1. Another split complex

The "Hodge" splitting induced by an inner product: Suppose that $(C_\bullet, \partial)$ is a chain complex of finite-dimensional, real (or complex) vector spaces with a positive-definite, symmetric (or Hermitian) inner product $(\cdot,\cdot)$: recall that means $(\cdot, w)$ is linear for all w, $(v,w) = \overline{(w,v)}$, and $(v,v) > 0$ for all $v \neq 0$.

(1.1) For each $k \in \mathbb{Z}$, show that there is a unique map $\delta_k : C_k \to C_{k+1}$ defined by the condition

$$(\delta_k(v), w) = (v, \partial_{k+1}(w))$$

for all $v \in C_k$ and $w \in C_{k+1}$. Show that $\delta_{k+1}\delta_k = 0$ for all k.

(1.2) Show that $C(_\bullet, \partial)$ is split.

(1.3) Let $\Lambda_k = \delta_{k-1}\partial_k + \partial_{k+1}\delta_k$, a map from C_k to itself. Show that there is an isomorphism

$$H_k(C_\bullet, \partial) \cong \ker \Lambda_k,$$

for all k.

2. Ext and Tor

(2.1) Let $R \to S$ be a homomorphism of commutative rings in such a way that S is a flat R-module.

(a) Show that, if M, N are R-modules, then there is a natural isomorphism of S-modules

$$\mathrm{Tor}^n_R(M,N) \otimes_R S \cong \mathrm{Tor}^n_S(M \otimes_R S, N \otimes_R S).$$

(b) Conclude that if R is a domain, then $\mathrm{Tor}^n_R(M,N)$ is a torsion R-module for all $n \geq 0$. [Hint: let S be the fraction field of R.]

(2.2) Show that for a fixed ring R, the following are equivalent:

(a) N is an injective R-module.

(b) $\mathrm{Ext}^p_R(M,N) = 0$ for all M and $p > 0$.

(c) $\mathrm{Ext}^1_R(M,N) = 0$ for all M.

(2.3) Let $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$, the p-adic integers, from the first assignment. Let $\mathbb{Z}/p^\infty\mathbb{Z} = \varinjlim \mathbb{Z}/p^n\mathbb{Z}$ (where the maps $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n+1}\mathbb{Z}$ are inclusions.) Prove that

$$\mathrm{Ext}^i(\mathbb{Z}/p^\infty\mathbb{Z}, \mathbb{Z}) \cong \begin{cases} 0 & \text{for } i \neq 1; \\ \mathbb{Z}_p & \text{for } i = 1. \end{cases}$$

[Hint: show $(\mathbb{Z}/p^\infty\mathbb{Z})^* \cong \mathbb{Z}_p$.]

3. $\varprojlim^1$ (optional)

(3.4) Consider the short exact sequence of towers

$$0 \longrightarrow \{\mathbb{Z}, p\} \longrightarrow \{\mathbb{Z}, 1\} \longrightarrow \{\mathbb{Z}/p^n\} \longrightarrow 0$$

where the notation $\{\mathbb{Z}, r\}$ means the tower with objects isomorphic to $\mathbb{Z}$, and all maps given by multiplication by r. Use the long exact sequence of $R\varprojlim$ to compute $\varprojlim^1\{\mathbb{Z}, p\}$.