GROTHENDIECK GROUPS OF TRIANGULATED CATEGORIES

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Contents

1. Grothendieck groups of general triangulated categories

[Gro77, Exposé VIII, 2] and [BGI71, Exposé I, 6.3 and Exposé IV, 1.1] each define additive functions on and Grothendieck groups of triangulated categories.

Definition 1.1. Let D be a triangulated category, and let G be an abelian group. A function $\operatorname{Ob} D \to G$ is said to be *additive* if for all $X, Y, Z \in \operatorname{Ob} D$ such that there is a distinguished triangle $X \to Y \to Z \to X[1]$, we have

$$f(Y) = f(X) + f(Z).$$

Write $\operatorname{Add}(D, F)$ for the abelian group of additive functions $\operatorname{Ob} D \to G$.

Claim 1.2. *The functor* $\operatorname{Add}(D, \cdot) : \mathrm{Ab} \to \mathrm{Ab}$ *is representable by an abelian group.*

Definition 1.3. Let D be a triangulated category. The *Grothendieck group* $k(D)$ is the abelian category representing the functor $\operatorname{Add}(D, \cdot) : \mathrm{Ab} \to \mathrm{Ab}$. It may be constructed as the quotient of the free abelian group generated by $\operatorname{Ob}(D)$ by the relations $Y = X + Z$ whenever there is a distinguished triangle $X \to Y \to Z \to X[1]$. There is a universal additive function $\mathrm{cl}_D : \operatorname{Ob}(D) \to k(D)$. When there is no confusion, we may also denote it by cl. Given an object $X \in \operatorname{Ob}(D)$, denote by $[X]$ the elemtn $\mathrm{cl}(X)$.

Lemma 1.4 ([Gro77, Exposé VIII 2])**.** *Let D be a triangulated category, let G be an abelian group, and let $f : D \to G$ be an additive function. We have the following:*

1. $f(0) = 0$.
2. $f(X[n]) = (-1)^f(X)$.
3. $f(X) = f(Y)$ *if X and Y are isomorphic.*
4. $f(X \oplus Y) = f(X) + f(Y)$.

Proof. These hold because the following triangles are distinguished:

$$X \xrightarrow{\mathrm{id}_X} X \to 0 \to X[1]$$
$$X \to 0 \to X[1] \to X[1]$$
$$X \xrightarrow{\cong} Y \to 0 \to X[1]$$
$$X \oplus Y \to X \to Y[1] \to (X \oplus Y)[1].$$

□

Definition 1.5. A functor $T : D_1 \to D_2$ between triangulated categories is called *exact* if it is additive, is translation preserving, and transforms distinguished triangles to distinguished triangles.

Proposition 1.6 ([Gro77, Exposé VIII, 3])**.** *Let D_1 and D_2 be two triangulated categories and let $T : D_1 \to D_2$ be an exact functor. There is a unique group homomorphism $k(T) : k(D_1) \to k(D_2)$ between the Grothendieck groups of D_1 and D_2 such that*

$$\begin{tikzcd} \mathrm{Ob}\, D_1 \arrow[r, "T"] \arrow[d, "\mathrm{cl}_{C_1}"] & \mathrm{Ob}\, D_2 \arrow[d, "\mathrm{cl}_{C_2}"] \\ k(C_1) \arrow[r, "k(T)"] & k(C_2) \end{tikzcd}$$

is commutative. Explicitly, $k(T)$ can be defined by $[X] \mapsto [TX]$ and extended linearly.

Proof. All that needs to be proven is that the proposed definition for $k(T)$ is well defined. In other words, we need to show that $k(T)$ sends relations on $k(C_1)$ to relations on $k(C_2)$. Suppose that there is a distinguished triangle $X \to Y \to Z \to X[1]$ in D_1 or equvialently that

$$[Y] = [X] + [Z]$$

in $k(C_1)$. Since T is assumed to be exact, there is a distinguished triangle $TX \to TY \to TZ \to TX[1]$ in D_2 and hence

$$[TY] = [TX] + [TZ]$$

in $k(C_2)$. □

2. Grothendieck groups of derived categories of abelian categories

Given an additive category A, write $K^b(A)$ for the bounded chain homotopy category of A; it is a triangulated category. Given an abelian category A, write $D^b(A)$ for the bounded derived category of A; it is also a triangulated category.

Lemma 2.1 ([BGI71, Exposé IV, Lemme 1.4])**.** *Let A be an additive (resp. abelian) category. Let f be an additive function on $\mathrm{Ob}\, K^b(A)$ (resp. $\mathrm{Ob}\, D^b(A)$). For any $E \in \mathrm{Ob}\, K^b(A)$ (resp.*

$\in \operatorname{Ob} D^b(A)$), *we have*

$$f(E) = \sum_i (-1)^i f(E^i)$$
$$f(E) = \sum_i (-1)^i f(H^i(E)).$$

References

[BGI71] Pierre Berthelot, Alexander Grothendieck, and Luc Illusie. *Théorie des Intersections et Théorème de Riemann-Roch (SGA6)*, volume 225 of *Lecture Notes in Mathematics*. Springer-Verlag, 1971.

[Gro77] Alexander Grothendieck. *Cohomologie l-adique et fonctions L Séminaire de Géométrie Algébrique due Bois-Marie 1965-1966 (SGA 5)*, volume 589 of *Springer Lecture Notes*. Springer-Verlag, 1977. Avec la collaboration de I. Bucur, C. Houzel, L. Illusie, J.-P. Jouanolou, et J.-P. Serre.