UWO MATH 9144B WINTER 2026

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Contents

0.1. Disclaimer 3
1. Basic category theory 3
1.1. Categories 3
1.2. Functors between categories 7
1.3. Natural transformation 10
1.3.1. Yoneda lemma 13
1.4. The category of categories 15
1.5. Miscellaneous categorical notions 16
2. Additive and abelian categories 17
2.1. Rings and modules 17
2.1.1. Extension/Restriction/co-extension of scalars 24
2.2. Limits and colimits in categories 27
2.2.1. Categorical products, coproducts, and direct sums 27
2.2.2. General limits and colimits 30
2.3. Additive categories 37
2.3.1. Additive and abelian categories 37
2.4. Additive functors between additive categories 41
2.4.1. Exact functors between abelian categories 42
2.5. Adjoint functors 48
2.5.1. Grothendieck's additional axioms for abelian categories 53
3. Chain complexes of objects in additive categories 56

3.1. Category of chain complexes as a functor category 58
3.2. Homology and cohomology of a chain complex 59
3.3. Singular homology groups of a topological space 64
3.4. Homotopy between morphisms of topological spaces 66
3.5. Chain homotopy between morphisms of chain complexes 67
3.6. Mapping cones of morphisms between topological spaces 68
3.7. Mapping cones of morphisms between chain complexes 68
4. Derived functors 68
4.1. δ-functors 69
4.2. Abelian categories with enough objects of a class and resolutions 70
4.3. Derived functors of right or left exact functors between abelian categories where the source category has enough projectives or injectives 78
4.4. Tor and Ext 82
4.5. Balancing Ext and Tor 83
4.5.1. Double complexes 83
4.5.2. Acyclic assembly lemma 85
5. Ext and Tor 89
6. Sheaves on topological spaces 89
6.0.1. Presheaves and sheaves on topological spaces 89
6.0.2. Sheafification of presheaves 92
6.0.3. Sheaf of rings, ringed spaces, and sheaves of modules 94
7. Left/right derived functors of right/left exact functors 94
8. 94
9. Assignment 1: Due Friday, Jan 23 95
10. Assignment 2: Due Feb 6 (Problems not yet complete) 97
11. Assignment 3: Due Feb 20 98
Appendix A. 99
Appendix B. Grothendieck topologies 101

Appendix C. Miscellaneous definitions 105

References 116

These are "notes" written for the course MATH 9144B Homological Algebra at the University of Western Ontario during the Winter 2026 term. The main text for the course is Weibel's *An Introduction to Homological Algebra* [Wei94], and I intend for the course to roughly follow parts of this text. This document aims to supplement the presentation of loc. cit. , which focuses much of its attention on left/right R-modules where R is an "associative ring"[1], by discussing general definitions.

Readers are recommended to read this document using a PDF reader (e.g. Adobe Acrobat/Reader) that supports Link Annotations (i.e. the boxes, usually red/green, that are rendered when using
`ref` and
`cite` commands).

My "notes" for Homological Algebra, Category Theory, Abstract Algebra, etc. may be of interest.

0.1. **Disclaimer.** The definitions and statements in this document are generally written less as basic introductions to any given concept and more so to present them in generality. As such, they are not necessarily linearly presented. Readers should judiciously decide what to read and what to skip.

Many statements are based on AI generated ones, and errors may be abound due to my own lack of care when verifying them. Readers are advised to exercise caution when citing claims, which may be erroneous, in these notes.

The contents of these notes may change constantly.

1. Basic category theory

1.1. Categories.

Definition 1.1.1 (Category)**.** A *category* $\mathcal{C}$ consists of the following data:

- A class of *objects* denoted $\mathrm{Ob}(\mathcal{C})$.
- For each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a class

$$\mathrm{Hom}_{\mathcal{C}}(X, Y)$$

[1] The introduction of [Wei94] assumes that the reader has a background in graduate algebra "based on a text such as *Jacobson's Basic Algebraic I*". Jacobson defines a ring to be both associative and unital (but not necessarily commutative), so ostensibly [Wei94] adopts this same convention.

of *morphisms* (also called *arrows* or *homs*). If the category $\mathcal{C}$ is clear, then this *hom-class* is also denoted by $\mathrm{Hom}(X, Y)$. It may also be denoted by $\mathrm{hom}_{\mathcal{C}}(X, Y)$ or $\mathrm{hom}(X, Y)$, especially to distinguish from other types of hom's (e.g. internal hom's)

- For each triple of objects X, Y, Z, a composition law

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z),$$

denoted $(g, f) \mapsto g \circ f$.

- For each object X, an *identity morphism*

$$\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X).$$

These data satisfy the following axioms:

- (Associativity) For all morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, and $h \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

- (Identity) For all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$,

$$\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X .$$

One often writes $X \in \mathcal{C}$ synonymously with $X \in \mathrm{Ob}(\mathcal{C})$, i.e. to denote that X is an object of of $\mathcal{C}$.

We may call a category as above an *ordinary category* to distinguish this notion from the notions of *categories enriched in monoidal categories* or higher/n-categories. (♠ TODO: TODO: define n-categories)

A category as defined above may be called called a *large category* or a *class category* to emphasize that the hom-classes may be proper classes rather than sets (note, however, that the possibility that hom-classes are sets is not excluded for large categories). Accordingly, a *category* may often refer to a locally small category (Definition 1.1.2), which is a category whose hom-classes are all sets.

Definition 1.1.2 (Locally small category)**.** A (large) category (Definition 1.1.1) $\mathcal{C}$ is called a *locally small category* if for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the collection $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms between them is a (small) *set* (as opposed to a proper class). In other words, each hom-class is a set and may even be called a *hom-set*.

In some contexts, a locally small category may simply be called a *category*, especially when genuinely large categories are not considered.

A category $\mathcal{C}$ is called a *small category* if it is a locally small category and the class $\mathrm{Ob}(\mathcal{C})$ of objects is a set.

Given a universe (Definition A.0.3) U, we can define the notion of a *U-locally small category* and of a *U-small category* similarly. More explicitly,

1. a U-locally small category is a category such that for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the collection $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms between them is a U-set.

2. a U-small category is a category such that $\mathrm{Ob}(\mathcal{C})$ is a U-set and for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the collection $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms between them is a U-set; in particular the collection of all objects and morhpisms in a U-small category is a U-set.

Remark 1.1.3. Many "concrete" categories considered in "classical mathematics" or outside of more "abstract" category theory tend to be locally small. For example, the categories of sets, groups, R-modules, vector spaces, topological spaces, schemes, manifolds, sheaves on "small enough" sites are all locally small.

Example 1.1.4. Here is an example of a "boring" category:

1. There is only one object, say X.
2. There is only one morphism, the identity $\mathrm{id}_X : X \to X$.

The composed morphism $\mathrm{id}_X \circ \mathrm{id}_X$ is then just id_X, and associativity automatically holds.

Example 1.1.5. Any poset (Definition C.0.17) $(P, \leq)$ induces a category (Definition 1.1.1) — let the objects be the elements of the set P, and let the morphisms/arrows be given as follows: there is a unique arrow $a \to b$ whenever $a \leq b$. Composition of arrows works as follows: given arrows $a \to b$ and $b \to c$, the composed arrow $a \to b \to c$ will be the unique arrow $a \to c$ corresponding to $a \leq c$. See Lemma C.0.18

Example 1.1.6. Here are common examples of categories:

1. The category of sets (Definition 1.1.7), whose objects are sets and whose morphisms are set maps/functions (Definition C.0.1).
2. The category of groups (Definition 1.1.8), whose objects are groups (Definition C.0.3) and whose morphisms are group homomorphisms (Definition C.0.4).
3. The category of abelian groups (Definition 1.1.8), whose objects are abelian groups (Definition C.0.3) and whose morphisms are group homomorphisms (Definition C.0.4).
4. The category of topological spaces (Definition 1.1.9), whose objects are topological spaces (Definition C.0.5) and whose morphisms are continuous maps between topological spaces (Definition C.0.6). .
5. The category of pointed topological spaces (Definition 1.1.10), whose objects (X, x) are pointed topological spaces (Definition 1.1.10) and whose morphisms $(X, x) \to (Y, y)$ are continuous maps (Definition C.0.6) $f : X \to Y$ such that $f(x) = y$.
6. The category of vector spaces (Definition 1.1.11) over a fixed field (Definition C.0.12) k whose objects are vector spaces (Definition C.0.14) over k and whose morphisms are k-linear maps (Definition C.0.15).
7. The category of finite dimensional vector spaces (Definition 1.1.11) over a fixed field (Definition C.0.12) k whose objects are finite dimensional (Definition C.0.16) vector spaces (Definition C.0.14) over k and whose morphisms are k-linear maps (Definition C.0.15).
8. Given a ring (Definition C.0.7) R, the category of (either left or right) R-modules (Definition 2.1.3) whose objects are R-modules (Definition 2.1.1) and whose morphisms are R-module homomorphisms (Definition 2.1.2).
9. The category of rings (Definition 1.1.12) whose objects are rings (Definition C.0.7) and whose morphisms are ring homomorphisms (Definition C.0.13).

10. The category of commutative rings (Definition 1.1.12) whose objects are commutative rings (Definition C.0.9) and whose morphisms are ring homomorphisms (Definition C.0.13).
11. The category of small categories (Definition 1.4.1), whose objects are the small categories (Definition 1.1.2) and whose morphisms are functors (Definition 1.2.2).
12. Given a fixed topological space X, the category $\mathrm{Open}(X)$ of open subsets of X (Definition 6.0.2), whose objects are the open subsets of X and whose morphisms $U \to V$ are given exactly by inclusions $U \subseteq V$. More precisely, for each inclusion $U \subseteq V$ of open subsets of X, there is a unique morphism $U \to V$, and the composition $U \to V \to W$ is the unique morphism $U \to W$ corresponding to the inclusion $U \subseteq W$.

Definition 1.1.7. The category of sets is the (locally small) (Definition 1.1.2) category (Definition 1.1.1)

- whose objects are sets, and
- whose morphisms $X \to Y$ are set functions (Definition C.0.1) $X \to Y$.

The category of sets is often denoted by notations such as Set, **Set**, Sets, **Sets**, (Set), (**Set**), (Sets), (**Sets**).

Definition 1.1.8. 1. The *category of groups* is the locally small (Definition 1.1.2) category (Definition 1.1.1) whose objects are groups (Definition C.0.3) and whose morphisms are group homomorphisms (Definition C.0.4). It is often denoted by notations such as **Grp**.
2. The *category of abelian groups* is the locally small (Definition 1.1.2) category (Definition 1.1.1) whose objects are abelian groups (Definition C.0.3) and whose morphisms are group homomorphisms (Definition C.0.4). It is often denoted by notations such as **Ab**.

Definition 1.1.9. The *category of topological spaces* is the (locally small) (Definition 1.1.2) category (Definition 1.1.1)

- whose objects are topological spaces (Definition C.0.5), and
- whose morphisms are continuous maps (Definition C.0.6).

The category of topological spaces is often denoted by notations such as Top, **Top**, etc.

Definition 1.1.10 (Pointed topological space)**.** Let X be a topological space (Definition C.0.5) and let $x_0 \in X$ be a chosen element of X. A *pointed/based (topological) space* is a pair (X, x_0) consisting of the space X together with the distinguished point x_0, called the *base point of X*. If the base point of a pointed space (X, x_0) is understood, then it may be suppressed from notation; in particular, X may be written as a pointed space as opposed to the full notation of (X, x_0).

A *morphism of pointed spaces* (or *based map*) or *continuous map* between pointed spaces (X, x_0) and (Y, y_0) is a continuous map (Definition C.0.6)

$$f : X \to Y$$

such that $f(x_0) = y_0$.

The collection of pointed spaces with their morphisms form a locally small (Definition 1.1.2) category (Definition 1.1.1), often called the *category of pointed spaces*. This category is often denoted by notations such as Top_*, $\mathrm{Top}_\bullet$, $\mathbf{Top}_*$, $\mathbf{Top}_\bullet$, etc. The set of continuous maps from pointed spaces X to Y may denoted by notations such as $C_*(X,Y)$, $C_\bullet(X,Y)$, $\mathrm{Top}_*(X,Y)$, $\mathrm{Top}_\bullet(X,Y)$, $\mathrm{Hom}_{\mathrm{Top}_\bullet}(X,Y)$, etc.

Definition 1.1.11. Let k be a field (Definition C.0.12). The *category of vector spaces over k* is the locally small (Definition 1.1.2) category (Definition 1.1.1)

- whose objects are vector spaces over k (Definition C.0.14), and
- whose morphisms are k-linear maps (Definition C.0.15).

The F-vector spaces of finite dimension (Definition C.0.16) form a full subcategory (Definition 1.3.7), called the *category of finite dimensional vector spaces over k*. Notations such as Vec_k or $\mathbf{Vec}_k$ are often used to denote either of these categories; when both categories are considered, notations such as FinVec_k or $\mathbf{FinVec}_k$ may be used to distinguish the category of finite dimensional k-vector spaces from the category of all k-vector spaces.

Definition 1.1.12.

1. The *category of rings* is the locally small (Definition 1.1.2) category (Definition 1.1.1) whose objects are rings (Definition C.0.7) R and whose morphisms $R \to S$ are ring homomorphisms (Definition C.0.13). The category of rings over R is often denoted by notations such as **Ring**.
2. The *category of commutative rings* is the full subcategory (Definition 1.3.7) of **Ring** consisting of the commutative rings (Definition C.0.9). It is denoted by notations such as **CommRing** or **CRing**.

Definition 1.1.13 (Isomorphism in a category)**.** Let $\mathcal{C}$ be a (large) category (Definition 1.1.1), and let $x, y \in \mathrm{Ob}(\mathcal{C})$. A morphism $f \in \mathcal{C}(x,y)$ is called an *isomorphism* if there exists a morphism $g \in \mathcal{C}(y,x)$ such that

$$g \circ f = 1_x \qquad \text{and} \qquad f \circ g = 1_y.$$

In this case, g is called the *inverse of f*, and x and y are said to be *isomorphic objects* in $\mathcal{C}$. It is standard to write $x \cong y$ if there exists an isomorphism $f : x \to y$.

In practice, isomorphisms in specific categories may be defined in different, yet equivalent, ways.

1.2. **Functors between categories.**

Definition 1.2.1 (Opposite category)**.** Let $\mathcal{C}$ be a (large) category (Definition 1.1.1). The *opposite category* of $\mathcal{C}$, denoted $\mathcal{C}^{\mathrm{op}}$, is defined as follows:

- The objects of $\mathcal{C}^{\mathrm{op}}$ are the same as those of $\mathcal{C}$.
- For any pair of objects $X, Y \in \mathcal{C}$, the morphisms from X to Y in $\mathcal{C}^{\mathrm{op}}$ are given by the morphisms from Y to X in $\mathcal{C}$:

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(Y,X).$$

- Composition in $\mathcal{C}^{\text{op}}$ is defined by reversing the order of composition in $\mathcal{C}$. That is, for morphisms $f \in \text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y)$ and $g \in \text{Hom}_{\mathcal{C}^{\text{op}}}(Y, Z)$, their composition is

$$g \circ_{\mathcal{C}^{\text{op}}} f := f \circ_{\mathcal{C}} g.$$

Intuitively, the category $\mathcal{C}^{\text{op}}$ thus "reverses" the direction of all morphisms in $\mathcal{C}$.

Definition 1.2.2. Let $\mathcal{C}$ and $\mathcal{D}$ be (large) categories (Definition 1.1.1).

1. A *functor $F : \mathcal{C} \to \mathcal{D}$ (from $\mathcal{C}$ to $\mathcal{D}$)* consists of :
   - For each object X in $\mathcal{C}$, an object $F(X)$ in $\mathcal{D}$.
   - For each morphism $f : X \to Y$ in $\mathcal{C}$, a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$,

   such that:

$$\begin{aligned} F(\text{id}_X) &= \text{id}_{F(X)} \quad \text{for all objects } X \text{ in } \mathcal{C}, \\ F(g \circ f) &= F(g) \circ F(f) \quad \text{for all } X, Y, Z \in \text{Ob}(\mathcal{C}) \text{ and all } f : X \to Y, g : Y \to Z \text{ in } \mathcal{C}. \end{aligned}$$

   Functors as defined above are also referred to as *covariant functors* to distinguish them from contravariant functors
2. A *contravariant functor from $\mathcal{C}$ to $\mathcal{D}$* refers to a covariant functor $F : \mathcal{C}^{\text{op}} \to \mathcal{D}$. Equivalently, such a functor consists of
   - For each object X in $\mathcal{C}$, an object $F(X)$ in $\mathcal{D}$.
   - For each morphism $f : X \to Y$ in $\mathcal{C}$, a morphism $F(f) : F(Y) \to F(X)$ in $\mathcal{D}$,

   such that:

$$\begin{aligned} F(\text{id}_X) &= \text{id}_{F(X)} \quad \text{for all objects } X \text{ in } \mathcal{C}, \\ F(g \circ f) &= F(f) \circ F(g) \quad \text{for all } X, Y, Z \in \text{Ob}(\mathcal{C}) \text{ and all } f : X \to Y, g : Y \to Z \text{ in } \mathcal{C}. \end{aligned}$$

   A synonym for a "contravariant functor from $\mathcal{C}$ to $\mathcal{D}$" is a "presheaf on $\mathcal{C}$ with values in $\mathcal{D}$ (Definition 6.0.1)".

Note that declarations such as "Let $F : \mathcal{C}^{\text{op}} \to \mathcal{D}$ be a contravariant functor" can be common; such declarations usually mean "Let F be a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$" as opposed to "Let F be a contravariant functor from $\mathcal{C}^{\text{op}}$ to $\mathcal{D}$". further note that a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is equivalent to a covariant functor from $\mathcal{C}^{\text{op}}$ to $\mathcal{D}$.

Example 1.2.3. Here are some examples of functors (Definition 1.2.2):

1. For any category $\mathcal{C}$, its identity functor (Definition 1.2.4).
2. "forgetful functors"; some forgetful functors (Definition 1.2.5) include
   (a) The forgetful functor $F : \textbf{Grp} \to \textbf{Sets}$ (Definition 1.1.8) (Definition 1.1.7) sending a group (Definition C.0.3) G to the underlying set of G, and sending a group homomorphism (Definition C.0.4) $G_1 \to G_2$ to the set function (Definition C.0.1) $G_1 \to G_2$. One can verify that

$$\begin{aligned} F(\text{id}_X) &= \text{id}_{F(X)} \quad \text{for all objects } G \text{ in } \textbf{Grp}, \\ F(g \circ f) &= F(g) \circ F(f) \quad \text{for all } G_1, G_2, G_3 \in \text{Ob}(\textbf{Grp}) \text{ and all } f : G_1 \to G_2, g : G_2 \to G_3 \text{ in } \textbf{Grp}. \end{aligned}$$

   (b) Similarly, the forgetful functor $F : \textbf{Top} \to \textbf{Sets}$ (Definition 1.1.9) (Definition 1.1.7) sending a topological space (Definition 1.1.9) X to the underlying

set of X, and sending a continuous map (Definition C.0.6) $X \to Y$ to the set function (Definition C.0.1) $X \to Y$.
   (c) The forgetful functor $F : \mathbf{Ab} \to \mathbf{Grp}$ (Definition 1.1.8) sending an abelian group A to itself, and sending a group homomorphism $A_1 \to A_2$ to itself.
3. "Free" functors
   (a) There is a functor $\mathbf{Sets} \to \mathbf{Grp}$ sending a set S to the free group (Definition C.0.19) $\langle S \rangle$ generated by S. The functor sends the morphism $f : S_1 \to S_2$ of sets, to the unique group homomorphism (Definition C.0.4) $\langle S_1 \rangle \to \langle S_2 \rangle$ given by sending $s \in S_1$ to $f(s) \in S_2$.
   (b) Similarly, there is a functor $\mathbf{Sets} \to \mathbf{Ab}$ sending a set S to the free abelian group (Definition C.0.20) $\mathbb{Z}S$ generated by S. The functor sends the morphism $f : S_1 \to S_2$ of sets, to the unique group homomorphism (Definition C.0.4) $\mathbb{Z}S_1 \to \mathbb{Z}S_2$ given by sending $s \in S_1$ to $f(s) \in S_2$.
4. The fundamental group functor $\mathbf{Top}_\bullet \to \mathbf{Grp}$ (Definition 1.1.10); given a pointed topological space (Definition 1.1.10) (X, x), its associated fundamental group (Definition C.0.21) $\pi_1(X, x)$ is the group of homotopy classes of loops $\gamma : [0, 1] \to X$. Given a morphism $(X, x) \to (Y, y)$ of pointed topological spaces, there is an (functorially) induced morphism $f : \pi_1(X, x) \to \pi_1(Y, Y)$ sending the homotopy class $[\gamma]$ of the loop $\gamma : [0, 1] \to X$ to the homotopy class $[f \circ \gamma]$.

Definition 1.2.4. Let $\mathcal{C}$ be a category (Definition 1.1.1). The *identity functor on $\mathcal{C}$* is the functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ (also denoted by $\mathrm{id}_{\mathcal{C}}$) defined by the following data:

- For every object $X \in \mathrm{Ob}(\mathcal{C})$, $1_{\mathcal{C}}(X) = X$.
- For every morphism $f : X \to Y$ in $\mathcal{C}$, $1_{\mathcal{C}}(f) = f$.

It satisfies the functor axioms trivially: $1_{\mathcal{C}}(f \circ g) = f \circ g = 1_{\mathcal{C}}(f) \circ 1_{\mathcal{C}}(g)$ and $1_{\mathcal{C}}(\mathrm{id}_X) = \mathrm{id}_X$.

Definition 1.2.5. Let $\mathcal{C}$ and $\mathcal{D}$ be categories (Definition 1.1.1). A functor (Definition 1.2.2) $U : \mathcal{C} \to \mathcal{D}$ is called a *forgetful functor* if it maps an object in $\mathcal{C}$ to an object in $\mathcal{D}$ by discarding some of its structure or properties, and maps morphisms accordingly. Common examples include the functor from the category of groups to the category of sets, or from the category of topological spaces to the category of sets.

Example 1.2.6. Here are some examples of contravariant functors (Definition 1.2.2).

1. The dual of a vector space: given a vector space (Definition C.0.14) V over a field k, the dual (Definition 2.1.12) is defined as $V^\vee := \mathrm{Hom}_k(V, k)$. The assignment $V \mapsto V^\vee$ specifies a contravariant functor $\mathrm{Vec}_k \to \mathrm{Vec}_k$ — given a linear map (Definition C.0.15) $f : V_1 \to V_2$, there is an induced linear map $V_2^\vee \to V_1^\vee$ given by sending the linear map $\phi : V_2 \to k$, which is an element of $V_2^\vee$, to the linear map $V_2 \circ f V_1 \to k$, which is an element of $V_1^\vee$.
2. A representable functor (Definition 1.3.12): given any locally small category (Definition 1.1.2) $\mathcal{C}$ and any object X of $\mathcal{C}$, there is a contravariant functor $h_X : \mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}, T \mapsto \mathrm{Hom}_{\mathcal{C}}(T, X)$; given a morphism $f : T_1 \to T_2$ in $\mathcal{C}$, there is an induced map $h_X(T_2) \to h_X(T_1)$ of sets, i.e. a map $\mathrm{Hom}_{\mathcal{C}}(T_2, X) \to \mathrm{Hom}_{\mathcal{C}}(T_1, X)$, given by $\phi \mapsto \phi \circ f$.

3. There is the contravariant power set functor $\mathcal{P} : \textbf{Sets}^{\text{op}} \to \textbf{Sets}$ that sends a set S to its power set (Definition C.0.28) $\mathcal{P}(S)$ and that sends a set morphism $f : S \to T$ to the set morphism $f^* : \mathcal{P}(T) \to \mathcal{P}(S)$ given by $B \mapsto f^{-1}(B)$.
4. The functor of continuous functions: let **Top** be the category of topological spaces and $\mathbb{R}$- Alg the category of $\mathbb{R}$-algebras (Definition C.0.29). The assignment $X \mapsto C(X, \mathbb{R})$ of a space to its algebra of continuous real-valued functions is a contravariant functor. For any continuous map $g : X \to Y$, the induced algebra homomorphism $g^* : C(Y, \mathbb{R}) \to C(X, \mathbb{R})$ is given by the pullback $g^*(\phi) = \phi \circ g$ for $\phi \in C(Y, \mathbb{R})$.

1.3. **Natural transformation.** One overarching philosophy in various categories is that we only really care about objects "up to equivalence"; intuitively, we consider objects to be equivalent when they are isomorphic (Definition 1.1.13). Similarly, we only really care about categories "up to equivalence" as well; there is a notion of equivalence (Definition 1.3.4) between categories. To define it, we first need to define the notion of natural transformations (Definition 1.3.1) between functors — a natural transformation is like a "morphism" between functors in a sense.

Definition 1.3.1. Let $\mathcal{C}$ and $\mathcal{D}$ be (large) categories (Definition 1.1.1). Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors (Definition 1.2.2).

A *natural transformation η between F and G* is a family of morphisms $\eta_X : F(X) \to G(X)$ in $\mathcal{D}$, one for each object X in $\mathcal{C}$, such that for every morphism $f : X \to Y$ in $\mathcal{C}$,

$$G(f) \circ \eta_X = \eta_Y \circ F(f)$$

in $\mathcal{D}$. In other words, the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

We write such a natural transformation by $\eta : F \Rightarrow G$.

If η_X is an isomorphism (Definition 1.1.13) for all objects X of $\mathcal{C}$, then η is said to be a *natural isomorphism*.

Example 1.3.2. Let k be a field (Definition C.0.12). Note that there is a "double dual" functor $((-)^\vee)^\vee : \mathrm{Vec}_k \to \mathrm{Vec}_k$ given by $V \mapsto (V^\vee)^\vee$ (Definition 2.1.12). Recall that (Example 1.2.6) $V \mapsto V^\vee$ is a contravariant functor (Definition 1.2.2), so the double dual functor is covariant. Recall that there is also the identity functor (Definition 1.2.4) $\mathrm{id}_{\mathrm{Vec}_k} : \mathrm{Vec}_k \to \mathrm{Vec}_k$ given by $V \mapsto V$.

For general vector spaces V, there is an injective k-linear map (Definition C.0.15)

$$\alpha_V : V \to (V^\vee)^\vee, \quad v \mapsto v^*,$$

where $v^* : V^\vee \to k$ is the k-linear map given by $\phi \mapsto \phi(v)$. We note that if V is finite dimensional (Definition C.0.16), then α_V is an isomorphism. In fact, α is a natural transformation

$\mathrm{id}_{\mathrm{Vec}_k} \Rightarrow ((-)^\vee)^\vee$ — one should verify that, for every k-linear map $f : V_1 \to V_2$, the following diagram commutes:

$$\begin{tikzcd} V_1 \arrow[r, "f"] \arrow[d, "\alpha_{V_1}"'] & V_2 \arrow[d, "\alpha_{V_2}"] \\ (V_1^\vee)^\vee \arrow[r, "(f^\vee)^\vee"'] & (V_2^\vee)^\vee \end{tikzcd}$$

Example 1.3.3. Let CRing be the category of commutative rings (Definition 1.1.12) and Group the category of groups (Definition 1.1.8). For a fixed $n \geq 1$, we have a covariant functor (Definition 1.2.2) $\mathrm{GL}_n : \mathrm{CRing} \to \mathrm{Group}$ assigning a ring R to the general linear group $\mathrm{GL}_n(R)$, and a functor $(\cdot)^\times : \mathrm{CRing} \to \mathrm{Group}$ assigning a ring to its group of units (Definition C.0.10).

The determinant $\det : \mathrm{GL}_n \Rightarrow (\cdot)^\times$ is a natural transformation (Definition 1.3.1). Its component at a ring R is the group homomorphism $\det_R : \mathrm{GL}_n(R) \to R^\times$. For any ring homomorphism $f : R \to S$, the following diagram commutes:

$$\begin{tikzcd} \mathrm{GL}_n(R) \arrow[r, "\mathrm{GL}_n(f)"] \arrow[d, "\det_R"'] & \mathrm{GL}_n(S) \arrow[d, "\det_S"] \\ R^\times \arrow[r, "f|_{R^\times}"'] & S^\times \end{tikzcd}$$

This commutativity expresses that the determinant is defined by the same universal formula (a polynomial in the matrix entries) regardless of the ring R, and is thus preserved by the "change of scalars" f.

Definition 1.3.4. An *equivalence of categories* between two (large) categories (Definition 1.1.1) $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of functors (Definition 1.2.2)

$$F : \mathcal{C} \to \mathcal{D} \quad \text{and} \quad G : \mathcal{D} \to \mathcal{C}$$

together with natural isomorphisms (Definition 1.3.1)

$$\eta : \mathrm{Id}_{\mathcal{C}} \xrightarrow{\sim} G \circ F \quad \text{and} \quad \epsilon : F \circ G \xrightarrow{\sim} \mathrm{Id}_{\mathcal{D}}.$$

(Definition 1.2.4) Such functors F and G may be called *(natural) inverses of each other*.

When $\mathcal{C}$ and $\mathcal{D}$ are locally small categories (Definition 1.1.2), F is an equivalence of categories if and only if F is fully faithful (Definition 1.3.5) and essentially surjective (Definition 1.3.9)

The “correct” notion for one category to embed into another is the notion of a fully faithful functor (Definition 1.3.5).

Definition 1.3.5. Let $\mathcal{C}$ and $\mathcal{D}$ be (large)) categories (Definition 1.1.1). Let $F : \mathcal{C} \to \mathcal{D}$ be a functor (Definition 1.2.2).

1. F is called *full* if for every pair of objects $x, y \in \mathrm{Ob}(\mathcal{C})$, the induced rule/assignment/class function

$$F_{x,y} : \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$$

on Hom-collections is “surjective”, i.e. for all morphisms $g : F(x) \to F(y)$, there exists some morphism $f : x \to y$ such that $F(f) = g$.
2. F is called *faithful* if for every pair of objects $x, y \in \mathrm{Ob}(\mathcal{C})$, the induced class function (assignment)
$$F_{x,y} : \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$$
on Hom-collections is “injective”, i.e., for any morphisms $f_1, f_2 \in \mathrm{Hom}_{\mathcal{C}}(x, y)$, if $F(f_1) = F(f_2)$ in $\mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$, then $f_1 = f_2$.
3. F is called *fully faithful* if it is both full and faithful.

Definition 1.3.6 (Subcategory)**.** Let $\mathcal{C}$ be a (large) category (Definition 1.1.1). A *subcategory* $\mathcal{D}$ of $\mathcal{C}$ consists of:

- a subclass of objects $\mathrm{Ob}(\mathcal{D}) \subseteq \mathrm{Ob}(\mathcal{C})$,
- for each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{D})$, a subclass of morphisms
$$\mathrm{Hom}_{\mathcal{D}}(X, Y) \subseteq \mathrm{Hom}_{\mathcal{C}}(X, Y),$$

such that

- for every object $X \in \mathrm{Ob}(\mathcal{D})$, the identity morphism id_X of X in $\mathcal{C}$ lies in $\mathrm{Hom}_{\mathcal{D}}(X, X)$,
- the composition of morphisms in $\mathcal{D}$ is inherited from $\mathcal{C}$ and is closed in $\mathcal{D}$: for morphisms $f \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{D}}(Y, Z)$, their composition $g \circ f$, which is a priori in $\mathrm{Hom}_{\mathcal{C}}(X, Z)$, is in $\mathrm{Hom}_{\mathcal{D}}(X, Z)$.

Definition 1.3.7 (Full subcategory)**.** Let $\mathcal{C}$ be a (large) category (Definition 1.1.1). A *full subcategory* $\mathcal{D}$ of $\mathcal{C}$ is a subcategory (Definition 1.3.6) such that for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{D})$, the morphism classes coincide:
$$\mathrm{Hom}_{\mathcal{D}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y).$$
In other words, a full subcategory includes all morphisms between its objects that exist in the ambient category $\mathcal{C}$.

The “correct” notion for one category to “surject” onto another is the notion of a essentially surjective functor (Definition 1.3.9).

Definition 1.3.8 (Essential image of a functor)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between (large) categories (Definition 1.1.1). The *essential image of F* is the full subcategory (Definition 1.3.7) of $\mathcal{D}$ whose objects are those $d \in \mathrm{Ob}(\mathcal{D})$ for which there exists an object $c \in \mathrm{Ob}(\mathcal{C})$ such that
$$F(c) \cong d.$$
(Definition 1.1.13) Equivalently, the essential image is given by
$$\mathrm{EssIm}(F) = \{\, d \in \mathrm{Ob}(\mathcal{D}) \mid \exists c \in \mathrm{Ob}(\mathcal{C}),\ F(c) \cong d \,\},$$
endowed with all morphisms $\mathcal{D}(d, d')$ between such objects.

Definition 1.3.9. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between (large) categories (Definition 1.1.1). It is said to be essentially surjective if its essential image (Definition 1.3.8) coincides with $\mathcal{D}$.

Example 1.3.10. Below are examples illustrating various properties of functors:

1. **Faithful but not full:** The forgetful functor (Definition 1.2.5) $U : \text{Group} \to \text{Set}$. It is faithful (Definition 1.3.5) because group homomorphisms are distinct if they are distinct as functions. It is not full because not every function between groups is a group homomorphism (e.g., the constant function $x \mapsto g$ for $g \neq e$).
2. **Full but not faithful:** The canonical functor $H : \mathsf{Top} \to h\mathsf{Top}$ from the category of topological spaces (Definition 1.1.9) to the homotopy category (Definition C.0.32) of topological spaces. It is the identity on objects and sends a continuous map f to its homotopy class $[f]$. This is full (Definition 1.3.5) by the definition of morphisms in $h\mathsf{Top}$, but not faithful (Definition 1.3.5) because it identifies distinct but homotopic maps (Definition C.0.30) (e.g., any two paths in $\mathbb{R}^n$ with the same endpoints).
3. **Fully faithful:** The inclusion functor $\iota : \text{Ab} \to \text{Group}$. It is faithful (Definition 1.3.5) (it is an embedding) and it is full (Definition 1.3.5) because any group homomorphism between two abelian groups is, by definition, a morphism in Ab.
4. **Essentially surjective:** Let $\mathbf{S}$ be the category whose objects are the standard sets $\underline{n} = \{0, \dots, n-1\}$ for each $n \in \mathbb{N}$ (and whose objects are the set functions between these sets), and **FinSet** be the category of all finite sets. The inclusion functor $I : \mathbf{S} \to \mathbf{FinSet}$ is essentially surjective because every finite set X is isomorphic to $\underline{n}$ where n is the cardinality of X.

Lemma 1.3.11. Let $\mathcal{C}$ and $\mathcal{D}$ be locally small categories (Definition 1.1.2), and let $F : \mathcal{C} \to \mathcal{D}$ be a functor (Definition 1.2.2).

F is an equivalence of categories (Definition 1.3.4) if and only if F is fully faithful (Definition 1.3.5) and essentially surjective (Definition 1.3.9)

1.3.1. *Yoneda lemma.* The Yoneda lemma basically expresses the idea that an object of a (locally small) category is essentially determined by its morphisms to other objects.

Definition 1.3.12. Let $\mathcal{C}$ be a locally small category (Definition 1.1.2). Given an object X of $\mathcal{C}$, the *functor of points* h_X is the functor (Definition 1.2.2)/presheaf (Definition 6.0.1) $\mathcal{C}^{\text{op}} \to \mathbf{Sets}$ given by

1. sending an object T of $\mathcal{C}$ to the set $\text{Hom}_{\mathcal{C}}(T, X)$, and
2. sending a morphism $f : T_1 \to T_2$ in $\mathcal{C}$ to the set map (Definition C.0.1)
$$\text{Hom}_{\mathcal{C}}(T_2, X) \to \text{Hom}_{\mathcal{C}}(T_1, X), \quad \phi \mapsto \phi \circ f.$$

A functor $\mathcal{C}^{\text{op}} \to \mathbf{Sets}$ (or equivalently, a presheaf on $\mathcal{C}$ valued in $\mathbf{Sets}$) is said to be *representable* if it is naturally isomorphic (Definition 1.3.1) to some functor h_X of points for an object X of $\mathcal{C}$.

Dually, a functor $\mathcal{C} \to \mathbf{Sets}$ is called *co-representable* if it is naturally isomorphic to a functor $h^X : \mathcal{C} \to \mathbf{Sets}$ given by $T \mapsto \text{Hom}_{\mathcal{C}}(X, T)$.

Note that the above notions of representability/co-representability are special caseiof those of Definition A.0.1, where the monoidal category $\mathcal{V}$ is the symmetric monoidal category (Definition A.0.2) **Sets** (Definition 1.1.7).

Theorem 1.3.13 (Yoneda Lemma)**.** Let $\mathcal{C}$ be a locally small category (Definition 1.1.2). Let A be an object of $\mathcal{C}$, and let $F : \mathcal{C} \to \mathbf{Set}$ be a covariant functor (Definition 1.2.2) to the category of sets (Definition 1.1.7). Let $h^A : \mathcal{C} \to \mathbf{Set}$ denote the covariant representable functor (Definition 1.3.12) defined by $h^A(X) = \mathrm{Hom}_{\mathcal{C}}(A, X)$.

There exists a bijection

$$y_{A,F} : \mathrm{Nat}(h^A, F) \xrightarrow{\cong} F(A)$$

between the set of natural transformations (Definition 1.3.1) from h^A to F and the set $F(A)$. This bijection is given by the mapping

$$\alpha \mapsto \alpha_A(\mathrm{id}_A),$$

where $\alpha : h^A \to F$ is a natural transformation, $\alpha_A : h^A(A) \to F(A)$ is its component at A, and $\mathrm{id}_A \in h^A(A) = \mathrm{Hom}_{\mathcal{C}}(A, A)$ is the identity morphism.

Furthermore, this isomorphism is natural in both A and F. Explicitly:

1. For any morphism $f : A \to B$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Nat}(h^B, F) \arrow[r, "y_{B,F}"] \arrow[d, "- \circ h^f"'] & F(B) \arrow[d, "F(f)"] \\
\mathrm{Nat}(h^A, F) \arrow[r, "y_{A,F}"] & F(A)
\end{tikzcd}$$

   where $h^f : h^B \to h^A$ is the natural transformation induced by pre-composition with f.
2. For any natural transformation $\eta : F \to G$, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Nat}(h^A, F) \arrow[r, "y_{A,F}"] \arrow[d, "\eta \circ -"'] & F(A) \arrow[d, "\eta_A"] \\
\mathrm{Nat}(h^A, G) \arrow[r, "y_{A,G}"] & G(A)
\end{tikzcd}$$

Corollary 1.3.14 (Yoneda Embedding)**.** Let $\mathcal{C}$ be a locally small category (Definition 1.1.2). The functor

$$h^\bullet : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathcal{C}}$$

(Definition 1.2.1) (Definition 2.2.6) defined on objects by $A \mapsto h^A = \mathrm{Hom}_{\mathcal{C}}(A, -)$ (Definition 1.3.12) and on morphisms by $f \mapsto h^f = (- \circ f)$ is fully faithful (Definition 1.3.5). That is, for any objects A, B in $\mathcal{C}$, the map

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Nat}(h^B, h^A)$$

given by sending a morphism $f : A \to B$ to the natural transformation (Definition 1.3.1) $h^f : h^B \to h^A$ (pre-composition by f) is a bijection.

Consequently, $\mathcal{C}^{\mathrm{op}}$ embeds as a full subcategory (Definition 1.3.7) of the functor category $\mathbf{Set}^{\mathcal{C}}$.

Theorem 1.3.15 (Contravariant Yoneda Lemma)**.** Let $\mathcal{C}$ be a locally small category (Definition 1.1.2). Let A be an object of $\mathcal{C}$, and let $G : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ be a contravariant functor (Definition 1.2.2) (i.e. a presheaf (Definition 6.0.1)). Let $h_A : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ denote the contravariant representable functor defined by $h_A(X) = \mathrm{Hom}_{\mathcal{C}}(X, A)$.

There exists a bijection natural in A and G:

$$\mathrm{Nat}(h_A, G) \cong G(A)$$

given by $\alpha \mapsto \alpha_A(\mathrm{id}_A)$.

Corollary 1.3.16 (Contravariant Yoneda Embedding)**.** Let $\mathcal{C}$ be a locally small category (Definition 1.1.2). The functor

$$h_\bullet : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$$

defined on objects by $A \mapsto h_A = \mathrm{Hom}_{\mathcal{C}}(-, A)$ and on morphisms by $f \mapsto h_f = (f \circ -)$ is fully faithful (Definition 1.3.5). That is, for any objects A, B in $\mathcal{C}$, the map

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Nat}(h_A, h_B)$$

given by sending a morphism $f : A \to B$ to the natural transformation (Definition 1.3.1) $h_f : h_A \to h_B$ (post-composition by f) is a bijection.

Consequently, $\mathcal{C}$ embeds as a full subcategory (Definition 1.3.7) of the category of presheaves $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$.

Example 1.3.17. The algebra of smooth functions (Definition C.0.35) $C^\infty(M)$ on a smooth manifold (Definition C.0.37) M is represented by the real line $\mathbb{R}$ in the category of smooth manifolds Man.

1. **The Functor:** Consider the covariant representable functor (Definition 1.3.12) $h^{\mathbb{R}} = \mathrm{Hom}_{\mathsf{Man}}(-, \mathbb{R})$. For any manifold M, we have
$$\mathrm{Hom}_{\mathsf{Man}}(M, \mathbb{R}) = C^\infty(M),$$
i.e. the representable functor $h^{\mathbb{R}}$ assigns M to the set of all smooth maps $f : M \to \mathbb{R}$, which is exactly $C^\infty(M)$.
2. **The Yoneda Correspondence:** By the Yoneda Lemma (Theorem 1.3.13), there is a natural bijection between the points of M and the natural transformations from h^M to $h^{\mathbb{R}}$:
$$\mathrm{Nat}(h^M, h^{\mathbb{R}}) \cong h^{\mathbb{R}}(M) = C^\infty(M)$$
3. **Geometric Interpretation:** This identifies a smooth function $\psi \in C^\infty(M)$ with a "natural" way to turn M-valued points of any other smooth manifold X into real-valued points. Specifically, if we have a "probe" $\alpha : X \to M$, the function ψ induces a map:
$$\alpha \mapsto \psi \circ \alpha$$
mapping $\mathrm{Hom}(X, M)$ to $\mathrm{Hom}(X, \mathbb{R})$.

This illustrates that the "global" algebraic data of a manifold (its functions) is entirely captured by its relationship to the "simplest" non-trivial manifold, $\mathbb{R}$.

1.4. **The category of categories.** Intuitively, one might think of a functor (Definition 1.2.2) as a “morphism” between two categories. Indeed, one can consider a category of categories in which morphisms are functors.

Definition 1.4.1. The *category of small categories* is the category defined by the following data:

- The objects are all small categories (Definition 1.1.2).
- The morphisms between two small categories $\mathcal{C}$ and $\mathcal{D}$ are the functors (Definition 1.2.2) $F : \mathcal{C} \to \mathcal{D}$.
- The composition of morphisms is the standard composition of functors.
- The identity morphism for each object $\mathcal{C}$ is the identity functor $1_{\mathcal{C}}$.

This category is a large category (Definition 1.1.1) and is denoted by **Cat**

If we allow ourselves to use Grothendieck universes (Definition A.0.3), then we can also talk about a category of categories in the sense of Definition A.0.7.

