

Eisenstein series

Speaker: [luo_yu](#)

The Siegel-Weil formula is pretty concrete. The following is some related exposition.

Consider the sum of squares problem. For instance, 5, 13, 17, and 29 are all expressible as sums of two integers squares. However, 3, 7, 11, 19, 23 are not.

Also, note that

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$$

Theorem

For $n > 0$, $n = x^2 + y^2$ for some integers x, y if and only if prime divisor $p \equiv 3 \pmod{4}$ of n has even degree.

Write $r(n) = \#\{(x, y) \in \mathbb{Z}^2 : n = x^2 + y^2\}$

Theorem

(Jacobi)

$$r(n) = 4 \left(\sum_{d \equiv 1 \pmod{4}} \frac{d|n}{1} 1 - \sum_{d \equiv 3 \pmod{4}} \frac{d|n}{1} 1 \right)$$

For instance, if $p \equiv 1 \pmod{4}$, then $r(p) = 8$ and if $p \equiv 3 \pmod{4}$, then $r(p) = 0$.

Now define the **Jacobi theta function**

$$\theta(q) = \sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + \cdots + 2q^{n^2} + \cdots$$

Then

$$\theta^2(q) = \sum_{n \in \mathbb{N}} \tau(n) q^n = 1 + 4q + \cdots$$

It turns out that $\theta^2(q) \in \mathcal{M}_1(\Gamma_1(4))$.

Now a very concrete fact about $\mathcal{M}_1(\Gamma_1(4))$ is that it is 1-dimensional.

There is a way to concretely construct the modular forms of any weight: with the Eisenstein series.

Let $\chi : (\mathbb{Z}/4\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ be the nontrivial character.

Then let

$$E_k^\chi(\tau) = \sum_{\Gamma_\infty \backslash \Gamma_1(4)} \frac{\chi(d)}{(c\tau + d)^k}$$

where $\Gamma_\infty = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. When $k = 1$, while this may not be absolutely convergent, we still have $E_1^\chi(\tau) \in \mathcal{M}_1(\Gamma_1(4))$.

We have a Fourier expansion

$$E_k^\chi(\tau) = 1 + c_k^\chi \sum_{n \geq 1} \left(\sum_{d|n} \chi(d) d^{k-1} \right) q^n$$

Since $\mathcal{M}_1(\Gamma_1(4)) = 1$, $\theta^2(q)$ and $E_1^\chi(\tau)$ are equal up to a scalar. By comparing the Fourier coefficients of both modular forms, we have

$$4 = C_1^\chi$$

$$\tau(n) = 4 \left(\sum_{d|n} \chi(d) \right)$$

and thus Jacobi's theorem above follows.

θ -function

Let V be a vector space over \mathbb{Q} with quadratic form $Q : V \rightarrow \mathbb{Q}$. Write H for the Heisenberg group. Fix an additive character $\psi : \mathbb{A} \rightarrow \mathbb{C}^\times$. Recall that there is a character $\psi_\infty e^{2\pi i x}$. Let H be the Heisenberg group. There is an action of $G(\mathbb{A}) \times H(\mathbb{A})$ on $S(V(\mathbb{A}))$.

Let $\varphi \in S(V(\mathbb{A}))$. Write

$$\theta(g, h; \varphi) = \sum_{x \in V(\mathbb{Q})} \omega(g) \varphi(h^{-1}x)$$

Define the θ -integral

$$\int_{H(\mathbb{Q}) \backslash H(\mathbb{A})} \theta(g, h; \varphi) dh$$

V is a positive definite quadratic form space. Let $\Lambda \subset V$ be a lattice.

Let $\varphi_p = \text{char}(\Lambda \otimes \mathbb{Z}_p)$ and $\varphi_\infty(x) = e^{-2\pi \mathbb{Q}[x]}$. Let $K \subset H(\mathbb{A})$ denote the stabilizer of Λ . Recall that there is a natural identification

$$H(\mathbb{Q}) \backslash H(\mathbb{A}) / K \xrightarrow{\sim} \text{Gen}(\Lambda)$$

where $\text{Gen}(\Lambda)$ is the set of isomorphism classes of lattices with the same genus as Λ . The identification sends $h \in H(\mathbb{Q}) \backslash H(\mathbb{A})/K$ to $h(\Lambda \otimes \hat{\mathbb{Z}}) \cap V$

Classically, it is known that $\text{Gen}(\Lambda)$ is finite. Write $\{h_i\}$ for representatives of $H(\mathbb{Q}) \backslash H(\mathbb{A})/K$ and $\{\Lambda_i\}$ for representatives of $\text{Gen}(\Lambda)$

Later on, we will use an example of $V = \mathbb{Q}^2$ where the quadratic form Q is given by $Q = x^2 + y^2$ where $(x, y) \in V$.

