STABLE HOMOTOPY CATEGORIES AND STABLE HOMOTOPY GROUPS

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Contents

The purpose of these notes is to concisely define stable homotopy categories and to state their relationship to stable homotopy groups.

1. Definitions

All spaces are assumed to be compactly generated.

X, Y: (Compactly generated) spaces. Unless otherwise specified, they will be based spaces. By default, $*$ denotes the base point of a based space.

Definition 1.0.1. Let X and B be topological spaces (Definition A.0.1) and let $p : X \to B$ be a continuous map (Definition A.0.2).

The map p is called a *fibration* or a *Serre fibration* if it satisfies the *Homotopy Lifting Property (HLP)*: for every space Y, every map $f : Y \to X$, every homotopy $H : Y \times I \to B$ with $H(y, 0) = p(f(y))$ for all $y \in Y$, and whenever a lift $\tilde{f}_0 : Y \to X$ of $H(\cdot, 0)$ is given, there exists a homotopy $\tilde{H} : Y \times I \to X$ such that

$$\tilde{H}(y, 0) = \tilde{f}_0(y) \quad \text{for all } y \in Y, p \circ \tilde{H} = H \quad \text{on } Y \times I.$$

Equivalently, for all Y, the map

$$\mathrm{Map}(Y, X) \to \mathrm{Map}(Y, B), \quad f \mapsto p \circ f$$

has the right lifting property with respect to the inclusion $\mathrm{Map}(Y, *) \hookrightarrow \mathrm{Map}(Y \times I, B)$ encoded by the HLP.

Definition 1.0.2. Let X be a topological space and let $A \subseteq X$ be a subspace. The inclusion map $i : A \hookrightarrow X$ is called a *cofibration* if it satisfies the *Homotopy Extension Property (HEP)*: for every space Y, every map $f : A \to Y$ and every homotopy $H : X \times I \to Y$ with $H(a,t) = f(a)$ for all $a \in A$ and all $t \in I$, there exists a homotopy $\tilde{H} : X \times I \to Y$ extending H such that $\tilde{H}|_{A\times I} = H|_{A\times I}$ and $\tilde{H}(x,0) = f(x)$ for all $x \in A$. Equivalently, the inclusion $i : A \hookrightarrow X$ has the left lifting property with respect to every map that is a fibration.

Definition 1.0.3. A pointed topological space (Definition A.0.7) X is said to be *nondegenerately based* or *well pointed* if the inclusion $* \hookrightarrow X$ is a cofibration (Definition 1.0.2) in the unbased sense, i.e. the inclusion satisfies the homotopy extension property.

Definition 1.0.4. Let X and Y be topological spaces (Definition A.0.1).

1. Write $\mathrm{Open}(Y)$ for the topology (Definition A.0.1) of Y and $\mathrm{Comp}(X)$ for the family of compact subsets of X. For $K \in \mathrm{Comp}(X)$ and $U \in \mathrm{Open}(Y)$, define the subset

$$\langle K, U\rangle := \{\, f \in C(X,Y) \mid f(K) \subseteq U \,\} \subseteq C(X,Y).$$

   where $C(X,Y)$ (Definition A.0.2) is the set of all continuous maps (Definition A.0.2) $X \to Y$. The collection of all such subsets is denoted by $\mathcal{S}_{\mathrm{co}}(X,Y)$.
2. The *compact-open topology on $C(X,Y)$* (Definition A.0.2) is the topology (Definition A.0.1) generated by (Definition A.0.4) the subbasis (Definition A.0.5) $\mathcal{S}_{\mathrm{co}}(X,Y)$. The resulting topological space is called the *function space from X to Y* and is commonly denoted by notations such as $\mathrm{Map}(X,Y)$, Y^X, $\mathrm{Fun}(X,Y)$, or $\mathrm{F}(X,Y)$. As a set, it equals $C(X,Y)$.
3. Let (X,x_0) and (Y,y_0) be based topological spaces (Definition A.0.7). The *based function space from (X,x_0) to (Y,y_0)* is the subspace of $\mathrm{Map}(X,Y)$ consisting of all based maps f with $f(x_0) = y_0$, equivalently $C_*((X,x_0),(Y,y_0))$ endowed with the subspace topology inherited from the compact-open function space $\mathrm{Map}(X,Y)$. It itself is a based topological space whose base point is given by the constant map $X \to Y$ with value y_0. Common notations for the based function space include those which include a star or bullet such as $\mathrm{Map}_*((X,x_0),(Y,y_0))$, $\mathrm{Map}_*(X,Y)$, or Y_*^X to emphasize that the spaces involved are pointed, or those which omit a star or bullet, such as $\mathrm{Map}(X,Y)$, and Y^X.

