ALGEBRAIC NUMBER THEORY

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Contents

1. Dedekind domains

Definition 1.0.1 (Noetherian conditions for a ring)**.** Let R be a ring. We say:

- R is *left-Noetherian* if every ascending chain of left ideals of R stabilizes, i.e., if for any chain
$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$
of left ideals, there exists n such that $I_m = I_n$ for all $m \geq n$.
- R is *right-Noetherian* if every ascending chain of right ideals of R stabilizes.
- R is *Noetherian* if it is both left-Noetherian and right-Noetherian.

(♠ TODO: finitely generated ideal) If R is commutative (Definition C.1.1), then R is Noetherian if and only if every ideal is finitely generated.

Definition 1.0.2 (Discrete valuation ring)**.** (♠ TODO: define principal ideal) A local integral domain (Definition C.1.2) $(R, \mathfrak{m})$ with maximal ideal (Definition C.1.4) $\mathfrak{m}$ is called a *discrete valuation ring (DVR)* if $\mathfrak{m}$ is principal and nonzero, and every nonzero ideal of R is of the form $\mathfrak{m}^n$ for some integer $n \geq 0$.

The fraction field of R then becomes a discrete valuation field (Definition B.2.3). (♠ TODO: explain how the discrete valuation works here.)

Theorem 1.0.3 (Correspondence between DVRs and discretely valued fields)**.** (♠ TODO: define field of fractions) Let K be a field equipped with a discrete valuation (Definition B.2.3) $v : K^\times \to \mathbb{Z}$. The valuation ring (Definition B.2.2)
$$R_v = \{x \in K \mid v(x) \geq 0\}$$
is a discrete valuation ring (Definition 1.0.2). Conversely, if $(R, \mathfrak{m})$ is a discrete valuation ring with field of fractions $K = \mathrm{Frac}(R)$, then there exists a discrete valuation
$$v : K^\times \to \mathbb{Z}$$
such that $R = R_v$ is the valuation ring associated to v.

In particular, the category of discrete valuation rings (up to isomorphism) corresponds bijectively to the category of fields equipped with a discrete valuation.

Definition 1.0.4 (Integral element over a ring)**.** Let R be a commutative ring with unity (Definition C.1.1).

1. Let A be an R-algebra. An element $a \in A$ is called *integral over R* if there exists a monic polynomial
$$p(x) = x^n + r_{n-1}x^{n-1} + \cdots + r_1 x + r_0$$
with coefficients $r_i \in R$ such that
$$p(a) = a^n + r_{n-1}a^{n-1} + \cdots + r_1 a + r_0 = 0 \quad \text{in } A.$$
2. Let A be an extension ring of R. The ring extension A/R is called an *integral extension* if every element of A is integral over R.
(♠ TODO: define subring)
3. Let A be an extension ring of R. The *integral closure of R in A*, sometimes denoted by $\widetilde{A}$, is the subring
$$\widetilde{A} = \{a \in A : a \text{ is integral over } R\}.$$

We say that R is integrally closed in A if $\widetilde{A}$ coincides with A (considered as a subring of R).

4. Let R be an integral domain with field of fractions $K = \mathrm{Frac}(R)$. We say that R is *integrally closed* if it is integrally closed as a subring of K.

Definition 1.0.5 (Dedekind domain)**.** An integral domain (Definition C.1.2) R is called a *Dedekind domain* if it satisfies the following equivalent conditions: (♠ TODO: define field of fractions)

- R is Noetherian (Definition 1.0.1), integrally closed (Definition 1.0.4) in its field of fractions, and every nonzero prime ideal (Definition C.1.4) of R is maximal (Definition C.1.4).
- Equivalently: for every nonzero prime ideal $\mathfrak{p}$ of R, the localization (Definition C.1.3) $R_{\mathfrak{p}}$ is a discrete valuation ring (Definition 1.0.2).

The following is immediately from the definitions:

Proposition 1.0.6. Every DVR (Definition 1.0.2) is a Dedekind domain (Definition 1.0.5). Conversely, every local (Definition C.1.5) Dedekind domain is a DVR.

Proposition 1.0.7. Let R be a Dedekind domain (Definition 1.0.5). It is a PID (Definition C.1.6) if and only if it is a UFD (Definition C.1.7).

