Shimura varieties and their special cycles - in the case of O(0,2), O(1,2), and O(2,2)

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Outline:

1. Generalities on Shimura v arieties
2. Orthogonal Shimura varieties
3. Special cycles
4. More examples

1 Shimura varieties

Definition (vague)

Let $(G, \mathbb{D})$ be a reductive algebraic group and a complx symmetric space. If this pair satisfies some conditions and G acts on $\mathbb{D}$ in some way, then this is a Shimura datum.
From a Shimura datum and given an open compact $K \subseteq G(\mathbb{A}_f)$, one can construct a double coset

$$X_K = G(\mathbb{Q}) \backslash \mathbb{D} \times G(\mathbb{A}_f)/K$$

The X_K's are very nice to study. Also the action is as follows:

$$g[x,h] = [gx, gh]$$
$$[x,h]k = [x, hk]$$

Properties

(Important)
X_K decomposes as a disjoint union

$$\coprod_j \Gamma_j \backslash \mathbb{D}^+$$

of connected; here, $\Gamma_j = G(\mathbb{Q})_+ \cap g_j K g_j^{-1}$ and the g_j's are double coset representatives of $G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K$

Theorem

(Reflex)
the X_K's are defined over a number field called the $\boxed{\textbf{reflex field}}$. Moreover, the reflex field does not depend on K.

Example

Let $G = \mathrm{GL}_2$, $K = \ker(\mathrm{GL}_2(\hat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}))$. Let
$\mathbb{D} = \{G(\mathbb{R})\text{-conjugacy classes of a map } h_0 : \mathbb{S} \to \mathrm{GL}_{2,\mathbb{R}}, \quad a + bi \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\}$
We get a decomposition

$$X_K = \coprod_{\mathbb{Z}/N\mathbb{Z}^\times} Y(\Gamma(N))$$

where $Y(\Gamma(N))$ is constructed as $\mathbb{H}/\Gamma(N)$

Computation

How do we know that the components correspond to elements of $\mathbb{Z}/N\mathbb{Z}^\times$? We need to see what the double coset representatives of $\mathrm{GL}_2(\mathbb{Q})_+ \backslash \mathrm{GL}_2(\mathbb{A}_f)/K$ are:

$$\begin{aligned} \mathrm{GL}_2(\mathbb{Q})_+ \backslash \mathrm{GL}_2(\mathbb{A}_f)/K &= \mathrm{GL}_2(\mathbb{Q})_+ \backslash \mathrm{GL}_2(\mathbb{Q}) \mathrm{GL}_2(\hat{\mathbb{Z}})/K \\ &= (\mathbb{Z}/N\mathbb{Z})^\times \end{aligned}$$

2. Orthogonal Shimura varieties

Here is the setup:

1. Let (V, Q) be an quadratic space of signature $(n, 2)$[1].
2. $C(V) = T(V)/(V^2 - Q(v))$ is the Clifford algebra. It decomposes as $C(V) = C^{\mathrm{even}} \oplus C^{\mathrm{odd}}(V)$, and we have an embedding $V \hookrightarrow C^{\mathrm{odd}}(V)$.

Write $G = \mathrm{GSpin}(V) = \{g \in C^{\mathrm{even}}(V)^\times : gVg^{-1} = V\}$

Table of accidental isomorphisms

$\dim V = n+2$	$C^{\mathrm{even}}(V)$	$\mathrm{GSpin}(V)$
$\dim V = 2$	Imaginary quadratic field K	$\mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_{K,m} \simeq K^\times$
$\dim V = 3$	Quaternion algebras over $\mathbb{Q}$	GL_2 or $B^\times$ (indeterminate quaternion algebra B)
$\dim V = 4$	Quaternion algebras over the center $Z((C^{\mathrm{even}}(V)))$	$\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$ or $\mathrm{GL}_{2,F}^{\det \in \mathbb{Q}^\times}$, $B^\times \times_{\mathbb{G}_m} B^\times$

