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Let $K=\mathbb{Q}$ and let $\mathbb{A}=\mathbb{A}_{\mathbb{Q}}$. Let W be a symplectic space over $\mathbb{Q}$. Given W a complete polarization $\langle,\rangle$. Fix a symplectic basis $\{e_i,f_j\}$ such that $\langle e_i,e_j\rangle=\langle f_i,f_j\rangle=0$ and $\langle e_i,f_j\rangle=\delta_{ij}$. Write X for the subspace generated by the e_i and write Y for the subspace generated by the f_i so that W decompoess as $W=X\oplus Y$.

We alos have a decomposition $W(\mathbb{A})=X(\mathbb{A})\oplus Y(\mathbb{A})$. Let $H(W)$ be the Heisenberg group $H(W)=W\oplus\mathbb{Q}$

Fix an additive character $\psi:\mathbb{Q}\backslash\mathbb{A}\to\mathbb{C}^*$. There is a character $\rho_\psi: H(W)(\mathbb{A})\to\mathcal{S}(X(\mathbb{A}))$[1].

Recall, for $f\in\mathcal{S}(X(\mathbb{A}))$, $x_0\in X(\mathbb{A})$, $x\in X(\mathbb{A})$, $y\in Y(\mathbb{A})$, and $t\in\mathbb{A}$ that

$$\rho_\psi(x)f(x_0)=f(x+x_0)$$
$$\rho_\psi(y)f(x_0)=\psi(\langle x_0,y\rangle)f(x_0)$$
$$\rho_\psi(t)f(x_0)=\psi(t)f(x_0)$$

There is also a distribution

$$\Theta:\mathcal{S}(X(\mathbb{A}))\to\mathbb{C}$$

with

$$\Theta(\phi):=\sum_{x\in X(\mathbb{Q})}\phi(x).$$

It is invariant under the action of $H(W)(\mathbb{Q})$

Theorem

(Weil) This is the unique up to scalar distribution invariant under $H(W)(\mathbb{Q})$.

Example

Let $X(\mathbb{A})=\mathbb{A}$, take $\phi={\bigotimes_v}'\phi_v$ where $\phi_p=1_{\mathbb{Z}_p}$ and ϕ_∞ is a Schwartz function on $\mathbb{R}$, i.e. $\phi_\infty\subset C^\infty(\mathbb{R})$ such that $\sup_{x\in\mathbb{R}}(1+|x|^m)\phi_\infty^{(n)}(x)<\infty$. For instance, compactly supported functions ϕ_∞ are Schwartz functions, and $\phi_\infty=e^{-\pi x^2}$ is also a Schwartz function.

In this case, $\Theta(\phi)$ is

$$\Theta(\phi)=\sum_{x\in\mathbb{Q}}\phi_\infty(x)\prod_p\phi_p(x)=\sum_{x\in\mathbb{Z}}\phi_\infty(x)$$

The second equality is true because $\phi_p(x)=1_{\mathbb{Z}_p}(x)$ by definition.

Back to the talk

There is an action of $\mathrm{Sp}(W)$ on $H(W)$ (cf kudla_nltc_notation_S_p_W_symplectic_group_of_nondegenerate_symplectic_vector_space_over_nonarchimedean_local_field_of_dimension_not_2): $A(g)$ acts by sending $(\rho_\psi,\mathcal{S}(X)(\mathbb{A}))$ to $(\rho_\psi^g,\mathcal{S}(X)(\mathbb{A})$.

This action induces a map

$$\mathrm{Sp}(W)(\mathbb{A})\to\mathrm{PGL}(\mathcal{S}(X)(\mathbb{A})).$$

By pulling this map back via the projection map $\mathrm{GL}(\mathcal{S}(X)(\mathbb{A}))\to\mathrm{PGL}(\mathcal{S}(X)(\mathbb{A}))$, we get the metaplectic group $\widetilde{\mathrm{Sp}(W)}(\mathbb{A})$.

