SET THEORY

November 27, 2025

Contents

1. ZFC Axioms

(♠ TODO: language of set theory)

Notation 1.0.1. Let the language of set theory consist of the binary relation symbol $\in$, intended to denote *membership*. The objects of discourse, called *sets*, are elements of a nonempty universe V, and all variables range over the elements of V.

Definition 1.0.2. (♠ TODO: first order language) The *axioms of Zermelo–Fraenkel set theory (ZF)* are the following statements, formulated in the first-order language with symbol $\in$:

1. **Axiom of Extensionality:** Two sets are equal if and only if they have the same elements.
$$\forall A\, \forall B\, (\forall x\, (x \in A \leftrightarrow x \in B) \to A = B).$$
2. **Axiom of Pairing:** For any sets x, y, there exists a set containing exactly x and y as elements.
$$\forall x\, \forall y\, \exists A\, \forall z\, [z \in A \leftrightarrow (z = x \lor z = y)].$$
3. **Axiom of Union:** For any set A, there exists a set that is the union of the elements of A.
$$\forall A\, \exists U\, \forall x\, [x \in U \leftrightarrow \exists B\, (x \in B \land B \in A)].$$

4. **Axiom of Power Set:** For any set A, there exists a set $\mathcal{P}(A)$ containing all subsets of A.
$$\forall A \, \exists P \, \forall B \, [B \in P \leftrightarrow B \subseteq A].$$
5. **Axiom of Infinity:** There exists an inductive set, that is, a set containing the empty set and closed under the successor operation.
$$\exists I \, [\varnothing \in I \land \forall x \, (x \in I \to x \cup \{x\} \in I)].$$
6. **Axiom Schema of Separation:** For any property $\varphi(x)$ expressible in the language of set theory and any set A, there is a subset of A consisting of the elements of A satisfying $\varphi(x)$.
$$\forall A \, \exists B \, \forall x \, [x \in B \leftrightarrow (x \in A \land \varphi(x))].$$
7. **Axiom Schema of Replacement:** For any definable function given by a formula $\varphi(x, y)$ ensuring that for each x there exists a unique y with $\varphi(x, y)$, the image of any set under this function is also a set.
$$\forall A \, [\forall x \in A \, \exists! y \, \varphi(x, y)] \to \exists B \, \forall y \, (y \in B \leftrightarrow \exists x \in A \, \varphi(x, y)).$$
8. **Axiom of Foundation (Regularity):** Every nonempty set A contains an element x that is disjoint from A.
$$\forall A \, [A \neq \varnothing \to \exists x \, (x \in A \land x \cap A = \varnothing)].$$

The *Axiom of Choice (AC)* asserts that for every set X of nonempty sets, there exists a function $f : X \to \bigcup X$ such that $f(A) \in A$ for all $A \in X$. Formally:
$$\forall X \, [(\forall A \in X \; A \neq \varnothing) \to \exists f \, (\mathrm{dom}(f) = X \land \forall A \in X \; f(A) \in A)] \, .$$
When ZF is augmented by the Axiom of Choice, the resulting system is called *Zermelo–Fraenkel set theory with Choice (ZFC)*.

Definition 1.0.3 (Small Set)**.** A *small set* refers to a set; the adjective of *small* is used to emphasize that the set is a set, rather than a proper class.

Alternatively, a *small set* is any object that belongs to a fixed universe of sets $\mathcal{U}$, called the *universe of small sets*. In other words, a set A is said to be *small* if and only if $A \in \mathcal{U}$. Elements of $\mathcal{U}$ may themselves be sets, but $\mathcal{U}$ is not assumed to be a set; it may be a proper class.

Definition 1.0.4 (Subset)**.** Let A and B be sets. The set A is said to be a *subset of B*, written as $A \subseteq B$ or $A \subset B$, if every element of A is also an element of B, that is,
$$A \subseteq B \iff \forall x \, (x \in A \Rightarrow x \in B).$$
If $A \subseteq B$ and $A \neq B$, then A is called a *proper subset of B*; this is commonly denoted by $A \subsetneq B$.

Lemma 1.0.5. Let A and B be sets. They are equal if and only if $A \subseteq B$ (Definition 1.0.4) and $B \subseteq A$.

Proof. This is by the axiom of extensionality (Definition 1.0.2). $\square$

Definition 1.0.6 (Power Set)**.** Let A be a set. The *power set of A*, denoted by $\mathcal{P}(A)$, is the set of all subsets of A:

$$\mathcal{P}(A) := \{\, B \mid B \subseteq A \,\}.$$

Equivalently, every element of $\mathcal{P}(A)$ is itself a set B satisfying $B \subseteq A$. Under the axiom of power set (Definition 1.0.2), note that the $\mathcal{P}(A)$ exists.