1.5. Miscellaneous categorical notions.

Definition 1.5.1 (Product Category of a Family of Categories)**.** Let $\{\mathcal{C}_i\}_{i\in I}$ be a family of (large) categories (Definition 1.1.1) indexed by a class I. The *product category of the family*, denoted

$$\prod_{i\in I} \mathcal{C}_i,$$

is the very large category (♠ TODO: define very large categories) defined as follows:

- The class of objects is
$$\mathrm{Ob}\Big(\prod_{i\in I} \mathcal{C}_i\Big) = \prod_{i\in I} \mathrm{Ob}(\mathcal{C}_i),$$
i.e., an object is a family $(A_i)_{i\in I}$ with $A_i \in \mathrm{Ob}(\mathcal{C}_i)$.
- For two objects $(A_i)_i$ and $(B_i)_i$, the morphism class is
$$\mathrm{Hom}_{\prod_{i\in I}\mathcal{C}_i}((A_i)_i, (B_i)_i) = \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}_i}(A_i, B_i).$$
In other words, a morphism $(f_i)_i : (A_i)_i \to (B_i)_i$ consists of morphisms $f_i : A_i \to B_i$ in each $\mathcal{C}_i$.
- For morphisms $(f_i)_i : (A_i)_i \to (B_i)_i$ and $(g_i)_i : (B_i)_i \to (C_i)_i$, composition is defined componentwise:
$$(g_i)_i \circ (f_i)_i = (g_i \circ_i f_i)_i.$$
- For each object $(A_i)_i$, the identity morphism is given by the family
$$(\mathrm{id}_{A_i})_i.$$

If I is a set, then $\prod_{i\in I} \mathcal{C}_i$ is a large category. If I is a set and if each $\mathcal{C}_i$ is locally small (Definition 1.1.2), then $\prod_{i\in I} \mathcal{C}_i$ is locally small.

In case that I is finite, the notation of $\times$ may be used for product categories, e.g. $\mathcal{C}_i \times \mathcal{C}_j$ denotes the product of two categories $\mathcal{C}_i \times \mathcal{C}_j$.

(♠ TODO: ordinal, U_α) If α is an ordinal such that $\mathcal{C}_i$ and I are U_α-large (i.e. they live in $U_{\alpha+1}$), then $\prod_{i\in I} \mathcal{C}_i$ is $U_{\alpha+1}$-large.

2. Additive and abelian categories

2.1. Rings and modules.

Definition 2.1.1. Let R be a not-necessarily commutative ring (Definition C.0.7).

1. A *left R-module* is an abelian group $(M, +)$ together with an operation $R \times M \to M$, denoted $(r, m) \mapsto rm$, such that for all $r, s \in R$ and $m, n \in M$:
   - $r(m + n) = rm + rn$,
   - $(r + s)m = rm + sm$,
   - $(rs)m = r(sm)$,
   - $1_R m = m$ where 1_R is the multiplicative identity of R.
2. A *right R-module* is defined similarly as an abelian group $(M, +)$ with an operation $M \times R \to M$, denoted $(m, r) \mapsto mr$, such that for all $r, s \in R$ and $m, n \in M$:
   - $(m + n)r = mr + nr$,
   - $m(r + s) = mr + ms$,
   - $m(rs) = (mr)s$,
   - $m1_R = m$.
3. Let R and S be (not necessarily commutative) rings (Definition C.0.7).
   An *R-S-bimodule* (or an *R-S-module* or an (R, S)-module, etc.) is an abelian group (Definition C.0.3) $(M, +)$ equipped with
   (a) a left action of R:
   $$R \times M \to M, \quad (r, m) \mapsto r \cdot m,$$
   making M a left R-module (Definition 2.1.1),
   (b) a right action of S:
   $$M \times S \to M, \quad (m, s) \mapsto m \cdot s,$$
   making M a right S-module,
   such that the left and right actions commute; that is, for all $r \in R$, $s \in S$, and $m \in M$,
   $$r \cdot (m \cdot s) = (r \cdot m) \cdot s.$$
4. A *two-sided R-module* (or *R-bimodule*) is an R-R-bimodule.

If R is a commutative ring (Definition C.0.9), then a left/right R-module can automatically be regarded as a two-sided R-module. As such, we simply talk about *R-modules* in this case.

Any abelian group is equivalent to a two-sided $\mathbb{Z}$-module. Moreover, any left R-module is equivalent to an $R - \mathbb{Z}$-bimodule (Definition 2.1.1) and any right R-module is equivalent to an $\mathbb{Z} - R$-bimodule (Definition 2.1.1). Given a left/right/two-sided R-module, its *natural bimodule structure* will refer to its structure as a R-$\mathbb{Z}$/$\mathbb{Z}$-R/R-R bimodule. In this way, many definitions associated with the notions of left/right/two-sided R-modules can be defined as special cases for definitions for R-S-bimodules.

Definition 2.1.2. Let R, S be (not-necessarily commutative) rings (Definition C.0.7).

1. Let M and N be R-S-bimodules (Definition 2.1.1). A function $\varphi : M \to N$ is called an *R-S-bimodule homomorphism* or *R-S-linear* if it is a group homomorphism (Definition C.0.4) of the underlying abelian groups of M and N and respects the scalar actions as follows: for all $m_1, m_2 \in M$, $r \in R$, and $s \in S$,
$$\varphi(r \cdot m_1) = r \cdot \varphi(m_1),$$
$$\varphi(m_1 \cdot s) = \varphi(m_1) \cdot s.$$
2. Let M and N be left/right/two-sided R-modules (Definition 2.1.1). A function $\varphi : M \to N$ is called a *left/right/two-sided R-module homomorphism* if it is an bimodule homomorphism on the natural bimodule structures (Definition 2.1.1) of M and N. Such a function is also called *R-linear*.

Modules and homomorphisms of a fixed type (i.e. R-S-bimodules or left/righ/two-sided R-modules) form a locally small (Definition 1.1.2) category (Definition 1.1.1).

Definition 2.1.3. Let R and S be (not necessarily commutative) rings (Definition C.0.7).

1. The *category of (R, S)-bimodules* (or R-S-bimodules), denoted by notations such as ${}_R\mathsf{Mod}_S$, is the category whose objects are (R, S)-bimodules (Definition 2.1.1) and whose R-S-bimodule homomorphisms (Definition 2.1.2).
2. The *category of left R-modules*, denoted by notations such as ${}_R\mathsf{Mod}$ or $R - \mathbf{Mod}$, is the category ${}_R\mathsf{Mod}_\mathbb{Z}$, i.e. the category whose objects are left R-modules (Definition 2.1.1) and whose morphisms are left R-linear maps (Definition 2.1.2).
3. The *category of right R-modules*, denoted by notations such as Mod_R or $\mathbf{Mod} - R$, is the category ${}_\mathbb{Z}\mathsf{Mod}_R$, i.e. the category whose objects are right R-modules (Definition 2.1.1) and whose morphisms are right R-linear maps (Definition 2.1.2).

The category of bimodules can be canonically identified with module categories over tensor product rings (Definition 2.1.10):

- ${}_R\mathsf{Mod}_S$ is isomorphic to the category of left modules over the ring $R \otimes_\mathbb{Z} S^{\mathrm{op}}$.
- ${}_R\mathsf{Mod}_S$ is isomorphic to the category of right modules over the ring $R^{\mathrm{op}} \otimes_\mathbb{Z} S$.

Consequently, standard module-theoretic concepts (such as projective objects, injective objects, and flat objects) in ${}_R\mathsf{Mod}_S$ correspond exactly to the respective concepts in ${}_{R\otimes S^{\mathrm{op}}}\mathsf{Mod}$.

Note that there are canonical isomorphisms of categories:
$${}_R\mathsf{Mod} \cong {}_R\mathsf{Mod}_\mathbb{Z} \quad \text{and} \quad \mathsf{Mod}_S \cong {}_\mathbb{Z}\mathsf{Mod}_S.$$
That is, left R-modules are exactly $(R, \mathbb{Z})$-bimodules, and right S-modules are exactly $(\mathbb{Z}, S)$-bimodules.

Definition 2.1.4. Let R, S be not-necessarily commutative rings (Definition C.0.7).

1. Let M be an R-S-bimodule whose abelian group (Definition C.0.3) structure is given by the operator $+$. An *R-S-submodule of M* is a subgroup $N \subseteq (M, +)$ if for all $r \in R$, $s \in S$, and $n \in N$, we have $rn \in N$ and $ns \in N$; in this case, N inherits an R-S bimodule structure from M.

2. If M is a left/right/two-sided R-module, then a *left/right/two-sided R-submodule of M* is a submodule of the natural bimodule structure (Definition 2.1.1) of M.

Definition 2.1.5. (♠ TODO: define coset) Let R, S be (not necessarily commutative) rings (Definition C.0.7).

1. Let M be an R-S-bimodule (Definition 2.1.1). Let $N \subseteq M$ be a submodule of M (Definition 2.1.4).

   The quotient group M/N, which is well defined as M is an abelian group (Definition C.0.3) and hence N is a normal subgroup, has the structure of an R-S-bimodule — the (abelian) group structure is simply the group structure of M/N, whereas the R-S-bimodule structure is given as follows: for $m \in M$, $r \in R$, $s \in S$, we have

   $$r \cdot (m + N) \cdot s = r \cdot m \cdot s + N.$$

   This R-S-bidmodule structure on M/N is called the *quotient R-S-bidmodule of M by N* and is also denoted as M/N.

   The canonical projection map

   $$\pi : M \to M/N, \quad m \mapsto m + N,$$

   is a surjective R-module homomorphism (Definition 2.1.2) with kernel (Definition 2.1.7) N.
2. Let M be a left/right/two-sided R-module. Let $N \subseteq M$ be a submodule of M. The *quotient R-module* M/N is the quotient of M by N for their respective natural bimodule structures (Definition 2.1.1).

Definition 2.1.6 (Submodule generated by elements in an (R, S)-bimodule)**.** Let R and S be (not necessarily commutative) rings (Definition C.0.7).

1. Let M be an (R, S)-bimodule (Definition 2.1.1).

   Given a subset $X \subseteq M$, the *sub-bimodule of M generated by X* is the smallest (R, S)-sub-bimodule of M containing X. It is often denoted by notations such as $\langle X \rangle = \langle X \rangle_{R,S}$ and is more explicitly the intersection

   $$\langle X \rangle_{R,S} = \bigcap_{X \subseteq T \subseteq M, T \text{ is a } (R,S)\text{-submodule of } M} T$$

   of al (R, S)-submodules of M containing X.

   Equivalently, $\langle X \rangle_{R,S}$ consists of all linear combinations of X.
2. If M is a left/right/two-sided R-module and given a subset $X \subseteq M$, the *submodule of M generated by X* is the submodule of the natural bimodule (Definition 2.1.1) of M generated by X. It is denoted by notations such as $\langle X \rangle = \langle X \rangle_R$.

Definition 2.1.7. Let R, S be (not-necessarily commutative) rings with unity (Definition C.0.7), and let M, N be R-S-bimodules (Definition 2.1.1). Let

$$\varphi : M \to N$$

be a homomorphism of R-S-bimodules (Definition 2.1.2). We define:

1. The *kernel of φ* is the submodule of M (Definition 2.1.4) given by

$$\ker(\varphi) := \{m \in M \mid \varphi(m) = 0\} \subseteq M.$$

2. The *image of φ* is the submodule of N given by

$$\operatorname{im}(\varphi) := \{\varphi(m) \mid m \in M\} \subseteq N.$$

3. The *cokernel of φ* is the quotient module of N (Definition 2.1.5) defined by

$$\operatorname{coker}(\varphi) := N/\operatorname{im}(\varphi).$$

4. The *coimage of φ* is the quotient module of M (Definition 2.1.5) defined by

$$\operatorname{coim}(\varphi) := M/\ker(\varphi).$$

It is not difficult to see that each of these are indeed R-S bimodules. In case M and N are left/right/two-sided R-modules, the *kernel, image, cokernel, and coimage* of a module homomorphism $\varphi : M \to N$ are respectively defined to be the kernel, image, cokernel, and coimage for the natural bimodule structures (Definition 2.1.1) of M and N.

The kerel, cokernel, image, and coimage of f are respectively the categorical kernel, cokernel (Definition 2.3.6), image, and coimage (Definition 2.3.8) (Lemma 2.3.9).

Two fundamental functors on categories of modules are given by Hom's (Definition 2.1.8) and tensor products (Definition 2.1.9)

Definition 2.1.8 (Hom of left/right/bi-modules)**.** Let R, S, T be (not necessarily commutative) rings (Definition C.0.7).

1. Let M and N be left R-modules (Definition 2.1.1). The *homomorphism group of left R-modules from M to N* is the abelian group

$$\operatorname{Hom}(M, N) = \operatorname{Hom}_R(M, N) := \{f : M \to N \mid f \text{ is a left } R\text{-module homomorphism}\}.$$

   (Definition 2.1.2)
2. Let M and N be right R-modules (Definition 2.1.1). The *homomorphism group of right R-modules from M to N* is the abelian group

$$\operatorname{Hom}(M, N) = \operatorname{Hom}_R(M, N) := \{f : M \to N \mid f \text{ is a right } R\text{-module homomorphism}\}.$$

3. Let S be a (not necessarily commutative ring) and let M and N be $R-S$-bimodules (Definition 2.1.1). The *homomorphism group of R-S-bimodules from M to N* is the abelian group

$$\operatorname{Hom}(M, N) = \operatorname{Hom}_{R-S}(M, N) \coloneqq \{f : M \to N | f \text{ is a } R-S\text{-bimodule homomorphism}\}$$

In each case, $\operatorname{Hom}(M, N)$ has a natural structure of an *abelian group* given by *pointwise addition*: for $f, g \in \operatorname{Hom}(M, N)$,

$$(f + g)(m) := f(m) + g(m),$$

and the zero morphism 0 given by $0(m) := 0_N$ acts as the identity element. The additive inverse $-f$ is defined by $(-f)(m) := -f(m)$. Moreover, depending on bi-module structures that M and N may be carrying, $\operatorname{Hom}(M, N)$ may itself carry additional module structures:

- In case that M is a $R-S$-bimodule and N is a $R-T$-bimodule, $\operatorname{Hom}_R(M, N)$, the group of left R-module homomorphisms, is an $S-T$-bimodule as follows:
$$(s \cdot f \cdot t)(m) = f(m \cdot s) \cdot t \quad f \in \operatorname{Hom}_R(M, N), s \in S, t \in T.$$
- Dually, in case that M is a $S-R$-bimodule and N is a $T-R$-bimodule, $\operatorname{Hom}_R(M, N)$, the group of right R-module homomorphisms, is an $S-T$-bimodule as follows:
$$(s \cdot f \cdot t)(m) = f(s \cdot m) \cdot t \quad f \in \operatorname{Hom}_R(M, N), s \in S, t \in T.$$

Some cases of interest may be when R, S, or T is in fact $\mathbb{Z}$ — these allow us to see module structures on $\operatorname{Hom}(M, N)$ even when M and N are one-sided modules.

(♠ TODO: state this as a theorem) We furthermore note that $\operatorname{Hom}_R(-, -)$ yields biadditive functors (Definition 2.4.2)

$$\operatorname{Hom}_R(-, -) : {}_R\mathbf{Mod}_S^{\mathrm{op}} \times {}_R\mathbf{Mod}_T \to {}_S\mathbf{Mod}_T$$

$$\operatorname{Hom}_R(-, -) : {}_S\mathbf{Mod}_R^{\mathrm{op}} \times {}_T\mathbf{Mod}_R \to {}_S\mathbf{Mod}_T.$$

(Definition 1.2.1) (Definition 2.1.3) (Theorem 2.5.11)

Definition 2.1.9 (Tensor product of bimodules)**.** Let R, S, T be (not necessarily commutative) rings (Definition C.0.7), let M be an R-S bimodule (Definition 2.1.1), and let N be an S-T bimodule. In the free abelian group (Definition C.0.20) $\mathbb{Z}[M \times N]$ generated by the Cartesian product $M \times N$ (Definition 2.2.2), let U be the subgroup generated by elements of the form (♠ TODO: subgroup generated)

$$\begin{aligned}
&(m + m', n) - (m, n) - (m', n), \\
&(m, n + n') - (m, n) - (m, n'), \\
&(m \cdot s, n) - (m, s \cdot n),
\end{aligned}$$

for all $m, m' \in M$, $n, n' \in N$, and $s \in S$. The *tensor product of M and N over S* is the quotient abelian group

$$M \otimes_S N := \mathbb{Z}[M \times N]/U.$$

The image of an element of the form $(m, n) \in M \times N$ in $M \otimes_S N$ is denoted $m \otimes n$ and called a *pure tensor*. In general, the elements of $M \otimes_S N$ are finite sums

$$\sum_{i=1}^{n} m_i \otimes n_i \quad m_i \in M, n_i \in N$$

of pure tensors. Thus, the pure tensors satisfy the following relations:

$$\begin{aligned}
(m + m') \otimes n &= m \otimes n + m' \otimes n \\
m \otimes (n + n') &= m \otimes n + m \otimes n' \\
(m \cdot s) \otimes n &= m \otimes (s \cdot n)
\end{aligned}$$

This tensor product becomes naturally an R-T bimodule with left action and right action defined by

$$r \cdot (m \otimes n) = (r \cdot m) \otimes n,$$
$$(m \otimes n) \cdot t = m \otimes (n \cdot t),$$

for all $r \in R$, $t \in T$, $m \in M$, and $n \in N$.

Inductively, given rings $R_0, \ldots, R_k$ and $R_{i-1} - R_i$-bimodules M_i for $i = 1, \ldots, k$, we may speak of the tensor product

$$M_0 \otimes_{R_1} M_1 \otimes_{R_2} \cdots \otimes_{R_{k-1}} M_k;$$

tensor products are associative(♠ TODO:), so parentheses are not strictly needed to notate them. Its *pure tensors* are elements of the form $m_0 \otimes m_1 \otimes \cdots \otimes m_k$ for $m_i \in M_i$, and its general elements are finite sums

$$\sum_{j=1}^{n} m_{0j} \otimes m_{1j} \otimes \cdots m_{kj} \quad m_{ij} \in M_i.$$

of pure tensors. It also has a natural $R_0 - R_k$-bimodule structure.

In general, $(M_0, \ldots, M_k) \mapsto M_0 \otimes_{R_1} M_1 \otimes_{R_2} \cdots \otimes_{R_{k-1}} M_k$ defines a $(k+1)$-ary additive functor (Definition 2.4.2)

$${}_{R_0}\mathbf{Mod}_{R_1} \times \cdots \times {}_{R_{k-1}}\mathbf{Mod}_{R_k} \to {}_{R_0}\mathbf{Mod}_{R_k}$$

(Theorem 2.5.11).

Given a ring R and a two-sided R-module M, we may also speak of the *n-fold tensor product* $M^{\otimes n} = M^{\otimes_R n}$

Definition 2.1.10. Let k be a not necessarily commutative ring (Definition C.0.7). Let R and S be k-rings (Definition C.0.13) (not necessarily commutative). Assume that at least one of R or S is a k-algebra (Definition C.0.29). The *tensor product ring* $R \otimes_k S$ is the k-module $R \otimes_k S$ (Definition 2.1.9) equipped with a multiplication defined on simple tensors by

$$(r_1 \otimes s_1) \cdot (r_2 \otimes s_2) = (r_1 r_2) \otimes (s_1 s_2)$$

and extended by linearity. This multiplication is well-defined and makes $R \otimes_k S$ into a k-ring under the ring homomorphism

$$k \to R \otimes_k S, \quad a \mapsto a \otimes 1 = 1 \otimes a.$$

The unit element is $1_R \otimes 1_S$.

In this ring, the subrings $R \otimes 1$ and $1 \otimes S$ commute with each other; that is, for all $r \in R$ and $s \in S$,

$$(r \otimes 1) \cdot (1 \otimes s) = r \otimes s = (1 \otimes s) \cdot (r \otimes 1).$$

If R and S are both k-algebras, then $R \otimes_k S$ is also a k-algebra.

Proposition 2.1.11 (Universal Property of the Tensor Product of Bimodules)**.** Let R, S, T be (not necessarily commutative) rings (Definition C.0.7). Let M be an R-S bimodule (Definition 2.1.1) and let N be an S-T bimodule. Let P be an R-T bimodule. Then for every R-T bilinear map

$$\beta : M \times N \to P,$$

that is, a map satisfying

$$\begin{aligned}
\beta(m + m', n) &= \beta(m, n) + \beta(m', n),\\
\beta(m, n + n') &= \beta(m, n) + \beta(m, n'),\\
\beta(r \cdot m, n) &= r \cdot \beta(m, n),\\
\beta(m, n \cdot t) &= \beta(m, n) \cdot t,\\
\beta(m \cdot s, n) &= \beta(m, s \cdot n),
\end{aligned}$$

for all $m, m' \in M$, $n, n' \in N$, $r \in R$, $s \in S$, $t \in T$, there exists a unique R-T bimodule homomorphism

$$\widetilde{\beta} : M \otimes_S N \to P$$

such that $\widetilde{\beta}(m \otimes n) = \beta(m, n)$ for all $m \in M$, $n \in N$.

Definition 2.1.12. Let R be a (not necessarily commutative) ring (Definition C.0.7). Depending on the module structure of M, we define its dual module as follows:

1. If M is a left R-module (Definition 2.1.1), then the *(right) dual module of M* is

   $$M^* = M^\vee := \mathrm{Hom}_R(M, R).$$

   (Definition 2.1.8) Note that it is a right R-module, as M is a $R - \mathbb{Z}$-bimodule and R is an $R - R$-bimodule.
2. If M is a right R-module (Definition 2.1.1), then the *(left) dual module of M* is

   $${}^*M = {}^\vee M := \mathrm{Hom}_R(M, R).$$

   (Definition 2.1.8) Note that it is a left R-module, as M is a $\mathbb{Z} - R$-bimodule and R is an $R - R$-bimodule.
3. If M is a two-sided R-module, then the *dual of M* usually refers to either the right or the left dual as above.

If a convention for the rightness or the leftness of modules is fixed, then the appropriate dual module is often denoted by $M^\vee$.

In any case, the functor $M \mapsto M^\vee$ is a contravariant functor (Definition 1.2.2) from the appropriate category of modules (Definition 2.1.3) to itself.

If R is a field (Definition C.0.12) F and V is an F-vector space (Definition C.0.14), then the dual module

$$V^* = V^\vee := \mathrm{Hom}_F(V, F)$$

is called the *dual vector space of V*.

2.1.1. *Extension/Restriction/co-extension of scalars.*

Definition 2.1.13. Let R and S be rings (Definition C.0.7) (not necessarily commutative), and let $f : R \to S$ be a ring homomorphism (Definition C.0.13). This homomorphism gives S the structure of an (R, R)-bimodule (Definition 2.1.1) via restriction of scalars (Definition 2.1.14). Let T be a ring (Definition C.0.7).

1. The *extension of scalars* (or *base change*) along f is defined separately for modules:
   - For an $(R-T)$-bimodule M, the extension of scalars of M along f is the $(S-T)$-bimodule $S \otimes_R M$ (Definition 2.1.9) where S is viewed as an (S, R)-bimodule. In particular, the action of $s' \in S$ on a simple tensor (Definition 2.1.9) $s \otimes m$ is given by $s' \cdot (s \otimes m) = (s's) \otimes m$.
   - For a $(T - R)$-bimodule M, the extension of scalars of M along f is the $(T - S)$-bimodule $M \otimes_R S$ where S is viewed as an (R, S)-bimodule. In particular, the action of $s' \in S$ on a simple tensor $m \otimes s$ is given by $(m \otimes s) \cdot s' = m \otimes (ss')$.
2. The *base change functor* (or *extension of scalars functor* or *induction functor*), denoted by f^*, $S \otimes_R -$, $- \otimes_R S$, or Ind_R^S, is given by:
   - For left R-modules:
   $$f^* : {}_R\mathsf{Mod}_T \to {}_S\mathsf{Mod}_T, \quad M \mapsto S \otimes_R M.$$
   (Definition 2.1.3) This is the left adjoint (Definition 2.5.1) to the restriction of scalars (Definition 2.1.14) functor $f_* : {}_S\mathsf{Mod}_T \to {}_R\mathsf{Mod}_T$.
   - For right R-modules:
   $$f^* : {}_T\mathsf{Mod}_R \to {}_T\mathsf{Mod}_S, \quad M \mapsto M \otimes_R S.$$
   This is the left adjoint to the restriction of scalars functor $f_* : {}_T\mathsf{Mod}_S \to {}_T\mathsf{Mod}_R$.
3. Let A be an R-ring (Definition C.0.13), i.e. a ring equipped with a ring homomorphism $R \to A$. Assume that S or A is an R-algebra (Definition C.0.29).
   The *base change of the algebra A along f* is the S-ring defined as
   $$A_S := S \otimes_R A$$
   (Definition 2.1.10) equipped with the natural homomorphism $S \to S \otimes_R A$ given by $s \mapsto s \otimes 1_A$. As a ring, the multiplication in A_S is determined by $(s_1 \otimes a_1)(s_2 \otimes a_2) = (s_1 s_2) \otimes (a_1 a_2)$ for $s_1, s_2 \in S$ and $a_1, a_2 \in A$.
4. Then the base change construction induces functors in the following situations:
   (a) If S is only an R-ring, then base change induces a functor $f^* : \mathbf{Alg}_R \to \mathbf{Ring}_S$.
   (b) If S is an R-algebra, then base change induces a functor $f^* : \mathbf{Ring}_R \to \mathbf{Ring}_S$ which restricts to a functor $f^* : \mathbf{Alg}_R \to \mathbf{Alg}_S$
   In either case, the base change functor is defined as follows:
   - On objects: For any R-algebra (A, φ), $f^*(A)$ is the S-algebra $S \otimes_R A$ defined above.
   - On morphisms: For any homomorphism of R-algebras $h : A \to B$, the image $f^*(h)$ is the map $\mathrm{id}_S \otimes h : S \otimes_R A \to S \otimes_R B$, defined by $s \otimes a \mapsto s \otimes h(a)$.
   (♠ TODO: comment on adjunction)

Definition 2.1.14. Let R and S be associative rings with identity (Definition C.0.7), and let $\varphi : R \to S$ be a unital ring homomorphism (Definition C.0.13). Let T be a ring.

1. Let $(M, +)$ be an abelian group (Definition C.0.3) equipped with either the structure of a $S-T$-bimodule (Definition 2.1.1) $(M, +, \cdot_S)$ or the structure of a $T-S$-bimodule $(M, +, \cdot_S)$.

   The *restriction of scalars of M along φ* is the R-module structure on the same underlying abelian group $(M, +)$ defined as follows:
   - If M is a $S-T$-bimodule, the restriction of scalars of M along φ, often denoted by $\varphi_* M$ (or ${}_R M$), is the $R-T$-bimodule whose R-action $\cdot_R : R \times M \to M$ is given by
   $$r \cdot_R m := \varphi(r) \cdot_S m$$
   for all $r \in R$ and $m \in M$.
   - If M is a $T-S$-bimodule, the restriction of scalars of M along φ, often denoted by $\varphi_* M$ (or M_R), is the $T-R$-bimodule whose R-action $\cdot_R : M \times R \to M$ is given by
   $$m \cdot_R r := m \cdot_S \varphi(r)$$
   for all $r \in R$ and $m \in M$.
2. Let ${}_S\mathrm{Mod}_T$ and ${}_T\mathrm{Mod}_S$ be the categories of $S-T$ and $T-S$-bimodules (Definition 2.1.3), respectively, and similarly for R.

   The *restriction of scalars functor for modules* is the covariant functor (Definition 1.2.2) induced by φ, defined for both left and right modules:
   - For left S-modules, it is the functor $\varphi_* : {}_S\mathrm{Mod}_T \to {}_R\mathrm{Mod}_T$ defined as follows:
     (a) On objects: For any left S-module M, $\varphi_*(M)$ is the left R-module obtained by restriction of scalars along φ.
     (b) On morphisms: For any homomorphism (Definition 2.1.2) of left S-modules $h : M \to N$, the image $\varphi_*(h) : \varphi_*(M) \to \varphi_*(N)$ is the map h itself, viewed as a homomorphism of left R-modules.
   - For right S-modules, it is the functor $\varphi_* : {}_T\mathrm{Mod}_S \to {}_T\mathrm{Mod}_R$ defined as follows:
     (a) On objects: For any right S-module M, $\varphi_*(M)$ is the right R-module obtained by restriction of scalars along φ.
     (b) On morphisms: For any homomorphism of right S-modules $h : M \to N$, the image $\varphi_*(h) : \varphi_*(M) \to \varphi_*(N)$ is the map h itself, viewed as a homomorphism of right R-modules.

   In either context, if φ is an inclusion map (making R a subring of S), this functor is often called the *forgetful functor*.
3. Let Ring_S (or S/Ring) denote the category of S-rings. Let Ring_R be defined similarly.

   The *restriction of scalars functor for rings*, denoted by $\varphi_* : \mathsf{Ring}_S \to \mathsf{Ring}_R$ is the functor defined as follows:
   - On objects: Let (A, ψ) be an S-ring. Then $\varphi_*(A)$ is the R-ring $(A, \psi \circ \varphi)$, where the structure map is the composition $R \xrightarrow{\varphi} S \xrightarrow{\psi} A$.
   - On morphisms: For any morphism of S-rings $h : (A, \psi_A) \to (B, \psi_B)$, the image $\varphi_*(h)$ is the map h itself, which satisfies $h \circ (\psi_A \circ \varphi) = \psi_B \circ \varphi$ and is thus a morphism of R-rings.

   This functor simply pre-composes the structure map with φ, effectively "forgetting" the factorization through S.

   The restriction of scalars functor restricts to a functor $\varphi_* : \mathbf{Alg}_S \to \mathbf{Alg}_R$ (Definition C.0.29).

Definition 2.1.15. Let R and S be rings (Definition C.0.7) (not necessarily commutative), and let $f : R \to S$ be a ring homomorphism (Definition C.0.13). The homomorphism f endows S with two bimodule structures — that of an (R, S)-bimodule and of an (S, R)-bimodule via restriction of scalars (Definition 2.1.14). Let T be a ring.

1. The *co-extension of scalars* (or simply *co-extension*) along f is defined separately for modules:
   - For an $R-T$-bimodule M, the co-extension of scalars of M along f is the $S-T$-bimodule $\mathrm{Hom}_R(S, M)$ (Definition 2.1.8) of left R-module homomorphisms from the $R-S$-bimodule S to M.
   - For a $T-R$-bimodule M, the co-extension of scalars of M along f is the $T-S$-bimodule $\mathrm{Hom}_R(S, M)$ of right R-module homomorphisms from the $S-R$-bimodule S to M.
2. The *co-extension of scalars functor* (or *coinduction functor*), denoted by $f^!$ or CoInd_R^S, is the functor given by:
   - For left R-modules:
   $$f^! : {}_R\mathsf{Mod}_T \to {}_S\mathsf{Mod}_T, \quad M \mapsto \mathrm{Hom}_R(S, M).$$
   This functor is the right adjoint to the restriction of scalars functor $f_* : {}_S\mathsf{Mod} \to {}_R\mathsf{Mod}$.
   - For right R-modules:
   $$f^! : {}_T\mathsf{Mod}_R \to {}_T\mathsf{Mod}_S, \quad M \mapsto \mathrm{Hom}_R(S, M).$$
   This functor is the right adjoint to the restriction of scalars functor $f_* : \mathsf{Mod}_S \to \mathsf{Mod}_R$.

Theorem 2.1.16 (Extension-Restriction and Restriction-Coextension adjunction for modules)**.** Let R and S be rings (Definition C.0.7) (not necessarily commutative), and let $f : R \to S$ be a ring homomorphism (Definition C.0.13). Let T be a ring.

1. **Extension-Restriction adjunction:**
   (a) **Restriction on the Left:** The extension of scalars functor (Definition 2.1.13)
   $$f^* = S \otimes_R - : {}_R\mathsf{Mod}_T \to {}_S\mathsf{Mod}_T$$
   (Definition 2.1.3) is left adjoint (Definition 2.5.1) to the restriction of scalars functor (Definition 2.1.14)
   $$f_* : {}_S\mathsf{Mod}_T \to {}_R\mathsf{Mod}_T.$$
   Explicitly, for any $R-T$-bimodule M and any $S-T$-bimodule N, there is a natural isomorphism:
   $$\mathrm{Hom}_{S-T}(S \otimes_R M, N) \cong \mathrm{Hom}_{R-T}(M, f_* N).$$
   (b) **Restriction on the Right:** The extension of scalars functor
   $$f^* = - \otimes_R S : {}_T\mathsf{Mod}_R \to {}_T\mathsf{Mod}_S$$
   is left adjoint to the restriction of scalars functor
   $$f_* : {}_T\mathsf{Mod}_S \to {}_T\mathsf{Mod}_R.$$

Explicitly, for any $T-R$-bimodule M and any $T-S$-bimodule N, there is a natural isomorphism:

$$\mathrm{Hom}_{T-S}(M \otimes_R S, N) \cong \mathrm{Hom}_{T-R}(M, f_* N).$$

2. **Restriction-Coextension adjunction:**
   (a) **Restriction on the Left:** The co-extension of scalars functor (Definition 2.1.15)

   $$f^! = \mathrm{Hom}_R(S, -) : {}_R\mathsf{Mod}_T \to {}_S\mathsf{Mod}_T$$

   is right adjoint (Definition 2.5.1) to the restriction of scalars functor

   $$f_* : {}_S\mathsf{Mod}_T \to {}_R\mathsf{Mod}_T.$$

   Explicitly, for any $S-T$-bimodule N and any $R-T$-bimodule M, there is a natural isomorphism:

   $$\mathrm{Hom}_{R-T}(f_* N, M) \cong \mathrm{Hom}_{S-T}(N, \mathrm{Hom}_R(S, M)).$$

   (b) **Restriction on the Right:** The co-extension of scalars functor

   $$f^! = \mathrm{Hom}_R(S, -) : {}_T\mathsf{Mod}_R \to {}_T\mathsf{Mod}_S$$

   is right adjoint to the restriction of scalars functor

   $$f_* : {}_T\mathsf{Mod}_S \to {}_T\mathsf{Mod}_R.$$

   Explicitly, for any $T-S$-bimodule N and any $T-R$-bimodule M, there is a natural isomorphism:

   $$\mathrm{Hom}_{T-R}(f_* N, M) \cong \mathrm{Hom}_{T-S}(N, \mathrm{Hom}_R(S, M))$$

   where here $\mathrm{Hom}_R(S, M)$ uses the left R-module structure on S and the right R-module structure on M.

2.2. **Limits and colimits in categories.**

2.2.1. *Categorical products, coproducts, and direct sums.* It is easier to first learn about products and coproducts (Definition 2.2.1) before learning about general limits and colimits (Definition 2.2.8).

One of the nice features about many algebraic structures such as groups and modules, is that simply taking their Cartesian product produces an object in the same category. In fact, the notion of product is a categorical one. Let us first define the categorical notion of product and the dual notion of coproducts:

Definition 2.2.1 (Product in a category). Let $\mathcal{C}$ be a category and let $\{X_i\}_{i \in I}$ be a family of objects in $\mathcal{C}$ indexed by a class I.

1. A *product of the family* $\{X_i\}$ is an object P of $\mathcal{C}$ together with a "universal" family of morphisms

   $$\pi_i : P \to X_i, \quad \text{for each } i \in I.$$

   More precisely, for any object Y and any family of morphisms $\{f_i : Y \to X_i\}_{i \in I}$, there exists a unique morphism

   $$f : Y \to P$$

making the following diagram commute for all $i \in I$, i.e. $\pi_i \circ f = f_i$:

$$\begin{tikzcd}
Y \arrow[d, dashed, "\exists! f"'] \arrow[dr, "f_i"] & \\
\prod X_i \arrow[r, "\pi_i"'] & X_i
\end{tikzcd}$$

Such a product is often denoted by $\prod_{i \in I} X_i$. If $\prod_{i \in I} X_i$ exists in $\mathcal{C}$, then it is unique up to unique isomorphism by the universal property described above.

Equivalently, the product $\prod_{i \in I} X_i$ is the limit (Definition 2.2.8) of the diagram (Definition 2.2.6) $I \to \mathcal{C}, i \mapsto X_i$, where I is made into a category whose objects are the members of I and whose morphisms are just the identity morphisms.

2. A *coproduct* (or synonymously *direct sum*) of the family $\{X_i\}$ is an object C of $\mathcal{C}$ together with a "universal" family of morphisms

$$\iota_i : X_i \to C, \quad \text{for each } i \in I.$$

More precisely, for any object Y and any family of morphisms $\{g_i : X_i \to Y\}_{i \in I}$, there exists a unique morphism

$$g : C \to Y$$

making the following diagram commute for all $i \in I$, i.e. $g \circ \iota_i = g_i$:

$$\begin{tikzcd}
X_i \arrow[r, "\iota_i"] \arrow[dr, "g_i"'] & \coprod X_i \arrow[d, dashed, "\exists! g"] \\
& Y
\end{tikzcd}$$

Such a coproduct is often denoted by $\coprod_{i \in I} X_i$ or $\oplus_{i \in I} X_i$. If $\coprod_{i \in I} X_i$ exists in $\mathcal{C}$, then it is unique up to unique isomorphism by the universal property described above.

Equivalently, the coproduct $\coprod_{i \in I} X_i$ is the colimit (Definition 2.2.8) of the diagram (Definition 2.2.6) $I \to \mathcal{C}, i \mapsto X_i$, where I is made into a category whose objects are the members of I and whose morphisms are just the identity morphisms.

For concrete categories (Definition A.0.8), cartesian products often result in the categorical products:

Definition 2.2.2 (Product of Sets)**.** Let I be a (possibly infinite but small) index set and let $\{A_i\}_{i \in I}$ be a family of sets indexed by I. The *Cartesian product of the family* $\{A_i\}_{i \in I}$, denoted by $\prod_{i \in I} A_i$, is defined as the set of all tuples/functions (Definition C.0.1)

$$\prod_{i \in I} A_i := \{(a_i)_{i \in I} \mid a_i \in A_i \text{ for all } i \in I\},$$

where $(a_i)_{i \in I}$ denotes a function from I to $\bigcup_{i \in I} A_i$ such that $(a_i)_{i \in I}(i) = a_i \in A_i$ for each $i \in I$.

The Cartesian product $\prod_{i \in I} A_i$ is the product (Definition 2.2.1) of the objects A_i in the category of sets (Definition 1.1.7).

The self product of a set A indexed by I is often denoted by A^I. Note that elements of A^I can be identified with functions (Definition C.0.1) $I \to A$. The finite self product of A taken

n times is often denoted by A^n. For finitely many sets $A_1, \ldots, A_n$, their Cartesian product is denoted by $A_1 \times \cdots \times A_n$. Elements of such a finite product may be written as $(a_1, \ldots, a_n)$.

Definition 2.2.3 (Product of Groups)**.** Let $\{G_i\}_{i \in I}$ be a family of groups indexed by a (possibly infinite but small) set I, each with group operation denoted multiplicatively and identity element e_i. The *(direct) product of the family* $\{G_i\}_{i \in I}$, denoted by $\prod_{i \in I} G_i$, is defined as the set of all tuples/functions

$$\prod_{i \in I} G_i := \{(g_i)_{i \in I} \mid g_i \in G_i \text{ for all } i \in I\},$$

equipped with the binary operation defined componentwise by

$$(g_i)_{i \in I} \cdot (h_i)_{i \in I} := (g_i h_i)_{i \in I}$$

for all $(g_i)_{i \in I}, (h_i)_{i \in I} \in \prod_{i \in I} G_i$. Then $\prod_{i \in I} G_i$ is a group with the identity element $(e_i)_{i \in I}$ and inverses given by

$$(g_i)_{i \in I}^{-1} = (g_i^{-1})_{i \in I}.$$

The product $\prod_{i=1} G_i$ is the product (Definition 2.2.1) of the objects G_i in the category of groups (Definition C.0.4). As a set, note that $\prod_{i \in I} G_i$ coincides with the product $\prod_{i \in I} G_i$ (Definition 2.2.2) of the G_i as sets.

A self product of a group G (indexed by a small set I), is often denoted by G^I. A finite self product of a group G taken n times is often denoted by G^n. In case that G is abelian, these may be written as $G^{\oplus I}$ and $G^{\oplus n}$ respectively.

The product of finitely many groups $G_1, \ldots, G_n$ is often denoted by $G_1 \times \cdots \times G_n$.

Definition 2.2.4 (Product of Modules)**.** Let R and S be (not necessarily commutative) rings (Definition C.0.7), and let $\{M_i\}_{i \in I}$ be a (possibly infinite but small) family of (R, S)-bimodules (Definition 2.1.1).

left R-modules, of right R-modules, of two-sided R-modules (Definition 2.1.1), or of

The *(direct) product of the family* $\{M_i\}_{i \in I}$ is defined, as a group (Definition C.0.3), as the product of sets (Definition 2.2.3):

$$\prod_{i \in I} M_i := \{(m_i)_{i \in I} \mid m_i \in M_i \text{ for all } i \in I\}.$$

$\prod_{i \in I} M_i$ inherits a natural R-S module structure defined componentwise by the following rules for all $(m_i)_{i \in I}, (n_i)_{i \in I} \in \prod_{i \in I} M_i$ and all scalars $r \in R$, $s \in S$:

$$(m_i)_{i \in I} + (n_i)_{i \in I} := (m_i + n_i)_{i \in I}, \qquad r \cdot (m_i)_{i \in I} \cdot s := (r \cdot m_i \cdot s)_{i \in I}.$$

The zero element of $\prod_{i \in I} M_i$ is the tuple $(0)_{i \in I}$, and additive inverses are given componentwise:

$$-(m_i)_{i \in I} := (-m_i)_{i \in I}.$$

Note that we can define the product of a family $\{M_i\}_{i \in I}$ of left/right/two-sided R-modules by taking the natural bimodule structure (Definition 2.1.1) of each module.

As usual (Definition 2.2.3), $\prod_{i\in I} M_i$ is the categorical product (Definition 2.2.1) of the objects M_i in the appropriate category of modules (Definition 2.1.3). Moreover, the product of finitely many modules $M_1, \dots, M_n$ is often written as $M_1 \times \cdots \times M_n$, which agrees with notation for the product of finitely many groups (Definition 2.2.3). We often write the self-product of a module M indexed by a (small) set I as M^I or by $M^{\oplus I}$. A finite self-product of a module M taken n times is often denoted by M^n or $M^{\oplus n}$; note that these all agree with the notations for abelian groups.

Definition 2.2.5 (Coproduct of Modules)**.** Let R and S be (not necessarily commutative) rings (Definition C.0.7), and let $\{M_i\}_{i\in I}$ be a (possibly infinite but small) family of (R, S)-bimodules.

The *coproduct (direct sum) of the family* $\{M_i\}_{i\in I}$, denoted by $\bigoplus_{i\in I} M_i$, is constructed as

$$\bigoplus_{i\in I} M_i := \left\{ (m_i)_{i\in I} \in \prod_{i\in I} M_i \mid m_i = 0 \text{ for all but finitely many } i \in I \right\}$$

(Definition 2.2.4) consisting of all tuples with only finitely many nonzero entries.

Addition and scalar multiplication in $\bigoplus_{i\in I} M_i$ are defined componentwise as in the direct product (Definition 2.2.4):

$$(m_i)_{i\in I} + (n_i)_{i\in I} := (m_i + n_i)_{i\in I}, \qquad r \cdot (m_i)_{i\in I} \cdot s := (r \cdot m_i \cdot s)_{i\in I}, \qquad r \in R,\ s \in S.$$

In all cases, the zero element is $(0)_{i\in I}$, and additive inverses are given by $-(m_i)_{i\in I} := (-m_i)_{i\in I}$.

Note that we can define the coproduct of a family $\{M_i\}_{i\in I}$ of left/right/two-sided R-modules by taking the natural bimodule structure (Definition 2.1.1) of each module.

