

ALGEBRAIC GROUPS

December 16, 2025

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1. DEFINITIONS

1.1. Algebraic groups and linear algebraic groups.

Definition 1.1.1. Let S be a scheme. An **algebraic group scheme over S** (or an **S -group scheme**) is a group object G in the category of schemes over S ; that is, G is an S -scheme equipped with S -morphisms: $m : G \times_S G \rightarrow G$ (*multiplication*), $i : G \rightarrow G$ (*inverse*), and $e : S \rightarrow G$ (*identity*), satisfying the group axioms expressed by the commutativity of the following diagrams:

1. **Associativity**

$$\begin{array}{ccc}
 G \times_S G \times_S G & \xrightarrow{m \times \text{id}} & G \times_S G \\
 \text{id} \times m \downarrow & & \downarrow m \\
 G \times_S G & \xrightarrow{m} & G
 \end{array}$$

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2. **Identity**

$$\begin{array}{ccc}
G \times_S S & \xrightarrow{\text{id} \times e} & G \times_S G \\
& \searrow \simeq & \downarrow m \\
& & G
\end{array}
\qquad
\begin{array}{ccc}
S \times_S G & \xrightarrow{e \times \text{id}} & G \times_S G \\
& \searrow \simeq & \downarrow m \\
& & G
\end{array}$$

3. **Inverse**

$$\begin{array}{ccc}
G & \xrightarrow{(\text{id}, i)} & G \times_S G \\
\text{id} \downarrow & & \downarrow m \\
G & \xrightarrow{e \circ \pi} & G
\end{array}$$

where $\pi : G \rightarrow S$ is the structure morphism and $e \circ \pi$ sends g to the identity section.

If G is affine over (Definition B.0.10) S , we call it an **affine group scheme over S** .

If the base scheme S is the spectrum of a field k , then we call G a **k -algebraic group** or an **algebraic group (scheme) over k** . If G is additionally a k -variety, then we call G a **k -group variety**.

Theorem 1.1.2 (Cartier; see [Mil17, Theorem 3.23]). Every affine algebraic group (Definition 1.1.1) over a field of characteristic zero is smooth.

Definition 1.1.3. Let S be a scheme, and let G be an algebraic group scheme over S . A **subgroup scheme** (or synonymously an **algebraic subgroup**) H of G is an S -subscheme

$$H \subseteq G$$

such that for every S -scheme T , the subset $H(T) \subseteq G(T)$ is a subgroup of the group $G(T)$. Equivalently, H is a group object in the category of S -schemes equipped with a monomorphism of group schemes $H \rightarrow G$.

(♠ TODO: ARe all group schemes over a characteristic zero field reduced?) (♠ TODO: ARe all subgroups of group schemes over a characteristic zero field closed?)

Over positive characteristic fields, not all subgroup schemes are closed:

Example 1.1.4 (A subgroup scheme that is not closed). Let k be an algebraically closed field of characteristic $p > 0$. Consider the additive group scheme $\mathbf{G}_a = \text{Spec}(k[t])$ over k . Define the subgroup functor H by

$$H(R) = \{x \in \mathbf{G}_a(R) \mid x^p = 0\}$$

for any k -algebra R . Then H is a subgroup scheme of \mathbf{G}_a known as the **infinitesimal subgroup of order p** .

However, H is not a closed subgroup scheme of \mathbf{G}_a because it corresponds to the ideal (t^p) in $k[t]$, which is not radical and hence does not define a closed subscheme.

Definition 1.1.5. Let S be a scheme, and let G and H be S -algebraic groups. A morphism of S -schemes $f : G \rightarrow H$ is a **homomorphism of algebraic groups** if f is a group homomorphism, i.e.,

- $f(m_G(x, y)) = m_H(f(x), f(y))$ for all $x, y \in G$,
- $f(i_G(x)) = i_H(f(x))$ for all $x \in G$, and

- $f(e_G) = e_H$.

It is called an *isomorphism of algebraic groups (over S)* if it is additionally an isomorphism of S -schemes. If there exists an isomorphism $f : G \rightarrow H$ of algebraic groups over S , then G and H are said to be *isomorphic S -algebraic groups*.

Definition 1.1.6 (General linear group). Let S be a base scheme and $n \geq 1$ an integer. The *general linear group over S* , denoted $\mathrm{GL}_{n,S}$ or GL_n/S or simply by GL_n if the base S is clear, is the group scheme over S defined by

$$\mathrm{GL}_{n,S}(T) = \{ \text{invertible } n \times n \text{ matrices with entries in } \mathcal{O}_T(T) \}$$

for any S -scheme T , where \mathcal{O}_T is the structure sheaf of T .

Equivalently, $\mathrm{GL}_{n,S}$ may be defined as the general linear group scheme $\mathrm{GL}(\mathcal{O}_S^{\oplus n})$ associated to the free \mathcal{O}_S -module of rank n (Definition 2.0.1).

Definition 1.1.7 (Linear algebraic group over a scheme). Let S be a base scheme. A *linear algebraic group over S* is an affine group scheme (Definition 1.1.1) G over S that is finitely presented and smooth over S , and such that for some integer $n \geq 1$, there exists a closed immersion of S -group schemes

$$G \hookrightarrow \mathrm{GL}_{n,S}.$$

(Definition 1.1.6)

1.2. Normal subgroup schemes.

Definition 1.2.1. Let S be a scheme, let G be an algebraic group over S (Definition 1.1.1), and let $H \subseteq G$ be a closed subgroup scheme (Definition 1.1.3) over S .

1. The subgroup scheme H is called a *normal subgroup scheme of G* if for every S -scheme T , the subgroup $H(T) \subseteq G(T)$ is normalized by $G(T)$, i.e.

$$ghg^{-1} \in H(T), \quad \forall g \in G(T), h \in H(T).$$

2. The subgroup scheme H is called a *characteristic subgroup scheme of G* if for every S -scheme T and automorphism $\varphi : G_T \rightarrow G_T$ of T -group schemes, one has

$$\varphi(H_T) = H_T.$$

In particular, all characteristic subgroup schemes are normal subgroup schemes.

Definition 1.2.2. 1. Let k be a field, and let G be a group scheme (Definition 1.1.1) over $\mathrm{Spec}(k)$ (Definition B.0.1). The *identity component or neutral component of G* , denoted G^0 , is the unique connected component of G containing the identity point $e \in G(k)$.

2. Let S be a scheme, and let G be an algebraic group scheme over S . The *identity component of G* , denoted G^0 , is the unique open and closed subgroup scheme of G such that for every geometric point $\bar{s} \rightarrow S$, the fiber $(G^0)_{\bar{s}}$ is the connected component of the identity element in the group scheme $G_{\bar{s}}$.

Note that G^0/S might not be connected if S is not connected

Lemma 1.2.3 (cf. [Mil17, Proposition 1.52]). (♠ **TODO: Does this hold for more general groups G/S ?**) Let G/k be an algebraic group (Definition 1.1.1) over a field. The identity component G^0 (Definition 1.2.2) is a characteristic subgroup (Definition 1.2.1) and hence a normal subgroup of G .

Definition 1.2.4 (Finite index subgroup scheme of an algebraic group scheme over a scheme). Let S be a scheme and G an algebraic group scheme over S (Definition 1.1.1). A (not necessarily normal) closed subgroup scheme $H \subseteq G$ (i.e., a monomorphism of group schemes $H \rightarrow G$ over S) is called a *finite index subgroup scheme* if the fppf (or equivalently, the fpqc) quotient sheaf G/H (Definition 1.2.6) is representable (Definition B.0.6) by a finite S -scheme.

Equivalently, H is of finite index in G if H is a closed subgroup scheme and the morphism $G \rightarrow G/H$ exhibits G as a finite fppf cover of G/H .

Lemma 1.2.5 (see e.g. [Mil17, Between Definitions 6.5 and 6.6]). Let G/k be an algebraic group scheme (Definition 1.1.1) and let H be a subgroup scheme (Definition 1.1.3).

1. The subgroup H has finite index (Definition 1.2.4) if and only if $\dim H = \dim G$.
2. If G is smooth, then H has finite index if and only if H contains G^0 (Definition 1.2.2).

Definition 1.2.6 (Quotient of an algebraic group scheme by an algebraic subgroup). Let S be a scheme. Let G be an algebraic group scheme over S (Definition 1.1.1), and let $H \subseteq G$ be a closed subgroup scheme over S , i.e., a monomorphism of group schemes $H \rightarrow G$ over S .

The *quotient sheaf* G/H is the fppf sheafification of the presheaf

$$T \mapsto G(T)/H(T)$$

on the category of S -schemes, where T is any S -scheme.

If G/H is representable by a scheme (Definition B.0.7) (or by an algebraic space (Definition B.0.8)) Q over S , then Q is called the *quotient of G by H* . In general, if H is normal (Definition 1.2.1), then G/H inherits the structure of a group scheme over S .

Otherwise, G/H is just a sheaf of sets (or algebraic space on $(\text{Sch}/S)_{\text{fppf}}$ (Definition B.0.9)) over S without a natural group structure.

Proposition 1.2.7 (e.g. [Mil17, Proposition 5.23]). Let G be an algebraic group (Definition 1.1.1) over a field and let H be an algebraic subgroup such that the quotient G/H (Definition 1.2.6) is representable by a scheme (Definition B.0.7). We have $\dim G = \dim H + \dim G/H$.

Definition 1.2.8 (Strongly connected algebraic group scheme over a scheme). Let S be a scheme. A group scheme G over S (Definition 1.1.1) is called a *strongly connected algebraic group scheme* if it has no proper algebraic subgroup over S of finite index.

