FOURIER TRANSFORMS

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The purpose of this document is to compile various notions of Fourier transforms and to compare and contrast them.

1. Fourier transform of a complex valued function on $\mathbb{R}$

We can first define the Fourier transform of functions $f \in L^1(\mathbb{R})$

Definition 1.0.1. Let $f : \mathbb{R} \to \mathbb{C}$ be a complex-valued function defined on the real line. The function f is said to be *integrable* if it belongs to the space $L^1(\mathbb{R})$, i.e., if

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty.$$

There are multiple conventions for defining the Fourier transform and developing its theory. We compile some common conventions below:

Definition 1.0.2. Let $f, g : \mathbb{R} \to \mathbb{C}$ be a measurable functions. The *(forward) Fourier transform of f* and the *inverse Fourier transform of f* are the functions $\mathcal{F}[f](t) = \hat{f}(t) : \mathbb{R} \to \mathbb{C}$ and $\mathcal{F}^{-1}[g](t) : \mathbb{R} \to \mathbb{C}$ defined by one of the following conventions (notation for the variables vary amongst the conventions as the variables represent different actors):

Convention name	Fourier Transform $\hat{f}(t)$	Inverse Fourier Transform $\mathcal{F}^{-1}[g](t)$
Standard mathematical convention	$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-ixt} dx$	$\mathcal{F}^{-1}[g](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(t) e^{ixt} dt$
Symmetric math convention	$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixt} dx$	$\mathcal{F}^{-1}[g](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t) e^{ixt} dt$
Time frequency convention	$\hat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$	$\mathcal{F}^{-1}[g](t) = \int_{-\infty}^{\infty} g(\omega) e^{-i\omega t} d\omega$
Position wavenumber convention	$\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$	$\mathcal{F}^{-1}[g](x) = \int_{-\infty}^{\infty} g(k) e^{ikx} dk$
Communications and signal processing convention	$\hat{f}(\nu) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \nu t} dt$	$\mathcal{F}^{-1}[g](t) = \int_{-\infty}^{\infty} g(\nu) e^{2\pi i \nu t} d\nu$

If $f \in L^1(\mathbb{R})$, then these are all well-defined.

Convention 1.0.3. We use the symmetric mathematical convention in the rest of the current section, which is the convention used in [Rud87, Chapter 9]; note that in loc. cit. , the notation

$$\int_{-\infty}^{\infty} f(x) dm(x)$$

is used to denote

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) dx$$

by letting m denote, unlike in other previous chapters of the book, the Lebesgue measure on $\mathbb{R}$ divided by $\sqrt{2\pi}$.

Remark 1.0.4. The Fourier transform and the inverse Fourier transform for the standard mathematical convention and the time frequency convention are reversed.

Theorem 1.0.5 (The Fourier inversion theorem, see e.g. [Rud87, Theorem 9.11])**.** Let $f \in L^1(\mathbb{R})$. If $\hat{f} \in L^1(\mathbb{R})$, then $\mathcal{F}^{-1}[\hat{f}]$ is a continuous function that vanishes at infinity (Definition D.0.1) and $f = g$ almost everywhere.

Theorem 1.0.6 (The Fourier transform uniqueness theorem, see e.g. [Rud87, Theorem 9.12])**.** If $f \in L^1(\mathbb{R})$ and $\hat{f}(t) = 0$ for all $t \in \mathbb{R}$, then $f(x) = 0$ a.e.

We now define the Fourier transform of a function $f \in L^2(\mathbb{R})$ by taking a limit of Fourier transforms of $L^1(\mathbb{R})$ functions that approximate f.

Definition 1.0.7. Let $f \in L^2(\mathbb{R})$. The *Fourier transform* $\hat{f}$ of f is defined as the limit in the L^2-norm of the Fourier transforms of a sequence $\{f_n\} \subset L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that

$$\lim_{n\to\infty} \|f - f_n\|_{L^2(\mathbb{R})} = 0;$$

such a sequence exists due to Corollary B.2.11. Specifically,

$$\hat{f} := \lim_{n\to\infty} \hat{f}_n \text{ in } L^2(\mathbb{R}).$$

where for each n, the Fourier transform $\hat{f}_n$ is given by Definition 1.0.2.

Convention 1.0.8. Unless otherwise stated, given by function $f \in L^2(\mathbb{R})$, $\hat{f}$ refers to the Fourier transform as defined in Definition 1.0.7 rather than the Fourier transform as defiend in Definition 1.0.2; recall that the latter notion is applicable to general measurable functions (although the integral defining the Fourier transform may not be well defined). Theorem 1.0.9 shows that the two definitions in fact coincide almost everywhere for $f \in L^2(\mathbb{R})$ under nice enough conditions.

Theorem 1.0.9 (see e.g. [Rud87, Theorem 9.14])**.** Let $f \in L^2(\mathbb{R})$. Let $\hat{f}$ be the Fourier transform $\hat{f}$ in the sense of Definition 1.0.7. If $\hat{f} \in L^1$, then

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixt} dx \quad \text{almost everywhere}$$

i.e. $\hat{f}$ almost everywhere coincides with the Fourier transform in the sense of Definition 1.0.2.

Plancherel's theorem shows that the Fourier transform is a unitary operator on $L^2(\mathbb{R})$.

Theorem 1.0.10 (Plancherel's theorem, see e.g. [Rud87, Theorem 9.13])**.** Let $f \in L^2(\mathbb{R})$.

1. $$\|f\|_{L^2(\mathbb{R})} = \|\hat{f}\|_{L^2(\mathbb{R})}.$$

2. The Fourier transform $f \mapsto \hat{f}$ is a Hilbert space (see Convention C.0.5) isomorphism $L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

Remark 1.0.11. As we are working with vector fields over $\mathbb{R}$ and $\mathbb{C}$ (in particular which are not of characteristic 2), saying that

$$\|f\|_{L^2(\mathbb{R})} = \|\hat{f}\|_{L^2(\mathbb{R})}$$

is equivalent to saying that the inner product is preserved under Fourier transforms, i.e.

$$(f, g) = (\hat{f}, \hat{g})$$

for all $f, g \in L^2(\mathbb{R})$. In other words, the Fourier transform is a unitary operator on $L^2(\mathbb{R})$.

2. Fourier transform of a function on a locally compact abelian group

We now extend the analytic notion of Fourier transform in Section 1 to apply to functions on locally compact abelian groups. In particular, the discussion in Section 1 is essentially a specialization of the discussion in this section in the case that the locally compact abelian group G is $\mathbb{R}$.

Definition 2.0.1. A *locally compact group* is an topological group that is locally compact Hausdorff. It is called a *locally compact abelian group* if it is abelian as well.

Definition 2.0.2. (♠ TODO: Define Borel sigma algebra, measure, regular Borel measure) Let G be a locally compact group (Definition 2.0.1) with Borel σ-algebra $\mathcal{B}(G)$. A measure $\mu : \mathcal{B}(G) \to [0, \infty]$ is called a *left Haar measure* if it satisfies:

1. μ is a regular Borel measure, finite on compact sets,
2. μ is left-translation invariant:

$$\mu(gS) = \mu(S) \quad \text{for all } g \in G,\, S \in \mathcal{B}(G),$$

   where $gS = \{g \cdot s : s \in S\}$.

The notion of a *right Haar measure* may be defined similarly. If μ also both a left and right Haar measure, then it is called *two-sidedinvariant* and may simply be referred to as a *Haar measure*.

Lemma 2.0.3. If G is a locally compact abelian group, then any left/right Haar measure is a Haar measure.

Lemma 2.0.4. Let G be a locally compact group.

1. There exists a left Haar measure on G. There exists a right Haar measure on G.
2. Any two left Haar measures on G are positive scalar multiples of each other. Any two right Haar measures on G are positive scalar multiples of each other

Notation 2.0.5. Given a locally compact group G, fix a left Haar measure on G; integration with respect to this Haar measure would often be denoted by $\int_G dg$

Definition 2.0.6. Let G be a locally compact group with a fixed left Haar measure μ. For functions $f_1, f_2 : G \to \mathbb{C}$, their *convolution* $f_1 * f_2 \in L^1(G)$ is defined by

$$(f_1 * f_2)(x) = \int_G f_1(y) f_2(y^{-1}x)\, d\mu(y)$$

If $f_1, f_2 \in L^1(G)$, then this integral converges absolutely for almost every $x \in G$.

(♠ TODO: TODO: define a Banach algebra)

Lemma 2.0.7. Let G be a locally compact group.

1. For $f_1, f_2 \in L^1(G)$, we have
$$\|f_1 * f_2\|_1 \leq \|f_1\|_1 \|f_2\|_1.$$
2. The convolution $*$ makes $L^1(G)$ into a Banach algebra (without unit).
3. If G is abelian, then $f_1 * f_2 = f_2 * f_1$ almost everywhere for $f_1, f_2 \in L^1(G)$.
4. If G is abelian, then $*$ makes $L^1(G)$ into a commutative Banach algebra (without unit).

Definition 2.0.8. Let G be a locally compact Hausdorff group.