Definition 1.0.7 (Set Complement)**.** Let U be a fixed set and let $A \subseteq U$ be a subset (Definition 1.0.4) of U. The *complement of A (with respect to U)*, denoted by A^c or $U \setminus A$, is defined as

$$A^c := \{x \in U \mid x \notin A\}.$$

Equivalently, for any $x \in U$, one has

$$x \in A^c \iff x \notin A.$$

When the set U is clear from context, the notation A^c may be used without explicit reference to U.

Definition 1.0.8 (Union of Sets)**.** Let I be a (possibly infinite (Definition 4.0.2) but small (Definition 1.0.3)) index set and let $\{A_i\}_{i\in I}$ be a family of sets indexed by I. The *union of the family $\{A_i\}_{i\in I}$*, denoted by $\bigcup_{i\in I} A_i$, is defined as

$$\bigcup_{i\in I} A_i := \{x \mid \exists i \in I \text{ such that } x \in A_i\}.$$

Under the axiom of union (Definition 1.0.2), note that $\bigcup_{i\in I} A_i$ exists. For finitely many sets $A_1, A_2, \dots, A_n$, their union is denoted by $A_1 \cup A_2 \cup \cdots \cup A_n$.

Definition 1.0.9 (Intersection of Sets)**.** Let I be a (possibly infinite (Definition 4.0.2) but small (Definition 1.0.3)) index set and let $\{A_i\}_{i\in I}$ be a family of sets indexed by I. The *intersection of the family $\{A_i\}_{i\in I}$*, denoted by $\bigcap_{i\in I} A_i$, is defined as

$$\bigcap_{i\in I} A_i := \{x \mid \forall i \in I,\, x \in A_i\}.$$

For a finite family of sets $A_1, A_2, \dots, A_n$, their intersection is denoted by $A_1 \cap \cdots \cap A_n$

2. Maps of sets

Definition 2.0.1. Let X and Y be sets. A *map* (or *function*) from X to Y is a rule f assigning to each element $x \in X$ exactly one element $f(x) \in Y$. We write $f : X \to Y$.

We say that X is the *domain* and that Y is the *codomain of f*.

Definition 2.0.2. The category of sets is the (locally small) (Definition 5.0.2) category (Definition 5.0.1)

- Whose objects are sets (Definition 1.0.2)
- Whose morphisms $X \to Y$ are set functions (Definition 2.0.1) $X \to Y$.

The category of sets is often denoted by notations such as Set, $\mathbf{Set}$, Sets, $\mathbf{Sets}$, (Set), $(\mathbf{Set})$, (Sets), $(\mathbf{Sets})$.

Definition 2.0.3. Let X be a set. The *identity function on X*, denoted by id_X, is the function (Definition 2.0.1) $\mathrm{id}_X : X \to X$ defined by

$$\mathrm{id}_X(x) = x \quad \text{for all } x \in X.$$

It is the unique function on X satisfying $f \circ \mathrm{id}_X = f = \mathrm{id}_X \circ f$ for every function $f : X \to Y$ and every function $f : Y \to X$. The identity function is the identity map (Definition 5.0.1) on the objects X in the category of sets (Definition 2.0.2).

Definition 2.0.4 (Preimage of a subset under a map of sets)**.** Let X and Y be sets (Definition 1.0.2) and let $f : X \to Y$ be a function (Definition 2.0.1). Let $B \subseteq Y$ be a subset (Definition 1.0.4) of the codomain Y. The *preimage of B under f* (also called the *inverse image of B under f*) is the subset of X defined by

$$f^{-1}(B) := \{x \in X : f(x) \in B\} \subseteq X.$$

If B is a singleton set $\{y\}$, then we often denote $f^{-1}(B)$ by $f^{-1}(y)$.

Definition 2.0.5. Let X and Y be sets and let $f : X \to Y$ be a function.

