Weil represntations and Θ functions

Speaker: Ryan Tamura

(i) Weil representations provide a generalization of Θ-functions whichcan be viewed as automorphic of $MP(w)$ and dual pairs of $Sp(w)$.

(ii) Θ-correspondence: correspondence between smooth irreducible rperesnetations of G and "" of Θ, where (Θ, Θ')

1. Heisenberg Groups and Representations

Setup: Local field F and a symplectic vector space W/F with a pairing: $(W, ()_W)$

Definition

(Heisenberg group):
$H(W) = W \times F$, twisted product $(w, f) \cdot (w', f') = (w + w', f + f' + \frac{1}{2}(w, w')_W)$

Example: take $F = \mathbb{R}$ with dimension 2 symplectic space over $\mathbb{R}$, and then

$$H(W) = \{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \}$$

Remark

$H(W)$ can be viewed as a central extension

$$1 \to F \to H(W) \to W \to 1$$

and $Z(H(W)) = ((0, f) : f \in F) \cong \mathcal{F}$

Philosophy: irreducible representations of $H(W)$ are given by Schwartz functions.

Theorem

(Stone-Von Neumann):
Start with a nontrivial additive character $\psi : (F, +) \to \mathbb{C}^*$

Then there is a unique (up to isomorphism) irreducible, smooth, admissible representation (p_ψ, S_ψ) where $Z(H(W))$ central character ψ, i.e. $(0, f) \circ V = \rho(f)_V$

Idea of the proof of existence

Remark

S_ψ concrete model of S_ψ Schrodinger model

Since W is a symplectic space, we can take a decomposition $W = X \oplus Y$ where X, Y are maximal isotropic subspaces

Define

$$\mathrm{Ind}_{Y \times F}^{H(W)} \psi = \{\varphi : H(W) \to \mathbb{C} \text{ satisfying the below listed conditions } \}$$

(i) $\varphi((y,f)\cdot h) = \psi(f)\circ\varphi(h)$

(ii) there exists an open compact neighborhood $v \subseteq H(W)$ such that for $(v,0)\in V$, for any $h\in H(W)$, we have $\varphi((v,0)h) = \varphi(h)$

There is an action: for $h \in H(W)$, let

$$(h\circ\varphi)(h') = \varphi(h'\circ h).$$

We also have a map

$$i : X \to H(W), \quad X \mapsto (x,0)$$

and a map

$$S_Y \to S(X) = \{f : X \to \mathbb{C}, \text{ compactly supported and locally constant }\}, \quad f \mapsto f\circ i$$

Let $(h,f)\in H(W)$. We want to define $(h,f)\circ\varphi(X)$

Decompose $W = X\oplus Y$, set $h = x+y$. So we are defining $(x+y,f)\circ\varphi(x)$:

$$(x+y,f)\circ\varphi(x_0) = \psi(f+(x_0,y)_W + \frac{1}{2}(x,y)_W)\circ\psi(x+x_0))$$

And then $Z(H(w)) = \{(0,f)\}$ acts by ψ.

The irreducible smooth representation of the Heisenberg group is in disguise the space of Schwarz functions (is what I think Ryan said).

Comment

Simon Marshall comments that Heisenberg is involved because this relates position and momentum; there is a quantum mechanical thing going on here.

Local Weil representation

Define a right action of $\mathrm{Sp}(W)$ on $H(W)$: for $\sigma\in\mathrm{Sp}(W)$,

$$(h,f)\sigma = (h\sigma, f)$$

Now define the twist of p_ψ by σ:

$(p^\sigma\psi, S_\psi), p^\sigma\psi((h,f)) = p_\psi((h,f)\circ\sigma)$

Proposition

The central characters of p_ψ and p_ψ^σ are the same

Theorem

$(p_\psi, S_\psi) \simeq (p_\psi^g, S_\psi)$, but this isomorphism is not canonical.

We have the Projective Weil representation

$$w : \mathrm{Sp}(W) \to \mathrm{PGL}(S_\psi) = \mathrm{GL}(S_\psi)/\mathbb{C}^*$$

where $w(\sigma) = [A(g)]$ such that

$$A(\sigma)p_\psi A(\sigma)^{-1} = p_\psi^\sigma$$

Remark

$H^2(\mathrm{Sp}(W), \mathbb{C}^*) \neq 0$

Definition

(Metaplectic group)
The Metaplectic group $\mathrm{Mp}(W)_\psi$ is the fiber product of $\mathrm{Sp}(w) \xrightarrow{w} \mathrm{PGL}(S_\psi)$ and $\pi : \mathrm{GL}(S) \to \mathrm{PGL}(S_\psi)$.

The induced map $\tilde{\omega} : \mathrm{Mp}(w)_\psi) \to \mathrm{GL}(S_\psi)$ is the Local Weil representation of ψ

We also have

$$\mathrm{Mp}_\psi(W) = \{(\sigma, \Phi), \Phi_\psi \Phi^{-1} = p_\psi^\omega\}$$

Theorem

(The metaplectic group doesn't depend on the non-trivial character ψ)

from $r : \mathrm{Sp}(W) \to \mathrm{Mp}(W)$, get an explicit 2 cocycle

$$(\sigma, r(\sigma)) \circ (\sigma_1', r(\sigma')) = (\sigma\sigma', c_2(\sigma\rho')r(\sigma\sigma'))$$

$$1 \to \mathbb{C}^* \to \mathrm{Mp}(W) \to \mathrm{Sp}(W) \to 1$$

$$\mathrm{Mp}_{\text{rao}}(w) = \mathrm{Sp}(W) \times \mathbb{C}^* = \{(\sigma, z)\} \text{ product } (\sigma, z)(\sigma', z') = (\sigma\sigma', c_{\text{rao}}(\sigma, \sigma')zz')$$

Examples

Take $\mathrm{Sp}_2(F) = \mathrm{SL}_2(F)$, we can take P to be the maximal parabolic subgroup, where $P = MN$ where

$$M = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}, N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}, a, b, \in F$$

there are then formulae

- (i)

$$\omega \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, z \right) \varphi(x) = z|a|^{1/2} \varphi(ax)$$

- (ii)

$$\omega \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, z \right) \varphi(x) = \psi \left(\frac{b}{2} x^2 \right) \varphi(x)$$

- (iii) $\omega \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, z \right) \varphi(x) = z S_V \circ \varphi(x)$

...

Take K to be a number field, $\psi : \mathbb{A}_K \to \mathbb{C}^*$. Take W/F to be a symplectic space

$$1 \to \mathbb{C}^* \to \mathrm{Mp}(W) \to \mathrm{Sp}(W) \to 1$$

(i) Take the restricted product

$$\tilde{\mathrm{Mp}}(W) = \prod_v{}' \mathrm{Mp}(W_V) = \{(g_v) : g_v \in \mathcal{O}_{K,v} \text{ for all but finitely many } v\}$$

Definition

(Adelic metaplectic group)
$\mathrm{Mp}(W) = \widetilde{\mathrm{Mp}(W)}/H$ where $H \leq \bigoplus \mathbb{C}$

✎ **Definition**

(Weil representation)
This is the representation

$$\bigoplus_v{}' S_{\psi,v} = \varinjlim_{S \text{ finite set of places}} \otimes_{v\in S} S_{\psi,V} = S \otimes (\mathbb{A})$$

See Also

Meta

References

Citations and Footnotes