Regularized theta lifting + theta CM values

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Lots of people have different ways to find problems to work on. For this paper, the initial goal was not the Gross-Zagier formula at all. It was to try to understand Schofer's thesis and whether one can use it to generalize/understand the Bruinier-Yang 2006 paper

Schofer's thesis was very nice, and then Bruinier proposed to Yang "let's try solving an easier problem".

Let $\Phi(z,h,f) = \int f(\tau)\theta_L(\theta,z,h)d\mu(\tau)$

Here, (z,h) live in a Shimura variety written X, and τ lives in the upper half plane τ. The integral is over a fundamental domain of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. Moreover, $f:\mathbb{H}\to\mathbb{C}$ is a weakly holomorphic modular form. For example, $j(z)=q^{-1}+\cdots$. However, a weakly holomorphic modular form blows up at $\tau=\infty$, so you need to regularize the integral

$\Phi(z,h,f) = \int^{\text{reg}} f(\tau)\theta_L(\theta,z,h)d\mu(\tau)$

We have $\Phi(\mathrm{CM}(E),f) = \sum_{(z,h)\in\mathrm{CM}(E)}\Phi(Z,h,f)$; here, $\mathrm{CM}(E)$ is a finite set in X.

Theorem

Schofer's theorem
$\Phi(\mathrm{CM}(z),f) = \sum_{p\text{ prime},p<\infty,p<d(Z)} a_p\log p$

Consider the special case[1] $f=j(\tau)-744$ and $X=Y(1)\times Y(1)=\mathrm{Sl}(\mathbb{Z})\backslash\mathbb{H}\times\mathrm{Sl}(\mathbb{Z})\backslash\mathbb{H}$. Here,

$$\Phi(z_1,z_2,f) = -\log|j(z_1)-j(z_2)|^2$$

Write $E=\mathbb{Q}(\sqrt{d_1},\sqrt{d_2})$. Then $\Phi(\mathrm{CM}(E),j-744)=\sum_{\mathrm{disc}(\tau_1)=d_1,\mathrm{disc}(\tau_2)=d_2}-\log|j(\tau_1)-j(\tau_2)|^2$ By Gross Zagier, this then equals $\sum_{p\le d_1d_2}a_p\log p$

We have an exact sequence

$$0\to M^!_{2-k}\hookrightarrow H_{2-k}\xrightarrow{\xi}S_k$$

$M^!_{2-k}$ is the space of weakly holomorphic forms, H_{2-k} is the space of Harmonic Maass forms, and S_k is the space of holomorphic cuspidal forms of weight k.

The work of Bruinier-Yang tries out the following ideas:

1. $f\in M_{2-k}$: do the same calculation (First four sections of the paper; Theorem 4.6 is the main theorem)

- The analogue of Schofer's theorem: $\Phi(\mathrm{CM}(E),f)=\sum_{p<\infty}a_p\log p + L'(\xi(f),\theta,\mathrm{Center})$

$\xi(f)$ is the Rankin-Selbert L-function

2. Conjecture: let $\hat{Z}(f)=(Z(f),\Phi(f))\in\widehat{\mathrm{CH}}^1(\mathscr{X})$. Then $\langle\hat{Z}(f),\mathrm{CM}(z)\rangle_{\mathrm{Fal}}=L'(\xi(f),\theta,\mathrm{center})$ verify low dimension case. The conjecture was eventually proven in 2015 by Bruinier-Howard-Yang
3. Surprise: give a proof of a variant of Gross-Zagier formula without doing any finite intersection!!

modularity: $\sum\hat{Z}(m)q^m$ modular (Bruinier-Howard-Kudla-Rapoport-Yang, circa 2020)

- Other CM values (2013) (Bruinier-Kudla-Yang)

Outline of paper

I: Weil representation, theta lifting, theta kernel

II: Shimura variety of orthogonal type $O(n,2)$ ($\mathrm{GSpin(V)} \to \mathrm{SO}(V)$) and Siegel-Weil formula

- Special divisors
- Small CM cycles

III: The main formula (quite simple / Soft)

Let (V,Q) be a quadratic space of sign $(n,2)$. Write $m = n+2$.

