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Let $K=\mathbb{Q}$ and let $\mathbb{A}=\mathbb{A}_\mathbb{Q}$. Let W be a symplectic space over \mathbb{Q} . Given W a complete polarization \langle,\rangle . Fix a symplectic basis $\{e_i,f_j\}$ such that $\langle e_i,e_j\rangle=\langle f_i,f_j\rangle=0$ and $\langle e_i,f_j\rangle=\delta_{ij}$. Write X for the subspace generated by the e_i and write Y for the subspace generated by the f_i so that W decomposes as $W=X\oplus Y$.

We also have a decomposition $W(\mathbb{A})=X(\mathbb{A})\oplus Y(\mathbb{A})$. Let H(W) be the Heisenberg group $H(W)=W\oplus \mathbb{Q}$

Fix an additive character $\psi: \mathbb{Q} \setminus \mathbb{A} \to \mathbb{C}^*$. There is a character $\rho_{\psi}: H(W)(\mathbb{A}) \to \mathcal{S}(X(\mathbb{A}))^{[1]}$.

Recall, for $f \in \mathcal{S}(X(\mathbb{A})), x_0 \in X(\mathbb{A}), x \in X(\mathbb{A}), y \in Y(\mathbb{A}), \text{ and } t \in \mathbb{A} \text{ that } t \in \mathbb{$

$$ho_{\psi}(x)f(x_0)=f(x+x_0) \
ho_{\psi}(y)f(x_0)=\psi(\langle x_0,y
angle)f(x_0) \
ho_{\psi}(t)f(x_0)=\psi(t)f(x_0)$$

There is also a distribution

$$\Theta: \mathcal{S}(X(\mathbb{A}))
ightarrow \mathbb{C}$$

with

$$\Theta(\phi) := \sum_{x \in X(\mathbb{Q})} \phi(x).$$

It is invariant under the action of $H(W)(\mathbb{Q})$

// Theorem

(Weil) This is the unique up to scalar distribution invariant under $H(W)(\mathbb{Q})$.

Example

Let $X(\mathbb{A})=\mathbb{A}$, take $\phi=\bigotimes_v{}'\phi_v$ where $\phi_p=1_{\mathbb{Z}_p}$ and ϕ_∞ is a Schwartz function on \mathbb{R} , i.e. $\phi_\infty\subset C^\infty(\mathbb{R})$ such that $\sup_{x\in\mathbb{R}}(1+|x|^m)\phi_\infty^{(n)}(x)<\infty$. For instance, compactly supported functions ϕ_∞ are Schwartz functions, and $\phi_\infty=e^{-\pi x^2}$ is also a Schwartz function.

In this case, $\Theta(\phi)$ is

$$\Theta(\phi) = \sum_{x \in \mathbb{Q}} \phi_{\infty}(x) \prod_p \phi_p(x) = \sum_{x \in \mathbb{Z}} \phi_{\infty}(x)$$

The second equality is true because $\phi_p(x) = 1_{\mathbb{Z}_p}(x)$ by definition.

Back to the talk

There is an action of $\mathrm{Sp}(W)$ on H(W) (cf

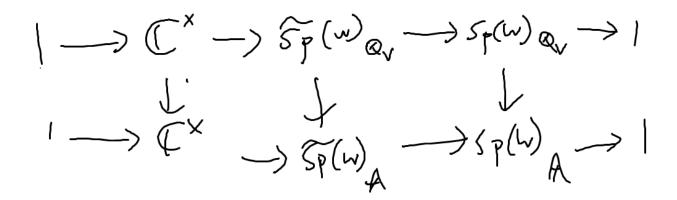
kudla_nltc_notation_S_p_W_symplectic_group_of_nondegenerate_symplectic_vector_space_over_nonarchimedean_local_field_of_dimension_not_2): A(g) acts by sending $(\rho_{\psi}, S(X)(\mathbb{A}))$ to $(\rho_{\psi}^g, S(X)(\mathbb{A}))$.

This action induces a map

$$\operatorname{Sp}(W)(\mathbb{A}) \to \operatorname{PGL}(\mathcal{S}(X)(\mathbb{A})).$$

By pulling this map back via the projection map $GL(S(X)(\mathbb{A})) \to PGL(S(X)(\mathbb{A}))$, we get the metaplectic group $\widetilde{Sp(W)}(\mathbb{A})$.

