1. Chain complexes and splitting

(1.1) Show that homology preserves coproducts in R-mod: that is, if $\{C_x\}_{x\in X}$ is a collection of chain complexes, then

$$H_n(\bigoplus_{x\in X} C_x) \cong \bigoplus_{x\in X} H_n C_x$$

for all n.

(1.2) Show that the short exact sequence (with the first map given by multiplication by p)

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/p \longrightarrow 0$$

is not split.

(1.3) Let $R = \mathbb{Z}[t]/(1-t^2)$. Let $A = R/(1-t)$ and $B = R/(1+t)$, regarded as R-modules. Show that $R \cong A \oplus B$ as $\mathbb{Z}$-modules (i.e. abelian groups). Show that the short exact sequence of R-modules

$$0 \longrightarrow B \longrightarrow R \longrightarrow A \longrightarrow 0$$

is not split, where the map $B \to R$ is given by $1 \mapsto 1-t$. Does it split if $\mathbb{Z}$ is replaced by $\mathbb{Q}$? by $\mathbb{F}_2$?

(1.4) Suppose $f\colon C. \to D.$ is a chain map. Prove that f is a quasiisomorphism if and only if the complex $\operatorname{cone}(f)$ is exact.

(1.5) Let C be a chain complex, and $1\colon C \to C$ the identity map. Let $\operatorname{cone}(C)$ denote the mapping cone of 1. Show that $\operatorname{cone}(1)$ is split exact.

(1.6) Suppose $f\colon C. \to D.$ is a chain map. Show that f is null-homotopic via a chain homotopy s if and only if f extends to a chain map $\operatorname{cone}(C) \to D$ given by $(-s, f)$.