Let $H = \mathrm{SO}(V) = \{h \in \mathrm{End}(V) : (hx, hy) = (x,y) \text{ for all } x, y \in V, \det h = 1\}$. Let $G = \mathrm{SP}_2 = \mathrm{SP}(W) = \mathrm{SL}_2$. We have a map

$$G \times H \to \mathrm{Sp}(V \otimes_{\mathbb{Q}} W)$$

that we can also take adelically (over $\mathbb{A}$). We have a map $\mathrm{Mp}(V \otimes_{\mathbb{Q}} W)/\mathbb{A} \to \mathrm{Sp}(V \otimes_{\mathbb{Q}} W)/\mathbb{A}$. We also have $\omega = \omega_\psi$ on $S(V_{\mathbb{A}}) = \otimes'_{p \le \infty} S(V_p)$.

Let $\psi : \mathbb{A} \to \mathbb{C}^1$ be an additive character.

$\omega_{V,\psi}$ is a representation of $G/\mathbb{A} \times H_{\mathbb{A}}$ on $S(\mathbb{A})$ given by

$$\omega_{V,4}(h)\varphi(x) = \varphi(h^{-1}x)$$

We also have formulae

$$\omega_{V,\psi}(n(b))\varphi(x) = \psi(Q(x))\varphi(x)$$

$$\omega_{V,\psi}(m(b))\varphi(x) = \chi(a)|a|_{\mathbb{A}}^{\frac{m}{2}}\varphi(x \cdot a)$$

$$\omega_{V,\psi}(w)\varphi(x) = \gamma(v_{\mathbb{A}}) \int_{V_{\mathbb{A}}} \varphi(y)\psi(-(x,y))dy$$

where $n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, $m(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\chi(a)$ isa quadratic character, and $\gamma(V_{\mathbb{A}})$ is roots of unity.

Theta kernel

For $\varphi \in S(V_{\mathbb{A}})$, define

$$\Theta(g,h,\varphi) = \sum_{x \in V} \omega(g)\varphi(h^{-1}x)$$

for $g \in G(\mathbb{A})$ and $h \in H(\mathbb{A})$. This converges and is $G(\mathbb{Q}) \times H(\mathbb{Q})$-invertible, i.e. it is a modular form on $[G] \times [H]$. Here, $[G]$

If f is a modular form on $[G]$, then $G(\mathbb{Q}) \backslash G(\mathbb{A}) \subset [G]$.

$$\Theta(h,f) = \int_{[G]} f(g)\Theta(g,h,\varphi)dg$$

is a modular form on $[H]$. Then you get that

$$\int_{[H]} f(h)\Theta(g,h)dh$$

is a modular form on $[G]$. We will modify $\Theta(h,f)$.

Siegel-Weil formula

The Siegel-Weil formula is the theta liting.

Let $f = 1$ on $[H]$. Then you get

$$\Theta(g,\varphi) = \int_{[H]} \Theta(g,h,\varphi)dh \cdot \frac{1}{\mathrm{vol}([H])} =?$$

What is this equal to? It turns out to be equal to an Eisenstein series by the Siegel-Weil formula:

$$= E(g, S_0, \varphi)$$

where $S_0 = \frac{\dim V - 2}{2}$.

Siegel did things where (V, Q) is a quadratic space of sign $(n, 0)$ instead, and then $\Theta(g, h, \varphi)$ apparently becomes a counting function (of numbers of ways to express integers are sums of squares).

Shimura variety

Given (V, Q), we get $H = \mathrm{SO}(V)$. Let $\mathbb{D} = \{\text{oriented negative 2-planes in } V_{\mathbb{R}}\} = \{Z \subset V | \dim Z = 2, Q|_Z > 0\}$.