(♠ TODO: submodule) Note that $\bigoplus_{i\in I} M_i$ is a submodule of $\prod_{i\in I} M_i$. Moreover, $\bigoplus_{i\in I} M_i$ is the coproduct (Definition 2.2.1) in the appropriate category of modules (Definition 2.1.3).

For finitely many modules $M_1, \dots, M_n$, the direct sum $\bigoplus_{j=1}^n M_j$, which may also be written as $M_1 \oplus \cdots \oplus M_n$, is simply the usual Cartesian product $\prod_{j=1}^n M_j$ (Definition 2.2.4) of the modules, as every tuple automatically has only finitely many nonzero entries.

2.2.2. *General limits and colimits.* It is probably best to keep in mind that the definitions listed in this subsubsection are intended as precise definitions, rather than intuitively helpful ones.

Definition 2.2.6 (Diagram in a category and category of diagrams)**.** Let $\mathcal{C}$ be a (large) category (Definition 1.1.1), and let I be a (large) category (Definition 1.1.1).

1. A *diagram of shape I in $\mathcal{C}$* is a functor (Definition 1.2.2) $D : I \to \mathcal{C}$. We often denote such a diagram by the family $\{D(i)\}_{i\in \mathrm{Ob}(I)}$ with transition maps given by the functorial image of morphisms in I.

   A diagram is also synonymously called a *system*. Moreover, the category I is called the *index category* or the *indexing category of the diagram D*.

2. Given two diagrams $D, E : I \to \mathcal{C}$, a *morphism of diagrams* is a simply a natural transformation (Definition 1.3.1) $D \Rightarrow E$ of the functors D and E.
3. The *category of I-shaped diagrams in $\mathcal{C}$* or simply *diagram category (of I-shaped diagrams in $\mathcal{C}$)*, often denoted $\mathcal{C}^I$, $[I, \mathcal{C}]$, or $\mathrm{Fun}(I, \mathcal{C})$, is the (large) category whose objects are functors $I \to \mathcal{C}$ (that is, diagrams of shape I in $\mathcal{C}$) and whose morphisms are natural transformations (Definition 1.3.1) between such functors. The category $\mathcal{C}^I$ is also called the *functor category of functors $I \to \mathcal{C}$*. Equivalently, the functor category $\mathcal{C}^I$ is the category $\mathrm{PreShv}(I^{\mathrm{op}}, \mathcal{C})$ of presheaves (Definition 6.0.1) on I^{op} with values in $\mathcal{C}$ and hence notations for presheaf categories are applicable as notations for functor categories.

   If $\mathcal{C}$ is locally small (Definition 1.1.2) and I is small, then $\mathcal{C}^I$ is locally small by Lemma 2.2.7.

Lemma 2.2.7. Let $\mathcal{C}$ be a small category (Definition 1.1.2) (resp. U-small category where U is some universe (Definition A.0.3)) and let $\mathcal{A}$ be a locally small (Definition 1.1.2) category (resp. U-locally small category). The presheaf category $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ (Definition 6.0.1) is locally small (resp. U-locally small).

Proof. A morphism $\mathcal{F} \to \mathcal{G}$ in $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ is a natural transformation (Definition 1.3.1) of the functors $\mathcal{F}, \mathcal{G} : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$. Such a natural transformation is encoded by a family $(\eta_C)_C$ of morphisms (satisfying certain conditions) $\eta_C : \mathcal{F}(C) \to \mathcal{G}(C)$ in $\mathcal{A}$ over objects C of $\mathcal{C}^{\mathrm{op}}$. The product $\prod_{C \in \mathrm{Ob}\, \mathcal{C}^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}(C), \mathcal{G}(C))$ is a product of (U-small) sets indexed by a (U-small) set, and the collection of natural transformations is a subset of this set. Therefore, $\mathrm{Hom}_{\mathrm{PreShv}(\mathcal{C}, \mathcal{A})}(\mathcal{F}, \mathcal{G})$ is a (U-small) set. $\square$

Definition 2.2.8 (Cones, limits and colimits in a category)**.** Let $\mathcal{C}$ be a (large) category (Definition 1.1.1), let I be a (large) category, and let $D : I \to \mathcal{C}$ be a diagram (Definition 2.2.6) (Definition 2.2.6).

1. A *cone to the diagram D* is an object $L \in \mathcal{C}$ together with a family of morphisms

$$\{\pi_i : L \to D(i)\}_{i \in I}$$

   such that for every morphism $f : i \to j$ in I, the diagram

$$\begin{tikzcd} & L \arrow[dl, "\pi_i"'] \arrow[dr, "\pi_j"] & \\ D(i) \arrow[rr, "D(f)"] & & D(j) \end{tikzcd}$$

   commutes, i.e. $D(f) \circ \pi_i = \pi_j$.
2. A cone $(L, \{\pi_i\})$ is called a *limit of D* if it satisfies the following "universal property": for any cone $(C, \{f_i\})$ over D, there exists a *unique* morphism $u : C \to L$ such that

$$\pi_i \circ u = f_i \quad \text{for all } i \in I.$$

   Visually, the following diagrams commute every morphism $f : i \to j$ in I:

$$\begin{tikzcd}
 & C \arrow[dl, bend right, "f_i"'] \arrow[dr, bend left, "f_j"] \arrow[d, dashed, "\exists! u"] & \\
 & L \arrow[dl, "\pi_i"] \arrow[dr, "\pi_j"'] & \\
D(i) \arrow[rr, "D(f)"] & & D(j).
\end{tikzcd}$$

If such a cone exists, then the object L is necessarily unique up to unique isomorphism by the universal property. In this case, L is denoted by $\lim_{i \in I} D$ or $\lim D$.

3. A *cocone from the diagram D* is an object $C \in \mathcal{C}$ together with a family of morphisms
$$\{\iota_i : D(i) \to C\}_{i \in I}$$
such that for every morphism $f : i \to j$ in I, the diagram
$$\begin{tikzcd}
D(i) \arrow[rr, "D(f)"] \arrow[dr, "\iota_i"'] & & D(j) \arrow[dl, "\iota_j"] \\
 & C &
\end{tikzcd}$$
commutes, i.e. $\iota_j \circ D(f) = \iota_i$.

4. A cocone $(L, \{\iota_i\})$ is called a *colimit of D* if it satisfies the following "universal property": for any cocone $(C, \{g_i\})$ under D, there exists a *unique* morphism $u : L \to C$ such that
$$u \circ \iota_i = g_i \quad \text{for all } i \in I.$$
Visually, the following diagrams commute every morphism $f : i \to j$ in I:
$$\begin{tikzcd}
D(i) \arrow[rr, "D(f)"] \arrow[dr, "\iota_i"] \arrow[ddr, bend right, "g_i"'] & & D(j) \arrow[dl, "\iota_j"'] \arrow[ddl, bend left, "g_j"] \\
 & L \arrow[d, dashed, "\exists! u"] & \\
 & C & .
\end{tikzcd}$$
If such a cocone exists, then the object L is necessarily unique up to unique isomorphism by the universal property. In this case, L is denoted by $\operatorname{colim}_{i \in I} D$ or $\operatorname{colim} D$.

A limit/colimit is called *finite* (resp. *small*) if the index category I is finite (resp. small).

Some authors use the terms *projective limit* or *inverse limit* to refer to what is defined here as a limit, Similarly, the terms *inductive limit* or *direct limit* are sometimes used to mean a colimit. However, these phrases can have more specific meanings to other authors: a *projective* or *inverse limit* may refer to a limit over a diagram indexed by a codirected poset (Definition C.0.17). Likewise, an *inductive* or *direct limit* may refer to a colimit over a directed poset (Definition C.0.17) (see Definition 2.2.13) .

Thus, while the terms are sometimes used interchangeably with "limit" and "colimit," they may also emphasize particular indexing shapes and directions, distinguishing them from general limits and colimits taken over arbitrary small categories.

Example 2.2.9 (Geometric and Algebraic Limits/Colimits)**.** Many fundamental constructions in topology and algebra are characterized by universal properties of limits and colimits.

1. **The p-adic Integers (Inverse Limit):** The ring of p-adic integers $\mathbb{Z}_p$ is the limit of the inverse system of finite cyclic rings $\mathbb{Z}/p^n\mathbb{Z}$ with quotient maps:
$$\mathbb{Z}_p \cong \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} = \{(a_n)_{n\in\mathbb{N}} \in \prod \mathbb{Z}/p^n\mathbb{Z} \mid a_{n+1} \equiv a_n \pmod{p^n}\}$$
2. **Absolute Galois Groups (Inverse Limit):** The absolute Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$ of a field K is the inverse limit of the Galois groups of all finite Galois extensions L/K contained in $\overline{K}$:
$$\mathrm{Gal}(\overline{K}/K) \cong \varprojlim_{L/K \text{ finite}} \mathrm{Gal}(L/K)$$
This gives G_K the structure of a profinite group.
3. **The Cone of a Space (Pushout):** Think of $X \times [0,1]$ as a cylinder. The map $X \to X \times [0,1]$ picks out the "top lid." The map $X \to \{*\}$ sends that entire lid to a single point. The **pushout** CX is the result of taking the cylinder and pinching the entire top lid into a single vertex.
$$\begin{tikzcd} X \arrow[r, "{(x,1)}"] \arrow[d] & X \times [0,1] \arrow[d] \\ \{*\} \arrow[r] & CX = (X \times [0,1])/(X \times \{1\}) \end{tikzcd}$$

(♠ TODO: examples of limits and colimits, including products, coproducts, equalizer and coequalizer)

Definition 2.2.10. Let $\mathcal{C}$ be a (large) category (Definition 1.1.1), let I be a small category (Definition 1.1.2), and let $D : I \to \mathcal{C}$ be a diagram (Definition 2.2.6).

A limit or colimit (Definition 2.2.8) is called *finite* (resp. *small*) if the indexing category I has finitely many objects and morphisms (resp. if I is a small category (Definition 1.1.2)).

Definition 2.2.11 (Complete and Cocomplete Category)**.** Let $\mathcal{C}$ be a category (Definition 1.1.1).

- The category $\mathcal{C}$ is called *complete* (resp. *finitely complete*) if all small limits (Definition 2.2.10) (resp. finite limits) exist in $\mathcal{C}$; that is, for every small diagram $D : J \to \mathcal{C}$ (with J a small category), the limit $\lim D$ exists and is an object of $\mathcal{C}$.
- The category $\mathcal{C}$ is called *cocomplete* (resp. *finitely cocomplete*) if all small colimits (Definition 2.2.10) (resp. finite colimits) exist in $\mathcal{C}$; that is, for every small diagram $D : J \to \mathcal{C}$, the colimit colim D exists and is an object of $\mathcal{C}$.

Definition 2.2.12 (Filtered category)**.** 1. A *filtered category* is a (nonempty, large) category $\mathcal{I}$ satisfying the following conditions:

- For every finite collection of objects $i_1, i_2, \ldots, i_n$ in $\mathcal{I}$, there exists an object j and morphisms
$$\phi_k : i_k \to j, \quad \text{for each } k = 1, \ldots, n.$$
- For every pair of morphisms $f, g : i \to j$ in $\mathcal{I}$, there exists an object k and a morphism
$$h : j \to k$$
that satisfies
$$h \circ f = h \circ g.$$

$$\begin{tikzcd}
i_1 \arrow[dr, dashed, "\phi_1"] & \\
& j \\
i_2 \arrow[ur, dashed, "\phi_2"'] &
\end{tikzcd}$$

FIGURE 1. *
Condition 1: Upper Bound

$$\begin{tikzcd}
i \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & j \arrow[r, dashed, "h"] & k
\end{tikzcd}$$

FIGURE 2. *
Condition 2: Coequalizing map

In other words, $\mathcal{I}$ is nonempty, any finite diagram of objects admits a cocone (Definition 2.2.8), and any pair of parallel morphisms become equal after post-composition with an appropriate morphism.

2. Dually, a *Cofiltered category* is a category whose opposite category (Definition 1.2.1) is filtered. More explicitly, A cofiltered category is a (nonempty, large) category $\mathcal{I}$ satisfying the following conditions:
   - For every finite collection of objects $i_1, i_2, \ldots, i_n$ in $\mathcal{I}$, there exists an object j and morphisms
   $$\phi_k : j \to i_k, \quad \text{for each } k = 1, \ldots, n.$$
   - For every pair of morphisms $f, g : j \to i$ in $\mathcal{I}$, there exists an object k and a morphism
   $$h : k \to j$$
   that satisfies
   $$f \circ h = g \circ h.$$

   In other words, $\mathcal{I}$ is nonempty, any finite diagram of objects admits a cone, and any pair of parallel morphisms become equal after pre-composition with an appropriate morphism.

Definition 2.2.13 (Special cases of limits)**.** Let $\mathcal{C}$ be a (large) category. Let I be a (large) category. Let $I \to \mathcal{C}$ be a diagram/system.

$$\begin{tikzcd}
 & i_1 \\
j \arrow[ur, dashed, "\phi_1"] \arrow[dr, dashed, "\phi_2"'] & \\
 & i_2
\end{tikzcd}$$

FIGURE 3. *
Condition 1: Lower Bound

$$\begin{tikzcd}
k \arrow[r, dashed, "h"] & j \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & i
\end{tikzcd}$$

FIGURE 4. *
Condition 2: Equalizing map

- Suppose that the system is a cofiltered system, i.e. I is a cofiltered category. A limit (Definition 2.2.8) of this diagram is often denoted by

$$\varprojlim_{i \in I} D(i)$$

and may be called a *cofiltered (inverse/projective) limit*. In case that the system is more specifically an inverse/projective system, i.e. I is a cofiltered poset (Definition C.0.17), the preferred term for such a limit is *inverse/projective limit*.

- Suppose that the system is a filtered system, i.e. I is a filtered category. A colimit of this diagram is often denoted by

$$\varinjlim_{i \in I} D(i)$$

and may be called a *filtered colimit* or a *direct/inductive/injective limit*. In case that the system is more specifically a direct/inductive system, i.e. I is a filtered poset (Definition C.0.17), the preferred term for such a limit is *direct/inductive limit*.

One noteworthy fact is that if a category has all (co)products (Definition 2.2.1) and (co)equalizers (Definition 2.2.14), then it has all (co)limits (Definition 2.2.8) (Theorem 2.2.15).

Definition 2.2.14 (Equalizer in a category)**.** Let $\mathcal{C}$ be a (large) category (Definition 1.1.1) and let $f, g : X \to Y$ be morphisms in $\mathcal{C}$.

1. An *equalizer of f and g* is an object E together with a morphism

$$e : E \to X$$

such that

$$f \circ e = g \circ e$$

and for any object Z with morphism $z : Z \to X$ satisfying

$$f \circ z = g \circ z,$$

there exists a unique morphism $u : Z \to E$ making the diagram commute:

$$e \circ u = z.$$

$$\begin{tikzcd}
Z \arrow[d, dashed, "\exists! u"'] \arrow[dr, "z"] & & \\
E \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}$$

If such an equalizer of f and g exists, then we say that the following *equalizer diagram is exact*:

$$\begin{tikzcd}
E \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}$$

2. A *coequalizer of f and g* is an object Q together with a morphism

$$q : Y \to Q$$

such that

$$q \circ f = q \circ g$$

and for any object Z with morphism $w : Y \to Z$ satisfying

$$w \circ f = w \circ g,$$

there exists a unique morphism $v : Q \to Z$ making the diagram commute:

$$v \circ q = w.$$

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] \arrow[dr, "w"'] & Q \arrow[d, dashed, "\exists! v"] \\
& & Z
\end{tikzcd}$$

If such a coequalizer of f and g exists, then we say that the following *coequalizer diagram is exact*:

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] & Q
\end{tikzcd}$$

Theorem 2.2.15. Let $\mathcal{C}$ be a category (Definition 1.1.1). Let $F : \mathcal{J} \to \mathcal{C}$ be a diagram (Definition 2.2.6) where $\mathcal{J}$ is a small category (Definition 1.1.2).

1. The limit (Definition 2.2.8) of F is constructed as the equalizer (Definition 2.2.14) of the pair of morphisms (s, t): assuming that the products (Definition 2.2.1) and equalizers below exist in $\mathcal{C}$, the limit $\lim F$ exists and

$$\lim F \cong \mathrm{eq}\left(\prod_{j \in \mathrm{Ob}(\mathcal{J})} F(j) \underset{t}{\overset{s}{\rightarrow}} \prod_{\alpha \in \mathrm{Mor}(\mathcal{J})} F(\mathrm{cod}(\alpha)) \right)$$

where $\mathrm{cod}(\alpha)$ stands for the codomain of the morphism α, and the morphisms s and t are induced by the universal property of the product, such that for any morphism $\alpha : i \to k$ in $\mathcal{J}$, the projection to the factor indexed by α is:
- $\pi_\alpha \circ s = F(\alpha) \circ \pi_i$
- $\pi_\alpha \circ t = \pi_k$

2. The colimit (Definition 2.2.8) of F is constructed as the coequalizer (Definition 2.2.14) of the pair of morphisms (s,t): assuming that the coproducts (Definition 2.2.1) and coequalizers below exist in $\mathcal{C}$, the colimit $\operatorname{colim} F$ exists and
$$\operatorname{colim} F \cong \operatorname{coeq}\left(\coprod_{\alpha\in\operatorname{Mor}(\mathcal{J})} F(\operatorname{dom}(\alpha)) \overset{s}{\underset{t}{\rightrightarrows}} \coprod_{j\in\operatorname{Ob}(\mathcal{J})} F(j)\right)$$
where $\operatorname{dom}(\alpha)$ stands for the domain of the morphism α, and the morphisms s and t are induced by the universal property of the coproduct, such that for any morphism $\alpha : i \to k$ in $\mathcal{J}$, the injection from the summand indexed by α is:
   - $s \circ \iota_\alpha = \iota_k \circ F(\alpha)$
   - $t \circ \iota_\alpha = \iota_i$

In particular,

1. If $\mathcal{C}$ has all nonempty finite (resp. small) products and equalizers, then $\mathcal{C}$ has all nonempty finite (resp. small) limits.
2. If $\mathcal{C}$ has all nonempty finite (resp. small) coproducts and coequalizers, then $\mathcal{C}$ has all nonempty finite (resp. small) colimits.
3. If $\mathcal{C}$ has all finite (resp. small) products and equalizers, then $\mathcal{C}$ has all finite (resp. small) limits.
4. If $\mathcal{C}$ has all finite (resp. small) coproducts and coequalizers, then $\mathcal{C}$ has all finite (resp. small) colimits.

2.3. **Additive categories.**

2.3.1. *Additive and abelian categories.* Given modules M and N, notice that $\operatorname{Hom}(M,N)$ (Definition 2.1.8) is not only a set, but also an abelian group. This, along with few other nice properties, makes the category of modules into an additive category (Definition 2.3.4).

Definition 2.3.1. Let $\mathcal{C}$ be a (large) category (Definition 1.1.1).

1. An object $I \in \mathcal{C}$ is called an *initial object* if for every object $X \in \mathcal{C}$ there exists a unique morphism
$$I \to X.$$
Equivalently, an initial object is a limit (Definition 2.2.8) of the empty diagram (Definition 2.2.6), if such a limit exists.
2. An object $F \in \mathcal{C}$ is called a *final object* (or *terminal object*) if for every object $X \in \mathcal{C}$ there exists a unique morphism
$$X \to F.$$
Equivalently, a final object is a colimit (Definition 2.2.8) of the empty diagram (Definition 2.2.6), if such a colimit exists.
3. An object $Z \in \mathcal{C}$ is called a *zero object* if Z is both initial and final in $\mathcal{C}$. In particular, for every object $X \in \mathcal{C}$ there exist unique morphisms
$$Z \to X \quad\text{and}\quad X \to Z.$$

In particular, if initial/final/zero objects exist in a cateogry, then they are unique up to unique isomorphism.

Definition 2.3.2. A *pointed category* is a category (Definition 1.1.1) that has a zero object (Definition 2.3.1).

Definition 2.3.3 (Zero morphism)**.** Let $\mathcal{C}$ be a pointed category (Definition 2.3.2). Let X and Y be objects in $\mathcal{C}$, and let 0 be a zero object (Definition 2.3.1) of $\mathcal{C}$. The *zero morphism* from X to Y is the unique composite morphism

$$0_{XY} : X \to 0 \to Y,$$

where $X \to 0$ is the unique morphism into the terminal object 0 and $0 \to Y$ is the unique morphism out of the initial object 0. This morphism is independent of the choice of the zero object 0.

Definition 2.3.4 (Additive category)**.** Let $\mathcal{A}$ be a locally small category (Definition 1.1.2).

1. $\mathcal{A}$ is said to be a *preadditive category* if the following hold:
   - For any two objects A, B in $\mathcal{A}$, the set $\mathrm{Hom}_{\mathcal{A}}(A, B)$ is an abelian group (Definition C.0.3), and composition of morphisms is bilinear.
   - There is a zero object (Definition 2.3.1) 0 in $\mathcal{A}$.
2. If $\mathcal{A}$ is preadditive, then it is called *additive* if it additionally satisfies the following:
   - For any two objects A, B in $\mathcal{A}$, there exists a product object $A \times B$ (Definition 2.2.1), often written $A \oplus B$, called the *direct sum of A and B*. In fact, $A \oplus B$ is not only a product but also a coproduct (Definition 2.2.5) of A and B (Lemma 2.3.5).

   Given a finite collection $\{A_i\}_i$ of objects A_i in an additive category $\mathcal{A}$, we may more generally speak of the *direct sum* $\bigoplus_i A_i$; it has canonical injections from and projections to each A_i.

Lemma 2.3.5. Let $\mathcal{A}$ be a preadditive category (Definition 2.3.4). Finite products (Definition 2.2.1) in $\mathcal{A}$ coincide with finite coproducts (Definition 2.2.1). More precisely, if $\{A_i\}_{i=1}^n$ is a finite collection of objects of $\mathcal{A}$, then

1. if $\prod_{i=1}^n A_i$ exists, then so does $\coprod_{i=1}^n A_i$ and these are naturally isomorphic.
2. if $\coprod_{i=1}^n A_i$, then so does $\prod_{i=1}^n A_i$ and these are naturally isomorphic.

Proof. (♠ TODO:) □

In fact, morphisms between modules have kernels, cokernels, and nice images and coimages

Definition 2.3.6. Let $\mathcal{C}$ be a (large) (Definition 1.1.1) pointed category (Definition 2.3.2), i.e. a category with a zero object (Definition 2.3.1) 0. Let $X, Y \in \mathrm{Ob}(\mathcal{C})$ be an object and let $f : X \to Y$ be a morphism.

1. A morphism $i : K \to X$ is called the *kernel of f* if:
   (a) $f \circ i = 0$, where 0 is the zero morphism (Definition 2.3.3) $K \to Y$,
   (b) for any morphism $g : Z \to X$ such that $f \circ g = 0$, there exists a unique morphism $u : Z \to K$ such that $g = i \circ u$.

The kernel, if it exists, is unique up to unique isomorphism (Definition 1.1.13). $\ker(f)$ denotes the object K determined (up to isomorphism) by a kernel of f.

Equivalently, $\ker(f)$ is the equalizer (Definition 2.2.14) of f and the 0 morphism $X \to Y$.

2. a morphism $p : Y \to Q$ is called the *cokernel of f* if:
   (a) $p \circ f = 0$, where 0 is the zero morphism (Definition 2.3.1) $X \to Q$,
   (b) for any morphism $g : Y \to Z$ such that $g \circ f = 0$, there exists a unique morphism $v : Q \to Z$ such that $g = v \circ p$.

   The cokernel, if it exists, is unique up to unique isomorphism. $\operatorname{coker}(f)$ denotes the object Q determined (up to isomorphism) by a cokernel of f.

   Equivalently, $\operatorname{coker}(f)$ is the coequalizer (Definition 2.2.14) of f and the 0 morphism $X \to Y$.

Definition 2.3.7 (Monomorphism and Epimorphism in Categories)**.** Let $\mathcal{C}$ be a category (Definition 1.1.1). For objects $A, B \in \mathcal{C}$, let $f : A \to B$ be a morphism in $\mathcal{C}$.

- The morphism f is called a *monomorphism* (or a *monic morphism*) if for every object X and every pair of morphisms $g_1, g_2 : X \to A$, the equality $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.
- The morphism f is called an *epimorphism* (or an *epic morphism*) if for every object Y and every pair of morphisms $h_1, h_2 : B \to Y$, the equality $h_1 \circ f = h_2 \circ f$ implies $h_1 = h_2$.

Definition 2.3.8. Let $\mathcal{C}$ be a category (Definition 1.1.1), and let $f : A \to B$ be a morphism in $\mathcal{C}$.

1. An *image of f* consists of an object $I \in \mathrm{Ob}(\mathcal{C})$ together with a factorization of f into two morphisms

$$A \xrightarrow{e} I \xrightarrow{m} B,$$

where e is an epimorphism (Definition 2.3.7) and m is a monomorphism (Definition 2.3.7), such that for any other factorization

$$A \xrightarrow{e'} I' \xrightarrow{m'} B$$

with e' epi and m' mono, there exists a unique isomorphism $\varphi : I \simeq I'$ satisfying $m = m'\varphi$ and $\varphi e = e'$.

$$\begin{tikzcd}
 & I \arrow[dr, hook, "m"] \arrow[dd, dashed, "\wr\ \exists!\varphi"] & \\
A \arrow[ur, two heads, "e"] \arrow[dr, two heads, "e'"'] & & B \\
 & I' \arrow[ur, hook, "m'"'] &
\end{tikzcd}$$

The monomorphism $m : I \to B$ (or equivalently its subobject class) is called the *image of f in $\mathcal{C}$*.

2. Let $\mathcal{C}$ be a category (Definition 1.1.1), and let $f : A \to B$ be a morphism in $\mathcal{C}$. A *coimage of f* consists of an object $C \in \mathrm{Ob}(\mathcal{C})$ together with a factorization of f into two morphisms
$$A \xrightarrow{e} C \xrightarrow{m} B,$$
where e is an epimorphism and m is a monomorphism, such that for any other factorization
$$A \xrightarrow{e'} C' \xrightarrow{m'} B$$
with e' epi and m' mono, there exists a unique isomorphism $\varphi : C \simeq C'$ satisfying $m = m'\varphi$ and $\varphi e = e'$.
$$\begin{tikzcd} & C \arrow[dr, hook, "m"] \arrow[dd, dashed, "\exists!\varphi", "\wr"'] & \\ A \arrow[ur, two heads, "e"] \arrow[dr, two heads, "e'"'] & & B \\ & C' \arrow[ur, hook, "m'"'] & \end{tikzcd}$$
The epimorphism $e : A \to C$ (or equivalently its quotient class) is called the *coimage of f in $\mathcal{C}$*.

Lemma 2.3.9. Let R, S be (not-necessarily commutative) rings with unity (Definition C.0.7), let M, N be R-S-bimodules (Definition 2.1.1), and let $f : M \to N$ be a module homomorphism (Definition 2.1.2). The kerel, cokernel, image, and coimage (Definition 2.1.7) of f are respectively the categorical kernel, cokernel (Definition 2.3.6), image, and coimage (Definition 2.3.8).

Definition 2.3.10 (Abelian category)**.** Let $\mathcal{A}$ be a category. The category $\mathcal{A}$ is an *abelian category* if:

- $\mathcal{A}$ is an additive category (Definition 2.3.4).
- Every morphism $f : A \to B$ has a kernel $\ker(f)$ and a cokernel $\mathrm{coker}(f)$ (Definition 2.3.6).
- For every morphism $f : A \to B$, the canonical morphism $\mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism, where
$$\mathrm{coim}(f) = \mathrm{coker}(\ker(f) \to A), \quad \mathrm{im}(f) = \ker(B \to \mathrm{coker}(f)).$$
(♠ TODO: I think I need to re-check this defintion) (♠ TODO: coimage)

.

It is also worth considering Grothendieck's additional axioms for abelian categories (Definition 2.5.7).

Proposition 2.3.11. Let $\mathcal{A}$ be a preadditive (Definition 2.3.4) (resp. additive (Definition 2.3.4), abelian (Definition 2.3.10)) category and let J be a small (Definition 1.1.2) category. The diagram category (Definition 2.2.6) $\mathcal{A}^J$ is preadditive (resp. additive, abelian).

2.4. **Additive functors between additive categories.**

Definition 2.4.1 (Additive functor)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be pre-additive categories. A functor

$$F : \mathcal{A} \to \mathcal{B}$$

is an *additive functor* if for every pair of objects $A, A' \in \mathcal{A}$, the induced map

$$F_{A,A'} : \mathrm{Hom}_{\mathcal{A}}(A, A') \to \mathrm{Hom}_{\mathcal{B}}(F(A), F(A'))$$

is a group homomorphism of abelian groups, or equvialently if it is enriched over the category Ab of abelian groups.

In the case that $\mathcal{A}$ and $\mathcal{B}$ are additive categories, this condition implies that F is is compatible with the additive structures on $\mathcal{A}$ and $\mathcal{B}$. More precisely,

- F sends the zero object $0_{\mathcal{A}}$ of $\mathcal{A}$ to the zero object $0_{\mathcal{B}}$ of $\mathcal{B}$, i.e.,
$$F(0_{\mathcal{A}}) = 0_{\mathcal{B}}.$$
- F preserves finite direct sums: For any finite family of objects $\{A_i\}_{i=1}^n$ in $\mathcal{A}$,
$$F\left(\bigoplus_{i=1}^n A_i\right) \cong \bigoplus_{i=1}^n F(A_i)$$
via the canonical isomorphism induced by F applied to the canonical injections and projections.

(♠ TODO: examples of additive functors)

We note that Hom's and tensor products induce bi-additive functors (Definition 2.4.2) on categories of modules.

Definition 2.4.2 (n-ary Additive Functor)**.** Let I be a finite set with $|I| = n$. Let $\{\mathcal{A}_i\}_{i\in I}$ be additive categories (Definition 2.3.4) and let $\mathcal{B}$ be an additive category. An *n-ary additive functor* (or *multilinear functor*)

$$F : \prod_{i\in I} \mathcal{A}_i \to \mathcal{B}$$

(Definition 1.5.1) is a functor such that for each fixed collection of all but one variable, the resulting functor in the remaining variable is additive (Definition 2.4.1). Equivalently, for every $j \in I$ and objects $(A_i)_{i\in I}$ and morphisms $f_1, f_2 : A_j \to A'_j$ in $\mathcal{A}_j$, we have

$$\begin{aligned} &F(A_1, \dots, A_{j-1}, f_1 + f_2, A_{j+1}, \dots, A_n) \\ =&F(A_1, \dots, A_{j-1}, f_1, A_{j+1}, \dots, A_n) \\ &+ F(A_1, \dots, A_{j-1}, f_2, A_{j+1}, \dots, A_n), \end{aligned}$$

and F preserves zero morphisms componentwise:

$$F(A_1, \dots, 0_{A_j, A'_j}, \dots, A_n) = 0_{F(A_1,\dots),F(A'_1,\dots)}.$$

A bifunctor that satisfies this property for $n = 2$ is simply called a *biadditive functor*.

2.4.1. *Exact functors between abelian categories.* One particularly nice kind of additive functor is an exact functor

Definition 2.4.3. Let $\mathcal{A}$ be an additive category (Definition 2.3.4). A sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

of moprhisms in $\mathcal{A}$ is called a *short exact sequence* if the morphisms satisfy:

- $f : A \to B$ is a monomorphism (Definition 2.3.7) and is the kernel of g (Definition 2.3.6),
- $g : B \to C$ is an epimorphism (Definition 2.3.7) and is the cokernel of f (Definition 2.3.6),
- the sequence is exact at (Definition 3.0.2) B, meaning $\mathrm{Im}(f) = \mathrm{Ker}(g)$ (Definition 2.3.8).

This means the sequence starts and ends with the zero object and is exact everywhere.

Definition 2.4.4. A short exact sequence (Definition 2.4.3) $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in an additive category (Definition 2.3.4) $\mathcal{A}$ is *split exact* if there exists a morphism $s : C \to B$ (called a *section*) such that $g \circ s = \mathrm{id}_C$, or equivalently (Theorem 2.4.5), a morphism $r : B \to A$ (called a *retraction*) such that $r \circ f = \mathrm{id}_A$.

Theorem 2.4.5. Let $\mathcal{A}$ be an additive category (Definition 2.3.4). For a short exact sequence (Definition 2.4.3) $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the following conditions are equivalent:

1. The sequence is split exact (Definition 2.4.4).
2. There exists a morphism (i.e. a section (Definition 2.4.4)) $s : C \to B$ such that $g \circ s = \mathrm{id}_C$.
3. There exists a morphism (i.e. a retraction (Definition 2.4.4)) $r : B \to A$ such that $r \circ f = \mathrm{id}_A$.
4. There exists an isomorphism $\varphi : B \xrightarrow{\cong} A \oplus C$ such that the following diagram commutes:
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "\varphi"'] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, "\iota_1"] & A \oplus C \arrow[r, "\pi_2"] & C \arrow[r] & 0 \end{tikzcd}$$
where ι_1 is the canonical injection and π_2 is the canonical projection.

Proposition 2.4.6. In an additive category $\mathcal{A}$, any split exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is preserved by any additive functor $F : \mathcal{A} \to \mathcal{B}$ to another additive category $\mathcal{B}$. That is, $0 \to F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \to 0$ is also a split exact sequence.

Definition 2.4.7. Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

1. Assume $\mathcal{C}$ admits all finite limits (Definition 2.2.8). A functor $F : \mathcal{C} \to \mathcal{D}$ is called *left exact* if it preserves all finite limits. Explicitly, for every finite diagram $D : \mathcal{J} \to \mathcal{C}$, the limit
$$\varprojlim(F \circ D)$$

exists and the canonical map

$$F(\varprojlim D) \xrightarrow{\cong} \varprojlim(F \circ D)$$

is an isomorphism.

2. Assume $\mathcal{C}$ admits all finite colimits (Definition 2.2.8). A functor $F : \mathcal{C} \to \mathcal{D}$ is called *right exact* if it preserves all finite colimits. Explicitly, for every finite diagram $D : \mathcal{J} \to \mathcal{C}$, the colimit
$$\varinjlim(F \circ D)$$
exists and the canonical map
$$\varinjlim(F \circ D) \xrightarrow{\cong} F(\varinjlim D)$$
is an isomorphism.
3. If $\mathcal{C}$ admits both finite limits and finite colimits, a functor F is called *exact* if it is both left exact and right exact.

Definition 2.4.8. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor (Definition 2.4.1) between abelian categories (Definition 2.3.10).

1. F is called *left exact* if it preserves all finite limits (Definition 2.2.8); since F is an additive functor between abelian categories, this is equivalent to saying that F preserves kernels (Definition 2.3.6)[2]

   Equivalently, for every left exact sequence in $\mathcal{A}$
$$0 \to A' \xrightarrow{f} A \xrightarrow{g} A''$$
the sequence
$$0 \to F(A') \xrightarrow{F(f)} F(A) \xrightarrow{F(g)} F(A'')$$
is exact at $F(A')$ and $F(A)$ (i.e., F preserves monomorphisms (Definition 2.3.7) and exactness at the first two terms).
2. Dually, F is called *right exact* if it preserves all finite colimits (Definition 2.2.8); since F is an additive functor between abelian categories, this is equivalent to saying that F preserves cokernels (Definition 2.3.6).

   Equivalently, for every right exact sequence in $\mathcal{A}$
$$A' \xrightarrow{f} A \xrightarrow{g} A'' \to 0,$$
the sequence
$$F(A') \xrightarrow{F(f)} F(A) \xrightarrow{F(g)} F(A'') \to 0$$
is exact at $F(A)$ and $F(A'')$ (i.e., F preserves epimorphisms (Definition 2.3.7) and exactness at the last two terms).

[2]To see this equivalence, note that
- finite products (Definition 2.2.1) coincide with finite direct sums (Definition 2.3.4) in abelian categories,
- additive functors preserve finite direct sums by definition,
- kernels (Definition 2.3.6) in a pointed category (Definition 2.3.2) are equalizers (Definition 2.2.14), and all equalizers in abelian categories are kernels,
- if all finite products and equalizers exist in a general category, then all finite limits exist in the category (Theorem 2.2.15)

3. F is called *exact* if it is both left and right exact.

The additive functor F is left/right exact if and only if it is left/right exact (Definition 2.4.7) in the more general sense, i.e. if it preserves all finite (Definition 2.2.10) limits/colimits (Definition 2.2.8)

Lemma 2.4.9. Let

$$F : \mathcal{A} \to \mathcal{B}$$

be an additive functor (Definition 2.4.1) between abelian categories (Definition 2.3.10). It is exact (Definition 2.4.8) if and only if it preserves kernels and cokernels (Definition 2.3.6).

Theorem 2.4.10 (Freyd-Mitchell Embedding Theorem)**.** Let $\mathcal{A}$ be a small (Definition 1.1.2) abelian category (Definition 2.3.10). There exists a ring (Definition C.0.7) R (which may not be commutative) and a functor $F : \mathcal{A} \to \mathrm{Mod}_R$ such that: (♠ TODO: Show thta exact functors preserve finite limits and colimits)

1. F is exact (Definition 2.4.8), meaning it preserves all finite limits and colimits (in particular, kernels, cokernels, and exact sequences).
2. F is fully faithful (Definition 1.3.5).

Consequently, any diagram-chasing argument valid for modules over a ring is also valid in any small abelian category, and by extension (using the fact that any exact diagram involves only a set of objects), in any abelian category.

The Snake lemma is an application to the Freyd-Mitchell embedding theorem (Theorem 2.4.10). More specifically, while one can prove the snake lemma for general abelian categories directly, one can alternatively first prove the snake lemma for the category of R-modules, then make use of the theorem to prove it in general.

Definition 2.4.11 (Reflecting a type of morphism)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between (large) categories (Definition 1.2.2), and let $\mathcal{P}$ be a property of morphisms (or more generally a property of sequences or families of morphisms) that is stable under isomorphism (Definition 1.1.13) (e.g. monomorphism, epimorphism (Definition 2.3.7), isomorphism, etc.). We say that F *reflects $\mathcal{P}$-morphisms* if for every morphism $f : x \to y$ in $\mathcal{C}$, whenever $F(f)$ has property $\mathcal{P}$ in $\mathcal{D}$, it follows that f has property $\mathcal{P}$ in $\mathcal{C}$.

One can speak of more general notions of reflection (e.g. reflection of limits (Definition 2.2.8), of colimits (Definition 2.2.8), of "exact sequences (Definition 3.0.2)", say for abelian categories, etc.)

Proposition 2.4.12. Let $F : \mathcal{A} \to \mathcal{B}$ be a full and faithful (Definition 1.3.5) additive functor (Definition 2.4.1) between abelian categories (Definition 2.3.10). Then F reflects exactness (Definition 2.4.11), i.e., given a sequence

$$0 \to A' \xrightarrow{f} A \xrightarrow{g} A'' \to 0$$

in $\mathcal{A}$, if the sequence

$$0 \to F(A') \xrightarrow{F(f)} F(A) \xrightarrow{F(g)} F(A'') \to 0$$

is exact in $\mathcal{B}$, then the original sequence is exact in $\mathcal{A}$. More generally, F reflects any exact sequence (Definition 3.0.2) and all limits and colimits that exist in $\mathcal{B}$ for diagrams coming from $\mathcal{A}$.

Lemma 2.4.13. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor (Definition 2.4.1) between additive categories (Definition 2.3.4). The functor F is faithful (Definition 1.3.5) if and only if for any morphism f in $\mathcal{A}$, we have $F(f) = 0$ exactly when $f = 0$.

Proof. To say that F is faithful means that for every pair of objects X and Y in $\mathcal{A}$, the induced abelian group (Definition C.0.3) homomorphism (Definition C.0.4)

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{B}}(F(X), F(Y))$$

on the Hom-sets is injective. Equivalently, this means that for every morphism $f : X \to Y$ in $\mathcal{A}$, we have $F(f) = 0$ if and only if $f = 0$. $\square$

Proposition 2.4.14. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor (Definition 2.4.8) between abelian categories (Definition 2.3.10). The functor F is faithful (Definition 1.3.5) if and only if it reflects (Definition 2.4.11) short exact sequences (Definition 2.4.3).

Proof. Suppose F is faithful. Let

$$A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3$$

be a sequence in $\mathcal{A}$ such that

$$F(A_1) \xrightarrow{F(f)} F(A_2) \xrightarrow{F(g)} F(A_3)$$

is exact in $\mathcal{B}$. Because F is exact, it preserves kernels and images, so

$$F(\ker(g)) = \ker(F(g)), \quad F(\mathrm{im}(f)) = \mathrm{im}(F(f)).$$

Exactness in $\mathcal{B}$ gives

$$\ker(F(g)) = \mathrm{im}(F(f)),$$

so

$$F(\ker(g)) = F(\mathrm{im}(f)).$$

Since F is faithful, it reflects (Definition 2.4.11) monomorphisms and epimorphisms (Definition 2.3.7) (Proposition 2.4.12), and morphisms that become equal via F must be equal in $\mathcal{A}$. Hence,

$$\ker(g) = \mathrm{im}(f).$$

Therefore, the sequence

$$A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3$$

is exact in $\mathcal{A}$, i.e., F reflects short exact sequences.

Conversely, suppose F reflects short exact sequences. Let $f : A \to B$ be a morphism in $\mathcal{A}$ such that

$$F(f) = 0.$$

Consider the sequence

$$0 \to A \xrightarrow{f} B.$$

Since $F(f) = 0$, we have the sequence

$$0 \to F(A) \xrightarrow{F(f)} F(B)$$

is exact at $F(A)$. Because F reflects short exact sequences, the original sequence must be exact at A, so f is a monomorphism with zero image, hence $f = 0$. Thus F is faithful by Lemma 2.4.13. $\square$

Lemma 2.4.15 (Snake lemma)**.** Let $\mathcal{A}$ be an abelian category (Definition 2.3.10). Let Given the commutative diagram with exact rows:

$$\begin{array}{ccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to & 0 \\ & & \downarrow a & & \downarrow b & & \downarrow c & & \\ 0 & \to & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & & \end{array}$$

there exists an exact sequence connecting the kernels and cokernels:

$$\ker(a) \xrightarrow{\bar{f}} \ker(b) \xrightarrow{\bar{g}} \ker(c) \xrightarrow{d} \text{coker}(a) \xrightarrow{\bar{f}'} \text{coker}(b) \xrightarrow{\bar{g}'} \text{coker}(c)$$

where d is the connecting homomorphism. Furthermore, if f is monic (Definition 2.3.7), then $\bar{f}$ is monic; if g' is epi (Definition 2.3.7), then $\bar{g}'$ is epi.

Proof. We first show this in the case that $\mathcal{A}$ is the category of R-modules (Definition 2.1.3) for any (not necessarily commutative) ring R. We perform a diagram chase on the elements of the modules.

1. **Construction of** d: Let $z \in \ker(c) \subseteq C$. Since g is surjective, there exists $y \in B$ such that $g(y) = z$. By commutativity, $g'(b(y)) = c(g(y)) = c(z) = 0$. Thus $b(y) \in \ker(g')$. By exactness of the bottom row, there exists a unique $x' \in A'$ such that $f'(x') = b(y)$. Define $d(z) = [x'] \in A'/\text{im}(a) = \text{coker}(a)$.

