1.5.2) as a morphism of locally ringed spaces (Definition 1.4.6):

$$f^{-1}(V) := \{x \in X \mid f(x) \in V\} \subseteq X.$$

This preimage is always an open subscheme of X with the structure sheaf inherited from $\mathcal{O}_X$, and is called the *scheme-theoretic preimage of V under f*.

Definition 1.5.4. Let R be a ring (Definition A.0.1) with unity, not necessarily commutative. The ring R is called a *local ring* if it has a unique maximal left ideal (Definition A.0.12). In this case, R also has a unique maximal right ideal, and these coincide with the Jacobson radical $J(R)$ of R. The unique maximal left (and right) ideal of a local ring R may sometimes be denoted by $\mathfrak{m}_R$.

Definition 1.5.5. Let R be an integral domain, and consider the set $R \times (R \setminus \{0\})$ as above. Define a relation $\sim$ on $R \times (R \setminus \{0\})$ by declaring that

$$(a, b) \sim (c, d) \quad \text{if and only if} \quad ad = bc,$$

for $a, c \in R$ and $b, d \in R \setminus \{0\}$. This relation is an equivalence relation. Its equivalence classes are denoted by $\frac{a}{b}$.

The set of equivalence classes

$$\left\{ \tfrac{a}{b} \,\middle|\, a \in R,\ b \in R \setminus \{0\} \right\}$$

under the relation $\sim$ defined above is called the *field of fractions of R*, and is denoted by $\mathrm{Frac}(R)$.

The operations on $\mathrm{Frac}(R)$ are defined by

$$\begin{aligned} \tfrac{a}{b} + \tfrac{c}{d} &= \tfrac{ad+bc}{bd}, \\ \tfrac{a}{b} \cdot \tfrac{c}{d} &= \tfrac{ac}{bd}, \end{aligned}$$

for $a, c \in R$ and $b, d \in R \setminus \{0\}$. With these operations, $\mathrm{Frac}(R)$ is a field (Definition E.0.38).

Equivalently, $\mathrm{Frac}(R)$ may be defined as the localization of R (Definition A.0.15) by the multiplicative subset $R \setminus \{0\}$ (Definition A.0.14).

Definition 1.5.6. 1. Let R be a local ring (Definition 1.5.4) with a unique maximal ideal (Definition A.0.12) $\mathfrak{m} \subseteq R$. The *residue field of R* is the field

$$\kappa(R) := R/\mathfrak{m}.$$

2. Let R be a commutative ring (Definition A.0.7) and let $\mathfrak{p} \subseteq R$ be a prime ideal (Definition A.0.12). The *residue field of R at $\mathfrak{p}$* is the field

$$\kappa(\mathfrak{p}) := \mathrm{Frac}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}),$$

(Definition 1.5.5) where $R_{\mathfrak{p}}$ is the localization (Definition A.0.15) of R at the prime $\mathfrak{p}$.

Definition 1.5.7. Let X be a scheme (Definition 1.4.7) and let $x \in X$ be a point. The *residue field of the scheme X at the point x* is defined as

$$\kappa(x) = \kappa(\mathcal{O}_{X,x}) := \mathcal{O}_{X,x}/\mathfrak{m}_x.$$

where $\mathcal{O}_{X,x}$ (Definition 1.1.8) is the local ring of X at x with maximal ideal $\mathfrak{m}_x$.

2. Affine and projective spaces

2.1. **Affine space.**

Definition 2.1.1. Let S be a scheme (Definition 1.4.7) and let $n \geq 0$ be an integer. We define the *affine space of dimension n over S*, denoted by $\mathbb{A}^n_S$, as follows:

1. If $S = \operatorname{Spec} A$ is an affine scheme (Definition 1.4.3), then $\mathbb{A}^n_S$ is the affine scheme defined by the polynomial ring in n variables over A:

$$\mathbb{A}^n_{\operatorname{Spec} A} = \operatorname{Spec}(A[T_1, \ldots, T_n]).$$

2. For a general scheme S, let $\{U_i = \operatorname{Spec} A_i\}_{i \in I}$ be an affine open covering of S. For each i, let $X_i = \mathbb{A}^n_{U_i} = \operatorname{Spec}(A_i[T_1, \ldots, T_n])$. Since polynomial rings behave well under localization, for any open immersion $U_{ij} = U_i \cap U_j \hookrightarrow U_i$, there is a canonical isomorphism on the overlaps:

$$\phi_{ij} : X_i|_{U_{ij}} \xrightarrow{\sim} X_j|_{U_{ij}}.$$

The scheme $\mathbb{A}^n_S$ is obtained by gluing the family $\{X_i\}_{i \in I}$ along these isomorphisms.

Alternatively, $\mathbb{A}^n_S$ can be defined globally as the relative spectrum (Definition 4.2.2) of the sheaf of polynomial algebras over $\mathcal{O}_S$:

$$\mathbb{A}^n_S = \mathbf{Spec}(\mathcal{O}_S[T_1, \ldots, T_n]).$$

2.2. **Projective space.**

2.2.1. *Graded rings.*

Definition 2.2.1. Let Γ be a monoid (Definition E.0.16) (often an abelian group (Definition E.0.17)). A *Γ-graded ring* is a ring (Definition A.0.1) R equipped with a decomposition into additive subgroups $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ (Definition E.0.18), such that the multiplication respects the grading:

$$R_\gamma \cdot R_\delta \subseteq R_{\gamma+\delta} \quad \text{for all } \gamma, \delta \in \Gamma.$$

Elements in a component R_γ are called *homogeneous elements of degree γ*. A general element $r \in R$ can be written uniquely as a finite sum $r = \sum_\gamma r_\gamma$ with $r_\gamma \in R_\gamma$, called the *homogeneous components of r*.

Definition 2.2.2. Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a graded ring (Definition 2.2.1). An ideal (Definition E.0.19) $I \subseteq R$ (left, right, or two-sided) is called *homogeneous* (or *graded*) if it satisfies any of the following equivalent conditions:

- I is generated by (Definition E.0.20) homogeneous elements (Definition 2.2.1).
- For every $r \in I$, all homogeneous components of r also belong to I.
- $I = \bigoplus_{\gamma\in\Gamma}(I \cap R_\gamma)$.

Definition 2.2.3. Let R be a (not necessarily commutative) graded ring (Definition 2.2.1).

- A *homogeneous prime ideal of R* a (necessarily two-sided) prime ideal (Definition A.0.12) that is homogeneous (Definition 2.2.2).
- A *homogeneous maximal ideal of R* is a homogeneous (Definition 2.2.2) proper (left/right/two-sided) ideal $\mathfrak{m} \subsetneq R$ such that there are no homogeneous (left/right/two-sided) ideals strictly contained between $\mathfrak{m}$ and R.

Definition 2.2.4. Let Γ be a monoid (Definition E.0.16). An element $z \in \Gamma$ is called an *absorbing zero* (or simply a *zero element*) if for all $\gamma \in \Gamma$:

$$z + \gamma = z \quad \text{and} \quad \gamma + z = z.$$

Such an element, if it exists, is unique and is typically denoted by 0 (using additive notation) or ∞ (in tropical contexts).

Definition 2.2.5. Let Γ be a monoid (Definition E.0.16) with an absorbing zero (Definition 2.2.4) $0 \in \Gamma$. A *positive cone in Γ* is a submonoid $\Gamma_+ \subseteq \Gamma$ such that: (♠ TODO: submonoid)

1. $0 \in \Gamma_+$ and the identity element (Definition E.0.24) $e \in \Gamma_+$ (if $0 \neq e$).
2. Γ_+ generates Γ as a group (if Γ is cancellative) or satisfies some appropriate generating condition relevant to the context.
3. Often, "positive" implies that the only invertible element in Γ_+ is the identity, or that there is a partial order $\leq$ on Γ such that $\gamma \in \Gamma_+ \iff \gamma \geq e$.

(Note: In the context of graded rings, one typically requires that Γ is a cancellative monoid and the grading is supported on Γ_+, but the "irrelevant ideal" definition below only strictly requires distinguishing "zero" degrees from "positive" degrees).

Definition 2.2.6. An *ordered monoid* is a monoid $(M, \cdot, e)$ equipped with a partial order (Definition E.0.21) $\leq$ that is compatible with the monoid operation. That is, for all $a, b, c \in M$, if $a \leq b$, then:

$$a \cdot c \leq b \cdot c \quad \text{and} \quad c \cdot a \leq c \cdot b.$$

If the order $\leq$ is a total order (Definition E.0.22), then M is called a *totally ordered monoid*.

Definition 2.2.7. Let Γ be a monoid (Definition E.0.16) with an absorbing zero (Definition 2.2.4) 0 (or let Γ be an ordered monoid with identity e). Let $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ be a Γ-graded ring (Definition 2.2.1). The *irrelevant ideal of R*, denoted by R_+, is the ideal (Definition E.0.19) generated by all homogeneous elements (Definition 2.2.1) of "positive" degree (i.e., degrees not equal to the neutral element of the grading monoid). Explicitly:

$$R_+ = \bigoplus_{\gamma \in \Gamma \setminus \{e\}} R_\gamma.$$

(If Γ is additive with identity 0, then $R_+ = \bigoplus_{\gamma \neq 0} R_\gamma$). In particular, we may speak of a the irrelevant ideal of a $\mathbb{Z}_{\geq 0}$-graded ring.

2.2.2. *Proj.*

Definition 2.2.8. Let $R = \bigoplus_{d \geq 0} R_d$ be a $\mathbb{Z}_{\geq 0}$-graded commutative ring. The *projective spectrum of R*, denoted by $\mathrm{Proj}(R)$, is the set of all homogeneous prime ideals (Definition 2.2.3) $\mathfrak{p} \subset R$ that do not contain the irrelevant ideal (Definition 2.2.7) $R_+ = \bigoplus_{d>0} R_d$.

The set $\mathrm{Proj}(R)$ is equipped with the *Zariski topology*, which is the topology (Definition B.0.1) where the closed sets are of the form

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Proj}(R) \mid \mathfrak{a} \subseteq \mathfrak{p}\}$$

for any homogeneous ideal (Definition 2.2.2) $\mathfrak{a} \subseteq R$.

The structure sheaf (Definition 1.4.1) $\mathcal{O}_{\mathrm{Proj}(R)}$ is defined on the basis (Definition E.0.23) of open sets

$$D_+(f) = \{\mathfrak{p} \in \mathrm{Proj}(R) \mid f \notin \mathfrak{p}\}$$

(where $f \in R_d$ is a homogeneous element of degree $d > 0$) by setting:

$$\mathcal{O}_{\mathrm{Proj}(R)}(D_+(f)) = R_{(f)},$$

where $R_{(f)}$ is the subring of degree zero elements in the localization (Definition A.0.15) R_f. This makes $(\mathrm{Proj}(R), \mathcal{O}_{\mathrm{Proj}(R)})$ into a scheme.

Definition 2.2.9. Let S be a scheme (Definition 1.4.7) and let $n \geq 0$ be an integer. The *projective space of dimension n over S*, denoted by $\mathbb{P}^n_S$, is constructed as follows:

1. If $S = \operatorname{Spec} A$ is an affine scheme (Definition 1.4.3), then $\mathbb{P}^n_S$ is the "usual" projective space over the ring A, defined as the projective spectrum (Definition 2.2.8) of the polynomial ring in $n+1$ variables with coefficients in A:

$$\mathbb{P}^n_{\operatorname{Spec} A} = \mathrm{Proj}(A[T_0, \ldots, T_n]);$$

here, each variable T_i is graded in degree 1, and elements of A are graded in degree 0.

2. For a general scheme S, let $\{U_i = \operatorname{Spec} A_i\}_{i \in I}$ be an affine open covering of S. For each i, let $X_i = \mathbb{P}^n_{U_i} = \operatorname{Proj}(A_i[T_0, \ldots, T_n])$. For any open immersion $U_{ij} = U_i \cap U_j \hookrightarrow U_i$, the restriction maps on coefficients induce canonical isomorphisms $X_i|_{U_{ij}} \cong X_j|_{U_{ij}}$. The scheme $\mathbb{P}^n_S$ is obtained by gluing the family $\{X_i\}$ along these isomorphisms.

Equivalently, $\mathbb{P}^n_S$ is the relative Proj (Definition 4.2.3) of the graded sheaf of polynomial algebras in $n+1$ variables over $\mathcal{O}_S$:

$$\mathbb{P}^n_S = \mathbf{Proj}(\mathcal{O}_S[T_0, \ldots, T_n]).$$

3. Properties of schemes and morphisms of schemes

3.1. **Properties of schemes.**

Definition 3.1.1 (Compact topological space)**.** A topological space $(X, \mathcal{T})$ is *compact* if every open cover of X admits a finite subcover; that is, for every collection $\{U_i\}_{i \in I}$ of open sets in $\mathcal{T}$ such that $X = \bigcup_{i \in I} U_i$, there exists a finite subcollection $\{U_{i_j}\}_{j=1}^n$ such that $X = \bigcup_{j=1}^n U_{i_j}$.

Some mathematicians, e.g. algebraic geometers, would refer to this property as *quasi-compactness*.

Definition 3.1.2 (Quasi-Compact Scheme)**.** A *quasi-compact scheme* X is a scheme (Definition 1.4.7) such that the underlying topological space (Definition B.0.1) of X is quasi-compact (Definition 3.1.1). That is, every open cover of X admits a finite subcover.

Equivalently, X is quasi-compact if it can be covered by finitely many affine open subschemes (Definition 1.4.9) $\operatorname{Spec} A_i$ (Definition 1.4.3).

Definition 3.1.3. Let R be a ring (Definition A.0.1). We say that R is a *reduced ring* if the only nilpotent element of R is the zero element. If R is commutative (Definition A.0.7), this is equivalent to the condition that the nilradical of R is the zero ideal, i.e., $\mathfrak{N}(R) = (0)$.

Definition 3.1.4. Let $(X, \mathcal{O}_X)$ be a scheme (Definition 1.4.7). The scheme X is said to be a *reduced scheme* if for every open subset $U \subseteq X$, the ring of sections $\mathcal{O}_X(U)$ contains no non-zero nilpotent elements. This condition is equivalent to requiring that for all $x \in X$, the local ring (Definition 1.5.4) $\mathcal{O}_{X,x}$ (Definition 1.1.8) is a reduced ring (Definition 3.1.3).

Definition 3.1.5. Let X be a topological space (Definition B.0.1). The space X is said to be an *irreducible topological space* if X is non-empty and cannot be expressed as the union of two proper closed subsets. That is, if $X = Z_1 \cup Z_2$ with Z_1, Z_2 closed in X, then either $Z_1 = X$ or $Z_2 = X$. Equivalently, X is irreducible if every non-empty open subset of X is dense in X.

Definition 3.1.6. Let X be a scheme (Definition 1.4.7). We say that X is an *integral scheme* if X is both a reduced scheme (Definition 3.1.4) and an irreducible topological space (Definition E.0.6). Equivalently, X is integral if for every non-empty open subset $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is an integral domain.

Definition 3.1.7. Let X be a scheme (Definition 1.4.7).

- The scheme X is called *normal* if for every point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is an integrally closed domain. If X is integral (Definition 3.1.6), this is equivalent to saying that for every affine open subset $U = \operatorname{Spec} A$, the ring A is an integrally closed domain.
- The scheme X is called *factorial* if for every point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a unique factorization domain (UFD).

Definition 3.1.8 (Noetherian conditions for a ring)**.** Let R be a ring (Definition A.0.1). We say:

- R is *left-Noetherian* if every ascending chain of left ideals (Definition E.0.19) of R stabilizes, i.e., if for any chain
$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$
of left ideals, there exists n such that $I_m = I_n$ for all $m \geq n$.
- R is *right-Noetherian* if every ascending chain of right ideals of R stabilizes.
- R is *Noetherian* if it is both left-Noetherian and right-Noetherian.

(♠ TODO: finitely generated ideal) If R is commutative (Definition A.0.7), then R is Noetherian if and only if every ideal is finitely generated.

Definition 3.1.9 (Locally Noetherian Scheme and Noetherian Scheme)**.** Let X be a scheme (Definition 1.4.7).

- X is called *locally Noetherian* if it admits an open cover $\{U_i\}$ such that for each i, the ring $\mathcal{O}_X(U_i)$ of regular functions on U_i is a Noetherian ring (Definition 3.1.8). Equivalently, X is locally Noetherian if it is covered by open affine subschemes $\operatorname{Spec} A_i$ with each A_i a Noetherian ring.
- X is called *Noetherian* if it is locally Noetherian and quasi-compact (Definition 3.1.2), i.e., X can be covered by finitely many affine opens $\operatorname{Spec} A_i$ where each A_i is Noetherian.

Definition 3.1.10. Let X be a topological space (Definition B.0.1). The *dimension of X* or *Krull dimension of X*, denoted by $\dim(X)$, is the supremum of the lengths of strictly ascending chains of distinct irreducible (Definition E.0.6) closed subsets of X. That is, $\dim(X)$ is the supremum of integers n such that there exists a chain
$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$
where each Z_i is a non-empty irreducible closed subset of X. If X is empty, its dimension is defined to be $-\infty$ (or sometimes -1).

Definition 3.1.11. Let X be a topological space (Definition B.0.1) and let $Y \subseteq X$ be an irreducible (Definition E.0.6) closed subset. The *codimension of Y in X*, denoted by $\operatorname{codim}(Y, X)$, is the supremum of lengths of chains of irreducible closed subsets containing Y. That is, it is the supremum of integers n such that there exists a chain
$$Y = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$
where each Z_i is an irreducible closed subset of X.

Proposition 3.1.12. If X is an irreducible (Definition E.0.6) scheme (Definition 1.4.7), and $Y \subseteq X$ is an irreducible closed subscheme, then $\operatorname{codim}(Y, X)$ (Definition 3.1.11) coincides with $\dim(\mathcal{O}_{X,\eta})$ (Definition 3.1.13) (Definition 1.1.8), where η is the generic point of Y (Definition E.0.12).

Definition 3.1.13. Let R be a commutative ring (Definition A.0.7). The *Krull dimension of R*, denoted by $\dim(R)$, is defined as the supremum of the lengths of strictly ascending chains of prime ideals (Definition A.0.12) of R. Explicitly, it is the supremum of integers n such that there exists a chain

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$$

where each $\mathfrak{p}_i$ is a prime ideal of R. This is equivalent to the dimension (Definition 3.1.14) of the (prime) spectrum $\operatorname{Spec}(R)$ (Definition 1.4.3) endowed with the Zariski topology.

Definition 3.1.14 (Dimension of a Scheme)**.** Let X be a scheme with underlying topological space $|X|$.

(♠ TODO: krull dimension)

- The *dimension at a point $x \in |X|$*, denoted $\dim_x(X)$, is the Krull dimension of the local ring (Definition 1.4.2) $\mathcal{O}_{X,x}$ (Definition 1.1.8). This is the supremum of the lengths n of chains of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subseteq \mathcal{O}_{X,x}.$$

- The *dimension of the scheme X* is defined as

$$\dim(X) := \sup_{x \in |X|} \dim_x(X).$$

Equivalently, it is the supremum of the lengths of chains of distinct irreducible closed subsets of $|X|$ ordered by inclusion.