Definition 1.2.9. Let S be a scheme and G an algebraic group scheme over S (Definition 1.1.1). The *strong identity component* G^{iso} of G is the intersection of the algebraic subgroups over S of finite index.

Proposition 1.2.10 (see [Mil17, Between Definition 6.9 and Proposition 6.10]). Let G/k be a smooth algebraic group scheme over a field. The strong identity component G^{so} (Definition 1.2.9) coincides with the identity component G^0 (Definition 1.2.2).

1.3. Major examples of linear algebraic groups.

Definition 1.3.1 (Special linear group). Let S be a base scheme and $n \geq 1$ an integer. The *special linear group* $\text{SL}_{n,S}$ is the subgroup scheme of $\text{GL}_{n,S}$ (Definition 1.1.6) defined by the condition

$$\text{SL}_{n,S}(T) = \{ g \in \text{GL}_{n,S}(T) : \det(g) = 1 \}.$$

Equivalently, a linear algebraic group over S may be regarded as an affine group scheme over S equipped with a representation on (Definition 2.0.2) the free \mathcal{O}_S -module of rank n .

Definition 1.3.2 (Orthogonal group). (♠ TODO: define quadratic form) Let S be a base scheme, $n \geq 1$ an integer, and let q be a quadratic form on the rank n free \mathcal{O}_S -module \mathcal{O}_S^n , i.e., a global section

$$q : \mathcal{O}_S^n \rightarrow \mathcal{O}_S,$$

satisfying the usual properties of a quadratic form. The *orthogonal group* $\text{O}(q)$ over S is the subgroup scheme of $\text{GL}_{n,S}$ (Definition 1.1.6) given by

$$\text{O}(q)(T) = \{ g \in \text{GL}_{n,S}(T) : q_T(g(v)) = q_T(v) \text{ for all } v \in \mathcal{O}_T^n \},$$

where q_T is the pullback of q to T .

Definition 1.3.3 (Special orthogonal group). (♠ TODO: quadratic form) Let S be a base scheme, $n \geq 1$ an integer, and let q be a quadratic form on the rank n free \mathcal{O}_S -module \mathcal{O}_S^n , i.e., a global section

$$q : \mathcal{O}_S^n \rightarrow \mathcal{O}_S,$$

satisfying the usual properties of a quadratic form. The *special orthogonal group* $\text{SO}(q)$ is the subgroup scheme of $\text{O}(q)$ consisting of elements with determinant 1:

$$\text{SO}(q)(T) = \{ g \in \text{O}(q)(T) : \det(g) = 1 \}.$$

(♠ TODO: check the definition of unitary group)

Definition 1.3.4 (Unitary group). Let S be a base scheme (Definition B.0.2). Suppose A is an \mathcal{O}_S -algebra (Definition B.0.18) equipped with an involution $*$: $A \rightarrow A$, and let h be a hermitian form (Definition A.0.11) on a locally free (Definition B.0.19) A -module M of finite rank over S . The *unitary group over S* is the subgroup scheme $\text{U}(h)$ of (♠ TODO: general linear group of sheaf algebra) $\text{GL}_A(M)$ defined by

$$\text{U}(h)(T) = \{ g \in \text{GL}_A(M)(T) : h_T(g(x), g(y)) = h_T(x, y) \text{ for all } x, y \in M_T \},$$

where h_T and M_T are the base changes of h and M to T .

Definition 1.3.5 (Special unitary group). Let S be a base scheme (Definition B.0.2). Suppose A is an \mathcal{O}_S -algebra (Definition B.0.18) equipped with an involution $*$: $A \rightarrow A$, and let h be a hermitian form (Definition A.0.11) on a locally free (Definition B.0.19) A -module M of finite rank over S . The *special unitary group* $\text{SU}(h)$ is the subgroup scheme of $\text{U}(h)$ (Definition 1.3.4) consisting of elements of reduced norm 1, i.e., (♠ TODO: reduced norm)

$$\text{SU}(h)(T) = \{ g \in \text{U}(h)(T) : \text{Nrd}(g) = 1 \},$$

where Nrd is the reduced norm.

Definition 1.3.6 (Multiplicative group scheme). Let S be a scheme (Definition B.0.2). The *multiplicative group scheme over S* , denoted $\mathbb{G}_{m,S}$, is the group scheme over S defined as the open subscheme

$$\mathbb{G}_{m,S} := \mathbb{A}_S^1 \setminus \{0\}_S$$

of the affine line (Definition B.0.28) \mathbb{A}_S^1 , equipped with the group law given by multiplication of functions:

$$m : \mathbb{G}_{m,S} \times_S \mathbb{G}_{m,S} \rightarrow \mathbb{G}_{m,S}, \quad (x, y) \mapsto xy.$$

The identity section is the morphism

$$e : S \rightarrow \mathbb{G}_{m,S}, \quad s \mapsto 1,$$

and the inversion morphism is given by

$$i : \mathbb{G}_{m,S} \rightarrow \mathbb{G}_{m,S}, \quad x \mapsto x^{-1}.$$

1.4. Unipotent groups, reductive groups, tori, and Borel subgroups.


Definition 1.4.1 (Unipotent element in a linear algebraic group). Let k be a field, and let $G \subseteq \text{GL}_n(k)$ be a linear algebraic group given by a faithful representation. An element $g \in G(k)$ is called a *unipotent element* if its image in $\text{GL}_n(k)$ is a unipotent matrix, i.e. if all eigenvalues of g in an algebraic closure \bar{k} of k are equal to 1. Equivalently, g is unipotent if $g - I_n$ is nilpotent.

Definition 1.4.2. Let R be a ring. The group of $n \times n$ square matrices over R that are upper triangular and whose diagonal entries are all 1 is called the *unitriangular group* or the *group of unit upper triangular matrices*. It is often denoted by U_n^1 or UT_n .

Definition 1.4.3. Let S be a scheme and let $n \geq 1$ be an integer. The *linear algebraic group of unit upper triangular $n \times n$ matrices over S* (Definition 1.1.7) is the subgroup scheme $U_{n,S} = UT_{n,S}$ (or $U_n/S = UT_n/S$ or $U_n = UT_n$ if the base S is clear) of $\text{GL}_{n,S}$ (Definition 1.1.6) defined by

$$UT_{n,S}(T) = \{A \in \text{GL}_{n,S}(T) : A \text{ is upper triangular with diagonal entries } 1\}$$

for any S -scheme T .

Definition 1.4.4 (Unipotent group). 1. An algebraic group scheme (Definition 1.1.1) U/k over a field is said to be *unipotent* if the following equivalent conditions hold: 
TODO: state some of these conditions more precisely)

- every nonzero representation of the group has a nonzero fixed vector.
- there exists a faithful representation (Definition 2.0.3) $U \hookrightarrow \text{GL}_n$ whose image consists entirely of unipotent matrices (Definition 1.4.1).
- U is isomorphic (Definition 1.1.5) to a closed subgroup scheme of $UT_n/\text{Spec } k$ for some integer $n \geq 1$.
- U is a linear algebraic group (Definition 1.1.7) and for any k -algebra R , the matrix group $U(R)$ consists entirely of unipotent matrices.

¹Note that this notation can be confused for or conflicts with notation for the unitary group (Definition 1.3.4).

2. Let U be an algebraic group scheme over a base scheme S . We say that U is **unipotent** if it satisfies one of the following equivalent conditions:

- For every geometric point $s \in S$, the fiber U_s is a unipotent algebraic group over the residue field $\kappa(s)$.
- There exists a faithful representation (Definition 2.0.3)

$$\rho : U \hookrightarrow \mathrm{GL}_n$$

over S such that for every S -algebra R , the image $\rho(U(R)) \subseteq \mathrm{GL}_n(R)$ consists entirely of unipotent matrices (i.e., matrices whose eigenvalues are all equal to 1).

- U admits a finite central composition series

$$1 = U_0 \triangleleft U_1 \triangleleft \cdots \triangleleft U_r = U,$$

where each successive quotient U_i/U_{i-1} is isomorphic to a subgroup scheme of the additive group scheme $\mathbb{G}_{a,S}$.

- U is isomorphic to a closed subgroup scheme of the group scheme UT_n of upper unitriangular $n \times n$ matrices over S for some integer n .

Example 1.4.5. Let k be a field.

(1) Characteristic zero case: Consider the additive group scheme \mathbb{G}_a over k . Over a field of characteristic zero, \mathbb{G}_a is a unipotent group scheme.

(2) Positive characteristic case: For k a field of characteristic $p > 0$, consider the finite group scheme α_p defined as the kernel of the Frobenius morphism on \mathbb{G}_a . Then α_p is a unipotent group scheme over k .

These examples illustrate fundamental instances of unipotent group schemes in different characteristic settings.

Definition 1.4.6. (♠ TODO: define smooth scheme, finitely presented, geometric fibers) Let G be a group scheme (Definition 1.1.1) that is smooth, finitely presented, with connected geometric fibers over a scheme S . The **unipotent radical of G** , denoted $\mathrm{Rad}_u(G)$, is the largest smooth, finitely presented, normal (Definition 1.2.1), unipotent subgroup of G with connected geometric fibers.

(♠ TODO: show why the unipotent radical exists)

Definition 1.4.7 (Reductive group). Let G be a linear algebraic group (Definition 1.1.7) that is smooth, finitely presented, and with connected geometric fibers over a scheme S .

- In the case that S is $\mathrm{Spec} k$ for a field k , the group G is called **reductive** if $\mathrm{Rad}_u(G)$ is trivial, i.e. $\mathrm{Rad}_u(G) = \{e\}$, and G is smooth, connected, and affine over S .
- For general S , the group G is called **reductive** if it is a smooth affine group scheme G/S whose geometric fibers are connected reductive algebraic groups.