   We show that $[x']$ is independent of the choice of y. Suppose we choose another lift $y_1 \in B$ such that $g(y_1) = z$. Then $g(y - y_1) = g(y) - g(y_1) = z - z = 0$. By the exactness of the top row at B, there exists some $x \in A$ such that $f(x) = y - y_1$. Applying b, we have $b(y - y_1) = b(f(x))$. By the commutativity of the first square, $b(f(x)) = f'(a(x))$. Let x' and x'_1 be the unique elements in A' such that $f'(x') = b(y)$ and $f'(x'_1) = b(y_1)$. Then $f'(x' - x'_1) = b(y) - b(y_1) = b(y - y_1) = f'(a(x))$. Since f' is injective (due to the exactness of the bottom row at A'), we must have $x' - x'_1 = a(x)$. This implies $x' \equiv x'_1 \pmod{\text{im}(a)}$. Therefore, $[x'] = [x'_1]$ in coker(a), proving that the definition of $d(z)$ is independent of the choice of the preimage y.
2. **Exactness at** $\ker(b)$: Clearly $\bar{g} \circ \bar{f} = 0$. If $y \in \ker(\bar{g})$, then $g(y) = 0$, so $y = f(x)$ for some $x \in A$. Then $f'(a(x)) = b(f(x)) = b(y) = 0$. Since f' is injective, $a(x) = 0$, so $x \in \ker(a)$ and $y \in \text{im}(\bar{f})$.
3. **Exactness at** $\ker(c)$: If $z = \bar{g}(y)$ for $y \in \ker(b)$, then in the construction of $d(z)$, we can choose this y. Since $b(y) = 0$, we have $x' = 0$, so $d(z) = 0$. Conversely, if $d(z) = 0$, then the lift x' is in $\text{im}(a)$, say $x' = a(x)$. Then $b(f(x)) = f'(a(x)) = f'(x') = b(y)$. Thus $y - f(x) \in \ker(b)$ and $g(y - f(x)) = g(y) = z$.
4. **Exactness at coker**(a): Similar diagram chasing confirms $\bar{f}' \circ d = 0$ and the corresponding inclusion.

We verify the exactness of the sequence at $\operatorname{coker}(a)$, specifically the sequence $\ker(c) \xrightarrow{d} \operatorname{coker}(a) \xrightarrow{\bar{f}'} \operatorname{coker}(b)$. Let $z \in \ker(c)$. By the construction of the connecting homomorphism, $d(z) = [x']$ where $f'(x') = b(y)$ for some $y \in B$ with $g(y) = z$. Applying $\bar{f}'$, we get $\bar{f}'([x']) = [f'(x')] = [b(y)]$. Since $b(y) \in \operatorname{im}(b)$, its class in $\operatorname{coker}(b)$ is zero. Thus $\bar{f}' \circ d = 0$. **Inclusion** $\ker(\bar{f}') \subseteq \mathbf{im}(d)$: Let $[x'] \in \operatorname{coker}(a)$ such that $\bar{f}'([x']) = [0]$ in $\operatorname{coker}(b)$. This implies $f'(x') \in \operatorname{im}(b)$, so there exists $y \in B$ such that $b(y) = f'(x')$. We apply g' to both sides: $g'(b(y)) = g'(f'(x'))$. Since the bottom row is exact, $g' \circ f' = 0$, so $g'(b(y)) = 0$. By commutativity of the second square, $c(g(y)) = g'(b(y)) = 0$. This means $g(y) \in \ker(c)$. Let $z = g(y)$. By the definition of the connecting homomorphism, $d(z)$ is found by lifting z to B (we choose y), applying b (we get $b(y) = f'(x')$), and taking the preimage under f' (which is x'). Thus, $d(z) = [x']$, which shows that $[x'] \in \operatorname{im}(d)$.

5. We verify the exactness of the sequence at $\operatorname{coker}(b)$, specifically looking at the sequence $\operatorname{coker}(a) \xrightarrow{\bar{f}'} \operatorname{coker}(b) \xrightarrow{\bar{g}'} \operatorname{coker}(c)$. Let $[x'] \in \operatorname{coker}(a)$ where $x' \in A'$. Then $\bar{f}'([x']) = [f'(x')]$. Applying $\bar{g}'$, we have $\bar{g}'([f'(x')]) = [g'(f'(x'))]$. Since the bottom row is exact, $g' \circ f' = 0$, thus $[g'(f'(x'))] = [0]$, so $\bar{g}' \circ \bar{f}' = 0$. Let $[y'] \in \operatorname{coker}(b)$ such that $\bar{g}'([y']) = [0]$ in $\operatorname{coker}(c)$. This implies $g'(y') \in \operatorname{im}(c)$, so there exists $z \in C$ such that $c(z) = g'(y')$. Since g is surjective, there exists $y \in B$ such that $g(y) = z$. By commutativity, $g'(b(y)) = c(g(y)) = c(z) = g'(y')$. Thus $g'(y' - b(y)) = 0$, which means $y' - b(y) \in \ker(g')$. By exactness of the bottom row, there exists $x' \in A'$ such that $f'(x') = y' - b(y)$. Transitioning to the cokernel, $[y'] = [f'(x') + b(y)] = [f'(x')] + [b(y)]$. Since $b(y) \in \operatorname{im}(b)$, its class $[b(y)] = 0$ in $\operatorname{coker}(b)$. Therefore, $[y'] = [f'(x')] = \bar{f}'([x'])$, showing $[y'] \in \operatorname{im}(\bar{f}')$.

Let $\mathcal{B}$ be the smallest full (Definition 1.3.7) abelian subcategory of $\mathcal{A}$ containing the objects A, B, C, A', B', C'; it is a small category. Let choose a ring R and a functor $F : \mathcal{B} \to \mathrm{Mod}_R$ that is exact (Definition 2.4.8) and fully faithful (Definition 1.3.5) by the Freyd-Mitchell Embedding Theorem (Theorem 2.4.10). Apply the functor F to the diagram. Since F is exact, it preserves kernels, cokernels, images, and the exactness of the rows. The result holds in Mod_R by the element-based proof above. Since F is a full and faithful embedding, the morphism $d : \ker(c) \to \operatorname{coker}(a)$ is present in $\mathcal{B}$. Moreover, by Proposition 2.4.12, F reflects (Definition 2.4.11) exactness, so the exact sequence in Mod_R remains exact in $\mathcal{B}$. In particular, the exact sequence exists and is exact in $\mathcal{A}$. $\square$

The five-lemma may also be proven using the Freyd-Mitchell embedding theorem

Lemma 2.4.16 (Five Lemma)**.** Consider the following commutative diagram in an abelian category (Definition 2.3.10) $\mathcal{A}$ with exact rows (Definition 2.4.3):

$$\begin{array}{ccccccccc} A & \xrightarrow{f_1} & B & \xrightarrow{f_2} & C & \xrightarrow{f_3} & D & \xrightarrow{f_4} & E \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \epsilon \\ A' & \xrightarrow{g_1} & B' & \xrightarrow{g_2} & C' & \xrightarrow{g_3} & D' & \xrightarrow{g_4} & E' \end{array}$$

where $A, B, C, D, E, A', B', C', D', E'$ are objects and $\alpha, \beta, \gamma, \delta, \epsilon$ are morphisms.

In the commutative diagram defined above, the following statements hold:

1. If β and δ are monomorphisms and α is an epimorphism, then γ is a monomorphism.
2. If β and δ are epimorphisms and ϵ is a monomorphism, then γ is an epimorphism.
3. If $\alpha, \beta, \delta, \epsilon$ are isomorphisms, then γ is an isomorphism.

2.5. **Adjoint functors.** Adjoint functors (Definition 2.5.1) enjoy nice properties, especially in homological algebra.

Definition 2.5.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories (Definition 1.1.1). Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors.

An *adjunction between F and G* consists of two natural transformations (Definition 1.3.1): $\eta : \mathrm{Id}_{\mathcal{C}} \implies GF$ (the *unit*), and $\varepsilon : FG \implies \mathrm{Id}_{\mathcal{D}}$ (the *counit*)

These must satisfy the triangle identities: For every object $X \in \mathcal{C}$ and $Y \in \mathcal{D}$,

$$\varepsilon_{FX} \circ F(\eta_X) = \mathrm{id}_{FX}$$

$$G(\varepsilon_Y) \circ \eta_{GY} = \mathrm{id}_{GY}.$$

In diagrammatic form, the triangle identities assert that the following are commutative diagrams:

$$\begin{tikzcd} F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "\mathrm{id}_{F(X)}"'] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \\ & F(X) \end{tikzcd} \qquad \begin{tikzcd} G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "\mathrm{id}_{G(Y)}"'] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] \\ & G(Y) \end{tikzcd}$$

We say that F is a *left adjoint to G* and G is a *right adjoint to F* (written $F \dashv G$).

In the case that $\mathcal{C}$ and $\mathcal{D}$ are locally small (Definition 1.1.2) categories (or U-locally small categories if a universe (Definition A.0.3) U is available), we have an adjunction $F \dashv G$ if and only if for every object X in $\mathcal{C}$ and Y in $\mathcal{D}$ there is a natural isomorphism (Definition 1.3.1)

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

that is natural in both X and Y. More precisely, there are two functors

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Sets}$$

given by

$$(X, Y) \mapsto \mathrm{Hom}_{\mathcal{D}}(F(X), Y)$$
$$(X, Y) \mapsto \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

respectively, and there is a natural isomorphism between them.

In this case, the *unit of the adjunction* is the natural transformation $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ such that,

1. for every $X \in \mathcal{C}$, the morphism $\eta_X : X \to GF(X)$ (each called a *unit morphism*) in $\mathcal{C}$ is obtained as the image of $\mathrm{id}_{F(X)}$ via the adjoint isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(F(X), F(X)) \cong \mathrm{Hom}_{\mathcal{C}}(X, GF(X)).$$

2. for every $Y \in \mathcal{D}$, the morphism $\epsilon_Y : FG(Y) \to Y$ (each called a *counit morphism*) in $\mathcal{D}$ is obtained as the image of $\mathrm{id}_{G(Y)}$ via the adjoint isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(G(Y), G(Y)) \cong \mathrm{Hom}_{\mathcal{D}}(FG(Y), Y).$$

(♠ TODO: examples of adjoint functors)

One of the most essential adjunction of functors in algebra would be the tensor-hom adjunction:

Theorem 2.5.2 (Tensor-Hom Adjunction for Bimodules)**.**

1. Let R, S, T be (not necessarily commutative) rings (Definition C.0.7). Let M be an R-S bimodule (Definition 2.1.1), let N be an S-T bimodule, and let P be an R-T bimodule. Then there is a natural isomorphism of abelian groups

$$\mathrm{Hom}_{R\text{-}T}(M \otimes_S N,\, P) \;\cong\; \mathrm{Hom}_{S\text{-}T}(N,\, \mathrm{Hom}_R(M, P))$$

(Definition 2.1.9) (Definition 2.1.8). Note that $\mathrm{Hom}_{R\text{-}T}$ is the abelian group of R-T bimodule homomorphisms, $\mathrm{Hom}_{S\text{-}T}$ is the abelian group of S-T bimodule homomorphisms, and $\mathrm{Hom}_R(M, P)$ is endowed with the structure of an S-T bimodule via

$$(s \cdot f)(m) = f(m \cdot s),$$
$$(f \cdot t)(m) = f(m) \cdot t,$$

for all $s \in S$, $t \in T$, $f \in \mathrm{Hom}_R(M, P)$, $m \in M$. Intuitively, this expresses that $M \otimes_S -$ is left adjoint (Definition 2.5.1) to $\mathrm{Hom}_R(M, -)$ in the category of bimodules.

2. Let R, S, T be (not necessarily commutative) rings (Definition C.0.7). Let M be an R-S bimodule (Definition 2.1.1), let N be an S-T bimodule, and let P be an R-T bimodule. Then there is a natural isomorphism of abelian groups

$$\mathrm{Hom}_{R\text{-}T}(M \otimes_S N,\, P) \;\cong\; \mathrm{Hom}_{R\text{-}S}(M,\, \mathrm{Hom}_T(N, P))$$

(Definition 2.1.9) (Definition 2.1.8).

Note that $\mathrm{Hom}_{R\text{-}T}$ is the abelian group of R-T bimodule homomorphisms, $\mathrm{Hom}_{R\text{-}S}$ is the abelian group of R-S bimodule homomorphisms, and $\mathrm{Hom}_T(N, P)$ is endowed with the structure of an R-S bimodule via

$$(r \cdot f)(n) = r \cdot f(n),$$
$$(f \cdot s)(n) = f(n \cdot s),$$

for all $r \in R$, $s \in S$, $f \in \mathrm{Hom}_T(N, P)$, and $n \in N$.

Intuitively, this expresses that $- \otimes N$ is left adjoint (Definition 2.5.1) to $\mathrm{Hom}_T(N, -)$ in the category of bimodules.

Another useful adjunction is a adjunction of limit and colimit functors:

Theorem 2.5.3. Let $\mathcal{C}$ be a locally small (Definition 1.1.2) category and $\mathcal{J}$ be a small index category (Definition 1.1.2).

1. If $\mathcal{C}$ admits all colimits (Definition 2.2.8) of shape (Definition 2.2.6) $\mathcal{J}$, then the colimit functor
$$\operatorname{colim} : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$$
is left adjoint (Definition 2.5.1) to the diagonal functor Δ. That is, for any functor $F : \mathcal{J} \to \mathcal{C}$ and any object $X \in \mathcal{C}$, there is a natural bijection:
$$\operatorname{Hom}_{\mathcal{C}}(\operatorname{colim} F, X) \cong \operatorname{Hom}_{\mathcal{C}^{\mathcal{J}}}(F, \Delta(X)).$$
2. If $\mathcal{C}$ admits all limits of shape $\mathcal{J}$, then the limit functor
$$\lim : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$$
is right adjoint (Definition 2.5.1) to the diagonal functor Δ. That is, for any object $X \in \mathcal{C}$ and any functor $F : \mathcal{J} \to \mathcal{C}$, there is a natural bijection:
$$\operatorname{Hom}_{\mathcal{C}}(X, \lim F) \cong \operatorname{Hom}_{\mathcal{C}^{\mathcal{J}}}(\Delta(X), F).$$

These adjunctions characterize limits and colimits via their universal properties.

Proposition 2.5.4. Let $\mathcal{A}, \mathcal{B}$ be abelian categories (Definition 2.3.10) and let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ be adjoint (Definition 2.5.1) functors (Definition 2.4.1), say with $F \dashv G$ (i.e. F is left adjoint to G).

1. The functors F and G are additive (Definition 2.4.1).
2. The left adjoint functor F is right exact (Definition 2.4.8) and and the right adjoint functor G is left exact (Definition 2.4.8)

Proof. For each pair of objects $X, Y \in \mathcal{A}$, let us write $\alpha = \alpha_{X,Y}$ for the isomorphism
$$\alpha = \alpha_{X,Y} : \operatorname{Hom}_{\mathcal{B}}(F(X), Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}}(X, G(Y))$$
given by the adjunction. We note that this isomorphism is a priori an isomorphism of sets, i.e. a bijection.

Given a morphism $f : X' \to X$ in $\mathcal{A}$, the fact that the $\alpha_{X,Y}$ are natural isomorphisms with respect to the first input means that we have the commuting diagrams

(A)
$$\begin{tikzcd} \operatorname{Hom}_{\mathcal{B}}(F(X), Y) \arrow[r, "\alpha_{X,Y}"] \arrow[d, "-\circ F(f)"] & \operatorname{Hom}_{\mathcal{A}}(X, G(Y)) \arrow[d, "-\circ f"] \\ \operatorname{Hom}_{\mathcal{B}}(F(X'), Y) \arrow[r, "\alpha_{X',Y}"] & \operatorname{Hom}_{\mathcal{A}}(X', G(Y)). \end{tikzcd}$$

1. We show that F is additive; that G is additive can be obtained dually. To do so, we need to show that for all objects $A_1, A_2 \in \mathcal{A}$, the induced map
$$F_{A_1,A_2} : \operatorname{Hom}_{\mathcal{A}}(A_1, A_2) \to \operatorname{Hom}_{\mathcal{B}}(F(A_1), F(A_2))$$
is a group homomorphism of abelian groups. In other words, given morphisms $f, g : A_1 \to A_2$, we need to show that $F(f + g) = F(f) + F(g)$.
Write $\eta : \operatorname{Id}_{\mathcal{A}} \to GF$ for the unit natural transformation; in particular, given each object $A \in \mathcal{A}$, we write $\eta_A : A \to GF(A)$ for the unit morphism (Definition 2.5.1).

For each $f : A_1 \to A_2$, we claim that

$$\text{(B)} \qquad \alpha(F(f)) = \eta_{A_2} \circ f.$$

To see this, we use (A) in the case of $X = A_2$, $Y = F(A_2)$, and $X' = A_1$. Start with $\mathrm{id}_{\mathcal{D}} \in \mathrm{Hom}_{\mathcal{D}}(F(A_2), F(A_2))$ in the upper left corner. Traversing to the right then down sends $\mathrm{id}_{\mathcal{D}}$ to the unit map η_{A_2} via α and then to $\eta_{A_2} \circ f$. Traversing down then to the right sends $\mathrm{id}_{\mathcal{D}}$ to $F(f)$ then to $\alpha(F(f))$. Therefore, (B) holds.

In particular,

$$\alpha(F(f+g)) = \eta_B \circ (f+g).$$

Since $f + g : A_1 \to A_2$ and $\eta_{A_2} : A_2 \to GF(A_2)$ are morphisms in the abelian category $\mathcal{B}$, where composition of morphisms is distributive over addition, we have

$$\eta_{A_2} \circ (f+g) = (\eta_{A_2} \circ f) + (\eta_{A_2} \circ g).$$

Applying (B), we have

$$\alpha(F(f+g)) = \alpha(F(f)) + \alpha(F(g)).$$

Since α is an isomorphism of sets, we conclude

$$F(f+g) = F(f) + F(g)$$

2. We show that F is right exact; that G is left exact can be obtained dually. It suffices to show that F preserves cokernels (Definition 2.3.6), i.e. that if $f : A_1 \to A_2$ is a morphism in $\mathcal{A}$, then $F(\operatorname{coker} f) \cong \operatorname{coker} F(f)$. Let $q : A_2 \to Q$ be the cokernel of f.

$$A_1 \xrightarrow{f} A_2 \xrightarrow{q} Q$$

We show that $F(q) : F(A_2) \to F(Q)$ is the cokernel of $F(f)$, i.e. that $F(q)$ possesses the universal property of $\operatorname{coker} F(f)$.

$$F(A_1) \xrightarrow{F(f)} F(A_2) \xrightarrow{F(q)} F(Q)$$

Equivalently, we show that $F(\operatorname{coker} f)$ possesses the universal property of $\operatorname{coker} F(f)$. Since F is additive, note that

$$F(q) \circ F(f) = F(q \circ f) = F(0) = 0.$$

Now suppose that $g : F(A_2) \to Z$ is a morphism such that $g \circ F(f) = 0$. We need to show that g factors uniquely through $F(q)$.

Apply the commutativity of (A) in the case that $X = A_2$, $Y = Z$, and $X' = A_1$; this yields

$$\alpha(g \circ F(f)) = \alpha(g) \circ f.$$

Since $g \circ F(f) = 0$, we have $\alpha(g) \circ f = 0$ in $\mathcal{A}$. By the universal property of $\operatorname{coker} f$, the map $\alpha(g)$ factors uniquely through $\operatorname{coker}(f)$, i.e. there is a unique map $\psi : Q \to G(Z)$ such that $\psi \circ q = \alpha(g)$.

$$\begin{tikzcd} A_1 \arrow[r, "f"] & A_2 \arrow[r, "q"] \arrow[dr, "\alpha(g)"'] & Q \arrow[d, dotted, "\exists!\psi"] \\ & & G(Z) \end{tikzcd}$$

Applying α^{-1}, we have
$$\alpha^{-1}(\psi \circ q) = g.$$
By the naturality of the adjunction isomorphism in the first variable again, we obtain
$$g = \alpha^{-1}(\psi \circ q) = \alpha^{-1}(\psi) \circ F(q).$$
In particular,
$$\begin{tikzcd} F(A_1) \arrow[r, "F(f)"] & F(A_2) \arrow[r, "F(q)"] \arrow[dr, "g"'] & F(Q) \arrow[d, dotted, "\alpha^{-1}(\psi)"] \\ & & Z \end{tikzcd}$$
commutes. In fact, $\alpha^{-1}(\psi)$ is the unique morphism making the above commute because α is a bijection and ψ is unique. Therefore, $F(q) : F(A_2) \to F(Q)$ is indeed the cokernel of $F(f)$ as desired.

□

Proposition 2.5.5. Let R, S, T be (not necessarily commutative) rings (Definition C.0.7). Recall that categories of modules are abelian (Definition 2.3.10) (Theorem 2.5.11).

1. Let M be an R-S-bimodule (Definition 2.1.1). The functor $M \otimes_S - : {}_S\mathbf{Mod}_T \to {}_R\mathbf{Mod}_T$ (Definition 2.1.9) is a right exact functor (Definition 2.4.8).
2. Let N be an S-T-bimodule. The functor $- \otimes_S N : {}_R\mathbf{Mod}_S \to {}_R\mathbf{Mod}_T$ is a right exact functor (Definition 2.4.8).
3. Let M be an R-S-bimodule. The functor
$$\mathrm{Hom}(M, -) : {}_R\mathbf{Mod}_T \to {}_S\mathbf{Mod}_T$$
(Definition 2.1.8) is a left exact functor (Definition 2.4.8).
Now let M be an S-R-bimodule. The functor
$$\mathrm{Hom}(M, -) : {}_T\mathbf{Mod}_R \to {}_S\mathbf{Mod}_T$$
is a left exact functor (Definition 2.4.8).
4. Let N be an R-T-bimodule. The functor
$$\mathrm{Hom}(-, N) : {}_R\mathbf{Mod}_S^{\mathrm{op}} \to {}_S\mathbf{Mod}_T$$
(Definition 1.2.1) is a left exact functor (Definition 2.4.8).
Now let N be an T-R-bimodule. The functor
$$\mathrm{Hom}(-, N) : {}_S\mathbf{Mod}_R^{\mathrm{op}} \to {}_S\mathbf{Mod}_T$$
is a left exact functor (Definition 2.4.8).

Proof. The right exactness of the tensor functors and the left exactness of $\mathrm{Hom}(M, -)$ follow from Theorem 2.5.2 and Proposition 2.5.4. (♠ TODO: prove left exactness of $\mathrm{Hom}(-, N)$). □

Corollary 2.5.6. (♠ TODO: this should be more true of more general categories) Let $\mathcal{A}$ be an abelian cateogry (Definition 2.3.10) and $\mathcal{J}$ be a small index category (Definition 1.1.2).

1. Assume that $\mathcal{A}$ admits all colimits (Definition 2.2.8) of shape $\mathcal{J}$ (e.g. $\mathcal{A}$ is Ab3 (Definition 2.5.7)). The colimit functor
$$\mathrm{colim} : \mathcal{A}^{\mathcal{J}} \to \mathcal{A}$$
(Definition 2.2.6)[3] is right exact (Definition 2.4.7).
2. Assume that $\mathcal{A}$ admits all limits (Definition 2.2.8) of shape $\mathcal{J}$ (e.g. $\mathcal{A}$ is Ab3$*$ (Definition 2.5.7)). The limit functor
$$\lim : \mathcal{A}^{\mathcal{J}} \to \mathcal{A}$$
(Definition 2.2.6) is left exact (Definition 2.4.7).

Proof. This follows from Theorem 2.5.3 and Proposition 2.5.4 □

2.5.1. *Grothendieck's additional axioms for abelian categories.* In fact, Grothendieck posed certain additional axioms that some abelian categories might enjoy.

Definition 2.5.7 (Grothendieck's axioms for abelian categories (Ab1–Ab5))**.** Let $\mathcal{A}$ be an abelian category (Definition 2.3.10).

Grothendieck introduced the following hierarchy of additional axioms to express stronger completeness and exactness properties in $\mathcal{A}$ — we note that Ab1, Ab2, and Ab2* are already satisfied for any abelian category:

- *Ab1*: Every morphism in $\mathcal{A}$ has a kernel and a cokernel (Definition 2.3.6).
- *Ab2*: Every monic (Definition 2.3.7) in $\mathcal{A}$ is the kernel of its cokernel.
- *Ab2**: Every epi in $\mathcal{A}$ is the cokernel of its kernel.
- *AB3*: The category $\mathcal{A}$ is cocomplete (Definition 2.2.11).
  - Since $\mathcal{A}$ is abelian (and hence admits equalizers (Definition 2.2.14) as kernels (Definition 2.1.7)), this is equivalent to requiring that $\mathcal{A}$ has all small coproducts (Definition 2.2.1) (direct sums).
- *AB4*: The category $\mathcal{A}$ satisfies AB3, and coproducts are exact (Definition 2.4.8) (when regarded as functors $\bigoplus_I : \mathcal{A}^I \to \mathcal{A}$).
  - That is, the coproduct of a family of short exact sequences is a short exact sequence. Explicitly, for any family of short exact sequences $0 \to A_i \to B_i \to C_i \to 0$ indexed by a set I, the sequence
$$0 \to \bigoplus_{i\in I} A_i \to \bigoplus_{i\in I} B_i \to \bigoplus_{i\in I} C_i \to 0$$
is exact in $\mathcal{A}$.
- *AB5*: The category $\mathcal{A}$ satisfies AB3, and filtered colimits (Definition 2.2.13) are exact (Definition 2.4.8) (when regarded as functors $\varinjlim_J : \mathcal{A}^J \to \mathcal{A}$).
  - Equivalently, for any filtered (Definition 2.2.12) index category J and any directed system of short exact sequences $0 \to A_j \to B_j \to C_j \to 0$, the colimit sequence
$$0 \to \varinjlim A_j \to \varinjlim B_j \to \varinjlim C_j \to 0$$
is exact.

[3]Recall that $\mathcal{A}^{\mathcal{J}}$ is abelian (Proposition 2.3.11)

  - Note: AB5 implies AB4. An abelian category satisfying AB5 and having a generator (Definition 2.5.8) is called a *Grothendieck category*.
- *AB6*: The category $\mathcal{A}$ satisfies AB3, and for any object X and any family of filtered subobjects $\{F_i\}_{i \in I}$ of X (where each F_i is a filter of subobjects), the intersection commutes with the limit:

$$\bigcap_{i \in I} (\varinjlim_{j \in F_i} U_{i,j}) = \varinjlim_{(j_i) \in \prod F_i} (\bigcap_{i \in I} U_{i,j_i}).$$

(This axiom is less commonly cited but appears in Grothendieck's Tohoku paper).
- *AB3***: The category $\mathcal{A}$ is complete (Definition 2.2.11) (i.e., has all small products).
- *AB4***: The category $\mathcal{A}$ satisfies AB3* and products are exact.
  - Note: This is rarely satisfied for module categories (e.g., it fails for Abelian groups), but it is satisfied for the category of sheaves on a space.
- *AB5***: The category $\mathcal{A}$ satisfies AB3* and filtered limits (inverse limits) are exact.

Notes:

- AB5 implies AB4, and AB4 implies AB3.
- AB5* implies AB4*, and AB4* implies AB3*.

Definition 2.5.8 (Generator of a category)**.** Let $\mathcal{C}$ be a category (Definition 1.1.1).

1. An object $G \in \mathcal{C}$ is called a *generator* (or *separator*) if for every pair of distinct morphisms $f, g : X \to Y$ in $\mathcal{C}$, there exists a morphism $h : G \to X$ such that

$$f \circ h \neq g \circ h.$$

   In case that $\mathcal{C}$ is locally small (Definition 1.1.2), this is equivalent to the condition that the representable functor (Definition A.0.1)

$$\mathrm{Hom}_{\mathcal{C}}(G, -) : \mathcal{C} \to \mathbf{Set}$$

   is faithful (Definition 1.3.5), which in turn is equivalent to the condition that for every object $X \in \mathcal{C}$, there exists an epimorphism

$$\bigoplus_{i \in I} G \twoheadrightarrow X$$

   for some indexing set I, where $\bigoplus$ denotes the coproduct (Definition 2.2.1) in $\mathcal{C}$.
2. A family $\{G_i\}_{i \in I}$ is called a *generating family* if for every pair of distinct morphisms $f, g : X \to Y$ in $\mathcal{C}$, there exists some index $i \in I$ and a morphism $h : G_i \to X$ such that

$$f \circ h \neq g \circ h.$$

   In case $\mathcal{C}$ is locally small, this is equivalent to the condition that the collection of representable functors

$$\{\mathrm{Hom}_{\mathcal{C}}(G_i, -) : \mathcal{C} \to \mathbf{Set}\}_{i \in I}$$

   is jointly faithful, which in turn is equivalent to the condition that for every object $X \in \mathcal{C}$, there exists a family of objects $\{G_i\}_{i \in J}$ from the generating set indexed by some set J, and an epimorphism

$$\bigoplus_{i\in J} G_i \twoheadrightarrow X.$$

Theorem 2.5.9 (Examples of Grothendieck Categories)**.** Examples of Grothendieck categories (Definition 2.5.7) include:

- The category of abelian groups,
- The category of R-S bimodules where R,S are (not necessarily commutative) rings (Definition C.0.7) (Theorem 2.5.11)
- The category of sheaves (Definition B.0.6) of abelian groups on a site (Definition B.0.4) with a small topologically generating family (Definition B.0.4),
- The category of sheaves of O_X-modules (Definition B.0.11) for any ringed space (Definition 6.0.8) (X, O_X).
- The category of quasi-coherent sheaves on a scheme or algebraic stack. (♠ TODO: quasi-coherent sheaves) (♠ TODO: I need to figure out if for sheaves of abelian groups/sheaves of $\mathcal{O}$-modules whether essentially smallness of the site is really necessary)
- The category of sheaves (Definition B.0.6) of abelian groups on an essentially small site (Definition B.0.4) (C, J).
- ([GV72, Exposé II, Proposition 6.7]) The category of sheaves of $\mathcal{O}$-modules on an essentially small site (or an essentially $\mathscr{U}$-small site if a universe $\mathscr{U}$ is available) (C, J).

Lemma 2.5.10. Let $\mathcal{C}$ be a category of sets with finitary operations (e.g. the category of sets (Definition 1.1.7), the category of modules over a ring (Definition 2.1.3), the category of monoids, the category of semigroups, the category of groups, the category of Lie algebras, the category of Lie algebras, etc.) (♠ TODO: define category of sets with finitary operations). Let $F : I \to \mathcal{C}$ be a small direct system (Definition 2.2.13). The direct limit (Definition 2.2.13) $\varinjlim_{i\in I} F(i)$ can be constructed as the quotient of the coproduct (Definition 2.2.1) $\coprod_{i\in I} F(i)$ given by the following equivalence relation: for $i, j \in I$, the elements $x_i \in F(i)$ and $x_j \in F(j)$ are equivalent if there exists some k with $i \leq k$ and $j \leq k$ such that $f_{ik}(x_i) = f_{jk}(x_j)$. In particular, the operations on the objects of $\mathcal{C}$ induce natural operations on this set making it into an object of $\mathcal{C}$.

Theorem 2.5.11. Let R, S be (not necessarily commutative) rings (Definition C.0.7). The category of (Definition 2.1.3) R-S-bimodules (Definition 2.1.1) is a Grothendieck (Definition 2.5.7) category and an $AB4*$ (Definition 2.5.7) category.

Proof. We handwave details.

Given R-S-bimodules M and N, the set $\mathrm{Hom}_{{}_R\mathrm{Mod}_S}(M, N)$ is an abelian group. Moreover, there is a 0-object, namely the zero module in ${}_R\mathrm{Mod}_S$. Therefore, ${}_R\mathrm{Mod}_S$ is preadditive (Definition 2.3.4). The direct sum of finitely many R-S-bimodules is their coproduct (Definition 2.2.5). Therefore, ${}_R\mathrm{Mod}_S$ is additive (Definition 2.3.4).

Given a morphism (Definition 2.1.2) $f : M \to N$ be R-S-bimodules, $\ker f$ and $\operatorname{coker} f$ (Definition 2.1.7) are the categorical kernel and cokernel (Definition 2.3.6) of f in ${}_R\mathrm{Mod}_S$

(Definition 2.1.3). Moreover, the monomorphisms (Definition 2.3.7) in ${}_R\mathrm{Mod}_S$ are the injective module homomorphisms $f : M \to N$; such an f is the kernel of its cokernel. In other words, ${}_R\mathrm{Mod}_S$ satisfies $AB1$ and $AB2$ and hence is an abelian category (Definition 2.3.10).

Moreover, small coproducts (Definition 2.2.1) exist in ${}_R\mathrm{Mod}_S$ (Definition 2.2.5), and it is easy to see that they are in fact exact, so ${}_R\mathrm{Mod}_S$ satisfies $AB3$ and $AB4$. To show that filtered colimits in ${}_R\mathrm{Mod}_S$ are exact, we first note that small (Definition 2.2.10) colimits (Definition 2.2.13) are (Corollary 2.5.6) right exact (Definition 2.4.8); for any small index category J and any system of short exact sequences $0 \to A_j \to B_j \to C_j \to 0$, the sequence $0 \to A \to B \to C \to 0$ of diagrams (Definition 2.2.6) is exact by Proposition 3.0.5, so applying the colimit functor yields a right exact sequence

$$\mathrm{colim}_{j\in J} A_j \to \mathrm{colim}_{j\in J} B_j \to \mathrm{colim}_{j\in J} C_j \to 0.$$

If J is additionally filtered (Definition 2.2.12) so that the system is a directed one, then we show that the sequence

$$\varinjlim_{j\in J} A_j \to \varinjlim_{j\in J} B_j \to \varinjlim_{j\in J} C_j \to 0. \tag{C}$$

is left exact as well. Take some element[4] of the kernel of $\varinjlim_{j\in J} A_j \to \varinjlim_{j\in J} B_j$; such an element is represented by some element $a_j \in A_j$ for some $j \in J$. Since its image is zero in $\varinjlim_{j\in J} B_j$, it must be zero as an element of B_k for some $k \in J$. Since J is filtered, there exists some $k' \in J$ so that there are arrows $j \to k'$ and $k \to k'$ and so that the image of a_j in $B_{k'}$ is 0. The image of a_j in $A_{k'}$ is then 0 due to the assumption that

$$0 \to A_{k'} \to B_{k'} \to C_{k'} \to 0$$

is exact. Therefore, a_j is 0 in $\varinjlim_{j\in J} A_j$, so (C) is left exact as claimed and ${}_R\mathrm{Mod}_S$ is $AB5$.

Similarly as how we argued that ${}_R\mathrm{Mod}_S$ is $AB3$ and $AB4$, one can argue that ${}_R\mathrm{Mod}_S$ is $AB3^*$ and $AB4^*$. Moreover, one can show that the R-S-bimodule $R \otimes_{\mathbb{Z}} S^{\mathrm{op}}$ (Definition 2.1.10) (Definition C.0.38) is a generator for ${}_R\mathrm{Mod}_S$. □

3. Chain complexes of objects in additive categories

Chain complexes (Definition 3.0.1) are made of a sequence of morphisms in an additive category such that the composition of any two conseuctive morphisms is 0. The first kind of chain complexes one should keep in mind are chain complexes of modules (Definition 2.1.1) over a ring.

Definition 3.0.1 (Chain complex in a preadditive category)**.** Let $\mathcal{A}$ be a preadditive category and let I be a totally ordered set (typically $\mathbb{Z}$, but $I \subseteq \mathbb{Z}$ is also allowed).

1. A *chain complex* $(K_\bullet, d_\bullet)$ in $\mathcal{A}$ indexed by I is the homological convention for sequences with decreasing degrees. It consists of:
   - Objects $\{K_i\}_{i\in I}$ in $\mathcal{A}$, called the *terms in degree i*,

[4] Recall ??

- Morphisms $d_i : K_i \to K_{i-1}$ in $\mathcal{A}$, called the *boundary maps* or *differentials in degree i*,

such that for every $i \in I$, $d_{i-1} \circ d_i = 0$. That is,

$$K_\bullet : \cdots \xrightarrow{d_{i+1}} K_i \xrightarrow{d_i} K_{i-1} \xrightarrow{d_{i-1}} K_{i-2} \to \cdots$$

with $d_{i-1}d_i = 0$ for all i. We typically use the notation $K_\bullet = (K_i, d_i)_{i\in I}$.

2. Dually, a *cochain complex* $(K^\bullet, d^\bullet)$ in $\mathcal{A}$ follows the *cohomological convention* with increasing degrees. It consists of objects $\{K^i\}_{i\in I}$ and *coboundary maps* $d^i : K^i \to K^{i+1}$ such that $d^{i+1} \circ d^i = 0$:

$$K^\bullet : \cdots \xrightarrow{d^{i-1}} K^i \xrightarrow{d^i} K^{i+1} \xrightarrow{d^{i+1}} K^{i+2} \to \cdots$$

We typically use the notation $K^\bullet = (K^i, d^i)_{i\in I}$.

3. Let $K_\bullet = (K_i, d_i^K)$ and $L_\bullet = (L_i, d_i^L)$ be chain complexes (Definition 3.0.1) in $\mathcal{A}$ indexed by the same set I. A *morphism of chain complexes* (or *chain map*)

$$f_\bullet : K_\bullet \to L_\bullet$$

consists of morphisms $f_i : K_i \to L_i$ for all $i \in I$, such that for every $i \in I$, the following diagram commutes:

$$\begin{array}{ccc} K_i & \xrightarrow{d_i^K} & K_{i-1} \\ \downarrow f_i & & \downarrow f_{i-1} \\ L_i & \xrightarrow{d_i^L} & L_{i-1} \end{array}$$

i.e., $d_i^L \circ f_i = f_{i-1} \circ d_i^K$.

A *morphism of cochain complexes* $f^\bullet : K^\bullet \to L^\bullet$ is defined similarly, satisfying the commutativity condition $d_L^i \circ f^i = f^{i+1} \circ d_K^i$.

The collection of these objects and morphisms forms a category. Notation for these categories is as follows:

- $\mathrm{Ch}(\mathcal{A})$ or $\mathbf{Ch}(\mathcal{A})$ is often used as a general term.
- To be explicit about the indexing convention, one uses $\mathrm{Ch}_\bullet(\mathcal{A})$ for chain complexes and $\mathrm{Ch}^\bullet(\mathcal{A})$ (or sometimes $\mathrm{CoCh}(\mathcal{A})$) for cochain complexes.
- The set of chain maps between two complexes is denoted by $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(K_\bullet, L_\bullet)$; it is an abelian group under pointwise addition $(f+g)_i = f_i + g_i$.

Definition 3.0.2 (Acyclic complex)**.** Let $\mathcal{A}$ be an additive category (Definition 2.3.4), and let $(C_\bullet, d_\bullet)$ be a complex (Definition 3.0.1) in $\mathcal{A}$:

$$\cdots \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots .$$

The complex $(C_\bullet, d_\bullet)$ is called *acyclic at C_n* (or sometimes synonymously *exact at C_n*) if we have $\ker d_n \cong \mathrm{im} d_{n+1}$ (Definition 2.3.6) (Definition 2.3.8).

If $\mathcal{A}$ is an abelian category (Definition 2.3.10), then this is equivalent to the condition that the (co)homology objects (Definition 3.2.2) $H^n(C_\bullet) := \ker d_n / \operatorname{im} d_{n+1}$ are zero in $\mathcal{A}$.

We furthermore say that the complex $(C_\bullet, d_\bullet)$ is *acyclic* or *exact* if it is acyclic/exact everywhere.

Example 3.0.3. All short exact sequences (Definition 2.4.3) in an additive category can be regarded as chain complexes, once we assign indices to the objects

(♠ TODO: simple examples of chain complexes)

Example 3.0.4. Some examples of chain complexes include (♠ TODO:)

1. The singular chain complex (Definition 3.3.4) of a topological space.

Proposition 3.0.5. Let $\mathcal{A}$ be a abelian category (Definition 2.3.10) and let J be a small category (Definition 1.1.2).

Given a sequence $A \to B \to C$ of objects in the diagram category (Definition 2.2.6) $\mathcal{A}^J$, the sequence is exact (Definition 3.0.2) at B if and only if all the sequences $A(j) \to B(j) \to C(j)$ are exact at $B(j)$ for every $j \in \mathrm{Ob}(J)$.

3.1. **Category of chain complexes as a functor category.**

Definition 3.1.1. A *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$, where:

- Q_0 is a collection of *vertices*.
- Q_1 is a collection of *arrows*.
- $s, t : Q_1 \to Q_0$ are functions assigning to each arrow $\alpha \in Q_1$ its *source* $s(\alpha)$ and its *target* $t(\alpha)$.

Definition 3.1.2. Let Q be a quiver (Definition 3.1.1). The *path category generated by Q*, denoted $\mathcal{F}(Q)$, is the category (Definition 1.1.1) defined as follows:

- The objects of $\mathcal{F}(Q)$ are the vertices Q_0.
- For any two objects $x, y \in Q_0$, the set of morphisms $\mathrm{Hom}_{\mathcal{F}(Q)}(x, y)$ consists of all paths from x to y — A *path of length $n \geq 1$ from x to y* is a sequence of arrows $\alpha_n \dots \alpha_1$ such that $s(\alpha_1) = x$, $t(\alpha_n) = y$, and $s(\alpha_{i+1}) = t(\alpha_i)$ for all $1 \leq i < n$. Additionally, for each vertex x, there is a path e_x of length 0, which serves as the identity morphism.
- Composition of morphisms is defined by the concatenation of paths.

Definition 3.1.3. Let Q be a quiver (Definition 3.1.1) whose collection of arrows is small.

The *preadditive category generated by Q*, denoted $\mathbb{Z}Q$, is the preadditive category (Definition 2.3.4), i.e. the category enriched over the category of abelian groups (Definition 1.1.8) defined as follows:

- The objects of $\mathbb{Z}Q$ are the vertices Q_0.
- For any objects $x, y \in Q_0$, the morphism set $\mathrm{Hom}_{\mathbb{Z}Q}(x, y)$ is the free abelian group generated by the set of all paths from x to y in Q.
- Composition is the unique bilinear extension of the path concatenation in $\mathcal{F}(Q)$. That is, for paths u, v, w where concatenation is defined, composition satisfies $(u + v) \circ w = u \circ w + v \circ w$ and $w \circ (u + v) = w \circ u + w \circ v$.

Definition 3.1.4. Let Q_{chain} be the quiver (Definition 3.1.1) with vertex set $Q_0 = \mathbb{Z}$ and arrow set $Q_1 = \{d_n : n \to n-1 \mid n \in \mathbb{Z}\}$. (♠ TODO: quotient of a category) The *walking chain complex category*, denoted $\mathcal{I}$, is the quotient of the preadditive category (Definition 2.3.4) $\mathbb{Z}Q_{\text{chain}}$ by the ideal generated by the relations $d_{n-1} \circ d_n = 0$ for all $n \in \mathbb{Z}$. Explicitly:

- Objects are the integers $\mathbb{Z}$.
- Morphisms are $\mathbb{Z}$-linear combinations of paths, subject to the relation that any path containing a subsegment $d_{n-1}d_n$ is identified with the zero morphism.

Definition 3.1.5. Let $\mathcal{A}$ and $\mathcal{B}$ be preadditive categories (Definition 2.3.4) (categories enriched over the category of abelian groups (Definition 1.1.8)). The *additive functor category* $\mathrm{Add}(\mathcal{A}, \mathcal{B})$ is the functor category where:

- Objects are additive functors $F : \mathcal{A} \to \mathcal{B}$. An additive functor (Definition 2.4.1) is a functor such that for any $x, y \in \mathrm{Ob}(\mathcal{A})$, the map $F : \mathrm{Hom}_{\mathcal{A}}(x, y) \to \mathrm{Hom}_{\mathcal{B}}(F(x), F(y))$ is a group homomorphism.
- Morphisms are natural transformations (Definition 1.3.1) between additive functors (Definition 2.4.1).

Proposition 3.1.6. Let $\mathcal{B}$ be a preadditive category (Definition 2.3.4). The category $\mathrm{Ch}(\mathcal{B})$ of chain complexes (Definition 3.0.1) in $\mathcal{B}$ is isomorphic to the category $\mathrm{Add}(\mathcal{I}, \mathcal{B})$ (Definition 3.1.5)(Definition 3.1.4).