2. Local and global fields

2.1. Local fields.

Definition 2.1.1 (Local field)**.** A *local field* is a field K with a nontrivial absolute value (Definition B.1.1) $|\cdot|$ such that K is locally compact (Definition C.2.1) under the topology induced by $|\cdot|$; we almost always treat a local field as a topological field with this topology.

The local field K is called *archimedean* if its absolute value is archimedean (Definition B.1.5) and is called *non-archimedean* if its absolute value is non-archimedean.

Convention 2.1.2. Given a local field (Definition 2.1.1) K with absolute value $|\cdot|$, we almost always equip it with the metric induced by $|\cdot|$ (Definition B.1.4), which is a complete metric (Theorem 2.1.3)

Theorem 2.1.3. Let K be a local field (Definition 2.1.1). K is complete under the metric induced by (Definition B.1.4) its absolute value.

Theorem 2.1.4. (♠ TODO: TODO: define Laurent seires) Up to isomorphism, local fields (Definition 2.1.1) consists precisely of:

- $\mathbb{R}$ and $\mathbb{C}$, which are the archimedean local fields),
- finite extensions of $\mathbb{Q}_p$ for a prime p, which are the nonarchimedean local fields of characteristic 0, and
- finite extensions of $\mathbb{F}_p((t))$, the field of formal Laurent series over a finite field, which are the non-archimedean local fields of positive characteristic.

2.2. **Global fields.**

Definition 2.2.1. A *global field* is a field K that is either:

- a finite extension of the field of rational numbers $\mathbb{Q}$ (i.e., a number field), or
- a finite extension of the field of rational functions $\mathbb{F}_q(t)$ in one variable over a finite field $\mathbb{F}_q$ (i.e., a global function field).

Definition 2.2.2. Two absolute values (Definition B.1.1) $|\cdot|_1$ and $|\cdot|_2$ on a field F are *equivalent* if there exists a positive real number $c > 0$ such that

$$|\cdot|_1 = |\cdot|_2^c.$$

Definition 2.2.3 (Place of a global field)**.** Let F be a global field. A *place of F* is an equivalence class (Definition 2.2.2) of absolute values (Definition B.1.1) on F.

If any (equivalently all) representatives of a place v of F is an archimedean absolute value (Definition B.1.5) (resp. non-archimedean absolute value), then we say that v is an *archimedean place* (resp. *non-archimedean place*). A representative of a place v is often denoted by $|\cdot|_v$.

Definition 2.2.4. Let K be a global field (Definition 2.2.1) and let v be a place (Definition 2.2.3) of K. Write $|\cdot|_v$ for an absolute value representing v. The *completion of K at v*, often denoted K_v, is the completion of K with respect to the metric induced by $|\cdot|_v$ (Definition B.1.4).

2.3. **Local fields are exactly the completions of global fields at places.**

Theorem 2.3.1. Let L be a global field (Definition 2.2.1) and let v be any place (Definition 2.2.3). The completion L_v (Definition 2.2.4) of L with respect to v is a local field (Definition 2.1.1).

Remark 2.3.2. Conversely, all local fields can be obtained as completions of global fields at places.

2.4. **Rings of integers of local and global fields.**

Definition 2.4.1 (Ring of Integers)**.** Let K be either a global field (Definition 2.2.1) or a nonarchimedean (Definition B.1.5) local field (Definition 2.1.1).

The *ring of integers of K* or *integer ring of K*, often denoted by notations including $\mathcal{O}_K$ and $\mathscr{O}_K$, is the set of elements of K that are integral over an appropriate base ring as follows:

- In the case where K is a number field, $\mathcal{O}_K$ is the integral closure (Definition 1.0.4) of $\mathbb{Z}$ in K, i.e.
$$\mathcal{O}_K := \{\, x \in K \mid x \text{ is a root of a monic polynomial } f(t) \in \mathbb{Z}[t] \,\}.$$
- In the case where K is a global function field over a finite field, i.e. a finite extension of $\mathbb{F}_q(t)$, $\mathcal{O}_K$ is the integral closure of the polynomial ring $\mathbb{F}_q[t]$ in K.
- In the case where K is a nonarchimedean local field with discrete valuation v, $\mathcal{O}_K$ is the valuation ring $\mathcal{O}_v$ (Definition B.2.2) of v, i.e.
$$\mathcal{O}_K := \mathcal{O}_v = \{x \in K \mid v(x) \geq 0\}.$$