Lemma 1.4.8. Let G/k be a reductive group (Definition 1.4.7) over a field. Any normal subgroup scheme (Definition 1.2.1) of G over k is a reductive group.

Proof. Let N be a normal subgroup scheme of G . The unipotent radical $\text{Rad}_u(N)$ (Lemma 1.4.9) is a characteristic subgroup of N (Lemma 1.4.9) and hence it is a normal subgroup of G . In particular, $\text{Rad}_u(N) \leq \text{Rad}_u(G) = 1$ and hence $\text{Rad}_u(N)$ is trivial, i.e. N is reductive. \square

Lemma 1.4.9. Let G/k be a smooth, finitely presented scheme over a field. The unipotent radical $\text{Rad}_u(G)$ (Definition 1.4.6) is a characteristic subgroup (Definition 1.2.1) of G .

Proof. For any k -scheme T , given a T -automorphism $\varphi : G \times_k T \rightarrow G \times_k T$, (♠ TODO: continue;) \square

Theorem 1.4.10. (♠ TODO: define the characteristic of a ring) The category of finite-dimensional representations of a reductive group (Definition 1.4.7) over a field of characteristic 0 is semisimple (Definition B.0.5)

Definition 1.4.11 (Borel subgroup scheme). 1. Let G be a connected reductive group over an algebraically closed field k . A **Borel subgroup** $B \subseteq G$ is a maximal connected closed solvable subgroup of G . Equivalently, a Borel subgroup is a minimal parabolic subgroup of G .
2. Let $G \rightarrow S$ be a connected reductive group scheme (Definition 1.4.7) over a base scheme S . A **Borel subgroup scheme of G** is a closed subgroup scheme $B \subseteq G$ such that:
(a) $B \rightarrow S$ is smooth and affine,
(b) for every geometric point $s \rightarrow S$, the fiber $B_s \subseteq G_s$ is connected and is a Borel subgroup of the connected reductive algebraic group G_s .

One also says **maximal Borel subgroup** to emphasize the maximality.

Proposition 1.4.12. Let G be a connected reductive group (Definition 1.4.7) over an algebraically closed field k . For any two Borel subgroups $B_1, B_2 \subseteq G$, there exists some $g \in G(k)$ such that $B_2 = gB_1g^{-1}$.

In certain contexts, maximal compact subgroups and maximal tori of reductive linear algebraic groups may be of interest; in nice enough cases, maximal compact subgroups of reductive groups are unique up to conjugation.

Proposition 1.4.13. 1. Let G be a connected reductive group (Definition 1.4.7) over $k = \mathbb{R}$ or \mathbb{C} . For any two maximal compact subgroups $K_1, K_2 \subseteq G$, there exists some $g \in G(k)$ such that $K_2 = gK_1g^{-1}$.

(♠ TODO: define a p -adic field)

2. Let F be a p -adic field and let \mathcal{O}_F be its ring of integers. For any maximal compact subgroup $K \subseteq \text{GL}_n/F$, there exists some $g \in \text{GL}_n(F)$ such that $\text{GL}_n(\mathcal{O}_F) = gKg^{-1}$.

Proposition 1.4.14. Let G be a connected reductive group (Definition 1.4.7) over an algebraically closed field k . for any two maximal tori $T_1, T_2 \subseteq G$, there exists some $g \in G(k)$ such that $T_2 = gT_1g^{-1}$.

1.4.1. Examples.

Example 1.4.15 (Examples of reductive linear algebraic groups). Some standard examples of reductive linear algebraic groups over fields include:

1. The general linear group GL_n (Definition 1.1.6), the group of invertible $n \times n$ matrices.
2. The special linear group SL_n (Definition 1.3.1), consisting of matrices with determinant 1, which is connected and semisimple.
3. The multiplicative group $\mathbb{G}_m \cong \mathrm{GL}_1$ (Definition 1.3.6), a one-dimensional torus.
4. Products of multiplicative groups, i.e., *algebraic tori* \mathbb{G}_m^r .
5. The special orthogonal groups SO_n (Definition 1.3.3), preserving nondegenerate quadratic forms (connected for $n \geq 3$). (♠ TODO: define the symplectic groups)
6. The *symplectic groups* Sp_{2n} , preserving nondegenerate alternating bilinear forms.

Non-examples include the additive group \mathbb{G}_a , since it is unipotent, and Borel subgroups that have nontrivial unipotent radical.

Example 1.4.16 (Borel subgroups of standard reductive groups). For each of the following reductive groups over an algebraically closed field k , a Borel subgroup can be described as follows:

1. For the *general linear group* $\mathrm{GL}_n(k)$, a Borel subgroup is the subgroup of *invertible upper triangular matrices*.
2. For the *special linear group* $\mathrm{SL}_n(k)$, the Borel subgroup is given by the subgroup of upper triangular matrices with determinant 1.
3. For the *multiplicative group* \mathbb{G}_m , the group itself is a torus and hence a Borel subgroup.
4. For products of multiplicative groups \mathbb{G}_m^r , Borel subgroups are the groups themselves since they are tori.
5. For the *special orthogonal group* $\mathrm{SO}_n(k)$, a Borel subgroup can be realized as the stabilizer of a suitable isotropic flag of subspaces, often realized by certain block upper-triangular matrices preserving the quadratic form.
6. For the *symplectic group* $\mathrm{Sp}_{2n}(k)$, a Borel subgroup is typically the subgroup preserving a full isotropic flag with respect to the symplectic form, formed by upper-triangular block matrices in a suitable basis.

Example 1.4.17 (Maximal compact subgroups of standard reductive groups). Some standard examples of maximal compact subgroups of reductive linear algebraic groups over \mathbb{R} and \mathbb{C} include:

1. **General linear group** GL_n :
 - Over \mathbb{R} , a maximal compact subgroup is the orthogonal group $\mathrm{O}(n)$.
 - Over \mathbb{C} , the maximal compact subgroup is the unitary group $\mathrm{U}(n)$.
2. **Special linear group** SL_n :
 - Over \mathbb{R} , a maximal compact subgroup is the special orthogonal group $\mathrm{SO}(n)$.
 - Over \mathbb{C} , the maximal compact subgroup is the special unitary group $\mathrm{SU}(n)$.
3. **Multiplicative group** $\mathbb{G}_m \cong \mathrm{GL}_1$:
 - Over \mathbb{R} , the maximal compact subgroup is $\{\pm 1\}$, the unit circle in \mathbb{R}^* .
 - Over \mathbb{C} , the maximal compact subgroup is the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.
4. **Algebraic tori** \mathbb{G}_m^r :
 - Over \mathbb{R} , a product of copies of $\{\pm 1\}$ and S^1 .
 - Over \mathbb{C} , the maximal compact subgroup is $(S^1)^r$.
5. **Special orthogonal groups** SO_n :

- Over \mathbb{R} , $\mathrm{SO}_n(\mathbb{R})$ itself is compact (for $n \geq 2$) and forms its own maximal compact subgroup.
 - Over \mathbb{C} , the maximal compact subgroup corresponds to the compact real form $\mathrm{SO}_n(\mathbb{R})$.
6. **Symplectic groups** Sp_{2n} :
- Over \mathbb{R} , the maximal compact subgroup is $\mathrm{Sp}(n)$, the compact symplectic group (quaternionic unitary group).
 - Over \mathbb{C} , the maximal compact subgroup corresponds to the compact real form isomorphic to $\mathrm{Sp}(n)$.

2. REPRESENTATIONS OF LINEAR ALGEBRAIC GROUPS

Definition 2.0.1. (♠ TODO: define locally free module on a scheme) Let S be a scheme (Definition B.0.2) and let \mathcal{E} be a locally free \mathcal{O}_S -module of finite rank n . The *general linear group scheme associated to \mathcal{E}* , denoted $\mathrm{GL}(\mathcal{E})$, is the affine group scheme over S representing the functor

$$\mathrm{GL}(\mathcal{E})(T) = \mathrm{Aut}_{\mathcal{O}_T}(\mathcal{E}_T)$$

for every S -scheme T , where $\mathcal{E}_T := \mathcal{E} \otimes_S \mathcal{O}_T$ and $\mathrm{Aut}_{\mathcal{O}_T}(\mathcal{E}_T)$ denotes the group of \mathcal{O}_T -linear automorphisms of \mathcal{E}_T .

Definition 2.0.2. Let S be a scheme (Definition B.0.2), let G be an affine group scheme (Definition 1.1.1) over S , and let \mathcal{E} be a locally free \mathcal{O}_S -module of finite rank. A *representation of G on \mathcal{E}* is a morphism of S -group schemes (Definition 1.1.5)

$$\rho : G \longrightarrow \mathrm{GL}(\mathcal{E}),$$

where $\mathrm{GL}(\mathcal{E})$ is the general linear group scheme associated to \mathcal{E} (Definition 2.0.1).

In the case that $S = \mathrm{Spec} k$ for a field k , note that a representation of G is necessarily a morphism of k -group schemes

$$\rho : G \rightarrow \mathrm{GL}_{n, \mathrm{Spec} k}$$

(Definition 1.1.6) for some $n \geq 1$.

Definition 2.0.3. Let S be a scheme, and let G be an affine group scheme (Definition 1.1.1) over S . Let \mathcal{E} be a locally free \mathcal{O}_S -module of finite rank. Let

$$\rho : G \longrightarrow \mathrm{GL}(\mathcal{E}),$$

(♠ TODO: define monomorphism of group schemes) be a representation of G (Definition 2.0.2). Such a representation ρ is called *faithful* if ρ is a monomorphism of group schemes, i.e., if for every S -scheme T , the induced group homomorphism

$$\rho_T : G(T) \longrightarrow \mathrm{GL}(\mathcal{E})(T)$$

is injective.