Let $K \subset H(\mathbb{A}_f)$ be compact open. Then we can produce a Shimura variety $X_K/\mathbb{Q}$ such that $X_K(\mathbb{C}) = H(\mathbb{Q})\backslash\mathbb{D} \times H(\mathbb{A}_f)/K = \coprod \Gamma_j\backslash\mathbb{D}$.

Shimura actually studied $\Gamma_j\backslash\mathbb{D}$ and then Deligne came along to study Shimura varieties as we talk about them today.

For

- $(0, 2)$,
  - we have CM points,
- $(1, 2)$
  - we have Shimura curves,
- $(2, 2)$,
  - we have products of Shimura curves and Hilbert modular surfaces.
- $(3, 2)$
  - We have Siegel 3-folds and probably other things

Given a lattice $L \subset V$, let

$$Z(m, L)" = "\Gamma\backslash\{z \in \mathbb{D} : \exists x \in L \text{ with } Q(x) = m, x \perp z\}.$$

This is of codimension 1.

Small CM cycle

Take $U \subset V$ a negative 2 subspace. Then $U_{\mathbb{R}}^{\pm} \in \mathbb{D}$. We call $z^{\pm}$ the points given by $U_{\mathbb{R}}$.

Also, $H_U = \mathrm{SO}(U) = k^1$ where $k = \mathbb{Q}(\sqrt{-\det U})$. Here,

$$Z(U) = \{Z_u^{\pm}\} \times k^1\backslash k_f^1/K_U = \text{small CM cycles defined over } \mathbb{Q}$$

Main result

Let $L \subset V$ be unimodular. We have a Θ function

$$\Theta(g, h, \varphi)$$

where $g \in \mathrm{SL}_2(\mathbb{A}) = \mathrm{SL}_2(\mathbb{Q})\,\mathrm{SL}_2(\hat{\mathbb{Z}})\,\mathrm{SL}_2(\mathbb{R})$. Also, $\mathrm{SL}_2(\mathbb{R})$ is $B \cdot \mathrm{SO}(2)$ where B consists of real matrices of the form $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$.

Write $\tau = u + iv \in \mathbb{H}$, and $g_\tau = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{u} & 0 \\ 0 & \frac{1}{\sqrt{u}} \end{pmatrix}$.

Also, $h \in \mathrm{SO}(V)(\mathbb{A})$

...

For $z \in \mathbb{D}$, $z \subset V_{\mathbb{R}}$. Decompose $V_{\mathbb{R}} = z \oplus z^{\perp}$, so given $x \in V_{\mathbb{R}}$, write it as $x = x_z + x_{z^\perp}$. We have

$$\varphi_\infty(z, x) = e^{-\pi((x_{z^\perp}, x_{z^\perp}) - (x_z, x_z))}$$

Then we have

$$\Theta(\tau, z, h, \varphi_f, \varphi_\infty) = \sum_{x \in V} \omega(g_z)\varphi_\infty(x)\varphi_f(h^{-1}x)$$

This is a modular form on τ of weight $\frac{n-2}{2}$

$f : \mathbb{H} \to \mathbb{C}$ weight $\frac{2-n}{2}$ Harmonoic Maass modular form.

$$\Phi(z, h, f) = \int_{\text{fundamental domain}}^{\text{reg}} f(z)\Theta(\tau, z, h)d\mu(\tau)$$

This Θ_{reg} function is a Green function of $Z(f) = \sum C_f(-m)Z(m, L)$ by the thesis of Bruinier, and by Bruinier-Funke

> **Theorem**
>
> (Bruinier-Yang 2009)
> $\Phi(Z(U), f) = \sum_p a_p \log p + L'(\xi(f), \theta_P, \text{center})$

where $V = U \oplus P$ where U and P are the $(0, 2)$ and $(n, 0)$ parts of V and $L \supset L_U \oplus J_\Gamma$...

Proof: ...

See Also

Meta

References

Citations and Footnotes

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1. Apparently, Schofer's theorem doesn't actually apply to this case.↩