- The function f is said to be *injective* (or *one-to-one*) if for all $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
- The function f is said to be *surjective* (or *onto*) if for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$.
- The map f is *bijective* if it is both injective and surjective. In this case, there exists a unique *inverse map* $f^{-1} : Y \to X$ such that for all $x \in X$ and $y \in Y$,

$$f^{-1}(f(x)) = x \text{ and } f(f^{-1}(y)) = y.$$

Definition 2.0.6 (Product of Sets)**.** Let I be a (possibly infinite (Definition 4.0.2) but small (Definition 1.0.3)) index set and let $\{A_i\}_{i \in I}$ be a family of sets indexed by I. The *Cartesian product of the family $\{A_i\}_{i \in I}$*, denoted by $\prod_{i \in I} A_i$, is defined as the set of all tuples/functions (Definition 2.0.1)

$$\prod_{i \in I} A_i := \{(a_i)_{i \in I} \mid a_i \in A_i \text{ for all } i \in I\},$$

where $(a_i)_{i \in I}$ denotes a function from I to $\bigcup_{i \in I} A_i$ such that $(a_i)_{i \in I}(i) = a_i \in A_i$ for each $i \in I$.

The Cartesian product $\prod_{i \in I} A_i$ is the product (Definition 5.0.8) of the objects A_i in the category of sets (Definition 2.0.2).

The self product of a set A indexed by I is often denoted by A^I. Note that elements of A^I can be identified with functions (Definition 2.0.1) $I \to A$. The finite self product of A taken n times is often denoted by A^n. For finitely many sets $A_1, \ldots, A_n$, their Cartesian product is denoted by $A_1 \times \cdots \times A_n$. Elements of such a finite product may be written as $(a_1, \ldots, a_n)$.

3. Relations

Definition 3.0.1. Let n be a positive integer. Let $X_1, X_2, \ldots, X_n$ be sets. An *n-ary relation* on these sets is a subset $R \subseteq X_1 \times X_2 \times \cdots \times X_n$. The integer n is called the *arity* (or *degree*) of the relation. The sets $X_1, \ldots, X_n$ are called the *domains of the relation*.

If $(x_1, x_2, \ldots, x_n) \in R$, we say that the elements $x_1, \ldots, x_n$ are *related* by R.

In the special case where $X_1 = X_2 = \cdots = X_n = X$, we say that R is an *n-ary relation on the set X*. In this case, $R \subseteq X^n$.

Specific arities have standard names:

- A *unary relation* on X is a subset of X (arity $n = 1$).
- A *binary relation* from X to Y is a subset of $X \times Y$ (arity $n = 2$).
- A *ternary relation* is a subset of $X \times Y \times Z$ (arity $n = 3$).

Definition 3.0.2. Let X and Y be sets. Let $R \subseteq X \times Y$ be a binary relation (Definition 3.0.1) from X to Y. If the pair (x, y) is an element of R, we say that x is *related to y by R*. This is denoted by the notation:

$$xRy \quad \text{or} \quad x \sim_R y.$$

Definition 3.0.3. Let $R \subseteq X \times Y$ be a binary relation. The *domain of R* is the set of all first coordinates in the relation:

$$\operatorname{dom}(R) = \{x \in X \mid \exists y \in Y, xRy\}.$$

The *range* (or *image*) of R is the set of all second coordinates in the relation:

$$\operatorname{ran}(R) = \{y \in Y \mid \exists x \in X, xRy\}.$$

Definition 3.0.4. Let $R \subseteq X \times Y$ be a binary relation (Definition 3.0.1). The *inverse relation* (or *converse relation*) of R, denoted by R^{-1} or R^{op}, is the binary relation from Y to X defined by reversing the pairs in R:

$$R^{-1} = \{(y, x) \in Y \times X \mid (x, y) \in R\}.$$

Equivalently, $yR^{-1}x$ if and only if xRy.

Definition 3.0.5. Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be binary relations (Definition 3.0.1). The *composition* of S with R, denoted by $S \circ R$, is the binary relation from X to Z defined by:

$$S \circ R = \{(x, z) \in X \times Z \mid \exists y \in Y \text{ such that } xRy \text{ and } ySz\}.$$

In infix notation, $x(S \circ R)z$ if and only if there exists an intermediate element $y \in Y$ such that xRy and ySz.

Definition 3.0.6. Let R be a binary relation (Definition 3.0.1) on a set X.

1. R is *reflexive* if for all $x \in X$, xRx.
2. R is *symmetric* if for all $x, y \in X$, xRy implies yRx.
3. R is *antisymmetric* if for all $x, y \in X$, $(xRy$ and $yRx)$ implies $x = y$.

4. R is *transitive* if for all $x, y, z \in X$, (xRy and yRz) implies xRz.
5. R is *total* (or *connected*) if for all distinct $x, y \in X$, either xRy or yRx.

Definition 3.0.7. An *equivalence relation on a set X* is a binary relation (Definition 3.0.1) on X that is reflexive, symmetric, and transitive (Definition 3.0.6).