Equivalently, K is the integral closure of $\mathbb{Z}_p$ or of $\mathbb{F}_p((t))$ under and identification of K with (Theorem 2.1.4) a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$

Theorem 2.4.2.
1. The ring of integers (Definition 2.4.1) of a nonarchimedean (Definition B.2.4) local field (Definition 2.1.1) is a DVR (Definition 1.0.2). In particular (Proposition 1.0.6), it is a Dedekind domains (Definition 1.0.5).
2. The ring of integers (Definition 2.4.1) of a global field (Definition 2.2.1)is a Dedekind domains (Definition 1.0.5).

3. Adèles and idèles of global fields

Definition 3.0.1. Let K be a global field. Write M_K for the set of all places (Definition 2.2.3) of K and write M_K^∞ for the set of archimedean places of K. Let $S \subseteq M_K$ be some subset of places of K (typically, S is a finite set). For each $v \in M_K$, write $\mathcal{O}_v$ for the ring of integers (Definition 2.4.1) in the completion K_v (Definition 2.2.4) (Theorem 2.3.1)

- The *adèle ring of K*, denoted $\mathbb{A}_K$, is the restricted direct product of the K_v (over all places v of K), with respect to the $\mathcal{O}_v$ at non-archimedean v:

$$\mathbb{A}_K = \left\{ (x_v)_v \in \prod_{v \in M_K} K_v \;\middle|\; x_v \in \mathcal{O}_v \text{ for all but finitely many non-archimedean } v \right\}.$$

- The *idèle group of K*, commonly denoted $\mathbb{A}_K^\times$ or $\mathbb{I}_K$, is the group of invertible elements of $\mathbb{A}_K$:

$$\mathbb{I}_K = \mathbb{A}_K^\times = \left\{ (x_v)_v \in \prod_{v \in M_K} K_v^\times \;\middle|\; x_v \in \mathcal{O}_v^\times \text{ for all but finitely many non-archimedean } v \right\},$$

  where $\mathcal{O}_v^\times$ denotes the group of units of $\mathcal{O}_v$ for non-archimedean v.
- The *adèle ring outside S of K*, commonly denoted $\mathbb{A}_K^S$ or $\mathbb{A}_{K,S}$, is the restricted product of the completions K_v over all places $v \in M_K \setminus S$, with respect to the rings of integers $\mathcal{O}_v$ at non-archimedean places:

$$\mathbb{A}_{K,S} = \mathbb{A}_K^S = \left\{ (x_v)_v \in \prod_{v \in M_K \setminus S} K_v \;\middle|\; x_v \in \mathcal{O}_v \text{ for all but finitely many non-archimedean } v \right\}.$$

- The *idèle group outside S of K*, commonly denoted $(\mathbb{A}_K^\times)^S$, $(\mathbb{A}_{K,S}^\times)$, $\mathbb{I}_K^S$, or $\mathbb{I}_{K,S}$ is the group of invertible elements of $\mathbb{A}_K^S$:

$$(\mathbb{A}_K^\times)^S = \left\{ (x_v)_v \in \prod_{v \in M_K \setminus S} K_v^\times \;\middle|\; x_v \in \mathcal{O}_v^\times \text{ for all but finitely many non-archimedean } v \right\}.$$

- The *ring of finite adèles of K*, commonly denoted $\mathbb{A}_{K,\mathrm{fin}}$, $\mathbb{A}_K^{\mathrm{fin}}$, $\mathbb{A}_{K,\mathrm{f}}$, $\mathbb{A}_K^{\mathrm{f}}$, is the adèle ring outside $S = M_K^\infty$, the set of archimedean places of K:

$$\mathbb{A}_{K,\mathrm{fin}} := \mathbb{A}_K^{M_K^\infty} = \left\{ (x_v)_v \in \prod_{v \notin M_K^\infty} K_v \;\middle|\; x_v \in \mathcal{O}_v \text{ for all but finitely many non-archimedean } v \right\}.$$

