1. Rings and Modules – review

(1.1) Show that "$2 \otimes 1$" is zero in $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2$ but not in $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2$ or $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3$.

(1.2) What can you say about the isomorphism type of the ring $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$?

(1.3) Suppose that k is a field and $\mathrm{k} \to A$ is a change of rings homomorphism. Assume A is commutative. Prove that $\mathrm{Hom}_A(M \otimes_{\mathrm{k}} A, N \otimes_{\mathrm{k}} A)$ is isomorphic to $\mathrm{Hom}_{\mathrm{k}}(M, N) \otimes_{\mathrm{k}} A$ as A-modules, for any finite-dimensional k-vector spaces M, N.

(1.4) Let I and J be ideals of a commutative ring R. Prove that

$$R/I \otimes_R R/J \cong R/(I + J)$$

via the homomorphism

$$(r \mod I) \otimes (r' \mod J) \mapsto (rr' \mod I + J).$$

(1.5) Let $\phi : R \to S$ be a change of rings map (that is, elements of $\phi(R)$ commute with all of S.) Recall that, if M is a left S-module, then ϕ makes M into a left R-module in a natural way. If we want to distinguish these two different module structures on M, we will write ϕ^*M to denote M thought of as an R-module. Let M, N be left S-modules.

(a) Show that $\mathrm{Hom}_R(\phi^*M, \phi^*N)$ is a left S-module if we define $s \cdot f$ for $s \in S$ and $f \in \mathrm{Hom}_R(\phi^*M, \phi^*N)$ by

$$(s \cdot f)(m) = sf(m)$$

for $m \in M$.

(b) Suppose further that M is a (S, S)-bimodule. Show that $\mathrm{Hom}_R(\phi^*M, \phi^*N)$ has another left S-module structure given by defining $s \cdot f$ (as above) by

$$(s \cdot f)(m) = f(ms).$$

(1.6) Consider the change of rings $i : \mathbb{Q} \hookrightarrow \mathbb{Q}[t]$. Regard $\mathbb{Q}[[t]]$ (the ring of formal power series) as a $\mathbb{Q}[t]$-module in the natural way.

(a) Consider $\mathbb{Q}$ as a $\mathbb{Q}[t]$-module by letting t act as zero. Check that $i^*\mathbb{Q} = \mathbb{Q}$, the one-dimensional vector space.

(b) Check that

$$\mathrm{Hom}_{\mathbb{Q}}(i^*\mathbb{Q}[t], \mathbb{Q}) \cong i^*\mathbb{Q}[[t]]$$

as $\mathbb{Q}$-vector spaces, by writing an isomorphism explicitly.

(c) Show that $\mathrm{Hom}_{\mathbb{Q}}(i^*\mathbb{Q}[t], \mathbb{Q}) \cong \mathbb{Q}[[t]]$ as $\mathbb{Q}[t]$-modules as well, by giving $\mathbb{Q}[[t]]$ a module structure induced by setting

$$t \cdot t^n = \begin{cases} t^{n-1} & \text{if } n \geq 1; \\ 0 & \text{otherwise.} \end{cases}$$

(1.7) Show that, if V and W are k-vector spaces with V finite-dimensional, then

$$\mathrm{Hom}_{\mathrm{k}}(V, W) \cong V^* \otimes_{\mathrm{k}} W.$$

(Find an explicit isomorphism.)

(1.8) If G is a group and A is a (left) $\mathrm{k}[G]$-module, let k be the trivial module, and

$$A^G := \{a \in A \colon ga = a \text{ for all } g \in G\}.$$

Prove that $A^G \cong \mathrm{Hom}_{\mathrm{k}[G]}(\mathrm{k}, A)$. *Corrected*

2. Categorical notions:

(2.1) Verify that the category R-mod has the structure of an abelian category. You can assume it is an additive category if you like.

(2.2) For any ring R and left R-module M, consider the functor

$$\mathrm{Hom}_R(M, -)\colon R\text{-mod} \to R\text{-mod}.$$

(a) Show that $\mathrm{Hom}_R(M, -)$ is additive.
(b) Show that $\mathrm{Hom}_R(M, -)$ is left exact.

(2.3) Fix a prime p. For each $n \geq 1$, $\mathbb{Z}/p^n\mathbb{Z}$ can be identified with a subgroup of $\mathbb{Z}/p^{n+1}\mathbb{Z}$, via the map

$$\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n+1}\mathbb{Z}$$

that sends $\overline{1}$ to $\overline{p}$.

Let $\mathbb{Z}_{p^\infty} = \varinjlim_n \mathbb{Z}/p^n\mathbb{Z}$ denote the colimit of the diagram

$$\mathbb{Z}/p\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^3\mathbb{Z} \xrightarrow{p} \cdots$$

Show that $\mathbb{Z}_{p^\infty} \cong \mathbb{Z}[p^{-1}]/\mathbb{Z}$, where $\mathbb{Z}[p^{-1}]$ denotes the subgroup of $\mathbb{Q}$ consisting of rational numbers whose denominator is a power of p.