Definition 3.0.8. If $\sim$ is an equivalence relation on a set X and $x \in X$, the *equivalence class of x*, denoted by $[x]$ or $[x]_\sim$, is the set defined by

$$[x] = \{y \in X \mid x \sim y\}.$$

The set of all equivalence classes is called the *quotient set of X by $\sim$*, denoted by $X/\sim$.

(♠ TODO:)

Definition 3.0.9. Let R be a relation on a set X.

1. R is a *partial order* if it is reflexive, antisymmetric, and transitive. A set X equipped with a partial order is called a *partially ordered set* or *poset*.
2. R is a *strict partial order* if it is irreflexive (for all $x \in X$, not xRx) and transitive.
3. R is a *total order* (or *linear order*) if it is a partial order that is total.
4. R is a *well-order* if it is a total order such that every non-empty subset of X has a least element with respect to R.

4. Ordinals and Cardinals

Notation 4.0.1. Let $\mathbb{N}$ denote the set of natural numbers, typically taken as $\mathbb{N} = \{1, 2, 3, \dots\}$. Elements of $\mathbb{N}$ will serve as indices for sequences whenever cardinality is compared to that of countably infinite sets.

Definition 4.0.2. Let A be a set.

- The set A is said to be *countably infinite* (or simply *countable*) if there exists a bijection $f : \mathbb{N} \to A$.
- The set A is said to be *finite* if there exists some $n \in \mathbb{N}$ and a bijection $g : \{1, 2, \dots, n\} \to A$.
- The set A is said to be *at most countable* if it is either finite or countably infinite.
- The set A is said to be *uncountable* if it is not at most countable.

Definition 4.0.3. A set α is an *ordinal number* (or simply an *ordinal*) if it satisfies the following two conditions:

1. α is *transitive*, meaning that every element of α is also a subset of α. That is, if $x \in \alpha$, then $x \subseteq \alpha$.
2. α is *strictly well-ordered* by the membership relation $\in$. That is, the relation $<$ on α defined by $x < y \iff x \in y$ is a strict total ordering, and every non-empty subset of α has a least element under this ordering.

The class of all ordinal numbers is denoted by Ord or $\mathbf{ON}$. For ordinals α and β, we write $\alpha < \beta$ when $\alpha \in \beta$, and we write and $\alpha \leq \beta$ when $\alpha \in \beta$ or $\alpha = \beta$.

Definition 4.0.4. Let α be an ordinal number (Definition 4.0.3). The ordinal number α is called a *successor ordinal* if there exists an ordinal β such that $\alpha = \beta \cup \{\beta\}$. In this case, α is the *immediate successor of* β and is denoted by $\beta + 1$.

Definition 4.0.5. An ordinal number (Definition 4.0.3) α is a *limit ordinal* if $\alpha \neq \emptyset$ and α is not a successor ordinal (Definition 4.0.4). Equivalently, α is a limit ordinal if for every $\beta \in \alpha$, the successor $\beta + 1$ is also in α. In this case, α is equal to the union of all smaller ordinals:

$$\alpha = \bigcup_{\beta<\alpha} \beta.$$

Definition 4.0.6. An ordinal number (Definition 4.0.3) α is a *finite ordinal* (or a *natural number*) if $\alpha = \emptyset$ or if α is a successor ordinal and every element $\beta \in \alpha$ is either $\emptyset$ or a successor ordinal. Ordinals that are not finite are called *infinite ordinals* or *transfinite ordinals*.

Definition 4.0.7. The set of all finite ordinals (Definition 4.0.6) is denoted by ω or $\mathbb{N}$. It is the smallest limit ordinal (Definition 4.0.5).

Definition 4.0.8. An ordinal number (Definition 4.0.3) κ is a *cardinal number* (or simply a *cardinal*) if for every ordinal $\alpha < \kappa$, there is no bijection (Definition 2.0.5) between α and κ. Equivalently, a cardinal is an initial ordinal—an ordinal that is not equinumerous with any smaller ordinal.

The *cardinality of an arbitrary set* X, denoted by $|X|$, $\text{card}(X)$, or $\#X$, is the unique cardinal number κ such that there exists a bijection between X and κ. (The existence of such a κ for every set requires the Axiom of Choice (Definition 1.0.2)).

Definition 4.0.9. Let κ be a cardinal number (Definition 4.0.8).

- κ is a *successor cardinal* if there exists a cardinal λ such that κ is the smallest cardinal strictly greater than λ. This successor is denoted by λ^+.
- κ is a *limit cardinal* if $\kappa \neq 0$ and κ is not a successor cardinal.