Now we have

$$\begin{aligned} \int_{H(\mathbb{Q}) \backslash H(\mathbb{A})} \theta dh &= \sum_i \int_{H(\mathbb{Q}) \backslash H(\mathbb{Q})/h_i K} \theta(g, h; \varphi) dh \\ &= \sum_i \int_{H(\mathbb{Q}) \backslash H(\mathbb{Q})h_i K h_i^{-1}} \theta(g, h h_i; \varphi) dh. \end{aligned}$$

We also have

$$H(\mathbb{Q}) \backslash H(\mathbb{Q})h_i K h_i^{-1} \simeq (H(\mathbb{Q}) \cap h_i K h_i^{-1}) \backslash h_i K h_i^{-1}$$

Also, $(H(\mathbb{Q}) \cap h_i K h_i^{-1}) = \text{Aut}(\Lambda_i)$

Now the summand

$$\int_{H(\mathbb{Q}) \backslash H(\mathbb{Q})h_i K h_i^{-1}} \theta(g, h h_i; \varphi) dh$$

from before equals

$$\begin{aligned} &= \frac{1}{\# \text{Aut}(\Lambda_i)} \int_{h_i K h_i^{-1}} \theta(g, h h_i; \varphi) dh \\ &= \frac{1}{\# \text{Aut}(\Lambda_i)} \int_K \theta(g, h_i h; \varphi) dh \end{aligned}$$

By strong approximation, write

$$g = g_i = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{\frac{1}{2}} & 0 \\ 0 & v^{-\frac{1}{2}} \end{pmatrix}$$

where $\tau = u + iv$.

Continuing on, we have

$$\int_K \theta(g_\tau, h_i h; \varphi) dh = \sum_K \sum_{x \in V(\mathbb{Q})} \omega(g_\tau) \varphi(h^{-1} h_i^{-1} x) dh$$

Now $h^{-1} h_i^{-1} x$ is in $\Lambda \otimes \mathbb{Z}_p$ if and only if $x \in h_i h \Lambda \otimes \mathbb{Z}_p = \Lambda_i \otimes \mathbb{Z}_p$. Thus, the above equals

$$\begin{aligned} & \int_K \sum_{x \in \Lambda_i} \omega_\infty(g_i) \varphi_\infty(x) dh \\ &= \text{vol}(K) \sum_{x \in \Lambda_i} \omega_\infty(g_i) \varphi_\infty(x). \end{aligned}$$

The summand $\omega_\infty(g_i) \varphi_\infty(x)$ equals

$$\begin{aligned} \omega_\infty(n(u)m(v^{\frac{1}{2}})) \varphi_\infty(x) &= \psi(uQ(x)) |v|^{\frac{m}{4}} \varphi_\infty(xv^{\frac{1}{2}}) \\ &= |v|^{\frac{m}{4}} e^{2\pi i u Q(x)} \cdot e^{2\pi v Q(x)} = |v|^{\frac{m}{4}} e^{2\pi i \tau Q(x)} \end{aligned}$$

where $m = \dim V$, so going back, we have

$$\text{vol}(K) |v|^{\frac{m}{4}} \sum_{x \in \Lambda_i} e^{2\pi i \tau Q(x)}.$$

Example 1

Now let us take the example of $V = \mathbb{Q}^2$, $\mathbb{Z}^2 \subset V$, and $Q[(x, y)] = x^2 + y^2$.

We have the square of the theta function

$$\sum_{a, b \in \mathbb{Z}^2} q^{a^2 + b^2} = \Theta^2(q)$$

Example 2

When we have $V = \mathbb{Q}$, $\mathbb{Z} \subset V$, and $Q[x] = x^2$, we have the θ function

$$\sum_{n \in \mathbb{Z}} q^{n^2} = \theta(q)$$

Summary

So the work that was done in this section shows that Theta integral is closely related to the sum $\sum_{x \in \Lambda_i} e^{2\pi i \tau Q(x)}$, which specializes in the two examples above to $\theta^2(q)$ and $\theta(q)$

Eisenstein series

Definition

$$I(s, \chi) = \text{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})} \chi_V | \cdot |^{s+1}$$

where $B(\mathbb{A})$ is the Borel group

There is a principal series representation $\Theta(g, s)$. We have

$$\Theta(n(b)m(a)g, x) = \chi(a) |a|^{s+1} \Phi(g)$$

[1]

Example

There is a map $\lambda : S(V(\mathbb{A})) \rightarrow I(s_0, \chi_v)$ sending φ to $\lambda(\varphi)$ where $\lambda(\varphi)(g) = (\omega(g)\varphi)(0)$.

Let us check that the conditions of $\lambda(\varphi)$ to be in the induced representation are satisfied:

$$\lambda \varphi(n(b)m(a)g, s_0) = \omega(n(b)m(a)g)\varphi(0) = \chi_v(a)|a|^{s_0+1}\omega(g)\varphi(0).$$

Here, $\lambda(\varphi)(g)$ is $\omega(g)$???

Back to the discussion

A section $\Phi(s)$ is called **standard** if $\Phi(s)|_{K_\infty K}$ is constant.

We define the **Eisenstein series**

$$E(g, s; \Phi) = \sum_{\gamma \in B(\mathbb{Q}) \backslash G(\mathbb{Q})} \Phi(\gamma g, s)$$

In the case that $\Phi = \lambda(\varphi)$, we have a Siegel Eisenstein series.