- The *finite idèle group of K*, commonly denoted $\mathbb{A}_{K,\mathrm{fin}}^\times$, $\mathbb{I}_{K,\mathrm{fin}}$, $\mathbb{I}_K^{\mathrm{fin}}$, $\mathbb{I}_{K,\mathrm{f}}$, $\mathbb{I}_K^{\mathrm{f}}$ etc. is the group of units of the ring of finite adèles:

$$\mathbb{A}_{K,\mathrm{fin}}^\times := (\mathbb{A}_K^\times)^{M_K^\infty} = \left\{ (x_v)_v \in \prod_{v \notin M_K^\infty} K_v^\times \;\middle|\; x_v \in \mathcal{O}_v^\times \text{ for all but finitely many non-archimedean } v \right\}.$$

All of these are equipped with the restricted product topology induced by the topologies of the local fields (Definition 2.1.1) K_v and the subspace topologies thereof.

Definition 3.0.2 (Idelic norm of a global field)**.** Let F be a global field. The *idelic norm* (also called the *module*) is the map

$$|\cdot| : \mathbb{A}_F^\times \to \mathbb{R}_{>0}$$

defined by (♠ TODO: definition of canonical absolute value of places)

$$|(x_\mathfrak{p})_\mathfrak{p}| := \prod_\mathfrak{p} |x_\mathfrak{p}|_\mathfrak{p},$$

where the product is well-defined as all but finitely many factors equal 1.

(♠ TODO: state the product formula for global fields, define idele class group) By the product formula for global fields, the idelic norm factors through the idele class group and satisfies

$$|a| = 1 \quad \text{for all } a \in F^\times,$$

where $F^\times$ is diagonally embedded in $\mathbb{A}_F^\times$.

4. Chebotarev density theorem

The Chebotarev density theorem is a statement that Frobenius elements $\mathrm{Frob}_\mathfrak{p}$ are equidistributed across the conjugacy classes of $\mathrm{Gal}(L/K)$ of a finite Galois extension of number fields, showing that the splitting behavior of primes in extensions is governed by uniform distribution with respect to the Galois group structure.

Theorem 4.0.1 (Chebotarev Density Theorem for number fields)**.** (♠ TODO: define unramified prime) Let L/K be a finite Galois extension (Definition A.0.1) of number fields (Definition 2.2.1) with Galois group $G = \mathrm{Gal}(L/K)$ (Definition A.0.1). For a conjugacy class $C \subseteq G$, let

$$\pi_C(x) = \#\{\, \mathfrak{p} \subseteq \mathcal{O}_K : N\mathfrak{p} \le x,\ \mathfrak{p} \text{ unramified in } L,\ \mathrm{Frob}_\mathfrak{p} \in C \,\}$$

be the number of prime ideals $\mathfrak{p}$ of K with norm at most x whose Frobenius conjugacy class $\mathrm{Frob}_\mathfrak{p}$ in G equals C.

(♠ TODO: define natural density of primes) Then the natural density of such primes exists and satisfies

$$\lim_{x\to\infty} \frac{\pi_C(x)}{\pi(x)} = \frac{|C|}{|G|},$$

where $\pi(x)$ is the number of prime ideals $\mathfrak{p} \subseteq \mathcal{O}_K$ with $N\mathfrak{p} \leq x$.

Theorem 4.0.2 (Chebotarev Density theorem for function fields, see e.g. [Ros02, Theorem 9.13A, 9.13B])**.** Let L/K be a Galois extension (Definition A.0.1) of global function fields (Definition 2.2.1) with Galois group $G = \mathrm{Gal}(L/K)$. Let $C \subset G$ be a conjugacy class in G and S'_K be the set of primes of K which are unramified in L. (♠ TODO: continue statement)

Appendix A. Galois theory

Definition A.0.1 (Galois Extension)**.** An extension L/K is called a *Galois extension* if it is both a normal extension and a separable extension. Its *Galois group*, usually denoted by $\mathrm{Gal}(L/K)$, is defined to be the automorphism group $\mathrm{Aut}(L/K)$.

Appendix B. Absolute values and valuations on fields

B.1. **Absolute values on fields.**

Definition B.1.1 (Absolute Value on a Field)**.** Let F be a field. An *absolute value on F* is a function

$$|\cdot| : F \to \mathbb{R}_{\geq 0}$$

that satisfies the following properties for all $a, b \in F$:

1. Non-negativity: $|a| \geq 0$,
2. Positive-definiteness: $|a| = 0 \iff a = 0$,
3. Multiplicativity: $|ab| = |a| \cdot |b|$,
4. Triangle inequality: $|a + b| \leq |a| + |b|$.