An additive functor $F : \mathcal{I} \to \mathcal{B}$ corresponds to the chain complex defined by $C_n = F(n)$ and differentials $\partial_n = F(d_n)$.

Lemma 3.1.7. Let $\mathcal{A}, \mathcal{B}$ be preadditive categories (Definition 2.3.4) with $\mathcal{A}$ small (Definition 1.1.2).

1. The additive functor category (Definition 3.1.5) $\mathrm{Add}(\mathcal{A}, \mathcal{B})$ is preadditive. If $\mathcal{B}$ is additionally additive (Definition 2.3.4)/abelian (Definition 2.3.10), then so is $\mathrm{Add}(\mathcal{A}, \mathcal{B})$.
2. If $\mathcal{B}$ is an abelian category with property ABn for $n = 3, 4, 5, 6$ or ABn^* for $n = 3, 4, 5$ (Definition 2.5.7), then $\mathrm{Add}(\mathcal{A}, \mathcal{B})$ possesses the same property.

Proposition 3.1.8. Let $\mathcal{A}$ be an additive category.

1. The category $\mathrm{Ch}(\mathcal{A})$ of chain complexes is itself and additive category.
2. If $\mathcal{A}$ is an abelian category, then $\mathrm{Ch}(\mathcal{A})$ is an abelian category.
3. If $\mathcal{A}$ is an abelian category satisfying Grothendieck's axiom ABn (resp. ABn^*) (Definition 2.5.7) for $n \in \{3, 4, 5, 6\}$, then $\mathrm{Ch}(\mathcal{A})$ also satisfies ABn (resp. ABn^*). If $\mathcal{A}$ is a Grothendieck abelian category (Definition 2.5.7), then so is $\mathrm{Ch}(\mathcal{A})$

Proof. Combine Proposition 3.1.6 and Lemma 3.1.7. □

3.2. **Homology and cohomology of a chain complex.**

Definition 3.2.1. Let $\mathcal{A}$ be an abelian category (Definition 2.3.10).

1. Let $C_\bullet = (\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \dots)$ be a chain complex (Definition 3.0.1) in $\mathcal{A}$. For each integer n, we define:
   - The object of *n-cycles*, denoted $Z_n(C)$, is the kernel (Definition 2.3.6) of the differential leaving C_n:
   $$Z_n(C) := \ker(d_n : C_n \to C_{n-1}).$$
   - The object of *n-boundaries*, denoted $B_n(C)$, is the image (Definition 2.3.8) of the differential entering C_n:
   $$B_n(C) := \operatorname{im}(d_{n+1} : C_{n+1} \to C_n).$$

   Since $d_n \circ d_{n+1} = 0$, there is a canonical monomorphism (Definition 2.3.7) $B_n(C) \hookrightarrow Z_n(C)$.
2. Let $C^\bullet = (\cdots \to C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \to \dots)$ be a *cochain complex in $\mathcal{A}$*. For each integer n, we define:
   - The object of *n-cocycles*, denoted $Z^n(C)$, is the kernel (Definition 2.3.6) of the differential leaving C^n:
   $$Z^n(C) := \ker(d^n : C^n \to C^{n+1}).$$
   - The object of *n-coboundaries*, denoted $B^n(C)$, is the image> (Definition 2.3.8) of the differential entering C^n:
   $$B^n(C) := \operatorname{im}(d^{n-1} : C^{n-1} \to C^n).$$

   Since $d^n \circ d^{n-1} = 0$, there is a canonical monomorphism (Definition 2.3.7) $B^n(C) \hookrightarrow Z^n(C)$.

Definition 3.2.2 (Chain complexes and their (co)homology objects)**.** Let $\mathcal{A}$ be an abelian category.

- For a cochain complex $K^\bullet$, its *cohomology object in degree i* is defined as the quotient of the object of i-cocycles by the object of i-coboundaries:
$$H^i(K^\bullet) := Z^i(K)/B^i(K) = \ker(d^i)/\operatorname{im}(d^{i-1}).$$
In fact, each H^i is a functor $H^i : \mathbf{Ch}^\bullet(\mathcal{A}) \to \mathcal{A}$ (Definition 3.0.1); given a morphism $f^\bullet : K^\bullet \to L^\bullet$ of cochain complexes, it is thus customary to write (Definition 1.2.2) $H^i(f^\bullet) : H^i(K^\bullet) \to H^i(L^\bullet)$ for the induced morphism on the ith cohomology objects.
- For a chain complex $K_\bullet$, its *homology object in degree i* is defined as the quotient of the object of i-cycles by the object of i-boundaries:
$$H_i(K_\bullet) := Z_i(K)/B_i(K). = \ker(d_i)/\operatorname{im}(d_{i+1}).$$
In fact, each H_i is a functor $H_i : \mathbf{Ch}_\bullet(\mathcal{A}) \to \mathcal{A}$ (Definition 3.0.1); given a morphism $f_\bullet : K_\bullet \to L_\bullet$ of cochain complexes, it is thus customary to write (Definition 1.2.2) $H_i(f_\bullet) : H_i(K_\bullet) \to H_i(L_\bullet)$ for the induced morphism on the ith homology objects. .

See Proposition 3.2.7

Example 3.2.3 (The Singular Complex of a Point)**.** The simplest non-zero geometric example is the chain complex of a single point $X = \{*\}$. In singular homology, there is exactly one map $\sigma_n : \Delta^n \to X$ for each $n \geq 0$. Thus, $C_n(X) \cong \mathbb{Z}$ for all $n \geq 0$. The boundary map $d_n : C_n \to C_{n-1}$ is given by the alternating sum of face maps, which results in:

$$\dots \xrightarrow{\text{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

The homology groups are $H_0(X) \cong \mathbb{Z}$ and $H_n(X) = 0$ for $n > 0$.

Example 3.2.4 (The Algebraic Complex of a Ring Element)**.** Let R be a ring and $x \in R$ be an element. We can define a chain complex $C_\bullet$ of R-modules concentrated in degrees 1 and 0:

$$0 \to R \xrightarrow{d_1} R \to 0$$

where $d_1(r) = xr$. Since $d_0 = 0$, the condition $d_0 \circ d_1 = 0$ is trivially satisfied. The homology groups are:

- $H_1(C_\bullet) = \ker(\cdot x) = \{r \in R \mid xr = 0\}$, the x-torsion of R.
- $H_0(C_\bullet) = \operatorname{coker}(\cdot x) = R/xR$.

Example 3.2.5 (Acyclic Complexes from Exact Sequences)**.** A chain complex $C_\bullet$ is said to be *acyclic* if $H_n(C_\bullet) = 0$ for all n. Any short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in an abelian category $\mathcal{A}$ can be viewed as an acyclic chain complex:

$$\cdots \to 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \to \dots$$

The condition $g \circ f = 0$ is guaranteed by the fact that $\operatorname{im}(f) = \ker(g)$, which simultaneously ensures that the homology at object B is zero.

Example 3.2.6 (The Simplicial Triangle)**.** Let K be a 2-simplex (a triangle) with vertices $\{v_0, v_1, v_2\}$. There is an associated chain complex

$$\cdots \to 0 \to C_2 \to C_1 \to C_0 \to 0 \to \cdots$$

of abelian groups. The chain groups are:

- $C_0 = \mathbb{Z}\langle v_0, v_1, v_2\rangle$ (rank 3)
- $C_1 = \mathbb{Z}\langle [v_0, v_1], [v_1, v_2], [v_0, v_2]\rangle$ (rank 3)
- $C_2 = \mathbb{Z}\langle [v_0, v_1, v_2]\rangle$ (rank 1)

The boundary map $d_2 : C_2 \to C_1$ is $d_2([v_0, v_1, v_2]) = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$. One can verify that $d_1 \circ d_2 = 0$ by calculating the boundaries of the resulting edges.

Proposition 3.2.7.

1. Let $f_\bullet : C_\bullet \to D_\bullet$ be a morphism of chain complexes (Definition 3.0.1) in an abelian category (Definition 2.3.10). This morphism sends boundaries to boundaries (Definition 3.2.1) and cycles to cycles (Definition 3.2.1). More preicsely, for all $i \in \mathbb{Z}$, $f_i : C_i \to D_i$ restricts to morphisms $Z_i(C_\bullet) \to Z_i(D_\bullet)$ and $B_i(C_\bullet) \to B_i(D_\bullet)$. Therefore, there are induced morphisms $H_i(f_\bullet) : H_i(C_\bullet) \to H_i(D_\bullet)$ which are in fact functorial, i.e. respect composition.
2. Let $f^\bullet : C^\bullet \to D^\bullet$ be a morphism of cochain complexes (Definition 3.0.1) in an abelian category (Definition 2.3.10). This morphism sends coboundaries to coboundaries (Definition 3.2.1) and cocycles to cocycles (Definition 3.2.1). More precisely, for all $i \in \mathbb{Z}$,

$f^i : C^i \to D^i$ restricts to morphisms $Z^i(C^\bullet) \to Z^i(D^\bullet)$ and $B^i(C^\bullet) \to B^i(D^\bullet)$. Therefore, there are induced morphisms $H^i(f^\bullet) : H^i(C^\bullet) \to H^i(D^\bullet)$ which are functorial, i.e. respect composition.

Proof. (♠ TODO: Theorem 2.4.10 (Freyd-Mitchell) should be usable) □

Definition 3.2.8 (Quasi-isomorphism)**.** Let $\mathcal{A}$ be an abelian category (Definition 2.3.10), and let

$$f_\bullet : (C_\bullet, d_\bullet^C) \to (D_\bullet, d_\bullet^D)$$

be a chain map between complexes (Definition 3.0.1) in $\mathcal{A}$.

The morphism $f_\bullet$ is called a *quasi-isomorphism* if it induces isomorphisms on all cohomology objects, i.e., for every integer n, the induced morphism on homology (Definition 3.2.2) (or cohomology, depending on the convention)

$$H_n(f_\bullet) : H_n(C_\bullet) \to H_n(D_\bullet)$$

is an isomorphism in $\mathcal{A}$.

Note that all of these notions are applicable to the cohomological convention as well.

Remark 3.2.9. Let $C_\bullet$ and $D_\bullet$ be chain complexes (Definition 3.0.1) in an abelian category (Definition 2.3.10). Even if $H_n(C_\bullet)$ and $H_n(D_\bullet)$ are isomorphic for all n, there need not exist a quasi-isomorphism (Definition 3.2.8) $C_\bullet \to D_\bullet$.

Theorem 3.2.10 (Long Exact Sequence in Homology)**.**

1. Let $(C_\bullet, d_\bullet^C)$, $(D_\bullet, d_\bullet^D)$, and $(E_\bullet, d_\bullet^E)$ be chain complexes (Definition 3.0.1) in an abelian category (Definition 2.3.10). Recall that $\mathbf{Ch}(\mathcal{A})$ (Definition 3.0.1) is itself an abelian category (Proposition 3.1.8). Assume that

$$0 \longrightarrow C_\bullet \xrightarrow{\alpha_\bullet} D_\bullet \xrightarrow{\beta_\bullet} E_\bullet \longrightarrow 0$$

is a short exact sequence (Definition 2.4.3) of chain complexes. Equivalently, for each integer n,

$$0 \to C_n \xrightarrow{\alpha_n} D_n \xrightarrow{\beta_n} E_n \to 0$$

is an exact sequence of R-modules.

Then there exists a natural *long exact sequence* in homology (Definition 3.2.2):

$$\cdots \longrightarrow H_{n+1}(E_\bullet) \xrightarrow{\delta_{n+1}} H_n(C_\bullet) \xrightarrow{H_n(\alpha)} H_n(D_\bullet) \xrightarrow{H_n(\beta)} H_n(E_\bullet) \xrightarrow{\delta_n} H_{n-1}(C_\bullet) \longrightarrow \cdots$$

The homomorphisms $\delta_n : H_n(E_\bullet) \to H_{n-1}(C_\bullet)$ are called the *connecting homomorphisms* induced by the short exact sequence of chain complexes.

Moreover, this long exact sequence is natural with respect to morphisms (Definition 3.0.1) of short exact sequences of chain complexes.

2. Let $(C^\bullet, d_C^\bullet)$, $(D^\bullet, d_D^\bullet)$, and $(E^\bullet, d_E^\bullet)$ be cochain complexes (Definition 3.0.1) in an abelian category (Definition 2.3.10). Recall that $\mathbf{Ch}(\mathcal{A})$ (Definition 3.0.1) is itself an abelian category (Proposition 3.1.8).

Assume that

$$0 \longrightarrow C^\bullet \xrightarrow{\alpha^\bullet} D^\bullet \xrightarrow{\beta^\bullet} E^\bullet \longrightarrow 0$$

is a short exact sequence (Definition 2.4.3) of cochain complexes. Equivalently, for each integer n,

$$0 \to C^n \xrightarrow{\alpha^n} D^n \xrightarrow{\beta^n} E^n \to 0$$

is an exact sequence of R-modules.

Then there exists a natural *long exact sequence* in cohomology (Definition 3.2.2):

$$\cdots \longrightarrow H^{n-1}(E^\bullet) \xrightarrow{\delta^{n-1}} H^n(C^\bullet) \xrightarrow{H^n(\alpha)} H^n(D^\bullet) \xrightarrow{H^n(\beta)} H^n(E^\bullet) \xrightarrow{\delta^n} H^{n+1}(C^\bullet) \longrightarrow \cdots$$

The morphisms $\delta^n : H^n(E^\bullet) \to H^{n+1}(C^\bullet)$ are called the *connecting homomorphisms* induced by the short exact sequence of cochain complexes.

Moreover, this long exact sequence is natural with respect to morphisms of short exact sequences of cochain complexes.

Proof. (♠ TODO: why kernel and cokernel are left/right exact) Consider the following commutative diagram in $\mathcal{A}$ with exact rows, where the vertical maps are the differentials of the respective complexes:

$$\text{(D)}\quad \begin{tikzcd} 0 \arrow[r] & C_n/\operatorname{im}(d^C_{n+1}) \arrow[r, "\bar{\alpha}_n"] \arrow[d, "\bar{d}^C_n"] & D_n/\operatorname{im}(d^D_{n+1}) \arrow[r, "\bar{\beta}_n"] \arrow[d, "\bar{d}^D_n"] & E_n/\operatorname{im}(d^E_{n+1}) \arrow[r] \arrow[d, "\bar{d}^E_n"] & 0 \\ 0 \arrow[r] & \ker(d^C_{n-1}) \arrow[r, "\alpha_{n-1}"] & \ker(d^D_{n-1}) \arrow[r, "\beta_{n-1}"] & \ker(d^E_{n-1}) \arrow[r] & 0 \end{tikzcd}$$

The rows are exact because the original sequence of complexes is exact and the functors $\ker(d_{n-1})$ and $\operatorname{coker}(d_{n+1})$ are respectively left and right exact. Specifically, the bottom row is exact at the right because β_n is surjective and the complexes satisfy the cycle condition. (♠ TODO: why is the bottom row exact at the right and why is the top row exact at the left)

We now apply the Snake Lemma to this diagram. To identify the resulting terms, we calculate the kernel and cokernel of the vertical map $\bar{d}^C_n$ (and similarly for D and E):

- **Kernel:** The kernel of $\bar{d}^C_n : C_n/\operatorname{im}(d^C_{n+1}) \to \ker(d^C_{n-1})$ consists of elements[5] $x + \operatorname{im}(d^C_{n+1})$ such that $d^C_n(x) = 0$, i.e. $x \in \ker(d^C_n)$. Therefore,
$$\ker(\bar{d}^C_n) = \ker(d^C_n)/\operatorname{im}(d^C_{n+1}) = H_n(C_\bullet).$$
- **Cokernel:** The cokernel of $\bar{d}^C_n$ is the quotient of the target by the image. The target is $\ker(d^C_{n-1})$ and the image is $\operatorname{im}(d^C_n)$. Thus,
$$\operatorname{coker}(\bar{d}^C_n) = \ker(d^C_{n-1})/\operatorname{im}(d^C_n) = H_{n-1}(C_\bullet).$$

Applying the Snake Lemma to (D) yields the exact sequence:

$$\ker(\bar{d}^C_n) \to \ker(\bar{d}^D_n) \to \ker(\bar{d}^E_n) \xrightarrow{\delta_n} \operatorname{coker}(\bar{d}^C_n) \to \operatorname{coker}(\bar{d}^D_n) \to \operatorname{coker}(\bar{d}^E_n)$$

[5] Since elements are invoked in this argument, the Freyd-Mitchell embedding theorem (Theorem 2.4.10) should technically be used for this argument; nevertheless, one can argue that $\ker(\bar{d}^C_n) = H_n(C_\bullet)$ purely categorically.

Substituting the homology groups identified above, we obtain:

$$H_n(C_\bullet) \xrightarrow{H_n(\alpha)} H_n(D_\bullet) \xrightarrow{H_n(\beta)} H_n(E_\bullet) \xrightarrow{\delta_n} H_{n-1}(C_\bullet) \xrightarrow{H_{n-1}(\alpha)} H_{n-1}(D_\bullet) \xrightarrow{H_{n-1}(\beta)} H_{n-1}(E_\bullet)$$

Since this construction exists for every $n \in \mathbb{Z}$, we can splice these sequences together to form the bi-infinite long exact sequence. The naturality of the sequence follows from the naturality of the Snake Lemma and the functoriality of the kernel and cokernel constructions. □

3.3. **Singular homology groups of a topological space.**

Definition 3.3.1. Let V be a real vector space of finite dimension. A *k-simplex in topology* (or a *geometric k-simplex*) is the convex hull of $k+1$ affinely independent points $v_0, v_1, \ldots, v_k \in V$, and is denoted by

$$[v_0, v_1, \ldots, v_k] := \left\{ \sum_{i=0}^{k} t_i v_i \;\middle|\; t_i \geq 0,\ \sum_{i=0}^{k} t_i = 1 \right\}.$$

It is also standard to talk of the *standard topological n-simplex* — the topological space $|\Delta^n|$ defined as the subset of Euclidean space $\mathbb{R}^{n+1}$ given by

$$|\Delta^n| = \left\{ (t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} t_i = 1, \text{ and } t_i \geq 0 \text{ for all } i \right\}$$

equipped with the induced topology from the usual Euclidean topology on $\mathbb{R}^{n+1}$.

(♠ TODO: comment on how $|\Delta^n|$ makes sense via a geometric realization)

(♠ TODO: comment on functoriality)

Definition 3.3.2 (Singular simplices)**.** Let X be a topological space (Definition C.0.5). For each integer $n \geq 0$, A *singular n-simplex in X* is a continuous map (Definition C.0.6)

$$\sigma : \Delta^n \to X.$$

where Δ^n is the standard topological n-simplex (Definition 3.3.1). The set of all singular n-simplices in X is denoted by $S_n(X)$.

Definition 3.3.3 (Singular chain group with coefficients)**.** Let X be a topological space (Definition C.0.5), let $S_n(X)$ be the set of singular n-simplices in X (Definition 3.3.2), and let R be a commutative ring (Definition C.0.9) with unity.

1. The *singular n-chain group of X with coefficients in R* is the free R-module $C_n(X; R)$ whose elements are finite formal linear combinations

$$\sum_i r_i \, \sigma_i, \quad \text{with } \sigma_i \in S_n(X),\ r_i \in R.$$

Elements of $C_n(X; R)$ are called *singular n-chains in X with coefficients in R*.

2. If $A \subseteq X$ is a subspace, the quotient groups

$$C_n(X, A; R) = C_n(X; R)/C_n(A; R)$$

may be referred to as the *relative singular n-chain groups of X in A with coefficients in R* and elements of this group may be referred to as *relative singular n-chains in X relative to A with coefficients in R*.

In either case, when $R = \mathbb{Z}$, the ring R may be suppressed from notation, so we may write $C_n(X)$ and $C_n(X, A)$ for $C_n(X, \mathbb{Z})$ and $C_n(X, A, \mathbb{Z})$ respectively.

Definition 3.3.4 (Singular chain complex with coefficients)**.** Let X be a topological space (Definition C.0.5), and let R be a commutative ring (Definition C.0.9) with unity.

For each $n \geq 1$, define the R-linear *boundary operator*

$$\partial_n : C_n(X; R) \to C_{n-1}(X; R)$$

(Definition 3.3.3) by

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \, \sigma \circ \delta_i,$$

where $\delta_i : \Delta^{n-1} \to \Delta^n$ is the i-th face inclusion. Extend ∂_n to $C_n(X; R)$ by R-linearity. Then $(C_n(X; R), \partial_n)$ forms a chain complex (Definition 3.0.1) in the abelian category of R-modules. This chain complex is called the *singular chain complex of X with coefficients in R*.

If $A \subseteq X$ is a subspace, then the boundary maps above induce maps

$$C_n(X, A; R) \to C_{n-1}(X, A; R)$$

on the relative chain groups $C_n(X, A; R)$, yielding a chain complex of R-modules; this chain complex may be called the *relative singular chain complex of the pair (X, A) with coefficients in R*.

Definition 3.3.5 (Singular homology with coefficients)**.** Let X be a topological space (Definition C.0.5) and R a commutative ring (Definition C.0.9) with 1. The *n-th singular homology group of X with coefficients in R* is the homology group (Definition 3.2.2)

$$H_n(X; R) = H_n(C_*(X; R))$$

where $C_*(X; R)$ is the singular chain complex (Definition 3.3.4) of X with coefficients in R.

Given a subspace $A \subseteq X$, the *n-th relative singular homology group of (X, A) with coefficients in R* is defined as

$$H_n(X, A; R) = H_n(C_*(X, A; R))$$

where $C_*(X, A; R)$ is the relative singular chain complex (Definition 3.3.4) of (X, A) with coefficients in R.

We may denote $H_n(X; \mathbb{Z})$ and $H_n(X, A; \mathbb{Z})$ by $H_n(X)$ and $H_n(X, A)$ respectively.

(♠ TODO: talk about CW-complexes and how singular homology is in practice computed with CW complex homology)

3.4. **Homotopy between morphisms of topological spaces.**

Definition 3.4.1 (Homotopy of maps of topological spaces)**.** Let X and Y be topological spaces and let $K \subseteq X$ be a subset. Let $C(X,Y)$ denote the set of all continuous maps $f : X \to Y$.

1. A *homotopy between two maps $f, g \in C(X,Y)$ relative to K* is a continuous map

$$H : X \times [0,1] \to Y$$

such that for all $x \in X$,

$$H(x,0) = f(x), \quad H(x,1) = g(x),$$

and for all $x \in K$ and $t \in [0,1]$,

$$H(x,t) = f(x) = g(x).$$

If such an H exists, we say f and g are *homotopic relative to K*, and we write $f \simeq g \text{ rel } K$; this is an equivalence relation.

A *homotopy between two maps $f, g \in C(X,Y)$* is simply a homotopy relative to $\emptyset$. We write we write $f \simeq g$ if a homotopy between them exists.

2. Let (X, x_0) and (Y, y_0) be pointed topological spaces (Definition 1.1.10) and let $K \subseteq X$ be a subset with $x_0 \in K$. Let $C_*(X,Y)$ denote the set of all continuous based maps $f : X \to Y$ satisfying $f(x_0) = y_0$.

A *homotopy of based maps $f, g \in C_*(X,Y)$ relative to K* is a continuous map

$$H : X \times [0,1] \to Y$$

such that for all $x \in X$,

$$H(x,0) = f(x), \quad H(x,1) = g(x),$$

and for all $k \in K$ and $t \in [0,1]$,

$$H(k,t) = f(k) = g(k),$$

in particular fixing the basepoint throughout,

$$H(x_0,t) = y_0 \quad \text{for all } t \in [0,1].$$

If such an H exists, we say f and g are *based homotopic relative to K*, and we write $f \simeq g \text{ rel } K$. This is an equivalence relation.

A *homotopy of based maps $f, g \in C_*(X,Y)$* without relative condition is the special case $K = \{x_0\}$ and is called a *homotopy of based maps* or *based homotopy*. We write $f \simeq g$ if such a homotopy exists.

(♠ TODO: homotopy of maps of topological spaces results in chain homotopy for singular homology)

3.5. **Chain homotopy between morphisms of chain complexes.**

Definition 3.5.1. Let $\mathcal{A}$ be an additive category (Definition 2.3.4).

1. Let $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ be chain maps between complexes (Definition 3.0.1) in $\mathcal{A}$. A *chain homotopy from $f_\bullet$ to $g_\bullet$* is a collection of morphisms $\{s_n : C_n \to D_{n+1}\}$ such that for all n,
$$f_n - g_n = d^D_{n+1} \circ s_n + s_{n-1} \circ d^C_n.$$
If such an $s_\bullet$ exists, we say that $f_\bullet$ and $g_\bullet$ are *chain homotopic* and write $f_\bullet \simeq g_\bullet$.
2. Let $f_\bullet : C_\bullet \to D_\bullet$ be a chain map between complexes (Definition 3.0.1) in $\mathcal{A}$. A *chain contraction* is a chain homotopy from $f_\bullet$ to the zero complex. The chain map $f_\bullet$ is said to be *null homotopic* if a chain contraction of $f_\bullet$ exists, i.e. $f_\bullet$ is chain homotopic to the 0 chain complex.
3. Let $f_\bullet : C_\bullet \to D_\bullet$ be a chain map between complexes. We say that $f_\bullet$ is a *chain homotopy equivalence* if there exists a chain map and $h_\bullet : D_\bullet \to C_\bullet$ such that
$$fg \simeq \mathrm{id}_{D_\bullet} \quad \text{and} \quad gf \simeq \mathrm{id}_{C_\bullet}.$$
In this case, it is appropriate to call f and g *chain homotopy inverses of each other*.

One similarly defines the above notions for cochain complexes and their morphisms.

Lemma 3.5.2. Let $\mathcal{A}$ be an additive category (Definition 2.3.4). Being chain homotopic (Definition 3.5.1) is an equivalence relation on chain complexes (Definition 3.0.1) of objects in $\mathcal{A}$. (♠ TODO: equivalence relation)

Lemma 3.5.3. Let $\mathcal{A}$ be an abelian category (Definition 2.3.10). Let $f_\bullet : C_\bullet \to D_\bullet$ (resp. $f^\bullet : C^\bullet \to D^\bullet$) be a morphism of chain complexes (Definition 3.0.1) (resp. of cochain complexes). If $f_\bullet$ is null homotopic (Definition 3.5.1) (resp. $f^\bullet$ is null homotopic), then the induced maps
$$H_n(f_\bullet) : H_n(C_\bullet) \to H_n(D_\bullet)$$
of homology (Definition 3.2.2) (resp.
$$H^n(f^\bullet) : H^n(C^\bullet) \to H^n(D^\bullet)$$
of cohomology) are zero.

Proof. (♠ TODO:) □

Lemma 3.5.4. Let $\mathcal{A}$ be an abelian category (Definition 2.3.10). Let $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ (resp. $f^\bullet, g^\bullet : C^\bullet \to D^\bullet$) be a morphism of chain complexes (Definition 3.0.1) (resp. of cochain complexes) that are chain homotopic (Definition 3.5.1). The induced maps
$$H_n(f_\bullet) : H_n(C_\bullet) \to H_n(D_\bullet)$$
and
$$H_n(g_\bullet) : H_n(C_\bullet) \to H_n(D_\bullet)$$
of homology (Definition 3.2.2) (resp.
$$H^n(f^\bullet) : H^n(C^\bullet) \to H^n(D^\bullet)$$
and
$$H^n(g^\bullet) : H^n(C^\bullet) \to H^n(D^\bullet)$$

of cohomology) are equal.

Proof. This follows from Lemma 3.5.3. □

3.6. Mapping cones of morphisms between topological spaces. (♠ TODO:)

3.7. Mapping cones of morphisms between chain complexes.

Definition 3.7.1. 1. Let $f : (C_\bullet, d_\bullet^C) \to (D_\bullet, d_\bullet^D)$ be a morphism of chain complexes (Definition 3.0.1) in an additive category $\mathcal{A}$ (Definition 2.3.4).

The *mapping cone of f*, denoted $\mathrm{Cone}(f)$, is the chain complex defined by:

- Objects: For each n,

$$\mathrm{Cone}(f)_n = D_n \oplus C_{n-1}.$$

- Differential: For each n, define

$$d_n^{\mathrm{Cone}(f)} : \mathrm{Cone}(f)_n \to \mathrm{Cone}(f)_{n-1}$$

by the matrix morphism

$$d_n^{\mathrm{Cone}(f)} = \begin{pmatrix} d_n^D & f_{n-1} \\ 0 & -d_{n-1}^C \end{pmatrix} : D_n \oplus C_{n-1} \to D_{n-1} \oplus C_{n-2}.$$

This construction defines a chain complex, i.e., $d_{n-1}^{\mathrm{Cone}(f)} \circ d_n^{\mathrm{Cone}(f)} = 0$.

2. Dually, let $g : (C^\bullet, d_C^\bullet) \to (D^\bullet, d_D^\bullet)$ be a morphism of cochain complexes (Definition 3.0.1) in $\mathcal{A}$.

The *mapping cone of g*, denoted $\mathrm{Cone}(g)$, is the cochain complex (Definition 3.0.1) defined by:

- Objects: For each n,

$$\mathrm{Cone}(g)^n = D^n \oplus C^{n+1}.$$

- Differential: For each n, define

$$d^n_{\mathrm{Cone}(g)} : \mathrm{Cone}(g)^n \to \mathrm{Cone}(g)^{n+1}$$

by the matrix morphism

$$d^n_{\mathrm{Cone}(g)} = \begin{pmatrix} d_D^n & g^{n+1} \\ 0 & -d_C^{n+1} \end{pmatrix} : D^n \oplus C^{n+1} \to D^{n+1} \oplus C^{n+2}.$$

This construction defines a cochain complex, i.e., $d^{n+1}_{\mathrm{Cone}(g)} \circ d^n_{\mathrm{Cone}(g)} = 0$.

4. Derived functors

This section roughly follows [Wei94, Section 2].

4.1. **δ-functors.** δ-functors are nice enough functors that yield long exact sequences of homology/cohomology from short exact sequences. The main general examples of (universal) δ-functors are left/right derived functors (Definition 4.3.1) of right/left exact functors (Definition 2.4.8) (Theorem 4.3.4). In particular, the homology/cohomology (Definition 3.2.2) functors $\mathbf{Ch}(\mathcal{A}) \to \mathcal{A}$ are the quintessential examples of (universal) δ-functors.

Definition 4.1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories (Definition 2.3.10).

1. A *homological δ functor from $\mathcal{A}$ to $\mathcal{B}$* is a pair $(T_n, \delta_n)_{n \geq 0}$ consisting of:
   - a sequence of additive functors (Definition 2.4.1) $T_n : \mathcal{A} \to \mathcal{B}$ for each integer $n \geq 0$, and
   - for every short exact sequence (Definition 2.4.3) $0 \to A' \xrightarrow{u} A \xrightarrow{v} A'' \to 0$ in $\mathcal{A}$, a *connecting morphism*
   $$\delta_n(A', A, A'') : T_n(A'') \to T_{n-1}(A')$$
   in $\mathcal{B}$ for each $n > 0$

   that make the induced sequence
   $$\cdots \to T_{n+1}(A'') \xrightarrow{\delta_{n+1}} T_n(A') \to T_n(A) \to T_n(A'') \xrightarrow{\delta_n} T_{n-1}(A') \to \cdots$$
   exact (Definition 3.0.2) in $\mathcal{B}$, and are natural in short exact sequences. That is, for any morphism of short exact sequences, the induced morphisms between these long exact sequences commute.
2. A *cohomological δ-functor from $\mathcal{A}$ to $\mathcal{B}$* is defined dually: it consists of additive functors (Definition 2.4.1) $T^n : \mathcal{A} \to \mathcal{B}$ for $n \geq 0$ and *connecting morphisms*
   $$\delta^n(A', A, A'') : T^n(A') \to T^{n+1}(A'')$$
   such that for each short exact sequence (Definition 2.4.3) $0 \to A' \to A \to A'' \to 0$ the resulting sequence
   $$0 \to T^0(A') \to T^0(A) \to T^0(A'') \xrightarrow{\delta^0} T^1(A') \to T^1(A) \to T^1(A'') \xrightarrow{\delta^1} \cdots$$
   is exact (Definition 3.0.2) and the construction is natural with respect to morphisms of short exact sequences.
3. Let (T_n, δ_n) and (S_n, ∂_n) be homological δ-functors from $\mathcal{A}$ to $\mathcal{B}$. A *morphism of (homological) δ-functors*
   $$\eta : T_\bullet \to S_\bullet$$
   is a family of natural transformations (Definition 1.3.1) $\eta_n : T_n \Rightarrow S_n$ for each $n \geq 0$ such that for every short exact sequence $0 \to A' \xrightarrow{u} A \xrightarrow{v} A'' \to 0$, the following diagram in $\mathcal{B}$ commutes for all $n > 0$:
   $$\begin{array}{ccc} T_n(A'') & \xrightarrow{\delta_n} & T_{n-1}(A') \\ \downarrow \eta_n & & \downarrow \eta_{n-1} \\ S_n(A'') & \xrightarrow{\partial_n} & S_{n-1}(A') \end{array}$$
   The dual notion (for cohomological δ-functors) is defined analogously, reversing the direction of the connecting morphisms.

(♠ TODO: universal delta functors)

4.2. **Abelian categories with enough objects of a class and resolutions.** To be able to define and compute left/right derived functors (Definition 4.3.1) of right/left exact functors, the source abelian category needs to "have enough objects (Definition 4.2.1)" of certain classes; in practice, these classes would be the classes of projective (Lemma 4.2.9) (or flat (Definition 4.2.7) objects when the functor considered is a kind of tensor product, e.g. Definition 2.1.9) or injective (Definition 4.2.2) objects. When the abelian category has enough objects, every object has an appropriate resolution (Definition 4.2.8), see Lemma 4.2.10.

Definition 4.2.1. Let $\mathcal{A}$ be an abelian category (Definition 2.3.10) and let $\mathcal{X}$ be a class of objects in $\mathcal{A}$.

We say that $\mathcal{A}$ *has enough objects of class $\mathcal{X}$ on the left (resp. on the right)* if for any object $M \in \mathcal{A}$, there exists an object X of the class $\mathcal{X}$ and an epimorphism (Definition 2.3.7) $X \twoheadrightarrow M$ (resp. a monomorphism $M \hookrightarrow X$).

The most basic classes of objects that we will use to consider resolutions are injective and projective objects

Definition 4.2.2 (Injective and Projective objects in a general category)**.** Let $\mathcal{C}$ be a category (Definition 1.1.1)

- An object $I \in \mathcal{C}$ is called *injective* if for every monomorphism (Definition 2.3.7) $m : A \to B$ in $\mathcal{C}$ and every morphism $f : A \to I$, there exists a morphism $\tilde{f} : B \to I$ such that the diagram
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "m"] & I \\ B \arrow[ur, dashed, "\tilde{f}"'] & \end{tikzcd}$$
commutes, i.e., $\tilde{f} \circ m = f$.
- Dually, an object $P \in \mathcal{C}$ is called *projective* if for every epimorphism (Definition 2.3.7) $e : X \to Y$ in $\mathcal{C}$ and every morphism $g : P \to Y$, there exists a morphism $\tilde{g} : P \to X$ such that the diagram
$$\begin{tikzcd} & P \arrow[d, "g"] \arrow[dl, dashed, "\tilde{g}"'] \\ X \arrow[r, two heads, "e"] & Y \end{tikzcd}$$
commutes, i.e., $e \circ \tilde{g} = g$.

Definition 4.2.3. Let $\mathcal{A}$ be an abelian category (Definition 2.3.10).

1. $\mathcal{A}$ is said to *have enough injectives* if for every object A in $\mathcal{A}$, there is an monomorphism (Definition 2.3.7) $A \to I$ with I an injective object (Definition 4.2.2) of $\mathcal{A}$. Equivalently, $\mathcal{A}$ has enough injectives if it has enough objects of the class of injectives on the right (Definition 4.2.1)
2. $\mathcal{A}$ is said to *have enough projectives* if for every object A in $\mathcal{A}$, there is a epimorphism (Definition 2.3.7) $P \to A$ with P a projective object (Definition 4.2.2) of $\mathcal{A}$. Equivalently, $\mathcal{A}$ has enough projectives if it has enough objects of the class of projectives on the left (Definition 4.2.1)

Theorem 4.2.4.
1. Examples of abelian categories with enough injectives include:
   - The category of abelian groups.
   - The category of modules over a ring.
   - The category of sheaves of abelian groups on a ringed space or on an essentially small site.
2. Examples of abelian categories with enough projectives include:
   - The category of modules over a ring with enough projectives (e.g., rings with unity and suitable properties). (♠ TODO: make this more precise)
   - The category of finitely generated modules over a semisimple ring.

Baer's criterion describes injective objects in a category of modules.

Theorem 4.2.5 (Baer's Criterion)**.** Let R be a (not necessarily commutative) ring (Definition C.0.7) with unit and let I be an R-module (Definition 2.1.1). Then I is an injective (Definition 4.2.2) R-module if and only if for every left ideal $J \subseteq R$, every R-module homomorphism (Definition 2.1.2)

$$f : J \to I$$

extends to an R-module homomorphism

$$\tilde{f} : R \to I.$$

There are multiple equivalent ways to describe projective objects in a category of modules.

Definition 4.2.6. Let R and S be (not necessarily commutative) rings (Definition C.0.7). A *projective R-S-bimodule* is an (R, S)-bimodule (Definition 2.1.1) P that satisfies any of the following equivalent conditions:

1. The functor
$$\mathrm{Hom}_{{}_R\mathsf{Mod}_S}(P, -) : {}_R\mathsf{Mod}_S \to \mathsf{Ab}$$
is an exact functor (Definition 2.4.8) between the abelian categories ${}_R\mathsf{Mod}_S$ (Definition 2.1.3) and Ab (Definition 1.1.8).
2. P is a projective left module over the ring $R \otimes_{\mathbb{Z}} S^{\mathrm{op}}$ (Definition 2.1.10) (Definition C.0.38).
3. P is a direct summand of a free (R, S)-bimodule. (A free (R, S)-bimodule is a direct sum of copies of the tensor product $R \otimes_{\mathbb{Z}} S$, equipped with the natural left R-action and right S-action).
4. P is a projective object (Definition 4.2.2) in the category ${}_R\mathsf{Mod}_S$. That is, for every surjective homomorphism of (R, S)-bimodules $f : M \to N$ and every homomorphism $g : P \to N$, there exists a homomorphism $h : P \to M$ such that $f \circ h = g$.

Being a projective bimodule is a strictly stronger condition than being projective as a left or right module.

- A bimodule ${}_RP_S$ may be projective as a left R-module (i.e., projective in ${}_R\mathsf{Mod}$) without being a projective bimodule.
- Similarly, it may be projective as a right S-module (i.e., projective in Mod_S) without being a projective bimodule.

- A bimodule that is projective on both sides is sometimes called *biprojective*, but this does not imply it is a projective object in ${}_R\mathsf{Mod}_S$. For example, if $R = S = \mathbb{Z}$, the bimodule $\mathbb{Z}$ is free (hence projective) on both sides, but it is *not* a projective $(\mathbb{Z}, \mathbb{Z})$-bimodule because $\mathbb{Z}$ is not a projective $\mathbb{Z}[\mathbb{Z}]$-module (the augmentation ideal is not projective).

When a tensor product is being considered, "flat" objects are used to construct resolutions; in particular, in a category of modules, the notion of flatness is as follows:

Definition 4.2.7 (Flat module over a ring)**.** Let R be a (not necessarily commutative) ring (Definition C.0.7).

1. Let M be a left R-module. The module M is said to be *flat (with respect to the left R-module structure)* if the functor
$$- \otimes_R M : \mathrm{Mod}_R \to \mathbf{Ab}$$
(Definition 2.1.9) from the category of right R-modules to abelian groups is exact; that is, for every exact sequence of right R-modules
$$0 \to N' \to N \to N'' \to 0,$$
the induced sequence
$$0 \to N' \otimes_R M \to N \otimes_R M \to N'' \otimes_R M \to 0$$
is exact.
(♠ TODO: tor) Equivalently, M is flat if $\mathrm{Tor}_1^R(-, M) = 0$.
2. Let M be a right R-module. The module M is said to be *flat (with respect to the right R-module structure)* if the functor
$$M \otimes_R - : {}_R\mathrm{Mod} \to \mathbf{Ab}$$
from the category of left R-modules to abelian groups is exact; that is, for every exact sequence of right R-modules
$$0 \to N' \to N \to N'' \to 0,$$
the induced sequence
$$0 \to M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0$$
is exact.

Intuitively speaking, resolutions are approximations of an object in an abelian category by acyclic sequences.

Definition 4.2.8. Let $\mathcal{A}$ be an abelian category (Definition 2.3.10) and let $\mathcal{X}$ be a class of objects in $\mathcal{A}$. Let M be an object of $\mathcal{A}$.

1. A *right resolution of M* is a cochain complex (Definition 3.0.1) $I^\bullet$ with $I^i = 0$ for $i < 0$ and a map $M \to I^0$ such that the augmented complex
$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$$
is exact (Definition 3.0.2).

2. A *left resolution of M* is a chain complex (Definition 3.0.1) $P_\bullet$ with $P_i = 0$ for $i < 0$ and a map $P_0 \to M$ such that the augmented complex

$$\cdots P_2 \to P_1 \to P_0 \to M \to 0$$

is exact (Definition 3.0.2).
3. An *$\mathcal{X}$-left resolution* of an object $M \in \mathcal{A}$ a left resolution (Definition 4.2.8) by objects of X, i.e. an exact complex

$$\cdots \to X_2 \to X_1 \to X_0 \to M \to 0$$

with each $X_i \in \mathcal{X}$.
4. An *$\mathcal{X}$-right resolution* of an object $M \in \mathcal{A}$ a right resolution (Definition 4.2.8) by objects of X, i.e. an exact complex

$$0 \to M \to X^0 \to X^1 \to X^2 \to \cdots$$

with each $X_i \in \mathcal{X}$.
5. A *projective resolution of M* is a left resolution $P^\bullet$ for which the objects P^i are all projective (Definition 4.2.2).
6. An *injective resolution of M* is a right resolution $I^\bullet$ for which the objects I^i are all injective (Definition 4.2.2).

Lemma 4.2.9. Let $\mathcal{A}$ be an abelian category (Definition 2.3.10).

1. A projective object (Definition 4.2.2) $\mathcal{P}$ always has a projective resolution (Definition 4.2.8) given by

$$\cdots \to 0 \to \mathcal{P} \xrightarrow{\text{id}} \mathcal{P} \to 0.$$

2. A injective object (Definition 4.2.2) $\mathcal{I}$ always has a injective resolution (Definition 4.2.8) given by

$$0 \to \mathcal{I} \xrightarrow{\text{id}} \mathcal{I} \to 0 \to \cdots .$$

Proof. This is clear. ☐

Lemma 4.2.10 (cf. [Wei94, Lemma 2.2.5, Lemma 2.3.6])**.** Let $\mathcal{A}$ be an abelian category (Definition 2.3.10) and let $\mathcal{X}$ be a class of objects in $\mathcal{A}$.

1. If $\mathcal{A}$ has enough objects of class $\mathcal{X}$ on the right (Definition 4.2.1), then for every object $A \in \mathcal{A}$ there exists an $\mathcal{X}$-right resolution of A (Definition 4.2.8).
2. If $\mathcal{A}$ has enough objects of class $\mathcal{X}$ on the left (Definition 4.2.1), then for every object $A \in \mathcal{A}$ there exists an $\mathcal{X}$-left resolution of A (Definition 4.2.8).