3. ALGEBRAIC GROUPOIDS

(♠ TODO: groupoid object)

Definition 3.0.1. A *groupoid* can be defined equivalently in categorical or set-theoretic terms:

1. **Categorical Definition:** A groupoid is a small category (Definition B.0.16) \mathcal{G} in which every morphism is an isomorphism. That is, for every morphism $f : x \rightarrow y$ in \mathcal{G} , there exists a morphism $g : y \rightarrow x$ such that $g \circ f = \text{id}_x$ and $f \circ g = \text{id}_y$.
2. **Set-Theoretic Definition:** A groupoid consists of a pair of sets (G_0, G_1) , called the *set of objects* and the *set of arrows* respectively, equipped with the following structure maps:
 - *Source* and *Target*: $s, t : G_1 \rightarrow G_0$,
 - *Identity*: $e : G_0 \rightarrow G_1$, assigning to each object $x \in G_0$ an identity arrow $e(x)$,
 - *Composition*: A partial map $m : G_1 \times_{s, G_0, t} G_1 \rightarrow G_1$, defined on the set of composable pairs

$$G_1 \times_{s, G_0, t} G_1 := \{(g, h) \in G_1 \times G_1 \mid s(g) = t(h)\}$$

and denoted by $m(g, h) = g \circ h$,

- *Inverse*: $i : G_1 \rightarrow G_1$, denoted by $i(g) = g^{-1}$.

These structure maps must satisfy the following axioms for all $g, h, k \in G_1$ and $x \in G_0$ where the operations are defined:

(a) **Source and Target Compatibility:**

$$s(g \circ h) = s(h), \quad t(g \circ h) = t(g).$$

(b) **Associativity:** If $s(g) = t(h)$ and $s(h) = t(k)$, then

$$(g \circ h) \circ k = g \circ (h \circ k).$$

(c) **Identity:**

$$\begin{aligned} s(e(x)) &= x, & t(e(x)) &= x, \\ g \circ e(s(g)) &= g, & e(t(g)) \circ g &= g. \end{aligned}$$

(d) **Inverse:**

$$\begin{aligned} s(g^{-1}) &= t(g), & t(g^{-1}) &= s(g), \\ g \circ g^{-1} &= e(t(g)), & g^{-1} \circ g &= e(s(g)). \end{aligned}$$

Definition 3.0.2. Let \mathcal{C} be a (large) category (Definition B.0.15).

A *groupoid object in \mathcal{C}* consists of two objects X_0 (the "object of objects") and X_1 (the "object of morphisms"), together with five structure morphisms:

- *Source* and *Target*: $s, t : X_1 \rightarrow X_0$, such that the fiber product (Definition B.0.14) $X_1 \times_{s, X_0, t} X_1$ of the morphisms s and t exists in \mathcal{C} ,
- *Identity*: $e : X_0 \rightarrow X_1$,

- **Composition:** $m : X_1 \times_{s, X_0, t} X_1 \rightarrow X_1$,
- **Inverse:** $i : X_1 \rightarrow X_1$,

such that the following conditions hold (expressing the axioms of a category where every morphism is invertible):

1. **Source/Target identities:**

$$s \circ e = \text{id}_{X_0}, \quad t \circ e = \text{id}_{X_0}$$

$$s \circ m = s \circ \pi_2, \quad t \circ m = t \circ \pi_1$$

2. **Associativity:** The following diagram of composition commutes:

$$m \circ (m \times \text{id}_{X_1}) = m \circ (\text{id}_{X_1} \times m)$$

3. **Unitality:**

$$m \circ (e \circ s, \text{id}_{X_1}) = \text{id}_{X_1}, \quad m \circ (\text{id}_{X_1}, e \circ t) = \text{id}_{X_1}$$

4. **Invertibility:**

$$m \circ (i, \text{id}_{X_1}) = e \circ s, \quad m \circ (\text{id}_{X_1}, i) = e \circ t$$

Definition 3.0.3. Let R be a commutative ring (Definition B.0.12).

A **Hopf algebra over R** is a R -module H equipped with the structure of a unital associative algebra (H, μ, η) and a counital coassociative coalgebra (H, Δ, ε) , along with a R -linear map $S : H \rightarrow H$ called the **antipode**, satisfying the following compatibility axioms:

1. Δ and ε are algebra homomorphisms.
2. The antipode condition:

$$\mu \circ (S \otimes \text{id}_H) \circ \Delta = \eta \circ \varepsilon = \mu \circ (\text{id}_H \otimes S) \circ \Delta$$

In Sweedler notation, if $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$, the antipode condition is expressed as:

$$\sum S(h_{(1)})h_{(2)} = \varepsilon(h)1_H = \sum h_{(1)}S(h_{(2)})$$

Definition 3.0.4. Let R be a commutative ring (Definition B.0.12).

A **Hopf algebroid over R** is a pair of commutative R -algebras (A, Γ) together with structure maps that form a cogroupoid object in the category of commutative R -algebras. Specifically, it consists of:

- **Left and Right Units** (dual to source/target): $\eta_L : A \rightarrow \Gamma$ and $\eta_R : A \rightarrow \Gamma$. These induce a left A -module structure on Γ via η_L and a right A -module structure on Γ via η_R .
- **Counit** (dual to identity): $\varepsilon : \Gamma \rightarrow A$, such that $\varepsilon \circ \eta_L = \text{id}_A = \varepsilon \circ \eta_R$.
- **Comultiplication** (dual to composition): $\Delta : \Gamma \rightarrow \Gamma \otimes_A \Gamma$, where the tensor product is formed using the right A -action on the left factor and the left A -action on the right factor.

- **Antipode** (dual to inverse): $S : \Gamma \rightarrow \Gamma$, which is an algebra anti-homomorphism (or homomorphism if Γ is commutative) satisfying the appropriate compatibility diagrams dual to the groupoid axioms.

The pair (A, Γ) represents the affine groupoid scheme with objects $\text{Spec}(A)$ and morphisms $\text{Spec}(\Gamma)$.

Definition 3.0.5. Let S be a scheme.

An **affine groupoid scheme over S** (or simply an **affine groupoid over S**) is a groupoid object (Definition 3.0.2) in the category of schemes affine (Definition B.0.10) over (Definition B.0.11) S . Explicitly, it consists of a pair of S -schemes (G_0, G_1) equipped with structural morphisms:

- **Source** and **Target**: $s, t : G_1 \rightarrow G_0$,
- **Identity**: $e : G_0 \rightarrow G_1$,
- **Multiplication** (or composition): $m : G_1 \times_{s, G_0, t} G_1 \rightarrow G_1$,
- **Inverse**: $i : G_1 \rightarrow G_1$,

such that:

1. The structure morphisms $G_1 \rightarrow S$ and $G_0 \rightarrow S$ are affine morphisms (Definition B.0.10).
2. The morphisms satisfy the standard axioms of a groupoid (associativity of m , unitality of e , invertibility via i).

If $S = \text{Spec}(R)$ for a commutative ring (Definition B.0.12) k , then $G_0 = \text{Spec}(A)$ and $G_1 = \text{Spec}(H)$ for some commutative R -algebras (Definition B.0.13) A and H , and the data is equivalent to a Hopf algebroid structure (Definition 3.0.4) on the pair (A, H) .

APPENDIX A. SESQUILINEAR AND HERMITIAN FORMS

Definition A.0.1. Let \mathcal{C} be a category. An **involution on an object $X \in \text{Ob}(\mathcal{C})$** is an endomorphism $i : X \rightarrow X$ such that $i \circ i = \text{id}_X$.

Definition A.0.2. Let R be a unital associative ring. An **involution on R** is a ring homomorphism (Definition B.0.25) that is an involution (Definition A.0.1), i.e. a map $\sigma : R \rightarrow R$ satisfying the following conditions for all $a, b \in R$:

1. $\sigma(a + b) = \sigma(a) + \sigma(b)$;
2. $\sigma(ab) = \sigma(b)\sigma(a)$ (anti-homomorphism property);
3. $\sigma(1) = 1$;
4. $\sigma(\sigma(a)) = a$.

In many contexts, an involution may simply be notated by $\cdot \mapsto \bar{\cdot}$. A **ring with involution** is a ring equipped with an involution.

Definition A.0.3. Let (R, σ) be a ring with involution (Definition A.0.2). Let M be a left R -module. A **σ -sesquilinear form** (or simply **sesquilinear form**) on M is a map $\phi : M \times M \rightarrow R$ such that for all $u, v, w \in M$ and $a \in R$:

1. $\phi(u + v, w) = \phi(u, w) + \phi(v, w)$ and $\phi(u, v + w) = \phi(u, v) + \phi(u, w)$;
2. $\phi(au, v) = a\phi(u, v)$ (linear in the first variable);
3. $\phi(u, av) = \phi(u, v)\sigma(a)$ (conjugate-linear in the second variable).

Definition A.0.4. Let (R, σ) be a ring with involution (Definition A.0.2) and M a left R -module (Definition B.0.20). A *hermitian form on M* is a σ -sesquilinear form (Definition A.0.3) $\phi : M \times M \rightarrow R$ satisfying the symmetry condition:

$$\phi(v, u) = \sigma(\phi(u, v))$$

for all $u, v \in M$. The pair (M, ϕ) is often called a *hermitian module*.

Definition A.0.5. Let (R, σ) be a ring with involution (Definition A.0.2). Let N be a left R -module (Definition B.0.20). The *twisted dual module*, denoted N^\vee or $\text{Hom}_R(N, R)_\sigma$, is the set of all R -linear maps (Definition B.0.26) $f : N \rightarrow R$, equipped with the left R -module structure given by

$$(r \cdot f)(n) = f(n)\sigma(r)$$

for all $r \in R, f \in N^\vee, n \in N$.