1. A *quasicharacter of G* is a continuous group homomorphism $\chi : G \to \mathbb{C}^\times$.
2. A quasicharacter χ is *unitary* if its image lies in the unit circle $S^1 \subset \mathbb{C}^\times$, i.e. $|\chi(g)| = 1$ for all $g \in G$. Such a quasicharacter is also simply called a *character*.
3. A (quasi)character is *finite* if its image is finite, i.e. its kernel has finite index in its domain.

When G is finite, G is usually equipped with the discrete topology — a character (and even a quasicharacter) is thus simply a group homomorphism $G \to \mathbb{C}^\times$ such that $|\chi(g)| = 1$ for all $g \in G$.

Definition 2.0.9. Let G be a locally compact abelian group. The *Pontryagin dual of G* is the group
$$\widehat{G} = \{\, \chi : G \to S^1 \mid \chi \text{ is a continuous group homomorphism} \,\},$$
equipped with pointwise multiplication and the compact-open topology.

Remark 2.0.10. A posteriori, we will equip $\widehat{G}$ with a Haar measure, see Convention 2.0.15

Lemma 2.0.11. Let G be a locally compact abelian group.

1. $\widehat{G}$ is a locally compact abelian group.
2. There is a canonical isomorphism
$$\begin{aligned} G &\xrightarrow{\sim} \widehat{\widehat{G}} \\ g &\mapsto (\chi \mapsto \chi(g)) \end{aligned}$$
of topological groups. This is usually called the *Pontryagin map*.

Notation 2.0.12. Let G be a locally compact abelian group. Given $g \in G$ and $\chi \in \widehat{G}$, write $\langle g, \chi \rangle$ for $\chi(g) \in \mathbb{C}$.

Definition 2.0.13. Let G be a locally compact abelian group with a fixed Haar measure μ.

For a function $f : G \to \mathbb{C}$, the *Fourier transform* $\mathcal{F}[f] = \hat{f} : \widehat{G} \to \mathbb{C}$ is defined by

$$\hat{f}(\chi) = \int_{x \in G} f(x)\overline{\langle x, \chi \rangle}\, d\mu(x),$$

(Notation 2.0.12) for all $\chi \in \widehat{G}$. If $f \in L^1(G)$, then the integral converges absolute and hence $\hat{f}(\chi)$ is well defined for all $\chi \in \hat{G}$. Moreover, If $f \in L^1(G)$, then $\hat{f} \in C_0(\widehat{G})$, i.e. is a function on $\widehat{G}$ that vanishes at infinity (Definition D.0.1).

For a function $g : \widehat{G} \to \mathbb{C}$, the *inverse Fourier transform* $\mathcal{F}^{-1}[g] : G \to \mathbb{C}$ is defined by

$$\mathcal{F}^{-1}[g](x) = \int_{\chi \in \widehat{G}} g(\chi)\langle x, \chi \rangle\, d\nu(\chi).$$

for $x \in G$, where ν is a Haar measure on $\widehat{G}$. Similarly as before, if $g \in L^1(\widehat{G})$, then the integral converges absolute and hence $\mathcal{F}^{-1}[g](x)$ is well defined for all $x \in G$. Moreover, If $g \in L^1(\widehat{G})$, then $\mathcal{F}^{-1}[g] \in C_0(G)$, i.e. is a function on G that vanishes at infinity (Definition D.0.1).

Theorem 2.0.14 (Fourier Inversion Formula)**.** Let G be a locally compact abelian group with Haar measure μ. There exists a unique Haar measure ν in $\widehat{G}$ for which the following holds: for any $f \in L^1(G)$, if $\hat{f} \in L^1(\widehat{G})$, then

$$f(x) = \mathcal{F}^{-1}[g](x)$$

and the integral defining $\mathcal{F}^{-1}[g](x)$ converges absolutely for almost every $x \in G$.

Convention 2.0.15. Given a locally compact abelian group G and a choice of Haar measure μ, we always equip $\widehat{G}$ with the unique Haar measure ν in Theorem 2.0.14.

(♠ TODO: TODO: state the Plancherel theorem)

3. Mellin transform of a complex valued function on the positive real numbers

Definition 3.0.1 (Mellin transform)**.** Let $f : \mathbb{R}_{>0} \to \mathbb{C}$ be a measurable function. The *Mellin transform of f* is the function $\mathcal{M}[f] : \mathbb{C} \to \mathbb{C}$ defined by

$$\mathcal{M}[f](s) := \int_0^\infty f(x)\, x^{s-1}\, dx,$$

whenever the integral exists (absolutely or as an improper integral).

Although this definition is stated for general measurable functions $f : \mathbb{R}_{>0} \to \mathbb{C}$, it is intuitively meaningful when in the case that f is a function into $\mathbb{C}^\times$ — the function f would be a function from $\mathbb{R}_{>0}$, the identity component of the multiplicative group $\mathbb{R}^\times$, to the multiplicative group $\mathbb{C}^\times$. It is further intuitively meaningful to express the integral as

$$\int_0^\infty f(x)\, x^s \, \frac{dx}{x},$$

and regarding $\frac{dx}{x}$ as a Haar measure (Definition 2.0.2) (♠ TODO: on what?)

(♠ TODO: TODO: sufficient conditions for the Mellin transform to converge) (♠ TODO: TODO: the inverse Mellin transform to converge)

Theorem 3.0.2 (Relation between Mellin and Fourier transforms)**.** Let $f : \mathbb{R}_{>0} \to \mathbb{C}$ be a measurable function such that the Mellin transform $\mathcal{M}[f]$ converges for $s = \sigma + it \in \mathbb{C}$ in some vertical strip $\sigma_1 < \sigma < \sigma_2$.

Let $g : \mathbb{R} \to \mathbb{C}$ be the function

$$g(u) := f(e^u).$$

Then the Fourier transform $\mathcal{F}[g](\omega)$ of g, defined by (♠ TODO: TODO: according to my convention, the Fourier transform is slightly different.)

$$\mathcal{F}[g](\omega) := \int_{-\infty}^\infty g(u)\, e^{-i\omega u}\, du,$$

exists (appropriately as an improper integral or in distributional sense) for $\omega \in \mathbb{R}$ in a corresponding domain, and satisfies the identity

$$\mathcal{F}[g](\omega) = \mathcal{M}[f]\big(\sigma - i\omega\big),$$

where the Mellin transform is analytically continued if necessary to the point $\sigma - i\omega$.

In other words, the Fourier transform of the function $u \mapsto f(e^u)$ is equal to the Mellin transform of f evaluated on the vertical line in the complex plane parameterized by $s = \sigma - i\omega$.

4. Arithmetic Fourier transform

4.1. **Deligne's Fourier transform of perverse sheaves on the additive group over a finite field.**

Definition 4.1.1. Let $\mathbb{F}_q$ be a finite field. Let $\psi : \mathbb{G}_a(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^*$ be a nontrivial character. Let $m : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$ be the scheme morphism given by $(x, y) \mapsto xy$.

The *Fourier transform functor asscciated to ψ* (also called *Deligne's (arithmetic) Fourier transform asscciated to ψ*) is the functor

$$\boxed{T_\psi : D_c^b(\mathbb{G}_a, \overline{\mathbb{Q}}_\ell) \to D_c^b(\mathbb{G}_a, \overline{\mathbb{Q}}_\ell)}$$

defined by

$$T_\psi(M) = R\pi_{1!}\left(\pi_2^*(M) \otimes m^*(\mathscr{L}(\psi))\right)[1]$$

where π_1 and π_2 are the projection morphisms $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ and $\mathscr{L}(\psi)$ is the Artin-Schreier sheaf of ψ.

(♠ TODO: TODO: Put in generalized definitions of Deligne's fourier transform functor)

4.2. **Fourier transform of the trace function of a cohomologically bounded complex of constructible sheaves on a comutative algebraic group over a finite field.** We give an abstract definition for the Fourier transform of a function on a finite group of rational points on an algebraic group. The case of interest (Definition 4.2.5) is when base field is finite and the function is the Frobenius trace function (Definition 4.2.3).

Definition 4.2.1. Let G/k be an algebraic group over a field. Let F be a field. Let $f : G(k) \to F$ be a function. Write $\widehat{G}(k)$ for the set of characters/group homomorphisms $\chi : G(k) \to F^\times$. Assuming that $G(k)$ is a finite set, define the *(discrete) Fourier transform of f* to be the function $\mathcal{F}[f] = \hat{f} : \widehat{G}(k) \to F^\times$ defined by

$$\hat{f}(\chi) = \sum_{x \in G(k)} \chi(x) f(x).$$

In studying arithmetic Fourier transforms for functions on algebraic groups over finite fields, the main functions of interest are Frobenius trace functions:

Definition 4.2.2. Let k be either a finite field $\mathbb{F}_q$ of q elements or the algebraic closure $\overline{\mathbb{F}}_q$ thereof and let $p = \operatorname{Char} k$.

1. For $n \geq 0$, the *p^nth power Frobenius automorphism on k* is the absolute p^nth Frobenius endomorphism of k; it is a field automorphism.
2. When k is a finite field and l is an algebraic extension of k, the *arithmetic Frobenius automorphism on l (over the base field k)* usually refers to the $|k|$th power Frobenius automorphism on l; it is an element of $\operatorname{Gal}(l/k)$ that we usually denote by $\operatorname{Frob}_{l/k}$. We often simply let the finite field k be the base field and also denote $\operatorname{Frob}_{l/k}$ by Frob_l.
3. the *geometric Frobenius automorphism on l* refers to the element $\operatorname{Frob}_{l/k}^{-1} \in \operatorname{Gal}(l/k)$.