For all places v, we have a commuting diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{C}^\times \arrow[r] \arrow[d] & \widetilde{\mathrm{Sp}}(W)_{\mathbb{Q}_v} \arrow[r] \arrow[d] & \mathrm{Sp}(W)_{\mathbb{Q}_v} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{C}^\times \arrow[r] & \widetilde{\mathrm{Sp}}(W)_{\mathbb{A}} \arrow[r] & \mathrm{Sp}(W)_{\mathbb{A}} \arrow[r] & 1
\end{tikzcd}$$

Theorem

(Weil)
There exists a canonical splitting on $\mathrm{Sp}(W)(\mathbb{Q})$:

$$\begin{tikzcd}
 & \widetilde{\mathrm{Sp}}(W)_{\mathbb{A}} \arrow[d] \\
\mathrm{Sp}(\mathbb{Q}) \arrow[ur] \arrow[r, hook] & \mathrm{Sp}(W)(\mathbb{A})
\end{tikzcd}$$

Let $\widetilde{\mathrm{Sp}(W)}(\mathbb{Q})$ be the preimage of $\mathrm{Sp}(W)(\mathbb{Q})$ under the metaplectic cover.

Let ω_ψ be the Weil representation of $\widetilde{\mathrm{Sp}(W)}_{\mathbb{A}}$ acting on $\mathcal{S}(X(\mathbb{A}))$. Now for arbitrary $g \in \widetilde{\mathrm{Sp}(W)}(\mathbb{Q})$ and for all $\phi \in \mathcal{S}(X(\mathbb{A}))$.

$$\sum_{x \in X(\mathbb{Q})} (w_\psi(g)\phi)(x)$$

is another distribution on $\mathcal{S}(X(\mathbb{A}))$.

In fact, this distribution is also invariant under $H(W)(\mathbb{Q})$, and Weil's theorem tells us that there is a $\lambda_g \in \mathbb{C}^*$ relating this distribution to Θ. This gives us a character $\widetilde{\mathrm{Sp}(W)}(\mathbb{Q}) \to \mathbb{C}^*$.

For $g \in \widetilde{\mathrm{Sp}(W)}_{\mathbb{A}}$, $\phi \in \mathcal{S}(X(\mathbb{A}))$, let

$$\Theta(g, \phi) := \sum_{x \in X(\mathbb{Q})} (\omega_\psi(g)\phi)(x).$$

It is invariant under $\mathrm{Sp}(w)(\mathbb{Q})$. In particular, $\Theta(\cdot, \phi)$ gives a map

$$\mathrm{Sp}(W)(\mathbb{Q}) \backslash \widetilde{\mathrm{Sp}(W)}_{\mathbb{A}} \to \mathbb{C}$$

Reductive dual pairs

A Reductive dual pair is a pair (G, G') of subgroups of $Sp(W)$ such that G_1 and G_2 are reductive groups,

- $G =$ Centralizer of G' in $\mathrm{Sp}(W)$
- $G' =$ Centralizer of G in $\mathrm{Sp}(W)$

The example pair is the pair of the orthogonal group and the symplectic group:

Example

Let $V, (,)$ be a quadratic space. Let $W, \langle , \rangle$ be a symplectic space. Write

$$\mathbb{W} := V \otimes W$$

which is also symplectic; the bilinear form is given by the tensor product:

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = (v_1, v_2)\langle w_1, w_2 \rangle$$

We have a map

$$O(V) \times \mathrm{Sp}(W) \to \mathrm{Sp}(\mathbb{W})$$

Back to the talk

If (G_1, G_1') is in $\mathrm{Sp}(W_1)$ and (G_2, G_2') is in $\mathrm{Sp}(W_2)$, then $(G_1 \times G_2, G_1' \times G_2')$ is in $\mathrm{Sp}(W_1 \oplus W_2)$

There are two types of reductive dual pairs:

- Type I: GG' on W is irreducible
- Type II: reducible

Type I: For any base field K and division algebra D/K,

$V, (,), \tau((v_1, v_2)) = (v_2, v_1)$

$W, \langle , \rangle \ \tau(\langle w_1, w_2 \rangle) = -\langle w_2, w_1 \rangle$

Type II: Given W_1, W_2 over K and a division algebra D/K,

$$W = W_1 \otimes W_2 \oplus W_1^* \oplus W_2^*$$

Fix (G, G') a dual pair in $\mathrm{Sp}(W)$ over $\mathbb{Q}$.