Note that this is a special case of Proposition 4.2.12 obtained by letting the complex $M^\bullet$ be the complex such that

$$M^i = \begin{cases} A & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}.$$

In particular,

- If $\mathcal{A}$ has enough injective objects (Definition 4.2.3), then for every object $A \in \mathcal{A}$ there exists an injective resolution of A (Definition 4.2.8).
- If $\mathcal{A}$ has enough projective objects (Definition 4.2.3), then for every object $A \in \mathcal{A}$ there exists a projective resolution of A (Definition 4.2.8).
- If $F : \mathcal{A} \to \mathcal{B}$ is a left (resp. right) exact functor (Definition 2.4.8) between abelian categories and $\mathcal{A}$ has enough F-acyclic objects on the right (resp. left), then for every object $A \in \mathcal{A}$, there exists an right (resp. left) F-acyclic resolution of A.

Proof.

1. Let $A \in \mathcal{A}$ be an object. Since $\mathcal{A}$ has enough objects of class $\mathcal{X}$ of the right, there is an object X_0 of $\mathcal{X}$ and a monomorphism $\varepsilon_0 : A \to X_0$. Let $A_0 = \operatorname{coker} \varepsilon_0$ (Definition 2.3.6). Inductively, given an object A_{n-1} of $\mathcal{A}$, choose an object X_n of $\mathcal{X}$ and a monomorphism $\varepsilon_n : A_{n-1} \hookrightarrow X_n$. Let $A_n = \operatorname{coker} \varepsilon_n$. In particular, there is a surjection $X_n \twoheadrightarrow A_n$. Let d_n be the composition
$$X_{n-1} \twoheadrightarrow A_{n-1} \xrightarrow{\varepsilon_n} X_n.$$
The chain complex
$$0 \to A \xrightarrow{\varepsilon_0} X_0 \xrightarrow{d_0} X_1 \xrightarrow{d_1} \cdots$$
is thus an $\mathcal{X}$-right resolution of A.
2. This is simply the dual statement of the next statement.

□

In fact, a generalization is possible: if the abelian category $\mathcal{A}$ has enough objects in a class $\mathcal{X}$ (on the right/left) *complex*, then the complex has a "resolution" by objects of $\mathcal{X}$.

Lemma 4.2.11. Abelian categories (Definition 2.3.10) are finitely complete and finitely cocomplete (Definition 2.2.11).

Proof. Abelian categories have finite products and finite coproducts (in the form of direct sums), empty products and coproducts (in the form of the zero object), and equalizers and coequalizers (Definition 2.2.14) (in the form of kernels and cokernels of morphisms), so Theorem 2.2.15 applies. □

Proposition 4.2.12. Let $\mathcal{A}$ be an abelian category (Definition 2.3.10) and let $\mathcal{X}$ be a class of objects in $\mathcal{A}$.

1. If $\mathcal{A}$ has enough objects of class $\mathcal{X}$ on the right (Definition 4.2.1), then for every bounded below complex $M^\bullet$ of objects in $\mathcal{A}$, there exists a bounded below complex $I^\bullet$ of objects in $\mathcal{X}$ and a quasi-isomorphism (Definition 3.2.8) $M^\bullet \to I^\bullet$.
2. If $\mathcal{A}$ has enough objects of class $\mathcal{X}$ on the left (Definition 4.2.1), then for every bounded above complex $M^\bullet$ of objects in $\mathcal{A}$, there exists a bounded above complex $P^\bullet$ of objects in $\mathcal{X}$ and a quasi-isomorphism (Definition 3.2.8) $P^\bullet \to M^\bullet$.

Proof. We prove that if $\mathcal{A}$ has enough objects of class $\mathcal{X}$ on the left, then there exists a complex $P^\bullet$ of objects in $\mathcal{X}$ and a quasi-isomorphism $P^\bullet \to M^\bullet$. The other statement can be proven basically symmetrically.

First suppose that $M^\bullet$ is bounded above; say that $M^i = 0$ for all $i > n$. We inductively construct $P^\bullet$ and the quasi-isomoprhism to $M^\bullet$. Choose an object P^n from $\mathcal{X}$ and a surjective morphism $\epsilon_n : P^n \twoheadrightarrow M^n$, and let $d : P^n \to P^{n+1}$ be the zero map. Assume inductively that we have constructed the complex $P^\bullet$ and maps $\epsilon_i : P^i \to M^i$ for $i = k+1, k+2, \ldots, n$. We want to construct P^k, the differential $d : P^k \to P^{k+1}$ and the map $\epsilon_k : P^k \to M^k$.

Let $L_k = Z^{k+1}(P) \times_{Z^{k+1}(M)} M^k$

$$\begin{tikzcd} L_k \arrow[r] \arrow[d] & Z^{k+1}(P) \arrow[d, "\epsilon_{k+1}"] \\ M^k \arrow[r, "d"] & Z^{k+1}(M) \end{tikzcd}$$

where Z^i denotes the ith cycle (Definition 3.2.1) of a complex (Definition 3.2.2); recall that abelian categories have finite limits by Lemma 4.2.11, so fiber products (Definition C.0.24) exist. Choose an object P^k from $\mathcal{X}$ and a surjective moprhism $\pi : P^k \twoheadrightarrow L_k$. Set the differential $d : P^k \to P^{k+1}$ to be $\mathrm{proj}_{Z^{k+1}(P)} \circ \pi$ and the map $\epsilon_k : P^k \to M^k$ to be $\mathrm{proj}_{M^k} \circ \pi$.

We verify that the square

$$\begin{tikzcd} P^k \arrow[r, "d"] \arrow[d, "\epsilon_k"] & P^{k+1} \arrow[d, "\epsilon_{k+1}"] \\ M^k \arrow[r, "d"] & M^{k+1} \end{tikzcd}$$

commutes, i.e. that $\epsilon_{k+1} \circ d = d \circ \varepsilon_k$. The left is $\epsilon_{k+1} \circ \mathrm{proj}_{Z^{k+1}(P)} \circ \pi$ and the right is $d \circ \mathrm{proj}_{M^k} \circ \pi$, which do indeed coincide.

We verify that the chain map ϵ induces isomorphisms $H^*(P) \xrightarrow{\sim} H^*(M)$ on cohomology objects. We first show that the induced maps on cohomology are epimorphisms. Let $u : Z^k(M) \to L_k = Z^{k+1}(P) \times_{Z^{k+1}(M)} M^k$ be the unique morphism corresponding to the morphisms $0 : Z^k(M) \to Z^{k+1}(P)$ and $Z^k(M) \hookrightarrow M^k$. Let Y be the pullback of π along u:

$$\begin{tikzcd} Y \arrow[r, "\tilde{\pi}"] \arrow[d, "\tilde{u}"] & Z^k(M) \arrow[d, "u"] \\ P^k \arrow[r, "\pi"] & L_k. \end{tikzcd}$$

Note that $\operatorname{im} \tilde{u}$ is a subobject of $Z^k(P) = \ker(d : P^k \to P^{k+1})$ because

$$d \circ \tilde{u} = \mathrm{proj}_{Z^{k+1}(P)} \circ \pi \circ \tilde{u} = \mathrm{proj}_{Z^{k+1}(P)} \circ u \circ \tilde{\pi} = 0.$$

Therefore, $\tilde{u}$ factors through $Z^k(P)$. Writing $[\tilde{u}]$ for the composition $Y \xrightarrow{\tilde{u}} Z^k(P) \twoheadrightarrow H^k(P)$, note that

$$H^k(\epsilon) \circ [\tilde{u}] = [\epsilon_k \circ \tilde{u}] = [\mathrm{proj}_{M^k} \circ \pi \circ \tilde{u}] = [\mathrm{proj}_{M^k} \circ u \circ \tilde{\pi}] = [(\mathrm{id} : Z^k(M) \to Z^k(M)) \circ \tilde{\pi}].$$

The right most expression is the composition

$$Y \xrightarrow{\tilde{\pi}} Z^k(M) \twoheadrightarrow H^k(M).$$

Since π is an epimorphism, $\tilde{\pi}$ is an epimorphism, so the above composition is an epimorphism. We have thus shown that $H^k(\epsilon) \circ [\tilde{u}]$ is an epimorphism, so $H^k(\epsilon)$ is an epimorphism.

We now show that $H^{k+1}(\epsilon) : H^{k+1}(P) \to H^{k+1}(M)$ is a monomorphism. Let K be the kernel of $Z^{k+1}(P) \xrightarrow{\epsilon_{k+1}} Z^{k+1}(M) \twoheadrightarrow H^{k+1}(M)$; this kernel coincides with "the (k+1)-cycles of P mapping to (k+1)-boundaries of M". More precisely, K can be regarded as the fiber product

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & Z^{k+1}(P) \arrow[d, "\epsilon_{k+1}"] \\
B^{k+1}(M) \arrow[r, hook] & Z^{k+1}(M),
\end{tikzcd}$$

and note that this Cartesian diagram displays K as a subobject of $Z^{k+1}(P)$. Furthe note that the morphism $d : M^k \to B^{k+1}(M)$ naturally induces a morphism $L_k \to K$; in fact, K is then the image of the projection map $\mathrm{proj}_{Z^{k+1}(P)} : L_k \to Z^{k+1}(P)$. On the other hand, by definition,

$$B^{k+1}(P) = \mathrm{im}(d : P^k \to Z^{k+1}(P)) = \mathrm{im}(\mathrm{proj}_{Z^{k+1}(P)} \circ \pi).$$

Since π is an epimorphism, this image in turn equals $\mathrm{im}(\mathrm{proj}_{Z^{k+1}(P)})$, which equals K as we have seen. Therefore, K coincides with $B^{k+1}(P)$, which means that the map $Z^{k+1}(P) \to H^{k+1}(M)$, whose kernel is K by definition, naturally induces a monomorphism $H^{k+1}(P) \to H^{k+1}(M)$ as desired.

□

Lemma 4.2.13 (cf. [Wei94, Porism 2.2.7])**.** Let $\mathcal{A}$ be an abelian category (Definition 2.3.10).

1. Let
$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$
be a chain complex (Definition 3.0.1) with P_i projective (Definition 4.2.2). For every left resolution (Definition 4.2.8) $Q_\bullet \to N$ of an object N, every map $M \to N$ lifts to a complex map (Definition 3.0.1) $P_\bullet \to Q_\bullet$ unique up to chain homotopy (Definition 3.5.1).
2. Let
$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$$
be a (co)chain complex (Definition 3.0.1) with I^i injective (Definition 4.2.2). For every right resolution (Definition 4.2.8) $N \to Q^\bullet$ of an object N, every map $N \to M$ lifts to a complex map (Definition 3.0.1) $Q^\bullet \to I^\bullet$ unique up to chain homotopy (Definition 3.5.1).

Proof. 1. The map $P_0 \to M \to N$ lifts to a map $P_0 \to Q_0$ because P_0 is projective and $Q_0 \to N$ is an epimorphism. Inductively suppose that there are morphisms $P_i \to Q_i$ for $0 \leq i \leq n$, where $n \geq 0$ that make

$$\begin{tikzcd}
P_n \arrow[r] \arrow[d] & P_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
Q_n \arrow[r] & Q_{n-1} \arrow[r] & \cdots \arrow[r] & Q_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

into a commuting diagram are established. The morphism $Q_n \to Q_{n-1}$ (where we let $Q_{-1} = N$ and $P_{-1} = M$ here in case that $n = 0$) acts as 0 when restricted to $\mathfrak{I} \coloneqq \mathrm{im}(P_{n+1} \to P_n \to Q_n)$ (Definition 2.3.8) because the composition

$$P_{n+1} \to P_n \to Q_n \to Q_{n-1}$$

equals the composition

$$P_{n+1} \to P_n \to P_{n-1} \to Q_{n-1}.$$

In other words, $\mathfrak{I}$ is a subobject of $\ker(Q_n \to Q_{n-1})$ (Definition 2.3.6), which is isomorphic to $\mathrm{im}(Q_{n+1} \to Q_n)$ by the acyclicity of the sequence of the Q_i's. Therefore, we have a map $P_{n+1} \twoheadrightarrow \mathfrak{I} \hookrightarrow \mathrm{im}(Q_{n+1} \to Q_n)$ along with an epimorphism $Q_{n+1} \twoheadrightarrow \mathrm{im}(Q_{n+1} \to Q_n)$. Since P_{n+1} is projective, the former map lifts to a map $P_{n+1} \to Q_{n+1}$ in a way that is compatible with the latter, i.e. the following commutes:

$$\begin{tikzcd}
P_{n+1} \arrow[d, dotted] \arrow[dr] & \\
Q_{n+1} \arrow[r] & \mathrm{im}(Q_{n+1} \to Q_n).
\end{tikzcd}$$

By induction, this shows thta $M \to N$ lifts to a morphism $P_\bullet \to Q_\bullet$ of complexes.

We show that the morphism of complexes is unique up to chain homotopy, i.e. if $f_1, f_2 : P_\bullet \to Q_\bullet$ are two morphisms of complexes, then $h \coloneqq f_1 - f_2$ is null homotopic. We construct a chain contraction (Definition 3.5.1) $\{s_n : P_n \to Q_{n+1}\}$ of h by induction on n. If $n < 0$, then set $s_n = 0$. If $n = 0$, note that the composition $P_0 \xrightarrow{h_0} Q_0 \to N$ equals the composition $P_0 \to M \xrightarrow{0} N$, so $\mathrm{im}(h_0)$ is a subobject of $\ker(Q_0 \to N) \cong \mathrm{im}(Q_1 \to Q_0)$. The projectivitiy of P_0 thus yields a lift $s_0 : P_0 \to Q_1$ such that h_0 equals the composition $P_0 \xrightarrow{s_0} X_1 \xrightarrow{d} Q_0$:

$$\begin{tikzcd}
 & P_0 \arrow[dl, dotted, "s_0"] \arrow[d, "h_0"] \\
Q_1 \arrow[r, "d"] & Q_0
\end{tikzcd}$$

Note moreover that $h_0 = ds_0 + s_{-1}d$ because $s_{-1} = 0$. Inductively suppose that we have maps s_i for $i \leq n$ such that $h_n = ds_n + s_{n-1}d$ or equivalently that $ds_n = h_n - s_{n-1}d$. Consider the map $h_{n+1} - s_n d : P_{n+1} \to Q_{n+1}$. Compute

$$d(h_{n+1} - s_n d) = dh_{n+1} - ds_n d = dh_{n+1} - (h_n - s_{n-1}d)d = (dh_{n+1} - h_n d) + s_{n-1}dd = 0$$

Therefore, $\mathrm{im}(h_{n+1} - s_n d)$ is a subobject of $\ker(Q_{n+1} \to Q_n) \cong \mathrm{im}(Q_{n+2} \to Q_{n+1})$, which is in turn a quotient of Q_{n+2}. Since P_{n+1} is projective, there is a morphism $s_{n+1} : P_{n+1} \to Q_{n+2}$ such that $ds_{n+1} = h_{n+1} - s_n d$.

$$\begin{tikzcd} & & P_{n+1} \arrow[d, "h_n - s_{n-1}d = ds_n"] \arrow[dll, dotted, "s_{n+1}"'] \\ Q_{n+2} \arrow[r, "d"] & \operatorname{im}(Q_{n+2} \to Q_{n+1}) \cong \ker(Q_{n+1} \to Q_n) \end{tikzcd}$$

The s_n thus form a chain contraction as needed.

2. This is simply dual to the previous part.

□

4.3. **Derived functors of right or left exact functors between abelian categories where the source category has enough projectives or injectives.**

Definition 4.3.1. (♠ TODO: I think that the definition of derived categories might be doable for more general kinds of resolutions? Perhaps it is that if I have a right exact functor F, then L^iF can be computed with resolutions of F-acyclic objects?) (♠ TODO: Apparently, left/right derived functors may be defined for functors that are additive and preserve finite coproducts, and not necessarily right/left exact; the exactness condition ensures that the zeroth derived functor agrees with F.) Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories (Definition 2.3.10), and let

$$F : \mathcal{A} \to \mathcal{B}$$

be an additive functor (Definition 2.4.1).

1. Suppose that the functor F is right exact (Definition 2.4.8) and suppose that $A \in \mathcal{A}$ is an object for which a projective resolution (Definition 4.2.8)

$$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0$$

exists in $\mathcal{A}$. We define the *left derived object* $L_nFA \in \mathcal{B}$ by applying F to obtain a complex

$$\cdots \to F(P_2) \to F(P_1) \to F(P_0) \to 0$$

and letting $L_nF(A)$ be the n-th homology object (Definition 3.2.2) of this complex in $\mathcal{B}$:

$$L_nF(A) := H_n(F(P_\bullet)).$$

The object $L_nF(A)$ is independent of the choice of projective resolution up to natural isomorphism (Proposition 4.3.3).

By convention, set $L_nF = 0$ for $n < 0$.

The *higher left derived objects* refer to the object $L_nF(A)$ for $n > 0$.

2. Suppose that the functor F is right exact (Definition 2.4.8) and that $\mathcal{A}$ has enough projectives (Definition 4.2.3). The *left derived functors* refer to the family of functors

$$L_nF : \mathcal{A} \to \mathcal{B}, \quad A \mapsto L_nF(A).$$

The *higher left derived functors* refer to the functors L_nF for $n > 0$.

3. Suppose that the functor F is right exact (Definition 2.4.8) and suppose that $A \in \mathcal{A}$ is an object for which a injective resolution (Definition 4.2.8)

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$$

exists in $\mathcal{A}$. We define the *right derived object* $R_nFA \in \mathcal{B}$, also often denoted by R^nFA, by applying F to obtain a complex

$$0 \to F(I^0) \to F(I^1) \to F(I^2) \to \cdots .$$

and letting $R_nF(A)$ be the n-th cohomology object (Definition 3.2.2) of this complex in $\mathcal{B}$:

$$R_nF(A) := H^n(F(I_\bullet)).$$

The object $R_nF(A)$ is independent of the choice of injective resolution up to natural isomorphism (Proposition 4.3.3).

By convention, set $R_nF = 0$ for $n < 0$.

The *higher right derived objects* refer to the object $R_nF(A)$ for $n > 0$.

4. Suppose that the functor F is right exact (Definition 2.4.8) and that $\mathcal{A}$ has enough injectives (Definition 4.2.3). The *right derived functors* refer to the family of functors

$$R_nF : \mathcal{A} \to \mathcal{B}, \quad A \mapsto R_nF(A).$$

The right derived functors are also often denoted by R^nF. The *higher right derived functors* refer to the functors R_nF for $n > 0$.

Lemma 4.3.2 (Horseshoe lemma, cf. [Wei94, Horsehoe Lemma 2.2.8])**.** Let $\mathcal{A}$ be an abelian category (Definition 2.3.10).

1. Suppose that

$$0 \to A' \xrightarrow{i_A} A \xrightarrow{\pi_A} A'' \to 0$$

is a short exact sequence in $\mathcal{A}$, and that $\varepsilon' : P'_\bullet \to A'$ and $\varepsilon'' : P''_\bullet \to A''$ are respectively projective resolutions.

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
\cdots P'_2 \arrow[r] & P'_1 \arrow[r] & P'_0 \arrow[r, "\varepsilon'"] & A' \arrow[r] \arrow[d, "i_A"] & 0 \\
 & & & A \arrow[d, "\pi_A"] & \\
\cdots P''_2 \arrow[r] & P''_1 \arrow[r] & P''_0 \arrow[r, "\varepsilon''"] & A'' \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

Let $P_\bullet = P'_\bullet \oplus P''_\bullet$. The complex $P_\bullet$ is a projective resolution of A, and the short exact sequence lifts to an exact esquence of complexes

$$0 \to P' \xrightarrow{i} P \xrightarrow{\pi} P'' \to 0$$

where $i_n : P'_n \to P_n$ and $\pi_n : P_n \to P''_n$ are the natural inclusion and projection respectively.

2. Suppose that
$$0 \to A' \xrightarrow{i_A} A \xrightarrow{\pi_A} A'' \to 0$$
is a short exact sequence in $\mathcal{A}$, and that $\eta' : A' \to I'^\bullet$ and $\eta'' : A'' \to I''^\bullet$ are respectively injective resolutions.
$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & A' \arrow[r, "\eta'"] \arrow[d, "i_A"] & I'^0 \arrow[r] & I'^1 \arrow[r] & I'^2 \cdots \\
 & A \arrow[d, "\pi_A"] & & & \\
0 \arrow[r] & A' \arrow[r, "\eta''"] \arrow[d] & I''^0 \arrow[r] & I''^1 \arrow[r] & I''^2 \cdots \\
 & 0 & & &
\end{tikzcd}$$
Let $I^\bullet = I'^\bullet \oplus I''^\bullet$. The complex $I^\bullet$ is an injective resolution of A, and the short exact sequence lifts to a short exact sequence of complexes
$$0 \to I'^\bullet \xrightarrow{i} I^\bullet \xrightarrow{\pi} I''^\bullet \to 0,$$
where $i^n : I'^n \to I^n$ and $\pi^n : I^n \to I''^n$ are the natural inclusion and projection at each degree n.

Proof. (♠ TODO:) □

Proposition 4.3.3 (cf.[Wei94, Lemma 2.4.1])**.** Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor (Definition 2.4.1) between abelian categories (Definition 2.3.10). Let A be an object of $\mathcal{A}$.

1. Suppose that F is right exact (Definition 2.4.8), and suppose that a projective resolution (Definition 4.2.8)
$$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0$$
of A exists in $\mathcal{A}$. Let
$$\cdots \to Q_2 \to Q_1 \to Q_0 \to A \to 0$$
be any projective resolution of A in $\mathcal{A}$. For all n, there are natural isomorphisms
$$H_n(F(P_\bullet)) \cong H_n(F(Q_\bullet)).$$
In other words, the left derived objects $L_nF(A)$ (Definition 4.3.1) is well defined.
2. Suppose that F is left exact (Definition 2.4.8), and suppose that a injective resolution (Definition 4.2.8)
$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$$
of A exists in $\mathcal{A}$. Let
$$0 \to A \to Q^0 \to Q^1 \to Q^2 \to \cdots$$

be any injective resolution of A in $\mathcal{A}$. For all n, there are natural isomorphisms

$$H_n(F(I^\bullet)) \cong H_n(F(Q^\bullet)).$$

In other words, the right derived objects $R_nF(A)$ (Definition 4.3.1) is well defined.

Proof. 1. By Lemma 4.2.13, there is a lift $f : P_\bullet \to Q_\bullet$ of the identity map $A \to A$ unique up to chain homotopy. There are then induced natural maps $H_n(F(f)) : H_n(F(P_\bullet)) \to H_n(F(Q_\bullet))$. There is also a lift $f' : Q_\bullet \to P_\bullet$ of the identity map $A \to A$ unique up to chain homotopy, and this also induces natural maps $H_n(F(f')) : H_n(F(Q_\bullet)) \to H_n(F(P_\bullet))$. The chain maps f and f' are in fact chain homotopy inverses (Definition 3.5.1) because Lemma 4.2.13 also implies that any lifts $P_\bullet \to P_\bullet$ and $Q_\bullet \to Q_\bullet$ of the identity map $A \to A$ are chain homotopic to the identity chain maps. Therefore, $H_n(F(f))$ and $H_n(F(f'))$ are inverses of each other as morphisms in $\mathcal{B}$. (♠ TODO: prove basic facts about the fucntoriality of homology/cohomology of chain complexes)

2. This is dual to the previous part.

□

Theorem 4.3.4 (cf. [Wei94, Theorem 2.4.6])**.** Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor (Definition 2.4.1) between abelian categories (Definition 2.3.10).

1. Suppose that F is right exact (Definition 2.4.8).
   (a) Let
   $$0 \to A' \to A \to A'' \to 0$$
   be a short exact sequence (Definition 2.4.3) in $\mathcal{A}$. Suppose that there exist projective resolutions (Definition 4.2.8) $P' \to A'$ and $P'' \to A''$. There exists a long exact sequence
   $$\cdots \xrightarrow{\partial} L_iF(A') \to L_iF(A) \to L_iF(A'') \xrightarrow{\partial} L_{i-1}F(A') \to L_{i-1}F(A) \to L_{i-1}F(A'') \xrightarrow{\partial} \cdots$$
   of derived objects (Definition 4.3.1) and this long exact sequence is natural.
   (b) If $\mathcal{A}$ has enough projectives (Definition 4.2.3), then the derived functor L_*F form a homological δ-functor (Definition 4.1.1).
2. Suppose that G is left exact (Definition 2.4.8).
   (a) Let
   $$0 \to A' \to A \to A'' \to 0$$
   be a short exact sequence (Definition 2.4.3) in $\mathcal{A}$. Suppose that there exist injective resolutions (Definition 4.2.8) $I' \to A'$, $I \to A$, and $I'' \to A''$. There exists a long exact sequence
   $$\cdots \xrightarrow{\partial} R^iG(A') \to R^iG(A) \to R^iG(A'') \xrightarrow{\partial} R^{i+1}G(A') \to R^{i+1}G(A) \to R^{i+1}G(A'') \xrightarrow{\partial} \cdots$$
   of derived objects (Definition 4.3.1), and this long exact sequence is natural.
   (b) If $\mathcal{A}$ has enough injectives (Definition 4.2.3), then the derived functors R^*G form a cohomological δ-functor (Definition 4.1.1).

Proof. 1. (a) By the Horseshoe lemma (Lemma 4.3.2), there is a projective resolution $P \to A$ fitting into a short exact sequence $0 \to P' \to P \to P'' \to 0$. Since the P''_n are projective, each sequence $0 \to P'_n \to P_n \to P''_n \to 0$ is split exact. (♠ TODO: show that SES's ending in projective objects are split). Since F is additive, each sequence

$$0 \to F(P'_n) \to F(P_n) \to F(P''_n) \to 0$$

is split exact in $\mathcal{B}$ (♠ TODO: show why). Therefore,

$$0 \to F(P') \to F(P) \to F(P'') \to 0$$

is a short exact sequence of chain complex. Its associated long exact sequence in homology (Theorem 4.3.4) is the desired long exact sequence.

We now show that the long exact sequence is natural. (♠ TODO: continue)

□

4.4. **Tor and Ext.** Recall by Proposition 2.5.4 that left/right adjoint functors between abelian categories are right/left exact. Also recall the tensor-hom adjunction (Theorem 2.5.2) of modules. We can thus define derived functors as follows:

Definition 4.4.1. Let R, S, T be (not necessarily commutative) rings (Definition C.0.7), let M be an R-S bimodule (Definition 2.1.1), and let N be an S-T bimodule. Let $n \geq 0$ be an integer. The *Tor of M and N*, denoted by $\mathrm{Tor}_n(M, N)$ or $\mathrm{Tor}_n^S(M, N)$ (or other notations such as $\mathrm{Tor}_n^S({}_RM_S, {}_SN_T))$ or ${}_R(\mathrm{Tor}_n^S(M, N))_T$ to further emphasize the module structures), is defined in the following ways:

1. as the nth left derived functor (Definition 4.3.1) $L_n(M \otimes_S -)(N)$ of the right exact functor (Definition 2.4.8) $M \otimes_S - : {}_S\mathbf{Mod}_T \to {}_R\mathbf{Mod}_T$ (Definition 2.1.3) (Definition 2.1.9) (Theorem 2.5.2) (Proposition 2.5.4)
2. as the nth left derived functor (Definition 4.3.1) $L_n(- \otimes_S N)(M)$ of the right exact functor (Definition 2.4.8) $- \otimes_S N : {}_R\mathbf{Mod}_S \to {}_R\mathbf{Mod}_T$ (Definition 2.1.3) (Definition 2.1.9) (Theorem 2.5.2) (Proposition 2.5.4).

The two definitions turn out to coincide up to natural isomorphism. (♠ TODO: ref) (♠ TODO: discuss how it can be computed by flat's or projectives)

Definition 4.4.2. Let R, S, T be (not necessarily commutative) rings (Definition C.0.7), let M be an R-S bimodule (Definition 2.1.1), and let N be an R-T bimodule. Let $n \geq 0$ be an integer. The *Ext of M and N*, denoted by $\mathrm{Ext}^n(M, N)$ or $\mathrm{Ext}_R^n(M, N)$ (or other notations such as $\mathrm{Ext}_R^n({}_RM_S, {}_RN_T)$ or ${}_S(\mathrm{Ext}_R^n(M, N))_T$ to further emphasize the module structures), is defined in the following ways:

1. as the nth right derived functor (Definition 4.3.1) $R^n \operatorname{Hom}_R(M, -)(N)$ of the left exact functor (Definition 2.4.8) $\operatorname{Hom}_R(M, -) : {}_R\mathbf{Mod}_T \to {}_S\mathbf{Mod}_T$ (Definition 2.1.3) (Theorem 2.5.2) (Proposition 2.5.4)
2. as the nth right derived functor (Definition 4.3.1) $R^n \operatorname{Hom}_R(-, N)(M)$ of the left exact functor (Definition 2.4.8) $\operatorname{Hom}_R(-, N) : ({}_R\mathbf{Mod}_S)^{\mathrm{op}} \to {}_S\mathbf{Mod}_T$ (Definition 2.1.3) (Definition 1.2.1) (Theorem 2.5.2) (Proposition 2.5.4).

The two definitions turn out to coincide up to natural isomorphism. (♠ TODO: ref)

4.5. **Balancing Ext and Tor.** The goal is to show that the two definitions of Ext (Definition 4.4.2) agree with each other and similarly for Tor (Definition 4.4.1)(♠ TODO: ref). Double complexes (Definition 4.5.1) are useful to do so.

4.5.1. *Double complexes.*

Definition 4.5.1. Let $\mathcal{A}$ be an additive category (Definition 2.3.4). A *double complex* (also called a *bicomplex*) in $\mathcal{A}$ is a collection of objects

$$\{A^{p,q}\}_{(p,q)\in\mathbb{Z}^2}$$

together with morphisms

$$d_A'^{\,p,q} : A^{p,q} \to A^{p+1,q}, \quad d_A''^{\,p,q} : A^{p,q} \to A^{p,q+1},$$

satisfying the identities

$$(d_A')^{p+1,q} \circ (d_A')^{p,q} = 0, \qquad (d_A'')^{p,q+1} \circ (d_A'')^{p,q} = 0,$$

and the *anti-commutativity relation*

$$(d_A'')^{p+1,q} \circ (d_A')^{p,q} + (d_A')^{p,q+1} \circ (d_A'')^{p,q} = 0.$$

An alternative (equivalent) convention defines a double complex $(A^{p,q}, d_A', d_A'')$ so that

$$(d_A'')^{p+1,q} \circ (d_A')^{p,q} - (d_A')^{p,q+1} \circ (d_A'')^{p,q} = 0.$$

This convention differs by a sign and corresponds to replacing d_A'' by $-d_A''$. Thus, both conventions are equivalent up to the natural isomorphism $A^{p,q} \mapsto A^{p,q}$, $d_A' \mapsto d_A'$, $d_A'' \mapsto -d_A''$.

Definition 4.5.2. Let $\mathcal{A}$ be an additive category (Definition 2.3.4), and let

$$(A^{p,q}, d_A', d_A'')_{(p,q)\in\mathbb{Z}^2}, \quad (B^{p,q}, d_B', d_B'')_{(p,q)\in\mathbb{Z}^2}$$

be double complexes (Definition 4.5.1) in $\mathcal{A}$.

A *morphism of double complexes* $f : A \to B$ is a collection of morphisms

$$f^{p,q} : A^{p,q} \to B^{p,q},$$

for all $(p,q) \in \mathbb{Z}^2$, such that the following diagrams commute:

$$d_B'^{\,p,q} \circ f^{p,q} = f^{p+1,q} \circ d_A'^{\,p,q},$$
$$d_B''^{\,p,q} \circ f^{p,q} = f^{p,q+1} \circ d_A''^{\,p,q}.$$

In other words, f respects both horizontal and vertical differentials of the double complexes.

The double complexes and their morphisms form a category, sometimes denoted by $\mathbf{DC}(\mathcal{A})$.

Definition 4.5.3. Let $(A^{p,q}, d_A', d_A'')$ be a double complex in an additive category $\mathcal{A}$.

- The double complex is called *bounded above* if there exist integers p_0, q_0 such that $A^{p,q} = 0$ whenever $p > p_0$ or $q > q_0$.

- The double complex is called *bounded below* if there exist integers p_0, q_0 such that $A^{p,q} = 0$ whenever $p < p_0$ or $q < q_0$.
- The double complex is called *bounded* if it is both bounded above and below.
- The double complex is said to be in the *first quadrant* (also called *first-quadrant double complex*) if $A^{p,q} = 0$ whenever $p < 0$ or $q < 0$. In particular, any first quadrant double complex is bounded below.
- The double complex is said to be in the *third quadrant* (also called *third-quadrant double complex*) if $A^{p,q} = 0$ whenever $p > 0$ or $q > 0$. In particular, any third quadrant double complex is bounded above.
- Let us say that the double complex is *locally finite along diagonals* or *locally bounded along diagonals*[6] if for each integer n, there exist at most finitely many pairs (p, q) with $p + q = n$ such thta $A^{p,q} \neq 0$.
- Let us say that the double complex is *bounded in total degree*[7] if there exist integers m and M such that $A^{p,q} = 0$ whenever $m \leq p + q \leq M$.

Definition 4.5.4. Let $(A^{p,q}, d'_A, d''_A)$ be a double complex (Definition 4.5.1) in an additive category (Definition 2.3.4) $\mathcal{A}$. For each integer n, define:

$$\mathrm{Tot}^n_{\oplus}(A) = \bigoplus_{p+q=n} A^{p,q},$$
$$\mathrm{Tot}^n_{\Pi}(A) = \prod_{p+q=n} A^{p,q}.$$

assuming that the direct sum $\bigoplus_{p+q=n} A^{p,q}$ and the product $\prod_{p+q=n} A^{p,q}$ respectively exist. These are called the *direct-sum total complex* and the *product total complex*, respectively.

Define the differentials categorically as follows:

- For the direct sum total complex, the differential $d^n : \mathrm{Tot}^n_{\oplus}(A) \to \mathrm{Tot}^{n+1}_{\oplus}(A)$ is the unique morphism such that, for each (p, q) with $p + q = n$, we have
$$d^n \circ \iota_{p,q} = \iota_{p+1,q} \circ {d'_A}^{p,q} + (-1)^p \, \iota_{p,q+1} \circ {d''_A}^{p,q},$$
where $\iota_{p,q} : A^{p,q} \to \mathrm{Tot}^n_{\oplus}(A)$ is the canonical inclusion.
- For the product total complex, the differential $d^n : \mathrm{Tot}^n_{\Pi}(A) \to \mathrm{Tot}^{n+1}_{\Pi}(A)$ is the unique morphism such that, for each (p, q) with $p + q = n + 1$, we have
$$\pi_{p,q} \circ d^n = {d'_A}^{p-1,q} \circ \pi_{p-1,q} + (-1)^{p-1} \, {d''_A}^{p,q-1} \circ \pi_{p,q-1},$$
where $\pi_{p,q} : \mathrm{Tot}^{n+1}_{\Pi}(A) \to A^{p,q}$ is the canonical projection.

Then $(\mathrm{Tot}^{\bullet}_{\oplus}(A), d)$ and $(\mathrm{Tot}^{\bullet}_{\Pi}(A), d)$ are chain complexes (Definition 3.0.1) in $\mathcal{A}$, whenever the corresponding sums or products exist. The complexes $\mathrm{Tot}^{\bullet}_{\oplus}(A)$ and $\mathrm{Tot}^{\bullet}_{\Pi}(A)$ are also denoted by $\mathrm{Tot}^{\oplus}(A)$ and $\mathrm{Tot}^{\Pi}(A)$.

[6] These do not seem to be standard teminology.

[7] This does not seem to be standard teminology.

Definition 4.5.5 (Double Complex associated to biadditive functor and chain complexes)**.** Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be additive categories (Definition 2.3.4), and suppose that

$$F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$$

is a biadditive functor (Definition 2.4.2). Let $X_\bullet$ and $Y_\bullet$ be chain complexes (Definition 3.0.1) of objects in $\mathcal{A}$ and $\mathcal{B}$ respectively.

Construct the double complex (Definition 4.5.1) $Z_{\bullet,\bullet} = (Z_{n,m}, d^h_{n,m}, d^v_{n,m})$ in C associated to F, $X_\bullet$, and $Y_\bullet$ as follows:

$$Z_{n,m} := F(X_n, Y_m),$$

with horizontal differentials

$$d^h_{n,m} := F(d^X_n, \mathrm{id}_{Y_m}) : Z_{n,m} \to Z_{n-1,m},$$

and vertical differentials

$$d^v_{n,m} := (-1)^n F(\mathrm{id}_{X_n}, d^Y_m) : Z_{n,m} \to Z_{n,m-1}.$$

These differentials indeed satisfy the double complex conditions:

$$d^h \circ d^h = 0, \quad d^v \circ d^v = 0, \quad \text{and} \quad d^h \circ d^v + d^v \circ d^h = 0.$$

We may often denote the double complex $Z_{\bullet,\bullet}$ by $F(X_\bullet, Y_\bullet)$. In particular, F induces a bifunctor

$$F : \mathbf{Ch}(\mathcal{A}) \times \mathbf{Ch}(\mathcal{B}) \to \mathbf{DC}(\mathcal{C})$$

(Definition 4.5.2) that is in fact a biadditive functor of additive categories (see Proposition 3.1.8) (♠ TODO: verify that we indeed get a biadditive functor)

In particular, we may speak of the total complexes (Definition 4.5.4) $\mathrm{Tot}^\oplus(F(X_\bullet, Y_\bullet))$ and $\mathrm{Tot}^\Pi(F(X_\bullet, Y_\bullet))$, and these specify biadditive functors

$$\mathrm{Tot}^\oplus(F(-,-)), \mathrm{Tot}^\Pi(F(-,-)) : \mathbf{Ch}(\mathcal{A}) \times \mathbf{Ch}(\mathcal{B}) \to \mathbf{Ch}(\mathcal{C}).$$

4.5.2. *Acyclic assembly lemma.*

Lemma 4.5.6. Let C be a double complex (Definition 4.5.1) of objects in an additive category (Definition 2.3.4) $\mathcal{A}$. If C is locally bounded along diagonals (Definition 4.5.3), then the complexes $\mathrm{Tot}^\bigoplus(A)$ and $\mathrm{Tot}^\prod(A)$ (Definition 4.5.4) are naturally isomorphic.

Proof. Since C is locally bounded along diagonals, the degreee n components

$$\left(\mathrm{Tot}^\oplus\right)^n(A) = \bigoplus_{p+q=n} A^{p,q},$$
$$\left(\mathrm{Tot}^\Pi\right)^n(A) = \prod_{p+q=n} A^{p,q}.$$

are finite direct sums and finite products respectively and hence are naturally isomorphic (Lemma 2.3.5). The differential maps of the two total complexes also naturally coincide. ☐

Lemma 4.5.7 (cf. [Wei94, Acyclic Assembly Lemma 2.7.3])**.** (♠ TODO: It may be the case that this is generalizable beyond first quadrant double complexes, but I don't have a slick way to show this. See the commented out code for the statements; also, it may be necessary to assume something like AB4∗ for such statements) Let $\mathcal{A}$ be an abelian category (Definition 2.3.10) for which (small) filtered colimits (Definition 2.2.13) which exist are exact (e.g. which holds if $\mathcal{A}$ satisfies Ab5 (Definition 2.5.7)). Let C be a double complex (Definition 4.5.1) in $\mathcal{A}$.

If C has exact columns or has exact rows and C is a bounded below or bounded above double complex (Definition 4.5.3), then $\mathrm{Tot}^{\Pi}(C)$ is an acyclic chain complex (Definition 3.0.2).

Proof. We show that if C has exact columns and C is bounded below, then $\mathrm{Tot}^{\Pi}(C)$ is an acyclic chain complex (Definition 3.0.2); it can then be argued symmetrically that if C has exact columns and C is bounded above, then $\mathrm{Tot}^{\Pi}(C)$ is acyclic. Moreover, the case of exact rows can be deduced by reflecting the rows and columns of double complexes.

Note that since C is assumed to be bounded below and hence is locally bounded along diagonals (Definition 4.5.3), $\mathrm{Tot}^{\Pi}(C)$ and $\mathrm{Tot}^{\oplus}(C)$ exist, are constructed by finite products (which are also finite coproducts), and are naturally isomorphic by Lemma 4.5.6. Further recall that finite coproducts in an abelian category are exact.

Define the sub-double complexes F^kC of C by

$$(F^kC)^{p,q} = \begin{cases} C^{p,q} & \text{if } p \leq k \\ 0 & \text{otherwise} \end{cases}.$$

This yields a filtration

$$\cdots \subseteq F^{k-1}C \subseteq F^kC \subseteq F^{k+1} \subseteq \cdots \subseteq C.$$

Moreover, for each n,

$$\left(\mathrm{Tot}^{\Pi}(F^kC)\right)^n = \prod_{p\leq k, p+q=n} C^{p,q}.$$

For each n, the above stabilizes as $k \to \infty$ to $\mathrm{Tot}^{\Pi}(C)^n$. Now let

$$D^k = F^kC/F^{k-1}C = \begin{cases} C^{p,q} & \text{if } p = k \\ 0 & \text{otherwise.} \end{cases}$$

Since each column $C^{k,*}$ is exact by assumption, the total complex $\mathrm{Tot}^{\Pi}(D^k)$ is acyclic. Note that we have short exact sequences

$$0 \to F^{k-1}C \to F^kC \to D^k \to 0$$

of double complexes. The totalization functor $\mathrm{Tot}^{\Pi}(-)$ in this case is exact because all of the double complexes are locally bounded along diagonals (Definition 4.5.3). We hence have a short exact sequence

$$0 \to \mathrm{Tot}^{\Pi}(F^{k-1}C) \to \mathrm{Tot}^{\Pi}(F^kC) \to \mathrm{Tot}^{\Pi}(D^k) \to 0.$$

Since $\operatorname{Tot}^{\Pi}(D^k)$ is acyclic (Definition 3.0.2), the long exact cohomology sequences (Theorem 4.3.4) yield isomorphisms

$$H^n(\operatorname{Tot}^{\Pi}(F^{k-1}C)) \cong H^n(\operatorname{Tot}^{\Pi}(F^kC)).$$

Since C is assumed to be bounded, the subcomplex F^kC is zero for sufficiently negative k, in which case $\operatorname{Tot}^{\Pi}(F^kC)$ is acyclic. By induction on k, $\operatorname{Tot}^{\Pi}(F^kC)$ remains acyclic for all k. Moroever, the filtered colimit $\varinjlim_k \operatorname{Tot}^{\Pi}(F^kC)$ is $\operatorname{Tot}^{\Pi}(C)$. The assumed exactness of filtered colimits in $\mathcal{A}$ concludes that $\operatorname{Tot}^{\Pi}(C)$ is acyclic.

By symmetry, if C instead has exact rows, then $\operatorname{Tot}^{\Pi}(C)$ is an acyclic chain complex. $\square$

Lemma 4.5.8. Let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a biadditive functor (Definition 2.4.2) of abelian categories (Definition 2.3.10). Assume that (small) filtered colimits which exist in $\mathcal{C}$ are exact (e.g. which holds if $\mathcal{C}$ satisfies Ab5 (Definition 2.5.7)).

Let $A \in \mathcal{A}$ and $B \in \mathcal{B}$ be objects.