Definition 1.0.5. Let X be a set and $\mathcal{A} \subseteq \mathsf{Top}(X)$. The *greatest lower bound* (coarsest topology below every member of $\mathcal{A}$) is $\bigcap \mathcal{A}$, which lies in $\mathsf{Top}(X)$. The *least upper bound* (finest topology above every member of $\mathcal{A}$) is the topology generated by $\bigcup \mathcal{A}$, namely $\tau(\bigcup \mathcal{A})$.

Definition 1.0.6. Let X be a set. The *indiscrete topology* on X is $\{\varnothing, X\}$ and is the coarsest topology on X. The *discrete topology* on X is $\mathcal{P}(X)$ and is the finest topology on X. When the underlying set is clear, one may write τ_{ind} and τ_{disc} for these topologies.

Proposition 1.0.7. [May99, Proposition Page 41] For compactly generated (Definition A.0.11) topological spaces (Definition A.0.1) X, Y, Z, the canonical bijection

$$Z^{(X\times Y)} \cong (Z^Y)^X$$

of function spaces (Definition 1.0.4) is a homeomorphism (Definition A.0.9).

(♠ TODO: continue going through here) $F(X, Y)$: The subspace of Y^X consisting of the based maps, with the constant based map as basepoint. We have a natural homeomorphism

$$F(X \wedge Y, Z) \cong F(X, F(Y, Z))$$

of based spaces for any based spaces X, Y, and Z.

Definition 1.0.8 (Homotopy groups)**.** For any pointed topological space (Definition A.0.7) (X, x_0) and integer $n \geq 0$, the *n-th homotopy group of X at x_0*, denoted $\pi_n(X, x_0)$, is defined as the set of all homotopy classes (rel. ∂I^n) (Definition A.0.8) of based maps

$$f : (I^n, \partial I^n) \to (X, x_0),$$

where $I^n = [0, 1]^n$. For $n \geq 1$, $\pi_n(X, x_0)$ is a group under concatenation of based maps, and for $n \geq 2$, it is abelian.

(♠ TODO: loop, path) The *fundamental group of (X, x_0)* refers to $\pi_1(X, x_0)$. Equivalently, it is the group of homotopy classes (rel. endpoints) of loops $\gamma : [0, 1] \to X$ satisfying $\gamma(0) = \gamma(1) = x_0$.

$\pi_n(X) = \pi_n(X, *)$: The *nth homotopy group of X*, defined by

$$\pi_n(X) = \pi_n(X, *) = [S^n, X].$$

Claim 1.0.9. $[X, Y]$ may be identified with $\pi_0(F(X, Y))$.

Definition 1.0.10. A space X is said to be *n-connected* if $\pi_q(X, x) = 0$ for $0 \leq q \leq n$ for all x.

ΣX: The *(reduced) suspension of X*, defined as

$$\Sigma = X \wedge S^1 = X \times S^1 / (\{*\} \times \S 1 \cup X \times \{1\}).$$

$\Sigma^n X$: The *n-fold suspension of X*.

ΩX: The *loop space of X*, defined as $F(S^1, X)$; the points are the loops in X at the basepoint.

$\Omega^n X$: The *n-fold loop space of X*.

Claim 1.0.11. There is an adjunction

$$F(\Sigma X, Y) \cong F(X, \Omega Y).$$

Applying π_0, we have an adjunction

$$[\Sigma X, Y] \cong [X, \Omega Y].$$

1.1. **Stable homotopy groups and the stable homotopy theory.**

$\Sigma : \pi_q(X) \to \pi_{q+1}(\Sigma X)$: The suspension homomorphism of X, defined by setting

$$\Sigma f = f \wedge \mathrm{id} : S^{q+1} \cong S^q \wedge S^1 \to X \wedge S^1.$$

Theorem 1.1.1 (Freudenthal suspension, see e.g. [May99, Chapter 11])**.** If X is nondegenerately based and $(n-1)$-connected where $n \geq 1$, then $\Sigma : \pi_q(X) \to \pi_{q+1}(\Sigma X)$ is a bijection if $q < 2n-1$ and a surjection if $q = 2n-1$.