Now we are not assuming that V is a positive definite quadratic space.

Theorem

(Siegel-Weil formula)

$$\frac{\alpha}{2} \int_{O(\mathbb{Q}) \backslash O(\mathbb{A})} \theta(g, h; \varphi) dh = E(g, s_0; \lambda(\varphi))$$

Simon's comments on the local theta correspondence

Let F be a p -adic local field. Let ψ be an additive character on F . Let W be a symplectic space. Let V be an even orthogonal space. Let $\mathbb{W} = W \otimes V$. Let $G = \mathrm{Sp}(W)$. Let $H = O(V)$. Let ω be the Weil representation of $\tilde{S}_p(\mathbb{W})$.

We say that $\pi_G \in \mathrm{Irr}(G)$ and $\pi_H \in \mathrm{Irr}(H)$ correspond if $\mathrm{Hom}_{G \times H}(\omega, \pi_G \otimes \pi_H) \neq 0$.

Theorem

(Mœglin-Vignéras-Waldspurger for $p > 2$; $p = 2$ is really hard and has only been done more recently, Howe for $f = \mathbb{R}$)

This correspondence is a bijection between a subset of $\text{Irr}(G)$ and $\text{Irr}(H)$

"A representation occurs in the Theta correspondence if it is a quotient of ω . The local theta correspondence is a partial bijection between two sets of representations"

The correspondence is "1. take π_G . 2. Look for representation π_H such that π_G and π_H correspond (i.e. such that $\text{Hom}_{G \times H}(\omega, \pi_G \otimes \pi_H) \neq 0$ ". And then this is apparently a bijection for some subset of $\text{Irr}(G)$ and $\text{Irr}(H)$

Write $\omega[\pi_G]$ for the maximal quotient of ω for which G acts π_G -isotypically.

Then $\omega[\pi_G] \simeq \pi_G \otimes \Theta(\pi_G)$.

Theorem

$\Theta(\pi_G)$ has a unique irreducible quotient as a representation of H . The quotient is $\Theta(\pi_G)$

Simon's comments on the global theta correspondence

We have an integral

$$\theta(g, \phi, \varphi) = \int_{[H]} \Theta(g, h, \varphi) \overline{\phi}(h) dh$$

One of the important things about this map is that the integral is H -invariant:

$$\int_{[H]} \Theta(g, h, h' \varphi) \overline{\phi}(h) dh$$

What this integral is doing is that it is giving an element of $\text{Hom}_{(G \times H)(\mathbb{A})}(\omega \otimes \pi_H^\vee, \mathcal{A}(G))$ -- in particular, said element is equivariant for the $(G \times H)(\mathbb{A})$ actions. Also, this Hom space is isomorphic to $\text{Hom}_{(G \times H)(\mathbb{A})}(\omega, \pi_H \otimes \mathcal{A}(G))$.

The image of the integral, which is in $\mathcal{A}(G)$, is apparently a quotient of $\Theta(\pi_H)$ by the local theta correspondence.

How to relate Eisenstein series with classical Eisenstein series

Let χ be the trivial character. Take ℓ to be an even number. Write

$k_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \in K_\infty = \text{SO}(2)$. Pick a function Φ_∞^ℓ so that

$$\Phi_\infty^\ell(gk_\theta) = e^{2\pi i \ell \theta} \Phi_\infty^\ell(g)$$

Writing $\tau = u + iv$, it suffices to understand the values at $g_\tau = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u^{\frac{1}{2}} & 0 \\ 0 & v^{-\frac{1}{2}} \end{pmatrix}$ (by something about strong approximation?).

Write $\Phi = \Phi_\infty^\ell \otimes \Phi_p$.

Also let

$$E(g_\tau, s, \Phi) = \sum_{\gamma \in B(\mathbb{Q} \backslash G(\mathbb{Q}))\Phi(\gamma g_\tau s)}$$

We have

$$B(\mathbb{Q}) \backslash G(\mathbb{Q}) \simeq \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z}) \simeq \mathbb{P}^1(\mathbb{Q}) \simeq \mathbb{P}^1(\mathbb{Z})$$

Lemma

For $\gamma = \begin{pmatrix} a & b \\ \cdot & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q})$,

Write $\gamma g_\tau = g_{\gamma\tau} \cdot k_\theta$ where k_θ lies in the upper half plane. Then

$$e^{i\theta} = \frac{c\bar{\tau} + d}{|c\tau + d|}.$$

Lemma

$$\mathrm{Im}(\gamma\tau) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2}$$

Lemma

$$\Phi_\infty^\ell(g_\tau, s) = \Phi_\infty^\ell \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{\frac{1}{2}} & 0 \\ 0 & v^{-\frac{1}{2}} \end{pmatrix}, s \right) = |v|^{\frac{1}{2}(s+1)}$$

...

You do a bunch of stuff and recover the original Eisenstein series with $\ell = 1$.

See Also

Meta

References

Citations and Footnotes

1. $m(a)$ denotes $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ and $n(b)$ denotes $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \leftarrow$