An example of why these happen for $n = 0$

Let (V, Q) is a quadratic space and say that V has an orthogonal basis $\{e_1, e_2\}$: $V = \mathbb{Q}e_1 \oplus \mathbb{Q}e_2$. The Clifford algebra decomposes as

$$C(V) \cong \mathbb{Q} \oplus \mathbb{Q}e_1 \oplus \mathbb{Q}e_2 \oplus \mathbb{Q}(e_1 \otimes e_2)$$

and

$$C^{\text{even}}(V) \cong \mathbb{Q}[X]/(X^2 + q_1 q_2)$$

where $q_i = Q(e_i)$. This isomorphism then tells you that

$$C^{\text{even}}(V)^\times \cong \mathbb{Q}(\sqrt{-q_1 q_2})^\times$$

> **Remark**
>
> Relationship between SO and GSpin: there is a short exact sequence
>
> $$1 \to \mathbb{G}_m \to \mathrm{GSpin}(V) \to \mathrm{SO}(V) \to 1;$$
>
> Here, $g \in \mathrm{GSpin}(V)$ is sent to the element of $\mathrm{SO}(V)$ which acts on V via gVg^{-1}. Moreover, in this case, $\mathrm{SO}(V)$ is the group of norm 1 elements of $K^\times$.

Example Orthogonal Sr associated to $O(0, 2)$

Let $k = \mathbb{Q}(\sqrt{d})$ with $d < 0$. Let $G = k^\times$. Let $\mathbb{D}$ be the Grassmannian of regular definite oriented 2 planes in $V(\mathbb{R})$. There are 2 orientations of $V_\mathbb{R}$, denoted by $z_0^\pm$. Let $K = \hat{\mathcal{O}}_k^\times \subseteq \mathbb{A}_{f,k}^\times$. Let We have $X_K = k^\times \backslash \{z_0^\pm\} \times \mathbb{A}_{f,k}^\times / \hat{\mathcal{O}}_k^\times = \{z_0^+\} \times \mathrm{Cl}(k)$.

Why is this?

We need to count the number of connected components.

We have $k^\times \backslash \mathbb{A}_{f,k}^\times / \hat{\mathcal{O}}_k^\times = \mathrm{Cl}(k)$, and then we can decompose X_K as

$$X_K = \coprod_{\mathrm{Cl}(k)} \Gamma_{y_j} \backslash \mathbb{D}^+ = \coprod_{\mathrm{Cl}(K)} \{z_0^+\}.$$

3. Special Cycles (Kudla 1997)

How do we construct special cycles?

> **Definition**
>
> Let (V, Q) be a quadratic space with signature $(n, 2)$. Pick $x \in V(\mathbb{Q})$ such that $Q(x) > 0$. Write $V_x = x^\perp$. It is a quadratic space of signature $(n-1, 2)$. Then $\mathbb{D}_x \subseteq \mathbb{D}$ is the

corresponding negative definite plane; it has a point z with $z \perp x$.

We have a map, where G_j is the subset of G of stabilizers of x.

$$G_x(\mathbb{Q})\backslash\mathbb{D}_x \times G_x(\mathbb{A}_f)/G(\mathbb{A}_f) \cap K \to X_K = G(\mathbb{Q})\backslash\mathbb{D} \times G(\mathbb{A}_f)/K$$

The image of this map is $Z(x; K)$

Shifted cycles

$Z(x, g, K)$ is defined as the image of the map

$$G_x(\mathbb{Q})\backslash\mathbb{D}_x \times G_x(\mathbb{A}_f)/K \cap G_x(\mathbb{A}_f) \to X_K$$

sending $[x, h] \mapsto [x, hg]$ where $g \in G(\mathbb{A}_f)$; this is the same kind of map as before, except you shift the connected component that you land one.

Special cycles

What are special cycles?

Definition

Let $\varphi \in \mathcal{S}(V(\mathbb{A}_f))^K$ be a K-invariant Schwarz function, let $m \in \mathbb{Q}$, and let $x \in V(\mathbb{Q})$. Then

$$Z(m, \varphi, K) = \sum_j \varphi(g_j^{-1}x) \cdot Z(x, g_j; K);$$

this is a finite sum over the representatives $g_1, \ldots, g_r$ of $K\backslash \operatorname{Supp}\varphi \cap \Omega_m(\mathbb{A}_f)$. Here, $\Omega_m(\mathbb{A}_f) = \{x \in V(\mathbb{A}_f) : Q(x) = m\}$.