Definition 1.1.2.

1. A *prespectrum* T is a sequence of based space T_n and based maps $\sigma = \sigma_n^T : \Sigma T_n \to T_{n+1}$. The space T_n may be referred to as the *degree n space of the prespectrum* and the maps $\sigma : \Sigma T_n \to T_{n+1}$ may be referred to as the *structure maps*. A *morphism $f : T \to T'$ of prespectra* is a sequence of maps $f_n : T_n \to T'_n$ such thta $\sigma'_n \circ \Sigma f_n = f_{n+1} \circ \sigma_n$ for all n.
2. A *spectrum* E is a prespectrum such that the adjoints $\tilde{\sigma} : E_n \to \Omega E_{n+1}$ to the structure maps $\sigma : \Sigma T_n \to T_{n+1}$ are homeomorphisms. A *morphism $f : E \to E'$ of spectra* is a morphism of prespectra.
3. The *suspension spectrum of a based space X* is the spectrum $\Sigma^\infty X = \Sigma^\infty(X)$ whose degree n space is $\Sigma^n X$.

$\pi_q^s(X) = \pi_q(\Sigma^\infty X)$: the *qth stable homotopy group of X*, defined by

$$\pi_q^s(X) = \pi_q(\Sigma^\infty X) \coloneqq \operatorname{colim} \pi_{q+n}(\Sigma^n X).$$

By the Freudenthal suspension Theorem 1.1.1, $\Sigma^n : \pi_q(X) \to \pi_{q+n}(\Sigma^n X)$ is an isomorphsim for $q < n-1$, so

$$\pi_q(\Sigma^\infty X) \cong \pi_{q+n}(\Sigma^n X)$$

for any $n > q+1$. In particular, note that $\pi_q(\Sigma^\infty X)$ is abelian.

Theorem 1.1.3. [May99, Theorem Chapter 22 Page 176] Let $\{T_n\}$ be a prespectrum such that T_n is $(n-1)$-connected and of the homotopy type of a CW complex for each n. Define

$$\tilde{E}_q(X) = \operatorname{colim}_n \pi_{q+n}(X \wedge T_n)$$

where the colimit is taken over the maps

$$\pi_{q+n}\left(X \wedge T_n\right) \xrightarrow{\Sigma} \pi_{q+n+1}\left(\Sigma\left(X \wedge T_n\right)\right) \cong \pi_{q+n+1}\left(X \wedge \Sigma T_n\right) \xrightarrow{\mathrm{id} \wedge \sigma} \pi_{q+n+1}\left(X \wedge T_{n+1}\right).$$

Then the functors $\tilde{E}_q$ define a reduced homology theory on based CW complexes.

Corollary 1.1.4. The stable homotopy groups $\pi_q(\Sigma^\infty X)$ give a reduced homology theory.

Definition 1.1.5. The reduced homology theory given by the functors $\{X \mapsto \pi_q(\Sigma^\infty X)\}_q$ is called *stable homotopy theory*.

More generally, we can define the stable homotopy group of a spectrum.

Definition 1.1.6 ([nLa25a, Definition 2.1])**.** Let E be a spectrum. For $q \in \mathbb{Z}$, the *qth (stable) homotopy group of E* is the colimit

$$\pi_q(E) := \operatorname{colim}_n \pi_{q+n}(E_n)$$

where the colimit is taken over the maps

$$\pi_{q+n}(E_n) \xrightarrow{\Sigma} \pi_{q+n+1}(\Sigma E_n) \xrightarrow{\pi_{q+n+1}(\sigma_n^E)} \pi_{q+n+1}(E_{n+1})$$

Recall that we notated $\pi_q(\Sigma^\infty X)$ as the qth stable homotopy group $\pi_q^s(X)$ of the based space X and this indeed coincides with the qth homotopy group of the suspension spectrum $\Sigma^\infty X$.

1.2. **Stable homotopy category.**

Definition 1.2.1 ([nLa25c, Definition 0.38])**.** Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism of spectra.