Definition 4.0.10. Let α be a limit ordinal (Definition 4.0.5). A subset $A \subseteq \alpha$ is *cofinal in* α if for every $\beta \in \alpha$, there exists $\gamma \in A$ such that $\beta \leq \gamma$. The *cofinality of* α, denoted by $\text{cf}(\alpha)$, is the least ordinal which is the order type of a cofinal subset of α. Equivalently, $\text{cf}(\alpha)$ is the smallest cardinality of a cofinal subset of α.

Definition 4.0.11. Let κ be a cardinal number (Definition 4.0.8).

- κ is a *singular cardinal* if it can be written as a union of fewer than κ sets, each of cardinality less than κ. Formally, κ is singular if its *cofinality* is strictly less than κ, i.e., $\text{cf}(\kappa) < \kappa$.
- κ is a *regular cardinal* if it is not singular. Formally, κ is regular if $\text{cf}(\kappa) = \kappa$.

Definition 4.0.12. The *aleph numbers* are the infinite cardinals (Definition 4.0.8) indexed by ordinals (Definition 4.0.3). They are defined recursively:

$$\begin{aligned}\aleph_0 &= \omega,\\ \aleph_{\alpha+1} &= \aleph_\alpha^+ \quad \text{(the successor cardinal of } \aleph_\alpha\text{)},\\ \aleph_\lambda &= \bigcup_{\beta<\lambda} \aleph_\beta \quad \text{if } \lambda \text{ is a limit ordinal.}\end{aligned}$$

(Definition 4.0.7) (Definition 4.0.9) (Definition 4.0.5) A cardinal κ is an *uncountable cardinal* if $\kappa > \aleph_0$.

5. Category Theorey

Definition 5.0.1 (Category)**.** A *category* $\mathcal{C}$ consists of the following data:

- A class of *objects*, denoted $\mathrm{Ob}(\mathcal{C})$.
- For each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a class

  $$\mathrm{Hom}_{\mathcal{C}}(X, Y)$$

  of *morphisms* (also called *arrows* or *homs*). If the category $\mathcal{C}$ is clear, then this *hom-class* is also denoted by $\mathrm{Hom}(X, Y)$. It may also be denoted by $\mathrm{hom}_{\mathcal{C}}(X, Y)$ or $\mathrm{hom}(X, Y)$, especially to distinguish from other types of hom's (e.g. internal hom's)
- For each triple of objects X, Y, Z, a composition law

  $$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z),$$

  denoted $(g, f) \mapsto g \circ f$.
- For each object X, an *identity morphism*

  $$\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X).$$

These data satisfy the following axioms:

- (Associativity) For all morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, and $h \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$,

  $$h \circ (g \circ f) = (h \circ g) \circ f.$$
- (Identity) For all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$,

  $$\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X .$$

One often writes $X \in \mathcal{C}$ synonymously with $X \in \mathrm{Ob}(\mathcal{C})$, i.e. to denote that X is an object of of $\mathcal{C}$.

We may call a category as above an *ordinary category* to distinguish this notion from the notions of *categories enriched in monoidal categories* or higher/n-categories. (♠ TODO: TODO: define n-categories)

A category as defined above may be called called a *large category* or a *class category* to emphasize that the hom-classes may be proper classes rather than sets (note, however, that the possibility that hom-classes are sets is not excluded for large categories). Accordingly, a *category* may often refer to a locally small category (Definition 5.0.2), which is a category whose hom-classes are all sets.

Definition 5.0.2 (Locally small category)**.** A (large) category (Definition 5.0.1) $\mathcal{C}$ is called a *locally small category* if for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the collection $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms between them is a (small (Definition 1.0.3)) *set* (as opposed to a proper class). In other words, each hom-class is a set and may even be called a *hom-set*.

In some contexts, a locally small category may simply be called a *category*, especially when genuinely large categories are not considered.

A category $\mathcal{C}$ is called a *small category* if it is a locally small category and the class $\mathrm{Ob}(\mathcal{C})$ of objects is a set.

Given a universe (Definition 6.2.1) U, we can define the notion of a *U-locally small category* and of a *U-small category* similarly.

Remark 5.0.3. Many "concrete" categories considered in "classical mathematics" or outside of more "abstract" category theory tend to be locally small. For example, the categories of sets, groups, R-modules, vector spaces, topological spaces, schemes, manifolds, sheaves on "small enough" sites are all locally small.

Definition 5.0.4. Let $\mathcal{C}$ and $\mathcal{D}$ be (large) categories (Definition 5.0.1).

