# Eisenstein series

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The Siegel-Weil formula is pretty concrete. The following is some related exposition.

Consider the sum of squares problem. For instance, $5$, $13$, $17$, and $29$ are all expressible as sums of two integers squares. However, $3, 7, 11, 19, 23$ are not.

Also, note that

$$(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2$$

**Theorem**

For $n > 0$, $n = x^2 + y^2$ for some integers $x, y$ if and only if prime divisor $p \equiv 3 \pmod 4$ of $n$ has even degree.

Write $r(n) = \#\{(x,y) \in \mathbb{Z}^2 : n = x^2 + y^2\}$

**Theorem**

(Jacobi)

$r(n) = 4\left(\sum_{\substack{d|n \\ d\equiv 1 \pmod 4}} 1 - \sum_{\substack{d|n \\ d\equiv 3 \pmod 4}} 1\right)$

For instance, if $p \equiv 1 \pmod 4$, then $r(p) = 8$ and if $p \equiv 3 \pmod 4$, then $r(p) = 0$.

Now define the **Jacobi theta function**

$$\theta(q) = \sum_{n\in\mathbb{Z}q^{n^2}} = 1 + 2q + 2q^4 + \cdots + 2q^{n^2} + \cdots$$

Then

$$\theta^2(q) = \sum_{n\in\mathbb{N}} \tau(n) q^n = 1 + 4q + \cdots$$

It turns out that $\theta^2(q) \in \mathcal{M}_1(\Gamma_1(4))$.

Now a very concrete fact about $\mathcal{M}_1(\Gamma_1(4))$ is that is it $1$-dimensional.

There is a way to concretely construct the modular forms of any weight: with the Eisenstein series.

Let $\chi : (\mathbb{Z}/4\mathbb{Z})^\times \to \mathbb{C}^\times$ be the nontrivial character.

Then let

$$E_k^\chi(\tau) = \sum_{\Gamma_\infty \backslash \Gamma_1(4)} \frac{\chi(d)}{(c\tau + d)^k}$$

where $\Gamma_\infty = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. When $k = 1$, while this may not be absolutely convergent, we still have $E_1^\chi(\tau) \in \mathcal{M}_1(\Gamma_1(4))$.

We have a Fourier expansion

$$E_k^\chi(\tau) = 1 + c_k^\chi \sum_{n \geq 1} \left( \sum_{d|n} \chi(d) d^{k-1} \right) q^n$$

Since $\mathcal{M}_1(\Gamma_1(4)) = 1$, $\theta^2(q)$ and $E_1^\chi(\tau)$ are equal up to a scalar. By comparing the Fourier coefficients of both modular forms, we have

$$4 = C_1^\chi$$

$$\tau(n) = 4 \left( \sum_{d|n} \chi(d) \right)$$

and thus Jacobi's theorem above follows.

# $\theta$-function

Let $V$ be a vector space over $\mathbb{Q}$ with quadratic form $Q : V \to \mathbb{Q}$. Write $H$ for the Heisenberg group. Fix an additive character $\psi : \mathbb{A} \to \mathbb{C}^\times$. Recall that there is a character $\psi_\infty e^{2\pi i x}$. Let $H$ be the Heisenberg group. There is an action of $G(\mathbb{A}) \times H(\mathbb{A})$ on $S(V(\mathbb{A}))$.

Let $\varphi \in S(V(\mathbb{A}))$. Write

$$\theta(g, h; \varphi) = \sum_{x \in V(\mathbb{Q})} \omega(g) \varphi(h^{-1} x)$$