1. Suppose that left resolutions (Definition 4.2.8) $P_{A,\bullet} \to A$ and $P_{B,\bullet} \to B$ exist such that $P_{A,i}$ and $P_{B,i}$ are flat with respect to F on the left and right respectively, i.e. $F(P_{A,i}, -) : \mathcal{B} \to \mathcal{C}$ and $F(-, P_{B,i}) : \mathcal{A} \to \mathcal{C}$ are exact for all i.
   The complexes $F(P_{A,\bullet}, B)$ and $F(A, P_{B,\bullet})$ are quasi-isomorphic (Definition 3.2.8) to the complex $\operatorname{Tot}(F(P_{A,\bullet}, P_{B,\bullet}))$ (Definition 4.5.4) (Definition 4.5.5).
2. Suppose that right resolutions (Definition 4.2.8) $A \to I^{A,\bullet}$ and $B \to I^{B,\bullet}$ exist such that $I^{A,i}$ and $I^{B,i}$ are flat with respect to F on the left and right respectively, $F(I^{A,i}, -) : \mathcal{B} \to \mathcal{C}$ and $F(-, I^{B,i}) : \mathcal{A} \to \mathcal{C}$ are exact for all i.
   The complexes $F(I^{A,\bullet}, B)$ and $F(A, I^{B,\bullet})$ are quasi-isomorphic (Definition 3.2.8) to the complex $\operatorname{Tot}(F(I^{A,\bullet}, I^{B,\bullet}))$ (Definition 4.5.4) (Definition 4.5.5).

Proof. We prove 1. The other part is the dual statement.

Choose resolutions $P_{A,\bullet} \xrightarrow{\varepsilon} A$ and $P_{B,\bullet} \xrightarrow{\eta} B$ such that $F(P_{A,i}, -) : \mathcal{B} \to \mathcal{C}$ and $F(-, P_{B,i}) : \mathcal{A} \to \mathcal{C}$ are exact for all i. Identifying A and B with complexes concentrated in degree 0, we can form (Definition 4.5.5) the three double complexes (Definition 4.5.1) $F(P_{A,\bullet}, P_{B,\bullet})$, $F(A, P_{B,\bullet})$, and $F(P_{A,\bullet}, B)$. Note that the augmentation morphisms ε and η induce morphisms $P_{A,\bullet} \otimes P_{B,\bullet} \to A \otimes P_{B,\bullet}, P_{A,\bullet} \otimes B$.

Let C be the double complex of objects in $\mathcal{C}$ obtained from $F(P_{A,\bullet}, P_{B,\bullet})$ by adding $F(A, P_{B,\bullet}[-1])$ in the column $p = -1$. One can show that the translate $\operatorname{Tot}(C)[1]$ is the mapping cone (Definition 3.7.1) of the map

$$\operatorname{Tot}(F(P_{A,\bullet}, P_{B,\bullet})) \xrightarrow{\varepsilon \otimes \mathrm{id}} \operatorname{Tot}(F(A, P_{B,\bullet})) = F(A, P_{B,\bullet}).$$

Moreover, since each $F(-, P_{B,i})$ is an exact functor, every row of C is exact, so $\operatorname{Tot}(C)$ is exact by Lemma 4.5.7. Therefore, $F(\varepsilon, \mathrm{id})$ is a quasi-isomorphism and hence

$$H_*(\operatorname{Tot}(F(P_{A,\bullet}, P_{B,\bullet}))) \xrightarrow{H_*(F(\varepsilon, P_{B,\bullet}))} H_*(F(A, P_{B,\bullet}))$$

is a natural isomorphism.

By symmetry, there is a natural isomorphism $H_*(\operatorname{Tot}(F(P_{A,\bullet} P_{B,\bullet}))) \to H_*(F(P_{A,\bullet}, B))$. $\square$

Theorem 4.5.9 (Balancing generalized derived functors of a biadditive functor of abelian categories computed via flat resolutions)**.** Let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a biadditive functor (Definition 2.4.2) of abelian categories (Definition 2.3.10). Assume that (small) filtered colimits which exist in $\mathcal{C}$ are exact (e.g. which holds if $\mathcal{C}$ satisfies Ab5 (Definition 2.5.7)).

Let $A \in \mathcal{A}$ and $B \in \mathcal{B}$ be objects.

1. Suppose that left resolutions (Definition 4.2.8) $P_{A,\bullet} \to A$ and $P_{B,\bullet} \to B$ exist such that $F(P_{A,i}, -) : \mathcal{B} \to \mathcal{C}$ and $F(-, P_{B,i}) : \mathcal{A} \to \mathcal{C}$ are exact for all i, i.e. $P_{A,\bullet}$ and $P_{B,\bullet}$ are flat resolutions of A and B respectively.
   (a) The objects $L_n^I F(A,B)$ and $L_n^{II} F(A,B)$ are naturally isomorphic.
   (b) The objects $L_n^I F(A,B)$ and $L_n^{II} F(A,B)$ are well defined (up to natural isomorphism), i.e. do not depend on the choice of left resolutions of A and B respectively.
2. Suppose that right resolutions (Definition 4.2.8) $A \to I^{A,\bullet}$ and $B \to I^{B,\bullet}$ exist such that $F(I^{A,i}, -) : \mathcal{B} \to \mathcal{C}$ and $F(-, I^{B,i}) : \mathcal{A} \to \mathcal{C}$ are exact for all i.
   (a) The objects $R_I^n F(A,B)$ and $R_{II}^n F(A,B)$ are naturally isomorphic.
   (b) The objects $R_I^n F(A,B)$ and $R_{II}^n F(A,B)$ are well defined (up to natural isomorphism), i.e. do not depend on the choice of left resolutions of A and B respectively.

Proof. We prove 1. The other part is the dual statement.

Choose resolutions $P_{A,\bullet} \xrightarrow{\varepsilon} A$ and $P_{B,\bullet} \xrightarrow{\eta} B$ such that $F(P_{A,i}, -) : \mathcal{B} \to \mathcal{C}$ and $F(-, P_{B,i}) : \mathcal{A} \to \mathcal{C}$ are exact for all i. As per Lemma 4.5.8, $F(\varepsilon, \mathrm{id})$ is a quasi-isomorphism and hence

$$H_*(\mathrm{Tot}(F(P_{A,\bullet}, P_{B,\bullet}))) \xrightarrow{H_*(F(\varepsilon, P_{B,\bullet}))} H_*(F(A, P_{B,\bullet}))$$

is a natural isomorphism. By symmetry, there is a natural isomorphism $H_*(\mathrm{Tot}(F(P_{A,\bullet} P_{B,\bullet}))) \to H_*(F(P_{A,\bullet}, B))$. Therefore, $L_n^I(A,B)$ and $L_n^{II}(A,B)$ are naturally isomorphic as claimed. In particular, $L_n^I(A,B)$ and $L_n^{II}(A,B)$ are independent of the choice of resolution of A and B respectively.

□

Theorem 4.5.10 (Balancing of Tor, cf. [Wei94, Theorem 2.7.2])**.** Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories (Definition 2.3.10), and let $\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a biadditive functor (Definition 2.4.2) that is right exact (Definition 2.4.8) in each variable. Assume that (small) filtered colimits which exist in $\mathcal{C}$ are exact (e.g. which holds if $\mathcal{C}$ satisfies Ab5 (Definition 2.5.7)).

Given $A \in \mathcal{A}$ and $B \in \mathcal{B}$ for which flat resolutions exist, let $\mathrm{Tor}_n^I(A,B)$ and $\mathrm{Tor}_n^{II}(A,B)$ respectively be the Tor objects $\mathrm{Tor}_n^{\mathcal{A}}(A,B)$ computed via flat resolutions of A in $\mathcal{A}$ and of B in $\mathcal{B}$.

1. $\mathrm{Tor}_n^I(A,B)$ and $\mathrm{Tor}_n^{II}(A,B)$ are naturally isomorphic.
2. $\mathrm{Tor}_n^I(A,B)$ and $\mathrm{Tor}_n^{II}(A,B)$ are independent of the choice of flat resolution of A and B respectively.

In particular, we may identify the objects $\mathrm{Tor}_n^I(A,B)$ and $\mathrm{Tor}_n^{II}(A,B)$ and simply write $\mathrm{Tor}_n(A,B)$ for either.

Proof. This follows from Theorem 4.5.9. $\square$

(♠ TODO: state the balancing theorems of Ext and Tor explicitly)

5. Ext and Tor

6. Sheaves on topological spaces

Sheaves (say on a topological space) of abelian groups form an abelian category with a rich theory.

6.0.1. *Presheaves and sheaves on topological spaces.*

Definition 6.0.1 (Presheaf on a category)**.** Let C and $\mathcal{A}$ be (large) categories (Definition 1.1.1).

1. A *presheaf $\mathcal{F}$ on C with values in $\mathcal{A}$* is a functor
$$\mathcal{F} : C^{\mathrm{op}} \to \mathcal{A}.$$
In other words, a presheaf $\mathcal{F}$ on C with values in $\mathcal{A}$ is simply a contravariant functor (Definition 1.2.2) from C to $\mathcal{A}$. Explicitly, for every object U in C, one has an object $\mathcal{F}(U)$ in $\mathcal{A}$ (called the *U-valued sections/sections evaluated at U of $\mathcal{F}$*), and for every morphism $f : V \to U$ in C, one has a morphism (called the *restriction map*)
$$\mathcal{F}(f) : \mathcal{F}(U) \to \mathcal{F}(V)$$
in $\mathcal{A}$, such that for all composable morphisms $W \xrightarrow{g} V \xrightarrow{f} U$ in C, the following diagram in $\mathcal{A}$ commutes:
$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\mathcal{F}(f)"] \arrow[rr, bend left, "\mathcal{F}(f\circ g)"] & \mathcal{F}(V) \arrow[r, "\mathcal{F}(g)"] & \mathcal{F}(W) \end{tikzcd}$$
That is,
$$\mathcal{F}(g) \circ \mathcal{F}(f) = \mathcal{F}(f \circ g),$$
and for every object U in C, $\mathcal{F}(\mathrm{id}_U) = \mathrm{id}_{\mathcal{F}(U)}$.
2. Let $\mathcal{F}, \mathcal{G} : C^{\mathrm{op}} \to \mathcal{A}$ be two presheaves on C with values in $\mathcal{A}$. A *morphism of presheaves*
$$\varphi : \mathcal{F} \to \mathcal{G}$$
is a natural transformation of functors (Definition 1.3.1): for each object U of C, one has a morphism
$$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

in $\mathcal{A}$, such that for every morphism $f : V \to U$ in C, the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\varphi_U"'] & \mathcal{F}(V) \arrow[d, "\varphi_V"] \\ \mathcal{G}(U) \arrow[r, "\mathcal{G}(f)"'] & \mathcal{G}(V) \end{tikzcd}$$

commutes, i.e.,

$$\varphi_V \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \varphi_U$$

for all objects and morphisms in C.

3. Given a universe (Definition A.0.3) U, a *U-presheaf on $\mathcal{C}$* typically refers to a presheaf of U-sets on C.
4. The *presheaf category/category of $\mathcal{A}$-valued presheaves on $\mathcal{C}$* is the (large) category whose objects are the presheaves on C with values in $\mathcal{A}$ and whose morphisms are the presheaf morphisms. Common notations for the presheaf category include, but are not limited to: $\mathcal{A}^{\mathcal{C}^{\text{op}}}$, $\text{PreShv}(\mathcal{C}, \mathcal{A})$, $[\mathcal{C}^{\text{op}}, \mathcal{A}]$. If the value category $\mathcal{A}$ is clear from context, then notations such as $\text{PreShv}(\mathcal{C})$ are also common. Note that the presheaf category $\text{PreShv}(\mathcal{C}, \mathcal{A})$ is equivalent to the category of functors (Definition 2.2.6) $\mathcal{C}^{\text{op}} \to \mathcal{A}$ and hence notations for the functor categories are applicable as notations for presheaf categories.

Definition 6.0.2 (Category of opens of a topological space)**.** Let X be a topological space (Definition C.0.5). The *category of opens of X*, sometimes denoted $\mathbf{Open}(X)$ (or $\text{Open}(X)$ or $\text{Ouv}(X)$ (for the French word "ouvert", meaning open), etc.), is the small (Definition 1.1.2) category (Definition 1.1.1) defined as follows:

- The objects are the open subsets $U \subseteq X$.
- For two open sets $U, V \subseteq X$, the morphism set is

$$\text{Hom}_{\mathbf{Open}(X)}(U, V) = \begin{cases} \{\iota_{U,V}\}, & \text{if } U \subseteq V, \\ \varnothing, & \text{otherwise,} \end{cases}$$

  where $\iota_{U,V}$ denotes the inclusion morphism $U \hookrightarrow V$.
- Composition of morphisms is given by composition of set-theoretic inclusions, i.e.

$$\iota_{V,W} \circ \iota_{U,V} = \iota_{U,W} \quad \text{whenever } U \subseteq V \subseteq W.$$

- The identity morphism on an object U is the inclusion $\iota_{U,U} = \text{id}_U$.

Definition 6.0.3. Let (X, τ_X) and (Y, τ_Y) be topological spaces (Definition C.0.5), and let $f : X \to Y$ be a continuous map (Definition C.0.6). Let $\text{Open}(X)$ and $\text{Open}(Y)$ be their respective categories of open sets (Definition 6.0.2) with inclusion morphisms, equipped with the canonical (Definition B.0.5) Grothendieck topologies (Definition B.0.4) given by open coverings.

Define the functor

$$f^{-1} : \text{Open}(Y) \to \text{Open}(X), \quad U \mapsto f^{-1}(U).$$

It is a continuous functor of sites from $\text{Open}(Y)$ to $\text{Open}(Y)$ which induces a site morphism

$$f : (\text{Open}(X), \text{can}) \to (\text{Open}(Y), \text{can})$$

.

Definition 6.0.4 (Presheaf on a topological space)**.** Let X be a topological space (Definition C.0.5). Let $\mathcal{D}$ be a category.

A *presheaf (of objects of $\mathcal{D}$/valued in $\mathcal{D}$) on X* is a rule $\mathcal{F}$ that assigns:

- to each open set $U \subseteq X$, an object $\mathcal{F}(U) \in \operatorname{Ob}\mathcal{D}$, called the *sections of $\mathcal{F}$ over U*,
- to each inclusion of open sets $V \subseteq U$, a morphism

$$\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V), \quad s \mapsto s|_V,$$

in the category $\mathcal{D}$ called the *restriction map* such that the following conditions hold:

- (Identity) For every open set $U \subseteq X$, the restriction map ρ_U^U is the identity on $\mathcal{F}(U)$.
- (Transitivity) For inclusions $W \subseteq V \subseteq U$ of open sets, one has

$$\rho_W^U = \rho_W^V \circ \rho_V^U.$$

For instance, we may speak of a *presheaf of sets/groups/rings/etc. on the topological space X*.

Equivalently, a presheaf on X (of objects in a category $\mathcal{D}$) is a functor (Definition 1.2.2)

$$\mathbf{Open}(X)^{\mathrm{op}} \to \mathcal{D}$$

from the opposite of the category $\mathbf{Open}(X)$ (Definition 6.0.2) of open subsets of X (see also Definition 6.0.1) .

Equivalently, a presheaf on X is a presheaf on the category $\mathbf{Open}(X)$ in the sense of Definition 6.0.1 .

The sections object $\mathcal{F}(U)$ is also denoted by $\Gamma(U, \mathcal{F})$. Moreover, the object $\mathcal{F}(X) = \Gamma(X, \mathcal{F})$ is called the *global sections object of $\mathcal{F}$*.

Definition 6.0.5 (Sheaf on a topological space)**.** Let X be a topological space (Definition C.0.5), let $\mathcal{D}$ be a category (Definition 1.1.1) with a terminal object, and let $\mathcal{F}$ be a presheaf valued in $\mathcal{D}$ on X (Definition 6.0.4). Then $\mathcal{F}$ is a *sheaf* if it satisfies the following additional condition (known as the *sheaf axioms*):

For every open set $U \subseteq X$ and every open cover $\{U_i\}_{i \in I}$ of U, let $\mathcal{J}$ be the diagram (Definition 2.2.6) in the category of opens (Definition 6.0.2) of U consisting of the inclusions $U_i \cap U_j \hookrightarrow U_i$ for all $i, j \in I$. Then $\mathcal{F}$ is a sheaf if the limit (Definition 2.2.8) of the diagram $\mathcal{F} \circ \mathcal{J}$ exists in $\mathcal{D}$ and the natural morphism

$$\mathcal{F}(U) \to \lim_{j \in \mathcal{J}} \mathcal{F}(j)$$

is an isomorphism. More precisely, $\mathcal{J} : J \to \mathrm{Open}(U)$ should be the functor whose index category J consists of

1. An object i for every $i \in I$ and an object (i, j) for every pair $i, j \in I$,
2. Morphisms $p_1 : (i, j) \to i$ and $p_2 : (i, j) \to j$ for every pair $i, j \in I$

and which sends the objects and morphisms as follows:

1. $\mathcal{J}(i) = U_i$
2. $\mathcal{J}(i,j) = U_i \cap U_j$
3. $\mathcal{J}(p_1) : U_i \cap U_j \hookrightarrow U_i$
4. $\mathcal{J}(p_2) : U_i \cap U_j \hookrightarrow U_j$.

In particular, taking $U = \emptyset$ and taking the empty open cover of the empty set, $\mathcal{F}(\emptyset)$ must be the terminal object of $\mathcal{D}$

In the case that $\mathcal{D}$ admits all small limits (Definition 2.2.10), the sheaf condition is equivalent to the following: For every open set $U \subset X$ and every open cover $\{U_i\}_{i \in I}$ of U, the following equalizer diagram is exact (Definition 2.2.14):

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j).$$

Here, the morphism $\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i)$ and the two morphisms $\prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$ are induced by the restriction maps (Definition 6.0.4) $\mathcal{F}(U) \to \mathcal{F}(U_i)$ and $\mathcal{F}(U_i) \to \mathcal{F}(U_i \cap U_j)$.

In the case that $\mathcal{D}$ is some subcategory (Definition 1.3.6) of the category of sets (Definition 1.1.7), the sheaf condition is equivalent to the following: For every open set $U \subseteq X$ and every open cover $\{U_i\}_{i \in I}$ of U,

- (Locality) If $s, t \in \mathcal{F}(U)$ are such that $s|_{U_i} = t|_{U_i}$ for all i, then $s = t$.
- (Gluing) If for each i there is $s_i \in \mathcal{F}(U_i)$ such that for all i, j one has $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all i.

Equivalently, a sheaf on a topological space X may be defined as a sheaf on (Definition B.0.6) the site (Definition B.0.4) of opens on X (Definition B.0.5).

(♠ TODO: examples of presheaves and sheaves on topological spaces)

Note that on a topological space X, it is natural to talk about open sets $\{U_i\}$ which cover an open set U of X. One can express more general notions of "coverings" of objects in a category through the notion of a "site" (Definition B.0.4), which consists of a category and a "Grothendieck topology" (Definition B.0.4) on the category. The notion of "sheaf on a topological space" (Definition 6.0.5) above is a specialization of this more general notion.

6.0.2. *Sheafification of presheaves.*

Definition 6.0.6. (♠ TODO: Move these notations to the defintions of presheaves and sheaves on topological spaces) Let X be a topological space (Definition C.0.5), and let $\mathcal{D}$ be a category (Definition 1.1.1) with a terminal object.

The presheaves on X valued in $\mathcal{D}$ (Definition 6.0.4), along with the morphisms thereof, form a (in general large) category (Definition 1.1.1) often denoted by notations such as

$\mathrm{PreShv}(X, \mathcal{D})$ (♠ TODO: include more notations) (or $\mathrm{PreShv}(X)$ if the category $\mathcal{D}$ is clear). If $\mathcal{D}$ is locally small (Definition 1.1.2), then so is $\mathrm{PreShv}(X, \mathcal{D})$.

Similarly, the sheaves on X valued in $\mathcal{D}$ (Definition 6.0.5), along with the morphisms thereof, form a (in general large) category (Definition 1.1.1) often denoted by notations such as $\mathrm{Shv}(X, \mathcal{D})$ (♠ TODO: include more notations) (or $\mathrm{Shv}(X)$ if the category $\mathcal{D}$ is clear). The category $\mathrm{Shv}(X, \mathcal{D})$ is a full subcategory (Definition 1.3.7) of $\mathrm{PreShv}(X, \mathcal{D})$.

Equivalently, the categories of presheaves and sheaves are the categories $\mathrm{PreShv}(\mathbf{Open}(X), \mathcal{D})$ and $\mathrm{Shv}(\mathbf{Open}(X), \mathcal{D})$ of presheaves (Definition 6.0.1) and sheaves (Definition B.0.6) where $\mathbf{Open}(X)$ (Definition 6.0.2) is the category of open subsets of X equipped with its usual (Definition B.0.5) Grothendieck pretopology.

Definition 6.0.7 (Sheaf associated to a presheaf)**.** Let X be a topological space, and let $\mathcal{D}$ be a category (Definition 1.1.1) admitting direct colimits (Definition 2.2.13) (e.g. the category of sets, groups, abelian groups, modules over rings, or vector spaces over fields). Let $\mathcal{P}$ be a presheaf on X with values in $\mathcal{D}$ (Definition 6.0.4).

The *sheaf associated to the presheaf $\mathcal{P}$* or the *sheaffification of the presheaf $\mathcal{P}$*, denoted $\mathcal{P}^+$ or sometimes by $a\mathcal{P}$, is a sheaf on X together with a morphism of presheaves

$$\eta : \mathcal{P} \to \mathcal{P}^+,$$

satisfying the following universal property: for every sheaf $\mathcal{F}$ on X (valued in $\mathcal{D}$), any morphism of presheaves

$$\varphi : \mathcal{P} \to \mathcal{F}$$

factors uniquely through η, i.e., there exists a unique morphism of sheaves

$$\widetilde{\varphi} : \mathcal{P}^+ \to \mathcal{F}$$

such that

$$\varphi = \widetilde{\varphi} \circ \eta.$$

Concretely, $\mathcal{P}^+$ can be constructed by assigning to each open set $U \subseteq X$ the set (or object in $\mathcal{D}$)

$$\mathcal{P}^+(U) := \left\{ s = (s_x)_{x \in U} \in \prod_{x \in U} \mathcal{P}_x \;\middle|\; \begin{array}{l} \forall x \in U, \\ \exists \text{ an open } V \subseteq U \text{ with } x \in V, \\ \exists t \in \mathcal{P}(V) \text{ such that} \\ \quad \forall y \in V, s_y = t_y \end{array} \right\}.$$

where $\mathcal{P}_x$ is the stalk of $\mathcal{P}$ at x, and t_y is the germ of t at y. In particular, $\mathcal{P}^+$ exists.

It is noteworthy that the assignment $\mathcal{P} \mapsto \mathcal{P}^+$ is a functor

$$\mathrm{PreShv}(X, \mathcal{D}) \to \mathrm{Shv}(X, \mathcal{D}).$$

(Definition 6.0.6) and that this functor is left adjoint to the inclusion functor

$$\mathrm{Shv}(X, \mathcal{D}) \hookrightarrow \mathrm{PreShv}(X, \mathcal{D})$$

Equivalently, the assignment $\mathcal{P} \mapsto \mathcal{P}^+$ is the sheafification functor as defined in Definition B.0.7.

6.0.3. *Sheaf of rings, ringed spaces, and sheaves of modules.*

Definition 6.0.8 (Ringed space)**.** A *ringed space* is a pair $(X, \mathcal{O}_X)$ where

- X is a topological space (Definition C.0.5), and
- $\mathcal{O}_X$ is a sheaf of (Definition 6.0.5) commutative rings (Definition C.0.9) on X.

Equivalently, a ringed space is a ringed site (Definition B.0.10) where the site is the site of opens (Definition B.0.5) of the topological space X The sheaf $\mathcal{O}_X$ may be suppressed from the notation and only X may be used to denote a ringed space. The sheaf $\mathcal{O}_X$, also commonly denoted by $\mathscr{O}_X$, is called the *structure sheaf of X*.

(♠ TODO: define module sheaf on a topological space)

(♠ TODO: define sheaf hom, sheaf tensor product)

7. Left/right derived functors of right/left exact functors

8.

9. Assignment 1: Due Friday, Jan 23

Problem 9.0.1. In the following categories, prove whether there are initial/final objects (Definition 2.3.1) and describe what they are.

1. The category of sets (Definition 1.1.7).
2. The category of groups (Definition 1.1.8).

Read the definition of a group object (Definition C.0.27).

Problem 9.0.2. Let $\mathcal{C}$ be a locally small category (Definition 1.1.2) with a final object (Definition 2.3.1) and let G be a group object of $\mathcal{C}$. Prove that the representable functor (Definition 1.3.12) $h_G : \mathcal{C} \to \mathbf{Sets}$ in fact factors through $\mathbf{Grp}$ (Definition 1.1.8). In other words, h_G can be regarded as a functor $\mathcal{C} \to \mathbf{Grp}$, and composing this functor with the forgetful functor $\mathbf{Grp} \to \mathbf{Sets}$ recovers the original functor $h_G : \mathcal{C} \to \mathbf{Sets}$.

Problem 9.0.3. Given a commutative ring (Definition C.0.9) R, one can construct a topological space (Definition C.0.5) $\operatorname{Spec} R$, see (Definition C.0.23) (focus on the construction of the topological space, and ignore the discussion on the structure sheaf).

1. Show that there is a functor $\operatorname{Spec} : \mathbf{CommRing}^{\mathrm{op}} \to \mathbf{Top}$ given by
   - sending a commutative ring R to $\operatorname{Spec} R$, and
   - sending a ring homomorphism $\varphi : R_1 \to R_2$ to the map $\varphi^* : \operatorname{Spec} R_2 \to \operatorname{Spec} R_1$ given by $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$.
2. Show that the above functor Spec is not faithful (Definition 1.3.5). (Hint: one should be able to find examples involving finite rings)

Read the definition of a fiber product (Definition C.0.24) of two objects in a category.

Problem 9.0.4. Let $f : X \to Z$ and $g : Y \to Z$ be morphisms in the category of sets (Definition 1.1.7). Prove that the fiber product (Definition C.0.24) $X \times_Z Y$ exists by explicitly constructing it (along with canonical morphisms from $X \times_Z Y$ to X, Y, Z), and verifying that your construction possesses the appropriate universal property.

Problem 9.0.5 (The Pullback Lemma)**.** Consider a commutative diagram in a category $\mathcal{C}$:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[d, "h"] \\ X \arrow[r, "p"] & Y \arrow[r, "q"] & Z \end{tikzcd}$$

Assume that the right-hand square (with corners B, C, Y, Z) is a Cartesian square (Definition C.0.24). Prove that the left-hand square (with corners A, B, X, Y) is a Cartesian square if and only if the outer rectangle (with corners A, C, X, Z) is a Cartesian square.

Read the definition of the category of opens (Definition 6.0.2) of a topological space.

Problem 9.0.6. Let $\mathbb{R}(x)$ denote the field of rational functions of x with coefficients in $\mathbb{R}$. Given an open subset U of $\mathbb{R}$, let $\mathcal{O}(U)$ denote

$$\mathcal{O}(U) = \{f \in \mathbb{R}(x) : f \text{ is defined at all points } x \in U\}.$$

Note that it is a commutative ring under pointwise addition and multiplication: for $f, g \in \mathbb{R}(U)$, we have $(f+g)(x) = f(x) + g(x)$ and $(f \cdot g)(x) = f(x) \cdot g(x)$.

Given an inclusion $U \subseteq V$ of open subsets, note that there is an injective ring homomorphism $\mathcal{O}(V) \hookrightarrow \mathcal{O}(U)$ and note that this describes a functor/diagram (Definition 2.2.6) $\mathcal{O} : \mathbf{Open}(\mathbb{R})^{\mathrm{op}} \to \mathbf{CommRings}$ (Definition 6.0.2) (Definition 1.2.1) (you do not need to prove this).

Fix a point $x \in \mathbb{R}$. There is a (full) subcategory (Definition 1.3.7) N_x of $\mathbf{Open}(\mathbb{R})$ whose objects are open neighborhoods of x, so there is an induced diagram $N_x^{\mathrm{op}} \to \mathbf{CommRings}$. Show that the direct limit (Definition 2.2.13)

$$\varinjlim_{U \in N_x^{\mathrm{op}}} \mathcal{O}(U)$$

of this diagram is isomorphic to the following ring:

$$\mathcal{O}_x = \{f \in \mathbb{R}(x) : f \text{ is defined at } x\}.$$

Problem 10.0.1. Let R and S be not necessarily commutative rings (Definition C.0.7). Let $\{M_i\}_{i\in I}$ be a small family of R-S-bimodules (Definition 2.1.1). Prove that $\prod_{i\in I} M_i$ and $\bigoplus_{i\in I} M_i$ as constructed in Definition 2.2.4 and Definition 2.2.5 are respectively the categorical product and coproduct (Definition 2.2.1) in the category of R-S-bimodule (Definition 2.1.3). In particular, note that when I is infinite, the product and coproduct do NOT coincide.

Problem 10.0.2. Let R and S be not necessarily commutative rings (Definition C.0.7). Show that the following categories are equivalent (Definition 1.3.4) by producing an explicit equivalence of categories and showing that it is indeed an equivalence of categories:

- The category ${}_R\mathbf{Mod}_S$ (Definition 2.1.3)
- The category $(R \otimes_{\mathbb{Z}} S^{\mathrm{op}}) - \mathbf{Mod}$ (Definition C.0.38) (Definition 2.1.10).

Problem 10.0.3. Let $\mathcal{C}$ be a locally small category (Definition 1.1.2). Let X and Y be objects of $\mathcal{C}$. Show that X and Y are isomorphic if and only if h_X and h_Y (Definition 1.3.12) are naturally isomorphic (Definition 1.3.1) functors (note that, dually, we can also say that X and Y are isomorphic if and only if h^X and h^Y are naturally isomorphic functors).

Problem 10.0.4. Let R, S, T, U be not necessarily commutative rings (Definition C.0.7). Let M be an $R-S$-bidmodule, let N be an $S-T$-bimodule, and let P be an $T-U$-bimodule. Show that the tensor products $(M \otimes_S N) \otimes_T P$ and $M \otimes_S (N \otimes_T P)$ are isomorphic using Problem 10.0.3 and the tensor-Hom adjunction (Theorem 2.5.2)[8]. In particular, do not write out an explicit isomorphism between the tensor products.

Problem 10.0.5. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be an adjoint pair (Definition 2.5.1) $F \dashv G$ of functors between locally small categories (Definition 1.1.2). Prove that the right adjoint G is faithful (Definition 1.3.5) if and only if the counit morphism (Definition 2.5.1) $\varepsilon_D : FG(D) \to D$ is an epimorphism (Definition 2.3.7) for every object D of $\mathcal{D}$.

(Dually, the left adjoint F is faithful if and only if the unit morphism $C \to GF(C)$ is a monomorphism for every object C of $\mathcal{D}$, but I am not asking you to prove this.)

Problem 10.0.6. Prove the following, called the short five lemma. Do not assume the general five lemma (Lemma 2.4.16).

Let $\mathcal{A}$ be an abelian category (Definition 2.3.10). Consider the following commutative diagram in $\mathcal{A}$ where the rows are short exact (Definition 2.4.3):

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0
\end{tikzcd}$$

[8] In fact, the tensor products are naturally isomorphic, and you should think about what that means precisely, but I am not telling you to prove that the isomorphism is naturally isomorphic

1. If α and γ are monomorphisms (Definition 2.3.7), then β is a monomorphism.
2. If α and γ are epimorphisms, then β is an epimorphism.
3. If α and γ are isomorphisms, then β is an isomorphism.

Problem 10.0.7. Show that the short exact sequence (Definition 2.4.3) (in the category of abelian groups (Definition 1.1.8))

$$0 \to \mathbb{Z} \xrightarrow{1 \mapsto p} \mathbb{Z} \to \mathbb{Z}/p \to 0$$

is not split (Definition 2.4.4).

Problem 10.0.8. Let $R = \mathbb{Z}[t]/(1-t^2)$. Let $A = R/(1-t)$ and let $B = R/(1+t)$. Show that the short exact sequence

$$0 \to B \xrightarrow{1 \mapsto 1-t} R \to A \to 0$$

is

1. split (Definition 2.4.4) as a short exact sequence of abelian groups (Definition C.0.3).
2. not split (Definition 2.4.4) as a short exact sequence of R-modules (Definition 2.1.1).

11. Assignment 3: Due Feb 20

Problem 11.0.1. Let $\mathcal{A}$ be an abelian category (Definition 2.3.10). Let $\{A_{\bullet,i}\}_{i\in I}$ be a small collection of chain complexes (Definition 3.0.1) in $\mathcal{A}$.

1. precisely show that there exists a "canonical" "comparison" morphism

$$\bigoplus_{i\in I} H_n(A_{\bullet,i}) \to H_n\left(\bigoplus_{i\in I} A_{\bullet,i}\right)$$

   coming from the functoriality (Proposition 3.2.7) of H_n, the canonical "inclusion" morphisms for coproducts (Definition 2.2.1)[9]
2. Show that if $\mathcal{A}$ satisfies AB4 (Definition 2.5.7), then

$$\bigoplus_{i\in I} H_n(A_{\bullet,i}) \cong H_n\left(\bigoplus_{i\in I} A_{\bullet,i}\right)$$

   (In fact, the canonical comparison morphism would be such an isomorphism, and if you solve this problem "correctly", then the isomorphism that you produce will be the canonical comparison morphism, but I am not asking you to carefully justify that the isomorphism that you produce is the canonical comparison morphism).

[9] the canonical morphisms for general coproducts refer to the morphisms $C_i \to \coprod_{i\in I} C_i$

Definition A.0.1. Let C be a category enriched in a monidal category $\mathcal{V}$. Given an object X of C, the *functor of points* h_X is the functor (Definition 1.2.2)/presheaf (Definition 6.0.1) $C^{\mathrm{op}} \to \mathcal{V}$ given by $T \mapsto \mathrm{Hom}_C(T, X)$. A functor $C^{\mathrm{op}} \to \mathcal{V}$ (or equivalently, a presheaf on C valued in $\mathcal{V}$) is said to be *representable* if it is naturally isomorphic (Definition 1.3.1) to some functor h_X of points for an object X of C.

Dually, a functor $C \to \mathcal{V}$ is called *co-representable* if it is naturally isomorphic to a functor $T \mapsto \mathrm{Hom}_C(X, T)$ for an object X in C.

For instance, we may speak of these notions when $\mathcal{V}$ is the monoidal category **Sets**, i.e. C is a locally small category (Definition 1.1.2).

Definition A.0.2. A *symmetric monoidal category* is a monoidal category $(\mathcal{C}, \otimes, \mathbb{I})$ together with a natural isomorphism (symmetry)

$$\gamma_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$$

for all $X, Y \in \mathcal{C}$, such that for all $X, Y, Z \in \mathcal{C}$ the following holds:

- $\gamma_{Y,X} \circ \gamma_{X,Y} = \mathrm{id}_{X \otimes Y}$ (involutivity);
- the **hexagon coherence diagrams** commute:

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "\alpha_{X,Y,Z}"] \arrow[d, "\gamma_{X,Y} \otimes \mathrm{id}_Z"'] & X \otimes (Y \otimes Z) \arrow[r, "\gamma_{X,Y \otimes Z}"] & (Y \otimes Z) \otimes X \\
(Y \otimes X) \otimes Z \arrow[rr, "\alpha_{Y,X,Z}"'] & & Y \otimes (X \otimes Z) \arrow[u, "\mathrm{id}_Y \otimes \gamma_{X,Z}"']
\end{tikzcd}$$

  and the analogous hexagon with inverse braiding:

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "\alpha^{-1}_{X,Y,Z}"] \arrow[d, "\mathrm{id}_X \otimes \gamma_{Y,Z}"'] & (X \otimes Y) \otimes Z \arrow[r, "\gamma_{X \otimes Y,Z}"] & Z \otimes (X \otimes Y) \\
X \otimes (Z \otimes Y) \arrow[rr, "\alpha^{-1}_{X,Z,Y}"'] & & (X \otimes Z) \otimes Y \arrow[u, "\gamma_{X,Z} \otimes \mathrm{id}_Y"']
\end{tikzcd}$$

- the **symmetry coherence diagram** commutes:

$$\begin{tikzcd}
X \otimes Y \arrow[r, "\gamma_{X,Y}"] \arrow[dr, "\mathrm{id}_{X \otimes Y}"'] & Y \otimes X \arrow[d, "\gamma_{Y,X}"] \\
& X \otimes Y
\end{tikzcd}$$

A *closed symmetric monoidal category* usually refers to a symmetric monoidal category that is closed as a monoidal category.

Definition A.0.3 (Grothendieck Universe)**.** Let U be a set. We say U is a *Grothendieck universe* (or just a *universe*) if the following conditions hold:

1. If $x \in U$ and $y \in x$, then $y \in U$ (transitivity).

2. If $x, y \in U$, then $\{x, y\} \in U$ (closed under pair formation).
3. If $x \in U$, then the power set $\mathcal{P}(x) \in U$.
4. If $I \in U$ and $(x_\alpha)_{\alpha \in I}$ is a family with each $x_\alpha \in U$, then $\bigcup_{\alpha \in I} x_\alpha \in U$.

A set X is called *U-small* or a *U-set* if $X \in U$.

Definition A.0.4. *Tarski–Grothendieck set theory*, denoted by TG, is the theory consisting of the axioms of ZFC together with *Tarski's Axiom of Universes*, which asserts that for every set x there exists a universe (Definition A.0.3) U such that $x \in U$:

$$\forall x\, \exists U\, (U \text{ is a Grothendieck universe} \wedge x \in U)$$

Definition A.0.5. A set x is a *hereditarily finite set* if its transitive closure $\mathrm{tc}(x)$ is a finite set.

The collection of all hereditarily finite sets is denoted by V_ω. It is a set.

Definition A.0.6. Given an axiom system containing the Tarski-Grothendieck set theory (Definition A.0.4) axioms, the *hierarchy of Grothendieck universes* consists of the Grothendieck universes (Definition A.0.3) U_n for $n \in \mathbb{N} \cup \{0\}$ constructed as follows:

1. Base case: $U_0 \coloneqq V_\omega$, the set of all hereditarily finite sets (Definition A.0.5).
2. Successor step: For each $n \geq 0$, U_{n+1} is the unique minimal Grothendieck universe such that $U_n \in U_{n+1}$.

In particular, $U_n \in U_{n+1}$ and $U_n \subset U_{n+1}$ for all n.

Definition A.0.7. Let $\mathcal{U}$ be a fixed Grothendieck universe (Definition A.0.3). The *category of categories (relative to $\mathcal{U}$)* or the *category of U-small categories* is the category defined by:

- The objects are all U-small (Definition 1.1.2) categories (Definition 1.1.1) $\mathcal{C}$.
- The morphisms are functors (Definition 1.2.2) between such categories.

This category is denoted by $U - \mathbf{Cat}$ or by $\mathbf{Cat}$, if U is understood.

In the case that Tarski-Grothendieck (Definition A.0.4) set theory is assumed (so in particular, the axiom of universes is assumed), one often adopts the convention of denoting $U_n - \mathbf{Cat}$ (Definition A.0.6) by $\mathbf{Cat}$ and $U_{n+1} - \mathbf{Cat}$ (Definition A.0.6) by $\mathbf{CAT}$. Thus, for $n = 1$, $\mathbf{Cat}$ would be the category of small categories, whereas $\mathbf{CAT}$ would serve as the "category of large categories" which consists of many common categories that are too large to be U_1-categories such as $\mathbf{Set}$, $\mathbf{Grp}$, $\mathbf{Top}$ or even $\mathbf{Cat}$.

Definition A.0.8. Let $\mathcal{X}$ be a category. A *concrete category over $\mathcal{X}$* is a pair $(\mathcal{C}, U)$ consisting of a category $\mathcal{C}$ and a faithful functor (Definition 1.3.5) $U\colon \mathcal{C} \to \mathcal{X}$ In this context, U is called the *underlying functor* (or *forgetful functor*) of the concrete category.

When $\mathcal{X} = \mathbf{Set}$ (the category of sets), the pair $(\mathcal{C}, U)$ is simply referred to as a *concrete category*. For any object A in $\mathcal{C}$, the set $U(A)$ is called the *underlying set* of A, and for any morphism $f : A \to B$, the function $U(f) : U(A) \to U(B)$ is called the *underlying function* of f.

Definition B.0.1 ([GV72, Exposé I Définition 4.1])**.** Let C be a (large) category (Definition 1.1.1).

1. A *sieve S on the category C* is a full subcategory (Definition 1.3.7) D of C such that for any object U of C there exists an object V of (♠ TODO: correctly parse the definiton)
2. A *sieve S on an object $U \in \mathrm{Ob}(C)$* is a collection of morphisms in C with codomain U that is closed under precomposition by any compatible morphism in C. In other words, S is a sieve if for every $f : V \to U$ in S and morphism $g : W \to V$ in C, the composition $f \circ g : W \to U$ is also in S.

   Given a morphism $f : V \to U$ in a sieve S, we also say that *f factors through U*.

Definition B.0.2. Let $\mathcal{C}$ be a category (Definition 1.1.1) and $U \in \mathcal{C}$ an object. Let $\mathcal{S} = \{f_i : U_i \to U\}_{i \in I}$ be a family of morphisms with codomain U.

The *sieve generated by $\mathcal{S}$*, denoted $(\mathcal{S})$ or $\langle \mathcal{S} \rangle$, is the smallest sieve on U (Definition B.0.1) containing all the morphisms in $\mathcal{S}$.

Explicitly, a morphism $h : V \to U$ belongs to the generated sieve if and only if h factors through some morphism in $\mathcal{S}$. That is, there exists an index $i \in I$ and a morphism $g : V \to U_i$ such that

$$h = f_i \circ g.$$

Definition B.0.3. Let C be a category, let $U \in \mathrm{Ob}(C)$, and let S be a sieve on U (Definition B.0.1). For a morphism $f : V \to U$ in C, the *pullback sieve* f^*S (or *basechange sieve* $S \times_U V$) on V is defined by

$$f^*S = \{g : W \to V \mid f \circ g \in S\}.$$

In other words, f^*S consists of all morphisms into V whose composite with f belongs to the sieve S on U.

Definition B.0.4 (Grothendieck topology)**.** Let $\mathscr{U}$ be a universe (Definition A.0.3).

1. (See [GV72, Exposé II, Définition 1.1]) Let $\mathcal{C}$ be a category (Definition 1.1.1). A *Grothendieck topology on $\mathcal{C}$* assigns to each object U of $\mathcal{C}$ a collection $J(U)$ of sieves (Definition B.0.1) $\{U_i \to U\}_{i \in I}$, each called a *covering sieve of U*, satisfying:
   (a) (Stability under "base change"): If $S \in J(U)$ is a covering sieve of an object U, and $f : V \to U$ is any morphism in $\mathcal{C}$, then the pullback sieve (Definition B.0.3) f^*S is a covering sieve of U.
   (b) (Local character condition) If S is a sieve on U, and if there exists a covering sieve $R \in J(U)$ such that for all $f : V \to U$ in R the pullback sieve (Definition B.0.3) f^*S is in $J(V)$, then $S \in J(U)$.
   (c) The maximal sieve is a covering sieve.

   Some will refer to a Grothendieck topology as simply a *topology*, not to be confused with the related, but less general, notion of a topology on a set (Definition C.0.5).
2. (See [GV72, Exposé II, 1.1.5]) A *site* is a category $\mathcal{C}$ equipped with a Grothendieck topology.

When we are working with a Grothendieck pretopology K on a category $\mathcal{C}$, we may regard $\mathcal{C}$ as a site by equipping it with the Grothendieck topology generated by K.

3. (See [GV72, Exposé II, Définition 1.2]) Let $(\mathcal{C}, J)$ be a site. A family of morphisms $(U_i \to U)_{i \in I}$ is called a *covering family of U (with respect to the site/topology)* or a *cover of U (with respect to the site/topology)* if the sieve generated by (Definition B.0.2) the family is a covering sieve of U.
4. (See [GV72, Exposé II, Définition 3.0.1]) Let $(\mathcal{C}, J)$ be a site (Definition B.0.4), where J is a Grothendieck topology on $\mathcal{C}$.