Definition 3.1.15. Let X be a scheme (Definition 1.4.7). (♠ TODO: irreducible component) We say that X is *pure dimensional* (or *equidimensional*) if every irreducible component of X has the same dimension. That is, if $X = \bigcup_{i \in I} Z_i$ is the decomposition of X into its irreducible components, there exists an integer d such that:

$$\dim(Z_i) = d \quad \text{for all } i \in I$$

where $\dim(Z_i)$ denotes the Krull dimension (Definition 3.1.10) of the topological space Z_i. (♠ TODO: define irreducible component, Krull dimension of a scheme)

Definition 3.1.16. Let $(A, \mathfrak{m})$ be a Noetherian (Definition 3.1.8) local ring (Definition 1.5.4). The *embedding dimension of A* is the dimension of the cotangent space $\mathfrak{m}/\mathfrak{m}^2$ (Definition 3.1.17) as a vector space over the residue field (Definition 1.5.6) $k = A/\mathfrak{m}$. The ring A is called a *regular local ring* if its global dimension is finite, or equivalently, if its embedding dimension equals its Krull dimension (Definition 3.1.13):

$$\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim(A).$$

Definition 3.1.17. Let $(A, \mathfrak{m})$ be a local ring (Definition 1.4.2) with residue field (Definition 1.5.6) $k = A/\mathfrak{m}$.

- The *Zariski cotangent space* of A is defined as the k-vector space $\mathfrak{m}/\mathfrak{m}^2$ (Definition E.0.7) (Definition E.0.8).
- The *Zariski tangent space* (or simply *tangent space*) of A, denoted by T_A or $T_{\mathfrak{m}}(A)$, is defined as the dual vector space (Definition E.0.9) over k to the cotangent space:

$$T_A = \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k).$$

Definition 3.1.18. Let X be a scheme (Definition 1.4.7) and let $x \in X$ be a point. Let $\mathcal{O}_{X,x}$ (Definition 1.1.8) be the local ring at x with maximal ideal (Definition A.0.12) $\mathfrak{m}_x$ and residue field (Definition 1.5.6) $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$.

- The *cotangent space of X at x*, denoted by T_x^*X, is the Zariski cotangent space (Definition 3.1.17) of the local ring $\mathcal{O}_{X,x}$, i.e., $T_x^*X = \mathfrak{m}_x/\mathfrak{m}_x^2$.
- The *tangent space of X at x*, denoted by T_xX, is the Zariski tangent space (Definition 3.1.17) of the local ring $\mathcal{O}_{X,x}$, given by

$$T_xX = \mathrm{Hom}_{\kappa(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, \kappa(x)).$$

Definition 3.1.19. Let A be a Noetherian ring (Definition 3.1.8). We say that A is a *regular ring* if for every prime ideal (Definition A.0.12) $\mathfrak{p} \subset A$, the localization (Definition A.0.15) $A_{\mathfrak{p}}$ is a regular local ring (Definition 3.1.16).

Definition 3.1.20. Let X be a locally Noetherian scheme (Definition 3.1.9).

- The scheme X is *regular at a point $x \in X$* (or *nonsingular at a point $x \in X$*) if the local ring (Definition 1.4.2) $\mathcal{O}_{X,x}$ (Definition 1.1.8) is a regular local ring (Definition 3.1.16). Otherwise, X is said to be *singular at x*, or synonymously at x is a *singularity of X*.
- The scheme X is called a *regular scheme* (or *nonsingular scheme*) if it is regular at every point $x \in X$. Otherwise, X is said to be *singular*.

3.2. **Properties of morphisms of schemes.**

Definition 3.2.1 (Quasi-compact morphism of schemes)**.** Let $f : X \to Y$ be a morphism of schemes (Definition 1.4.8).

The morphism f is *quasi-compact* if for every open affine subset (Definition 1.4.9) $V = \mathrm{Spec}(B) \subseteq Y$ (Definition 1.4.3), the preimage $f^{-1}(V)$ (Definition 1.5.3) can be covered by finitely many affine open subsets of X.

Equivalently, f is quasi-compact if the inverse image of every quasi-compact (Definition 3.1.2) open subset of Y is quasi-compact in X.

Definition 3.2.2 (Diagonal morphism of a morphism of schemes)**.** Let $f : X \to Y$ be a morphism of schemes (Definition 1.4.8).

The *diagonal morphism asscciated to f* is the morphism $\Delta_f : X \to X \times_Y X$ (Definition 3.2.8) which is the diagonal morphism associated to the morphism f in the category of schemes.

In other words, Δ_f is defined as the unique morphism induced by the universal property of the fiber product (Definition 3.2.8) making the following diagram commute:

$$\begin{tikzcd}
X, \arrow[dr, "\Delta_f"] \arrow[drr, bend left, "\mathrm{id}_X"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\
& X \times_Y X \arrow[r, "p_2"] \arrow[d, "p_1"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

where p_1 and p_2 are the natural projections from the fiber product.

In other words, Δ_f is given by the pair of identity morphisms $(\mathrm{id}_X, \mathrm{id}_X)$ over Y:

$$\Delta_f := (\mathrm{id}_X, \mathrm{id}_X) : X \to X \times_Y X.$$

Definition 3.2.3 (Quasi-separated morphism of schemes)**.** Let $f : X \to Y$ be a morphism of schemes (Definition 1.4.8).

(♠ TODO: closed immersion) The morphism f is *quasi-separated* (resp. *separated*) if the diagonal morphism (Definition 3.2.2)

$$\Delta_f : X \to X \times_Y X$$

is quasi-compact (Definition 3.2.1) (resp. a closed immersion). In this case, we can also say that X is a *quasi-separated Y-scheme* or a *quasi-separated scheme over Y*. A *quasi-separated scheme* refers to a quasi-separated scheme over $\operatorname{Spec} \mathbb{Z}$ (Definition 1.4.3).

Intuitively, f is quasi-separated if the intersection of two affine open subsets in X which map into an affine open of Y can be covered by finitely many affine opens.

Definition 3.2.4. A scheme (Definition 1.4.7) is called *qcqs* (or *coherent*) if it is both quasi-compact (Definition 3.1.2) and quasi-separated (Definition 3.2.3).

Theorem 3.2.5. The following classes of schemes are coherent (Definition 3.2.4) (qcqs):

1. Any affine scheme (Definition 1.4.3) $X = \operatorname{Spec}(R)$ for any commutative ring R.
2. Any Noetherian scheme (Definition 3.1.9).
3. Any scheme of finite type (Definition 3.2.7) over a coherent base (e.g., varieties (Definition 5.0.1) over a field).

Definition 3.2.6 (Locally of finite presentation morphism of schemes)**.** Let $f : X \to Y$ be a morphism of schemes.

1. We say that f is *locally of finite presentation* if for every affine open subset (Definition 1.4.9) $\operatorname{Spec}(B) \subseteq Y$ (Definition 3.2.28), and every affine open subset $\operatorname{Spec}(A) \subseteq f^{-1}(\operatorname{Spec}(B))$ (Definition 1.5.3), the induced ring homomorphism $B \to A$ presents A as a B-algebra of finite presentation (Definition A.0.6); that is, A is isomorphic to a quotient of a polynomial ring in finitely many variables over B by a finitely generated ideal:

$$A \cong B[x_1, \ldots, x_n]/(f_1, \ldots, f_m),$$

for some finite n, m.

2. A morphism of schemes $f : X \to Y$ is of *finite presentation* if it is locally of finite presentation and quasi-compact (Definition 3.2.1) and quasi-separated (Definition 3.2.3); in particular, f can be covered by finitely many affine opens satisfying the finite presentation condition above.

Definition 3.2.7. Let $f : X \to Y$ be a morphism of schemes (Definition 1.4.8). We say that f is a *finite type morphism* if for every affine open (Definition 1.4.9) $V = \operatorname{Spec} B \subseteq Y$ with $U = f^{-1}(V)$ affine, say $U = \operatorname{Spec} A$, the ring A is a finitely generated B-algebra (Definition A.0.5).

When X is equipped with a finite type morphism $f : X \to Y$, we say that X is a *finite type scheme over Y* or a *finite type Y-scheme* or a *Y-scheme of finite type* (Definition 1.4.10), etc.

Definition 3.2.8. Let $\mathcal{C}$ be a category (Definition C.0.1), let Z be an object, and let X, Y be objects of $\mathcal{C}$ over (Definition C.0.7) Z, i.e. morphisms $X \to Z$ and $Y \to Z$ are fixed. A *cartesian product of X and Y over Z in $\mathcal{C}$* (or *fiber product* or *pullback diagram*) is an object, often denoted by $X \times_Z Y$, with *projection morphisms* $X \times_Z Y \to X$ and $X \times_Z Y \to Y$ that are universal. More precisely, for any object T of $\mathcal{C}$ and morphisms $f_X : T \to X$, $f_Y : T \to Y$, there exists a unique morphism $u : T \to X \times_Z Y$ such that the following diagram commutes:

$$
\begin{tikzcd}
T \arrow[dr, dotted, "u"] \arrow[drr, bend left, "f_X"] \arrow[ddr, bend right, "f_Y"'] & & \\
& X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d] \\
& Y \arrow[r] & Z
\end{tikzcd}
$$

Equivalently, $X \times_Z Y$ is the limit (Definition C.1.1) of the diagram

$$
\begin{tikzcd}
& X \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}
$$

in $\mathcal{C}$.

The commutative diagram

$$
\begin{tikzcd}
X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}
$$

may be referred to as a *cartesian square*.

Definition 3.2.9. Let $\mathcal{C}$ be a category. Let $f : X \to Y$ be a morphism in $\mathcal{C}$, and let $g : Y' \to Y$ be any morphism in $\mathcal{C}$ such that the fiber product $X \times_Y Y'$ (Definition 3.2.8) exists. The *base change of f by g* is the morphism $f' : X \times_Y Y' \to Y'$ obtained by the pullback of f along g. In particular,

$$\begin{array}{ccc} X \times_Y Y' & \xrightarrow{p_1} & X \\ \downarrow f' & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

is a Cartesian diagram (or pullback square) in $\mathcal{C}$.

Theorem 3.2.10. Let S be a scheme. The category Sch/S of schemes over S (Definition 1.4.10) is closed under fiber products. More precisely, for any morphisms $X \to Z$ and $Y \to Z$ in Sch/S, there exists a fiber product $X \times_Z Y$ (Definition 3.2.8).

Definition 3.2.11. Let $f : X \to Y$ be a map of topological spaces (Definition B.0.2).

- The map f is called an *open map* if for every open subset $U \subseteq X$, the image $f(U)$ is an open subset of Y.
- The map f is called a *closed map* if for every closed subset $Z \subseteq X$, the image $f(Z)$ is a closed subset of Y.

Definition 3.2.12. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of schemes (Definition 1.4.8).

- The morphism f is called an *open immersion* if the underlying map of topological spaces induces a homeomorphism (Definition E.0.3) from X onto an open subset $V \subseteq Y$, and the induced map of sheaves (Definition D.0.3) $f^\sharp|_V : \mathcal{O}_Y|_V \to f_*\mathcal{O}_X$ (Definition 1.4.6) is an isomorphism of sheaves of rings on V.
  (♠ TODO: surjective map of sheaves of sets)
- The morphism f is called a *closed immersion* if the underlying map of topological spaces induces a homeomorphism from X onto a closed subset $Z \subseteq Y$, and the induced map of sheaves $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective.

Definition 3.2.13. Let $\mathcal{C}$ be a site (Definition D.0.2). Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves (Definition D.0.3) of sets on $\mathcal{C}$, and let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves.

- The morphism ϕ is called *injective* if for every object U in $\mathcal{C}$, the map of sets $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective (Definition E.0.1). Equivalently, ϕ is a monomorphism (Definition E.0.5) in the category $\mathrm{Sh}(\mathcal{C}, \mathbf{Sets})$ of sheaves on $\mathcal{C}$ valued in sets.
- The morphism ϕ is called *surjective* if for every object U in $\mathcal{C}$ and every section $s \in \mathcal{G}(U)$, there exists a covering $\{U_i \to U\}_{i \in I}$ in the topology of $\mathcal{C}$ such that for each i, the restriction $s|_{U_i}$ is in the image of $\phi_{U_i} : \mathcal{F}(U_i) \to \mathcal{G}(U_i)$. Equivalently, ϕ is an epimorphism (Definition E.0.5) in the category $\mathrm{Sh}(\mathcal{C}, \mathbf{Sets})$ of sheaves on $\mathcal{C}$ valued in sets.

Definition 3.2.14. Let X be a topological space (Definition B.0.1). A subset $Z \subseteq X$ is called a *locally closed subset* if Z can be written as the intersection $U \cap C$, where U is an open subset of X and C is a closed subset of X. Equivalently, Z is a locally closed subset if it is an open subset of its closure $\overline{Z}$ endowed with the subspace topology (Definition E.0.4).

Definition 3.2.15. 1. Let $f : X \to Y$ be a morphism of schemes (Definition 1.4.8). The morphism f is called a *locally closed immersion* (or simply an *immersion*) if it can be factored as the composition $f = j \circ i$, where $i : X \to Z$ is a closed immersion (Definition 3.2.12) into a scheme Z, and $j : Z \to Y$ is an open immersion (Definition 3.2.12). Equivalently, f induces an isomorphism of X onto a locally closed subscheme of Y.
2. Let X be a scheme. A *locally closed subscheme of X* is a scheme Z equipped with a morphism $i : Z \to X$ that is a locally closed immersion.

Definition 3.2.16. Let X be a scheme (Definition 1.4.7).

- An *open subscheme of X* is a subscheme U whose structure morphism $i : U \to X$ is an open immersion (Definition 3.2.12). The underlying topological space of U is an open subset of X, and its structure sheaf is the restriction of $\mathcal{O}_X$ to U.
- A *closed subscheme of X* is a subscheme Z whose structure morphism $i : Z \to X$ is a closed immersion (Definition 3.2.12). The underlying topological space of Z is a closed subset of X, and the morphism $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective.

Definition 3.2.17. A morphism of schemes (Definition 1.4.8) $f : X \to Y$ is said to be *universally closed* if it is a closed map (Definition 3.2.11) of topological spaces, and for every morphism $Y' \to Y$, the base (Definition 3.2.9) change $f' : X \times_Y Y' \to Y'$ is also a closed map. That is, for any scheme Y' over Y, the image of any closed subset of $X \times_Y Y'$ under the projection to Y' is closed in Y'.

Definition 3.2.18. Let $f : X \to Y$ be a morphism of schemes (Definition 1.4.8). The morphism f is said to be *proper* if it satisfies the following three conditions:

1. f is separated (Definition 3.2.3),
2. f is of finite type (Definition 3.2.7),
3. f is universally closed (Definition 3.2.17).

Definition 3.2.19 (Tensor product of bimodules)**.** Let R, S, T be (not necessarily commutative) rings (Definition A.0.1), let M be an R-S bimodule (Definition A.0.8), and let N be an S-T bimodule. In the free abelian group $\mathbb{Z}[M \times N]$ generated by the Cartesian product $M \times N$, let U be the subgroup generated by elements of the form (♠ TODO: subgroup generated)

$$\begin{aligned}
&(m + m', n) - (m, n) - (m', n),\\
&(m, n + n') - (m, n) - (m, n'),\\
&(m \cdot s, n) - (m, s \cdot n),
\end{aligned}$$

for all $m, m' \in M$, $n, n' \in N$, and $s \in S$. The *tensor product of M and N over S* is the quotient abelian group

$$M \otimes_S N := \mathbb{Z}[M \times N]/U.$$

The image of an element of the form $(m, n) \in M \times N$ in $M \otimes_S N$ is denoted $m \otimes n$ and called a *pure tensor*. In general, the elements of $M \otimes_S N$ are finite sums

$$\sum_{i=1}^{n} m_i \otimes n_i \quad m_i \in M, n_i \in N$$

of pure tensors. Thus, the pure tensors satisfy the following relations:

$$(m + m') \otimes n = m \otimes n + m' \otimes n$$
$$m \otimes (n + n') = m \otimes n + m \otimes n'$$
$$(m \cdot s) \otimes n = m \otimes (s \cdot n)$$

This tensor product becomes naturally an R-T bimodule with left action and right action defined by

$$r \cdot (m \otimes n) = (r \cdot m) \otimes n,$$
$$(m \otimes n) \cdot t = m \otimes (n \cdot t),$$

for all $r \in R$, $t \in T$, $m \in M$, and $n \in N$.

Inductively, given rings $R_0, \dots, R_k$ and $R_{i-1} - R_i$-bimodules M_i for $i = 1, \dots, k$, we may speak of the tensor product

$$M_0 \otimes_{R_1} M_1 \otimes_{R_2} \cdots \otimes_{R_{k-1}} M_k;$$

tensor products are associative(♠ TODO:), so parentheses are not strictly needed to notate them. Its *pure tensors* are elements of the form $m_0 \otimes m_1 \otimes \cdots \otimes m_k$ for $m_i \in M_i$, and its general elements are finite sums

$$\sum_{j=1}^{n} m_{0j} \otimes m_{1j} \otimes \cdots m_{kj} \quad m_{ij} \in M_i.$$

of pure tensors. It also has a natural $R_0 - R_k$-bimodule structure.

Given a ring R and a two-sided R-module M, we may also speak of the *n-fold tensor product $M^{\otimes n} = M^{\otimes_R n}$*

Definition 3.2.20 (Flat module over a ring)**.** Let R be a (not necessarily commutative) ring (Definition A.0.1).

1. Let M be a left R-module. The module M is said to be *flat (with respect to the left R-module structure)* if the functor

   $$- \otimes_R M : \mathrm{Mod}_R \to \mathbf{Ab}$$

   (Definition 3.2.19) from the category of right R-modules to abelian groups is exact; that is, for every exact sequence of right R-modules

   $$0 \to N' \to N \to N'' \to 0,$$

   the induced sequence

   $$0 \to N' \otimes_R M \to N \otimes_R M \to N'' \otimes_R M \to 0$$

   is exact.
   (♠ TODO: tor) Equivalently, M is flat if $\mathrm{Tor}_1^R(-, M) = 0$.
2. Let M be a right R-module. The module M is said to be *flat (with respect to the right R-module structure)* if the functor

   $$M \otimes_R - : {}_R\mathrm{Mod} \to \mathbf{Ab}$$

from the category of left R-modules to abelian groups is exact; that is, for every exact sequence of right R-modules

$$0 \to N' \to N \to N'' \to 0,$$

the induced sequence

$$0 \to M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0$$

is exact.

Definition 3.2.21 (Faithfully flat module)**.** Let R be a (not necessarily commutative) ring (Definition 3.2.22), and let M be a left/right R-module (Definition A.0.8).

We say that M is a *faithfully flat left/right R-module* if:

1. M is flat (Definition 3.2.20) as a left/right R-module, i.e. the appropriate functor $- \otimes_R M : \mathbf{Mod}_R \to \mathrm{Ab}$ (Definition 3.2.19) or $M \otimes_R - : {}_R\mathbf{Mod} \to \mathrm{Ab}$ is exact.
2. The functor $- \otimes_R M : \mathbf{Mod}_R \to \mathrm{Ab}$ or $M \otimes_R - : {}_R\mathbf{Mod} \to \mathrm{Ab}$ is faithful. Since the flatness of M means that the tensor functors are exact, the faithfulness of these functors is equivalent to saying that for any left/right R-module N, if $M \otimes_R N = 0$ or $N \otimes_R M = 0$ as appropriate, then $n = 0$.

Equivalently, the functor $M \otimes_R -$ or $- \otimes_R M$ as appropriate reflects exactness and injectivity (Definition E.0.1), making M a generator in the category of left/right R-modules.