Definition A.0.6. Let (R, σ) be a ring with involution (Definition A.0.2) and M a left R -module (Definition B.0.20) equipped with a sesquilinear form (Definition A.0.3) $\phi : M \times M \rightarrow R$. The form ϕ is *nondegenerate* if the induced adjoint map

$$\hat{\phi} : M \rightarrow M^\vee, \quad u \mapsto (v \mapsto \phi(v, u))$$

is injective.

Definition A.0.7. Let (R, σ) be a ring with involution and M a left R -module equipped with a sesquilinear form $\phi : M \times M \rightarrow R$. The *adjoint map of ϕ* is the morphism of R -modules

$$\hat{\phi} : M \rightarrow M^\vee, \quad u \mapsto (v \mapsto \phi(v, u))$$

where $M^\vee = \text{Hom}_R(M, R)_\sigma$ is the twisted dual module (Definition A.0.5).

Proposition A.0.8. Let (R, σ) be a ring with involution (Definition A.0.2) and $\phi : M \times M \rightarrow R$ a sesquilinear form (Definition A.0.3). If M is a finitely generated (Definition B.0.23) projective (Definition B.0.24) R -module, then the adjoint map $\hat{\phi} : M \rightarrow \text{Hom}_R(M, R)_\sigma$ is an isomorphism if and only if ϕ is non-degenerate (Definition A.0.6).

Definition A.0.9. Let (\mathcal{S}, τ) be a site (Definition B.0.21) equipped with a Grothendieck topology. Let \mathcal{O} be a sheaf (Definition B.0.22) of rings on \mathcal{S} . An *involution on \mathcal{O}* is a morphism of sheaves of rings $\sigma : \mathcal{O} \rightarrow \mathcal{O}^{\text{op}}$ such that $\sigma \circ \sigma = \text{id}_{\mathcal{O}}$, i.e. an involution (Definition A.0.1) in the category of sheaves of rings on \mathcal{S} .

Definition A.0.10. Let (\mathcal{S}, τ) be a site (Definition B.0.21). Let (\mathcal{O}, σ) be a sheaf of rings (Definition B.0.22) on \mathcal{S} with involution (Definition A.0.9) $\sigma : \mathcal{O} \rightarrow \mathcal{O}$. Let \mathcal{E} be a sheaf of left \mathcal{O} -modules (Definition B.0.27) on \mathcal{S} .

A *sesquilinear form on \mathcal{E}* is a morphism of sheaves (Definition B.0.22) of abelian groups

$$\phi : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{O}$$

such that for every object $U \in \mathcal{S}$, the induced map on sections

$$\phi_U : \mathcal{E}(U) \times \mathcal{E}(U) \rightarrow \mathcal{O}(U)$$

is a σ_U -sesquilinear form (Definition A.0.3) on the $\mathcal{O}(U)$ -module $\mathcal{E}(U)$, where $\sigma_U : \mathcal{O}(U) \rightarrow \mathcal{O}(U)$ is the involution (Definition A.0.2) on global sections.

Definition A.0.11. Let (\mathcal{O}, σ) be a sheaf (Definition B.0.22) of rings with involution (Definition A.0.9) on a site (Definition B.0.21) \mathcal{S} . Let \mathcal{E} be a sheaf of left \mathcal{O} -modules (Definition B.0.27).

A *hermitian form on \mathcal{E}* is a morphism of sheaves of sets

$$\phi : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{O}$$

such that for every object U in \mathcal{S} , the map on sections

$$\phi_U : \mathcal{E}(U) \times \mathcal{E}(U) \rightarrow \mathcal{O}(U)$$

is a hermitian form (Definition A.0.4) on the $\mathcal{O}(U)$ -module $\mathcal{E}(U)$ with respect to the involution (Definition A.0.2) $\sigma_U : \mathcal{O}(U) \rightarrow \mathcal{O}(U)$.

APPENDIX B. MISCELLANEOUS DEFINITIONS

Definition B.0.1 (Affine scheme). Let A be a commutative ring with unity (Definition B.0.12). Define the set $\text{Spec}(A)$ to be the set of all prime ideals of A . Equip it with the *Zariski topology*, which is the topology whose closed sets are given by *vanishing loci*

$$V(I) = \{\mathfrak{p} \in \text{Spec}(A) : I \subseteq \mathfrak{p}\}$$

for ideals $I \subseteq A$. Define the sheaf $\mathcal{O}_{\text{Spec}(A)}$, called the *structure sheaf of $\text{Spec } A$* , by

$$\mathcal{O}_{\text{Spec}(A)}(U) = \{ \text{locally defined fractions of elements of } A \text{ on } U \},$$

for each open set $U \subseteq \text{Spec}(A)$. It is the case that the stalk at $\mathfrak{p} \in \text{Spec}(A)$ is canonically the localization $A_{\mathfrak{p}}$. Then $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$ is a locally ringed space, called the *affine scheme associated to A* .

Moreover, given $f \in A$, we define the locus $D(f)$ by

$$D(f) = \text{Spec } A \setminus V((f)) = \{\mathfrak{p} \in \text{Spec } A : f \notin \mathfrak{p}\}$$

Definition B.0.2 (Scheme). A *scheme* is a locally ringed space (X, \mathcal{O}_X) that admits an open cover $\{U_i\}_{i \in I}$ such that each $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic (as a locally ringed space) to an affine scheme $(\text{Spec}(A_i), \mathcal{O}_{\text{Spec}(A_i)})$ (Definition B.0.1) for some commutative ring A_i . In other words, a scheme is a locally ringed space locally isomorphic to affine schemes.

Definition B.0.3 (Additive category). Let \mathcal{A} be a locally small category (Definition B.0.16).

1. \mathcal{A} is said to be a *preadditive category* if the following hold:
 - For any two objects A, B in \mathcal{A} , the set $\text{Hom}_{\mathcal{A}}(A, B)$ is an abelian group, and composition of morphisms is bilinear.
 - There is a zero object 0 in \mathcal{A} .
2. If \mathcal{A} is preadditive, then it is called *additive* if it additionally satisfies the following:
 - For any two objects A, B in \mathcal{A} , there exists a product object $A \times B$, often written $A \oplus B$, called the *direct sum of A and B* . In fact, $A \oplus B$ is not only a product but also a coproduct of A and B .

Given a finite collection $\{A_i\}_i$ of objects A_i in an additive category \mathcal{A} , we may more generally speak of the **direct sum** $\bigoplus_i A_i$; it has canonical injections from and projections to each A_i .

Definition B.0.4. Let \mathcal{C} be an additive category (Definition B.0.3). An object $X \in \text{Ob}(\mathcal{C})$ is called **semisimple** if it is isomorphic to a finite direct sum (Definition B.0.3) of simple objects in \mathcal{C} .

Definition B.0.5. An additive category (Definition B.0.3) \mathcal{C} is called a **semisimple category** if every object of \mathcal{C} is semisimple (Definition B.0.4).

Definition B.0.6. Let C be a category enriched in a monoidal category \mathcal{V} . Given an object X of C , the **functor of points** h_X is the functor/presheaf $C^{\text{op}} \rightarrow \mathcal{V}$ given by $T \mapsto \text{Hom}_C(T, X)$. A functor $C^{\text{op}} \rightarrow \mathcal{V}$ (or equivalently, a presheaf on C valued in \mathcal{V}) is said to be **representable** if it is naturally isomorphic to some functor h_X of points for an object X of C .

Dually, a functor $C \rightarrow \mathcal{V}$ is called **co-representable** if it is naturally isomorphic to a functor $T \mapsto \text{Hom}_C(X, T)$ for an object X in C .

For instance, we may speak of these notions when \mathcal{V} is the monoidal category **Sets**, i.e. C is a locally small category (Definition B.0.16).

Definition B.0.7. (♠ TODO: make this definition be for more general presheaves valued in more general caetgories) Let S be a scheme and let $f : F \rightarrow G$ be a morphism of presheaves valued in sets on Sch/S . We say that f is **representable by schemes** if for every scheme T over S and every morphism $h_T \rightarrow G$ of sheaves (i.e., every T -point of G), the fiber product

$$h_T \times_G F$$

is representable (Definition B.0.6) by a scheme over T , i.e. there exists a scheme X over T such that $h_T \times_G F \cong h_X$ as presheaves.

Definition B.0.8. (♠ TODO: TODO: read) (♠ TODO: TODO: state that a sheaf gives a stack) (♠ TODO: TODO: find out if it is equivalent to have U be an algebraic space instead.) Let S be a scheme (Definition B.0.2). Let τ be a Grothendieck topology (Definition B.0.21) on Sch/S . Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of stacks in groupoids over the site $(Sch/S)_\tau$. We say that f is **representable by algebraic spaces** if for every scheme U over S and every 1-morphism $y : U \rightarrow \mathcal{Y}$ (i.e., every object $y \in \mathcal{Y}(U)$), the 2-fiber product

$$\mathcal{X} \times_{\mathcal{Y}} U$$

is equivalent (as a stack over $(Sch/U)_\tau$) to an algebraic space over U (Definition B.0.9).

Definition B.0.9. Let S be a scheme. Let τ be a Grothendieck topology (Definition B.0.21) on Sch/S (Definition B.0.7) for which surjectivity and étaleness are preserved under base change and τ -local on the base. An **algebraic space over S for the topology τ** is a sheaf F of sets for $(Sch/S)_\tau$ such that

1. The diagonal morphism $F \rightarrow F \times F$ is representable by schemes (Definition B.0.7);
2. There exists a scheme U over S and a morphism of sheaves $h_U \rightarrow F$ which is surjective and étale.