Here, 0 denotes the additive identity of the field F, and the codomain $\mathbb{R}_{\geq 0}$ consists of non-negative real numbers.

Definition B.1.2 (Discrete absolute value)**.** Let K be a field equipped with an absolute value (Definition B.1.1)$|\cdot|$. The absolute value $|\cdot|$ is called *discrete* if its image $|K^\times| = \{|x| : x \in K^\times\} \subseteq \mathbb{R}_{>0}$ is a discrete subgroup of the multiplicative group $\mathbb{R}_{>0}$ (with the usual topology).

Definition B.1.3 (Trivial absolute value)**.** Let K be a field. The *trivial absolute value on K* is the absolute value (Definition B.1.1)

$$|\cdot|_{\mathrm{triv}} : K \to \mathbb{R}_{\geq 0}$$

defined by

$$|x|_{\mathrm{triv}} := \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \neq 0. \end{cases}$$

Definition B.1.4. Let K be a field with an absolute value $|\cdot|$. By the metric induced by $|\cdot|$ we mean the metric induced by $|\cdot|$ regarded as a norm (Definition C.2.5) on the 1-dimensional K-vector space K with absolute value $|\cdot|_v$.

Definition B.1.5. Let F be a field. An absolute value on F is said to be *non-archimedean* if

$$|x+y| \leq \max(|x|, |y|) \quad \text{for all } x, y \in F$$

and is said to be *archimedean* otherwise.

Definition B.1.6 (Topology on a field induced by an absolute value)**.** Let K be a field equipped with an absolute value (Definition B.1.1) $|\cdot| : K \to \mathbb{R}_{\geq 0}$. The *topology induced by $|\cdot|$ on K* is defined by declaring a subset $U \subseteq K$ to be open if for every $x \in U$, there exists $\varepsilon > 0$ such that

$$B(x, \varepsilon) := \{y \in K : |y - x| < \varepsilon\} \subseteq U.$$

The collection of all such open sets forms a topology on K, making $(K, \mathcal{T}_{|\cdot|})$ a topological field. The set $B(x, \varepsilon)$ is called the *open ball of radius ε around x*.

Equivalently, the topology induced by $|\cdot|$ on K is the topology induced by the metric induced by $|\cdot|$ as a norm (Definition C.2.5) on the 1-dimensional K-vector space K.

B.2. **Valuations on fields.**

Definition B.2.1 (Valuation on a field)**.** Let K be a field. A *valuation on K* is a function

$$v : K \to \Gamma \cup \{\infty\},$$

where $(\Gamma, +, \leq)$ is a totally ordered abelian group and ∞ is an element greater than all elements of Γ, satisfying for all $x, y \in K$:

1. $v(x) = \infty$ if and only if $x = 0$,
2. $v(xy) = v(x) + v(y)$,
3. $v(x + y) \geq \min\{v(x), v(y)\}$.

Alternatively (and essentially equivalently), a valuation on K is also defined as a function $v : K^\times \to \Gamma$ with properties (2) and (3) and is extended into a function $v : K \to \Gamma \cup \{\infty\}$ by setting $v(0) = \infty$.

The pair (K, v) is called a *valued field*.

Definition B.2.2 (Valuation Ring of a Valued Field)**.** Let K be a field equipped with a valuation (Definition B.2.1) $v : K \to \Gamma$, where Γ is a totally ordered abelian group.

The *valuation ring of the valued field (K, v)* is the subring

$$\mathcal{O}_v := \{x \in K \mid v(x) \geq 0\},$$

where 0 is the neutral element of Γ. This ring $\mathcal{O}_v$ is a local ring with maximal ideal

$$\mathfrak{m}_v := \{x \in K \mid v(x) > 0\}.$$

Its *residue field* is defined as $\kappa_v := \mathcal{O}_v / \mathfrak{m}_v$.

Definition B.2.3 (Discrete valuation)**.** Let (K, v) be a valued field. The valuation v is called *discrete* if the value group $\Gamma = v(K^\times)$ is isomorphic, as an ordered group, to $\mathbb{Z}$ (the integers with the usual order).