A corresponding automorphism $\operatorname{Spec} l \to \operatorname{Spec} l$ may also be referred to as an *arithmetic/geometric Frobenius automorphism* as appropriate.

Definition 4.2.3 (e.g. [FFK24, Section A.4]). (♠ TODO: TODO: notate the bounded derived category of constructible sheaves) (♠ TODO: TODO: define stalks) Let X be a finite type and separated scheme over a finite field $\mathbb{F}_q$, let $\ell \neq \operatorname{Char} \mathbb{F}_q$ be a prime number, and let $M \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$. For points $x \in X(\mathbb{F}_q)$, write $\overline{x}$ for a geometric point above x.

The *Frobenius trace function of M over $\mathbb{F}_{q^n}$* is the function $G(\mathbb{F}_{q^n}) \to \overline{\mathbb{Q}}_\ell$ defined by

$$t_M(x; \mathbb{F}_{q^n}) = \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{Tr}(\operatorname{Frob}_{q^n} | H^i(M)_{\overline{x}})$$

where $\operatorname{Frob}_{q^n}$ is the geometric Frobenius action of the sheaf $H^i(M)$ at the stalk at $\overline{x}$ (Definition D.0.6). This is independent of the choice of geometric point $\overline{x}$ above x.

Notation 4.2.4. Given an algebraic group G over a finite field $\mathbb{F}_q$, and when a prime $\ell \neq \operatorname{Char} \mathbb{F}_q$ is understood in context, let $\widehat{G}(\mathbb{F}_{q^n})$ denote the set of (unitary) characters $\chi : G \to \overline{\mathbb{Q}}_\ell^\times$. Let $\widehat{G}$ denote the union $\bigcup_{n \geq 1} \widehat{G}(\mathbb{F}_{q^n})$.

Definition 4.2.5. Let G be an algebraic group over a finite field $\mathbb{F}_q$, let $\ell \neq \operatorname{Char} \mathbb{F}_q$ be a prime number, and let $M \in D_c^b(G, \overline{\mathbb{Q}}_\ell)$.

The *arithmetic Fourier transform function of M* is the function $S(M, -) : \widehat{G} \to \overline{\mathbb{Q}}_\ell$ (Notation 4.2.4) defined by Fourier transforms (Definition 4.2.1) of $t_M(-; \mathbb{F}_q)$ (Definition 4.2.3), i.e. if $\chi \in \widehat{G}(\mathbb{F}_{q^n})$, then

$$S(M, \chi) = \sum_{x \in G(\mathbb{F}_{q^n})} \chi(x) t_M(x; \mathbb{F}_{q^n}).$$

4.3. Lang's isogeny theorem and Kummer sheaves associated to characters on connected commutative algebraic groups over finite fields.

Definition 4.3.1. Let G be an algebraic group over some scheme S, and let $F : G \to G$ be an endomorphism. The *Lang map associated to F* is the morphism $L : G \to G$ defined by

$$L(g) = g^{-1} F(g)$$

for every $g \in G$.

In case that S is the spectrum of a finite field and F is the qth power relative Frobenius morphism on $G/\mathbb{F}_q$ (Definition D.0.4), the Lang map associated to F is also called the *Lang isogeny of G*.

Theorem 4.3.2 (See e.g. [Bor12, Chapter V Corollary 16.4])**.** Let $\mathbb{F}_q$ be a finite field of characteristic p with q elements, let G be a connected algebraic group over $\mathbb{F}_q$, and let F be the relative qth power Frobenius endomorphism on $G/\mathbb{F}_q$ (Definition D.0.4). The Lang map associated to F (Definition 4.3.1) is a separable Galois étale (surjective) isogeny whose kernel is precisely $G(\mathbb{F}_q)$, the group of $\mathbb{F}_q$-points of G. (♠ TODO: TODO: Borel's book states the Lang isogeny as separable, and surjective, but I don't know what shows the Galoisness and étaleness)

Definition 4.3.3. Let $G/\mathbb{F}_q$ be a conected commutative algebraic group. Let $\ell \neq \operatorname{Char} \mathbb{F}_q$ be a prime number. Let $n \geq 1$. By Lang's theorem (Theorem 4.3.2), given a connected commutative algebraic group $G/\mathbb{F}_q$, we have we have a short exact sequence

$$1 \to G(\mathbb{F}_{q^n}) \to G_{\mathbb{F}_{q^n}} = G \times_{\mathbb{F}_q} \mathbb{F}_{q^n} \xrightarrow{L} G_{\mathbb{F}_{q^n}} \to 1$$

where L is the Lang isogeny (Definition 4.3.1). Since this identifies $G_{\mathbb{F}_{q^n}}$ with a finite étale $G(\mathbb{F}_{q^n})$-cover of $G_{\mathbb{F}_{q^n}}$, we have a corresponding surjective group homomorphism $\pi_1^{\text{ét}}(G_{\mathbb{F}_{q^n}}, e) \to G(\mathbb{F}_{q^n})$.

Let $\chi : G(\mathbb{F}_{q^n}) \to \overline{\mathbb{Q}}_\ell^\times$ be a character. There is a lisse sheaf $\mathcal{L}_\chi$ of rank 1 on $G_{\mathbb{F}_{q^n}} = G \times_{\mathbb{F}_q} \mathbb{F}_{q^n}$ corresponding to the composed group homomorphism

$$\pi_1^{\text{ét}}(G_{\mathbb{F}_{q^n}}, e) \to G(\mathbb{F}_{q^n}) \xrightarrow{\chi} \overline{\mathbb{Q}}_\ell^\times.$$

We might often call this lisse sheaf the *Kummer sheaf associated to χ*.

Definition 4.3.4. Let G be a connected commutative algebraic group over a finite field $\mathbb{F}_q$, and let $\ell \neq \operatorname{Char} \mathbb{F}_q$ be a prime. Let $\chi \in \widehat{G}(\mathbb{F}_{q^n})$ be a character for some $n \geq 1$.

- Let $M \in D_c^b(G, \overline{\mathbb{Q}}_\ell)$. The *twist* M_χ of M by χ is the object

$$M_\chi = M_{\mathbb{F}_{q^n}} \otimes \mathcal{L}_\chi$$

  where $M_{\mathbb{F}_{q^n}}$ is the pullback of M under the base change map $G_{\mathbb{F}_{q^n}} \to G$ and $\mathcal{L}_\chi$ is the Kummer sheaf associated to χ (Definition 4.3.3).
- More generally, let $\pi : X \to G_{\mathbb{F}_{q^n}}$ be some morphism of schemes over $\mathbb{F}_{q^n}$ and let $M \in D_c^b(X, \overline{\mathbb{Q}}_\ell)$. The twist M_χ of M by χ is the object

$$M_\chi = M \otimes \mathcal{L}_\chi.$$

Lemma 4.3.5 ([FFK24, Lemma 1.17])**.** (♠ TODO: re-verify this citation for the final version of FFK) Let $f : X \to G$ be a morphism from an algebraic variety X to a connected commutative algebraic group G, both defined over $\mathbb{F}_q$. Let $\chi \in \widehat{G}$ be a character. Then the functor $M \mapsto M_\chi$ on $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ or $\mathrm{D}_c^{\mathrm{b}}\left(X_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell\right)$ is t-exact for the standard and perverse t-structures. In particular, if M is perverse (resp. semiperverse) then so is M_χ.

Proposition 4.3.6. Let G be a connected commutative algebraic group over a finite field $\mathbb{F}_q$, let $\ell \neq \operatorname{Char} \mathbb{F}_q$ be a prime, let $M, N \in D_c^b(G, \overline{\mathbb{Q}}_\ell)$, and let $n \geq 1$. Write $1_n \in \widehat{G}(\mathbb{F}_{q^n})$ for the trivial character.

(♠ TODO: TODO: define $*_!$, negligible objects, generic subset)

1. $S(M, \chi) = S(M_\chi, 1_n)$ for any $\chi \in \widehat{G}(\mathbb{F}_{q^n})$.
2. Given any homomorphism $f : G \to H$ of connected commutative algebraic groups over $\mathbb{F}_q$ and for any $n \geq 1$, we have

$$S(Rf_! M, 1_{H,n}) = S(M, 1_{G,n})$$

   where $1_{G,n} \in \widehat{G}(\mathbb{F}_{q^n})$ and $1_{H,n} \in \widehat{H}(\mathbb{F}_{q^n})$ denote the trivial characters.
3. $S(M *_! N, \chi) = S(M, \chi) \cdot S(N, \chi)$ for any $\chi \in \widehat{G}(\mathbb{F}_{q^n})$.
4. If M is negligible, then $S(M, \chi) = 0$ for χ in a generic subset of $\widehat{G}$. The converse holds if M is perverse.