1. We say that f is a strict weak equivalence if each component $f_n : X_n \to Y_n$ is a weak homotopy equivalence (i.e. a weak equivalence for the classical model category structure for topological spaces).
2. We say that f is a strict fibration if each component $f_n : X_n \to Y_n$ is a Serre fibration (i.e. a fibration for the classical model category structure for topological spaces).
3. We say that f is a strict cofibration if $f_0 : X_0 \to Y_0$ and the maps

$$(f_{n+1}, \sigma_n^Y) : X_{n+1} \coprod_{S^1 \wedge X_n} (S^1 \wedge Y_n) \to Y_{n+1}$$

for $n > 1$ are retracts of relative cell complexes (i.e. cofibrations for the classical model category structure for topological spaces).

Theorem 1.2.2 ([nLa25c, Theorem 0.40])**.** The classes (i.e. strict weak equivalences, strict fibrations, and strict cofibrations) of morphisms in Definition 1.2.1 give the category of spectra the structure of a (closed) model category called the *strict/level model structure on topological spectra*, denoted by $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$.

Definition 1.2.3 ([nLa25c, Definition 0.14])**.** A morphism $f : X \to Y$ of spectra is called a *stable weak homotopy equivalence* if its image

$$\pi_\bullet(f) : \pi_\bullet(X) \xrightarrow{\simeq} \pi_\bullet(Y)$$

under the stable homotopy group functor (Definition 1.1.6) is an isomorphism.

Theorem 1.2.4 ([nLa25c, Theorem 0.70])**.** The left Bousfield localization of the strict model structure on spectra at the clsas of stable weak homotopy equivalences exists.

Definition 1.2.5. The left Bousfield localization in Theorem 1.2.4 is a model category called the *stable model structure on topological spectra*, denoted by $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$:

$$\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} \rightleftarrows \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}.$$

The *stable homotopy category*, often denoted by $\mathcal{SH}$ or $\mathrm{Ho}(\mathrm{Spectra})$, is the homotopy category of the model category $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$.

Claim 1.2.6 ([nLa25d, Properties], [nLa25b, Proposition 4.14])**.**

1. The smash product of spectra makes $\mathcal{SH}$ into a symmetric monoidal category.
2. SH has the structure of a triangulated category, where the translation functor is the canonical suspension functor $\Sigma : \mathcal{SH} \to \mathcal{SH}$ and the distinguished triangles are the closures under isomorphisms of triangles of the images (under localization $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} \to \mathcal{SH}$) of the canonical long homotopy cofiber seqeunces

$$A \xrightarrow{f} B \to \mathrm{hocofib}(f) \to \Sigma A.$$

3. $\mathcal{SH}$ is an additive category — there is an abelian group structure on the pointed hom-sets $[X, Y]$ for X, Y in $\mathcal{SH}$.

Appendix A. Miscellaneous definitions

Definition A.0.1 (Topology)**.** Let X be a set. A *topology on X* is a collection $\mathcal{T}$ of subsets of X such that:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
2. For any collection $\{U_i\}_{i\in I} \subseteq \mathcal{T}$ (with I arbitrary), the union $\bigcup_{i\in I} U_i \in \mathcal{T}$,
3. For any finite collection $\{U_1, \dots, U_n\} \subseteq \mathcal{T}$, the intersection $U_1 \cap \cdots \cap U_n \in \mathcal{T}$.

If $\mathcal{T}$ is a topology on X, the pair $(X, \mathcal{T})$ is called a *topological space*. Members of $\mathcal{T}$ are called *open sets*.

A subset $C \subseteq X$ is *closed* if its complement $X \setminus C$ is an open set in $\mathcal{T}$

One very often refers to X as a topological spcae, omitting the notation of the topology $\mathcal{T}$.

The collection of all topologies on a set X may be denoted by notations such as $\mathrm{Top}(X)$, $\mathbf{Top}(X)$, or $\mathsf{Top}(X)$.

Definition A.0.2. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces (Definition A.0.1). A map $f : X \to Y$ is called *continuous* if for every open set $V \in \mathcal{T}_Y$, the preimage $f^{-1}(V)$ is an open set in X, that is,

$$\forall V \in \mathcal{T}_Y,\ f^{-1}(V) \in \mathcal{T}_X.$$

Equivalently, f is continuous if and only if for every closed set $C \subseteq Y$, the preimage $f^{-1}(C)$ is closed in X.

A *map of topological spaces* usually refers to a continuous map between the topological spaces.