Define the $\theta$-integral

$$\int_{H(\mathbb{Q}) \backslash H(\mathbb{A})} \theta(g, h; \varphi) dh$$

$V$ is a positive definite quadratic form space. Let $\Lambda \subset V$ be a lattice.
Let $\varphi_p = \mathrm{char}(\Lambda \otimes \mathbb{Z}_p)$ and $\varphi_\infty(x) = e^{-2\pi \mathbb{Q}[x]}$. Let $K \subset H(\mathbb{A})$ denote the stabilizer of $\Lambda$. Recall that there is a natural identfication

$$H(\mathbb{Q}) \backslash H(\mathbb{A}) / K \xrightarrow{\sim} \mathrm{Gen}(\Lambda)$$

where $\mathrm{Gen}(\Lambda)$ is the set of isomorphism classes of lattices with the same genus as $\Lambda$. The identification sends $h \in H(\mathbb{Q})\backslash H(\mathbb{A})/K$ to $h(\Lambda \otimes \hat{\mathbb{Z}}) \cap V$

Classically, it is known that $\mathrm{Gen}(\Lambda)$ is finite. Write $\{h_i\}$ for representatives of $H(\mathbb{Q})\backslash H(\mathbb{A})/K$ and $\{\Lambda_i\}$ for representatives of $\mathrm{Gen}(\Lambda)$

> Later on, we will use an example of $V = \mathbb{Q}^2$ where the quadratic form $Q$ is given by $Q = x^2 + y^2$ where $(x, y) \in V$.

Now we have

$$\begin{aligned}\int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \theta dh &= \sum_i \int_{H(\mathbb{Q})\backslash H(\mathbb{Q})/h_i K} \theta(g, h; \varphi) dh \\ &= \sum_i \int_{H(\mathbb{Q})\backslash H(\mathbb{Q}) h_i K h_i^{-1}} \theta(g, hh_i; \varphi) dh.\end{aligned}$$

We also have

$$H(\mathbb{Q})\backslash H(\mathbb{Q}) h_i K h_i^{-1} \simeq (H(\mathbb{Q}) \cap h_i K h_i^{-1})\backslash h_i K h_i^{-1}$$

Also, $(H(\mathbb{Q}) \cap h_i K h_i^{-1}) = \mathrm{Aut}(\Lambda_i)$

Now the summand

$$\int_{H(\mathbb{Q})\backslash H(\mathbb{Q}) h_i K h_i^{-1}} \theta(g, hh_i; \varphi) dh$$

from before equals

$$\begin{aligned}&= \frac{1}{\#\,\mathrm{Aut}(\Lambda_i)} \int_{h_i K h_i^{-1}} \theta(g, hh_i, \varphi) dh \\ &= \frac{1}{\#\,\mathrm{Aut}(\Lambda_i)} \int_K \theta(g, h_i h; \varphi) dh\end{aligned}$$

By strong approximation, write

$$g = g_i = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{\frac{1}{2}} & 0 \\ 0 & v^{-\frac{1}{2}} \end{pmatrix}$$

where $\tau = u + iv$.

Continuing on, we have

$$\int_K \theta(g_\tau, h_i h; \varphi) dh = \sum_K \sum_{x \in V(\mathbb{Q})} \omega(g_\tau) \varphi(h^{-1} h_i^{-1} x) dh$$

Now $h^{-1} h_i^{-1} x$ is in $\Lambda \otimes \mathbb{Z}_p$ if and only if $x \in h_i h \Lambda \otimes \mathbb{Z}_p = \Lambda_i \otimes \mathbb{Z}_p$. Thus, the above equals

$$\int_K \sum_{x\in\Lambda_i} \omega_\infty(g_i)\varphi_\infty(x)dh$$
$$= \mathrm{vol}(K) \sum_{x\in\Lambda_i} \omega_\infty(g_i)\varphi_\infty(x).$$

The summand $\omega_\infty(g_i)\varphi_\infty(x)$ equals

$$\omega_\infty(n(u)m(v^{\frac{1}{2}}))\varphi_\infty(x) = \psi(uQ(x))|v|^{\frac{m}{4}}\varphi_\infty(xv^{\frac{1}{2}})$$
$$= |v|^{\frac{m}{4}} e^{2\pi i u Q(x)} \cdot e^{2\pi v} Q(x) \qquad = |v|^{\frac{m}{4}} e^{2\pi i \tau Q(x)}$$

where $m = \dim V$, so going back, we have

$$\mathrm{vol}(K)|v|^{\frac{m}{4}} \sum_{x\in\Lambda_i} e^{2\pi i \tau Q(x)}.$$

# Example 1

Now let us take the example of $V = \mathbb{Q}^2$, $\mathbb{Z}^2 \subset V$, and $Q[(x, y)] = x^2 + y^2$.