Proof. 1.

$$\begin{aligned}
S(M_\chi, 1_n) &= \sum_{x \in G(\mathbb{F}_{q^n})} t_{M_\chi}(x; \mathbb{F}_{q^n}) \cdot 1_n(x) \\
&= \sum_{x \in G(\mathbb{F}_{q^n})} t_{M \otimes \mathcal{L}_\chi}(x; \mathbb{F}_{q^n}) \\
&= \sum_{x \in G(\mathbb{F}_{q^n})} t_M(x; \mathbb{F}_{q^n}) \cdot t_{\mathcal{L}_\chi}(x; \mathbb{F}_{q^n}) \\
&= \sum_{x \in G(\mathbb{F}_{q^n})} t_M(x; \mathbb{F}_{q^n}) \cdot \chi(x) \\
&= S(M, 1_n).
\end{aligned}$$

2.

$$\begin{aligned}
S(Rf_! M, 1_n) &= \sum_{y \in H(\mathbb{F}_{q^n})} t_{Rf_! M}(y; \mathbb{F}_{q^n}) \\
&= \sum_{y \in H(\mathbb{F}_{q^n})} \sum_{\substack{x \in G(\mathbb{F}_{q^n}) \\ f(x)=y}} t_M(x; \mathbb{F}_{q^n}) \\
&= \sum_{x \in G(\mathbb{F}_{q^n})} t_M(x; \mathbb{F}_{q^n}) \\
&= S(M, 1_n)
\end{aligned}$$

3.

$$\begin{aligned}
S(M *_! N, \chi) &= \sum_{x \in G(\mathbb{F}_{q^n})} t_{M *_! N}(x; \mathbb{F}_{q^n}) \cdot \chi(x) \\
&= \sum_{x \in G(\mathbb{F}_{q^n})} \left(\sum_{y \in G(\mathbb{F}_{q^n})} t_M(y; k_n) \cdot t_N(y^{-1}x; k_n) \right) \cdot \chi(x) \\
&= \sum_{x \in G(\mathbb{F}_{q^n})} \left(\sum_{y \in G(\mathbb{F}_{q^n})} t_M(y; k_n) \cdot \chi(y) \cdot t_N(y^{-1}x; k_n) \cdot \chi(y^{-1}x) \right) \\
&= \sum_{y \in G(\mathbb{F}_{q^n})} \sum_{x \in G(\mathbb{F}_{q^n})} t_M(y; k_n) \cdot \chi(y) \cdot t_N(y^{-1}x; k_n) \cdot \chi(y^{-1}x) \\
&= \sum_{y \in G(\mathbb{F}_{q^n})} \left(t_M(y; k_n) \cdot \chi(y) \sum_{x \in G(\mathbb{F}_{q^n})} t_N(y^{-1}x; k_n) \cdot \chi(y^{-1}x) \right)
\end{aligned}$$

using the change of variables $z = y^{-1}x$, the above equals

$$\begin{aligned}
&\sum_{y\in G(\mathbb{F}_{q^n})} \left(t_M(y;k_n)\cdot\chi(y) \sum_{z\in G(\mathbb{F}_{q^n})} t_N(z;k_n)\cdot\chi(z) \right) \\
=& \left(\sum_{y\in G(\mathbb{F}_{q^n})} t_M(y;k_n)\cdot\chi(y) \right) \cdot \left(\sum_{z\in G(\mathbb{F}_{q^n})} t_N(z;k_n)\cdot\chi(z) \right) \\
=& S(M,\chi)\cdot S(N,\chi).
\end{aligned}$$

4. For generic $\chi \in \widehat{G}$, we have $H^i_c(G_{\overline{\mathbb{F}}_q}, M_\chi) = 0$ for all $i > 0$ by the generic vanishing theorem. The Grothendieck-Lefschetz trace formula shows that

$$S(M_\chi, 1_n) = \sum_{x\in G_{\mathbb{F}_{q^n}}} t_{M_\chi}(x;\mathbb{F}_{q^n}) = \sum_{i\in\mathbb{Z}} (-1)^i \operatorname{tr}(\mathrm{Frob}_{q^n} | H^i_c(G_{\overline{\mathbb{F}}_q}, M_\chi)).$$

Therefore,

$$S(M_\chi, 1_n) = \operatorname{tr}(\mathrm{Frob}_{q^n} | H^0_c(G_{\overline{\mathbb{F}}_q}, M_\chi))$$

for generic χ. This value is generically 0 if and only if $H^0_c(G_{\overline{\mathbb{F}}_q}, M_\chi))$ is generically 0.

If M is perverse, then this is true if and only if M is negligible. If M is a more general negligible object of $D^b_c(G, \overline{\mathbb{Q}}_\ell)$, then its perverse cohomology objects ${}^p\mathcal{H}^i(M)$ are all negligible by definition. Recall that $[M] = \sum_{i\in\mathbb{Z}}(-1)^i[{}^p\mathcal{H}^i(M)]$ in the Grothendieck group $K(D^b_c(G, \overline{\mathbb{Q}}_\ell))$. Moreover, $S(-,\chi)$ is additive in distinguished triangles because $t(-;\mathbb{F}_{q^n})$ is additive in distinguished triangles. Therefore,

$$S(M,\chi) = \sum_{i\in\mathbb{Z}} (-1)^i S({}^p\mathcal{H}^i(M),\chi) = 0.$$

□

Lemma 4.3.7. Let $\{G_i\}_i$ be a finite collction of connected commutative algebraic groups over a finite field $\mathbb{F}_q$. Write $G = \prod_i G_i$. Let $\ell \neq \operatorname{Char}\mathbb{F}_q$ be a prime. Let $n \geq 1$.

1. The characters $\chi \in \widehat{G}(\mathbb{F}_{q^n})$ are in bijective correspondence with tuples $(\chi_i)_i$ of characters $\chi_i \in \widehat{G}_i(\mathbb{F}_{q^n})$ under the inverse maps

$$\begin{aligned}
\widehat{G}(\mathbb{F}_{q^n}) &\to \prod_i \widehat{G}_i(\mathbb{F}_{q^n}) \\
\chi &\mapsto (\chi\circ\iota_i)_i \\
\left((g_i)_i \mapsto \prod_i \chi_i(g_i) \right) &\leftarrow\!\shortmid (\chi_i)_i
\end{aligned}$$

where $\iota_i : G_i(\mathbb{F}_{q^n}) \to G(\mathbb{F}_{q^n})$ sends $g_i \in G_i(\mathbb{F}_{q^n})$ to the point of $G(\mathbb{F}_{q^n})$ whose ith coordinate is g_i and whose other coordinates are all identity elements.

2. Let $\chi \in \widehat{G}(\mathbb{F}_{q^n})$ be a character and let $(\chi_i)_i \in \prod_i \widehat{G}_i(\mathbb{F}_{q^n})$ be the corresponding tuple of characters. There is a natural isomorphism

$$\mathcal{L}_\chi \cong \bigotimes_i \mathrm{pr}_i^* \mathcal{L}_{\chi_i}$$

where $\mathrm{pr}_i : G \to G_i$ is the projection morphism.

Proof. 1. This is clear.

2. Since inverse images preserve tensor products, it suffices to show this in the case that the collction $(G_i)_i$ consists of two groups G_1 and G_2. It also suffices to show this in the case of $n = 1$ by base change. Take the commutative diagram

$$\begin{tikzcd} G_1 \times G_2 \arrow[r, "\varphi"] \arrow[d] & G_1 \times G_2 \arrow[r] \arrow[d, "\psi"] & G_2 \arrow[d, "L_2"] \\ G_1 \times G_2 \arrow[r] \arrow[d] & G_1 \times G_2 \arrow[r] \arrow[d] & G_2 \arrow[d] \\ G_1 \arrow[r, "L_1"] & G_1 \arrow[r] & \operatorname{Spec} \mathbb{F}_q \end{tikzcd}$$

whose squares are all Cartesian where we have written $L_i : G_i \to G_i$ to denote the Lang isogeny for $i = 1, 2$. The morphisms labeled $\varphi, \psi : G_1 \times G_2 \to G_1 \times G_2$ in the diagram are base changes of L_1 and L_2 respectively and hence are Galois étale of Galois groups $G_1(\mathbb{F}_q)$ and $G_2(\mathbb{F}_q)$ respectively. In fact, these Galois étale covers correspond to the surjective group homomorphisms

$$\text{(A)} \qquad \begin{aligned} \pi_1^{\text{ét}}(G_1 \times G_2, e) \to \pi_1^{\text{ét}}(G_1, e) \twoheadrightarrow G_1(\mathbb{F}_q) \\ \pi_1^{\text{ét}}(G_1 \times G_2, e) \to \pi_1^{\text{ét}}(G_2, e) \twoheadrightarrow G_2(\mathbb{F}_q) \end{aligned}$$

where the maps from $\pi_1^{\text{ét}}(G_1 \times G_2, e)$ are induced by the natural projection maps on the group schemes. Note that base changing the Lang isogeny exact sequence

$$1 \to G(\mathbb{F}_q) \to G \xrightarrow{L} G \to 1$$

to $\overline{\mathbb{F}}_q$ results in the exact sequence

$$1 \to G(\mathbb{F}_q) \to G_{\overline{\mathbb{F}}_q} \xrightarrow{L} G_{\overline{\mathbb{F}}_q} \to 1$$

describing a Galois étale cover of connected group schemes, so the homomorphisms (A) are in fact surjective even when restricted the subgroup $\pi_1^{\text{ét}}((G_1 \times G_2)_{\overline{\mathbb{F}}_q}, e)$ of $\pi_1^{\text{ét}}(G_1 \times G_2, e)$.