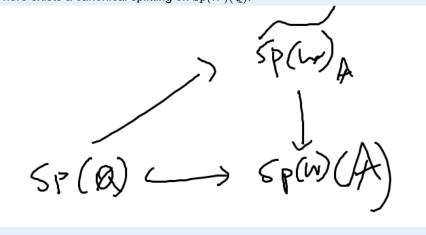
For all places v, we have a commuting diagram



// Theorem

(Weil)

There exists a canonical splitting on $Sp(W)(\mathbb{Q})$:



Let $\widetilde{\mathrm{Sp}(W)}(\mathbb{Q})$ be the preimage of $\mathrm{Sp}(W)(\mathbb{Q})$ under the metaplectic cover.

Let ω_{ψ} be the Weil representation of $\widetilde{\mathrm{Sp}(W)}_{\mathbb{A}}$ acting on $\mathcal{S}(X(\mathbb{A}))$. Now for arbitrary $g \in \widetilde{\mathrm{Sp}(W)}(\mathbb{Q})$ and for all $\phi \in \mathcal{S}(X(\mathbb{A}))$.

$$\sum_{x\in X(\mathbb{O})}(w_{\psi}(g)\phi)(x)$$

is another distribution on $\mathcal{S}(X(\mathbb{A}))$.

In fact, this distribution is also invariant under $H(W)(\mathbb{Q})$, and Weil's theorem tells us that there is a $\lambda_g \in \mathbb{C}^*$ relating this distribution to Θ . This gives us a character $\widetilde{\mathrm{Sp}(W)}(\mathbb{Q}) \to \mathbb{C}^*$.

For
$$g\in \widetilde{\mathrm{Sp}(W)}_{\mathbb{A}}$$
, $\phi\in \mathcal{S}(X(\mathbb{A}))$, let

$$\Theta(g,\phi) := \sum_{x \in X(\mathbb{Q})} (\omega_{\psi}(g)\phi)(x).$$

It is invariant under $\mathrm{Sp}(w)(\mathbb{Q})$. In particular, $\Theta(\cdot,\phi)$ gives a map

$$\mathrm{Sp}(W)(\mathbb{Q}) \backslash \widetilde{\mathrm{Sp}(W)}_{\mathbb{A}} o \mathbb{C}$$

Reductive dual pairs

A Reductive dual pair is a pair (G,G') of subgroups of Sp(W) such that G_1 and G_2 are reductive groups,

- G =Centralizer of G' in Sp(W)
- G' =Centralizer of G in Sp(W)

The example pair is the pair of the orthogonal group and the symplectic group:

Example

Let V, (,) be a quadratic space. Let W, \langle, \rangle be a symplectic space. Write

$$\mathbb{W} := V \otimes W$$

which is also symplectic; the bilinear form is given by the tensor product:

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = (v_1, v_2) \langle w_1, w_2 \rangle$$

We have a map

$$O(V) imes \mathrm{Sp}(W) o \mathrm{Sp}(\mathbb{W})$$

Back to the talk

If (G_1,G_1') is in $\mathrm{Sp}(W_1)$ and (G_2,G_2') is in $\mathrm{Sp}(W_2)$, then $(G_1\times G_2,G_1'\times G_2')$ is in $\mathrm{Sp}(W_1\oplus W_2)$

There are two types of reductive dual pairs:

- Type I: GG' on W is irreducible
- Type II: reducible

Type I: For any base field K and division algebra D/K,

$$V,(,), \tau((v_1,v_2))=(v_2,v_1)$$

$$W,\langle,
angle\ au(\langle w_1,w_2
angle)=-\langle w_2,w_1
angle$$

Type II: Given W_1, W_2 over K and a division algebra D/K,

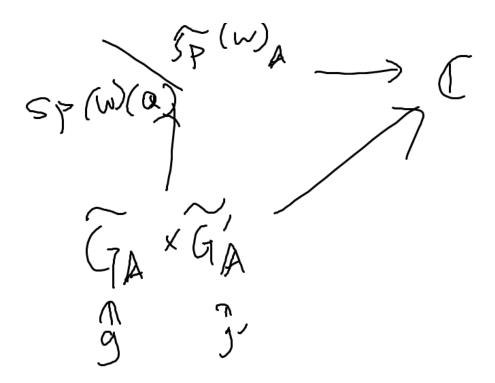
$$W=W_1\otimes W_2\oplus W_1^*\oplus W_2^*$$

Fix (G, G') a dual pair in Sp(W) over \mathbb{Q} .