Definition 3.2.22 (Flat ring homomorphism)**.** (♠ TODO: flat module) Let R and S be (not necessarily commutative) rings (Definition A.0.1), and let $\varphi : R \to S$ be a ring homomorphism (Definition A.0.3).

1. We may say that S is *flat as a left module* or that φ is *flat (as a left module)* if $- \otimes_R S : \mathrm{Mod}_R \to \mathrm{Mod}_S$ (Definition 3.2.19) is exact.
2. We may say that S is *flat as a right module* or that φ is *flat (as a right module)* if $S \otimes_R - : {}_R\mathrm{Mod}_R \to {}_S\mathrm{Mod}$ (Definition 3.2.19) is exact.

These two notions of flatness are different in general. However, if φ equips S with the structure of an R-algebra (Definition A.0.4), then these two notions of flatness coincide. In this case, we may speak of the notion of *flatness of R-algebras* and say that S is *flat over R*.

If S is faithfully flat (Definition 3.2.21) as a left/right R-module, then we may say that φ is *faithfully flat (as a left/right R-module)*.

Definition 3.2.23 (Flat morphism of schemes)**.** (♠ TODO: flat module) Let $f : X \to Y$ be a morphism of schemes (Definition 1.4.8).

1. Let $x \in X$ be a point and let $y = f(x)$. We say that f is *flat at x* if the induced ring homomorphism (Definition 1.4.5) on local rings
$$\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$$
makes $\mathcal{O}_{X,x}$ into a flat $\mathcal{O}_{Y,y}$-module.
2. We say f is *flat* if it is flat at every point $x \in X$.
3. f is *faithfully flat* if it is flat and surjective (Definition E.0.1).

Proposition 3.2.24. Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:

1. f is faithfully flat (Definition 3.2.23), i.e. flat and surjective
   (♠ TODO: pullback respects quasi-coherence)
2. functor $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ on quasi-coherent sheaves (Definition 4.0.8) is both exact and faithful.
3. For every quasi-coherent $\mathcal{O}_Y$-module $\mathcal{M}$, if $f^*(\mathcal{M}) = 0$, then $\mathcal{M} = 0$.
4. Given any affine open (Definition 1.4.9) $\operatorname{Spec} A \subseteq Y$ (Definition 1.4.3) and affine open $\operatorname{Spec} B \subseteq f^{-1}(\operatorname{Spec} A)$ (Definition 1.5.3), the ring homomorphism $A \to B$ induced by $f$is a faithfully flat (Definition 3.2.22) ring homomorphism.

Definition 3.2.25 (Faithfully flat morphism of schemes)**.** Let $f : X \to Y$ be a morphism of schemes.

The morphism f is *faithfully flat* if it is *flat* and *surjective* on the underlying topological spaces.

More precisely:

- f is *flat*, meaning for every $x \in X$ with $y = f(x)$, the local ring homomorphism
  $$\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$$
  makes $\mathcal{O}_{X,x}$ a flat $\mathcal{O}_{Y,y}$-module.
- f is *surjective* at the level of topological spaces, i.e., the continuous map $f : |X| \to |Y|$ is surjective.

Equivalently, f is faithfully flat if the functor $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ on quasi-coherent sheaves is both exact and faithful.

Definition 3.2.26 (Weakly unramified morphism of schemes)**.** Let $f : X \to Y$ be a morphism of schemes (Definition 1.4.8).

The morphism f is *weakly unramified* if the diagonal morphism (Definition 3.2.2)
$$\Delta_f : X \to X \times_Y X$$
is *locally quasi-finite*; that is, for every point $x \in X$, there exists an open neighborhood $U \subseteq X$ of x such that $\Delta_f|_U : U \to U \times_Y U$ is quasi-finite.

In particular, this condition is weaker than requiring Δ_f to be an immersion or a monomorphism, but still controls the ramification behavior of f via the diagonal.

Definition 3.2.27 (Unramified morphism of schemes)**.** (♠ TODO: sheaf of relative differentials) A morphism of schemes (Definition 1.4.8) $f : X \to Y$ is *unramified* if it is locally of finite type (Definition 3.2.7) and the sheaf of relative differentials $\Omega_{X/Y}$ is zero. Equivalently:

- For every $x \in X$, the induced ring map on stalks $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is of finite type (Definition A.0.5),
- and the module of Kähler differentials $\Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}}$ is 0.

Definition 3.2.28. Let $f : X \to Y$ be a morphism of schemes (Definition 1.4.8). We say that f is an *affine morphism* if for every affine open (Definition 1.4.9) $V = \operatorname{Spec} B \subseteq Y$, the preimage $U = f^{-1}(V)$ (Definition 1.5.3) is an affine scheme (Definition 1.4.3).

Definition 3.2.29. Let $B \to A$ be a B-algebra (Definition A.0.4). We say that A is a *finite B-algebra* if A is finitely generated (Definition A.0.5) as a B-module.

Definition 3.2.30. Let $f : X \to Y$ be an affine morphism of schemes (Definition 3.2.28). We say that f is a *finite morphism* if for every affine open (Definition 1.4.9) $V = \operatorname{Spec} B \subseteq Y$ with $U = f^{-1}(V) = \operatorname{Spec} A$, the ring A is a finite B-algebra (Definition 3.2.29).

Definition 3.2.31 (Quasifinite morphism)**.** Let $f \colon X \to Y$ be a morphism of schemes (Definition 1.4.8). We say that f is *quasifinite* if f is of finite type (Definition 3.2.7) and has finite fibers, meaning that for every point $y \in Y$, the fiber $X_y = X \times_Y \operatorname{Spec} \kappa(y)$ (Definition 1.5.7) (Definition 1.4.3) is a finite scheme (i.e., X_y is a scheme with finitely many points and $\Gamma(X_y, \mathcal{O}_{X_y})$ is a finite-dimensional (Definition E.0.34) $\kappa(y)$-algebra (Definition 4.2.1)).

Theorem 3.2.32. Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:

1. f is finite (Definition 3.2.30).
2. f is affine (Definition 3.2.28) and proper (Definition 3.2.18).
3. f is quasifinite (Definition 3.2.31) and proper (Definition 3.2.18).

Definition 3.2.33 (Smooth Morphism of Schemes)**.** Let $f : X \to S$ be a morphism of schemes (Definition 1.4.7).

We say that f is *smooth*, and that X is a *smooth scheme over S*, if it satisfies the following conditions:

(♠ TODO: residue field)

- f is locally of finite presentation (Definition 3.2.6): for every point $x \in X$, there exists an open neighborhood $U \subseteq X$ of x and an open neighborhood $V \subseteq S$ of $f(x)$ such that the restriction $f|_U : U \to V$ corresponds to a morphism of rings $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ that is finitely presented.
- f is flat (Definition 3.2.23): the induced map on local rings is flat.
- For every point $x \in X$, the fiber $X_{f(x)} = X \times_S \operatorname{Spec} \kappa(f(x))$ is a smooth variety over the residue field $\kappa(f(x))$, equivalently, the sheaf of relative Kähler differentials $\Omega_{X/S}$ is locally free of finite rank.

Informally, a smooth morphism behaves like a submersion in differential geometry, providing "nice" fiber structures and descent properties.

Given a scheme S, the *category of smooth schemes over S* is the following locally small (Definition C.0.2) category (Definition C.0.1):

- The objects are smooth morphisms $X \to S$.
- The morphisms betewen objects $X_1 \to S$ and $X_2 \to S$ are S-morphisms (Definition 1.4.10) $X_1 \to X_2$ such that the following commutes:

$$\begin{tikzcd} X_1 \arrow[rr] \arrow[dr] & & X_2 \arrow[dl] \\ & S & \end{tikzcd}$$

The category of smooth schemes over S is often denoted by notations such as Sm/S, $\mathbf{Sm}/S$, Sm_S, $\mathbf{Sm}_S$ etc.

Definition 3.2.34. (♠ TODO: integral, noeterhain scheme, regular, proper, dominant, function fields, generically finite) Let X be an integral noetherian scheme. An *alteration of X* is a morphism of schemes (Definition 1.4.8) $f\colon X' \to X$ satisfying the following conditions:

- The scheme X' is integral and regular.
- The morphism f is proper and dominant.
- The extension of function fields $[k(X') : k(X)]$ induced by f is finite; i.e., f is generically finite.

Definition 3.2.35. A morphism of schemes (Definition 1.4.8) $f : X \to Y$ is called *étale* if it satisfies the following conditions: (♠ TODO: sheaf of relative differentials)

- f is locally of finite presentation (Definition 3.2.6),
- f is flat (Definition 3.2.23),
- f is unramified (Definition 3.2.27), i.e., the sheaf of relative differentials $\Omega_{X/Y}$ equals 0.

(♠ TODO: relative dimension) Equivalently, a morphism of schemes is étale if and only if it is smooth (Definition 3.2.33) of relative dimension 0. A finite (Definition 3.2.30) étale morphism is synonymously called a *finite étale cover*.

Definition 3.2.36. Let X be a scheme (Definition 1.4.7) and let A be an quasi-coherent sheaf (Definition 4.0.8) of $\mathcal{O}_X$ (Definition 4.2.1)-algebras (equivalently, a quasi-coherent sheaf of $\mathcal{O}_X$-algebras). We say that A is an *étale X-algebra* if the associated morphism $f\colon \underline{\mathrm{Spec}} A \to X$ from the relative spectrum of A (Definition 4.2.2) is an étale morphism of schemes (Definition 3.2.35).

Definition 3.2.37. Let R be a commutative ring (Definition A.0.7). An R-algebra B is called an *étale R-algebra* if B is flat over (Definition 3.2.22) R and the module of Kähler differentials satisfies $\Omega^1_{B/R} = 0$. (♠ TODO: module of kahler differentials) Equivalently, B is an étale R-algebra if and only if the corresponding morphism of affine schemes (Definition 1.4.3) $\mathrm{Spec}\, B \to \mathrm{Spec}\, R$ is étale (Definition 3.2.35). (♠ TODO: morphism of Spec's)

Definition 3.2.38. Let R be a commutative ring (Definition A.0.7). An R-algebra B is called *ind-étale over R* if B is a filtered colimit (Definition C.1.4) of étale R-algebras (Definition 3.2.37), i.e. $B = \varinjlim_{i\in I} B_i$ where $\{B_i\}_{i\in I}$ is a filtered system (Definition C.1.3) of étale R-algebras.

Definition 3.2.39. (♠ TODO: (affine) open cover) Let X be a scheme. A morphism of schemes $f\colon Y \to X$ is called *ind-étale* if for every affine open (Definition 1.4.9) $U = \mathrm{Spec}\, R \subseteq X$, the inverse image (Definition 1.5.3) $f^{-1}(U)$ admits an affine open cover $\{V_j = \mathrm{Spec}\, B_j\}$

such that each R-algebra B_j is ind-étale over (Definition 3.2.38) R. Equivalently, f is locally of the form $\underline{\mathrm{Spec}}(\mathcal{A}) \to X$ (Definition 4.2.2) where $\mathcal{A}$ is a quasi-coherent (Definition 4.0.8) $\mathcal{O}_X$-algebra (Definition 4.2.1) that is locally (on affine opens of X) a filtered colimit (Definition C.1.4) of étale $\mathcal{O}_X$-algebras (Definition 3.2.36).

Definition 3.2.40. Let X be a scheme.

1. A morphism of schemes $f : Y \to X$ is called a *profinite-étale cover* (or *pro-finite cover*) if Y can be expressed as an inverse limit (Definition C.1.4)

$$Y = \varprojlim_{i \in I} Y_i$$

where $\{Y_i \to X\}_{i \in I}$ is an inverse system of finite étale (Definition 3.2.35) morphisms indexed by a directed set (Definition C.1.2) I, with transition maps $Y_j \to Y_i$ for $j \geq i$, which are necessarily affine (♠ TODO: show why). Note by Theorem 3.3.1 that inverse limits of such inverse systems exist.
2. The *universal profinite-étale/pro-finite cover of X* is the inverse limit of the inverse system $\{Y_i \to X\}_{i \in I}$ of all finite étale morphisms to X.

Definition 3.2.41 (Weakly étale morphism of schemes, [BS15c, Definition 2.2])**.** A morphism $f : X \to Y$ of schemes (Definition 1.4.8) is *weakly étale* if it and its diagonal (Definition 3.2.2) are both flat (Definition 3.2.23).

3.3. **Inverse limits of schemes by affine morphisms.**

Theorem 3.3.1 (Category of schemes admits projective limits)**.** Let S be a scheme (Definition 1.4.7). The category of S-schemes (Definition 1.4.10) admits all cofiltered limits (Definition C.1.4) whose transition morphisms are affine (Definition 3.2.28). More precisely, given any cofiltered system (Definition C.1.3) $\{X_i\}_{i \in I}$ of S-schemes such that $X_i \to X_j$ is an affine morphism for any arrow $i \in j$ in I, the inverse limit (Definition C.1.4) $\varprojlim_{i \in I} X_i$ exists.

Example 3.3.2. Here are some examples of inverse limits of schemes under affine morphisms:

1. The universal profinite cover (Definition 3.2.40) of a scheme.
2. The infinite dimensional affine space (Definition 3.3.3) $\mathbb{A}^\infty_S$ over a base scheme S can be regarded as the inverse limit of the inverse system

$$\cdots \to \mathbb{A}^2_S \to \mathbb{A}^1_S$$

where the corresponding morphisms $\mathcal{O}_S[x_1, \ldots, x_n] \to \mathcal{O}_S[x_1, \ldots, x_n, x_{n+1}]$ are given by the natural inclusions.
3. (♠ TODO: Perfectoid spaces)

Definition 3.3.3 (Infinite dimensional affine space)**.** Let S be a scheme (Definition 1.4.3) and let I be an arbitrary index set. The *affine space over S indexed by I*, denoted $\mathbb{A}^I_S$ or $\mathbb{A}^{(I)}_S$, is defined as the scheme

$$\mathbb{A}^I_S = \underline{\mathrm{Spec}}\mathcal{O}_S[x_i \mid i \in I] = S \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Z}[x_i \mid i \in I],$$

(Definition 4.2.2) where $\mathcal{O}_S[x_i \mid i \in I]$ denotes the symmetric algebra of the free $\mathcal{O}_S$-module with basis indexed by I. Equivalently, $\mathbb{A}^I_S$ represents the functor that assigns to each S-scheme T the set of families $(f_i)_{i \in I}$ of global sections $f_i \in \Gamma(T, \mathcal{O}_T)$. When $I = \mathbb{N}$ or I is countably infinite, we may write $\mathbb{A}^\infty_S$ for the infinite dimensional affine space over S.

3.4. **Geometric properties.**

Definition 3.4.1. Let $f : X \to Y$ be a morphism of schemes (Definition 1.4.8). For any point $y \in Y$, we define the *fiber of f at y*, denoted X_y, as the scheme-theoretic fiber product (Definition 3.2.8):

$$X_y := X \times_Y \operatorname{Spec}(\kappa(y))$$

where $\kappa(y)$ is the residue field (Definition 1.5.7) of Y at y. Explicitly, the underlying topological space of X_y is the subspace $f^{-1}(y) \subset X$ (Definition E.0.35) equipped with the induced topology.

Definition 3.4.2. Let S be a scheme. A *geometric point of S* is a morphism $\bar{s} : \operatorname{Spec}(\Omega) \to S$ where Ω is an algebraically closed field.

Definition 3.4.3. Let $f : X \to Y$ be a morphism of schemes. For any geometric point (Definition 3.4.2) $\bar{y} : \operatorname{Spec}(\Omega) \to Y$, the *geometric fiber of f at $\bar{y}$*, denoted $X_{\bar{y}}$, is the base change (Definition 3.2.9):

$$X_{\bar{y}} := X \times_Y \operatorname{Spec}(\Omega)$$

Note that this depends on the choice of the geometric point $\bar{y}$, not just on the image point $y \in Y$.

Definition 3.4.4. Let k be a field. A *geometric property of a k-scheme* X (or a *property "holding geometrically"*) is a property that holds for the base change $X_{\bar{k}} := X \times_k \operatorname{Spec}(\bar{k})$, where $\bar{k}$ is an algebraic closure (Definition E.0.37) of k. In particular, we may speak of the following properties of X:

1. geometrically connected (Definition 3.4.6)
2. geometrically irreducible
3. geometrically reduced (Definition 3.1.4)
4. geometrically integral (Definition 3.1.6) (♠ TODO: clarify that this is equivalent to being both geometrically reduced and geometrically irreducible)
   (♠ TODO: go thorugh the below)
5. geometrically normal
6. geometrically regular (♠ TODO: Note: for schemes of finite type over a perfect field, this is equivalent to being regular)
7. geometrically unibranch
8. geometrically Cohen-Macaulay
9. geometrically locally factorial (or geometrically parafactorial)
10. geometrically satisfying Serre's condition S_n
11. geometrically satisfying Serre's condition R_n

Definition 3.4.5. Let $f : X \to Y$ be a morphism of schemes (Definition 1.4.8). We say that *f has geometrically connected fibers* if for every geometric point (Definition 3.4.2) $\bar{y}$ of Y, the geometric fiber (Definition 3.4.3) $X_{\bar{y}}$ is a connected topological space (Definition E.0.33).

Definition 3.4.6. If X is a scheme over (Definition 1.4.10) a field k, we say X is *geometrically connected over k* if the structure morphism $X \to \operatorname{Spec}(k)$ has geometrically connected fibers; equivalently, if $X \times_k \operatorname{Spec}(\bar{k})$ is connected.

4. Quasi-coherent and Coherent sheaves

Definition 4.0.1. 1. Let $\mathcal{C}$ be a site (Definition D.0.2), and let $\mathcal{A}$ and $\mathcal{B}$ be sheaves (Definition D.0.3) of (not necessarily commutative) rings (Definition A.0.1) on $\mathcal{C}$.

(a) An *$(\mathcal{A},\mathcal{B})$-bimodule* (or a *bimodule over $(\mathcal{A},\mathcal{B})$*) is a sheaf (Definition D.0.3) $\mathcal{M}$ of abelian groups on $\mathcal{C}$ equipped with a left $\mathcal{A}$-module structure given by a morphism of sheaves (Definition D.0.3) of sets

$$\lambda : \mathcal{A} \times \mathcal{M} \longrightarrow \mathcal{M},$$

and a right $\mathcal{B}$-module structure given by a morphism of sheaves of sets

$$\rho : \mathcal{M} \times \mathcal{B} \longrightarrow \mathcal{M},$$

such that the actions are compatible. Specifically, for every object U in $\mathcal{C}$, every section $m \in \mathcal{M}(U)$, every $a \in \mathcal{A}(U)$, and every $b \in \mathcal{B}(U)$, the equality

$$\lambda_U(a, \rho_U(m,b)) = \rho_U(\lambda_U(a,m), b)$$

holds in $\mathcal{M}(U)$. In standard multiplicative notation where $\lambda(a,m)$ is denoted $a \cdot m$ and $\rho(m,b)$ is denoted $m \cdot b$, this condition is the associativity axiom

$$(a \cdot m) \cdot b = a \cdot (m \cdot b).$$

In particular, for every object $U \in \mathcal{C}$, the abelian group $\mathcal{M}(U)$ has the structure of an $\mathcal{A}(U) - \mathcal{B}(U)$-bimodule (Definition A.0.8).