Therefore, the product group homomorphism

$$\text{(B)} \qquad \pi_1^{\text{ét}}(G_1 \times G_2, e) \to G_1(\mathbb{F}_q) \times G_2(\mathbb{F}_q)$$

induced by the two group homomorphisms of (A) is surjective and corresponds to the composed cover

$$G_1 \times G_2 \xrightarrow{\varphi} G_1 \times G_2 \xrightarrow{\psi} G_1 \times G_2.$$

Note that $\varphi = (L_1, \mathrm{id}_{G_2})$ and $\psi = (\mathrm{id}_{G_1} \times L_2)$, so the composed covering morphism above is (L_1, L_2), which is exactly the Lang isogeny of $(G_1 \times G_2)$ over $\mathbb{F}_q$.

By construction, the lisse sheaf $\mathcal{L}_{\chi_i}$ corresponds to the composed group homomorphism

$$\pi_1^{\text{ét}}(G_i, e) \to G_i(\mathbb{F}_q) \xrightarrow{\chi_i} \overline{\mathbb{Q}}_\ell^\times$$

and hence its pullback $\mathrm{pr}_i^* \mathcal{L}_{\chi_i}$ to $G_1 \times G_2$ corresponds to the composition

$$\pi_1^{\text{ét}}(G_1 \times G_2, e) \to \pi_1^{\text{ét}}(G_i, e) \twoheadrightarrow G_i(\mathbb{F}_q) \xrightarrow{\chi_i} \overline{\mathbb{Q}}_\ell^\times$$

of the group homomorphism in (A) with χ_i. The tensor product $\mathrm{pr}_1^* \mathcal{L}_{\chi_1} \times \mathrm{pr}_2^* \mathcal{L}_{\chi_2}$ thus corresponds to the homomorphism

$$\text{(C)} \quad \pi_1^{\text{ét}}(G_1 \times G_2, e) \to \pi_1^{\text{ét}}(G_1, e) \times \pi_1^{\text{ét}}(G_2, e) \twoheadrightarrow G_1(\mathbb{F}_q) \times G_2(\mathbb{F}_q) \xrightarrow{(g_1,g_2)\mapsto\chi_1(g_1)\cdot\chi_2(g_2)} \overline{\mathbb{Q}}_\ell^\times.$$

On the other hand, $\mathcal{L}_\chi$ corresponds to the group homomorphism

$$\text{(D)} \quad \pi_1^{\text{ét}}(G_1 \times G_2, e) \to G_1(\mathbb{F}_q) \times G_2(\mathbb{F}_q) \cong (G_1 \times G_2)(\mathbb{F}_q) \xrightarrow{\chi} \overline{\mathbb{Q}}_\ell$$

where the first homomorphism is (B). Since $\chi(g_1, g_2) = \chi_1(g_1) \cdot \chi_2(g_2)$, and since the homomoprhism (B) is the product of the homomorphisms of (A), the two homomorphisms (C) and (D) coincide. The two lisse sheaves $\mathcal{L}_\chi$ and $\mathrm{pr}_1^* \mathcal{L}_{\chi_1} \otimes \mathrm{pr}_2^* \mathcal{L}_{\chi_2}$ are thus naturally isomorphic.

□

Proposition 4.3.8. Let $\{G_i\}_i$ be a finite collction of connected commutative algebraic groups over a finite field $\mathbb{F}_q$. Write $G = \prod_i G_i$ and write $\mathrm{pr}_i : G \to G_i$ for the projection morphism. Let $\ell \neq \mathrm{Char}\, \mathbb{F}_q$ be a prime. Let $M \in \mathrm{Perv}(G)$ be arithmetically semisimple and pure of weight 0. Suppose for each i that $R\,\mathrm{pr}_{i!}\, M$ is arithmetically isomorphic to a direct sum $\delta_{x_i} \oplus N_i$ where δ_{x_i} is a skyscraper sheaf of rank 1 supported at a point $x_i \in G_i(\mathbb{F}_q)$ and N_i is a negligible object on G_i. Further suppose that there exists some i_0 such that $H_c^*(G_{i_0,\overline{\mathbb{F}}_q}, N_{i_0}) = 0$.

The object M itself must be direct sum of a skyscraper sheaf of rank 1 and a negligible object.

Proof. The hypotheses and conclusion are preserved under base change by a finite extension of $\mathbb{F}_q$; we may thus apply such base changes and replace $\mathbb{F}_q$ with appropriate finite extensions in the process.

Say that $R\,\mathrm{pr}_{i!}\, M$ is the direct sum $\delta_{x_i} \oplus N_i$ of a skyscraper sheaf δ_{x_i} supported at a point $x_i \in G_i(\mathbb{F}_q)$ and a negligible object N_i on G_i.

Given a character $\chi_i \in \widehat{G}_i(\mathbb{F}_{q^n})$, Proposition 4.3.6 shows that

$$S(M, \mathrm{pr}_i^* \chi_i) = S(M_{\mathrm{pr}_i^* \chi_i}, 1_n) = S(R\,\mathrm{pr}_{i!}(M_{\mathrm{pr}_i^* \chi_i}), 1_n).$$

By the projection formula (♠ TODO: cite the projection formula), this equals

$$\begin{aligned} S((R\,\mathrm{pr}_{i!}\, M)_{\chi_i}, 1_n) &= S(R\,\mathrm{pr}_{i!}\, M, \chi_i) \\ &= S(\delta_{x_i} \oplus N_i, \chi_i) \\ &= S(\delta_{x_i}, \chi_i) \oplus S(N_i, \chi_i). \end{aligned}$$

For generic $\chi_i \in \widehat{G}_i$, the value $S(N_i, \chi_i)$ equals 0, so the above equals $S(\delta_{x_i}, \chi_i)$. Since δ_{x_i} is skyscraper, we have

$$S(\delta_{x_i}, \chi_i) = \sum_{y_i \in G_i(\mathbb{F}_{q^n})} t_{\delta_{x_i}}(y_i; \mathbb{F}_{q^n}) \cdot \chi_i(y_i) = t_{\delta_{x_i}}(x_i; \mathbb{F}_{q^n}) \cdot \chi_i(x_i) = S(\delta_{x_i}, 1_n) \cdot \chi_i(x_i).$$

In turn, this equals

$$S(\delta_{x_i}, 1_n) \cdot \chi_i(x_i) = S(R\,\mathrm{pr}_{i!}\, M, 1_n) \cdot \chi_i(x_i) = S(M, 1_n) \cdot \chi_i(x_i).$$

We have thus shown that

$$\text{(E)} \qquad S(M, \mathrm{pr}_i^* \chi_i) = S(M, 1_n) \cdot \chi_i(x_i)$$

holds for generic $\chi_i \in \widehat{G}_i$.

Let $x \in G(\mathbb{F}_q)$ be the point $(x_i)_i$. For each i, the skyscraper sheaf δ_{x_i} corresponds to a continuous character $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$; in particular, the Frobenius element must be sent to a root of unity. Replace $\mathbb{F}_q$ by a finite extension by base change so that each skyscraper sheaf δ_{x_i} corersponds to the trivial character $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$. Let δ_x be the trivial skyscraper sheaf of rank 1 at x. By assumption, there is some i_0 such that $H_c^*(G_{i_0,\overline{\mathbb{F}}_q}, N_{i_0}) = 0$, so in particular $S(N_{i_0}, 1_n) = 0$ for every $n \geq 1$ by the Grothendieck-Lefschetz trace formula. Further note that

$$\text{(F)} \qquad \begin{aligned} S(M, 1_n) &= S(R\,\mathrm{pr}_{i_0!}\, M, 1_n) \\ &= S(\delta_{x_{i_0}} \oplus N_{i_0}, 1_n) \\ &= S(\delta_{x_{i_0}}, 1_n) + S(N_{i_0}, 1_n) \\ &= S(\delta_{x_{i_0}}, 1_n) + 0 \\ &= S(R\,\mathrm{pr}_{i_0!}\, \delta_x, 1_n) \\ &= S(\delta_x, 1_n). \end{aligned}$$

The ideas used to show (E) show that

$$\text{(G)} \qquad S(\delta_x, \mathrm{pr}_i^* \chi_i) = S(\delta_x, 1_n) \cdot \chi_i(x_i)$$

for any $\chi_i \in \widehat{G}_i(\mathbb{F}_{q^n})$.