   A family G of objects $\mathcal{C}$ is called a *topologically generating family of the site/topology* or a *generating family/collection of the site/topology* if for every object $X \in \mathcal{C}$, there is a covering family $\{X_\alpha \to X\}_{\alpha \in A}$ of X such that every X_α is a member of G.

   Equivalently, the Grothendieck topology J is the smallest Grothendieck topology containing all covers of the U_i. Also equivalently, for any $S \in J(X)$, the sieve S contains a covering family $\{V_i \to X\}$ such that each morphism $V_i \to X$ factors through some member of G. (♠ TODO: Verify that these claimed equivalences are indeed equivalences)
5. (See [GV72, Exposé II, Définition 3.0.2]) A *$\mathscr{U}$-site* is a site whose underlying category $\mathcal{C}$ is $\mathscr{U}$-locally small (Definition 1.1.2) and which has a $\mathscr{U}$-small topologically generating family. A $\mathscr{U}$-site is called *$\mathscr{U}$-small* if its underlying category is $\mathscr{U}$-small. Similarly, a *small site* is a site whose underlying category is a set and a *locally small site* is a site whose underlying category is locally small (Definition 1.1.2).

Definition B.0.5. Let (X, τ_X) be a topological space. The *small site associated to X* or *the site of open covers of X* or *the canonical site on* $\operatorname{Open} X$ is the category $\operatorname{Open}(X)$ of open subsets (Definition 6.0.2) of X with inclusion morphisms, equipped with the canonical Grothendieck topology (Definition B.0.4) generated by the Grothendieck pretopology whose covering families $\{U_i \to U\}_{i \in I}$, for $U \in \operatorname{Open}(X)$ are families of morphisms in $\operatorname{Open}(X)$ such that $\bigcup_{i \in I} U_i = U$. In other words, $\{U_i \to U\}_{i \in I}$ is a covering for the pretopology if it is an open coverings.

Definition B.0.6 (Sheaf on a site)**.** Let $(\mathcal{C}, J)$ be a site (Definition B.0.4). Let $\mathcal{A}$ be a (large) category (Definition 1.1.1).

1. A presheaf (Definition 6.0.1) $\mathcal{F} : \mathcal{C}^{\text{op}} \to \mathcal{A}$ (Definition 1.2.1) is called a *sheaf on the site $(\mathcal{C}, J)$ valued in $\mathcal{A}$* if, for every object U of $\mathcal{C}$ and every covering sieve (Definition B.0.4) $S \in J(U)$, the limit (Definition 2.2.8)
$$\varprojlim_{(V \to U) \in (\mathcal{D}_S)^{\text{op}}} \mathcal{F}|_{\mathcal{D}_S}(V),$$
   exists and the canonical natural morphism
$$\mathcal{F}(U) \to \varprojlim_{(V \to U) \in (\mathcal{D}_S)^{\text{op}}} \mathcal{F}|_{\mathcal{D}_S}(V)$$
   is an isomorphism. Here, $\mathcal{D}_S \hookrightarrow \mathcal{C}/U$ is the full downward-closed subcategory such that $\operatorname{Ob}(\mathcal{D}_S) = \{(f : V \to U) : f \in S(V)\}$,

   In particular, when we are working with a Grothendieck pretopology K on a category $\mathcal{C}$, we may speak of sheaves on the site whose Grothendieck topology is the one generated by K.

2. Given sheaves $\mathcal{F}, \mathcal{G} : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$ on the site $(\mathcal{C}, J)$, a *morphism between the sheaves* is a morphism (Definition 6.0.1) between $\mathcal{F}$ and $\mathcal{G}$ as presheaves.
3. Let U be a universe (Definition A.0.3). A *U-sheaf* typically refers to a U-presheaf that is a sheaf for a U-site. In other words, a U-sheaf is a sheaf on a site whose underlying category is U-locally small (Definition 1.1.2) and which has a U-small topologically generating family such that the sheaf is valued in U-sets.
4. The *sheaf category/category of $\mathcal{A}$-valued sheaves on $\mathcal{C}$* is the (large) category defined as the full subcategory of $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ whose objects are the sheaves on $\mathcal{C}$ with values in $\mathcal{A}$. Common notations for the sheaf category include $\mathrm{Shv}(\mathcal{C}, \mathcal{A})$, $\mathrm{Shv}(\mathcal{C}, J, \mathcal{A})$, $\mathrm{Sh}(\mathcal{C}, \mathcal{A})$, $\mathrm{Sh}(\mathcal{C}, J, \mathcal{A})$. If the value category $\mathcal{A}$ is clear from context, then notations such as $\mathrm{Shv}(\mathcal{C})$, $\mathrm{Shv}(\mathcal{C}, J)$, $\mathrm{Sh}(\mathcal{C})$, $\mathrm{Sh}(\mathcal{C}, J)$ are also common.

Definition B.0.7. Let $\mathcal{C}$ be a site (Definition B.0.4) and let $\mathcal{A}$ be a (large) category (Definition 1.1.1).

Assuming that the presheaf (Definition 6.0.1) category $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ (and hence the sheaf (Definition B.0.6) category $\mathrm{Shv}(\mathcal{C}, \mathcal{A})$) is locally small (Definition 1.1.2) (or U-locally small if a Grothendieck universe (Definition A.0.3) U is available), a *sheafification functor* refers to a functor

$$a : \mathrm{PreShv}(\mathcal{C}, \mathcal{A}) \to \mathrm{Shv}(\mathcal{C}, \mathcal{A})$$

that is left adjoint (Definition 2.5.1) to the inclusion functor

$$i : \mathrm{Shv}(\mathcal{C}, \mathcal{A}) \hookrightarrow \mathrm{PreShv}(\mathcal{C}, \mathcal{A}).$$

If such a sheafification functor exists, then it is unique up to unique natural isomorphism. Given a presheaf P, the sheafification $a(P)$ is also sometimes called the *sheaf associated to P*. See Theorem B.0.8 for common conditions under which sheafification exists.

Theorem B.0.8. cf. [GV72, Exposé II, Théorème 3.4]

1. Let U be a universe. Let $\mathcal{C}$ be a U-site (Definition B.0.4). A sheafification functor (Definition 6.0.7)

$$a : \mathrm{PreShv}(\mathcal{C}, U\mathbf{-Sets}) \to \mathrm{Shv}(\mathcal{C}, U\mathbf{-Sets}).$$

   exists.
2. Let $\mathcal{C}$ be a site whose underlying category is locally small (Definition 1.1.2) and which has a topologically generating family (Definition B.0.4) that is a set (rather than a proper class). A sheafification functor

$$a : \mathrm{PreShv}(\mathcal{C}, \mathbf{Sets}) \to \mathrm{Shv}(\mathcal{C}, \mathbf{Sets})$$

   exists.

   (♠ TODO: It may be the case that I need to assume $\mathcal{A}$ to be locally small; if so, recheck all the statements that use this.)
3. (see e.g. [nLa25f, 3], [?, Theorem 17.4.7]) Let $(\mathcal{C}, J)$ be a site (Definition B.0.4) on an essentially small category $\mathcal{C}$. Suppose that the category $\mathcal{A}$ is complete, cocomplete (Definition 2.2.11), that small filtered colimits (Definition 2.2.13) in $\mathcal{A}$ are exact (Definition 2.4.7), and that $\mathcal{A}$ satisfies the IPC-property. A sheafification functor (Definition B.0.7)

$$a : \mathrm{PreShv}(\mathcal{C}, \mathcal{A}) \to \mathrm{Shv}(\mathcal{C}, \mathcal{A})$$

exists.

(♠ TODO: state as a fact that the category of groups and k-algebras are complete, cocomplete, with small filtered colimits that are exact) This is true for instance of $\mathcal{A} = \mathbf{Set}$, $\mathbf{Grp}$, $k - \mathbf{Alg}$ for a field k, or $\mathbf{Mod}_R$ for a (not necessarily commutative unital) ring R (Definition C.0.7).

Remark B.0.9. If the presheaf is valued in nice "algebraic category", e.g. groups, abelian groups, rings, modules over a ring, etc., then the sheafification is again valued in that category. (♠ TODO: Make this more precise.)

Definition B.0.10. (♠ TODO: there are places where sites and sheaves of rings on them are used, but it would be better to just have them be ringed sites.)

A *ringed site* is a site (Definition B.0.4) $(\mathcal{C}, J)$ with a small topological generating family (Definition B.0.4) equipped with a sheaf (Definition B.0.6) of (not necessarily commutative) rings $\mathcal{O}$. If the Grothendieck topology J is clear in context, one may even write that $(C, \mathcal{O})$ is a ringed site.

A *morphism of ringed sites*

$$((\mathcal{C}, J), \mathcal{O}) \to ((\mathcal{C}', J'), \mathcal{O}')$$

consists of a morphism of sites $f : (\mathcal{C}, J) \to (\mathcal{C}', J')$ and a morphism of sheaves (Definition B.0.6) of rings $f^\# : \mathcal{O}' \to f_*\mathcal{O}$.

Definition B.0.11. 1. Let $\mathcal{C}$ be a site (Definition B.0.4), and let $\mathcal{A}$ and $\mathcal{B}$ be sheaves (Definition B.0.6) of (not necessarily commutative) rings (Definition C.0.7) on $\mathcal{C}$.

(a) An *$(\mathcal{A}, \mathcal{B})$-bimodule* (or a *bimodule over $(\mathcal{A}, \mathcal{B})$*) is a sheaf (Definition B.0.6) $\mathcal{M}$ of abelian groups on $\mathcal{C}$ equipped with a left $\mathcal{A}$-module structure given by a morphism of sheaves (Definition B.0.6) of sets

$$\lambda : \mathcal{A} \times \mathcal{M} \longrightarrow \mathcal{M},$$

and a right $\mathcal{B}$-module structure given by a morphism of sheaves of sets

$$\rho : \mathcal{M} \times \mathcal{B} \longrightarrow \mathcal{M},$$

such that the actions are compatible. Specifically, for every object U in $\mathcal{C}$, every section $m \in \mathcal{M}(U)$, every $a \in \mathcal{A}(U)$, and every $b \in \mathcal{B}(U)$, the equality

$$\lambda_U(a, \rho_U(m, b)) = \rho_U(\lambda_U(a, m), b)$$

holds in $\mathcal{M}(U)$. In standard multiplicative notation where $\lambda(a, m)$ is denoted $a \cdot m$ and $\rho(m, b)$ is denoted $m \cdot b$, this condition is the associativity axiom

$$(a \cdot m) \cdot b = a \cdot (m \cdot b).$$

In particular, for every object $U \in \mathcal{C}$, the abelian group $\mathcal{M}(U)$ has the structure of an $\mathcal{A}(U) - \mathcal{B}(U)$-bimodule (Definition 2.1.1).

(b) Let $\mathcal{M}$ and $\mathcal{N}$ be $(\mathcal{A}, \mathcal{B})$-bimodules. A *homomorphism of $(\mathcal{A}, \mathcal{B})$-bimodules* (or an *$(\mathcal{A}, \mathcal{B})$-linear morphism*) is a morphism of sheaves of abelian groups $f : \mathcal{M} \to \mathcal{N}$ such that for every object U of $\mathcal{C}$, every section $m \in \mathcal{M}(U)$, every $a \in \mathcal{A}(U)$, and every $b \in \mathcal{B}(U)$, the following compatibility conditions hold:

$$f_U(a \cdot m) = a \cdot f_U(m) \quad \text{and} \quad f_U(m \cdot b) = f_U(m) \cdot b.$$

We denote the category of $(\mathcal{A},\mathcal{B})$-bimodules, with morphisms being morphisms of sheaves of abelian groups compatible with both the left $\mathcal{A}$-action and the right $\mathcal{B}$-action, by $\mathcal{A}$-$\mathcal{B}$-Mod or sometimes by ${}_{\mathcal{A}}\mathsf{Mod}_{\mathcal{B}}$ (♠ TODO: talk about how bimodules can be identifies with left/right modules)

2. Let $(\mathcal{C}, J)$ be a site (Definition B.0.4). Let $\mathcal{O}$ be a sheaf of (not necessarily commutative) rings on $(\mathcal{C}, J)$ (Definition B.0.6), i.e. $((\mathcal{C}, J), \mathcal{O})$ is a ringed site (Definition B.0.10).
   (a) An *(left/right/two-sided) $\mathcal{O}$-module* consists of the following data:
   - A sheaf $\mathcal{F}$ of abelian groups on $(\mathcal{C}, J)$,
   - for every object $U \in \mathcal{C}$, the structure of an (left/right/two-sided) $\mathcal{O}(U)$-module on $\mathcal{F}(U)$,

   such that for every morphism $f : V \to U$ in $\mathcal{C}$, the restriction map
   $$\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$$
   is $\mathcal{O}(U)$-linear when the $\mathcal{O}(U)$-action on $\mathcal{F}(V)$ is defined via the natural ring homomorphism
   $$\mathcal{O}(U) \to \mathcal{O}(V)$$
   induced by f.

   (b) Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}$-modules (Definition B.0.11).
   A *morphism of $\mathcal{O}$-modules* $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves (Definition B.0.6) of abelian groups such that, for every object $U \in \mathcal{C}$, the component map
   $$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$
   is $\mathcal{O}(U)$-linear, i.e. it satisfies
   $$\varphi_U(r \cdot s) = r \cdot \varphi_U(s) \quad \text{for all } r \in \mathcal{O}(U),\ s \in \mathcal{F}(U).$$
   The collection of all $\mathcal{O}$-modules together with their morphisms of $\mathcal{O}$-modules forms the *category of $\mathcal{O}$-modules*, denoted $\mathbf{Mod}(\mathcal{O})$.

   In case that a sheafification functor (Definition B.0.7)
   $$\mathrm{PreShv}(\mathcal{C}, \mathbf{Rings}) \to \mathrm{Shv}(\mathcal{C}, \mathbf{Rings})$$
   exists, a left, right, two-sided $\mathcal{O}$-module (and morphisms thereof) is equivalent to a $(\mathcal{O}, \mathbb{Z})$-bimodule, $(\mathbb{Z}, \mathcal{O})$-bimodule, and $(\mathcal{O}, \mathcal{O})$-bimodule (and morphisms thereof) respectively, where $\mathbb{Z}$ is the constant sheaf of the integer ring $\mathbb{Z}$.

Appendix C. Miscellaneous definitions

Definition C.0.1. Let X and Y be sets. A *map* (or *function*) from X to Y is a rule f assigning to each element $x \in X$ exactly one element $f(x) \in Y$. We write $f : X \to Y$.

We say that X is the *domain* and that Y is the *codomain of f*.

Definition C.0.2 (Monoid)**.** A *monoid* is a semigroup $(M, \cdot)$ such that there exists an element $e \in M$, called the *identity element*, with the property:
$$e \cdot a = a \cdot e = a \quad \text{for all } a \in M.$$

Equivalently, a monoid is a monoid object (Definition C.0.26) in the category of sets (Definition 1.1.7).

Definition C.0.3 (Groups)**.** A *group* is a pair $(G, \cdot)$ where G is a set and $\cdot : G \times G \to G$ is a binary operation, subject to the following conditions:

1. (Associativity) For all $g, h, k \in G$ one has
$$(g \cdot h) \cdot k = g \cdot (h \cdot k).$$
2. (Identity element) There exists an element $e \in G$ such that for all $g \in G$,
$$e \cdot g = g \cdot e = g.$$
3. (Inverse element) For all $g \in G$ there exists an element $g^{-1} \in G$ such that
$$g \cdot g^{-1} = g^{-1} \cdot g = e.$$

The element e is called the *identity element of G*, and g^{-1} is called the *inverse of g*.

Equivalently, a group is a monoid (Definition C.0.2) with inverse elements.

Equivalently, a group is a group object (Definition C.0.27) in the category of sets (Definition 1.1.7).

A group $(G, \cdot)$ is often simply written as G, when the notation for the binary operation $\cdot$ is clear.

An *abelian group* or synonymously, a *commutative group*, is a group $(G, \cdot)$ whose binary operation $\cdot$ is *abelian* or *commutative*, i.e. satisfies
$$g \cdot h = h \cdot g$$
for all $g, h \in G$.

An abelian group is equivalent to a $\mathbb{Z}$-module.

Definition C.0.4 (Group homomorphism)**.** Let $(G, \cdot)$ and $(H, *)$ be groups (Definition C.0.3). A map $f : G \to H$ is called a *group homomorphism* if for all $g_1, g_2 \in G$ one has
$$f(g_1 \cdot g_2) = f(g_1) * f(g_2).$$

The collection of all groups with the group homomorphisms forms a locally small (Definition 1.1.2) category (Definition 1.1.1), called the *category of groups*.

If f is bijective, then f is called a *group isomorphism*. Equivalently, a group isomorphism is an isomorphism (Definition 1.1.13) in the category of groups .

Definition C.0.5 (Topology)**.** Let X be a set. A *topology on X* is a collection $\mathcal{T}$ of subsets of X such that:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
2. For any collection $\{U_i\}_{i \in I} \subseteq \mathcal{T}$ (with I arbitrary), the union $\bigcup_{i \in I} U_i \in \mathcal{T}$,
3. For any finite collection $\{U_1, \ldots, U_n\} \subseteq \mathcal{T}$, the intersection $U_1 \cap \cdots \cap U_n \in \mathcal{T}$.

If $\mathcal{T}$ is a topology on X, the pair $(X, \mathcal{T})$ is called a *topological space*. Members of $\mathcal{T}$ are called *open sets*.

A subset $C \subseteq X$ is *closed* if its complement $X \setminus C$ is an open set in $\mathcal{T}$

One very often refers to X as a topological spcae, omitting the notation of the topology $\mathcal{T}$.

The collection of all topologies on a set X may be denoted by notations such as $\mathrm{Top}(X)$, $\mathbf{Top}(X)$, or $\mathsf{Top}(X)$.

Definition C.0.6. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces (Definition C.0.5). A map $f : X \to Y$ is called *continuous* if for every open set $V \in \mathcal{T}_Y$, the preimage $f^{-1}(V)$ is an open set in X, that is,

$$\forall V \in \mathcal{T}_Y,\ f^{-1}(V) \in \mathcal{T}_X.$$

Equivalently, f is continuous if and only if for every closed set $C \subseteq Y$, the preimage $f^{-1}(C)$ is closed in X.

A *map of topological spaces* usually refers to a continuous map between the topological spaces.

The set of continuous maps from X to Y is sometimes denoted by $C(X, Y)$. Other standard notation include $\mathrm{Hom}_{\mathrm{Top}}(X, Y)$ or $\mathrm{Top}(X, Y)$ coming from more general notation for morphisms between objects in a category (Definition 1.1.1).

Definition C.0.7. A *ring* is a triple $(R, +, \cdot)$ where

1. $(R, +)$ is a commutative group (Definition C.0.3), and
2. $(R, \cdot)$ is a monoid (Definition C.0.2).
3. $\cdot$ is distributive over $+$, i.e. for all $a, b, c \in R$, we have

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and} \quad (a + b) \cdot c = a \cdot c + b \cdot c.$$

Equivalently, a ring is a triple $(R, +, \cdot)$ where $+, \cdot : R \times R \to R$ are binary operations satisfying

1. $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$ for all $a, b, c \in R$
2. There exists an element $0 \in R$ such that $a + 0 = a = 0 + a$ for all $a \in R$.
3. For every $a \in R$, there exists an element $-a \in R$ such that $a + (-a) = 0 = (-a) + a$ for all $a \in R$.
4. There exists an element $1 \in R$ such that $a \cdot 1 = a = 1 \cdot a$ for all $a \in R$.
5. For all $a, b, c \in R$, we have

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and} \quad (a + b) \cdot c = a \cdot c + b \cdot c.$$

The operation $+$ is often called *addition* and the operation $\cdot$ is often called *multiplication*. Accordingly, the identity element 0 of $+$ is often called the *additive identity* and the identity element 1 of $\cdot$ is often called the *multiplicative identity*.

Remark C.0.8. Some writers might not require a ring to have a multiplicative identity element, i.e. would define a ring so that $(R, +)$ is a commutative group, $(R, \cdot)$ is a semigroup, and $\cdot$ is distributive over $+$. Such writers would call the notion of ring in Definition C.0.7 a *unitary ring* to emphasize the existence of the multiplicative identity 1.

Definition C.0.9. A *commutative (unital) ring* is a ring (Definition C.0.7) $(R, +, \cdot)$ such that $\cdot$ is a commutative operation, i.e. $a \cdot b = b \cdot a$.

For many writers (e.g. "commutative" algebraists or number theorists), a *ring* refers to a commutative ring as above.

Definition C.0.10. Let $(R, +, \cdot)$ be a not-necessarily commutative ring (Definition C.0.7). A *unit* or *invertible element of R* is an element $u \in R$ such that there exist an element $v \in R$ such that

$$uv = 1 = vu.$$

Such an element v is called the *multiplicative inverse of u* and is often denoted by u^{-1}. If it exists, then it is unique.

The set of units of R forms a group (Definition C.0.3), often denoted by $R^\times$ or R^*, under the multiplication operation $\cdot$. It is called the *group of units* or *unit group* of R.

Definition C.0.11. Let $(R, +, \cdot)$ be a not-necessarily commutative ring (Definition C.0.7). It is called a *division ring*, a *skew field*, or an *sfield*, if it is a nontrivial ring in which every nonzero element $a \in R$ is a unit (Definition C.0.10).

Definition C.0.12 (Field)**.** A *field* is commutative division (Definition C.0.11) ring (Definition C.0.9). In other words, a field is a commutative ring for which all nonzero elements have a multiplicative inverse (Definition C.0.10).

Definition C.0.13. Let $(R, +, \cdot)$ and $(S, +, \cdot)$ be rings (Definition C.0.7), not assumed to be commutative. A function $f : R \to S$ is called a *ring homomorphism* if for all $r_1, r_2 \in R$ the following properties hold:

1. $f(r_1 + r_2) = f(r_1) + f(r_2)$,
2. $f(r_1 r_2) = f(r_1) f(r_2)$,
3. $f(1_R) = 1_S$ where 1_R and 1_S denote the multiplicative identities in R and S, respectively.

A ring homomorphism is said to be a *ring isomorphism* if it is invertible as a map of sets.

An *R-ring* refers to a ring S equipped with a ring homomorphism $f : R \to S$.

We note that a ring homomorphism $f : R \to S$ yields a natural left R-module (Definition 2.1.1) structure on S and a natural right R-module structure on S respectively as follows for $r \in R$ and $s \in S$:

$$r \cdot s = f(r) \cdot s$$
$$s \cdot r = s \cdot f(r).$$

However, these left and right module structures need not yield a two-sided R–module structure.

Definition C.0.14 (Vector space over a field)**.** Let $(k, +, \cdot)$ be a field (Definition C.0.12). A *vector space over k* or a *k-vector space* is a triple $(V, +, \cdot)$[10] where

[10]Note that $+$ and $\cdot$ are abuse of notation here as these are already used for the addition and multiplication of $\cdot$

1. $(V, +)$ is an abelian group, and
2. $\cdot$ is a map $k \times V \to V$, called *scalar multiplication*

such that the following axioms hold for all $a, b \in k$ and all $u, v \in V$:

1. (Compatibility with field multiplication)
$$(ab) \cdot v = a \cdot (b \cdot v).$$
2. (Identity scalar)
$$1 \cdot v = v.$$
3. (Distributivity over vector addition)
$$a \cdot (u + v) = a \cdot u + a \cdot v.$$
4. (Distributivity over scalar addition)
$$(a + b) \cdot v = a \cdot v + b \cdot v.$$

Definition C.0.15. Let F be a field (Definition C.0.12), and let V and W be F-vector spaces (Definition C.0.14). A function $T : V \to W$ is called a *(homo)morphism of vector spaces over F*, or an *F-linear map*, if for all $u, v \in V$ and all $a, b \in F$, we have
$$T(au + bv) = aT(u) + bT(v).$$
The set of all such morphisms from V to W is denoted by
$$\mathrm{Hom}_F(V, W).$$

Definition C.0.16. Let F be a field (Definition C.0.12), and let V be an F-vector space (Definition C.0.14). A subset $B \subseteq V$ is called a *basis of V* if: (i) B is linearly independent over F, and (ii) B spans V.

If B is a basis, we define the *dimension of V over F* (or *rank of V over F*), denoted by
$$\dim_F(V),$$
(♠ TODO: cardinality) to be the cardinality of B. This value is uniquely determined by V and F.

Definition C.0.17 (Partially ordered set)**.**

1. A *partially ordered set* (or *poset*), or *ordered set* is a pair $(P, \leq)$ where P is a set and
$$\leq : P \times P \to \{\text{true}, \text{false}\}$$
is a binary relation on P satisfying the following axioms for all $a, b, c \in P$:
   - *Reflexivity:* $a \leq a$,
   - *Antisymmetry:* if $a \leq b$ and $b \leq a$, then $a = b$,
   - *Transitivity:* if $a \leq b$ and $b \leq c$, then $a \leq c$.

   The relation $\leq$ is called an *order* or a *partial order*
2. A partially ordered set $(P, \leq)$ is called a *directed partially ordered set* if for every pair $a, b \in P$, there exists $c \in P$ such that
$$a \leq c \quad \text{and} \quad b \leq c.$$

3. A partially ordered set $(P, \leq)$ is called a *codirected partially ordered set* (or *downward directed poset*) if for every pair $a, b \in P$, there exists $d \in P$ such that

$$d \leq a \quad \text{and} \quad d \leq b.$$

Lemma C.0.18. Let $(P, \leq)$ be a nonempty poset (Definition C.0.17).

1. Regarding P as a category whose objects are the elements of P and such that there is a unique arrow $a \to b$ if and only if $a \leq b$, the category is filtered.
2. Every nonempty small (Definition 1.1.2) thin filtered category (Definition 2.2.12) corresponds to a poset in this way.
3. Moreover, the poset P is directed (Definition C.0.17) if and only if the category is filtered. The poset P is codirected (Definition C.0.17) if and only if the category is cofiltered.

Definition C.0.19. Let S be a set. The *free group generated by S* is a pair $(F(S), \iota)$ consisting of a group (Definition C.0.3) $F(S)$ and a function $\iota : S \to F(S)$, satisfying the following universal property: for any group G and any function $f : S \to G$, there exists a unique group homomorphism $\varphi : F(S) \to G$ such that the diagram commutes (i.e., $\varphi \circ \iota = f$). The standard notation for the free group on S is $F(S)$ or sometimes $\langle S \rangle$ Elements of $F(S)$ are uniquely represented as reduced words in the alphabet $S \cup \{s^{-1} \mid s \in S\}$.

Definition C.0.20. Let S be a set. The *free abelian group generated by S* is the abelian group consisting of all formal linear combinations of elements of S with integer coefficients, such that only finitely many coefficients are nonzero. This group is denoted by $\mathbb{Z}[S]$ or alternatively as the direct sum

$$\mathbb{Z}^{(S)} := \bigoplus_{s \in S} \mathbb{Z}s$$

It satisfies the universal property that for any abelian group A and any function $f : S \to A$, there exists a unique group homomorphism $\psi : \mathbb{Z}[S] \to A$ extending f.

Definition C.0.21 (Homotopy groups)**.** For any pointed topological space (Definition 1.1.10) (X, x_0) and integer $n \geq 0$, the *n-th homotopy group of X at x_0*, denoted $\pi_n(X, x_0)$, is defined as the set of all homotopy classes (rel. ∂I^n) (Definition C.0.31) of based maps

$$f : (I^n, \partial I^n) \to (X, x_0),$$

where $I^n = [0, 1]^n$. For $n \geq 1$, $\pi_n(X, x_0)$ is a group under concatenation of based maps, and for $n \geq 2$, it is abelian.

The *fundamental group of (X, x_0)* refers to $\pi_1(X, x_0)$. Equivalently, it is the group of homotopy classes (rel. endpoints) of loops $\gamma : [0, 1] \to X$ satisfying $\gamma(0) = \gamma(1) = x_0$.

Definition C.0.22. Let R be a (not necessarily commutative) ring (Definition C.0.7). A proper two-sided ideal $P \trianglelefteq R$ is called a *prime ideal* if the following equivalent conditions holds:

1. If I, J are left ideals and $IJ \subset P$, then $I \subset P$ or $J \subset P$.
2. If I, J are right ideals and $IJ \subset P$, then $I \subset P$ or $J \subset P$.
3. If I, J are two-sided idaels and $IJ \subset P$, then $I \subset P$ or $J \subset P$.

4. If $x, y \in R$ with $xRy \subset \mathfrak{p}$, then $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

A proper left/right/two-sided ideal $M \subsetneq R$ is called *maximal* if there exists no other left/right/two-sided ideal $J \trianglelefteq R$ such that $M \subsetneq J \subsetneq R$. Equivalently,

- a left/right ideal M of R is maximal if and only if the quotient module R/M (Definition 2.1.5) is a simple left/right R-module.
- a two-sided ideal M of R is maximal if and only if the quotient ring R/M is a simple ring.

Definition C.0.23 (Affine scheme)**.** Let A be a commutative ring with unity (Definition C.0.9). Define the set $\operatorname{Spec}(A)$ to be the set of all prime ideals (Definition C.0.22) of A. Equip it with the *Zariski topology*, which is the topology (Definition C.0.5) whose closed sets are given by *vanishing loci*

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec}(A) : I \subseteq \mathfrak{p}\}$$

for ideals $I \subseteq A$. Define the sheaf $\mathcal{O}_{\operatorname{Spec}(A)}$, called the *structure sheaf of* $\operatorname{Spec} A$, by

$$\mathcal{O}_{\operatorname{Spec}(A)}(U) = \{ \text{ locally defined fractions of elements of } A \text{ on } U \ \},$$

for each open set $U \subseteq \operatorname{Spec}(A)$. It is the case that the stalk at $\mathfrak{p} \in \operatorname{Spec}(A)$ is canonically the localization $A_{\mathfrak{p}}$. Then $(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)})$ is a locally ringed space, called the *affine scheme associated to* A.

Moreover, given $f \in A$, we define the locus $D(f)$ by

$$D(f) = \operatorname{Spec} A \setminus V((f)) = \{\mathfrak{p} \in \operatorname{Spec} A : f \notin \mathfrak{p}\}$$

Definition C.0.24. Let $\mathcal{C}$ be a category (Definition 1.1.1), let Z be an object, and let X, Y be objects of $\mathcal{C}$ over Z, i.e. morphisms $X \to Z$ and $Y \to Z$ are fixed. A *cartesian product of X and Y over Z in $\mathcal{C}$* (or *fiber product* or *pullback diagram*) is an object, often denoted by $X \times_Z Y$, with *projection morphisms* $X \times_Z Y \to X$ and $X \times_Z Y \to Y$ that are universal. More precisely, for any object T of $\mathcal{C}$ and morphisms $f_X : T \to X$, $f_Y : T \to Y$ such that the compositions $T \xrightarrow{f_X} X \to Z$ and $T \xrightarrow{f_Y} Y \to Z$ agree, there exists a unique morphism $u : T \to X \times_Z Y$ such that the following diagram commutes:

$$\begin{tikzcd}
T \arrow[drr, bend left, "f_X"] \arrow[dr, dotted, "u"] \arrow[ddr, bend right, "f_Y"'] & & \\
& X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d] \\
& Y \arrow[r] & Z
\end{tikzcd}$$

Equivalently, $X \times_Z Y$ is the limit (Definition 2.2.8) of the diagram (Definition 2.2.6)

$$\begin{tikzcd}
& X \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}$$

in $\mathcal{C}$.

The commutative diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

may be referred to as a *cartesian square*.

Definition C.0.25. Let $\mathcal{C}$ be a (large) category (Definition 1.1.1). A *semigroup object in $\mathcal{C}$* is an object $A \in \mathcal{C}$ such that the product (Definition 2.2.1) $A \times A$ exists in $\mathcal{C}$ together with a morphism

$$\mu : A \times A \to A,$$

called the *multiplication morphism* such that the associativity diagram

$$\begin{tikzcd} A \times A \times A \arrow[r, "\mu \times \mathrm{id}_A"] \arrow[d, "\mathrm{id}_A \times \mu"'] & A \times A \arrow[d, "\mu"] \\ A \times A \arrow[r, "\mu"'] & A \end{tikzcd}$$

commutes.

The semigroup object structure (A, μ, η, ι) is said to be *abelian* or *commutative* if the morphisms $\mu : A \times A \to A$ and $\mu \circ \tau_{A,A} : A \times A \to A$ coincide, where $\tau_{A,A} : A \times A \to A \times A$ is the symmetry morphism swapping the two factors.

Definition C.0.26. Let $\mathcal{C}$ be a (large) category (Definition 1.1.1) with a final object (Definition 2.3.1). A *monoid object in $\mathcal{C}$* is a semigroup object (Definition C.0.25) (A, μ) together with a *unit morphism*

$$\eta : 1 \to A$$

such that the products (Definition 2.2.1) $1 \times A$ and $A \times 1$ exist and the unit diagrams

$$\begin{tikzcd} 1 \times A \arrow[rr, "\eta \times \mathrm{id}_A"] \arrow[dr, "\mathrm{pr}_2"'] & & A \times A \arrow[dl, "\mu"] \\ & A & \end{tikzcd}$$

$$\begin{tikzcd} A \times 1 \arrow[rr, "\mathrm{id}_A \times \eta"] \arrow[dr, "\mathrm{pr}_1"'] & & A \times A \arrow[dl, "\mu"] \\ & A & \end{tikzcd}$$

commute.

Definition C.0.27. Let $\mathcal{C}$ be a (large) category (Definition 1.1.1) with a final object (Definition 2.3.1). A *group object in $\mathcal{C}$* is a monoid object (Definition C.0.26) (A, μ, η) together with a *inverse morphism*

$$\iota : A \to A$$

such that the diagrams

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[dr, "\eta\circ !_A"'] & A \times A \arrow[d, "\mu\circ(\mathrm{id}_A \times \iota)"] \\ & A \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[dr, "\eta\circ !_A"'] & A \times A \arrow[d, "\mu\circ(\iota \times \mathrm{id}_A)"] \\ & A \end{tikzcd}$$

commute, where $\Delta : A \to A \times A$ is the diagonal and $!_A : A \to 1$ is the unique morphism.

Definition C.0.28 (Power Set)**.** Let A be a set. The *power set of A*, denoted by $\mathcal{P}(A)$, is the set of all subsets of A:

$$\mathcal{P}(A) := \{ B \mid B \subseteq A \}.$$

Equivalently, every element of $\mathcal{P}(A)$ is itself a set B satisfying $B \subseteq A$. Under the axiom of power set, note that the $\mathcal{P}(A)$ exists.

Definition C.0.29. Let R be a (not-necessarily commutative) ring with unity (Definition C.0.7). An *R-algebra* is a ring A together with a ring homomorphism (Definition C.0.13)

$$\varphi : R \to A$$

into the center $Z(A)$ of A (so that $\varphi(r)$ commutes with every element of A for all $r \in R$), such that $\varphi(1_R) = 1_A$. The ring homomorphism φ is called the *structure map* of the algebra.

Equivalently, an R-algebra consists of a ring A endowed with a two-sided R-module (Definition 2.1.1) structure for which the scalar multiplication satisfies

$$r \cdot (ab) = (r \cdot a)b = a(r \cdot b) \quad \text{for all } r \in R,\ a, b \in A.$$

In particular, any ring homomorphism between commutative rings (Definition C.0.9) specifies an algebra structure.

Definition C.0.30 (Homotopy of maps of topological spaces)**.** Let X and Y be topological spaces and let $K \subseteq X$ be a subset. Let $C(X, Y)$ denote the set of all continuous maps $f : X \to Y$.

1. A *homotopy between two maps $f, g \in C(X, Y)$ relative to K* is a continuous map

   $$H : X \times [0, 1] \to Y$$

   such that for all $x \in X$,

   $$H(x, 0) = f(x), \quad H(x, 1) = g(x),$$

and for all $x \in K$ and $t \in [0,1]$,

$$H(x,t) = f(x) = g(x).$$

If such an H exists, we say f and g are *homotopic relative to K*, and we write $f \simeq g \text{ rel } K$; this is an equivalence relation.

A *homotopy between two maps $f, g \in C(X,Y)$* is simply a homotopy relative to $\emptyset$. We write we write $f \simeq g$ if a homotopy between them exists.

2. Let (X, x_0) and (Y, y_0) be pointed topological spaces (Definition 1.1.10) and let $K \subseteq X$ be a subset with $x_0 \in K$. Let $C_*(X,Y)$ denote the set of all continuous based maps $f : X \to Y$ satisfying $f(x_0) = y_0$.

   A *homotopy of based maps $f, g \in C_*(X,Y)$ relative to K* is a continuous map

$$H : X \times [0,1] \to Y$$

   such that for all $x \in X$,

$$H(x,0) = f(x), \quad H(x,1) = g(x),$$

   and for all $k \in K$ and $t \in [0,1]$,

$$H(k,t) = f(k) = g(k),$$

   in particular fixing the basepoint throughout,

$$H(x_0,t) = y_0 \quad \text{for all } t \in [0,1].$$

   If such an H exists, we say f and g are *based homotopic relative to K*, and we write $f \simeq g \text{ rel } K$. This is an equivalence relation.

   A *homotopy of based maps $f, g \in C_*(X,Y)$* without relative condition is the special case $K = \{x_0\}$ and is called a *homotopy of based maps* or *based homotopy*. We write $f \simeq g$ if such a homotopy exists.

Definition C.0.31 (Homotopy class of maps relative to a subset)**.** Let X and Y be topological spaces (Definition C.0.5) and let $K \subseteq X$. Let $C(X,Y)$ denote the set of all continuous maps (Definition C.0.6) $f : X \to Y$.

1. Two maps $f, g \in C(X,Y)$ are said to be in the same *homotopy class relative to K* if there exists a homotopy relative to K (Definition C.0.30)

$$H : X \times [0,1] \to Y$$

   such that

$$H(x,0) = f(x), \quad H(x,1) = g(x),$$

   and

$$H(k,t) = f(k) = g(k) \quad \text{for all } k \in K, t \in [0,1].$$

   The *homotopy class of maps relative to K* containing a map $f : X \to Y$ is denoted by $[f]_K$.

   Two maps $f, g \in C(X,Y)$ are said to be in the same *homotopy class* if they are in the same homotopy class relative to $\emptyset$.

   The *homotopy class of maps* containing a map $f : X \to Y$ is denoted by $[f]$.

   The set of homotopy classes of maps may often be denoted by $[X,Y]$.

2. Let (X, x_0) and (Y, y_0) be pointed topological spaces (Definition 1.1.10) and let $K \subseteq X$ be a subset containing x_0. Let $C_*(X, Y)$ denote the set of all continuous based maps $f : X \to Y$ with $f(x_0) = y_0$.

   Two based maps $f, g \in C_*(X, Y)$ are said to be in the same *homotopy class relative to K* if there exists a homotopy of based maps relative to K

$$H : X \times [0, 1] \to Y$$

   such that for all $x \in X$,

$$H(x, 0) = f(x), \quad H(x, 1) = g(x),$$

   and for all $k \in K$ and $t \in [0, 1]$,

$$H(k, t) = f(k) = g(k),$$

   particularly ensuring the basepoint is fixed throughout,

$$H(x_0, t) = y_0 \quad \text{for all } t \in [0, 1].$$

   The *homotopy class relative to K* containing $f : (X, x_0) \to (Y, y_0)$ is denoted by $[f]_K$.

   Two based maps $f, g \in C_*(X, Y)$ are said to be in the same *homotopy class* if they are in the same homotopy class relative to $\{x_0\}$.

   The *homotopy class* containing a map $f : (X, x_0) \to (Y, y_0)$ is denoted by $[f]$.

   The set of homotopy classes of pointed maps $(X, x_0) \to (Y, y_0)$ may often be denoted by $[(X, x_0), (Y, y_0)]$ or by $[X, Y]$ if the base points are clear.

Definition C.0.32. The *homotopy category of topological spaces*, denoted hTop, is the category whose objects are topological spaces (Definition C.0.5) and whose morphisms are homotopy classes (Definition C.0.31) of continuous maps (Definition C.0.6). In other words, for objects X and Y, the set of morphisms is defined as $\mathrm{Hom}_{\mathrm{hTop}}(X, Y) = [X, Y] = C(X, Y)/\simeq$.

Proposition C.0.33. Composition in the homotopy category of topological spaces (Definition C.0.32) is well-defined. If $f_1, f_2 : X \to Y$ are homotopic and $g_1, g_2 : Y \to Z$ are homotopic, then the compositions $g_1 \circ f_1$ and $g_2 \circ f_2$ are homotopic as maps from X to Z. That is, $[g] \circ [f] = [g \circ f]$ is independent of the choice of representatives.

Theorem C.0.34. There exists a canonical functor $Q : \mathrm{Top} \to \mathrm{hTop}$ which is the identity on objects and maps each continuous map f to its homotopy class $[f]$. This functor is full and essentially surjective.

Definition C.0.35. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ be fixed. Let $(M, \mathcal{A}_M)$ and $(N, \mathcal{A}_N)$ be C^k-manifolds with boundary (Definition C.0.37) of dimensions n and m, respectively, where M, N are topological manifolds with boundary and $\mathcal{A}_M$ and $\mathcal{A}_N$ are C^k-atlases whose charts map to open subsets of the closed half-spaces $\mathbb{H}^n$ and $\mathbb{H}^m$.

A *C^k-morphism* (or *C^k-map*) between M and N is a continuous map (Definition C.0.6)

$$f : M \to N$$

such that for every $p \in M$ there exist charts $(U, \varphi) \in \mathcal{A}_M$ with $p \in U$ and $(V, \psi) \in \mathcal{A}_N$ with $f(p) \in V$ satisfying

$$\psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \to \psi(V)$$

is a C^k-map between open subsets of the closed half-spaces $\mathbb{H}^n$ and $\mathbb{H}^m$, i.e.,

$$\psi \circ f \circ \varphi^{-1} \in C^k(\varphi(U \cap f^{-1}(V)), \psi(V)).$$

If f is a homeomorphism and its inverse $f^{-1} : N \to M$ is also a C^k-morphism, then f is called a *C^k-diffeomorphism*. We let $C^k(M, N)$ denote the space of C^k-maps $M \to N$. We let $C^k(M)$ denote the space of *C^k-functions*, i.e., the C^k-maps $M \to \mathbb{R}$.

In particular, we may speak of these notions when M and N are C^k-manifolds without boundary (Definition C.0.37).

Remark C.0.36. The notations $C^k(M, N)$ (and $C^k(M)$) agrees with the usual notations $C^k(M, N)$ and $C^k(M)$ in the case that M is an open subset of $\mathbb{R}^n$.

Definition C.0.37. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ be fixed. An *n-dimensional C^k/k-differentiable-(real)manifold with boundary (resp. without boundary)* is a pair $(M, \mathcal{A})$, where M is a topological manifold with boundary (resp. without boundary) of dimension n and $\mathcal{A}$ is a C^k-atlas on M.

The atlas $\mathcal{A}$ is usually taken to be maximal with respect to C^k-compatibility, meaning it contains every C^k-chart compatible with all charts in $\mathcal{A}$.

Note that a C^0-manifold is simply a topological manifold and that a C^∞-manifold is synonymously referred to as a *smooth/differentiable (real) manifold*.

Definition C.0.38 (Opposite ring)**.** Let $R = (R, +, \cdot, 0, 1)$ be a ring (Definition C.0.7) with addition $+$, multiplication $\cdot$, additive identity 0, and multiplicative identity 1 (not necessarily commutative).

The *opposite ring of R*, denoted R^{op}, is the ring with the same underlying set R and the same addition $+$ and additive identity 0, but with multiplication defined by

$$r \star s := s \cdot r$$

for all $r, s \in R$.

That is, multiplication in R^{op} is the multiplication of R reversed in order.

If R is commutative (Definition C.0.9), then R and R^{op} are naturally isomorphic to each other.

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