Let $\tilde{G}_{\mathbb{A}}, \tilde{G}'_{\mathbb{A}}$ be the preimages of $G(\mathbb{A})$ and $G'(\mathbb{A})$ under the metaplectic covering

$$\widetilde{Sp}(W)_{\mathbb{A}} \to Sp(W)(\mathbb{A})$$

Now note that we have

$$\begin{tikzcd}
Sp(W)(\mathbb{Q})\backslash\widetilde{Sp}(W)_{\mathbb{A}} \arrow[r] & \mathbb{C} \\
\tilde{G}_{\mathbb{A}} \times \tilde{G}'_{\mathbb{A}} \arrow[u] \arrow[ur] & \\
(g, g') \arrow[u, phantom, sloped, "\in"] &
\end{tikzcd}$$

and then $\Theta(g, g', \phi)$ is called the Theta kernel

Theta lift for $\tilde{G}_{\mathbb{A}}$ to $\tilde{G}'_{\mathbb{A}}$

Let f be a (cuspidal) automorphic form on $G(\mathbb{Q})\backslash\tilde{G}_{\mathbb{A}}$

Let

$$\Theta(f, \phi)(g') = \int_{G(\mathbb{Q})\backslash\tilde{G}_{\mathbb{A}}} \Theta(g, g', \phi)\overline{f(g)}dg$$

Example

Fix $\psi : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$, let $\psi_\infty : \mathbb{R} \to \mathbb{C}^*$ be the usual exponential function $x \mapsto e^{2\pi i x}$, and let $\psi_p : \mathbb{Q}_p \to \mathbb{C}^\times$ be given by $x \mapsto e^{-2\pi i(x - \lfloor x \rfloor_p)}$

Let $(V, (,))$ be quadratic over $\mathbb{Q}$ with $\dim V = 2m$. Define the quadratic form $q(x) = \frac{1}{2}(x, x)$. Note that $(x, y) = q(x + y) - q(x) - q(y)$. Choose a $\mathbb{Z}$-lattice $\Lambda \subset V$. For almost all places p, the lattice $\Lambda_p = \Lambda \otimes_p \mathbb{Z}_p \subset V_p$ and $\Lambda_p^\perp = \Lambda_p$.

Let K_p be the stabilizer of Λ_p in $O(V)(\mathbb{Q}_p)$. It is maximum compact.

Write $K_f = \prod K_p \subset O(V)(\mathbb{A}_f)$, $K_\infty = O(V)(\mathbb{R})$, and $K = K_\infty K_f$.

Take $\mathbb{1}$ on $G(\mathbb{Q})\backslash G(\mathbb{A})$. We know that $G(\mathbb{Q})\backslash G(\mathbb{A})/K = \mathrm{gen}(\Lambda)$; the genus of a lattice means the equivalence class of lattices that are locally isomorphic everywhere.

Given $g = (g_p)_p$, we have the action $\Lambda \mapsto \bigcap_p (g_p\Lambda_p \cap V(\mathbb{Q}) = g(\Lambda)$

So $G(\mathbb{Q})\backslash G(\mathbb{A})$ acts on $\mathrm{gen}(\Lambda)$

Decompose $G(\mathbb{A}) = \coprod_i G(\mathbb{Q})g_i K$ and in particular,

$$\mathbb{1} = \bigcup_i \mathbb{1}_{G(\mathbb{Q})\backslash G(\mathbb{Q})g_i K}$$

Let us describe the action of $\mathrm{SL}_2(\mathbb{A})$ on $\mathcal{S}(V(\mathbb{A}))$:

For $a \in \mathbb{A}^\times$ and $b \in \mathbb{A}$,

$$\omega_\psi\left(\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}\right)\phi(x) = \chi_V(a)|a|^m\varphi(ax)$$

$$\omega_\psi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)\phi(x) = \psi(bq(x))\phi(x)$$

$$\omega_\psi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\phi(x) = \gamma\hat{\phi}(x)$$

$$\omega_\psi(g)\phi(x) = \phi(g^{-1}x)$$

for $g \in G(\mathbb{A})$

See Also

Meta

References

Citations and Footnotes

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1. cf kudla_nltc_notation_rho_psi_S_unique_smooth_irreducible_representation_of_H_W_with_central_character_psi↩