Let $\chi \in \widehat{G}(\mathbb{F}_{q^n})$ be a character and let it correspond to the tuple $(\chi_i)_i$ of $\prod_i \widehat{G}_i(\mathbb{F}_{q^n})$ via Lemma 4.3.7. (E) and (G) show that

$$\begin{aligned} S(M, \mathcal{L}_\chi) &= S(M, 1_n) \cdot \prod_i \chi_i(x_i) \quad \text{for generic } \chi \in \widehat{G} \\ S(\delta_x, \mathcal{L}_\chi) &= S(\delta_x, 1_n) \cdot \prod_i \chi_i(x_i) \quad \text{for all } \chi \in \widehat{G}(Fqn). \end{aligned}$$

By (F), $S(M, \mathcal{L}_\chi)$ and $S(\delta_x, \mathcal{L}_\chi)$ thus coincide for generic $\chi \in \widehat{G}$. The generic Fourier invertibility theorem [FFK24, Theorem 6.11] (♠ TODO: ref the invertibility theorem) thus concludes that M and δ_x are arithmetically isomorphic as objects of $\mathbf{P}_{\mathrm{int}}(G)$. (♠ TODO: re-verify this citation for the final version of FFK) ☐

5. Fourier-Mukai equivalence

(♠ TODO: TODO: Read the statements in this section to verify)

Definition 5.0.1. Let X and Y be smooth projective varieties over a field, and let

$$K \in D^b(\mathrm{Coh}(X \times Y))$$

be an object in the bounded derived category of coherent sheaves on the product $X \times Y$.

Denote by

$$p : X \times Y \to Y, \quad q : X \times Y \to X$$

the projection maps onto the second and first factors, respectively.

The *Fourier–Mukai transform* associated to the kernel K is the exact functor

$$\Phi_K : D^b(\mathrm{Coh}(X)) \to D^b(\mathrm{Coh}(Y))$$

defined by

$$\Phi_K(\mathcal{F}) = Rp_*\big(q^*\mathcal{F} \otimes^{\mathbf{L}} K\big),$$

where

- q^* is the (derived) pullback functor,
- $\otimes^{\mathbf{L}}$ denotes the derived tensor product,
- Rp_* is the derived pushforward functor.

This construction generalizes the classical Fourier transform to the setting of derived categories and algebraic geometry.

Definition 5.0.2. Let X and Y be smooth projective varieties over a field k. Denote by $D^b(\mathrm{Coh}(X))$ and $D^b(\mathrm{Coh}(Y))$ the bounded derived categories of coherent sheaves on X and Y, respectively. Let $K \in D^b(\mathrm{Coh}(X \times Y))$. The *Fourier–Mukai transform* with kernel K is the exact functor

$$\Phi_K : D^b(\mathrm{Coh}(X)) \to D^b(\mathrm{Coh}(Y)), \quad \Phi_K(\mathcal{F}) = Rp_*\big(q^*\mathcal{F} \otimes^{\mathbf{L}} K\big),$$

where $q : X \times Y \to X$ and $p : X \times Y \to Y$ are the natural projection morphisms.

Proposition 5.0.3 (Composition of Fourier–Mukai Functors)**.** Let X, Y, Z be smooth projective varieties, and let $K \in D^b(\mathrm{Coh}(X \times Y))$, $L \in D^b(\mathrm{Coh}(Y \times Z))$. Define the object $K \star L \in D^b(\mathrm{Coh}(X \times Z))$ by

$$K \star L = Rp_{13*}\big(p_{12}^*K \otimes^{\mathbf{L}} p_{23}^*L\big),$$

where p_{12}, p_{23}, p_{13} are the projections from $X \times Y \times Z$ onto the corresponding factors. Then

$$\Phi_L \circ \Phi_K \cong \Phi_{K \star L}.$$

Theorem 5.0.4 (Orlov's Representability Theorem for Fully Faithful Functors)**.** Let X and Y be smooth projective varieties over a field. Every exact fully faithful functor

$$F : D^b(\mathrm{Coh}(X)) \to D^b(\mathrm{Coh}(Y))$$

is isomorphic to a Fourier–Mukai functor Φ_K for some $K \in D^b(\mathrm{Coh}(X \times Y))$.

Theorem 5.0.5 (Equivalences as Fourier–Mukai Transforms)**.** Let X and Y be smooth projective varieties over a field. If $\Phi_K : D^b(\mathrm{Coh}(X)) \to D^b(\mathrm{Coh}(Y))$ is an exact equivalence of derived categories, then its quasi-inverse is isomorphic to a Fourier–Mukai functor $\Phi_{K'}$, where

$$K' = R\mathcal{H}om_{X\times Y}(K, p_X^! \mathcal{O}_X)$$

(up to dualization and permutation of factors), and $\Phi_{K'} \circ \Phi_K \cong \mathrm{Id}$, $\Phi_K \circ \Phi_{K'} \cong \mathrm{Id}$.

Theorem 5.0.6 (Mukai's Equivalence Theorem)**.** Let A be an abelian variety of dimension g over an algebraically closed field, and let $\widehat{A} = \mathrm{Pic}^0(A)$ denote its dual abelian variety. Let $\mathcal{P}$ denote the normalized Poincaré line bundle on $A \times \widehat{A}$. The Fourier–Mukai functor

$$\Phi_{\mathcal{P}} : D^b(\mathrm{Coh}(A)) \to D^b(\mathrm{Coh}(\widehat{A}))$$

is an exact equivalence of triangulated categories, where $p : A \times \widehat{A} \to A$ and $\widehat{p} : A \times \widehat{A} \to \widehat{A}$ are the projections.

Proposition 5.0.7 (Parseval Formula for Fourier–Mukai Functors)**.** Let X and Y be smooth projective varieties over a field, let $K \in D^b(\mathrm{Coh}(X \times Y))$, and let $\Phi_K : D^b(\mathrm{Coh}(X)) \to D^b(\mathrm{Coh}(Y))$ be the associated Fourier–Mukai functor. For all $\mathcal{F}, \mathcal{G} \in D^b(\mathrm{Coh}(X))$ and all $i \in \mathbb{Z}$,

$$\mathrm{Hom}_{D^b(\mathrm{Coh}(X))}(\mathcal{F}, \mathcal{G}[i]) \cong \mathrm{Hom}_{D^b(\mathrm{Coh}(Y))}(\Phi_K(\mathcal{F}), \Phi_K(\mathcal{G})[i])$$

if Φ_K is fully faithful. In particular, this isomorphism holds if Φ_K is an equivalence.

Theorem 5.0.8 (Generic Vanishing for Abelian Varieties)**.** Let A be a complex abelian variety, and $\mathcal{F}$ a coherent sheaf on A. For each $i \geq 0$, define the *cohomological support loci*

$$V^i(\mathcal{F}) = \{\alpha \in \widehat{A} : H^i(A, \mathcal{F} \otimes \mathcal{P}_\alpha) \neq 0\},$$

where $\mathcal{P}_\alpha$ is the restriction of the Poincaré bundle to $A \times \{\alpha\}$. Then for a generic $\alpha \in \widehat{A}$,

$$H^i(A, \mathcal{F} \otimes \mathcal{P}_\alpha) = 0 \quad \text{for all } i > \dim \mathrm{Supp}\, \mathcal{F}.$$

Theorem 5.0.9 (Index Theorem for the Fourier–Mukai Transform)**.** Let A be an abelian variety over an algebraically closed field with dual $\widehat{A}$ and normalized Poincaré bundle $\mathcal{P}$. Then for any $x \in A$, consider the skyscraper sheaf $\mathcal{O}_x$. Its image under the Fourier–Mukai equivalence $\Phi_{\mathcal{P}}$ is a line bundle:

$$\Phi_{\mathcal{P}}(\mathcal{O}_x) \cong \mathcal{P}_x[-g],$$

where $\mathcal{P}_x$ is the restriction of $\mathcal{P}$ to $\{x\} \times \widehat{A}$, and $g = \dim A$. In particular,

$$H^i(A, \mathcal{L}) = 0 \text{ for all } i \neq \mathrm{ind}(\mathcal{L})$$

for $\mathcal{L}$ a sufficiently ample line bundle, where $\mathrm{ind}(\mathcal{L})$ is the unique i for which $H^i(A, \mathcal{L}) \neq 0$.

Appendix A. Comparisons

Appendix B. Analysis and L^p-norms

B.1. **(Extended) norms and metrics.**

Definition B.1.1 (Absolute Value on a Field)**.** Let F be a field. An *absolute value on F* is a function

$$|\cdot| : F \to \mathbb{R}_{\geq 0}$$

that satisfies the following properties for all $a, b \in F$:

1. Non-negativity: $|a| \geq 0$,
2. Positive-definiteness: $|a| = 0 \iff a = 0$,
3. Multiplicativity: $|ab| = |a| \cdot |b|$,
4. Triangle inequality: $|a + b| \leq |a| + |b|$.

Here, 0 denotes the additive identity of the field F, and the codomain $\mathbb{R}_{\geq 0}$ consists of non-negative real numbers.

Remark B.1.2. For our analytic discussions, the valued field will usually be $\mathbb{R}$ or $\mathbb{C}$ with the usual absolute values. In particular, for $a + bi \in \mathbb{C}$ with $a, b \in bbR$, $|a + bi| = \sqrt{a^2 + b^2}$.