1. A *functor $F : \mathcal{C} \to \mathcal{D}$ (from $\mathcal{C}$ to $\mathcal{D}$)* consists of :
   - For each object X in $\mathcal{C}$, an object $F(X)$ in $\mathcal{D}$.
   - For each morphism $f : X \to Y$ in $\mathcal{C}$, a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$,

   such that:
   $$F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \text{for all objects } X \text{ in } \mathcal{C},$$
   $$F(g \circ f) = F(g) \circ F(f) \quad \text{for all } X, Y, Z \in \mathrm{Ob}(\mathcal{C}) \text{ and all } f : X \to Y, g : Y \to Z \text{ in } \mathcal{C}.$$
   Functors as defined above are also referred to as *covariant functors* to distinguish them from contravariant functors
2. A *contravariant functor from $\mathcal{C}$ to $\mathcal{D}$* refers to a covariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. Equivalently, such a functor consists of
   - For each object X in $\mathcal{C}$, an object $F(X)$ in $\mathcal{D}$.
   - For each morphism $f : X \to Y$ in $\mathcal{C}$, a morphism $F(f) : F(Y) \to F(X)$ in $\mathcal{D}$,

   such that:
   $$F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \text{for all objects } X \text{ in } \mathcal{C},$$
   $$F(g \circ f) = F(f) \circ F(g) \quad \text{for all } X, Y, Z \in \mathrm{Ob}(\mathcal{C}) \text{ and all } f : X \to Y, g : Y \to Z \text{ in } \mathcal{C}.$$

Note that declarations such as "Let $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ be a contravariant functor" can be common; such declarations usually mean "Let F be a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$" as opposed to "Let F be a contravariant functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$". further note that a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is equivalent to a covariant functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$.

Definition 5.0.5 (Diagram in a category and category of diagrams)**.** Let $\mathcal{C}$ be a (large) category (Definition 5.0.1), and let I be a (large) category (Definition 5.0.1).

1. A *diagram of shape I in $\mathcal{C}$* is a functor (Definition 5.0.4) $D : I \to \mathcal{C}$. We often denote such a diagram by the family $\{D(i)\}_{i \in \mathrm{Ob}(I)}$ with transition maps given by the functorial image of morphisms in I.

   A diagram is also synonymously called a *system*. Moreover, the category I is called the *index category* or the *indexing category of the diagram D*.
2. Given two diagrams $D, E : I \to \mathcal{C}$, a *morphism of diagrams* is a simply a natural transformation $D \Rightarrow E$ of the functors D and E.
3. The *category of I-shaped diagrams in $\mathcal{C}$*, often denoted $\mathcal{C}^I$, $[I, \mathcal{C}]$, or $\mathrm{Fun}(I, \mathcal{C})$, is the (large) category whose objects are functors $I \to \mathcal{C}$ (that is, diagrams of shape I in $\mathcal{C}$) and whose morphisms are natural transformations between such functors. The category $\mathcal{C}^I$ is also called the *functor category of functors $I \to \mathcal{C}$*.

   If $\mathcal{C}$ is locally small (Definition 5.0.2) and I is small, then $\mathcal{C}^I$ is locally small by Lemma 5.0.6.

Lemma 5.0.6. Let $\mathcal{C}$ be a small category (Definition 5.0.2) (resp. U-small category where U is some universe (Definition 6.2.1)) and let $\mathcal{A}$ be a locally small category (resp. U-locally small category). The presheaf category $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ is locally small (resp. U-locally small).

Proof. A morphism $\mathcal{F} \to \mathcal{G}$ in $\mathrm{PreShv}(\mathcal{C}, \mathcal{A})$ is a natural transformation of the functors $\mathcal{F}, \mathcal{G} : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$. Such a natural transformation is encoded by a family $(\eta_C)_C$ of morphisms (satisfying certain conditions) $\eta_C : \mathcal{F}(C) \to \mathcal{G}(C)$ in $\mathcal{A}$ over objects C of $\mathcal{C}^{\mathrm{op}}$. The product $\prod_{C \in \mathrm{Ob}\, \mathcal{C}^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}(C), \mathcal{G}(C))$ is a product of (U-small) sets indexed by a (U-small) set, and the collection of natural transformations is a subset of this set. Therefore, $\mathrm{Hom}_{\mathrm{PreShv}(\mathcal{C}, \mathcal{A})}(\mathcal{F}, \mathcal{G})$ is a (U-small) set. $\square$

Definition 5.0.7 (Cones, limits and colimits in a category)**.** Let $\mathcal{C}$ be a (large) category (Definition 5.0.1), let I be a (large) category, and let $D : I \to \mathcal{C}$ be a diagram (Definition 5.0.5) (Definition 5.0.5).