Definition B.0.10. Let $f : X \rightarrow Y$ be a morphism of schemes. We say that f is an **affine morphism** if for every affine open $V = \operatorname{Spec} B \subseteq Y$, the preimage $U = f^{-1}(V)$ is an affine scheme (Definition B.0.1).

Definition B.0.11 (Scheme over a scheme). Let (S, \mathcal{O}_S) be a scheme. A **scheme over S** (or an **S -scheme**) is a scheme (X, \mathcal{O}_X) together with a morphism of schemes

$$\pi : (X, \mathcal{O}_X) \rightarrow (S, \mathcal{O}_S).$$

This morphism π is called the **structure morphism of the scheme X over S** .

If $S = \operatorname{Spec}(R)$ is an affine scheme for a commutative ring R , then an S -scheme is synonymously called an **R -scheme** or a **scheme over R** .

Let (S, \mathcal{O}_S) be a scheme, and let (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) be schemes over S with structure morphisms

$$\pi_X : X \rightarrow S, \quad \pi_Y : Y \rightarrow S.$$

A **morphism of S -schemes** (or synonymously a **S -scheme morphism**) is a morphism of schemes

$$(f, f^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$$

such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \pi_X \downarrow & & \downarrow \pi_Y \\ S & = & S \end{array}$$

In other words,

$$\pi_Y \circ f = \pi_X.$$

Given a fixed scheme S , there is a category, often denoted by **Sch $_S$** , **Sch $_{/S}$** , **Sch/ S** , **Sch $_S$** , **Sch $_{/S}$** , **Sch/ S** etc. whose objects are schemes T over S and whose morphisms $T_1 \rightarrow T_2$ are morphisms of schemes over S . If $S = \operatorname{Spec} R$ for some commutative ring R , then we may instead write **Sch $_R$** to denote $\operatorname{Sch}_{\operatorname{Spec} R}$, etc. It is noteworthy that $\operatorname{Sch}_{\mathbb{Z}}$ coincides with the category Sch of all schemes. In other words, a \mathbb{Z} -scheme can be identified simply with a scheme.

Definition B.0.12. A **commutative (unital) ring** is a ring $(R, +, \cdot)$ such that \cdot is a commutative operation, i.e. $a \cdot b = b \cdot a$.

For many writers (e.g. “commutative” algebraists or number theorists), a **ring** refers to a commutative ring as above.

Definition B.0.13. Let R be a (not-necessarily commutative) ring with unity. An **R -algebra** is a ring A together with a ring homomorphism (Definition B.0.25)

$$\varphi : R \rightarrow A$$

into the center $Z(A)$ of A (so that $\varphi(r)$ commutes with every element of A for all $r \in R$), such that $\varphi(1_R) = 1_A$. The ring homomorphism φ is called the **structure map** of the algebra.

Equivalently, an R -algebra consists of a ring A endowed with a two-sided R -module (Definition B.0.20) structure for which the scalar multiplication satisfies

$$r \cdot (ab) = (r \cdot a)b = a(r \cdot b) \quad \text{for all } r \in R, a, b \in A.$$

In particular, any ring homomorphism between commutative rings (Definition B.0.12) specifies an algebra structure.

Definition B.0.14. Let \mathcal{C} be a category (Definition B.0.15), let Z be an object, and let X, Y be objects of \mathcal{C} over Z , i.e. morphisms $X \rightarrow Z$ and $Y \rightarrow Z$ are fixed. A **cartesian product of X and Y over Z in \mathcal{C}** (or **fiber product** or **pullback diagram**) is an object, often denoted by $X \times_Z Y$, with **projection morphisms** $X \times_Z Y \rightarrow X$ and $X \times_Z Y \rightarrow Y$ that are universal. More precisely, for any object T of \mathcal{C} and morphisms $f_X : T \rightarrow X$, $f_Y : T \rightarrow Y$, there exists a unique morphism $u : T \rightarrow X \times_Z Y$ such that the following diagram commutes:

$$\begin{array}{ccccc} T & & & & \\ & \searrow^{f_X} & & & \\ & & X \times_Z Y & \longrightarrow & X \\ & \swarrow_{f_Y} & \downarrow & & \downarrow \\ & & Y & \longrightarrow & Z \end{array}$$

(Note: In the original image, a dotted arrow labeled u points from T to $X \times_Z Y$.)

Equivalently, $X \times_Z Y$ is the limit of the diagram

$$\begin{array}{ccc} & X & \\ & \downarrow & \\ Y & \longrightarrow & Z \end{array}$$

in \mathcal{C} .

The commutative diagram

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

may be referred to as a **cartesian square**.

Definition B.0.15 (Category). A **category** \mathcal{C} consists of the following data:

- A class of **objects** denoted $\text{Ob}(\mathcal{C})$.
- For each pair of objects $X, Y \in \text{Ob}(\mathcal{C})$, a class

$$\text{Hom}_{\mathcal{C}}(X, Y)$$

of *morphisms* (also called *arrows* or *homs*). If the category \mathcal{C} is clear, then this *hom-class* is also denoted by $\text{Hom}(X, Y)$. It may also be denoted by $\text{hom}_{\mathcal{C}}(X, Y)$ or $\text{hom}(X, Y)$, especially to distinguish from other types of hom's (e.g. internal hom's)

- For each triple of objects X, Y, Z , a composition law

$$\circ : \text{Hom}_{\mathcal{C}}(Y, Z) \times \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{C}}(X, Z),$$

denoted $(g, f) \mapsto g \circ f$.

- For each object X , an *identity morphism*

$$\text{id}_X \in \text{Hom}_{\mathcal{C}}(X, X).$$

These data satisfy the following axioms:

- (Associativity) For all morphisms $f \in \text{Hom}_{\mathcal{C}}(X, Y)$, $g \in \text{Hom}_{\mathcal{C}}(Y, Z)$, and $h \in \text{Hom}_{\mathcal{C}}(Z, W)$,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

- (Identity) For all $f \in \text{Hom}_{\mathcal{C}}(X, Y)$,

$$\text{id}_Y \circ f = f = f \circ \text{id}_X.$$

One often writes $X \in \mathcal{C}$ synonymously with $X \in \text{Ob}(\mathcal{C})$, i.e. to denote that X is an object of \mathcal{C} .

We may call a category as above an *ordinary category* to distinguish this notion from the notions of *categories enriched in monoidal categories* or higher/ n -categories. (♠ TODO: define n -categories)

A category as defined above may be called called a *large category* or a *class category* to emphasize that the hom-classes may be proper classes rather than sets (note, however, that the possibility that hom-classes are sets is not excluded for large categories). Accordingly, a *category* may often refer to a locally small category (Definition B.0.16), which is a category whose hom-classes are all sets.

Definition B.0.16 (Locally small category). A (large) category (Definition B.0.15) \mathcal{C} is called a *locally small category* if for every pair of objects $X, Y \in \text{Ob}(\mathcal{C})$, the collection $\text{Hom}_{\mathcal{C}}(X, Y)$ of morphisms between them is a (small) *set* (as opposed to a proper class). In other words, each hom-class is a set and may even be called a *hom-set*.

In some contexts, a locally small category may simply be called a *category*, especially when genuinely large categories are not considered.

A category \mathcal{C} is called a *small category* if it is a locally small category and the class $\text{Ob}(\mathcal{C})$ of objects is a set.

Remark B.0.17. Many “concrete” categories considered in “classical mathematics” or outside of more “abstract” category theory tend to be locally small. For example, the categories of sets, groups, R -modules, vector spaces, topological spaces, schemes, manifolds, sheaves on “small enough” sites are all locally small.

Definition B.0.18. Let (\mathcal{C}, J) be a site (Definition B.0.21). Let \mathcal{O} be a sheaf of commutative rings on (\mathcal{C}, J) (Definition B.0.22), i.e., $((\mathcal{C}, J), \mathcal{O})$ is a ringed site.

1. An **\mathcal{O} -algebra** consists of the following data:
 - A sheaf \mathcal{A} of (not necessarily commutative) rings on (\mathcal{C}, J) ,
 - A morphism of sheaves of rings $\eta : \mathcal{O} \rightarrow \mathcal{A}$ such that for every object $U \in \mathcal{C}$, the image of $\eta_U : \mathcal{O}(U) \rightarrow \mathcal{A}(U)$ is contained in the center of $\mathcal{A}(U)$.

This makes $\mathcal{A}(U)$ an $\mathcal{O}(U)$ -algebra for every $U \in \mathcal{C}$, such that for every morphism $f : V \rightarrow U$ in \mathcal{C} , the restriction map

$$\rho_{U,V} : \mathcal{A}(U) \rightarrow \mathcal{A}(V)$$

is a homomorphism of $\mathcal{O}(U)$ -algebras (where the $\mathcal{O}(U)$ -algebra structure on $\mathcal{A}(V)$ is induced via restriction $\mathcal{O}(U) \rightarrow \mathcal{O}(V)$).

2. Let \mathcal{A} and \mathcal{B} be \mathcal{O} -algebras (Definition B.0.18).

A **morphism of \mathcal{O} -algebras** $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is a morphism of sheaves (Definition B.0.22) of rings such that, for every object $U \in \mathcal{C}$, the component map

$$\varphi_U : \mathcal{A}(U) \rightarrow \mathcal{B}(U)$$

is a homomorphism of $\mathcal{O}(U)$ -algebras, i.e., it is a ring homomorphism that commutes with the structure maps $\eta_{\mathcal{A}}$ and $\eta_{\mathcal{B}}$:

$$\varphi_U(\eta_{\mathcal{A},U}(r)) = \eta_{\mathcal{B},U}(r) \quad \text{for all } r \in \mathcal{O}(U).$$

The collection of all \mathcal{O} -algebras together with their morphisms forms the **category of \mathcal{O} -algebras**, denoted by notations such as **$\mathbf{Alg}(\mathcal{O})$** .