We have the square of the theta function

$$\sum_{a,b\in\mathbb{Z}^2} q^{a^2+b^2} = \Theta^2(q)$$

# Example 2

When we have $V = \mathbb{Q}$, $Z \subset V$, and $Q[x] = x^2$, we have the $\theta$ function

$$\sum_{n\in\mathbb{Z}} q^{n^2} = \theta(q)$$

# Summary

So the work that was done in this section shows that Theta integral is closely related to the sum $\sum_{x\in\Lambda_i} e^{2\pi i \tau Q(x)}$, which specializes in the two examples above to $\theta^2(q)$ and $\theta(q)$

# Eisenstein series

**Definition**

$I(s, \chi) = \mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})} \chi_V| \cdot |^{s+1}$
where $B(\mathbb{A})$ is the Borel group

There is a principal series representation $\Theta(g, s)$. We have

$$\Theta(n(b)m(a)g, x) = \chi(a)|a|^{s+1|}\Phi(g)$$

[1]

# Example

There is a map $\lambda : S(V(\mathbb{A})) \to I(s_0, \chi_v)$ sending $\varphi$ to $\lambda(\varphi)$ where $\lambda(\varphi)(g) = (\omega(g)\varphi)(0)$.

Let us check that the conditions of $\lambda(\varphi)$ to be in the induced representation are satisfied:

$$\lambda 9\varphi)(n(b)m(a)g, s_0) = \omega(n(b)m(a)g)\varphi(0) = \chi_v(a)|a|^{s_0+1}\omega(g)\varphi(0).$$

Here, $\lambda(\varphi)(g)$ is $\omega(g)$???

# Back to the discussion

A section $\Phi(s)$ is called $\boxed{\textbf{standard}}$ if $\Phi(s)|_{K_\infty K}$ is constant.

We define the $\boxed{\textbf{Eisenstein series}}$

[

$$E(g, s; \Phi) = \sum -\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})\Phi(\gamma g, s)$$

]

In the case that $\Phi = \lambda(\varphi)$, we have a Siegel Eisenstein series.

Now we are not assuming that $V$ is a positive definite quadratic space.

**Theorem**

(Siegel-Weil formula)

$$\frac{\alpha}{2}\int_{O(\mathbb{Q})\backslash O(\mathbb{A})} \theta(g, h; \varphi)dh = E(g, s_0; \lambda(\varphi))$$

# Simon's comments on the local theta correspondence

Let $F$ be a $p$-adic local field. Let $\psi$ be an additive character on $F$. Let $W$ be a symplectic space. Let $V$ be an even orthogonal space. Let $\mathbb{W} = W \otimes V$. Let $G = \mathrm{Sp}(W)$. Let $H = O(V)$. Let $\omega$ be the Weil representation of $\tilde{S}_p(\mathbb{W})$.

We say that $\pi_G \in \mathrm{Irr}(G)$ and $\pi_H \in \mathrm{Irr}(H)$ correspond if $\mathrm{Hom}_{G\times H}(\omega, \pi_G \otimes \pi_H) \neq 0$.