Lemma 4.1 / kudla proposition 5.4

If $G(\mathbb{A}_f) = G(\mathbb{Q})_+ K$, then $Z(m, \varphi; K) = \sum_{x \in \Gamma_K\backslash\Omega_m(\mathbb{Q})} \varphi(X) \operatorname{pr}(\mathbb{D}_x, 1)$ where $\Gamma_K = G(\mathbb{Q})_+ \cap K$ and $\operatorname{pr} : \mathbb{D}_x \times G(\mathbb{A}_f) \to X_K$

This is useful for computations.

Examples

There is a general procedure by which you can construct special cycles.

General procedure

Typically, you are going to take

1. A (even integral) lattice $L \subseteq V(\mathbb{Q})$
2. $\varphi = \sum_{\mu \in \text{Orbits in } L^\vee/L} \text{char}(\mu + \hat{L}) \in \mathcal{S}(V(\mathbb{A}_f))^K$
3. Get "nice" special cycles $Z(m, \varphi; K)$

Examples

Heegner points on $X_0(N)$

Let $V = M_2(\mathbb{Q})^{\text{Tr}=0}$, let $Q(x) = N \cdot \det(x)$. Then $\text{GSpin}(V) \cong \text{GL}_2$, and $\text{GL}_2(\mathbb{Q})$ acts on V via $g \cdot x = gxg^{-1}$. Moreover, $\text{SO}(V) = \text{PGL}_2$.

Now consider the biholomorphic map

$$\mathbb{H}^\pm \to \mathbb{D}, \quad x + iy = z \mapsto \begin{pmatrix} z & z^2 \\ 1 & -z \end{pmatrix}$$

We have

$$K = \prod K_p \subseteq G(\mathbb{A}_f)$$

where $K_p = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}_p) : c \in N \cdot \mathbb{Z}_p\}$.

Fact

$G(\mathbb{A}_f) = G(\mathbb{Q})K$

We also have

$$X_K = K \cap G(\mathbb{Q})_+ \backslash \mathbb{D}^+ = \Gamma_0(N) \backslash \mathbb{H} = Y_0(N)$$

Classical Heegner divisors

Take $P_{D,r} + P_{D,-r}$ on $X_0(N)$; here,

$$P_{D,r} = \{z \in X_0(N) : az^2 + bz + c = 0, a > 0, a \equiv 0 \pmod{N}, b \equiv 0 \pmod{-r} \pmod{2N}, b^2 - 4a$$

[2]

(This is Gross-Kohren-Zagier)

Claim

$Z(\cdot, \varphi; K) = P_{D,r} + P_{D,-r}$

Step 1: Make a Schwarz function

Let $L = \{\begin{pmatrix} b & -q/N \\ c & -b \end{pmatrix} : a, b, c \in \mathbb{Z}\}$. It turns out that $L^\vee/L \cong \mathbb{Z}/2N\mathbb{Z}$ where $r \in \mathbb{Z}/2N\mathbb{Z}$ correspnods to $\begin{pmatrix} \frac{r}{2N} & 0 \\ 0 & -\frac{r}{2N} \end{pmatrix} r$.

Step 2:

Claim: K acts on $L^\vee/L$ trivially.

Step 3:

So $\varphi = \mathrm{char}(\begin{pmatrix} \frac{r}{2N} & 0 \\ 0 & -\frac{r}{2N} \end{pmatrix} + \hat{L})$

There is a lemma from Bruinier-Yang that says that

$$Z(m, \varphi, K) = \sum_{x \in \Gamma_K \backslash \Omega_m(\mathbb{Q})} \varphi(x) \, \mathrm{pr}(\mathbb{D}_x, 1)$$

$\mathbb{D}_x$ consists of

$$0 = (x, \omega(z)) = -N \, \mathrm{Tr} \begin{pmatrix} \frac{r}{2N} & \frac{1}{N} \\ \frac{D-r^2}{4N} & -\frac{r}{2N} \end{pmatrix} \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} = \cdots = \frac{D-r^2}{4} z^2 - rz - 1$$

Here, $D = -4Nm \in \mathbb{Z}$

Last thing to check/compute

$$\Gamma_K \backslash \Omega_m(\mathbb{Q}) \cap \mathrm{Supp}(\varphi) \ni x$$

See Also

Meta

References

Citations and Footnotes

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1. It turns out that only signature $(n, 2)$ quadratic spaces give Shimura varieties↩
2. Here, we are regarding z as representing a point in $\mathbb{H} \subseteq \mathbb{C}$.↩