1. A *cone to the diagram D* is an object $L \in \mathcal{C}$ together with a family of morphisms
$$\{\pi_i : L \to D(i)\}_{i \in I}$$
   such that for every morphism $f : i \to j$ in I, the diagram
$$D(f) \circ \pi_i = \pi_j$$
   commutes.
2. A *limit of D*, often denoted by $\lim_{i \in I} D$ or $\lim D$, is a cone $(L, \{\pi_i\})$ to D such that for every object $C \in \mathcal{C}$ with a cone $\{f_i : C \to D(i)\}_{i \in I}$, there exists a unique morphism $u : C \to L$ in $\mathcal{C}$ making all triangles commute:
$$\pi_i \circ u = f_i \quad \text{for all } i \in I.$$
3. A *cocone from the diagram D* is an object $C \in \mathcal{C}$ together with a family of morphisms
$$\{\iota_i : D(i) \to C\}_{i \in I}$$
   such that for every morphism $f : i \to j$ in I, the diagram
$$\iota_j \circ D(f) = \iota_i$$
   commutes.

4. A *colimit of D*, often denoted by $\operatorname{colim}_{i\in I} D$ or $\operatorname{colim} D$, is a cocone $(C, \{\iota_i\})$ from D such that for every object $L \in \mathcal{C}$ with a cocone $\{g_i : D(i) \to L\}_{i\in I}$, there exists a unique morphism $v : C \to L$ in $\mathcal{C}$ making all triangles commute:

$$v \circ \iota_i = g_i \quad \text{for all } i \in I.$$

Since the morphisms u and v descibing the "universal properties" are unique, limits and colimits are unique up to unique isomorphism if they exist.

Some authors use the terms *projective limit* or *inverse limit* to refer to what is defined here as a limit, Similarly, the terms *inductive limit* or *direct limit* are sometimes used to mean a colimit. However, these phrases can have more specific meanings to other authors: a *projective* or *inverse limit* may refer to a limit over a diagram indexed by a codirected poset. Likewise, an *inductive* or *direct limit* may refer to a colimit over a directed poset.

Thus, while the terms are sometimes used interchangeably with "limit" and "colimit," they may also emphasize particular indexing shapes and directions, distinguishing them from general limits and colimits taken over arbitrary small categories.

Definition 5.0.8 (Product in a category)**.** Let $\mathcal{C}$ be a category and let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$ indexed by a class I.

1. A *product of the family $\{X_i\}$* is an object P of $\mathcal{C}$ together with a family of morphisms

$$\pi_i : P \to X_i, \quad \text{for each } i \in I$$

such that for any object Y and any family of morphisms $\{f_i : Y \to X_i\}_{i\in I}$, there exists a unique morphism

$$f : Y \to P$$

making the following diagram commute for all $i \in I$:

$$\pi_i \circ f = f_i .$$

Such a product is often denoted by $\prod_{i\in I} X_i$. Equivalently, it is the limit (Definition 5.0.7) of the diagram $I \to \mathcal{C}, i \mapsto X_i$, where I is made into a category whose objects are the members of I and whose morphisms are just the identity morphisms.

2. A *coproduct* (or synonymously *direct sum*) of the family $\{X_i\}$ is an object C of $\mathcal{C}$ together with a family of morphisms

$$\iota_i : X_i \to C, \quad \text{for each } i \in I$$

such that for any object Y and any family of morphisms $\{g_i : X_i \to Y\}_{i\in I}$, there exists a unique morphism

$$g : C \to Y$$

making the following diagram commute for all $i \in I$:

$$g \circ \iota_i = g_i .$$

Such a coproduct is often denoted by $\coprod_{i\in I} X_i$ or $\oplus_{i\in I} X_i$. Equivalently, it is the colimit (Definition 5.0.7) of the diagram $I \to \mathcal{C}, i \mapsto X_i$, where I is made into a category whose objects are the members of I and whose morphisms are just the identity morphisms.

6.1. **First order logic.**

6.2.

Definition 6.2.1 (Grothendieck Universe)**.** Let U be a set. We say U is a *Grothendieck universe* (or just a *universe*) if the following conditions hold:

1. If $x \in U$ and $y \in x$, then $y \in U$ (transitivity).
2. If $x, y \in U$, then $\{x, y\} \in U$ (closed under pair formation).
3. If $x \in U$, then the power set $\mathcal{P}(x) \in U$.
4. If $I \in U$ and $(x_\alpha)_{\alpha \in I}$ is a family with each $x_\alpha \in U$, then $\bigcup_{\alpha \in I} x_\alpha \in U$.