Let $\tilde{G}_{\mathbb{A}}, \tilde{G}'_{\mathbb{A}}$ be the preimages of $G(\mathbb{A})$ and $G'(\mathbb{A})$ under the metaplectic covering

$$\widetilde{Sp}(W)_{\mathbb{A}} o Sp(W)(\mathbb{A})$$

Now note that we have



and then $\Theta(g, g', \phi)$ is called the Theta kernel

Theta lift for $ilde{G}_{\mathbb{A}}$ to $ilde{G}'_{\mathbb{A}}$

Let f be a (cuspidal) automorphic form on $G(\mathbb{Q}) ackslash ilde{G}_{\mathbb{A}}$

Let

$$\Theta(f,\phi)(g') = \int_{G(\mathbb{Q}) \setminus ilde{G}_{\mathbb{A}}} \Theta(g,g',\phi) \overline{f(g)} dg$$

Example

Fix $\psi: \mathbb{Q} \setminus \mathbb{A} \to \mathbb{C}^{\times}$, let $\psi_{\infty}: \mathbb{R} \to \mathbb{C}^{*}$ be the usual exponential function $x \mapsto e^{2\pi i x}$, and let $\psi_{p}: \mathbb{Q}_{p} \to \mathbb{C}^{\times}$ be given by $x \mapsto e^{-2\pi i (x - \lfloor x \rfloor_{p})}$

Let (V,(,)) be quadratic over $\mathbb Q$ with $\dim V=2m$. Define the quadratic form $q(x)=\frac{1}{2}(x,x)$. Note that (x,y)=q(x+y)-q(x)-q(y). Choose a $\mathbb Z$ -lattice $\Lambda\subset V$. For almost all places p, the lattice $\Lambda_p=\Lambda\otimes_p\mathbb Z_p\subset V_p$ and $\Lambda_p^\perp=\Lambda_p$.

Let K_p be the stabilizer of Λ_p in $O(V)(\mathbb{Q}_p)$. It is maximum compact.

Write
$$K_f = \prod K_p \subset O(V)(\mathbb{A}_f)$$
, $K_\infty = O(V)(\mathbb{R})$, and $K = K_\infty K_f$.

Take 1 on $G(\mathbb{Q})\backslash G(\mathbb{A})$. We know that $G(\mathbb{Q})\backslash G(\mathbb{A})/K=\mathrm{gen}(\Lambda)$; the genus of a lattice means the equivalence class of lattices that are locally isomorphic everywhere.

Given $g=(g_p)_p$, we have the action $\Lambda\mapsto \bigcap_p (g_p\Lambda_p\cap V(\mathbb{Q})=g(\Lambda)$

So $G(\mathbb{Q})\backslash G(\mathbb{A})$ acts on $gen(\Lambda)$

Decompose $G(\mathbb{A}) = \coprod_i G(\mathbb{Q})g_iK$ and in particular,

$$\mathbb{1} = \bigcup_i \mathbb{1}_{G(\mathbb{Q}) \setminus G(\mathbb{Q})g_iK}$$

Let us describe the action of $SL_2(\mathbb{A})$ on $\mathcal{S}(V(\mathbb{A}))$:

For $a \in \mathbb{A}^{\times}$ and $b \in \mathbb{A}$,

$$egin{align} \omega_{\psi}\left(inom{a}{a^{-1}}
ight)\phi(x) &= \chi_{V}(a)|a|^{m}arphi(ax) \ &\omega_{\psi}\left(inom{1}{b}
ight)\phi(x) &= \psi(bq(x))\phi(x) \ &\omega_{\psi}\left(inom{0}{1}
ight)\phi(x) &= \gamma\hat{\phi}(x) \ &\omega_{\psi}(g)\phi(x) &= \phi(g^{-1}x) \ \end{pmatrix}$$

for $g \in G(\mathbb{A})$

See Also

Meta

References

Citations and Footnotes

1. cf kudla_nltc_notation_rho_psi_S_unique_smooth_irreducible_representation_of_H_W_with_central_character_psi↔