Definition B.0.19. Let $((\mathcal{C}, J), \mathcal{O})$ be a ringed site, where \mathcal{O} is a sheaf (Definition B.0.22) of rings on the site (\mathcal{C}, J) .

1. Let I be an indexing set. The **free sheaf of \mathcal{O} -modules of rank I** (or simply a **free sheaf**), denoted by $\mathcal{O}^{\oplus I}$, is the sheaf associated to the presheaf $U \mapsto \mathcal{O}(U)^{\oplus I}$. If I is finite with cardinality n , we usually write $\mathcal{O}^{\oplus n}$.
2. An \mathcal{O} -module (Definition B.0.27) \mathcal{F} is called **locally free of rank n** (for an integer $n \geq 0$) if there exists a covering $\{U_i \rightarrow X\}_{i \in I}$ in the topology (Definition B.0.21) J (where X is the terminal object of \mathcal{C} if it exists, or more generally this condition holds restricted to each object of \mathcal{C} for a suitable cover) such that for each i , the restriction $\mathcal{F}|_{U_i}$ is isomorphic to the free sheaf $(\mathcal{O}|_{U_i})^{\oplus n}$ as an $\mathcal{O}|_{U_i}$ -module.

We might call a locally free \mathcal{O} -module of rank n an **algebraic vector bundle of rank n** . A **(algebraic) line bundle** or **invertible sheaf** is then an algebraic vector bundle of rank 1.

Definition B.0.20. Let R be a not-necessarily commutative ring.

1. A **left R -module** is an abelian group $(M, +)$ together with an operation $R \times M \rightarrow M$, denoted $(r, m) \mapsto rm$, such that for all $r, s \in R$ and $m, n \in M$:
 - $r(m + n) = rm + rn$,
 - $(r + s)m = rm + sm$,
 - $(rs)m = r(sm)$,
 - $1_R m = m$ where 1_R is the multiplicative identity of R .

2. A **right R -module** is defined similarly as an abelian group $(M, +)$ with an operation $M \times R \rightarrow M$, denoted $(m, r) \mapsto mr$, such that for all $r, s \in R$ and $m, n \in M$:
 - $(m + n)r = mr + nr$,
 - $m(r + s) = mr + ms$,
 - $m(rs) = (mr)s$,
 - $m1_R = m$.

3. Let R and S be (not necessarily commutative) rings.

An **R - S -bimodule** (or an **R - S -module** or an (R, S) -module, etc.) is an abelian group $(M, +)$ equipped with

- (a) a left action of R :

$$R \times M \rightarrow M, \quad (r, m) \mapsto r \cdot m,$$

making M a left R -module (Definition B.0.20),

- (b) a right action of S :

$$M \times S \rightarrow M, \quad (m, s) \mapsto m \cdot s,$$

making M a right S -module,

such that the left and right actions commute; that is, for all $r \in R$, $s \in S$, and $m \in M$,

$$r \cdot (m \cdot s) = (r \cdot m) \cdot s.$$

4. A **two-sided R -module** (or **R -bimodule**) is an R - R -bimodule.

If R is a commutative ring (Definition B.0.12), then a left/right R -module can automatically be regarded as a two-sided R -module. As such, we simply talk about **R -modules** in this case.

Any abelian group is equivalent to a two-sided \mathbb{Z} -module. Moreover, any left R -module is equivalent to an $R - \mathbb{Z}$ -bimodule (Definition B.0.20) and any right R -module is equivalent to an $\mathbb{Z} - R$ -bimodule (Definition B.0.20). Given a left/right/two-sided R -module, its **natural bimodule structure** will refer to its structure as a $R - \mathbb{Z} / \mathbb{Z} - R / R - R$ bimodule. In this way, many definitions associated with the notions of left/right/two-sided R -modules can be defined as special cases for definitions for R - S -bimodules.

Definition B.0.21 (Grothendieck topology). Let \mathcal{U} be a universe and let \mathcal{C} be a locally small category (Definition B.0.16).

1. **(Grothendieck Topology via Sieves)** A **Grothendieck topology** J on \mathcal{C} is an assignment to each object $U \in \mathcal{C}$ of a collection $J(U)$ of sieves on U , called **covering sieves**, satisfying:
 - (a) (Maximality) The maximal sieve $\{f : V \rightarrow U \mid V \in \mathcal{C}\}$ is in $J(U)$.
 - (b) (Stability) If $S \in J(U)$ and $f : V \rightarrow U$ is any morphism, then the pullback sieve f^*S is in $J(V)$.
 - (c) (Transitivity/Local Character) If S is a sieve on U and there exists a covering sieve $R \in J(U)$ such that for every morphism $f : V \rightarrow U$ in R , the pullback sieve f^*S is in $J(V)$, then $S \in J(U)$.
2. A **site** is a pair (\mathcal{C}, J) consisting of a category \mathcal{C} and a Grothendieck topology J .
3. A family of objects $\mathcal{G} = \{G_\alpha\}$ in a site (\mathcal{C}, J) is called a **topologically generating family** if for every object $X \in \mathcal{C}$, there exists a covering sieve $S \in J(X)$ generated by

morphisms with domains in \mathcal{G} . Equivalently, every object X admits a cover $\{U_i \rightarrow X\}$ where each $U_i \in \mathcal{G}$.

4. A \mathcal{U} -site is a site whose underlying category is \mathcal{U} -locally small and which admits a \mathcal{U} -small topologically generating family.

Definition B.0.22 (Sheaf on a site). (♠ TODO: There might be some need to say that \mathcal{A} is a category for which sheaves on the site “can be defined”) (♠ TODO: go through statements using the notion of sheaves and make sure that the value categories have small products and that the categories have small generating families.) Let (\mathcal{C}, J) be a site (Definition B.0.21) with a small topological generating family (Definition B.0.21) (or a U -small topologically generating family if a universe U is available) and let \mathcal{A} be a (large) category (Definition B.0.15) that has all small (Definition B.0.16) products (Some common examples of categories that have small products and thus play the role of \mathcal{A} here include $\mathcal{A} = \mathbf{Set}, \mathbf{Ab}, R\text{-}\mathbf{mod}$ for a fixed ring R , rings).

1. A presheaf $\mathcal{F} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{A}$ is a *sheaf on the site (\mathcal{C}, J) valued in \mathcal{A}* if, for every object U of \mathcal{C} and every covering $\{U_i \rightarrow U\}_{i \in I}$ in J , the following sequence is an equalizer in \mathcal{A} :

$$\mathcal{F}(U) \rightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$$

where the first map sends s to $(\mathcal{F}(U_i \rightarrow U)(s))_i$ and the arrows to $(\mathcal{F}(U_i \times_U U_j \rightarrow U_i)(s_i))_{i,j}$ and $(\mathcal{F}(U_i \times_U U_j \rightarrow U_j)(s_j))_{i,j}$, respectively.

2. A *morphism of sheaves* $\mathcal{F} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{A}$ is a morphism as presheaves.
3. Let U be a universe. A *U -sheaf* typically refers to a U -presheaf that is a sheaf for a U -site. In other words, a U -sheaf is a sheaf on a site whose underlying category is U -locally small (Definition B.0.16) and which has a U -small topologically generating family such that the sheaf is valued in U -sets.
4. The *sheaf category/category of \mathcal{A} -valued sheaves on \mathcal{C}* is the (large) category defined as the full subcategory of $\mathbf{PreShv}(\mathcal{C}, \mathcal{A})$ whose objects are the sheaves on \mathcal{C} with values in \mathcal{A} . Common notations for the sheaf category include $\mathbf{Shv}(\mathcal{C}, \mathcal{A})$, $\mathbf{Shv}(\mathcal{C}, J, \mathcal{A})$, $\mathbf{Sh}(\mathcal{C}, \mathcal{A})$, $\mathbf{Sh}(\mathcal{C}, J, \mathcal{A})$. If the value category \mathcal{A} is clear from context, then notations such as $\mathbf{Shv}(\mathcal{C})$, $\mathbf{Shv}(\mathcal{C}, J)$, $\mathbf{Sh}(\mathcal{C})$, $\mathbf{Sh}(\mathcal{C}, J)$ are also common.

Definition B.0.23 (Finitely generated modules and bimodules). Let R and S be (not necessarily commutative) rings.

1. An R - S -bimodule M is *finitely generated* if it has a finite spanning set.
2. A left/right/two-sided R -module is *finitely generated* if has a finite spanning set, or equivalently if its natural bimodule structure (Definition B.0.20) is finitely generated.

Definition B.0.24. Let R and S be (not necessarily commutative) rings. A *projective R - S -bimodule* is an (R, S) -bimodule (Definition B.0.20) P that satisfies any of the following equivalent conditions:

1. The functor

$$\mathbf{Hom}_{R\mathbf{Mod}_S}(P, -) : R\mathbf{Mod}_S \rightarrow \mathbf{Ab}$$

is an exact functor between the abelian categories $R\mathbf{Mod}_S$ and \mathbf{Ab} .

2. P is a projective left module over the ring $R \otimes_{\mathbb{Z}} S^{\text{op}}$.

3. P is a direct summand of a free (R, S) -bimodule. (A free (R, S) -bimodule is a direct sum of copies of the tensor product $R \otimes_{\mathbb{Z}} S$, equipped with the natural left R -action and right S -action).
4. P is a projective object in the category ${}_R\mathbf{Mod}_S$. That is, for every surjective homomorphism of (R, S) -bimodules $f : M \rightarrow N$ and every homomorphism $g : P \rightarrow N$, there exists a homomorphism $h : P \rightarrow M$ such that $f \circ h = g$.