The collection of topological spaces along with continuous maps form a locally small category, usually called the *category of topological spaces* and often denoted by notations such as Top, **Top**, etc. The set of continuous maps from X to Y is sometimes denoted by $C(X, Y)$. Other standard notation include $\mathrm{Hom}_{\mathrm{Top}}(X, Y)$ or $\mathrm{Top}(X, Y)$ coming from more general notation for morphisms between objects in a category.

Definition A.0.3. Let X be a set.

1. Let τ_1, τ_2 be topologies on (Definition A.0.1) X. Say that τ_1 is *coarser than* (equivalently, *smaller than*) τ_2 if $\tau_1 \subseteq \tau_2$, and that τ_1 is *finer than* (equivalently, *larger than*) τ_2 if $\tau_2 \subseteq \tau_1$. These relations are denoted by $\tau_1 \preceq \tau_2$ for "τ_1 coarser than τ_2" and $\tau_1 \succeq \tau_2$ for "τ_1 finer than τ_2"; their strict versions are $\tau_1 \prec \tau_2$ and $\tau_1 \succ \tau_2$, meaning proper inclusion.
2. Let $\mathcal{C}$ be some family of topologies on X. A topology $\tau \in \mathcal{C}$ is the *coarsest* (or *smallest*) element of $\mathcal{C}$ if for every $\sigma \in \mathcal{C}$ one has $\tau \subseteq \sigma$ (equivalently, $\tau \preceq \sigma$ for all $\sigma \in \mathcal{C}$). Dually, $\tau \in \mathcal{C}$ is the *finest* (or *largest*) element of $\mathcal{C}$ if for every $\sigma \in \mathcal{C}$ one has $\sigma \subseteq \tau$ (equivalently, $\sigma \preceq \tau$ for all $\sigma \in \mathcal{C}$).

Definition A.0.4. Let X be a set and $S \subseteq \mathcal{P}(X)$. The *topology generated by S*, often denoted by notations such as $\tau(S)$ and $\mathcal{T}_S$, is

$$\tau(S) := \bigcap \{ \mathcal{T} \in \mathsf{Top}(X) \mid S \subseteq \mathcal{T} \},$$

(Definition A.0.1) which is the coarsest (Definition A.0.3) topology on X that contains S.

Definition A.0.5. Let X be a set and let $S \subseteq \mathcal{P}(X)$ be a family of subsets of X.

1. The family S is a *subbasis (on X)* if $\bigcup S = X$.
2. If τ is a topology on X, and S is a subbasis on X, then we say that S is a *subbasis for τ* if and only if $\tau = \tau(S)$ (Definition A.0.4); in this case, members of S are called *subbasic open sets* of (X, τ).

Definition A.0.6. Let X be a set and let $\mathcal{B}$ be a collection of subsets of X. The collection $\mathcal{B}$ is called a *basis* (or *base*) for a topology (Definition A.0.1) on X if the following two conditions hold:

1. For every $x \in X$, there exists at least one $B \in \mathcal{B}$ such that $x \in B$.
2. For any $B_1, B_2 \in \mathcal{B}$ and any $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Given such a collection $\mathcal{B}$, the collection $\mathcal{T}$ of all unions of elements of $\mathcal{B}$ defines a topology on X, and it coincides with $\mathcal{T}_{\mathcal{B}}$, the topology generated by $\mathcal{B}$ (Definition A.0.4). In other words,

$$\mathcal{T}_{\mathcal{B}} = \{U \subseteq X : \text{for every } x \in U, \text{ there exists } B \in \mathcal{B} \text{ with } x \in B \subseteq U\}.$$

Definition A.0.7 (Pointed topological space)**.** Let X be a topological space (Definition A.0.1) and let $x_0 \in X$ be a chosen element of X. A *pointed/based (topological) space* is a pair (X, x_0) consisting of the space X together with the distinguished point x_0, called the *base point of X*. If the base point of a pointed space (X, x_0) is understood, then it may be suppressed from notation; in particular, X may be written as a pointed space as opposed to the full notation of (X, x_0).

A *morphism of pointed spaces* (or *based map*) or *continuous map* between pointed spaces (X, x_0) and (Y, y_0) is a continuous map (Definition A.0.2)

$$f : X \to Y$$

such that $f(x_0) = y_0$.