A set X is called *U-small* or a *U-set* if $X \in U$.

Definition 6.2.2. *Tarski–Grothendieck set theory*, denoted by TG, is the theory consisting of the axioms of ZFC (Definition 1.0.2) together with *Tarski's Axiom of Universes*:

$$\forall x\, \exists U\, (U \text{ is a Grothendieck universe} \wedge x \in U)$$

Definition 6.2.3. Let $\mathcal{L}_{\text{Class}}$ be a two-sorted first-order language with variables for sets (lowercase $x, y, z, \ldots$) and variables for classes (uppercase $X, Y, Z, \ldots$). We define a class X to be *proper* if it is not equal to any set (i.e., $\neg\exists x(X = x)$, treating sets as classes that are elements of other classes).

Definition 6.2.4. (♠ TODO: two-sorted first-order language) Let $\mathcal{L}_{\text{Class}}$ be a two-sorted first-order language with variables for sets (lowercase $x, y, z, \ldots$) and variables for classes (uppercase $X, Y, Z, \ldots$).

von Neumann–Bernays–Gödel set theory, denoted NBG, is the theory in $\mathcal{L}_{\text{Class}}$ comprised of the following axioms: (♠ TODO: theory)

1. **Extensionality:** Classes with the same elements are equal.
2. **Foundation:** Every non-empty class of sets has an $\in$-minimal element.
3. **Class Comprehension (Predicative):** For any formula ϕ in which quantification occurs only over sets, there exists a class X such that:
$$\forall x\, (x \in X \iff \phi(x))$$
4. **Limitation of Size:** A class X is a set if and only if there is no bijection between X and the class of all sets V:
$$\exists x(X = x) \iff \neg(X \cong V)$$
5. **Set Axioms:** The standard ZFC (Definition 1.0.2) axioms of Pairing, Power Set, Union, Infinity, and Choice, relativized to sets.

Definition 6.2.5. Let $\mathcal{L}_{\text{Class}}$ be a two-sorted first-order language with variables for sets (lowercase $x, y, z, \ldots$) and variables for classes (uppercase $X, Y, Z, \ldots$).

Morse–Kelley set theory, denoted MK, consists of the axioms of NBG (Definition 6.2.4) but with the *Impredicative Class Comprehension Schema*. That is, for *any* formula ϕ in $\mathcal{L}_{\text{Class}}$ (allowing quantification over classes), there exists a class X such that:

$$\forall x \, (x \in X \iff \phi(x))$$

Definition 6.2.6. (♠ TODO: formula, first-order language) Let $\mathcal{L}_{\in}$ be a first-order language with a single binary relation symbol $\in$. Let ϕ be a formula in the language $\mathcal{L}_{\in}$. A function $\sigma : \text{Vars}(\phi) \to \mathbb{N}$ mapping variables in ϕ to natural numbers is called a *stratification* if for every atomic formula $x \in y$ occurring in ϕ, we have $\sigma(y) = \sigma(x) + 1$, and for every atomic formula $x = y$, we have $\sigma(x) = \sigma(y)$.

A formula ϕ is called *stratified* if it admits a stratification.

Definition 6.2.7. (♠ TODO: formula, first-order language) Let $\mathcal{L}_{\in}$ be a first-order language with a single binary relation symbol $\in$. *New Foundations with Urelements*, denoted NFU, is a theory allowing for *urelements* (objects distinct from the empty set that have no elements) and includes the following axioms:

1. **Weak Extensionality:** Non-empty sets with the same elements are equal.
2. **Stratified Comprehension:** For any stratified formula $\phi(x)$ where y does not occur free, the set of all x satisfying ϕ exists:

$$\exists y \forall x \, (x \in y \iff \phi(x))$$

Unlike ZFC, NFU allows for a universal set V such that $V \in V$, as the formula $x = x$ is stratified.

Definition 6.2.8. *Constructive Zermelo–Fraenkel set theory*, denoted CZF, is a theory based on *intuitionistic logic* (rejecting the Law of Excluded Middle) comprising the following modifications to standard ZFC:

1. **Restricted Separation:** The Separation schema is restricted to bounded (Δ_0) formulas.
2. **Subset Collection:** Replaces the Power Set axiom (which is too strong constructively) with a schema stating that for any sets A, B, there exists a set C of "multivalued functions" covering all total relations from A to B.
3. **Strong Collection:** A constructive replacement for the Replacement and Collection schemas.