(b) Let $\mathcal{M}$ and $\mathcal{N}$ be $(\mathcal{A},\mathcal{B})$-bimodules. A *homomorphism of $(\mathcal{A},\mathcal{B})$-bimodules* (or an *$(\mathcal{A},\mathcal{B})$-linear morphism*) is a morphism of sheaves of abelian groups $f : \mathcal{M} \to \mathcal{N}$ such that for every object U of $\mathcal{C}$, every section $m \in \mathcal{M}(U)$, every $a \in \mathcal{A}(U)$, and every $b \in \mathcal{B}(U)$, the following compatibility conditions hold:

$$f_U(a \cdot m) = a \cdot f_U(m) \quad \text{and} \quad f_U(m \cdot b) = f_U(m) \cdot b.$$

We denote the category of $(\mathcal{A},\mathcal{B})$-bimodules, with morphisms being morphisms of sheaves of abelian groups compatible with both the left $\mathcal{A}$-action and the right $\mathcal{B}$-action, by $\mathcal{A}\text{-}\mathcal{B}\text{-}\mathsf{Mod}$ or sometimes by ${}_{\mathcal{A}}\mathsf{Mod}_{\mathcal{B}}$ (♠ TODO: talk about how bimodules can be identifies with left/right modules)

2. Let $(\mathcal{C}, J)$ be a site (Definition D.0.2). Let $\mathcal{O}$ be a sheaf of (not necessarily commutative) rings on $(\mathcal{C}, J)$ (Definition D.0.3), i.e. $((\mathcal{C}, J), \mathcal{O})$ is a ringed site (Definition 4.0.2).

(a) An *(left/right/two-sided) $\mathcal{O}$-module* consists of the following data:
- A sheaf $\mathcal{F}$ of abelian groups on $(\mathcal{C}, J)$,
- for every object $U \in \mathcal{C}$, the structure of an (left/right/two-sided) $\mathcal{O}(U)$-module on $\mathcal{F}(U)$,

such that for every morphism $f : V \to U$ in $\mathcal{C}$, the restriction map

$$\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$$

is $\mathcal{O}(U)$-linear when the $\mathcal{O}(U)$-action on $\mathcal{F}(V)$ is defined via the natural ring homomorphism

$$\mathcal{O}(U) \to \mathcal{O}(V)$$

induced by f.

(b) Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}$-modules (Definition 4.0.1).
A *morphism of $\mathcal{O}$-modules* $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves (Definition D.0.3) of abelian groups such that, for every object $U \in \mathcal{C}$, the component map

$$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

is $\mathcal{O}(U)$-linear, i.e. it satisfies

$$\varphi_U(r \cdot s) = r \cdot \varphi_U(s) \quad \text{for all } r \in \mathcal{O}(U),\ s \in \mathcal{F}(U).$$

The collection of all $\mathcal{O}$-modules together with their morphisms of $\mathcal{O}$-modules forms the *category of $\mathcal{O}$-modules*, denoted $\mathbf{Mod}(\mathcal{O})$.
See also Definition 4.2.1.

In case that a sheafification functor (Definition D.0.4)

$$\mathrm{PreShv}(\mathcal{C}, \mathbf{Rings}) \to \mathrm{Shv}(\mathcal{C}, \mathbf{Rings})$$

exists, a left, right, two-sided $\mathcal{O}$-module (and morphisms thereof) is equivalent to a $(\mathcal{O}, \mathbb{Z})$-bimodule, $(\mathbb{Z}, \mathcal{O})$-bimodule, and $(\mathcal{O}, \mathcal{O})$-bimodule (and morphisms thereof) respectively, where $\mathbb{Z}$ is the constant sheaf of the integer ring $\mathbb{Z}$.

Definition 4.0.2. (♠ TODO: there are places where sites and sheaves of rings on them are used, but it would be better to just have them be ringed sites.)

A *ringed site* is a site (Definition D.0.2) $(\mathcal{C}, J)$ with a small topological generating family (Definition D.0.2) equipped with a sheaf (Definition D.0.3) of (not necessarily commutative) rings $\mathcal{O}$. If the Grothendieck topology J is clear in context, one may even write that $(C, \mathcal{O})$ is a ringed site.

A *morphism of ringed sites*

$$((\mathcal{C}, J), \mathcal{O}) \to ((\mathcal{C}', J'), \mathcal{O}')$$

consists of a morphism of sites $f : (\mathcal{C}, J) \to (\mathcal{C}', J')$ and a morphism of sheaves (Definition D.0.3) of rings $f^{\#} : \mathcal{O}' \to f_*\mathcal{O}$.

Definition 4.0.3. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site (Definition 4.0.2). An $\mathcal{O}$-module sheaf (Definition 4.0.1) $\mathcal{F}$ is *quasi-coherent* if there exists a covering $\{U_i\}$ in $\mathcal{C}$ such that for each i, there is an exact sequence (Definition E.0.2) of $\mathcal{O}|_{U_i}$-modules

$$\mathcal{O}_{U_i}^{(I_i)} \to \mathcal{O}_{U_i}^{(J_i)} \to \mathcal{F}|_{U_i} \to 0,$$

with I_i, J_i index sets.

Definition 4.0.4. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site (Definition 4.0.2). An $\mathcal{O}$-module sheaf (Definition 4.0.1) $\mathcal{F}$ is *coherent* if it is quasi-coherent (Definition 4.0.3) and one can choose the index sets I_i, J_i to be finite for each i. Equivalently, $\mathcal{F}$ is locally finitely presented as an $\mathcal{O}$-module sheaf. (♠ TODO: locally finitely presented as a module sheaf)

Definition 4.0.5. Let $(X, \mathcal{O}_X)$ be a ringed space (Definition 1.4.1). A sheaf of $\mathcal{O}_X$-modules (Definition 4.0.1) $\mathcal{F}$ is *quasi-coherent* if for every point $x \in X$, there exists an open neighborhood $U \subseteq X$ of x and an exact sequence of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^{(I)} \to \mathcal{O}_U^{(J)} \to \mathcal{F}|_U \to 0,$$

where I and J are index sets (possibly infinite), and $\mathcal{O}_U^{(I)}, \mathcal{O}_U^{(J)}$ denote direct sums of copies of the structure sheaf.

Equivalently, a quasi-coherent sheaf on the ringed space $(X, \mathcal{O}_X)$ is a quasi-coherent sheaf (Definition 4.0.3) on $(X, \mathcal{O}_X)$ as a (Definition 1.4.1) ringed site (Definition 4.0.2).

Definition 4.0.6. Let $(X, \mathcal{O}_X)$ be a ringed space (Definition 1.4.1). An $\mathcal{O}_X$-module sheaf $\mathcal{F}$ is *coherent* if:

1. $\mathcal{F}$ is of finite type, i.e., locally there exists a surjection $\mathcal{O}_U^n \twoheadrightarrow \mathcal{F}|_U$ for some finite n, and
2. the kernel of any morphism $\mathcal{O}_U^m \to \mathcal{F}|_U$ is also of finite type, i.e., $\mathcal{F}$ is locally finitely presented.

Equivalently, $\mathcal{F}$ is locally the cokernel of a morphism between finite free $\mathcal{O}_U$-modules.

Equivalently, a coherent sheaf on the ringed space $(X, \mathcal{O}_X)$ is a coherent sheaf (Definition 4.0.4) on $(X, \mathcal{O}_X)$ as a (Definition 1.4.1) ringed site (Definition 4.0.2).

Definition 4.0.7. Let A be a commutative ring (Definition A.0.7) and $X = \operatorname{Spec} A$ its spectrum (Definition 1.4.3) equipped with the structure sheaf (Definition 1.4.3) $\mathcal{O}_X$.

1. For any A-module (Definition A.0.8) M, we define the corresponding sheaf (Definition 1.1.5) $\widetilde{M}$ on X, called the *sheaf associated to M*, by

   $$\widetilde{M}(D(f)) = M_f \text{ for all } f \in A,$$

   where $D(f) = \{\mathfrak{p} \in \operatorname{Spec} A : f \notin \mathfrak{p}\}$ (Definition 1.4.3) and M_f is the localization of M at f. (♠ TODO: localization of a module) It is a sheaf of $\mathcal{O}_X$-modules (Definition 4.0.1).
2. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called *quasi-coherent* if there exists an A-module M such that $\mathcal{F} \cong \widetilde{M}$.

Equivalently, a quasi-coherent sheaf on the affine scheme (Definition 1.4.3) $\operatorname{Spec} A$ is a quasi-coherent sheaf (Definition 4.0.5) on $(X, \mathcal{O}_X)$ as a (Definition 1.4.7) ringed space (Definition 1.4.1).

Definition 4.0.8. Let X be a scheme (Definition 1.4.7) and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules (Definition 4.0.1). The sheaf $\mathcal{F}$ is called *quasi-coherent* if for every open affine subset $U = \operatorname{Spec} A$ of X, there exists an A-module M_U such that the restriction $\mathcal{F}|_U$ is isomorphic to the sheaf $\widetilde{M_U}$ (Definition 4.0.7) associated to M_U.

For a scheme X, we denote by

$$\mathrm{QCoh}(X)$$

the full subcategory of $\mathrm{Mod}(\mathcal{O}_X)$ consisting of all quasi-coherent $\mathcal{O}_X$-modules.

Equivalently, a quasi-coherent sheaf on the scheme (Definition 1.4.7) $\operatorname{Spec} A$ is a quasi-coherent sheaf (Definition 4.0.5) on $(X, \mathcal{O}_X)$ as a (Definition 1.4.7) ringed space (Definition 1.4.1).

Definition 4.0.9. (♠ TODO: finitely generated module) (♠ TODO: locally of finite type sheaf of OX modules) Let X be a scheme. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules (Definition 4.0.1) is called *coherent* if it satisfies the following two conditions:

1. $\mathcal{F}$ is quasi-coherent (Definition 4.0.8).
2. For every affine open subset (Definition 1.4.9) $U = \operatorname{Spec} A$ of X, there exists a finitely generated A-module M_U such that $\mathcal{F}|_U \cong \widetilde{M_U}$.

Equivalently, $\mathcal{F}$ is locally of finite type and for every open subset $U \subseteq X$, the kernel of any morphism $\mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$ is of finite type.

Equivalently, a coherent sheaf on the scheme (Definition 1.4.7) $(X, \mathcal{O}_X)$ is a coherent sheaf (Definition 4.0.6) on $(X, \mathcal{O}_X)$ as a (Definition 1.4.7) ringed space (Definition 1.4.1).

For a scheme X, denote by

$$\mathrm{Coh}(X)$$

the full subcategory of $\mathrm{QCoh}(X)$ consisting of all coherent $\mathcal{O}_X$-modules.

(♠ TODO: Take the general definition of qc sheaves on a general scheme)

4.1. **Vector bundles on schemes.**

Definition 4.1.1. Let $((\mathcal{C}, J), \mathcal{O})$ be a ringed site (Definition 4.0.2), where $\mathcal{O}$ is a sheaf (Definition D.0.3) of rings on the site $(\mathcal{C}, J)$.

1. Let I be an indexing set. The *free sheaf of $\mathcal{O}$-modules of rank I* (or simply a *free sheaf*), denoted by $\mathcal{O}^{\oplus I}$, is the sheaf associated to the presheaf $U \mapsto \mathcal{O}(U)^{\oplus I}$. If I is finite with cardinality n, we usually write $\mathcal{O}^{\oplus n}$.
2. An $\mathcal{O}$-module (Definition 4.0.1) $\mathcal{F}$ is called *locally free of rank n* (for an integer $n \geq 0$) if there exists a covering $\{U_i \to X\}_{i \in I}$ in the topology (Definition D.0.2) J (where X is the terminal object of $\mathcal{C}$ if it exists, or more generally this condition holds restricted to each object of $\mathcal{C}$ for a suitable cover) such that for each i, the restriction $\mathcal{F}|_{U_i}$ is isomorphic to the free sheaf $(\mathcal{O}|_{U_i})^{\oplus n}$ as an $\mathcal{O}|_{U_i}$-module.

We might call a locally free $\mathcal{O}$-module of rank n an *algebraic vector bundle of rank n*. A *(algebraic) line bundle* or *invertible sheaf* or is then an algebraic vector bundle of rank 1.

Definition 4.1.2. Let X be a scheme. An *algebraic vector bundle* (or simply a *vector bundle*) of rank n on X is a locally free sheaf (Definition 4.1.1) of $\mathcal{O}_X$-modules (Definition 4.0.1) $\mathcal{E}$ of constant rank n. That is, $\mathcal{E}$ is a coherent sheaf (Definition 4.0.9) of $\mathcal{O}_X$-modules such that X can be covered by open sets U_i where the restriction $\mathcal{E}|_{U_i}$ is isomorphic to the free sheaf $\mathcal{O}_{U_i}^{\oplus n}$ for each i.

A *(algebraic) line bundle* or *invertible sheaf* or is then an algebraic vector bundle of rank 1.

4.2. **Relative Spec and Proj.**

Definition 4.2.1. Let $(\mathcal{C}, J)$ be a site (Definition D.0.2). Let $\mathcal{O}$ be a sheaf of commutative rings on $(\mathcal{C}, J)$ (Definition D.0.3), i.e., $((\mathcal{C}, J), \mathcal{O})$ is a ringed site (Definition 4.0.2).

1. An *$\mathcal{O}$-algebra* consists of the following data:
   - A sheaf $\mathcal{A}$ of (not necessarily commutative) rings on $(\mathcal{C}, J)$,
   - A morphism of sheaves of rings $\eta : \mathcal{O} \to \mathcal{A}$ such that for every object $U \in \mathcal{C}$, the image of $\eta_U : \mathcal{O}(U) \to \mathcal{A}(U)$ is contained in the center of $\mathcal{A}(U)$.

   This makes $\mathcal{A}(U)$ an $\mathcal{O}(U)$-algebra for every $U \in \mathcal{C}$, such that for every morphism $f : V \to U$ in $\mathcal{C}$, the restriction map
   $$\rho_{U,V} : \mathcal{A}(U) \to \mathcal{A}(V)$$
   is a homomorphism of $\mathcal{O}(U)$-algebras (where the $\mathcal{O}(U)$-algebra structure on $\mathcal{A}(V)$ is induced via restriction $\mathcal{O}(U) \to \mathcal{O}(V)$).
2. Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{O}$-algebras (Definition 4.2.1).
   A *morphism of $\mathcal{O}$-algebras* $\varphi : \mathcal{A} \to \mathcal{B}$ is a morphism of sheaves (Definition D.0.3) of rings such that, for every object $U \in \mathcal{C}$, the component map
   $$\varphi_U : \mathcal{A}(U) \to \mathcal{B}(U)$$
   is a homomorphism of $\mathcal{O}(U)$-algebras, i.e., it is a ring homomorphism that commutes with the structure maps $\eta_{\mathcal{A}}$ and $\eta_{\mathcal{B}}$:
   $$\varphi_U(\eta_{\mathcal{A},U}(r)) = \eta_{\mathcal{B},U}(r) \quad \text{for all } r \in \mathcal{O}(U).$$

The collection of all $\mathcal{O}$-algebras together with their morphisms forms the *category of $\mathcal{O}$-algebras*, denoted by notations such as $\mathbf{Alg}(\mathcal{O})$.

Definition 4.2.2. Let S be a scheme (Definition 1.4.7) and let $\mathcal{A}$ be a quasi-coherent sheaf (Definition 4.0.8) of $\mathcal{O}_S$-algebras (Definition 4.2.1). The *relative spectrum of $\mathcal{A}$*, denoted by notations such as $\mathbf{Spec}(\mathcal{A})$, $\mathcal{S}pec(\mathcal{A})$, or $\underline{\mathrm{Spec}}(\mathcal{A})$, is the unique S-scheme $f : X \to S$ such that for every open subset $U \subseteq S$, there is a natural isomorphism
$$\mathrm{Hom}_{\mathrm{Sch}/U}(T, f^{-1}(U)) \cong \mathrm{Hom}_{\mathcal{O}_S(U)\text{-alg}}(\mathcal{A}(U), \Gamma(T, \mathcal{O}_T))$$
(Definition 1.5.3) (Definition 1.1.3) for any scheme T over U. Affine locally, if $S = \operatorname{Spec} R$ and $\mathcal{A} = \widetilde{A}$ (Definition 4.0.7), then $\mathbf{Spec}(\mathcal{A}) \cong \operatorname{Spec} A$.

Definition 4.2.3. Let S be a scheme (Definition 1.4.7) and let $\mathcal{S} = \bigoplus_{d \geq 0} \mathcal{S}_d$ be a quasi-coherent (Definition 4.0.8) graded sheaf of $\mathcal{O}_S$-algebras (Definition 4.2.1) such that $\mathcal{S}_0 = \mathcal{O}_S$ and $\mathcal{S}$ is locally generated by $\mathcal{S}_1$ as an $\mathcal{S}_0$-algebra. The *relative Proj of $\mathcal{S}$*, denoted by notations such as $\mathbf{Proj}(\mathcal{S})$, $\mathcal{P}roj(\mathcal{S})$, or $\underline{\mathrm{Proj}}(\mathcal{A})$, is an S-scheme (Definition 1.4.10) $\pi : \mathbf{Proj}(\mathcal{S}) \to S$ constructed as follows:

1. For every affine open (Definition 1.4.9) subset $U = \operatorname{Spec} A \subseteq S$, let $S(U) = \Gamma(U, \mathcal{S})$ be the graded A-algebra of sections over U. We define the scheme X_U to be the projective spectrum of this ring:
   $$X_U = \mathrm{Proj}(S(U)).$$

This comes with a natural structure morphism $\pi_U : X_U \to U$.

2. For any inclusion of affine open subsets $V \subseteq U$, the restriction map $\mathcal{S}(U) \to \mathcal{S}(V)$ induces a canonical isomorphism of schemes $\pi_U^{-1}(V) \cong X_V$.
3. The global scheme $\mathbf{Proj}(\mathcal{S})$ is obtained by gluing the family $\{X_U\}$ for all affine open $U \subseteq S$ along these canonical isomorphisms.

The scheme $\mathbf{Proj}(\mathcal{S})$ comes equipped with a canonical invertible sheaf (Definition 4.1.2) $\mathcal{O}_{\mathbf{Proj}(\mathcal{S})}(1)$, which is defined locally on each $X_U = \mathrm{Proj}(S(U))$ as the sheaf associated to the twisted module $S(U)(1)$.

Definition 4.2.4. Let S be a scheme.

- A *geometric vector bundle of rank n on S* is an S-scheme $V \to S$ isomorphic to $\mathbf{Spec}(\mathrm{Sym}^\bullet(\mathcal{E}^\vee))$, where $\mathcal{E}$ is a locally free sheaf of rank n on S and $\mathcal{E}^\vee$ is its dual.
- A *geometric line bundle on S* is a geometric vector bundle of rank 1. This corresponds to an invertible sheaf $\mathcal{L}$ on S, and the total space is $\mathbf{Spec}(\mathrm{Sym}^\bullet(\mathcal{L}^\vee))$.