Definition B.2.4. Let K be a field and let $v : K \to \mathbb{Z}$ be a discrete valuation (Definition B.2.3). If a positive real number $c < 1$. The function

$$|x|_v := \begin{cases} c^{v(x)} & \text{if } x \in K^\times \\ 0 & \text{if } x = 0 \end{cases}$$

is a nonarchimedean absolute value on K (Definition B.1.5). Different values of c yield equivalent absolute values and thus do not change the topology induced by $|\cdot|_v$ (Definition B.1.6)

Appendix C. Miscellaneous definitions

C.1. **Abstract algebra.**

Definition C.1.1. A *commutative (unital) ring* is a ring $(R, +, \cdot)$ such that $\cdot$ is a commutative operation, i.e. $a \cdot b = b \cdot a$.

For many writers (e.g. "commutative" algebraists or number theorists), a *ring* refers to a commutative ring as above.

Definition C.1.2. Let $(R, +, \cdot)$ be a not-necessarily commutative ring.

1. An element $a \in R$ is a *left zero-divisor* if there exists a nonzero $x \in R$ such that $ax = 0$. Otherwise, a is called *left regular* or *left cancellable*.
2. An element $a \in R$ is a *right zero-divisor* if there exists a nonzero $x \in R$ such that $xa = 0$. Otherwise, a is called *right regular* or *right cancellable*.
3. An element $a \in R$ is a *zero-divisor* if it is a left zero-divisor or a right zero-divsor.
4. An element $a \in R$ is a *two-sided zero-divisor* if it is both a left zero-divisor and a right zero-divsor.
5. An element $a \in R$ is *regular*, *cancellable*, or a *non-zero-divisor* if it is both left and right regular.

A zero-divisor of any kind that is not itself 0 is said to be a *nonzero zero divisor* or a *nontrivial zero divisor* of its kind.

A non-zero ring with no nontrivial zero divisors is called a *domain*. A domain that it also a commutative ring (Definition C.1.1) is also called an *integral domain*.

Definition C.1.3. Let R be a commutative ring with unity (Definition C.1.1) and let $S \subseteq R$ be a multiplicative subset. The *localization of R at S*, denoted by $S^{-1}R$, is the ring whose elements are equivalence classes of pairs $(r, s) \in R \times S$ under the equivalence relation

$$(r, s) \sim (r', s') \iff \exists u \in S \text{ such that } u(sr' - s'r) = 0.$$

Write $\frac{r}{s}$ for the equivalence class of (r, s). Addition and multiplication on representatives are defined by

$$\frac{r}{s} + \frac{r'}{s'} = \frac{rs' + r's}{ss'},$$
$$\frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}.$$

The map $r \mapsto \frac{r}{1}$ defines a ring homomorphism; therefore, $S^{-1}R$ is naturally an R-algebra.

If P is a prime ideal of R (Definition C.1.4), then $R_P := S^{-1}R$ with $S = R \setminus P$ is called the *localization of R at P*. It is a local ring (Definition C.1.5) whose maximum ideal is given by

$$S^{-1}P = \{\frac{p}{s} \in R_P : p \in P\}.$$

Definition C.1.4. Let R be a (not necessarily commutative) ring. A proper two-sided ideal $P \trianglelefteq R$ is called a *prime ideal* if the following equivalent conditions holds:

1. If I, J are left ideals and $IJ \subset P$, then $I \subset P$ or $J \subset P$.
2. If I, J are right idaels and $IJ \subset P$, then $I \subset P$ or $J \subset P$.
3. If I, J are two-sided idaels and $IJ \subset P$, then $I \subset P$ or $J \subset P$.
4. If $x, y \in R$ with $xRy \subset \mathfrak{p}$, then $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

A proper left/right/two-sided ideal $M \subsetneq R$ is called *maximal* if there exists no other left/right/two-sided ideal $J \trianglelefteq R$ such that $M \subsetneq J \subsetneq R$. Equivalently,

- a left/right ideal M of R is maximal if and only if the quotient module R/M is a simple left/right R-module.
- a two-sided ideal M of R is maximal if and only if the quotient ring R/M is a simple ring.

