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We will be interested in $O(0,2)$, $O(1,2)$, and $O(2,2)$.

$O(0,2)$ correspond to algebraic points, $O(1,2)$ corresponds to modular curves, and $O(2,2)$ corresponds to modular surfaces.

We need to understand automorphic forms

1. Review of modular forms

Definition

(Congruence subgroup) $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a congruence subgroup if it contains some principal congruence subgroup $\Gamma(N) := \ker \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.

Examples:

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) :\equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N \right\}$$
$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) :\equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\}$$

Definition

1. $\mathbb{H} = \{\tau \in \mathbb{C} : \mathrm{im}(\tau) > 0\}$
2. $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ by $\tau \mapsto \frac{a\tau+b}{c\tau+d}$
3. $Y(\Gamma) \cong \Gamma \backslash \mathbb{H}$, $Y_0(N) = Y(\Gamma_0(N)), Y_1(N) = Y(\Gamma_1(N))$
4. $X(\Gamma) = \Gamma\backslash\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$

Examples:

1. $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \xrightarrow{j} \mathbb{A}^1; \tau \mapsto \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}) \mapsto j(E_\tau)$
2. $\Gamma(2)\backslash\mathbb{H} \to \mathbb{P}^1 - \{0,1,\infty\}; \tau \mapsto \mathbb{C}/(\mathbb{Z}+2\mathbb{Z}) \mapsto y^2 = x(x-1)(x-\lambda)$

Definition

$k \in \mathbb{Z}$, $\Gamma = \Gamma(N), \Gamma_0(N), \Gamma_1(N)$
$f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight kk and level Γ if $f|_k\gamma(\tau) = g(\tau)$ for any $\gamma \in \Gamma$; here, $f|_k\gamma(\tau) = f(\gamma\tau) \cdot j(\gamma\tau)^{-k}$ where $j(\gamma\tau) = \det\gamma^{-\frac{1}{2}}(c\tau+d)$

There is also a notion of f being holomorphic at a cusp;

Say that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$. We have $f|_k\gamma(\tau) = f(\tau+1)$, so we have a Fourier series $f(\tau) = \sum_{n=-\infty}^{\infty} a(n)q^n$ where $q = e^{2\pi i\tau}$.

f is holomorphic at a cusp if $a(n) = 0$ for $n < 0$ and it is cuspidal it is $a(0) = 0$.

Examples

1. Eisenstein series $\Gamma_\infty = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$

$$G_k(\tau) = \frac{1}{2}\sum_{\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} 1|_k\gamma = \frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}} \frac{1}{(cz+d)^k} \in M_k(\Gamma(1))$$

$$E_k(\tau) = \zeta(k)G_k(\tau) = \frac{1}{2}\sum \frac{1}{(c\tau+d)^k}$$

2. $\Delta(\tau) = \frac{1}{1728}(G_4^3 - G_6^2) \in S_{12}(\Gamma(1))$
3. Jacobi theta function $\theta(q) = \sum_{n\in\mathbb{Z}} q^{n^2}$

$\theta(\tau+1) = \theta(\tau)$

$$\theta\left(-\frac{1}{4\tau}\right) = (-2\pi i\tau)^{\frac{1}{2}}\theta(\tau)$$

$$\theta(\tau) \in M_{\frac{1}{2}}(\Gamma_1(4))$$

Why do we care about modular forms?

Say that f is a Hecke eigenform, normalized, cuspidal, newform of weight 2. We can then construct the L-function

$$L(f,s) = \sum_{n\geq 1} a(n)n^{-s} = \prod_{p|N} L_p(f,s)\cdot\prod_{p\nmid N}(1-a(p)p^s+p^{1-2s})^{-1}$$

Such a modular form corresponds to an elliptic curve $E/\mathbb{Q}$; also, $a(p) = p+1-\#E(\mathbb{F}_p)$

Comment

$M(\Gamma_0(N),\chi)$ where $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}$, consists of the holomorphic functions such that

$$f\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\tau\right) = \chi(d)(c\tau+d)^k f(\tau)$$

Note that $M(\Gamma_0(N),\chi) \subset M(\Gamma_1(N))$ and that $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})$.

We have the decomposition

$$M(\Gamma_1(N)) = \bigoplus_\chi M(\Gamma_0(N),\chi)$$

For any congruence subgroup Γ, there is some $\alpha\in \mathrm{GL}_2(\mathbb{Q})$ such that $\alpha\Gamma\alpha^{-1}\subset\Gamma_1(N)$.

2. Automorphic forms

Let $G = \mathrm{GL}_2$, let $F = \mathbb{Q}$, and let $\mathbb{A} = \mathbb{A}_F$. Let p be a place of F. Let $\mathbb{R} = F_\infty$

In general, we can replace G with a nice reductive group, and F with a general number field.