(♠ TODO: correspondence between algebraic and geometric vector bundles)

Definition 4.2.5. Let S be a scheme and let $\mathcal{A} = \bigoplus_{d\in\mathbb{Z}} \mathcal{A}_d$ be a quasi-coherent graded sheaf (Definition 4.0.8) of $\mathcal{O}_S$-algebras (Definition 4.0.1). Let $\mathcal{M} = \bigoplus_{d\in\mathbb{Z}} \mathcal{M}_d$ be a graded $\mathcal{A}$-module. For any integer $n \in \mathbb{Z}$, the *n-th shifted graded module of $\mathcal{M}$*, denoted by $\mathcal{M}(n)$, is the graded $\mathcal{A}$-module defined by shifting the grading of $\mathcal{M}$ such that the degree d component is given by:

$$\mathcal{M}(n)_d = \mathcal{M}_{n+d}.$$

The module structure is compatible with the shift: if $a \in \mathcal{A}_k$ and $m \in \mathcal{M}(n)_d = \mathcal{M}_{n+d}$, then the product $a \cdot m$ lies in $\mathcal{M}_{n+d+k} = \mathcal{M}(n)_{d+k}$.

Definition 4.2.6. Let S be a scheme (Definition 1.4.7) and let $\mathcal{A} = \mathcal{O}_S[T_0, \dots, T_m]$ be the sheaf of polynomial algebras defining the projective space (Definition 2.2.9) $\mathbb{P}^m_S = \mathbf{Proj}(\mathcal{A})$.

The *twisting sheaf of Serre of degree n* is the quasi-coherent sheaf $\mathcal{O}_{\mathbb{P}^m_S}(n)$ (Definition 4.2.5) on $\mathbb{P}^m_S$ associated to the graded $\mathcal{A}$-module $\mathcal{A}(n)$.

Explicitly, on the standard affine open cover $U_i = D_+(T_i) \cong \mathbb{A}^m_S$ (for $i = 0, \dots, m$), the sheaf is trivial and generated by T_i^n. The sections over an open set $V \subseteq U_i$ are given by:

$$\mathcal{O}_{\mathbb{P}^m_S}(n)(V) = \left\{ \frac{f}{T_i^d} \in \mathcal{O}_{\mathbb{P}^m_S}(V) \cdot T_i^n \right\}.$$

(♠ TODO: transition function) The transition function between the charts U_i and U_j is multiplication by $(T_j/T_i)^n$. This sheaf is an invertible sheaf (Definition 4.1.2) (a line bundle) and is called the *line bundle of degree n*.

5. Varieties

Definition 5.0.1. Let k be a field (Definition E.0.38). A *(algebraic) variety over k* is an integral (Definition 3.1.6), separated (Definition 3.2.3) scheme of finite type (Definition 3.2.7) over k.

Definition 6.0.1. Let k be a field. An *algebraic curve over k* is a scheme (Definition 1.4.7) C that is separated (Definition 3.2.3), of finite type (Definition 3.2.7) over k, and purely of dimension 1. (♠ TODO: define pure dimension of a scheme)

A curve is called *complete* if it is proper (Definition 3.2.18) over k. Many writers will typically assume algebraic curves to be complete and geometrically connected (Definition 3.4.6).

Definition 6.0.2. Let C be a proper (Definition 3.2.18) algebraic curve (Definition 6.0.1) over a field k. The *arithmetic genus of C*, denoted $p_a(C)$, is defined using the Euler characteristic of the structure sheaf:

$$p_a(C) := 1 - \chi(\mathcal{O}_C) = 1 - \dim_k H^0(C, \mathcal{O}_C) + \dim_k H^1(C, \mathcal{O}_C)$$

(♠ TODO: geometrically reduced) If C is geometrically connected (Definition 3.4.6) and geometrically reduced, we have $H^0(C, \mathcal{O}_C) \cong k$, and the formula simplifies to $p_a(C) = \dim_k H^1(C, \mathcal{O}_C)$. (♠ TODO: define cohomology of coherent sheaves, Euler characteristic χ)

7. Hirzebruch Surfaces

Definition 7.0.1. (♠ TODO: projective bundle,) Let k be a field and let $n \geq 0$ be an integer. Let $\mathbb{P}^1_k$ denote the projective line (Definition 2.2.9) over k, equipped with its structure sheaf $\mathcal{O}_{\mathbb{P}^1}$ and the twisting sheaf (Definition 4.2.6) $\mathcal{O}_{\mathbb{P}^1}(n)$ (the line bundle of degree n). The *Hirzebruch surface of degree n*, denoted by $\mathbb{F}_n$ (or sometimes Σ_n), is the projective bundle associated to the rank-2 vector bundle (Definition 4.1.2) $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$. That is,

$$\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) = \operatorname{Proj}\left(\operatorname{Sym}^\bullet(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))\right).$$

The natural projection $\pi : \mathbb{F}_n \to \mathbb{P}^1$ gives $\mathbb{F}_n$ the structure of a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.

Theorem 7.0.2 (Intersection Pairing on $\mathbb{F}_n$)**.** Let $\mathbb{F}_n$ be the Hirzebruch surface of degree $n \geq 0$. The intersection form on the Picard group $\operatorname{Pic}(\mathbb{F}_n) \cong \mathbb{Z}f \oplus \mathbb{Z}s$ (where f is the fiber class and s is the class of the unique section with $s^2 = -n$) is given by the matrix:

$$\begin{pmatrix} f^2 & f \cdot s \\ s \cdot f & s^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -n \end{pmatrix}.$$

Consequently, the intersection pairing is unimodular, and the determinant of the intersection form is -1.

Theorem 7.0.3 (Canonical Divisor of $\mathbb{F}_n$)**.** The canonical divisor class $K_{\mathbb{F}_n}$ of the Hirzebruch surface $\mathbb{F}_n$ is given in terms of the basis $\{f, s\}$ by the formula:

$$K_{\mathbb{F}_n} \sim -2s - (n+2)f.$$

Theorem 7.0.4 (Cohomology of Line Bundles on $\mathbb{F}_n$)**.** Let $D \equiv as + bf$ be a divisor on $\mathbb{F}_n$ with $a, b \in \mathbb{Z}$. The dimension of the space of global sections $H^0(\mathbb{F}_n, \mathcal{O}(D))$ is given by:

$$h^0(\mathbb{F}_n, \mathcal{O}(as + bf)) = \sum_{i=0}^{a} \max(0, b - ni + 1), \quad \text{if } a \geq 0.$$

If $a < 0$, then $h^0(\mathbb{F}_n, \mathcal{O}(D)) = 0$.

Proposition 7.0.5 (Curve Classes)**.** Let $C \subset \mathbb{F}_n$ be an irreducible curve effectively equivalent to $as + bf$.

- If $C = s$ (the unique negative section), then $a = 1, b = 0$.
- If $C \neq s$, then $a \geq 0$ and $b \geq na$.

This implies that the cone of effective curves (the Mori cone) $\overline{NE}(\mathbb{F}_n)$ is generated by the classes of the negative section s and the fiber f.

Theorem 7.0.6 (Structure of Hirzebruch Surfaces)**.** Let $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ be a Hirzebruch surface over an algebraically closed field k, where $n \geq 0$. Let $\pi : \mathbb{F}_n \to \mathbb{P}^1$ be the ruling.

1. **Picard Group:** The Picard group $\mathrm{Pic}(\mathbb{F}_n)$ is free abelian of rank 2, generated by the class of a fiber f of π and the class of a section s satisfying $s^2 = -n$. Explicitly,
$$\mathrm{Pic}(\mathbb{F}_n) \cong \mathbb{Z}f \oplus \mathbb{Z}s,$$
where $f \cdot f = 0$, $f \cdot s = 1$, and s corresponds to the quotient line bundle $\mathcal{O}_{\mathbb{P}^1}(n)$ of the defining vector bundle.
2. **Unique Negative Section:** If $n > 0$, there exists a unique irreducible curve $C_0 \subset \mathbb{F}_n$ with negative self-intersection. This curve is a section of π and satisfies $C_0^2 = -n$. It corresponds to the class s in the Picard group.
3. **Toric Description:** $\mathbb{F}_n$ is the complete toric surface associated to the fan $\Sigma_n \subset \mathbb{R}^2$ generated by the rays
$$v_1 = (1, 0), \quad v_2 = (0, 1), \quad v_3 = (-1, n), \quad v_4 = (0, -1).$$

Proposition 7.0.7 (Isomorphism Classification)**.** Two Hirzebruch surfaces $\mathbb{F}_n$ and $\mathbb{F}_m$ are isomorphic as abstract algebraic varieties if and only if $n = m$.

Theorem 7.0.8 (Rationality and Exceptional Cases)**.**
- $\mathbb{F}_0$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.
- $\mathbb{F}_1$ is isomorphic to the blow-up of $\mathbb{P}^2$ at a single point. The unique negative section C_0 (with $C_0^2 = -1$) is the exceptional divisor of the blow-up.
- All Hirzebruch surfaces $\mathbb{F}_n$ are rational surfaces.

Appendix A. Commutative algebra

Definition A.0.1. A *ring* is a triple $(R, +, \cdot)$ where

1. $(R, +)$ is a commutative group (Definition E.0.17), and
2. $(R, \cdot)$ is a monoid (Definition E.0.16).
3. $\cdot$ is distributive over $+$, i.e. for all $a, b, c \in R$, we have
$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and} \quad (a + b) \cdot c = a \cdot c + b \cdot c.$$

Equivalently, a ring is a triple $(R, +, \cdot)$ where $+, \cdot : R \times R \to R$ are binary operations satisfying

1. $(a+b)+c = a+(b+c)$ and $(ab)c = a(bc)$ for all $a, b, c \in R$
2. There exists an element $0 \in R$ such that $a + 0 = a = 0 + a$ for all $a \in R$.
3. For every $a \in R$, there exists an element $-a \in R$ such that $a + (-a) = 0 = (-a) + a$ for all $a \in R$.
4. There exists an element $1 \in R$ such that $a \cdot 1 = a = 1 \cdot a$ for all $a \in R$.
5. For all $a, b, c \in R$, we have

$$a \cdot (b+c) = a \cdot b + a \cdot c \quad \text{and} \quad (a+b) \cdot c = a \cdot c + b \cdot c.$$

The operation $+$ is often called *addition* and the operation $\cdot$ is often called *multiplication*. Accordingly, the identity element 0 of $+$ is often called the *additive identity* and the identity element 1 of $\cdot$ is often called the *multiplicative identity*.

Remark A.0.2. Some writers might not require a ring to have a multiplicative identity element, i.e. would define a ring so that $(R, +)$ is a commutative group, $(R, \cdot)$ is a semigroup, and $\cdot$ is distributive over $+$. Such writers would call the notion of ring in Definition A.0.1 a *unitary ring* to emphasize the existence of the multiplicative identity 1.

Definition A.0.3. Let $(R, +, \cdot)$ and $(S, +, \cdot)$ be rings (Definition A.0.1), not assumed to be commutative. A function $f : R \to S$ is called a *ring homomorphism* if for all $r_1, r_2 \in R$ the following properties hold:

1. $f(r_1 + r_2) = f(r_1) + f(r_2)$,
2. $f(r_1 r_2) = f(r_1) f(r_2)$,
3. $f(1_R) = 1_S$ where 1_R and 1_S denote the multiplicative identities in R and S, respectively.

A ring homomorphism is said to be a *ring isomorphism* if it is invertible as a map of sets.

An *R-ring* refers to a ring S equipped with a ring homomorphism $f : R \to S$.

We note that a ring homomorphism $f : R \to S$ yields a natural left R-module (Definition A.0.8) structure on S and a natural right R-module structure on S respectively as follows for $r \in R$ and $s \in S$:

$$r \cdot s = f(r) \cdot s$$
$$s \cdot r = s \cdot f(r).$$

However, these left and right module structures need not yield a two-sided R–module structure.

Definition A.0.4. Let R be a (not-necessarily commutative) ring with unity (Definition A.0.1). An *R-algebra* is a ring A together with a ring homomorphism (Definition A.0.3)

$$\varphi : R \to A$$

into the center $Z(A)$ of A (so that $\varphi(r)$ commutes with every element of A for all $r \in R$), such that $\varphi(1_R) = 1_A$. The ring homomorphism φ is called the *structure map* of the algebra.

Equivalently, an R-algebra consists of a ring A endowed with a two-sided R-module (Definition A.0.8) structure for which the scalar multiplication satisfies

$$r \cdot (ab) = (r \cdot a)b = a(r \cdot b) \quad \text{for all } r \in R,\ a, b \in A.$$

In particular, any ring homomorphism between commutative rings (Definition A.0.7) specifies an algebra structure.

Definition A.0.5. Let R be a (not-necessarily commutative) ring (Definition A.0.1) and let A be an R-algebra (Definition A.0.4). We say that A is a *finitely generated R-algebra* or synoymously that the ring homomorphism (Definition A.0.3) $R \to A$ is *of finite type* if there exists a finite subset $\{a_1, \ldots, a_n\} \subseteq A$ such that R-subalgebra of A generated by $\{a_1, \ldots, a_n\}$ is equal to A. Equivalently, every element of A can be expressed, using a finite composition of ring operations and R-linear combinations, from the generators $\{a_1, \ldots, a_n\}$.

Definition A.0.6 (Finitely presented algebra over a ring)**.** Let R be a (not necessarily commutative) ring (Definition A.0.1). An R-algebra (Definition A.0.4) A is said to be *finitely presented* if there exists an integer $n \geq 0$ and a surjective R-algebra homomorphism

$$\varphi : R\langle x_1, \ldots, x_n \rangle \twoheadrightarrow A$$

where $R\langle x_1, \ldots, x_n \rangle$ is the free R-algebra on n generators, such that the kernel $\ker(\varphi)$ is a finitely generated two-sided ideal (Definition E.0.19) of $R\langle x_1, \ldots, x_n \rangle$.

In other words, A admits a presentation as

$$A \cong R\langle x_1, \ldots, x_n \rangle / I,$$

where I is a finitely generated (Definition E.0.20) two-sided ideal.

If R and A are commutative rings, this recovers the usual definition of a finitely presented commutative R-algebra by replacing $R\langle x_1, \ldots, x_n \rangle$ with the polynomial ring $R[x_1, \ldots, x_n]$ and I a finitely generated ideal.

Definition A.0.7. A *commutative (unital) ring* is a ring (Definition A.0.1) $(R, +, \cdot)$ such that $\cdot$ is a commutative operation, i.e. $a \cdot b = b \cdot a$.

For many writers (e.g. "commutative" algebraists or number theorists), a *ring* refers to a commutative ring as above.

Definition A.0.8. Let R be a not-necessarily commutative ring (Definition A.0.1).

1. A *left R-module* is an abelian group $(M, +)$ together with an operation $R \times M \to M$, denoted $(r, m) \mapsto rm$, such that for all $r, s \in R$ and $m, n \in M$:
   - $r(m + n) = rm + rn$,
   - $(r + s)m = rm + sm$,
   - $(rs)m = r(sm)$,
   - $1_R m = m$ where 1_R is the multiplicative identity of R.
2. A *right R-module* is defined similarly as an abelian group $(M, +)$ with an operation $M \times R \to M$, denoted $(m, r) \mapsto mr$, such that for all $r, s \in R$ and $m, n \in M$:
   - $(m + n)r = mr + nr$,
   - $m(r + s) = mr + ms$,
   - $m(rs) = (mr)s$,
   - $m1_R = m$.
3. Let R and S be (not necessarily commutative) rings (Definition A.0.1).

   An *R-S-bimodule* (or an *R-S-module* or an (R, S)-module, etc.)is an abelian group (Definition E.0.17) $(M, +)$ equipped with

(a) a left action of R:

$$R \times M \to M, \quad (r, m) \mapsto r \cdot m,$$

making M a left R-module (Definition A.0.8),

(b) a right action of S:

$$M \times S \to M, \quad (m, s) \mapsto m \cdot s,$$

making M a right S-module,

such that the left and right actions commute; that is, for all $r \in R$, $s \in S$, and $m \in M$,

$$r \cdot (m \cdot s) = (r \cdot m) \cdot s.$$

4. A *two-sided R-module* (or *R-bimodule*) is an R-R-bimodule.

If R is a commutative ring (Definition A.0.7), then a left/right R-module can automatically be regarded as a two-sided R-module. As such, we simply talk about *R-modules* in this case.

Any abelian group is equivalent to a two-sided $\mathbb{Z}$-module. Moreover, any left R-module is equivalent to an $R - \mathbb{Z}$-bimodule (Definition A.0.8) and any right R-module is equivalent to an $\mathbb{Z} - R$-bimodule (Definition A.0.8). Given a left/right/two-sided R-module, its *natural bimodule structure* will refer to its structure as a R-$\mathbb{Z}$/$\mathbb{Z}$-R/R-R bimodule. In this way, many definitions associated with the notions of left/right/two-sided R-modules can be defined as special cases for definitions for R-S-bimodules.

Definition A.0.9. Let R, S be (not-necessarily commutative) rings (Definition A.0.1).

1. Let M be a R-S-bimodule (Definition A.0.8). Given elements $x_1, \ldots, x_n \in M$, $r_1, \ldots, r_n \in R$, and $s_1, \ldots, s_n \in R$, an element of the form

$$\sum_{i=1}^{n} r_i x_i s_i \in M$$

is called a *(R-S-)linear combination of $x_1, \ldots, x_n$*. Given an arbitrary subset $\{m_i\}_{i \in I} \subseteq M$, a *(R-S-)linear combination of the m_i over R* refers to a linear combination of some finite subset of $\{m_i\}_{i \in I}$ over R.

2. Let M be a left/right/two-sided R-module (Definition A.0.8). A *llinear combination of elements $\{m_i\}_{i \in I} \subseteq M$* refers to a linear combination of (a finite subset of) $\{m_i\}_{i \in I}$ for the natural bimodule structure (Definition A.0.8) of M. In case that M is a left/right R-module, and given finitely many $m_1, \ldots, m_n \in M$, a linear combination of these m_i is equivalently an element of the form

$$r_1 x_1 + r_2 x_2 + \cdots + r_n x_n$$

$$x_1 r_1 + x_2 r_2 + \cdots + x_n r_n$$

for $r_1, \ldots, r_n \in R$ respectively.

Definition A.0.10. Let R and S be (not necessarily commutative) rings (Definition A.0.1).

1. If M is an R-S-bidmodule (Definition A.0.8), then a subset $\{m_i\}_{i \in I} \subseteq M$ is said to *span M (as an R-S-bimodule)* if every element $m \in M$ is a linear combination (Definition A.0.9) of $\{m_i\}_{i \in I}$.

2. If M is a left/right/two-sided R-module, then a subset $\{m_i\}_{i\in I} \subseteq M$ is said to *span M (as a left/right/two-sided R-module)* if $\{m_i\}_{i\in}$ spans its natural bimodule structure (Definition A.0.8).

In each case, such a set $\{m_i\}_{i\in I}$ is called a *generating set* or *spanning set of M over R*.

In each case, such a set S is called a *generating set* or *spanning set of M over R*.

Definition A.0.11 (Finitely generated modules and bimodules)**.** Let R and S be (not necessarily commutative) rings (Definition A.0.1).

1. An R-S-bimodule M is *finitely generated* if it has a finite spanning set (Definition A.0.10).
2. A left/right/two-sided R-module is *finitely generated* if has a finite spanning set (Definition A.0.10), or equivalently if its natural bimodule structure (Definition A.0.8) is finitely generated.