Being a projective bimodule is a strictly stronger condition than being projective as a left or right module.

- A bimodule ${}_R P_S$ may be projective as a left R -module (i.e., projective in ${}_R\mathbf{Mod}$) without being a projective bimodule.
- Similarly, it may be projective as a right S -module (i.e., projective in \mathbf{Mod}_S) without being a projective bimodule.
- A bimodule that is projective on both sides is sometimes called *biprojective*, but this does not imply it is a projective object in ${}_R\mathbf{Mod}_S$. For example, if $R = S = \mathbb{Z}$, the bimodule \mathbb{Z} is free (hence projective) on both sides, but it is *not* a projective (\mathbb{Z}, \mathbb{Z}) -bimodule because \mathbb{Z} is not a projective $\mathbb{Z}[\mathbb{Z}]$ -module (the augmentation ideal is not projective).

Definition B.0.25. Let $(R, +, \cdot)$ and $(S, +, \cdot)$ be rings, not assumed to be commutative. A function $f : R \rightarrow S$ is called a *ring homomorphism* if for all $r_1, r_2 \in R$ the following properties hold:

1. $f(r_1 + r_2) = f(r_1) + f(r_2)$,
2. $f(r_1 r_2) = f(r_1) f(r_2)$,
3. $f(1_R) = 1_S$ where 1_R and 1_S denote the multiplicative identities in R and S , respectively.

A ring homomorphism is said to be a *ring isomorphism* if it is invertible as a map of sets.

An *R -ring* refers to a ring S equipped with a ring homomorphism $f : R \rightarrow S$.

We note that a ring homomorphism $f : R \rightarrow S$ yields a natural left R -module (Definition B.0.20) structure on S and a natural right R -module structure on S respectively as follows for $r \in R$ and $s \in S$:

$$r \cdot s = f(r) \cdot s$$

$$s \cdot r = s \cdot f(r).$$

However, these left and right module structures need not yield a two-sided R -module structure.

Definition B.0.26. Let R, S be (not-necessarily commutative) rings.

1. Let M and N be R - S -bimodules (Definition B.0.20). A function $\varphi : M \rightarrow N$ is called an *R - S -bimodule homomorphism* or *R - S -linear* if it is a group homomorphism of the underlying abelian groups of M and N and respects the scalar actions as follows: for

all $m_1, m_2 \in M$, $r \in R$, and $s \in S$,

$$\varphi(r \cdot m_1) = r \cdot \varphi(m_1),$$

$$\varphi(m_1 \cdot s) = \varphi(m_1) \cdot s.$$

2. Let M and N be left/right/two-sided R -modules (Definition B.0.20). A function $\varphi : M \rightarrow N$ is called a *left/right/two-sided R -module homomorphism* if it is an bimodule homomorphism on the natural bimodule structures (Definition B.0.20) of M and N . Such a function is also called *R -linear*.

Modules and homomorphisms of a fixed type (i.e. R - S -bimodules or left/right/two-sided R -modules) form a locally small (Definition B.0.16) category (Definition B.0.15).

Definition B.0.27. 1. Let \mathcal{C} be a site (Definition B.0.21), and let \mathcal{A} and \mathcal{B} be sheaves (Definition B.0.22) of (not necessarily commutative) rings on \mathcal{C} .

- (a) An *$(\mathcal{A}, \mathcal{B})$ -bimodule* (or a *bimodule over $(\mathcal{A}, \mathcal{B})$*) is a sheaf (Definition B.0.22) \mathcal{M} of abelian groups on \mathcal{C} equipped with a left \mathcal{A} -module structure given by a morphism of sheaves (Definition B.0.22) of sets

$$\lambda : \mathcal{A} \times \mathcal{M} \longrightarrow \mathcal{M},$$

and a right \mathcal{B} -module structure given by a morphism of sheaves of sets

$$\rho : \mathcal{M} \times \mathcal{B} \longrightarrow \mathcal{M},$$

such that the actions are compatible. Specifically, for every object U in \mathcal{C} , every section $m \in \mathcal{M}(U)$, every $a \in \mathcal{A}(U)$, and every $b \in \mathcal{B}(U)$, the equality

$$\lambda_U(a, \rho_U(m, b)) = \rho_U(\lambda_U(a, m), b)$$

holds in $\mathcal{M}(U)$. In standard multiplicative notation where $\lambda(a, m)$ is denoted $a \cdot m$ and $\rho(m, b)$ is denoted $m \cdot b$, this condition is the associativity axiom

$$(a \cdot m) \cdot b = a \cdot (m \cdot b).$$

In particular, for every object $U \in \mathcal{C}$, the abelian group $\mathcal{M}(U)$ has the structure of an $\mathcal{A}(U) - \mathcal{B}(U)$ -bimodule (Definition B.0.20).

- (b) Let \mathcal{M} and \mathcal{N} be $(\mathcal{A}, \mathcal{B})$ -bimodules. A *homomorphism of $(\mathcal{A}, \mathcal{B})$ -bimodules* (or an *$(\mathcal{A}, \mathcal{B})$ -linear morphism*) is a morphism of sheaves of abelian groups $f : \mathcal{M} \rightarrow \mathcal{N}$ such that for every object U of \mathcal{C} , every section $m \in \mathcal{M}(U)$, every $a \in \mathcal{A}(U)$, and every $b \in \mathcal{B}(U)$, the following compatibility conditions hold:

$$f_U(a \cdot m) = a \cdot f_U(m) \quad \text{and} \quad f_U(m \cdot b) = f_U(m) \cdot b.$$

We denote the category of $(\mathcal{A}, \mathcal{B})$ -bimodules, with morphisms being morphisms of sheaves of abelian groups compatible with both the left \mathcal{A} -action and the right \mathcal{B} -action, by $\mathcal{A}\text{-}\mathcal{B}\text{-Mod}$ or sometimes by ${}_{\mathcal{A}}\text{Mod}_{\mathcal{B}}$ (♠ TODO: talk about how bimodules can be identifies with left/right modules)

2. Let (\mathcal{C}, J) be a site (Definition B.0.21). Let \mathcal{O} be a sheaf of (not necessarily commutative) rings on (\mathcal{C}, J) (Definition B.0.22), i.e. $((\mathcal{C}, J), \mathcal{O})$ is a ringed site.
 - (a) An *(left/right/two-sided) \mathcal{O} -module* consists of the following data:
 - A sheaf \mathcal{F} of abelian groups on (\mathcal{C}, J) ,
 - for every object $U \in \mathcal{C}$, the structure of an (left/right/two-sided) $\mathcal{O}(U)$ -module on $\mathcal{F}(U)$,

such that for every morphism $f : V \rightarrow U$ in \mathcal{C} , the restriction map

$$\rho_{U,V} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$$

is $\mathcal{O}(U)$ -linear when the $\mathcal{O}(U)$ -action on $\mathcal{F}(V)$ is defined via the natural ring homomorphism

$$\mathcal{O}(U) \rightarrow \mathcal{O}(V)$$

induced by f .

- (b) Let \mathcal{F} and \mathcal{G} be \mathcal{O} -modules (Definition B.0.27).

A **morphism of \mathcal{O} -modules** $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves (Definition B.0.22) of abelian groups such that, for every object $U \in \mathcal{C}$, the component map

$$\varphi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$$

is $\mathcal{O}(U)$ -linear, i.e. it satisfies

$$\varphi_U(r \cdot s) = r \cdot \varphi_U(s) \quad \text{for all } r \in \mathcal{O}(U), s \in \mathcal{F}(U).$$

The collection of all \mathcal{O} -modules together with their morphisms of \mathcal{O} -modules forms the **category of \mathcal{O} -modules**, denoted **Mod(\mathcal{O})**.

See also Definition B.0.18.

In case that a sheafification functor

$$\text{PreShv}(\mathcal{C}, \mathbf{Rings}) \rightarrow \text{Shv}(\mathcal{C}, \mathbf{Rings})$$

exists, a left, right, two-sided \mathcal{O} -module (and morphisms thereof) is equivalent to a $(\mathcal{O}, \mathbb{Z})$ -bimodule, $(\mathbb{Z}, \mathcal{O})$ -bimodule, and $(\mathcal{O}, \mathcal{O})$ -bimodule (and morphisms thereof) respectively, where \mathbb{Z} is the constant sheaf of the integer ring \mathbb{Z} .

Definition B.0.28. Let S be a scheme (Definition B.0.2) and let $n \geq 0$ be an integer. We define the **affine space of dimension n over S** , denoted by \mathbb{A}_S^n , as follows:

1. If $S = \text{Spec } A$ is an affine scheme (Definition B.0.1), then \mathbb{A}_S^n is the affine scheme defined by the polynomial ring in n variables over A :

$$\mathbb{A}_{\text{Spec } A}^n = \text{Spec}(A[T_1, \dots, T_n]).$$

2. For a general scheme S , let $\{U_i = \text{Spec } A_i\}_{i \in I}$ be an affine open covering of S . For each i , let $X_i = \mathbb{A}_{U_i}^n = \text{Spec}(A_i[T_1, \dots, T_n])$. Since polynomial rings behave well under localization, for any open immersion $U_{ij} = U_i \cap U_j \hookrightarrow U_i$, there is a canonical isomorphism on the overlaps:

$$\phi_{ij} : X_i|_{U_{ij}} \xrightarrow{\sim} X_j|_{U_{ij}}.$$

The scheme \mathbb{A}_S^n is obtained by gluing the family $\{X_i\}_{i \in I}$ along these isomorphisms.

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