Definition C.1.5. Let R be a ring with unity, not necessarily commutative. The ring R is called a *local ring* if it has a unique maximal left ideal (Definition C.1.4). In this case, R also has a unique maximal right ideal, and these coincide with the Jacobson radical $J(R)$ of R. The unique maximal left (and right) ideal of a local ring R may sometimes be denoted by $\mathfrak{m}_R$.

Definition C.1.6 (Principal ideal ring/domain (PID))**.** Let R be a commutative unital ring (Definition C.1.1). Then R is a *principal ideal ring (PIR)* if every ideal of R is principal. If R is additionally an integral domain (Definition C.1.2), then R is said to be a *principal ideal domain (PID)*

Definition C.1.7 (Unique factorization domain (UFD) / Factorial ring)**.** (♠ TODO: define irreducible elements) An integral domain (Definition C.1.2) R is called a *unique factorization domain (UFD)* or *factorial ring* if every nonzero nonunit element of R can be factored as a product of irreducible elements uniquely up to order and units.

C.2. **Absolute values and norms.**

Definition C.2.1 (Locally compact)**.** Let $(X, \mathcal{T})$ be a topological space. X is *locally compact* if for every $x \in X$, there exists an open set $U \in \mathcal{T}$ containing x and a compact set $K \subseteq X$ such that $x \in U \subseteq K$.

Definition C.2.2 (Extended Metric Induced by an Extended Norm)**.** Let V be a vector space over a field F equipped with an absolute value (Definition B.1.1)

$$|\cdot| : F \to [0, \infty),$$

and let $\|\cdot\| : V \to [0, \infty]$ be an extended norm on V (Definition C.2.5). Then the *extended metric induced by the extended norm* is the function

$$d : V \times V \to [0, \infty]$$

defined by

$$d(x, y) := \|x - y\|.$$

It is indeed an extended metric. If $\|\cdot\|$ is a norm, then d is a metric.

Definition C.2.3 (Topology induced by an extended metric)**.** Let (X, d) be an extended metric space. The *topology induced by d on X* is defined by declaring a subset $U \subseteq X$ to be open if for every $x \in U$, there exists $\varepsilon > 0$ such that the open ball

$$B(x, \varepsilon) := \{y \in X : d(x, y) < \varepsilon\}$$

is contained in U. The collection of all such open sets forms a topology on X. The set $B(x, \varepsilon)$ is called the *open ball of radius ε centered at x*.

Definition C.2.4 (Topology induced by a norm)**.** Let V be a vector space over a field K with absolute value (Definition B.1.1) $|\cdot|$, and let $\|\cdot\|$ be an extended norm on V (Definition C.2.5). The *topology induced by the extended norm $\|\cdot\|$ on V* is defined by declaring a subset $U \subseteq V$ to be open if for every $x \in U$, there exists $\varepsilon > 0$ such that

$$B(x, \varepsilon) := \{y \in V : \|y - x\| < \varepsilon\}$$

is contained in U. The set $B(x, \varepsilon)$ is called the *open ball of radius ε around x*. The collection of all such open sets forms a topology on V.

Equivalently, the topology on V induced by the extended norm $\|\cdot\|$ is the topology on V induced by (Definition C.2.3) the extended metric $d : V \times V \to [0, \infty]$ induced by $\|\cdot\|$ (Definition C.2.2).

Definition C.2.5 (Extended Norm)**.** Let V be a vector space over a field F equipped with an absolute value (Definition B.1.1)

$$|\cdot| : F \to [0, \infty).$$

An *extended norm on V* is a function

$$\|\cdot\| : V \to [0, \infty]$$

satisfying for all $x, y \in V$ and all scalars $\alpha \in F$:

1. **Positive definiteness:** $\|x\| = 0$ if and only if $x = 0$.

2. **Homogeneity:** $\|\alpha x\| = |\alpha| \cdot \|x\|$.
3. **Triangle inequality:** $\|x + y\| \leq \|x\| + \|y\|$,

where arithmetic is extended to allow sums involving ∞ with the convention that $a+\infty = \infty$ for any $a \in [0, \infty]$. A vector space with an extended norm over a field with an absolute value is called an *extended normed space*.

If the range of the extended norm is contained in $[0, \infty)$, then the extended norm is a *norm* in the usual sense and V may be called a *normed space*.

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