Definition A.0.12. Let R be a (not necessarily commutative) ring (Definition A.0.1). A proper two-sided ideal $P \trianglelefteq R$ (Definition E.0.19) is called a *prime ideal* if the following equivalent conditions holds:

1. If I, J are left ideals and $IJ \subset P$ (Definition E.0.8), then $I \subset P$ or $J \subset P$.
2. If I, J are right ideals and $IJ \subset P$, then $I \subset P$ or $J \subset P$.
3. If I, J are two-sided idaels and $IJ \subset P$, then $I \subset P$ or $J \subset P$.
4. If $x, y \in R$ with $xRy \subset \mathfrak{p}$, then $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

A proper left/right/two-sided ideal $M \subsetneq R$ is called *maximal* if there exists no other left/right/two-sided ideal $J \trianglelefteq R$ such that $M \subsetneq J \subsetneq R$. Equivalently,

- a left/right ideal M of R is maximal if and only if the quotient module R/M (Definition E.0.7) is a simple left/right R-module.
- a two-sided ideal M of R is maximal if and only if the quotient ring R/M is a simple ring.

Definition A.0.13. Let $(R, \mathfrak{m}_R)$ and $(S, \mathfrak{m}_S)$ be local rings (Definition 1.5.4), not necessarily commutative. A ring homomorphism (Definition A.0.3) $\varphi : R \to S$ is called a *local morphism* (or *local homomorphism*) if $\varphi(\mathfrak{m}_R) \subseteq \mathfrak{m}_S$.

Definition A.0.14. Let R be a not necessarily commutative ring (Definition A.0.1). A subset $S \subseteq R$ is called a *multiplicative subset* if

- $1 \in S$ (assuming R has unity),
- For all $s, t \in S$, one has $st \in S$.

Definition A.0.15. Let R be a commutative ring with unity (Definition A.0.7) and let $S \subseteq R$ be a multiplicative subset (Definition A.0.14). The *localization of R at S*, denoted by $S^{-1}R$, is the ring whose elements are equivalence classes of pairs $(r, s) \in R \times S$ under the equivalence relation

$$(r, s) \sim (r', s') \iff \exists u \in S \text{ such that } u(sr' - s'r) = 0.$$

Write $\frac{r}{s}$ for the equivalence class of (r, s). Addition and multiplication on representatives are defined by

$$\frac{r}{s} + \frac{r'}{s'} = \frac{rs' + r's}{ss'},$$
$$\frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}.$$

The map $r \mapsto \frac{r}{1}$ defines a ring homomorphism (Definition A.0.3); therefore, $S^{-1}R$ is naturally an R-algebra (Definition A.0.4).

Given an element $f \in R$, the localization of R at $S = \{f^n : n \geq 0\}$ is denoted by R_f.

If P is a prime ideal of R (Definition A.0.12), then $R_P := S^{-1}R$ with $S = R \setminus P$ is called the *localization of R at P*. It is a local ring (Definition 1.5.4) whose maximal ideal (Definition A.0.12) is given by

$$S^{-1}P = \{\frac{p}{s} \in R_P : p \in P\}.$$

Appendix B. Topological spaces

Definition B.0.1 (Topology)**.** Let X be a set. A *topology on X* is a collection $\mathcal{T}$ of subsets of X such that:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
2. For any collection $\{U_i\}_{i \in I} \subseteq \mathcal{T}$ (with I arbitrary), the union $\bigcup_{i \in I} U_i \in \mathcal{T}$,
3. For any finite collection $\{U_1, \ldots, U_n\} \subseteq \mathcal{T}$, the intersection $U_1 \cap \cdots \cap U_n \in \mathcal{T}$.

If $\mathcal{T}$ is a topology on X, the pair $(X, \mathcal{T})$ is called a *topological space*. Members of $\mathcal{T}$ are called *open sets*.

A subset $C \subseteq X$ is *closed* if its complement $X \setminus C$ is an open set in $\mathcal{T}$

One very often refers to X as a topological spcae, omitting the notation of the topology $\mathcal{T}$.

The collection of all topologies on a set X may be denoted by notations such as $\mathrm{Top}(X)$, $\mathbf{Top}(X)$, or $\mathsf{Top}(X)$.

Definition B.0.2. Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open subsets, and let $k \in \mathbb{N}_0 \cup \{\infty\}$. A *continuous morphism* (or a *continuous map* or simply a *map between topological spaces*) from U to V is a function $f : U \to V$ that satisfies one of the following equivalent characterizations:

1. f is a continuous map from U to V as topological spaces.
2. for every point $x \in U$, for every open neighborhood W of $f(x)$ in V, there exists an open neighborhood O of x in U satisfying
$$f(O) \subseteq W.$$

We write $C(U, V) = C^0(U, V)$ for the set of continuous maps $U \to V$.

Definition C.0.1 (Category)**.** A *category* *category* $\mathcal{C}$ consists of the following data:

- A class of *objects* denoted $\mathrm{Ob}(\mathcal{C})$.
- For each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a class
$$\mathrm{Hom}_{\mathcal{C}}(X, Y)$$
of *morphisms* (also called *arrows* or *homs*). If the category $\mathcal{C}$ is clear, then this *hom-class* is also denoted by $\mathrm{Hom}(X, Y)$. It may also be denoted by $\mathrm{hom}_{\mathcal{C}}(X, Y)$ or $\mathrm{hom}(X, Y)$, especially to distinguish from other types of hom's (e.g. internal hom's)
- For each triple of objects X, Y, Z, a composition law
$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z),$$
denoted $(g, f) \mapsto g \circ f$.
- For each object X, an *identity morphism*
$$\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X).$$

These data satisfy the following axioms:

- (Associativity) For all morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, and $h \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$,
$$h \circ (g \circ f) = (h \circ g) \circ f.$$
- (Identity) For all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$,
$$\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X .$$

One often writes $X \in \mathcal{C}$ synonymously with $X \in \mathrm{Ob}(\mathcal{C})$, i.e. to denote that X is an object of of $\mathcal{C}$.

We may call a category as above an *ordinary category* to distinguish this notion from the notions of *categories enriched in monoidal categories* or higher/n-categories. (♠ TODO: TODO: define n-categories)

A category as defined above may be called called a *large category* or a *class category* to emphasize that the hom-classes may be proper classes rather than sets (note, however, that the possibility that hom-classes are sets is not excluded for large categories). Accordingly, a *category* may often refer to a locally small category (Definition C.0.2), which is a category whose hom-classes are all sets.

Definition C.0.2 (Locally small category)**.** A (large) category (Definition C.0.1) $\mathcal{C}$ is called a *locally small category* if for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the collection $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms between them is a (small) *set* (as opposed to a proper class). In other words, each hom-class is a set and may even be called a *hom-set*.

In some contexts, a locally small category may simply be called a *category*, especially when genuinely large categories are not considered.

A category $\mathcal{C}$ is called a *small category* if it is a locally small category and the class $\mathrm{Ob}(\mathcal{C})$ of objects is a set.

Remark C.0.3. Many "concrete" categories considered in "classical mathematics" or outside of more "abstract" category theory tend to be locally small. For example, the categories of sets, groups, R-modules, vector spaces, topological spaces, schemes, manifolds, sheaves on "small enough" sites are all locally small.

Definition C.0.4. Let $\mathcal{C}$ and $\mathcal{D}$ be (large) categories (Definition C.0.1).

1. A *functor $F : \mathcal{C} \to \mathcal{D}$ (from $\mathcal{C}$ to $\mathcal{D}$)* consists of :
   - For each object X in $\mathcal{C}$, an object $F(X)$ in $\mathcal{D}$.
   - For each morphism $f : X \to Y$ in $\mathcal{C}$, a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$,

   such that:
   $$F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \text{for all objects } X \text{ in } \mathcal{C},$$
   $$F(g \circ f) = F(g) \circ F(f) \quad \text{for all } X, Y, Z \in \mathrm{Ob}(\mathcal{C}) \text{ and all } f : X \to Y, g : Y \to Z \text{ in } \mathcal{C}.$$

   Functors as defined above are also referred to as *covariant functors* to distinguish them from contravariant functors
2. A *contravariant functor from $\mathcal{C}$ to $\mathcal{D}$* refers to a covariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. Equivalently, such a functor consists of
   - For each object X in $\mathcal{C}$, an object $F(X)$ in $\mathcal{D}$.
   - For each morphism $f : X \to Y$ in $\mathcal{C}$, a morphism $F(f) : F(Y) \to F(X)$ in $\mathcal{D}$,

   such that:
   $$F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \text{for all objects } X \text{ in } \mathcal{C},$$
   $$F(g \circ f) = F(f) \circ F(g) \quad \text{for all } X, Y, Z \in \mathrm{Ob}(\mathcal{C}) \text{ and all } f : X \to Y, g : Y \to Z \text{ in } \mathcal{C}.$$

   A synonym for a "contravariant functor from $\mathcal{C}$ to $\mathcal{D}$" is a "presheaf on $\mathcal{C}$ with values in $\mathcal{D}$ (Definition D.0.1)".

Note that declarations such as "Let $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ be a contravariant functor" can be common; such declarations usually mean "Let F be a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$" as opposed to "Let F be a contravariant functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$". further note that a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is equivalent to a covariant functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$.

Definition C.0.5. Let $\mathcal{C}$ and $\mathcal{D}$ be (large) categories (Definition C.0.1). Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors (Definition C.0.4).

A *natural transformation η between F and G* is a family of morphisms $\eta_X : F(X) \to G(X)$ in $\mathcal{D}$, one for each object X in $\mathcal{C}$, such that for every morphism $f : X \to Y$ in $\mathcal{C}$,

$$G(f) \circ \eta_X = \eta_Y \circ F(f)$$

in $\mathcal{D}$. In other words, the following diagram commutes:

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\
G(X) \arrow[r, "G(f)"'] & G(Y)
\end{tikzcd}$$

We write such a natural transformation by $\eta : F \Rightarrow G$.

If η_X is an isomorphism for all objects X of $\mathcal{C}$, then η is said to be a *natural isomorphism*.

Definition C.0.6 (Full subcategory)**.** Let $\mathcal{C}$ be a (large) category (Definition C.0.1). A *full subcategory* $\mathcal{D}$ of $\mathcal{C}$ is a subcategory such that for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{D})$, the morphism classes coincide:

$$\mathrm{Hom}_{\mathcal{D}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

In other words, a full subcategory includes all morphisms between its objects that exist in the ambient category $\mathcal{C}$.

Definition C.0.7 (Category of objects over a fixed object)**.** Let $\mathcal{C}$ be a category (Definition C.0.1) and let $X \in \mathrm{Ob}(\mathcal{C})$ be a fixed object.

1. The *category of objects over X* (or synonymously the *slice category of X in $\mathcal{C}$* or the *over category of X in $\mathcal{C}$*), commonly denoted $\mathcal{C}/X$ or $\mathcal{C}_{/X}$, is the category defined as follows:
   - An object of $\mathcal{C}/X$ is a morphism $f\colon A \to X$ in $\mathcal{C}$, where $A \in \mathrm{Ob}(\mathcal{C})$.
   - A morphism from $f\colon A \to X$ to $g\colon B \to X$ in $\mathcal{C}/X$ is a morphism $h\colon A \to B$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[dr, "f"'] & B \arrow[d, "g"] \\ & X \end{tikzcd}$$

   i.e. such that $g \circ h = f$.
   - The identity morphisms and composition in $\mathcal{C}/X$ are inherited from $\mathcal{C}$.
2. The *category of objects under X* (or synonymously the *coslice category of X in $\mathcal{C}$* or the *under category of X in $\mathcal{C}$*), commonly denoted $X/\mathcal{C}$, $X\backslash\mathcal{C}$ or $\mathcal{C}_{X/}$, is the category defined as follows:
   - An object of $X/\mathcal{C}$ is a morphism $f\colon X \to A$ in $\mathcal{C}$, where $A \in \mathrm{Ob}(\mathcal{C})$.
   - A morphism from $f\colon X \to A$ to $g\colon X \to B$ in $X/\mathcal{C}$ is a morphism $h\colon A \to B$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & A \arrow[d, "h"] \\ & B \end{tikzcd}$$

   i.e. such that $h \circ f = g$.
   - The identity morphisms and composition in $X/\mathcal{C}$ are inherited from $\mathcal{C}$.

C.1. **Limits and colimits.**

Definition C.1.1 (Cones, limits and colimits in a category)**.** Let $\mathcal{C}$ be a (large) category (Definition C.0.1), let I be a (large) category, and let $D : I \to \mathcal{C}$ be a diagram.

1. A *cone to the diagram D* is an object $L \in \mathcal{C}$ together with a family of morphisms

$$\{\pi_i : L \to D(i)\}_{i \in I}$$

such that for every morphism $f : i \to j$ in I, the diagram

$$\begin{tikzcd} & L \arrow[dl, "\pi_i"'] \arrow[dr, "\pi_j"] & \\ D(i) \arrow[rr, "D(f)"] & & D(j) \end{tikzcd}$$

commutes, i.e. $D(f) \circ \pi_i = \pi_j$.

2. A cone $(L, \{\pi_i\})$ is called a *limit of D* if it satisfies the following "universal property": for any cone $(C, \{f_i\})$ over D, there exists a *unique* morphism $u : C \to L$ such that

$$\pi_i \circ u = f_i \quad \text{for all } i \in I.$$

Visually, the following diagrams commute every morphism $f : i \to j$ in I:

$$\begin{tikzcd} & C \arrow[d, dashed, "\exists! u"] \arrow[ddl, bend right, "f_i"'] \arrow[ddr, bend left, "f_j"] & \\ & L \arrow[dl, "\pi_i"'] \arrow[dr, "\pi_j"] & \\ D(i) \arrow[rr, "D(f)"] & & D(j). \end{tikzcd}$$

If such a cone exists, then the object L is necessarily unique up to unique isomorphism by the universal property. In this case, L is denoted by $\lim_{i \in I} D$ or $\lim D$.

3. A *cocone from the diagram D* is an object $C \in \mathcal{C}$ together with a family of morphisms

$$\{\iota_i : D(i) \to C\}_{i \in I}$$

such that for every morphism $f : i \to j$ in I, the diagram

$$\begin{tikzcd} D(i) \arrow[rr, "D(f)"] \arrow[dr, "\iota_i"'] & & D(j) \arrow[dl, "\iota_j"] \\ & C & \end{tikzcd}$$

commutes, i.e. $\iota_j \circ D(f) = \iota_i$.

4. A cocone $(L, \{\iota_i\})$ is called a *colimit of D* if it satisfies the following "universal property": for any cocone $(C, \{g_i\})$ under D, there exists a *unique* morphism $u : L \to C$ such that

$$u \circ \iota_i = g_i \quad \text{for all } i \in I.$$

Visually, the following diagrams commute every morphism $f : i \to j$ in I:

$$
\begin{tikzcd}
D(i) \arrow[rr, "D(f)"] \arrow[dr, "\iota_i"] \arrow[ddr, bend right, "g_i"'] & & D(j) \arrow[dl, "\iota_j"'] \arrow[ddl, bend left, "g_j"] \\
& L \arrow[d, dashed, "\exists! u"] & \\
& C &
\end{tikzcd}
.
$$

If such a cocone exists, then the object L is necessarily unique up to unique isomorphism by the universal property. In this case, L is denoted by $\operatorname{colim}_{i \in I} D$ or $\operatorname{colim} D$.

A limit/colimit is called *finite* (resp. *small*) if the diagram category I is finite (resp. small).

Some authors use the terms *projective limit* or *inverse limit* to refer to what is defined here as a limit, Similarly, the terms *inductive limit* or *direct limit* are sometimes used to mean a colimit. However, these phrases can have more specific meanings to other authors: a *projective* or *inverse limit* may refer to a limit over a diagram indexed by a codirected poset (Definition E.0.21). Likewise, an *inductive* or *direct limit* may refer to a colimit over a directed poset (Definition E.0.21) (see Definition C.1.4) .

Thus, while the terms are sometimes used interchangeably with "limit" and "colimit," they may also emphasize particular indexing shapes and directions, distinguishing them from general limits and colimits taken over arbitrary small categories.

Definition C.1.2 (Filtered category)**.**

1. A *filtered category* is a (nonempty, large) category $\mathcal{I}$ satisfying the following conditions:
   - For every finite collection of objects $i_1, i_2, \ldots, i_n$ in $\mathcal{I}$, there exists an object j and morphisms
     $$\phi_k : i_k \to j, \quad \text{for each } k = 1, \ldots, n.$$
   - For every pair of morphisms $f, g : i \to j$ in $\mathcal{I}$, there exists an object k and a morphism
     $$h : j \to k$$
     (♠ TODO: equalizer) that is an equalizer of f and g, i.e. satisfies
     $$h \circ f = h \circ g.$$

   In other words, $\mathcal{I}$ is nonempty, any finite diagram of objects admits a cocone (Definition C.1.1), and any pair of parallel morphisms become equal after post-composition with an appropriate morphism.
2. Dually, a *Cofiltered category* is a category whose opposite category is filtered. More explicitly, A cofiltered category is a (nonempty, large) category $\mathcal{I}$ satisfying the following conditions:
   - For every finite collection of objects $i_1, i_2, \ldots, i_n$ in $\mathcal{I}$, there exists an object j and morphisms
     $$\phi_k : j \to i_k, \quad \text{for each } k = 1, \ldots, n.$$

$$\begin{tikzcd} i_1 \arrow[dr, dashed, "\phi_1"] & \\ & j \\ i_2 \arrow[ur, dashed, "\phi_2"'] & \end{tikzcd}$$

Figure 1. *
Condition 1: Upper Bound

$$\begin{tikzcd} i \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & j \arrow[r, dashed, "h"] & k \end{tikzcd}$$

Figure 2. *
Condition 2: Equalizer

- For every pair of morphisms $f, g : j \to i$ in $\mathcal{I}$, there exists an object k and a morphism

  $$h : k \to j$$

  that is a coequalizer of f and g, i.e. satisfies

  $$f \circ h = g \circ h.$$

$$\begin{tikzcd} i_1 & \\ & j \arrow[ul, dashed, "\phi_1"'] \arrow[dl, dashed, "\phi_2"] \\ i_2 & \end{tikzcd}$$

Figure 3. *
Condition 1: Lower Bound

$$\begin{tikzcd} k \arrow[r, dashed, "h"] & j \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & i \end{tikzcd}$$

Figure 4. *
Condition 2: Equalizer

In other words, $\mathcal{I}$ is nonempty, any finite diagram of objects admits a cone, and any pair of parallel morphisms become equal after pre-composition with an appropriate morphism.

Definition C.1.3 (Systems in a category)**.** Let $\mathcal{C}$ be a (large) category. Let I be a (large) category.

1. A diagram/system $I \to C$ is called *filtered* (resp. *cofiltered*) if I is a filtered (Definition C.1.2) (resp. cofiltered (Definition C.1.2)) category.
2. A diagram/system $I \to C$ is called *directed* (resp. *codirected* if I is small and thing, i.e. is regardable/comes from a directed (resp. codirected) partially ordered set (Definition E.0.21). A *direct system* or *inductive system* is synonymous for a directed system and a *inverse system* or *projective system* is synonymous for a codirected system.

One might also speak of a *filtered direct/inductive system* synonymously for a filtered system to emphasize that the indexing caetgory is a general filtered category, rather than a directed poset.

Definition C.1.4 (Special cases of limits)**.** Let $\mathcal{C}$ be a (large) category. Let I be a (large) category. Let $I \to \mathcal{C}$ be a diagram/system.