The collection of pointed spaces with their morphisms form a locally small category, often called the *category of pointed spaces*. This category is often denoted by notations such as Top_*, $\mathrm{Top}_\bullet$, $\mathbf{Top}_*$, $\mathbf{Top}_\bullet$, etc. The set of continuous maps from pointed spaces X to Y may denoted by notations such as $C_*(X, Y)$, $C_\bullet(X, Y)$, $\mathrm{Top}_*(X, Y)$, $\mathrm{Top}_\bullet(X, Y)$, $\mathrm{Hom}_{\mathrm{Top}_\bullet}(X, Y)$, etc.

Definition A.0.8 (Homotopy class of maps relative to a subset)**.** Let X and Y be topological spaces (Definition A.0.1) and let $K \subseteq X$. Let $C(X, Y)$ denote the set of all continuous maps (Definition A.0.2) $f : X \to Y$.

1. Two maps $f, g \in C(X, Y)$ are said to be in the same *homotopy class relative to K* if there exists a homotopy relative to K

$$H : X \times [0,1] \to Y$$

such that

$$H(x,0) = f(x), \quad H(x,1) = g(x),$$

and

$$H(k,t) = f(k) = g(k) \quad \text{for all } k \in K, t \in [0,1].$$

The *homotopy class of maps relative to K* containing a map $f : X \to Y$ is denoted by $[f]_K$.

Two maps $f, g \in C(X, Y)$ are said to be in the same *homotopy class* if they are in the same homotopy class relative to $\emptyset$.

The *homotopy class of maps* containing a map $f : X \to Y$ is denoted by $[f]$.

The set of homotopy classes of maps may often be denoted by $[X, Y]$.

2. Let (X, x_0) and (Y, y_0) be pointed topological spaces (Definition A.0.7) and let $K \subseteq X$ be a subset containing x_0. Let $C_*(X, Y)$ denote the set of all continuous based maps $f : X \to Y$ with $f(x_0) = y_0$.

Two based maps $f, g \in C_*(X, Y)$ are said to be in the same *homotopy class relative to K* if there exists a homotopy of based maps relative to K

$$H : X \times [0,1] \to Y$$

such that for all $x \in X$,

$$H(x,0) = f(x), \quad H(x,1) = g(x),$$

and for all $k \in K$ and $t \in [0,1]$,

$$H(k,t) = f(k) = g(k),$$

particularly ensuring the basepoint is fixed throughout,

$$H(x_0,t) = y_0 \quad \text{for all } t \in [0,1].$$

The *homotopy class relative to K* containing $f : (X, x_0) \to (Y, y_0)$ is denoted by $[f]_K$.

Two based maps $f, g \in C_*(X, Y)$ are said to be in the same *homotopy class* if they are in the same homotopy class relative to $\{x_0\}$.

The *homotopy class* containing a map $f : (X, x_0) \to (Y, y_0)$ is denoted by $[f]$.

The set of homotopy classes of pointed maps $(X, x_0) \to (Y, y_0)$ may often be denoted by $[(X, x_0), (Y, y_0)]$ or by $[X, Y]$ if the base points are clear.

Definition A.0.9. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces (Definition A.0.1). A function $f : X \to Y$ is called a *homeomorphism* if it satisfies all of the following:

1. f is bijective;
2. f is continuous (Definition A.0.2) with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$; and
3. the inverse map $f^{-1} : Y \to X$ is also continuous.

If such a function exists, the spaces X and Y are said to be *homeomorphic*.

Definition A.0.10 (Compactness)**.** Let $(X, \mathcal{T})$ be a topological space. A subset $K \subseteq X$ is *compact* if for every collection $\{U_i\}_{i\in I}$ of open sets such that $K \subseteq \bigcup_{i\in I} U_i$, there exists a finite subcollection $\{U_{i_j}\}_{j=1}^n$ with $K \subseteq \bigcup_{j=1}^n U_{i_j}$.

Definition A.0.11. (♠ TODO: final topology) A topological space (Definition A.0.1) X is said to be *compactly generated* (or a *k-space*) if a subset $U \subseteq X$ is open whenever for every compact subset $K \subseteq X$, the intersection $U \cap K$ is open in the subspace K. Equivalently, X is compactly generated if and only if the topology of X is the final topology with respect to the collection of inclusions $K \hookrightarrow X$ for compact $K \subseteq X$.

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