Definition B.1.3 (Extended Norm)**.** Let V be a vector space over a field F equipped with an absolute value (Definition B.1.1)

$$|\cdot| : F \to [0, \infty).$$

An *extended norm on V* is a function

$$\|\cdot\| : V \to [0, \infty]$$

satisfying for all $x, y \in V$ and all scalars $\alpha \in F$:

1. **Positive definiteness:** $\|x\| = 0$ if and only if $x = 0$.
2. **Homogeneity:** $\|\alpha x\| = |\alpha| \cdot \|x\|$.
3. **Triangle inequality:** $\|x + y\| \leq \|x\| + \|y\|$,

where arithmetic is extended to allow sums involving ∞ with the convention that $a + \infty = \infty$ for any $a \in [0, \infty]$. A vector space with an extended norm over a field with an absolute value is called an *extended normed (vector) space*.

If the range of the extended norm is contained in $[0, \infty)$, then the extended norm is a *norm* in the usual sense and V may be called a *normed (vector) space*.

Definition B.1.4 (Topology induced by a norm)**.** Let V be a vector space over a field K with absolute value (Definition B.1.1) $|\cdot|$, and let $\|\cdot\|$ be an extended norm on V (Definition B.1.3). The *topology induced by the extended norm $\|\cdot\|$ on V* is defined by declaring a subset $U \subseteq V$ to be open if for every $x \in U$, there exists $\varepsilon > 0$ such that

$$B(x, \varepsilon) := \{y \in V : \|y - x\| < \varepsilon\}$$

is contained in U. The set $B(x, \varepsilon)$ is called the *open ball of radius ε around x*. The collection of all such open sets forms a topology on V.

Definition B.1.5 (Extended Metric)**.** Let M be a set. An *extended metric* on M is a function

$$d : M \times M \to [0, \infty]$$

such that for all $x, y, z \in M$:

1. **Non-negativity:** $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$.
2. **Symmetry:** $d(x, y) = d(y, x)$.
3. **Triangle inequality:** $d(x, z) \leq d(x, y) + d(y, z)$,

again adopting the convention that sums involving ∞ behave so that $a + \infty = \infty$. A set equipped with an extended metric is called an *extended metric space*.

If the range of the extended metric is contained in $[0, \infty)$, then the extended metric is a *metric* in the usual sense and V may be called a *metric space*.

Remark B.1.6. If the codomain of an extended norm or an extended metric is $[0, \infty)$, then it is a (usual) norm or metric.

Definition B.1.7 (Extended Metric Induced by an Extended Norm)**.** Let V be a vector space over a field F equipped with an absolute value (Definition B.1.1)

$$|\cdot| : F \to [0, \infty),$$

and let $\|\cdot\| : V \to [0, \infty]$ be an extended norm on V (Definition B.1.3). Then the *extended metric induced by the extended norm* is the function

$$d : V \times V \to [0, \infty]$$

defined by

$$d(x, y) := \|x - y\|.$$

It is indeed an extended metric. If $\|\cdot\|$ is a norm, then d is a metric.

Definition B.1.8 (Convergence and Limits in an Extended Metric Space)**.** Let (X, d) be an extended metric space (Definition B.1.5). A sequence $(x_n)_{n=1}^{\infty}$ in X *converges to a point $x \in X$* if for every $\varepsilon > 0$, there exists an integer $N \in \mathbb{N}$ such that for all $n \geq N$,

$$d(x_n, x) < \varepsilon.$$

In that case, x is called the *limit of the sequence (x_n)*, and we write

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x$$

B.2. **L^p-norms and L^p-spaces on measure spaces.**

Definition B.2.1. Let $(X, \mathcal{A}, \mu)$ be a measure space.

1. Let $1 \leq p < \infty$. Given a measurable function $f : X \to \mathbb{C}$, the *L^p-norm* is the norm (Definition B.1.3) defined by

$$\|f\|_p = \left(\int_X |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}}.$$

2. Let $p = \infty$. Given a measurable function $f : X \to \mathbb{C}$, the *L^∞-norm* is the norm (Definition B.1.3) defined by

$$\|f\|_\infty = \inf\{M \geq 0 : |f(x)| \leq M \quad \text{for } \mu\text{-almost every } x \in X\}.$$

3. Let $1 \leq p \leq \infty$. The *L^p space on X* is defined as the set

$$L^p(X) = L^p(X, \mu) = \{f : X \to \mathbb{C} \mid f \text{ is measurable and } \|f\|_p < \infty\}.$$

An element $f \in L^p(X)$ is called an *L^p-function on the measure space X.*

Remark B.2.2. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $1 \leq p \leq \infty$. The L^p-norm $\|\cdot\|_p$ is an extended norm on the space of all measurable functions $f : X \to \mathbb{C}$. It thus induces an extended metric on the space of all measurable functions.

Theorem B.2.3 (See for example [Rud87, Theorem 3.11])**.** Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $1 \leq p \leq \infty$. The L^p-norm $\|\cdot\|_p$ induces a complete metric on $L^p(X)$.

Convention B.2.4. Let $1 \leq p \leq \infty$. Unless otherwise specified,

1. the L^p-space on a measure space is assumed to be made into a metric space by equipping it with the metric induced by (Definition B.1.7) the L^p-norm (Definition B.2.1).
2. the space of all measurable functions on X is assumed to be made into an extended metric space by equipping it with the extended metric induced by the L^p-norm.

Lemma B.2.5. Let $(X, \mathcal{A}, \mu)$ be a measure space. Any (complex) measurable simple function $s : X \to \mathbb{C}$ on X such that

$$\mu(\{x : s(x) \neq 0\}) < \infty$$

is in $L^p(X, \mu)$.

For finite measure spaces, we have the following inclusions of the $L^p(X, \mu)$ for varying $1 \leq p$; however, we are not concerned with such inclusions as we are concerned with L^1 and L^2-functions on the infinite measure space $\mathbb{R}$.

Proposition B.2.6. Let $(X, \mathcal{A}, \mu)$ be a measure space with finite measure $\mu(X) < \infty$. For $1 \leq p < q \leq \infty$, the following inclusion holds:

$$L^q(X, \mu) \subseteq L^p(X, \mu)$$

and there exists a constant $C > 0$ such that for all $f \in L^q(X, \mu)$,

$$\|f\|_{L^p} \leq C \|f\|_{L^q}.$$

We also can speak of measurable functions converging, with respect to an L^p-norm for some p, to a measurable function.

Definition B.2.7. Let $(X, \mathcal{A}, \mu)$ be a measure space and $1 \leq p \leq \infty$.

A sequence of measurable functions $\{f_n\}_{n=1}^{\infty}$, where each $f_n : X \to \mathbb{R}$ (or $f_n : X \to \mathbb{C}$), *converges to a function $f : X \to \mathbb{R}$ (or $f : X \to \mathbb{C}$) in the L^p norm* if and only if the sequence $(f_n)_{n=1}^{\infty}$ converges to f in the sense of (see Definition B.1.8) the extended metric space which (see Remark B.2.2) is the space of all measurable functions equipped with the metric induced by the L^p norm, i.e.

$$\lim_{n\to\infty} \|f_n - f\|_p = 0.$$

We may thus also say that f is the *limit of $(f_n)_{n=1}^{\infty}$ for the L^p norm on X*, and write

$$f = \lim_{n\to\infty} f_n \text{ in } L^p$$

or

$$f_n \to f \text{ in } L^p.$$

Lemma B.2.8. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions $f_n : X \to \mathbb{R}$ (or $f_n : X \to \mathbb{C}$), and let $1 \leq p \leq \infty$.

1. If there exists some function $f : X \to \mathbb{R}$ (or $f : X \to \mathbb{C}$) such that $f_n \to f$ in L^p, then f is measurable.
2. If f, g are functions $X \to \mathbb{R}$ (or $X \to \mathbb{C}$ such that $f_n \to f$ and $f_n \to g$ in L^p, then $f = g$ μ-almost everywhere.

Any element in the L^p-space $L^p(X, \mu)$ of a general measure space can be L^p-approximated by simple functions with finite measure support. We formulate this idea in terms of *density* of subspaces of $L^p(X, \mu)$ with respect to the topology induced by (Definition B.1.4) the L^p-norm (Definition B.2.1)

Notation B.2.9. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $C_c(X)$ be the class of all continuous (complex) functions on X whose support is compact.

Theorem B.2.10 (See for example [Rud87, Theorems 3.13, 3.14])**.** Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $1 \leq p < \infty$.

1. Let S be the class of all (complex), measurable, simple functions $s : X \to \mathbb{C}$ on X such that
$$\mu(\{x : s(x) \neq 0\}) < \infty.$$
The class S is dense in $L^P(\mu)$.
2. The class $C_c(X)$ is dense in $L^P(\mu)$.

Corollary B.2.11. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $1 \leq p, q < \infty$. The space $L^p(X, \mu) \cap L^q(X, \mu)$ is dense in $L^p(X, \mu)$. In particular, for any $f \in L^p(X, \mu)$, there exists a sequence $\{f_n\}_{n=1}^{\infty}$ functions $f_n \in L^q(x, \mu)$ such that $f^n \to f$ in L^p.