$G(\mathbb{A}) = G(\mathbb{R})\cdot G(\mathbb{A}_f) = G(\mathbb{R})\cdot\prod' G(\mathbb{Q}_p)$

Say that $K_f\subset G(\mathbb{A}_f)$ is an open compact subgroup. Equivalently, we can write $K_f = \prod K_p$ where $K_p = G(\mathcal{O}_p)$ for all but finitely many p.

Theorem

(Strong approximation) Say that $\det(K_f) = \hat{\mathbb{Z}}^\times$. In this case, $G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})^{\oplus}K_f$.

Another way to say this is that $G(\mathbb{Q})$ is dense $G(\mathbb{A})$

Let $K_0(N) = \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} : c\equiv 0 \pmod{N}\right\}$; $K_0(N)$ has matrices with entries over $\mathbb{A}_f$.

By strong approximation, we then have

$$G(\mathbb{Q})\backslash G(\mathbb{A})/K_0(N) \simeq \Gamma_0(N)\backslash G(\mathbb{R})^+$$

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Also, note that $G(\mathbb{R})^+ = Z(\mathbb{R}) \cdot \mathrm{SL}_2(\mathbb{R})$, and there is an isomorphism $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \xrightarrow{\sim} \mathbb{H}$. Here, Z is the center of G. In this case, Z consists of the scalar matrices. Therefore, this tells us that $f : \mathbb{H} \to \mathbb{C}$ should come from $\phi_f : \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$.

First let's define $\phi_f : G(\mathbb{R})^+ \to \mathbb{C}$, and then the isomorphism given by strong approximation will give us the map $\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$.

Define $\phi_f(g_\infty) = f(g_\infty i) j(g_\infty, i)^{-k}$

Proposition

Say that $f \in M_k(\Gamma_0(N))$ Then

1. $j(g_1 g_2, \tau) = j(g_1 g_2(\tau)) j(g_2 \tau)$,

$j(zk\theta, i) = \mathrm{sgn}(z) e^{-i\theta}$ where $z \in Z(\mathbb{R})$, $k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$j(z, i) = \mathrm{sgn}(z) j(k_\theta, i) = e^{-i\theta}$

2. For $\gamma \in \Gamma_0(N)$, $\phi(\gamma g_\infty) = \phi(g_\infty)$
3. $\phi_f(g_\infty, zk_\theta) = \phi(g_\infty)\,\mathrm{sgn}(z)^k (e^{i\theta})^k$

Let $\mathfrak{g}$ be the Lie algebra of $\mathrm{GL}_2(\mathbb{R})^+$. Write $\hat{L} = \begin{bmatrix} & 0 \\ 1 & \end{bmatrix}$, $\hat{R} = \begin{bmatrix} & 1 \\ 0 & \end{bmatrix}$, $\hat{H} = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$. Construct the universal enveloping algebra $U(\mathfrak{g})$. The center $Z(U(\mathfrak{g}))$ is an algebra generated by two elements; $Z(U(\mathfrak{g})) \cong \mathbb{C}[Z, \Delta]$, where Δ is the Casimir operator defined by $-\frac{1}{4}(\hat{H}^2 + 2\hat{R}\hat{L} + 2\hat{L}\hat{R})$.

We have a Lie algebra action of $\mathfrak{g}$ on automorphic forms: for $x \in \mathfrak{g}$,

$$x\phi_f(g_\infty) = \frac{d}{dt}\phi_f(g_\infty(I + Xt))|_{t=0}$$

We then have

$$Z\phi_f = 0$$

$$\Delta\phi_f = -\frac{1}{4}k(k-2)\phi_f$$

these two equations correspond to/explain the central character and holomorphicity

3. Automorphic representation

Let $\omega : \mathbb{Q}^\times \backslash \mathbb{A}^\times \to \mathbb{C}$ be a character of the ideles.

Definition

The space $\mathcal{A}(G, \omega)$ of automorphic forms is the space "smooth" functions $f : G(\mathbb{Q}) \to \mathbb{C}$ (in the archimedean places, smoothness is the usual notion; in the nonarchimedean places, smoothness means locally constant) such that

1. for $z \in Z(\mathbb{A})$, $f(zg) = \omega(z) f(g)$[1] and for $\gamma \in G(\mathbb{Q})$, we have $f(\gamma g) = f(g)$
2. Let $K_\infty = \mathrm{SO}_2(\mathbb{R})$; the action of K_∞ on the right of f is finite, i.e. $\dim_{\mathbb{C}} K_\infty f < \infty$.
3. There exists an open compact subgroup $K_f \subset \mathrm{GL}(\mathbb{A}_f)$ such that $K_f f = f$.
4. The action of $Z(U(g))$ is also finite.

5. Analytic growth condition

See Also

Meta

References

Citations and Footnotes

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1. This is analogous to $f(\gamma\tau) = \chi(d)f(\tau)$↩