**Theorem**

(Moeglin-Vigneras-Waldspurger for $p>2$; $p=2$ is really hard and has only been done more recently, Howe for $f=\mathbb{R}$)
This correspondence is a bijection is a bijection between a subset of $\mathrm{Irr}(G)$ and $\mathrm{Irr}(H)$

"A representation occurs in the Theta correspondence if it is a quotient of $\omega$. The local theta correspondence is a partial bijection between two sets of representations"

The correspondence is "1. take $\pi_G$. 2. Look for representation $\pi_H$ such that $\pi_G$ and $\pi_H$ correspond (i.e. such that $\mathrm{Hom}_{G\times H}(\omega, \pi_G \otimes \pi_H)) \neq 0$". And then this is apparently a bijection for some subset of $\mathrm{Irr}(G)$ and $\mathrm{Irr}(H)$

Write $\omega[\pi_G]$ for the maximal quotient of $\omega$ for which $G$ acts $\pi_G$-isotypically.

Then $\omega[\pi_G] \simeq \pi_G \otimes \Theta(\pi_a)$.

**Theorem**

$\Theta(\pi_G)$ has a unique irreducible quotient as a representation of $H$. The quotient is $\Theta(\pi_G)$

# Simon's comments on the global theta correspondence

We have an integral

$$\theta(g, \phi, \varphi) = \int_{[H]} \Theta(g, h, \varphi)\overline{\phi}(h)dh$$

One of the important things about this map is that the integral is $H$-invariant:
$\int_{[H]} \Theta(g, h, h'\varphi)h'\overline{\phi}(h)dh$

What this integral is doing is that it is giving an element of $\mathrm{Hom}_{(G\times H)(\mathbb{A})}(\omega \otimes \pi_H^\vee, \mathcal{A}(G))$ -- in particular, said element is equivariant for the $(G \times H)(\mathbb{A})$ actions. Also, this Hom space is isomorphic to $\mathrm{Hom}_{(G\times H)(\mathbb{A})}(\omega, \pi_H \otimes \mathcal{A}(G))$.

The image of the integral, which is in $\mathcal{A}(G)$, is apparently a quotient of $\Theta(\pi_H)$ by the local theta correspondence.

# How to relate Eisenstein series with classical Eisenstein series

Let $\chi$ be the trivial character. Take $\ell$ to be an even number. Write
$k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in K_\infty = \mathrm{SO}(2)$. Pick a function $\Phi_\infty^\ell$ so that

$$\Phi_\infty^\ell(gk_\theta) = e^{2\pi i\ell\theta}\Phi_\infty^\ell(g)$$

Writing $\tau = u + iv$, it suffices to understand the values at $g_\tau = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u^{\frac{1}{2}} & 0 \\ 0 & v^{-\frac{1}{2}} \end{pmatrix}$ (by something about strong approximation?).

Write $\Phi = \Phi_\infty^\ell \otimes \Phi_p$.

Also let

$$E(g_\tau, s, \Phi) = \sum \gamma \in B(\mathbb{Q} \backslash G(\mathbb{Q}) \Phi(\gamma g_\tau s)$$

We have
$B(\mathbb{Q}) \backslash G(\mathbb{Q}) \simeq \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z}) \simeq \mathbb{P}^1(\mathbb{Q}) \simeq \mathbb{P}^1(\mathbb{Z})$

**Lemma**

For $\gamma = \begin{pmatrix} a & b \\ \cdot & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q})$,
Write $\gamma g_\tau = g_{\gamma\tau} \cdot k_\theta$ where $k_\theta$ lies in the upper half plane. Then

$$e^{i\theta} = \frac{c\overline{\tau} + d}{|c\tau + d|}.$$

**Lemma**

$\mathrm{Im}(\gamma\tau) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2}$

**Lemma**

$\Phi_\infty^\ell(g_\tau, s) = \Phi_\infty^\ell\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{\frac{1}{2}} & 0 \\ 0 & v^{-\frac{1}{2}} \end{pmatrix}, s\right) = |v|^{\frac{1}{2}(s+1)}$

...

You do a bunch of stuff and recover the original Eisenstein series with $\ell = 1$.

# See Also

# Meta

## References

## Citations and Footnotes

1. $m(a)$ denotes $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ and $n(b)$ denotes $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ↩