Proof. By Lemma B.2.5, all compactly supported simple measurable functions on X are in $L^p(X, \mu) \cap L^q(X, \mu)$. By Theorem B.2.10, $L^p(X, \mu) \cap L^q(X, \mu)$ is thus dense in $L^p(X, \mu)$. □

Appendix C. Hilbert spaces and Banach spaces

Definition C.0.1. An *inner product on a vector space V over a field* $\mathbb{F}$ (either $\mathbb{R}$ or $\mathbb{C}$) is a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$$

satisfying, for all $u, v, w \in V$ and all scalars $\alpha \in \mathbb{F}$:

1. **Conjugate symmetry:** $\quad \langle u, v \rangle = \overline{\langle v, u \rangle}$
2. **Linearity in the first argument:** $\quad \langle \alpha u + v, w \rangle = \alpha \langle u, w \rangle + \langle v, w \rangle$

3. **Positive-definiteness:** $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0$ if and only if $v = 0$

A vector space over $\mathbb{F}$ equipped with such an inner product is called an *inner product space*.

Definition C.0.2. Let V be a vector space over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $\langle \cdot, \cdot \rangle$ be an inner product (Definition C.0.1). The *norm induced by* $\langle \cdot, \cdot \rangle$ is the norm (Definition B.1.3)

$$\| \cdot \| : V \to \mathbb{R}_{\geq 0}$$

defined by

$$\|v\| = \sqrt{\langle v, v\rangle}$$

for any $v \in V$.

Definition C.0.3. (♠ TODO: is a hilbert space definable over a more general field) A *Hilbert space* $\mathcal{H}$ is a vector space over $\mathbb{R}$ or $\mathbb{C}$ equipped with an inner product such that metric induced by the norm induced by the inner product makes $\mathcal{H}$ a complete metric space.

Proposition C.0.4. Let $(X, \mathcal{A}, \mu)$ be a measure space. The space $L^2(X, \mu)$ is a Hilbert space when equipped with the inner product

$$(f, g) = \int_X f\overline{g} d\mu$$

and the normed induced by this inner product is simply the L^2-norm $\| \cdot \|_2$.

Convention C.0.5. Given a measure space $(X, \mathcal{A}, \mu)$, assume that $L^2(X, \mu)$ is equipped with the inner product of Proposition C.0.4, unless otherwise specified.

Definition C.0.6 (Banach space over a complete valued field)**.** Let k be a field equipped with a non-trivial absolute value $| \cdot | : k \to \mathbb{R}_{\geq 0}$ such that k is complete with respect to the metric induced by $| \cdot |$. A *Banach space over* k is a vector space X over k equipped with a norm $\| \cdot \| : X \to \mathbb{R}_{\geq 0}$ satisfying:

$$\begin{aligned} \|x\| = 0 &\iff x = 0, \\ \|x + y\| \leq \|x\| + \|y\| &\quad \text{for all } x, y \in X, \\ \|\lambda x\| = |\lambda| \cdot \|x\| &\quad \text{for all } \lambda \in k, x \in X, \end{aligned}$$

such that the metric $d(x, y) := \|x - y\|$ makes (X, d) into a complete metric space.

Proposition C.0.7.
1. All Hilbert spaces are Banach spaces.
2. For any measure space $(X, \mathcal{A}, \mu)$, $L^p(X, \mu)$ is a Banach space when equipped with the L^p-norm $\| \cdot \|_p$.

Appendix D. Miscellaneous definitions

Definition D.0.1. A function $f : X \to \mathbb{C}$ on a locally compact Hausdorff space X is said to *vanish at infinity* if to every $\epsilon > 0$ there exists a compact set $K \subset X$ such that $|f(x)| < \epsilon$ for all x not in K.

Let $C_0(X)$ denote the space of functions $f : X \to \mathbb{C}$ that vanish at infinity.

Definition D.0.2. Let S be a scheme of prime characteristic p. The *Artin–Schreier morphism* is the scheme morphism

$$\begin{aligned} \mathbb{G}_{a,S} &\to \mathbb{G}_{a,S} \\ T &\mapsto T^p - T \end{aligned}$$

sometimes denoted by AS whose corresponding map $\mathrm{AS}^\sharp : k[T] \to k[T]$ of k-algebras is given by $T \mapsto T^p - T$.

Definition D.0.3. Let $\mathbb{F}_q$ be a finite field. Let $\psi : \mathbb{G}_a(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^*$ be a nontrivial character. Note that the Artin-Schreier morphism on $\mathbb{A}^1/\mathbb{F}_q$ corresponds to a surjective group homomorphism $\pi_1^{\text{ét}}(\mathbb{G}_a, 0) \to \mathbb{G}_a(\mathbb{F}_q)$. The rank 1 local system on $\mathbb{G}_a/\mathbb{F}_q$ corresponding to the composition

$$\pi_1^{\text{ét}}(\mathbb{G}_a, 0) \to \mathbb{G}_a(\mathbb{F}_q) \xrightarrow{\psi} \overline{\mathbb{Q}}_\ell^*$$

(♠ TODO: TODO: hyperlink a correspondence between local systems, applicable for $\overline{\mathbb{Q}}_\ell$-coefficients and representations of etale fundamental groups) is called the *Artin-Schreier sheaf of ψ* or *Artin-Schreier local system of ψ*. It may commonly be denoted by $\mathcal{L}_\psi$.

Definition D.0.4. Let S be a scheme of prime characteristic p. Let X be an S-scheme and let $\varphi : X \to S$ be a structure morphism. Let $n \in \mathbb{Z}$. The *relative p^nth power Frobenius morphism of X/S* is the morphism

$$(F_{X,p^n}, \varphi) : X \to X \times_S S_{F,p^n} = X^{(p^n)/S}$$

where

- $X^{(p^n)/S}$ denotes the extension of scalars of X/S by the p^nth power Frobenius
- S_{F,p^n} denotes the restriction of scalars of S (as an S-scheme), and
- φ is regarded as a morphism $X \to S_{F,p^n}$ by regarding S_{F,p^n} as a copy of S.

The relative Frobenius morphism above may commonly be denoted by notations such as $F_{X/S,p^n}$, $F_{X|S,p^n}$, $Frob_{X/S,p^n}$, and $Frob_{X|S,p^n}$. For $n < 0$, note that this definition is only well defined under the assumption that $F_{S,p^{|n|}}$ is invertible.

In the case that $n = 1$, this is referred to as the *relative Frobenius morphism of X/S* and may commonly be denoted by notations such as $F_{X/S}$, $F_{X|S}$, $Frob_{X/S}$, and $Frob_{X|S}$.

Definition D.0.5. Let S be a scheme. An algebraic group scheme over S (or an S-group scheme) is a group object G in the category of schemes over S; that is, G is an S-scheme equipped with S-morphisms: $m : G \times_S G \to G$ (*multiplication*), $i : G \to G$ (*inverse*), and $e : S \to G$ (*identity*), satisfying the group axioms expressed by the commutativity of the following diagrams:

1. **Associativity**
$$\begin{tikzcd} G \times_S G \times_S G \arrow[r, "m\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times m"'] & G \times_S G \arrow[d, "m"] \\ G \times_S G \arrow[r, "m"] & G \end{tikzcd}$$

2. **Identity**

$$\begin{tikzcd} G \times_S S \arrow[r, "\mathrm{id}\times e"] \arrow[dr, "\simeq"'] & G \times_S G \arrow[d, "m"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} S \times_S G \arrow[r, "e\times \mathrm{id}"] \arrow[dr, "\simeq"'] & G \times_S G \arrow[d, "m"] \\ & G \end{tikzcd}$$

3. **Inverse**

$$\begin{tikzcd} G \arrow[r, "(\mathrm{id}, i)"] \arrow[d, "\mathrm{id}"'] & G \times_S G \arrow[d, "m"] \\ G \arrow[r, "e\circ\pi"] & G \end{tikzcd}$$

where $\pi : G \to S$ is the structure morphism and $e \circ \pi$ sends g to the identity section.

If G is affine over S, we call it an *affine group scheme over S*.

If the base scheme S is the spectrum of a field k, then we call G a *k-algebraic group* or an *algebraic group (scheme) over k*. If G is additionally a k-variety, then we call G a *k-group variety*.

Definition D.0.6. Let X be a scheme of characteristic p. Let $x \in |X|$ be a closed point whose residue field is a finite field. Let $\overline{x}$ be a geometric point over X. Let $K \in D(X)$ be an object in the derived category of sheaves of abelian groups on X.

The geometric Frobenius automorphism in $\mathrm{Gal}(\overline{\kappa(x)}/\kappa(x))$ can be identified as an automorphism

$$\operatorname{Spec} \overline{\kappa(x)} \to \operatorname{Spec} \overline{\kappa(x)}$$

over $\operatorname{Spec} \kappa(x)$ and hence induces an action

$$\mathrm{Frob}_{\overline{x}} : K_{\overline{x}} \to K_{\overline{x}}$$

on the stalk. Up to isomorphism this action only depends on the closed point x and not the choice of the geometric point $\overline{x}$ above x.

This automorphism may be called the *geometric Frobenius action on K at the stalk at $\overline{x}$*. In case that the size of the residue field $\kappa(x)$ is q (or is considered as q, say by virtue of changing the base field over which X is defined) we may also denote the geometric Frobenius action using notations such as $\mathrm{Frob}_{\overline{x},q}$ or Frob_q to emphasize the size of $\kappa(x)$.

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