- Suppose that the system is a cofiltered system (Definition C.1.3), i.e. I is a cofiltered category. A limit (Definition C.1.1) of this diagram is often denoted by
$$\varprojlim_{i \in I} D(i)$$
and may be called a *cofiltered (inverse/projective) limit*. In case that the system is more specifically an inverse/projective system (Definition C.1.3), i.e. I is a cofiltered poset, the preferred term for such a limit is *inverse/projective limit.*
- Suppose that the system is a filtered system, i.e. I is a filtered category. A colimit of this diagram is often denoted by
$$\varinjlim_{i \in I} D(i)$$
and may be called a *filtered colimit* or a *direct/inductive/injective limit*. In case that the system is more specifically a direct/inductive system, i.e. I is a filtered poset, the preferred term for such a limit is *direct/inductive limit.*

Appendix D. General notions of presheaves and sheaves

Definition D.0.1 (Presheaf on a category)**.** Let C and $\mathcal{A}$ be (large) categories (Definition C.0.1).

1. A *presheaf $\mathcal{F}$ on C with values in $\mathcal{A}$* is a functor
$$\mathcal{F} : C^{\text{op}} \to \mathcal{A}.$$
In other words, a presheaf $\mathcal{F}$ on C with values in $\mathcal{A}$ is simply a contravariant functor (Definition C.0.4) from C to $\mathcal{A}$. Explicitly, for every object U in C, one has an object $\mathcal{F}(U)$ in $\mathcal{A}$ (called the *U-valued sections/sections evaluated at U of $\mathcal{F}$*, cf. Definition 1.1.4), and for every morphism $f : V \to U$ in C, one has a morphism (called the restriction map)
$$\mathcal{F}(f) : \mathcal{F}(U) \to \mathcal{F}(V)$$
in $\mathcal{A}$, such that for all composable morphisms $W \xrightarrow{g} V \xrightarrow{f} U$ in C, the following diagram in $\mathcal{A}$ commutes:
$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\mathcal{F}(f)"] \arrow[rr, bend left, "\mathcal{F}(f \circ g)"] & \mathcal{F}(V) \arrow[r, "\mathcal{F}(g)"] & \mathcal{F}(W) \end{tikzcd}$$
That is,
$$\mathcal{F}(g) \circ \mathcal{F}(f) = \mathcal{F}(f \circ g),$$

and for every object U in C, $\mathcal{F}(\mathrm{id}_U) = \mathrm{id}_{\mathcal{F}(U)}$.

2. Let $\mathcal{F}, \mathcal{G} : C^{\mathrm{op}} \to \mathcal{A}$ be two presheaves on C with values in $\mathcal{A}$. A *morphism of presheaves*
$$\varphi : \mathcal{F} \to \mathcal{G}$$
is a natural transformation of functors (Definition C.0.5): for each object U of C, one has a morphism
$$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$
in $\mathcal{A}$, such that for every morphism $f : V \to U$ in C, the diagram
$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\varphi_U"'] & \mathcal{F}(V) \arrow[d, "\varphi_V"] \\ \mathcal{G}(U) \arrow[r, "\mathcal{G}(f)"'] & \mathcal{G}(V) \end{tikzcd}$$
commutes, i.e.,
$$\varphi_V \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \varphi_U$$
for all objects and morphisms in C.
3. Given a universe U, a *U-presheaf on $\mathcal{C}$* typically refers to a presheaf of U-sets on C.
4. The *presheaf category/category of $\mathcal{A}$-valued presheaves on $\mathcal{C}$* is the (large) category whose objects are the presheaves on C with values in $\mathcal{A}$ and whose morphisms are the presheaf morphisms. Common notations for the presheaf category include, but are not limited to: $\mathcal{A}^{\mathcal{C}^{\mathrm{op}}}$, $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$, $[\mathcal{C}^{\mathrm{op}}, \mathcal{A}]$. If the value category $\mathcal{A}$ is clear from context, then notations such as $\mathrm{PreShv}(\mathcal{C})$ are also common.

Definition D.0.2 (Grothendieck topology)**.** Let $\mathscr{U}$ be a universe and let $\mathcal{C}$ be a locally small category (Definition C.0.2).

1. **(Grothendieck Topology via Sieves)** A *Grothendieck topology* J on $\mathcal{C}$ is an assignment to each object $U \in \mathcal{C}$ of a collection $J(U)$ of sieves on U, called *covering sieves*, satisfying:
   (a) (Maximality) The maximal sieve $\{f : V \to U \mid V \in \mathcal{C}\}$ is in $J(U)$.
   (b) (Stability) If $S \in J(U)$ and $f : V \to U$ is any morphism, then the pullback sieve f^*S is in $J(V)$.
   (c) (Transitivity/Local Character) If S is a sieve on U and there exists a covering sieve $R \in J(U)$ such that for every morphism $f : V \to U$ in R, the pullback sieve f^*S is in $J(V)$, then $S \in J(U)$.
2. A *site* is a pair $(\mathcal{C}, J)$ consisting of a category $\mathcal{C}$ and a Grothendieck topology J.
3. A family of objects $\mathcal{G} = \{G_\alpha\}$ in a site $(\mathcal{C}, J)$ is called a *topologically generating family* if for every object $X \in \mathcal{C}$, there exists a covering sieve $S \in J(X)$ generated by morphisms with domains in $\mathcal{G}$. Equivalently, every object X admits a cover $\{U_i \to X\}$ where each $U_i \in \mathcal{G}$.
4. A *$\mathscr{U}$-site* is a site whose underlying category is $\mathscr{U}$-locally small and which admits a $\mathscr{U}$-small topologically generating family.

Definition D.0.3 (Sheaf on a site)**.** (♠ TODO: There might be some need to say that $\mathcal{A}$ is a category for which sheaves on the site "can be defined") (♠ TODO: go through statements using the notion of sheaves and make sure that the value categories have small products and that the categories have small generating families.) Let $(\mathcal{C}, J)$ be a site (Definition D.0.2) with a small topological generating family (Definition D.0.2) (or a U-small topologically generating family if a universe U is available) and let $\mathcal{A}$ be a (large) category (Definition C.0.1) that has all small (Definition C.0.2) products (Definition E.0.26) (Some common examples of categories that have small products and thus play the role of $\mathcal{A}$ here include $\mathcal{A} = \text{Set}$, Ab, $R-\mathbf{mod}$ for a fixed ring R, rings).

1. A presheaf (Definition D.0.1) $\mathcal{F} : C^{\text{op}} \to \mathcal{A}$ is a *sheaf on the site $(\mathcal{C}, J)$ valued in $\mathcal{A}$* if, for every object U of $\mathcal{C}$ and every covering $\{U_i \to U\}_{i \in I}$ in J, the following sequence is an equalizer in $\mathcal{A}$:
$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$$
where the first map sends s to $(\mathcal{F}(U_i \to U)(s))_i$ and the arrows to $(\mathcal{F}(U_i \times_U U_j \to U_i)(s_i))_{i,j}$ and $(\mathcal{F}(U_i \times_U U_j \to U_j)(s_j))_{i,j}$, respectively.
2. A *morphism of sheaves* $\mathcal{F} : \mathcal{C}^{\text{op}} \to \mathcal{A}$ is a morphism as presheaves (Definition D.0.1).
3. Let U be a universe. A *U-sheaf* typically refers to a U-presheaf that is a sheaf for a U-site. In other words, a U-sheaf is a sheaf on a site whose underlying category is U-locally small (Definition C.0.2) and which has a U-small topologically generating family such that the sheaf is valued in U-sets.
4. The *sheaf category/category of $\mathcal{A}$-valued sheaves on $\mathcal{C}$* is the (large) category defined as the full subcategory of $\text{PreShv}(\mathcal{C}, \mathcal{A})$ whose objects are the sheaves on C with values in $\mathcal{A}$. Common notations for the sheaf category include $\text{Shv}(\mathcal{C}, \mathcal{A})$, $\text{Shv}(\mathcal{C}, J, \mathcal{A})$, $\text{Sh}(\mathcal{C}, \mathcal{A})$, $\text{Sh}(\mathcal{C}, J, \mathcal{A})$. If the value category $\mathcal{A}$ is clear from context, then notations such as $\text{Shv}(\mathcal{C})$, $\text{Shv}(\mathcal{C}, J)$, $\text{Sh}(\mathcal{C})$, $\text{Sh}(\mathcal{C}, J)$ are also common.

Definition D.0.4. Let $\mathcal{C}$ be a site (Definition D.0.2) and let $\mathcal{A}$ be a (large) category (Definition C.0.1) A *sheafification functor* refers to a functor
$$a : \text{PreShv}(\mathcal{C}, \mathcal{A}) \to \text{Shv}(\mathcal{C}, \mathcal{A})$$
that is left adjoint to the inclusion functor
$$i : \text{Shv}(\mathcal{C}, \mathcal{A}) \hookrightarrow \text{PreShv}(\mathcal{C}, \mathcal{A}).$$
If such a sheafification functor exists, then it is unique up to unique natural isomorphism. Given a presheaf P, the sheafification $a(P)$ is also sometimes called the *sheaf assciated to P*.

Appendix E. Miscellaneous definitions

Definition E.0.1. Let X and Y be sets and let $f : X \to Y$ be a function.

- The function f is said to be *injective* (or *one-to-one*) if for all $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
- The function f is said to be *surjective* (or *onto*) if for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$.

- The map f is *bijective* if it is both injective and surjective. In this case, there exists a unique *inverse map* $f^{-1} : Y \to X$ such that for all $x \in X$ and $y \in Y$,

$$f^{-1}(f(x)) = x \text{ and } f(f^{-1}(y)) = y.$$

Definition E.0.2 (Acyclic complex)**.** Let $\mathcal{A}$ be an additive category, and let $(C_\bullet, d_\bullet)$ be a complex in $\mathcal{A}$:

$$\cdots \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots .$$

The complex $(C_\bullet, d_\bullet)$ is called *acyclic at* C_n (or sometimes synonymously *exact at* C_n) if we have $\ker d_n \cong \mathrm{im} d_{n+1}$.

If $\mathcal{A}$ is an abelian category, then this is equivalent to the condition that the (co)homology objects $H^n(C_\bullet) := \ker d_n / \operatorname{im} d_{n+1}$ are zero in $\mathcal{A}$.

We furthermore say that the complex $(C_\bullet, d_\bullet)$ is *acyclic* or *exact* if it is acyclic/exact everywhere.

Definition E.0.3. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces (Definition B.0.1). A function $f : X \to Y$ is called a *homeomorphism* if it satisfies all of the following:

1. f is bijective (Definition E.0.1);
2. f is continuous with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$; and
3. the inverse map $f^{-1} : Y \to X$ (Definition E.0.1) is also continuous.

If such a function exists, the spaces X and Y are said to be *homeomorphic*.

Definition E.0.4 (Subspace topology and subspace)**.** Let $(X, \mathcal{T})$ be a topological space and $Y \subseteq X$ a subset. The topology induced on Y by $\mathcal{T}$, called the *subspace topology*, is defined as

$$\mathcal{T}_Y := \{U \cap Y : U \in \mathcal{T}\}.$$

The pair $(Y, \mathcal{T}_Y)$ is called a *subspace of* $(X, \mathcal{T})$.

Definition E.0.5 (Monomorphism and Epimorphism in Categories)**.** Let $\mathcal{C}$ be a category (Definition C.0.1). For objects $A, B \in \mathcal{C}$, let $f : A \to B$ be a morphism in $\mathcal{C}$.

- The morphism f is called a *monomorphism* (or a *monic morphism*) if for every object X and every pair of morphisms $g_1, g_2 : X \to A$, the equality $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.
- The morphism f is called an *epimorphism* (or an *epic morphism*) if for every object Y and every pair of morphisms $h_1, h_2 : B \to Y$, the equality $h_1 \circ f = h_2 \circ f$ implies $h_1 = h_2$.

Definition E.0.6. Let X be a topological space (Definition B.0.1). The space X is said to be an *irreducible topological space* if X is non-empty and cannot be expressed as the union of two proper closed subsets. That is, if $X = Z_1 \cup Z_2$ with Z_1, Z_2 closed in X, then either $Z_1 = X$ or $Z_2 = X$. Equivalently, X is irreducible if every non-empty open subset of X is dense in X.

Definition E.0.7. (♠ TODO: define coset, kernel of R-module homomorphism) Let R, S be (not necessarily commutative) rings (Definition A.0.1).

1. Let M be an R-S-bimodule (Definition A.0.8). Let $N \subseteq M$ be a submodule of M.
   The quotient group M/N, which is well defined as M is an abelian group (Definition E.0.17) and hence N is a normal subgroup, has the structure of an R-S-bimodule — the (abelian) group structure is simply the group structure of M/N, whereas the R-S-bimodule structure is given as follows: for $m \in M$, $r \in R$, $s \in S$, we have
   $$r \cdot (m + N) \cdot s = r \cdot m \cdot s + N.$$
   This R-S-bidmodule structure on M/N is called the *quotient R-S-bidmodule of M by N* and is also denoted as M/N.
   The canonical projection map
   $$\pi : M \to M/N, \quad m \mapsto m + N,$$
   is a surjective R-module homomorphism with kernel N.
2. Let M be a left/right/two-sided R-module. Let $N \subseteq M$ be a submodule of M. The *quotient R-module* M/N is the quotient of M by N for their respective natural bimodule structures (Definition A.0.8).

Definition E.0.8. Let R be a ring, and let I, J be left ideals of R (Definition E.0.19) (resp. right ideals, resp. two-sided ideals).

The *product ideal* IJ is the left (resp. right, resp. two-sided) ideal defined as the additive subgroup of R generated by (Definition E.0.20) all products xy where $x \in I$ and $y \in J$. In other words,

$$IJ = \left\{ \sum_{k=1}^{n} x_k y_k : n \geq 0, x_k \in I, y_k \in J \right\}.$$

More generally, we may speak of the product of finitely many ideals of R.

Definition E.0.9. Let R be a (not necessarily commutative) ring (Definition A.0.1). Depending on the module structure of M, we define its dual module as follows:

1. If M is a left R-module (Definition A.0.8), then the *(right) dual module of M* is
   $$M^* = M^\vee := \mathrm{Hom}_R(M, R).$$
   Note that it is a right R-module, as M is a $R-\mathbb{Z}$-bimodule and R is an $R-R$-bimodule.
2. If M is a right R-module (Definition A.0.8), then the *(left) dual module of M* is
   $${}^*M = {}^\vee M := \mathrm{Hom}_R(M, R).$$
   Note that it is a left R-module, as M is a $\mathbb{Z}-R$-bimodule and R is an $R-R$-bimodule.
3. If M is a two-sided R-module, then the *dual of M* usually refers to either the right or the left dual as above.

In any case, the functor $M \mapsto M^\vee$ is a contravariant functor (Definition C.0.4) from the appropriate category of modules to itself.

If R is a field (Definition E.0.38) F and V is an F-vector space, then the dual module

$$V^* = V^\vee := \operatorname{Hom}_F(V, F)$$

is called the *dual vector space of* V.

Definition E.0.10 (Closure of a subset)**.** Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$ a subset. The *closure of* A *in* X, denoted by $\overline{A}$, is defined as the intersection of all closed sets containing A, i.e.,

$$\overline{A} := \bigcap \{C \subseteq X : C \text{ is closed and } A \subseteq C\}.$$

Equivalently, $\overline{A}$ is the smallest closed set containing A.

Definition E.0.11 (Dense subset)**.** Let $(X, \mathcal{T})$ be a topological space. A subset $D \subseteq X$ is called *dense in* X if the closure of D equals X, i.e.,

$$\overline{D} = X.$$

Equivalently, D is dense in X if every non-empty open set $U \in \mathcal{T}$ intersects D, that is,

$$\forall U \in \mathcal{T},\ U \neq \emptyset \implies U \cap D \neq \emptyset.$$

Definition E.0.12. Let X be a topological space (Definition B.0.1). A point $\eta \in X$ is called a *generic point of* X if the closure (Definition E.0.10) of the singleton set $\{\eta\}$ is the entire space X, i.e., $\overline{\{\eta\}} = X$. More generally, if Z is an irreducible closed subset of X, a point $\eta \in Z$ is called a *generic point of* Z if $\overline{\{\eta\}} = Z$. In the context of schemes, every irreducible closed subset has a unique generic point.

Definition E.0.13. Let $\mathcal{C}$ be a (large) category (Definition C.0.1) with a final object (Definition E.0.28). A *monoid object in* $\mathcal{C}$ is a semigroup object (Definition E.0.25) (A, μ) together with a *unit morphism*

$$\eta : 1 \to A$$

such that the products (Definition E.0.26) $1 \times A$ and $A \times 1$ exist and the unit diagrams

$$\begin{tikzcd} 1 \times A \arrow[rr, "\eta \times \mathrm{id}_A"] \arrow[dr, "\mathrm{pr}_2"'] & & A \times A \arrow[dl, "\mu"] \\ & A & \end{tikzcd}$$

$$\begin{tikzcd} A \times 1 \arrow[rr, "\mathrm{id}_A \times \eta"] \arrow[dr, "\mathrm{pr}_1"'] & & A \times A \arrow[dl, "\mu"] \\ & A & \end{tikzcd}$$

commute.

Definition E.0.14. Let $\mathcal{C}$ be a (large) category (Definition C.0.1) with a final object (Definition E.0.28). A *group object in $\mathcal{C}$* is a monoid object (Definition E.0.13) (A, μ, η) together with a *inverse morphism*

$$\iota : A \to A$$

such that the diagrams

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[dr, "\eta\circ !_A"'] & A \times A \arrow[d, "\mu\circ(\mathrm{id}_A\times\iota)"] \\ & A \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[dr, "\eta\circ !_A"'] & A \times A \arrow[d, "\mu\circ(\iota\times\mathrm{id}_A)"] \\ & A \end{tikzcd}$$

commute, where $\Delta : A \to A \times A$ is the diagonal and $!_A : A \to 1$ is the unique morphism.

Definition E.0.15. The category of sets is the (locally small) (Definition C.0.2) category (Definition C.0.1)

- whose objects are sets, and
- whose morphisms $X \to Y$ are set functions $X \to Y$.

The category of sets is often denoted by notations such as Set, **Set**, Sets, **Sets**, (Set), (**Set**), (Sets), (**Sets**).

Definition E.0.16 (Monoid)**.** A *monoid* is a semigroup $(M, \cdot)$ such that there exists an element $e \in M$, called the *identity element*, with the property:

$$e \cdot a = a \cdot e = a \quad \text{for all } a \in M.$$

Equivalently, a monoid is a monoid object (Definition E.0.13) in the category of sets (Definition E.0.15).

Definition E.0.17 (Groups)**.** A *group* is a pair $(G, \cdot)$ where G is a set and $\cdot : G \times G \to G$ is a binary operation, subject to the following conditions:

1. (Associativity) For all $g, h, k \in G$ one has
$$(g \cdot h) \cdot k = g \cdot (h \cdot k).$$
2. (Identity element) There exists an element $e \in G$ such that for all $g \in G$,
$$e \cdot g = g \cdot e = g.$$
3. (Inverse element) For all $g \in G$ there exists an element $g^{-1} \in G$ such that
$$g \cdot g^{-1} = g^{-1} \cdot g = e.$$

The element e is called the *identity element of G*, and g^{-1} is called the *inverse of g*.

Equivalently, a group is a monoid (Definition E.0.16) with inverse elements.

Equivalently, a group is a group object (Definition E.0.14) in the category of sets (Definition E.0.15).

A group $(G, \cdot)$ is often simply written as G, when the notation for the binary operation $\cdot$ is clear.

An *abelian group* or synonymously, a *commutative group*, is a group $(G, \cdot)$ whose binary operation $\cdot$ is *abelian* or *commutative*, i.e. satisfies

$$g \cdot h = h \cdot g$$

for all $g, h \in G$.

An abelian group is equivalent to a $\mathbb{Z}$-module.

Definition E.0.18 (Coproduct of Modules)**.** Let R and S be (not necessarily commutative) rings (Definition A.0.1), and let $\{M_i\}_{i\in I}$ be a (possibly infinite but small) family of (R, S)-bimodules.

The *coproduct (direct sum) of the family* $\{M_i\}_{i\in I}$, denoted by $\bigoplus_{i\in I} M_i$, is constructed as

$$\bigoplus_{i\in I} M_i := \left\{ (m_i)_{i\in I} \in \prod_{i\in I} M_i \mid m_i = 0 \text{ for all but finitely many } i \in I \right\}$$

consisting of all tuples with only finitely many nonzero entries.

Addition and scalar multiplication in $\bigoplus_{i\in I} M_i$ are defined componentwise as in the direct product:

$$(m_i)_{i\in I} + (n_i)_{i\in I} := (m_i + n_i)_{i\in I}, \qquad r \cdot (m_i)_{i\in I} \cdot s := (r \cdot m_i \cdot s)_{i\in I}, \qquad r \in R,\ s \in S.$$

In all cases, the zero element is $(0)_{i\in I}$, and additive inverses are given by $-(m_i)_{i\in I} := (-m_i)_{i\in I}$.

Note that we can define the coproduct of a family $\{M_i\}_{i\in I}$ of left/right/two-sided R-modules by taking the natural bimodule structure (Definition A.0.8) of each module.

(♠ TODO: submodule) Note that $\bigoplus_{i\in I} M_i$ is a submodule of $\prod_{i\in I} M_i$. Moreover, $\bigoplus_{i\in I} M_i$ is the coproduct (Definition E.0.26) in the appropriate category of modules.

For finitely many modules $M_1, \ldots, M_n$, the direct sum $\bigoplus_{j=1}^{n} M_j$, which may also be written as $M_1 \oplus \cdots \oplus M_n$, is simply the usual Cartesian product $\prod_{j=1}^{n} M_j$ of the modules, as every tuple automatically has only finitely many nonzero entries.

Definition E.0.19. Let R be a (not necessarily commutative, possibly nonunital) ring (Definition A.0.1). A *left ideal of R* is a subset $I \subseteq R$ such that

- $(I, +)$ is an additive subgroup of $(R, +)$,
- $RI \subseteq I$, i.e., for all $r \in R$ and $x \in I$, one has $rx \in I$.

Similarly, a *right ideal of R* is a subset $I \subseteq R$ such that

- $(I, +)$ is an additive subgroup of $(R, +)$,
- $IR \subseteq I$, i.e., for all $r \in R$ and $x \in I$, one has $xr \in I$.

A *two-sided ideal* (or simply an *ideal*) of R is a subset $I \subseteq R$ which is both a left ideal and a right ideal of R. We denote by $I \trianglelefteq R$ the relation expressing that I is a two-sided ideal of R.

Equivalently, an left/right/two-sided ideal of R is a submodule of R as an R-module (Definition A.0.8).

A left/right/two-sided ideal is said to be *proper* if it is strictly contained in R.

Note that every left or right ideal of a commutative ring is a two-sided ideal.

Definition E.0.20. Let R be a (not necessarily commutative) ring (Definition A.0.1), and let $X \subseteq R$ be a subset.

The *left ideal generated by X* is the smallest left ideal (Definition E.0.19) of R containing X; it equals the set of all finite sums of elements of the form rx with $r \in R$ and $x \in X$.

Similarly, the *right ideal generated by X* is the smallest right ideal of R containing X; it equals the set of all finite sums of elements of the form xr with $x \in X$ and $r \in R$.

The *two-sided ideal generated by X* is the smallest two-sided ideal of R containing X; it equals the set of all finite sums of elements of the form rxs with $r, s \in R$ and $x \in X$.

A left/right/two-sided ideal I of R is said to be *finitely generated* if there exists some finite subset X of R such that I equals the left/right/two-sided ideal generated by X. Moreover, I is said to be *principal* if there exists some subset X of R of cardinality 1 such that I equals the left/right/two-sided ideal generated by X.

Definition E.0.21 (Partially ordered set).

1. A *partially ordered set* (or *poset*), or *ordered set* is a pair $(P, \leq)$ where P is a set and
$$\leq : P \times P \to \{\text{true}, \text{false}\}$$
is a binary relation on P satisfying the following axioms for all $a, b, c \in P$:
   - *Reflexivity:* $a \leq a$,
   - *Antisymmetry:* if $a \leq b$ and $b \leq a$, then $a = b$,
   - *Transitivity:* if $a \leq b$ and $b \leq c$, then $a \leq c$.

   The relation $\leq$ is called an *order* or a *partial order*
2. A partially ordered set $(P, \leq)$ is called a *directed partially ordered set* if for every pair $a, b \in P$, there exists $c \in P$ such that
$$a \leq c \quad \text{and} \quad b \leq c.$$
3. A partially ordered set $(P, \leq)$ is called a *codirected partially ordered set* (or *downward directed poset*) if for every pair $a, b \in P$, there exists $d \in P$ such that
$$d \leq a \quad \text{and} \quad d \leq b.$$

Definition E.0.22 (Total order on a set)**.** (♠ TODO: relation) Let X be a set, and let $\leq$ be a binary relation on X, that is, a subset of $X \times X$. The relation $\leq$ is called a *total order* (or *linear order*) on X if for all $x, y, z \in X$ the following conditions hold:

1. (Reflexivity) $x \leq x$.
2. (Antisymmetry) If $x \leq y$ and $y \leq x$, then $x = y$.
3. (Transitivity) If $x \leq y$ and $y \leq z$, then $x \leq z$.
4. (Totality) Either $x \leq y$ or $y \leq x$ (possibly both).

In this situation, the pair $(X, \leq)$ is called a *totally ordered set*.

Definition E.0.23. Let X be a set and let $\mathcal{B}$ be a collection of subsets of X. The collection $\mathcal{B}$ is called a *basis* (or *base*) for a topology (Definition B.0.1) on X if the following two conditions hold:

1. For every $x \in X$, there exists at least one $B \in \mathcal{B}$ such that $x \in B$.
2. For any $B_1, B_2 \in \mathcal{B}$ and any $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Given such a collection $\mathcal{B}$, the collection $\mathcal{T}$ of all unions of elements of $\mathcal{B}$ defines a topology on X, and it coincides with $\mathcal{T}_{\mathcal{B}}$, the topology generated by $\mathcal{B}$. In other words,

$$\mathcal{T}_{\mathcal{B}} = \{U \subseteq X : \text{for every } x \in U, \text{ there exists } B \in \mathcal{B} \text{ with } x \in B \subseteq U\}.$$

Definition E.0.24. Let S be a set equipped with a binary operation $\cdot : S \times S \to S$. An element $e \in S$ is called a *left identity element* if $e \cdot x = x$ for all $x \in S$. Similarly, an element $e \in S$ is called a *right identity element* if $x \cdot e = x$ for all $x \in S$. An element $e \in S$ is called an *identity element* (or *neutral element*) if it is both a left and a right identity element. If an identity element exists, it is unique.

Definition E.0.25. Let $\mathcal{C}$ be a (large) category (Definition C.0.1). A *semigroup object in $\mathcal{C}$* is an object $A \in \mathcal{C}$ such that the product (Definition E.0.26) $A \times A$ exists in $\mathcal{C}$ together with a morphism

$$\mu : A \times A \to A,$$

called the *multiplication morphism* such that the associativity diagram

$$\begin{tikzcd}
A \times A \times A \arrow[r, "\mu \times \mathrm{id}_A"] \arrow[d, "\mathrm{id}_A \times \mu"'] & A \times A \arrow[d, "\mu"] \\
A \times A \arrow[r, "\mu"'] & A
\end{tikzcd}$$

commutes.

The semigroup object structure (A, μ, η, ι) is said to be *abelian* or *commutative* if the morphisms $\mu : A \times A \to A$ and $\mu \circ \tau_{A,A} : A \times A \to A$ coincide, where $\tau_{A,A} : A \times A \to A \times A$ is the symmetry morphism swapping the two factors.

Definition E.0.26 (Product in a category)**.** Let $\mathcal{C}$ be a category and let $\{X_i\}_{i \in I}$ be a family of objects in $\mathcal{C}$ indexed by a class I.

1. A *product of the family* $\{X_i\}$ is an object P of $\mathcal{C}$ together with a "universal" family of morphisms
$$\pi_i : P \to X_i, \quad \text{for each } i \in I.$$
More precisely, for any object Y and any family of morphisms $\{f_i : Y \to X_i\}_{i\in I}$, there exists a unique morphism
$$f : Y \to P$$
making the following diagram commute for all $i \in I$, i.e. $\pi_i \circ f = f_i$:
$$\begin{tikzcd} Y \arrow[d, dashed, "\exists! f"] \arrow[dr, "f_i"] & \\ \prod X_i \arrow[r, "\pi_i"'] & X_i \end{tikzcd}$$
Such a product is often denoted by $\prod_{i\in I} X_i$. If $\prod_{i\in I} X_i$ exists in $\mathcal{C}$, then it is unique up to unique isomorphism by the universal property described above.
Equivalently, the product $\prod_{i\in I} X_i$ is the limit (Definition C.1.1) of the diagram $I \to \mathcal{C}, i \mapsto X_i$, where I is made into a category whose objects are the members of I and whose morphisms are just the identity morphisms.
2. A *coproduct* (or synonymously *direct sum*) of the family $\{X_i\}$ is an object C of $\mathcal{C}$ together with a "universal" family of morphisms
$$\iota_i : X_i \to C, \quad \text{for each } i \in I.$$
More precisely, for any object Y and any family of morphisms $\{g_i : X_i \to Y\}_{i\in I}$, there exists a unique morphism
$$g : C \to Y$$
making the following diagram commute for all $i \in I$, i.e. $g \circ \iota_i = g_i$:
$$\begin{tikzcd} X_i \arrow[r, "\iota_i"] \arrow[dr, "g_i"'] & \coprod X_i \arrow[d, dashed, "\exists! g"] \\ & Y \end{tikzcd}$$
Such a coproduct is often denoted by $\coprod_{i\in I} X_i$ or $\oplus_{i\in I} X_i$. If $\coprod_{i\in I} X_i$ exists in $\mathcal{C}$, then it is unique up to unique isomorphism by the universal property described above.
Equivalently, the coproduct $\coprod_{i\in I} X_i$ is the colimit (Definition C.1.1) of the diagram $I \to \mathcal{C}, i \mapsto X_i$, where I is made into a category whose objects are the members of I and whose morphisms are just the identity morphisms.

Definition E.0.27. Let S be a scheme. An algebraic group scheme over S (or an S-group scheme) is a group object G in the category of schemes over S; that is, G is an S-scheme equipped with S-morphisms: $m : G \times_S G \to G$ (*multiplication*), $i : G \to G$ (*inverse*), and $e : S \to G$ (*identity*), satisfying the group axioms expressed by the commutativity of the following diagrams:

1. **Associativity** $$\begin{tikzcd} G \times_S G \times_S G \arrow[r, "m\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times m"'] & G \times_S G \arrow[d, "m"] \\ G \times_S G \arrow[r, "m"] & G \end{tikzcd}$$
2. **Identity** $$\begin{tikzcd} G \times_S S \arrow[r, "\mathrm{id}\times e"] \arrow[dr, "\simeq"'] & G \times_S G \arrow[d, "m"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} S \times_S G \arrow[r, "e\times\mathrm{id}"] \arrow[dr, "\simeq"'] & G \times_S G \arrow[d, "m"] \\ & G \end{tikzcd}$$
3. **Inverse** $$\begin{tikzcd} G \arrow[r, "(\mathrm{id},i)"] \arrow[d, "\mathrm{id}"'] & G \times_S G \arrow[d, "m"] \\ G \arrow[r, "e\circ\pi"] & G \end{tikzcd}$$ where $\pi : G \to S$ is the structure morphism and $e \circ \pi$ sends g to the identity section.

Equivalently, a group scheme over S is a group object (Definition E.0.14) in the category of S-schemes (Definition 1.4.10)

If G is affine over (Definition 3.2.28) S, we call it an *affine group scheme over S*.

If the base scheme S is the spectrum of a field k, then we call G a *k-algebraic group* or an *algebraic group (scheme) over k*. If G is additionally a k-variety, then we call G a *k-group variety*.

Definition E.0.28. Let $\mathcal{C}$ be a (large) category (Definition C.0.1).

1. An object $I \in \mathcal{C}$ is called an *initial object* if for every object $X \in \mathcal{C}$ there exists a unique morphism
$$I \to X.$$
Equivalently, an initial object is a limit (Definition C.1.1) of the empty diagram, if such a limit exists.
2. An object $F \in \mathcal{C}$ is called a *final object* (or *terminal object*) if for every object $X \in \mathcal{C}$ there exists a unique morphism
$$X \to F.$$
Equivalently, a final object is a colimit (Definition C.1.1) of the empty diagram, if such a colimit exists.
3. An object $Z \in \mathcal{C}$ is called a *zero object* if Z is both initial and final in $\mathcal{C}$. In particular, for every object $X \in \mathcal{C}$ there exist unique morphisms
$$Z \to X \quad \text{and} \quad X \to Z.$$

In particular, if initial/final/zero objects exist in a cateogry, then they are unique up to unique isomorphism.

Definition E.0.29. Let $\mathcal{C}$ be a category (Definition C.0.1) and let G be a semigroup object (Definition E.0.25) in $\mathcal{C}$. Let X be an object of $\mathcal{C}$ such that the product (Definition E.0.26) $G \times X$ (and hence $X \times G$) exists in $\mathcal{C}$.

- A *left action of G on X* is a morphism $\sigma : G \times X \to X$ such that the following diagram commutes (associativity):
$$\sigma \circ (m \times \mathrm{id}_X) = \sigma \circ (\mathrm{id}_G \times \sigma).$$
- A *right action of G on X* is a morphism $\rho : X \times G \to X$ such that:
$$\rho \circ (\mathrm{id}_X \times m) = \rho \circ (\rho \times \mathrm{id}_G).$$

Definition E.0.30. Let $\mathcal{C}$ be a category (Definition C.0.1) with a terminal object (Definition E.0.28) 1. Let (M, m, e) be a monoid object (Definition E.0.13) in $\mathcal{C}$, where $m : M \times M \to M$ is the multiplication and $e : 1 \to M$ is the unit morphism. Let X be an object of $\mathcal{C}$ such that $M \times X$ (and hence $X \times M$) exists in $\mathcal{C}$.

- A *left action of M on X* is a morphism $\sigma : M \times X \to X$ such that the following two diagrams commute:
  1. **Associativity:**
  $$\sigma \circ (m \times \mathrm{id}_X) = \sigma \circ (\mathrm{id}_M \times \sigma).$$
  2. **Unitality:**
  $$\sigma \circ (e \times \mathrm{id}_X) = \mathrm{id}_X,$$
  where we identify $1 \times X \cong X$ via the canonical isomorphism.
- A *right action of M on X* is a morphism $\rho : X \times M \to X$ such that:
  1. **Associativity:**
  $$\rho \circ (\mathrm{id}_X \times m) = \rho \circ (\rho \times \mathrm{id}_M).$$
  2. **Unitality:**
  $$\rho \circ (\mathrm{id}_X \times e) = \mathrm{id}_X,$$
  where we identify $X \times 1 \cong X$ via the canonical isomorphism.

In particular, an action by a monoid object is an action (Definition E.0.30) of the monoid object as a semigroup object (Definition E.0.25).

Definition E.0.31. Let $\mathcal{C}$ be a category (Definition C.0.1) with a terminal object (Definition E.0.28) 1. Let G be a group object (Definition E.0.14) in $\mathcal{C}$ and let X be an object of $\mathcal{C}$ such that $G \times X$ (and hence $X \times G$) exists in $\mathcal{C}$. A *left/right action of G on X* is a left/right action (Definition E.0.30) of G as a monoid object (Definition E.0.13) on X.

Definition E.0.32. Let S be a scheme and let G be a group scheme over S with multiplication $m : G \times_S G \to G$ and identity section $e : S \to G$. An *action of the group scheme G on an S-scheme X* is defined as follows:

- A *left action* is a morphism of S-schemes $\sigma : G \times_S X \to X$ satisfying:
  1. **Associativity:** The following diagram commutes:
  $$\sigma \circ (m \times_S \mathrm{id}_X) = \sigma \circ (\mathrm{id}_G \times_S \sigma).$$
  2. **Unitality:** The identity element acts trivially:
  $$\sigma \circ (e \times_S \mathrm{id}_X) = \mathrm{id}_X,$$
  where we identify $S \times_S X \cong X$.

- A *right action* is a morphism of S-schemes $\rho : X \times_S G \to X$ satisfying:
  1. **Associativity:** The following diagram commutes:
  $$\rho \circ (\mathrm{id}_X \times_S m) = \rho \circ (\rho \times_S \mathrm{id}_G).$$
  2. **Unitality:** The identity element acts trivially:
  $$\rho \circ (\mathrm{id}_X \times_S e) = \mathrm{id}_X,$$
  where we identify $X \times_S S \cong X$.

Eqiuvalently, An *left/right action of the group scheme G on an S-scheme X* is a left/right action (Definition E.0.31) of G on X as objects of the category of S-schemes (Definition 1.4.10).

Definition E.0.33 (Connected topological space)**.** Let X be a topological space (Definition B.0.1). The space X is called a *connected topological space*, or simply *connected*, if it cannot be written as a disjoint union $X = U \sqcup V$ of two nonempty open subsets $U, V \subseteq X$. Equivalently, X is connected if the only subsets of X that are both open and closed are $\varnothing$ and X itself.

Definition E.0.34. Let F be a field (Definition E.0.38), and let V be an F-vector space. A subset $B \subseteq V$ is called a *basis of V* if: (i) B is linearly independent over F, and (ii) B spans (Definition A.0.10) V.

If B is a basis, we define the *dimension of V over F* (or *rank of V over F*), denoted by

$$\dim_F(V),$$

(♠ TODO: cardinality) to be the cardinality of B. This value is uniquely determined by V and F.

Definition E.0.35 (Fiber of a map of topological spaces)**.** Let X and Y be topological spaces (Definition B.0.1) and let $f : X \to Y$ be a continuous map. For a point $y \in Y$, the *fiber of f over y* is the inverse image $f^{-1}(y) = f^{-1}(\{y\})$ endowed with the subspace topology (Definition E.0.4) induced from X. The fiber is also denoted by notations such as $\mathrm{Fib}_f(y)$ or X_y.

Definition E.0.36. A field (Definition E.0.38) F is said to be *algebraically closed* if every nonconstant polynomial $f(x) \in F[x]$ has a root in F, i.e., for every such $f(x)$ there exists $a \in F$ with $f(a) = 0$.

Definition E.0.37. Let K be a field. A field extension L/K is called an *algebraic closure of K* if the following two conditions hold:

1. Every element $a \in L$ is algebraic over K.
2. The field L is algebraically closed (Definition E.0.36).

One often writes an algebraic closure of K by $\overline{K}$.

Definition E.0.38 (Field)**.** A *field* is commutative division ring (Definition A.0.7). In other words, a field is a commutative ring for which all nonzero elements have a multiplicative inverse.

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